Rota's Conjecture, the Missing Axiom, and Prime Cycles in Toric Varieties

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Abstract Rota's conjecture predicts that the coefficients of the characteristic polynomial of a matroid form a log-concave sequence. I will outline a proof for representable matroids using Milnor numbers and the Bergman fan. The same approach to the conjecture in the general case (for possibly non-representable matroids) leads to several intriguing questions on higher codimension algebraic cycles in toric varieties.

Attempting to solve the four color problem, Birkhoff introduced a polynomial associated to a graph which coherently encodes the answers to the analogous q-color problem for all natural numbers q [1]. This polynomial, called the chromatic polynomial, is a fundamental invariant of graphs. Any other numerical invariant of a simple graph which can be recursively computed by deletion and contraction of edges is a specialization of the chromatic polynomial.

In previous work [5] it is proved that the coefficients of the chromatic polynomial form a log-concave sequence for any graph, thus resolving a conjecture of Ronald Read [5]. An important step in the proof was to construct a complex algebraic variety associated to a graph and ask a more general question on the characteristic class of the algebraic variety. It turned out that the property of the characteristic class responsible for log-concavity is that it is *realizable*, meaning that the homology class is the class of an irreducible subvariety.

Proposition 1 ([5]) *Let ξ be a homology class in a product of projective spaces*

$$\xi = \sum_i d_i[\mathbb{P}^{k-i} \times \mathbb{P}^i] \in H_{2k}(\mathbb{P}^m \times \mathbb{P}^n; \mathbb{Z}).$$

Then some positive multiple of ξ is realizable if and only if $\{d_i\}$ form a log-concave sequence of nonnegative integers with no internal zeros.

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In general, for any complex projective variety X, one may define the space of prime cycles of X as a closed subset of $H_{2k}(X;\mathbb{R})$ which consists of limits of homology classes of irreducible k-dimensional subvarieties up to a positive multiple:

$$\mathcal{P}_k(X) := \overline{\{\xi \mid n\cdot\xi = [V] \text{ for some } n \in \mathbb{N} \text{ and } V \subseteq X\}} \subseteq H_{2k}(X;\mathbb{R}).$$

This subset showing asymptotic distribution of primes in the homology of X played a key role in the solution to the graph theory problem.

A motivating observation for further investigation is that, even for very simple toric varieties such as the one above, the orderly structure of the space of prime cycles becomes visible only after allowing positive multiples of homology classes. For example, there is no irreducible subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ which has the homology class

$$1[\mathbb{P}^5\times\mathbb{P}^0]+2[\mathbb{P}^4\times\mathbb{P}^1]+3[\mathbb{P}^3\times\mathbb{P}^2]+4[\mathbb{P}^2\times\mathbb{P}^3]+2[\mathbb{P}^1\times\mathbb{P}^4]+1[\mathbb{P}^0\times\mathbb{P}^5],$$

although $(1,2,3,4,2,1)$ is a log-concave sequence with no internal zeros [6]. One may expect that the same holds for the space of prime cycles of any smooth projective toric variety.

Conjecture 2 For any smooth projective toric variety X, the space of prime cycles $\mathcal{P}_k(X)$ is a closed semialgebraic subset of $H_{2k}(X;\mathbb{R})$.

Read's conjecture on graphs was later extended by Gian-Carlo Rota to combinatorial geometries, also called matroids, whose defining axioms are modeled on the relation of linear independence in a vector space. The above mentioned algebro-geometric proof does not work in this more general setting for one very interesting reason: not every matroid is realizable as a configuration of vectors in a vector space. Mathematicians since David Hilbert, who found a finite projective plane which is not coordinatizable over any field, have been interested in this tension between the axioms of combinatorial geometry and algebraic geometry. After numerous unsuccessful quests for the "missing axiom" which guarantees realizability, logicians found that one cannot add finitely many new axioms to matroid theory to resolve the tension [8, 9]. On the other hand, computer experiments revealed that numerical invariants of small matroids behave as if they were realizable, confirming Rota's conjecture in particular for all matroids within the range of our computational capabilities.

I believe that the discrepancy can be properly explained in the framework of algebraic geometry through a better understanding of the space of prime cycles. Here a matroid M of rank $k+1$ on $n+1$ elements can be viewed as a k-dimensional integral homology class Δ_M in the toric variety X_n constructed from the n-dimensional permutohedron. This homology class is the Bergman fan of M, and the realizability of M translates to the statement that the Bergman of M is realizable as an *integral* homology class of X_n. The toric variety X_n is equipped with a natural map

$$X_n \longrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

and the chromatic (characteristic) polynomials appear through the pushforward [7]

$$\Delta_M \longmapsto (\text{coefficients of the chromatic polynomial of } M).$$

Under this translation, Rota's log-concavity conjecture and its numerical evidences suggest an intriguing possibility that *any* matroid is realizable as a *real* homology class.

Conjecture 3 For any matroid M of rank $k+1$ on $n+1$ elements, we have

$$\Delta_M \in \mathcal{P}_k(X_n).$$

If true, this will not only prove the log-concavity conjecture but also explain the subtle discrepancy between combinatorial geometry and algebraic geometry.

Little is known on the space of prime cycles, even for very simple toric varieties such as X_n. What is needed as a first step in understanding the space of prime cycles is a systematic study of positivity of homology classes in toric varieties. More precisely, one needs to understand relations between the nef cone, the pseudoeffective cone, and the cone of movable cycles in a smooth projective toric variety X.

Question 4 (Nef Implies Effective) Is it true that the nef cone of k-dimensional cycles in X is contained in the pseudoeffective cone of k-dimensional cycles in X?

Question 5 (Toric Moving Lemma) Is it true that some positive multiple of a nef and effective cycle in X is homologous to an effective cycle intersecting all torus orbits of X properly?

Using the fact that the Bergman fan of a matroid spans an extremal ray of the nef cone of the toric variety X_n, one can show that affirmative answers to both questions for X_n implies Conjecture 3 for that n. An affirmative answer to the first question can be deduced from results of [3, 4]. The first question has a negative answer when X is not necessarily a toric variety [2].

In the special case when the dimension or the codimension is one, the above questions have an affirmative answer, and these statements form a basic part of the well-established theory relating the nef cone, the pseudoeffective cone, and the movable cone of a toric variety. Concrete results on cycles of higher dimension and codimension in toric varieties will guide the development of an analogous theory on positivity of algebraic cycles in more general algebraic varieties.

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