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Combinatorial Methods in Topology and Algebra

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Combinatorial Methods in Topology and Algebra

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Part II

Introduction

Bruno Benedetti, Emanuele Delucchi, and Luca Moci

Combinatorics and discrete geometry have been studied since the beginning of mathematics. Yet it is only in the last 50 years that combinatorics has flourished, with striking structural developments and a growing field of applications. Part of the reason for this blossoming may lie in the startling developments of computer science, which have taught us to look at mathematics with algorithmic eyes.

Moreover, many new connections between combinatorics and classical areas of mathematics, such as algebra and geometry, have emerged since the 70s. With no claim of completeness, let us provide (without references) five examples.

Hyperplane Arrangements A finite collection of linear one-codimensional subspaces in a complex vector space V is called an arrangement of hyperplanes. The intersection pattern of these hyperplanes gives rise to a rich combinatorial structure (see below under "Matroid") bearing a subtle relationship with the topology of the space obtained by removing the hyperplanes from V. Classical objects such as configuration spaces arise as special instances of these spaces which, in general, enjoy some nice topological properties—for instance, they are *minimal* (e.g., they

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have the homotopy type of CW complexes where, in every dimension, the number of cells equals the rank of the homology).

From the point of view of algebraic geometry, an arrangement is defined by a product $p(z)$ of degree-one homogeneous polynomials. One of the main topics of current research in this field is the study of the *Milnor fiber* $p^{-1}(1)$ of the arrangement.

Coxeter Groups A Coxeter group W is any group presented as

$$\langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \dots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \dots}_{m_{ij} \text{ terms}} \rangle$$

with $m_{ii} = 1$ and $2 \le m_{ij} \le \infty$ for all $i \neq j$, where $1 \le i, j \le n$ and $m_{ij} = \infty$ means that no condition on $s_i s_j$ is imposed. Symmetric groups and dihedral groups are of this type: indeed, the name of these groups reveals their origin in the study of regular polytopes by H.S.M. Coxeter. The combinatorics of Coxeter groups is very rich and deeply connected with the representation theory of Lie algebras, the algebraic geometry of flag varieties, and the topology of (real) reflection arrangements. With every pair of elements in W one can associate a *Kazhdan-Lusztig polynomial*. The coefficients of these polynomials are non-negative, and (when W is finite) they can be expressed in terms of the intersection cohomology Schubert varieties.

Removing the restriction $m_{ii} = 1$ from the presentation of a Coxeter group, we obtain the associated *Artin group*. A theorem of Deligne shows that the orbit space of the action of a finite-type Coxeter group on the complement of the hyperplane arrangement defined by (the complexification of) its reflection hyperplanes is a classifying space for the associated Artin group.

Matroids Matroids are certain types of set systems whose study was initiated by H. Whitney in the 1930s as an abstract common generalization of properties of linear algebra and graph theory. One possible definition is the following. A *matroid* on a finite set E is a nonempty collection $\mathfrak{B}$ of subsets of E such that the *exchange axiom* holds:

- If $A, B \in \mathfrak{B}$, for any $a \in A \setminus B$ there exists an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ is in $\mathfrak{B}$.

The elements of $\mathfrak{B}$ are called *bases*, and from this definition the connection to linear algebra should be apparent: if E is any finite subset of a $\mathbb{K}$-vector space V, the maximal linearly independent subsets of E form a matroid, which in this case is said to be *realizable over* $\mathbb{K}$. However, not all matroids arise in this way: some matroids are realizable only over some field, and some are not realizable over any field. Characterization of realizability is one of the main areas of research in matroid theory. The connection with graphs is as follows: if E is the edge set of a connected graph, the set of all edge sets of spanning trees satisfies the above definition and thus forms a matroid on E.

The interplay of matroid theory with algebraic geometry and commutative algebra has undergone thriving development in recent years, one of the main bridges being the language of *tropical geometry*.

Stanley–Reisner Ideals Let Δ be a simplicial complex on n vertices. The *Stanley–Reisner ideal* I_Δ is defined by

$$I_\Delta \stackrel{\text{def}}{=} \bigcap_{F \text{ facet of } \Delta} (x_i \,:\, i \notin F).$$

Since the ideals on the right-hand side are monomial and prime, I_Δ is monomial and radical. The uniqueness of prime decompositions of ideals implies that I_Δ determines Δ uniquely. Interestingly, every radical monomial ideal I is of the form I_Δ for a suitable complex Δ. This shows that there is a one-to-one correspondence between simplicial complexes (on n vertices) and radical monomial ideals (in n variables)—they are thus essentially the same thing. This allows transfer of properties back and forth between the two worlds: for example, one can characterize topologically the simplicial complexes Δ for which the ring S/I_Δ is Cohen–Macaulay. A well studied combinatorial property implying Cohen–Macaulayness is, for example, *shellability*.

Face Vectors of Polytopes Given a simplicial complex Δ, we denote by f_i the number of i-dimensional faces; by convention $f_{-1} = 1$. The f-*vector* of C is the vector $(f_{-1}, f_0, \ldots, f_d)$. The h-vector $(h_0, \ldots, h_d)$ is defined by the polynomial equality

$$\sum_{i=0}^{d} h_i X^{d-i} = \sum_{i=0}^{d} f_{i-1}(X-1)^{d-i}.$$

A natural question is: What integer vectors can arise as f-vectors of triangulated spheres?

When the sphere is the boundary of a polytope, the question was settled by the so-called g-theorem, proved by Billera-Lee and by Stanley in 1979 using commutative algebra and toric varieties and thereby giving a new stimulus to these fields. In the general case, progress was made by S. Murai, who in 2007 proved that a large family of shellable spheres, most of which are non-polytopal, satisfies the "Hard Lefschetz property", proved by Stanley for polytopes as a crucial step in his contribution to the g-theorem.

A consequence of the g-theorem is that h-numbers form a *unimodal* sequence, i.e., $h_0 \leq h_1 \leq \ldots \leq h_{\lfloor d/2 \rfloor} \geq \ldots \geq h_d$. Recently, Murai and Nevo proved the *generalized lower bound conjecture*, which claims the following: if in the h-vector of a d-polytope P one sees $h_{k-1} = h_k$ for some $k \leq d/2$, then there is a triangulation of P without simplices of dimension $\leq d-k$.

The present volume arises from a workshop on these interactions that was especially devoted to promoting outstanding young researchers. This INdAM Conference, entitled "Combinatorial Methods in Topology and Algebra" (or CoMeTA for short) took place in Cortona in September 2013. The detailed program of the conference is available at the website www.cometa2013.org.

About this Book

In the first part we have collected short surveys, which may be viewed as written and expanded versions of the talks given by the various speakers. Since the quality level of the lectures was very high, we believe that the inclusion of such material may be of great help for future studies. These surveys cover various topics:

(i) Hyperplane arrangements.
(ii) Matroids.
(iii) Polytopes and geometric combinatorics.
(iv) f-vectors of cellular complexes and triangulations.
(v) Combinatorial commutative algebra.
(vi) Coxeter groups and Kazhdan–Lusztig and Eulerian polynomials.
(vii) Combinatorial approaches to physics and analysis.

The second part consists of three peer-reviewed full research papers.

- The first sheds new light on positive sum systems. If $x_1, \dots, x_n$ are real numbers summing to zero, consider the family P^+ of all subsets $J \subseteq [n] := \{1, 2, \dots, n\}$ such that $\sum_{j \in J} x_j > 0$. Björner proves that the order complex of P^+, viewed as a poset under set containment, triangulates a shellable ball, whose f-vector depends only on n, and whose h-polynomial is the classical Eulerian polynomial.
- The second investigates an unexpected action by the group S_{n+1} on the minimal projective De Concini–Procesi model associated to the braid arrangements of type A_{n-1}. The action naturally arises from the fact that this model is isomorphic to the moduli space $\overline{M}_{0,n+1}$ of genus 0 stable curves with $n+1$ marked points.
- The third contribution focuses on Stanley's 1977 conjecture that the h-vectors of matroids are pure O-sequences. The conjecture is shown to hold in a few special cases, for example when the Cohen-Macaulay type is less than or equal to 3.

Part I

Extremal Graph Theory and Face Numbers of Flag Triangulations of Manifolds

Michał Adamaszek

Abstract We indicate how tools of extremal graph theory, mainly the stability method for Turán graphs, can be applied to derive upper bounds for face numbers of flag triangulations of spheres and manifolds.

1 Introduction

If $G = (V, E)$ is an arbitrary simple, finite, undirected graph, we denote by $\mathrm{Cl}(G)$ the *clique complex* of G, which is a simplicial complex defined as follows. The vertices of $\mathrm{Cl}(G)$ are the vertices of G and the faces of $\mathrm{Cl}(G)$ are those vertex sets which induce a clique (a complete subgraph) of G. The simplicial complexes which arise in this way are also known as *flag complexes*. This family includes for instance order complexes of posets and it appears in Gromov's theory of non-positive simplicial curvature [3].

A typical problem studied in enumerative combinatorics is to describe the f-vectors of interesting families of simplicial complexes. The f*-vector* $(f_0(K), \dots, f_d(K))$ of a d-dimensional complex K has as its ith entry, $f_i(K)$, the number of i-dimensional faces of K. The full classification of f-vectors of all flag complexes is probably impossible, although they are known to satisfy a number of non-trivial constraints [6]. We study this problem for the family of flag complexes which triangulate spheres and, more generally, homology manifolds and pseudomanifolds.

Note that G is the 1-skeleton of $\mathrm{Cl}(G)$. If we denote by $c_i(G)$ the number of cliques of cardinality i in G, then $f_i(\mathrm{Cl}(G)) = c_{i+1}(G)$, in particular $f_0(\mathrm{Cl}(G)) = |V(G)|$ and $f_1(\mathrm{Cl}(G)) = |E(G)|$. Our problem is thus equivalent to asking for the relations between clique numbers of graphs which satisfy some topological hypotheses.

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2 Upper Bounds

Let $K(n,s)$ denote the n-vertex balanced complete s-partite graph. It is uniquely determined by the requirement that the vertices can be split into s parts of sizes $\lfloor \frac{n}{s} \rfloor$ or $\lceil \frac{n}{s} \rceil$ each, with all possible edges between the parts and no edges within any part (see Fig. 1). By Turán's theorem this graph maximizes the number of edges among n-vertex graphs G which satisfy $c_{s+1}(G) = 0$.

Now let $J(n,s)$ be defined in the same way, except that we require each of the s parts to induce a cycle. We call this graph the *join* of s cycles of (almost) equal lengths. Note that $\mathrm{Cl}(J(n,s))$ is homeomorphic to the sphere S^{2s-1} (it is a topological join of s copies of S^1). The meta-statement we wish to advertise is that, provided n is large enough, $J(n,s)$ maximizes the number of edges among n-vertex graphs G for which $\mathrm{Cl}(G)$ is manifold-like in any reasonable sense.

More precisely, in [1] we show the following upper bound.

Theorem 1 *Let G be a graph with n vertices such that* $\mathrm{Cl}(G)$ *is a weak pseudomanifold of odd dimension* $d = 2s - 1$, *which satisfies the middle Dehn-Sommerville equation. If n is sufficiently large then we have*

$$|E(G)| \le \frac{s-1}{2s} n^2 + n \quad \big(\approx |E(J(n,s))| \big).$$

In particular, the conclusion holds when $\mathrm{Cl}(G)$ *is a* $(2s-1)$*-dimensional homology manifold.*

If $d = 3$ and $\mathrm{Cl}(G)$ is a homology 3-manifold the same bound was shown to hold for all n in [7]. A simplicial complex of dimension d is a *weak pseudomanifold* if every $(d-1)$-dimensional face belongs to exactly two facets. The graph G which achieves the upper bound of the theorem is $J(n,s)$. Moreover, one can expect this extremum to be stable, in the sense that the graphs for which $|E(G)|$ is close enough to the upper bound will be similar to $J(n,s)$. In [2] this was shown in dimension $d = 3$ in the form of the next theorem, which also contributes to the classification problem for f-vectors of flag 3-spheres.

Theorem 2 *Suppose G is a graph with n vertices such that* $\mathrm{Cl}(G)$ *is a* 3*-dimensional homology manifold and*

$$\frac{1}{4}n^2 + \frac{1}{2}n + \frac{17}{4} < |E(G)| \le \frac{1}{4}n^2 + n.$$

If n is sufficiently large then G is a join of two cycles of lengths $\frac{1}{2}n \pm O(\sqrt{n})$.

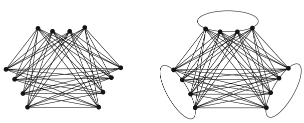

Fig. 1 Illustration for the definition of $K(n,s)$ (*left*) and $J(n,s)$ (*right*)

By Zykov's extension of Turán's theorem the graph $K(n, s)$ maximizes not only the number of edges, but in fact all clique numbers $c_2(G), \ldots, c_s(G)$ among graphs G with n vertices and $c_{s+1}(G) = 0$. It is likely that the methods used for Theorems 1 and 2 can be extended to prove an analogous statement about maximality of higher face numbers of $\mathrm{Cl}(J(n, s))$. At the time of writing it appears that the following conjecture (also stated in [8]) can be turned into a theorem.

Conjecture 3 If G is a graph with n vertices such that $\mathrm{Cl}(G)$ is a homology manifold of odd dimension $d = 2s - 1$ and n is sufficiently large, then

$$c_k(G) \le c_k(J(n, s))$$

for all $1 \le k \le s$.

3 Proofs

The technique we use to study dense flag manifold triangulations was developed in [2]. It relies on the similarity between the graphs $K(n, s)$ and $J(n, s)$ coupled with the special role played by Turán's graphs $K(n, s)$ in extremal graph theory. A typical application of this technique goes along the following lines.

1. Suppose G is a graph with n vertices such that $\mathrm{Cl}(G)$ is a homology manifold of dimension $d = 2s - 1$. In the first step we use the middle Dehn-Sommerville equation for $\mathrm{Cl}(G)$ to conclude that $c_{s+1}(G)$ is a linear combination of the numbers $1, c_1(G), \ldots, c_s(G)$. In particular $c_{s+1}(G) = O(n^s)$.
2. Now assume that G is as dense as $K(n, s)$, i.e. it has approximately $\frac{s-1}{2s}n^2$ or more edges. In a "typical" or "random" graph with this edge density a constant fraction of $(s + 1)$-tuples of vertices would span a clique. However, G has much fewer $(s+1)$-cliques, namely just $O(n^s)$. This has a consequence for the structure of G, which must be "similar" to $K(n, s)$, meaning that it can be obtained from $K(n, s)$ by adding or removing a relatively small number of edges (which can be $o(n^2)$ or even $O(n)$ depending on the specific problem). This step is known as the *stability method* in extremal graph theory, its origins going back to [5].
3. A graph which is similar to $K(n, s)$ is also similar to $J(n, s)$. The additional geometric properties of $\mathrm{Cl}(G)$, such as being a pseudomanifold or having homology spheres as face links, provide extra restrictions on the structure of G. They can now be used to rigidify G and conclude that it must be a join of cycles.

Let us mention that in the even dimensions $d = 2s$ this strategy is complicated by the fact that the extremal examples are (conjecturally) highly non-unique: one can take a join of $s - 1$ cycles with an *arbitrary* flag triangulation of S^2.

4 Conclusion

Much more is conjectured, than known, about the f-vectors of flag triangulations of spheres. In dimensions $d = 1$ and $d = 2$ they are easy to describe. In dimension $d = 3$ they can be classified up to, possibly, a finite number of exceptions, by the combined results of [2, 4, 7, 9]. The classification will be complete if Theorem 2 holds for all values of n, not just for sufficiently large ones. Note that already for $d = 3$ the classification relies on the deep result of Davis and Okun [4], namely the three-dimensional Charney-Davis conjecture. Full classification is available in dimension $d = 4$ (see [9]). In higher dimensions the only non-trivial restrictions known to hold are the upper bounds of Theorem 1. For more conjectures in this area see [7, 10].

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Combinatorial Stratifications and Minimality of Two-Arrangements

Karim A. Adiprasito

Abstract I present a result according to which the complement of any affine 2-arrangement in $\mathbb{R}^d$ is *minimal*, that is, it is homotopy equivalent to a cell complex with as many i-cells as its ith Betti number. To this end, we prove that the Björner–Ziegler complement complexes, induced by combinatorial stratifications of any essential 2-arrangement, admit perfect discrete Morse functions. This result extend previous work by Falk, Dimca–Papadima, Hattori, Randell, and Salvetti–Settepanella, among others.

A c-arrangement is a finite collection of distinct affine subspaces of $\mathbb{R}^d$, all of codimension c, with the property that the codimension of the non-empty intersection of any subset of $\mathfrak{A}$ is a multiple of c. For example, after identifying $\mathbb{C}$ with $\mathbb{R}^2$, any collection of hyperplanes in $\mathbb{C}^d$ can be viewed as a 2-arrangement in $\mathbb{R}^{2d}$. However, not all two-arrangements arise this way, cf. [10, Sect. III, 5.2] and [22]. In this paper, we study the complement $\mathfrak{A}^c := \mathbb{R}^d \setminus \mathfrak{A}$ of any 2-arrangement $\mathfrak{A}$ in $\mathbb{R}^d$.

Subspace arrangements $\mathfrak{A}$ and their complements $\mathfrak{A}^c$ have been extensively studied in several areas of mathematics. Thanks to the work by Goresky and MacPherson [10], the homology of $\mathfrak{A}^c$ is well understood; it is determined by the *intersection poset* of the arrangement, which is the set of all nonempty intersections of its elements, ordered by reverse inclusion. In fact, the intersection poset determines even the homotopy type of the compactification of $\mathfrak{A}$ [23]. On the other hand, it does not determine the homotopy type of the complement of $\mathfrak{A}^c$, even if we restrict ourselves to complex hyperplane arrangements [3, 16, 17], and understanding the homotopy type of $\mathfrak{A}^c$ remains challenging.

A standard approach to study the homotopy type of a topological space X is to find a *model* for it, that is, a CW complex homotopy equivalent to it. By cellular homology any model of a space X must use at least $\beta_i(X)$ i-cells for each i, where β_i is the ith (rational) Betti number. A natural question arises: Is the complement of an arrangement *minimal*, i.e., does it have a model with *exactly* $\beta_i(X)$ i-cells for all

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i? Studying minimality is not without its motivations; it appears, for instance, in the study of abelian covers of X [14].

Building on previous work by Hattori [12], Falk [8], and Cohen–Suciu [6], around 2000 Dimca–Papadima [7] and Randell [15] independently showed that the complement of any complex hyperplane arrangement is a minimal space. Roughly speaking, the idea is to consider the distance to a complex hyperplane in general position as a Morse function on the Milnor fiber to establish a Lefschetz-type hyperplane theorem for the complement of the arrangement. An elegant inductive argument completes their proof.

On the other hand, the complement of an arbitrary subspace arrangement is, in general, *not* minimal. In fact, complements of subspace arrangements might have arbitrary torsion in cohomology (cf. [10, Sect. III, Theorem. A]). This naturally leads to the following question:

Problem 1 (Minimality) Is the complement $\mathfrak{A}^{\mathrm{c}}$ of every c-arrangement $\mathfrak{A}$ minimal?

The interesting case is $c = 2$. In fact, if c is not 2, the complements of c-arrangements, and even c-arrangements of pseudospheres (cf. [5, Sects. 8 and 9]), are easily shown to be minimal. In 2007, Salvetti–Settepanella [19] proposed a combinatorial approach to Problem 1, based on Forman's discretization of Morse theory [9]. Discrete Morse functions are defined on regular CW complexes rather than on manifolds; instead of critical points, they have combinatorially-defined *critical faces*. Any discrete Morse function with c_i critical i-faces on a complex C yields a model for C with exactly c_i i-cells. Salvetti–Settepanella studied discrete Morse functions on the *Salvetti complexes* [18], which are models for complements of complexified real arrangements. Remarkably, they found that all Salvetti complexes admit *perfect* discrete Morse functions, that is, functions with exactly $\beta_i(\mathfrak{A}^{\mathrm{c}})$ critical i-faces. Formans's Theorem now yields the desired minimal models for $\mathfrak{A}^{\mathrm{c}}$.

This tactic does not extend to the generality of complex hyperplane arrangements. However, models for complex arrangements, and even for c-arrangements, have been introduced and studied by Björner and Ziegler [5]. In the case of complexified-real arrangements, their models contain the Salvetti complex as a special case. While our notion of the combinatorial stratification is slightly more restrictive than Björner–Ziegler's, it still includes most of the combinatorial stratifications studied in [5]. For example, we still recover the $\mathbf{s}^{(1)}$-stratification which gives rise to the Salvetti complex. With these tools at hand, we can tackle Problem 1 combinatorially:

Problem 2 (Optimality of Classical Models) Are there perfect discrete Morse functions on the Björner–Ziegler models for the complements of arbitrary two-arrangements?

We are motivated by the fact that discrete Morse theory provides a simple yet powerful tool to study stratified spaces. On the other hand, there are several difficulties to overcome. In fact, Problem 2 is more ambitious than Problem 1 in many respects:

- Few regular CW complexes, even among the minimal ones, admit perfect discrete Morse functions. For example, many 3-balls [4] and many contractible 2-complexes [21] are not collapsible.
- There are few results in the literature predicting the existence of perfect Morse functions. For example, it is not known whether any subdivision of the 4-simplex is collapsible, cf. [13, Problem. 5.5].
- Solving Problem 2 could help in obtaining a more explicit picture of the attaching maps for the minimal model; compare Salvetti–Settepanella [19] and Yoshinaga [20].

We answer both problems in the affirmative.

Theorem 3 ([1]) *Any complement complex of any* 2*-arrangement* $\mathfrak{A}$ *in* S^d *or* $\mathbb{R}^d$ *admits a perfect discrete Morse function.*

Corollary 4 ([1]) *The complement of any affine* 2*-arrangement in* $\mathbb{R}^d$*, and the complement of any* 2*-arrangement in* S^d*, is a minimal space.*

A crucial step on the way to the proof of Theorem 3 is the proof of a Lefschetz-type hyperplane theorem for the complements of two-arrangements. The lemma we actually need is a bit technical, but roughly speaking, the result can be phrased in the following way:

Theorem 5 ([1]) *Let* $\mathfrak{A}^c$ *denote the complement of any affine* 2*-arrangement* $\mathfrak{A}$ *in* $\mathbb{R}^d$*, and let* H *be any hyperplane in* $\mathbb{R}^d$ *in general position with respect to* $\mathfrak{A}$*. Then* $\mathfrak{A}^c$ *is homotopy equivalent to* $H \cap \mathfrak{A}^c$ *with finitely many* e*-cells attached, where* $e = \lceil d/2 \rceil = d - \lfloor d/2 \rfloor$.

An analogous theorem holds for complements of c-arrangements ($c \neq 2$, with $e = d - \lfloor d/c \rfloor$); it is an immediate consequence of the analogue of Corollary 4 for c-arrangements, $c \neq 2$. Theorem 5 extends a result on complex hyperplane arrangements, which follows the classical Lefschetz theorem, applied to the Milnor fiber [7, 11, 15]. The main ingredients to our study are:

- The formula to compute the homology of subspace arrangements in terms of the intersection lattice, due to Goresky and MacPherson [10].
- The study of combinatorial stratifications as initiated by Björner and Ziegler [5].
- The study of the collapsibility of complexes whose geometric realizations satisfy certain geometric constraints, as discussed previous work of Benedetti and Adiprasito, cf. [2].
- The idea of Alexander duality for Morse functions, in particular the elementary notion of "out-j collapse".
- The notion of (Poincaré) duality of discrete Morse functions, which goes back to Forman [9]. This is used to establish discrete Morse functions on complement complexes.

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Random Triangular Groups

Sylwia Antoniuk, Tomasz Łuczak, and Jacek Świątkowski

Abstract Let $\langle S|R\rangle$ denote a group presentation, where S is a set of n generators while R is a set of relations consisting of distinct cyclically reduced words of length three. The above presentation is called a triangular group presentation and the group it generates is called a triangular group. We study the following model $\Gamma(n,p)$ of a random triangular group. The set of relations R in $\Gamma(n,p)$ is chosen randomly, namely every relation is present in R independently with probability p. We study how certain properties of a random group $\Gamma(n,p)$ change with respect to the probability p. In particular, we show that there exist constants $c, C > 0$ such that if $p < \frac{c}{n^2}$, then a.a.s. a random group $\Gamma(n,p)$ is a free group and if $p > C\frac{\log n}{n^2}$, then a.a.s. this group has Kazhdan's property (T). What is more interesting, we show that there exist constants $c', C' > 0$ such that if $\frac{C'}{n} < p < c'\frac{\log n}{n^2}$, then a.a.s. a random group $\Gamma(n,p)$ is neither free, nor has Kazhdan's property (T). We prove the above statements using random graphs and random hypergraphs.

The notion of a random group goes back to Gromov [3], who studied groups given by random group presentations. Let $\langle S|R\rangle$ denote a group presentation with a set of generators S and a set of relations R consisting of distinct cyclically reduced words of length three, that is R consists of words of the form abc, where $a, b, c \in S \cup S^{-1}$ and $a \neq b^{-1}, b \neq c^{-1}, c \neq a^{-1}$. The above presentation is called a *triangular group presentation* and the group it generates is called a *triangular group*.

The subject of our interest is the following model of a random triangular group.

Definition 1 Let $\Gamma(n,p)$ denote a model of a random triangular group given by a random group presentation $\langle S|R\rangle$ with n generators, in which the set of relations R is chosen randomly in the following way: each cyclically reduced word of length three over the alphabet $S \cup S^{-1}$ is present in R independently with probability p.

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We are especially interested in the asymptotic properties of groups in the $\Gamma(n,p)$ model. In particular, we say that the random group $\Gamma(n,p)$, where $p = p(n)$ is some function of n, has a given property *asymptotically almost surely (a.a.s.)*, if the probability that $\Gamma(n,p)$ has this property tends to 1 as $n \to \infty$.

In [4] Żuk investigated threshold functions for specific and important properties of random triangular groups, such as Kazhdan's property (T), the property of being a free group or the property of being a trivial group. However, we should mention that Żuk studied a slightly different model of a random triangular group in which, rather than picking every relation independently, we choose uniformly at random the whole set of relations R among all the sets of prescribed size. Żuk's results stated for the $\Gamma(n,p)$ model read as follows.

Theorem 2 (Żuk [4]) *Let* $\epsilon > 0$.

1. *If* $p < n^{-2-\epsilon}$, *then a.a.s.* $\Gamma(n,p)$ *is a free group.*
2. *If* $n^{-2+\epsilon} < p < n^{-3/2-\epsilon}$, *then a.a.s.* $\Gamma(n,p)$ *is infinite, hyperbolic and has Kazhdan's property (T).*
3. *If* $p > n^{-3/2+\epsilon}$, *then a.a.s.* $\Gamma(n,p)$ *is trivial.*

In our work we managed to determine threshold functions more precisely than just up to the $n^{o(1)}$ factor. Our main results are captured in the following two theorems.

Theorem 3 (Antoniuk et al. [2])
There exist constants $c, c', C, C' > 0$, *such that:*

1. *If* $p < \frac{c}{n^2}$, *then a.a.s* $\Gamma(n,p)$ *is a free group.*
2. *If* $\frac{c'}{n^2} < p < \frac{C' \log n}{n^2}$, *then a.a.s.* $\Gamma(n,p)$ *is neither free, nor has Kazhdan's property (T).*
3. *If* $p > \frac{C \log n}{n^2}$, *then a.a.s.* $\Gamma(n,p)$ *has Kazhdan's property (T).*

Here, it is worth mentioning that we managed not only to improve bounds on the critical probability, but what is more interesting, we discovered a new period in the evolution of a random triangular group, in which a.a.s. this group is neither free, nor has Kazhdan's property (T).

Theorem 4 (Antoniuk et al. [1]) *There exists a constant* $C > 0$, *such that for* $p > Cn^{-3/2}$ *a.a.s.* $\Gamma(n,p)$ *is trivial.*

The proof of the first part of Theorem 3 relies on the fact that if p is small enough, then the expected number of relations in a random presentation $\langle S|R\rangle$ is also small and therefore we can find a generator $a \in S$ such that a and a^{-1} are present in at most one relation. Consequently, using Tietze movements we can eliminate generators from presentation one by one obtaining in the end a presentation without any relations.

For the second part of the proof of Theorem 3 we use the fact that if $p > \frac{c'}{n^2}$ for sufficiently large constant $c' > 0$, the presentation complex C_P of a random presentation $\langle S|R\rangle$ is a.a.s. aspherical and therefore it is the classifying space for

$\Gamma = \Gamma(n, p)$. Since it is easy to see that a.a.s. the Euler characteristic of Γ is non-positive, Γ cannot be a free group. On the other hand, if $p < \frac{C' \log n}{n^2}$ for sufficiently small constant $C' > 0$, then a.a.s. the presentation $\langle S|R\rangle$ contains a generator $a \in S$ such that neither a nor a^{-1} appears in any relation from R and one can show that a.a.s. Γ splits nontrivially as the free product $\Gamma = \langle a\rangle * \langle S \setminus \{a\}\rangle$. Thus a.a.s. Γ does not have property (T).

Finally, the third part of Theorem 3 follows from the result of Żuk who in [4] gave spectral conditions for property (T). Żuk's argument is based on the study of the spectrum of the Laplacian of an auxiliary graph called the *link graph*.

Definition 5 Let Γ be a group given by a presentation $\langle S|R\rangle$. The *link graph* $L = L_\Gamma$ is a graph whose vertices are elements from $S \cup S^{-1}$ and such that for every relation $abc \in R$ we place in L three edges: ab^{-1}, bc^{-1} and ca^{-1}.

Definition 6 Let G be a graph and $A = (a_{vw})_{v,w\in V}$ be its adjacency matrix. The *normalized Laplacian* of G is a symmetric matrix $\mathscr{L}(G) = (b_{vw})_{v,w\in V}$, where

$$b_{vw} = \begin{cases} 1, & \text{if } v = w \text{ and } d(v) > 0,\\ -a_{vw}/\sqrt{d(v)d(w)}, & \text{if } \{v, w\} \in G,\\ 0, & \text{otherwise.}\end{cases}$$

Here $d(v)$ denotes the degree of a vertex v.

Theorem 7 (Żuk [4]) *Let Γ be a group generated by a finite group presentation $\langle S|T\rangle$ and let $L = L_\Gamma$. Next, let $\lambda_1[\mathscr{L}(L)] \leq \lambda_2[\mathscr{L}(L)] \leq \cdots \leq \lambda_n[\mathscr{L}(L)]$ be the eigenvalues of the normalized Laplacian $\mathscr{L}(L)$. If L is connected and $\lambda_2[\mathscr{L}(L)] > 1/2$, then Γ has Kazhdan's property (T).*

Therefore, in order to prove the third part of Theorem 3, it is enough to show that if $p > \frac{C \log n}{n^2}$ for sufficiently large constant $C > 0$, then a.a.s. the second eigenvalue of $\mathscr{L}(L_\Gamma)$ is greater than $1/2$.

The proof of Theorem 4 involves the use of more advanced graph structures, i.e. random intersection graphs.

Definition 8 A *random intersection graph* $G_{n,m,\rho}$ is a graph with a set of vertices V of size n and a set of features W of size m such that for every vertex $v \in V$ and every feature $w \in W$, we assign feature w to vertex v independently with probability ρ. Two vertices $v_1, v_2 \in V$ are adjacent in $G_{n,m,\rho}$ if and only if they share a common feature $w \in W$.

Although the edges in the random intersection graph $G_{n,m,\rho}$ do not appear independently, it turns out that this graph behaves in a similar manner as the Erdős and Rényi random graph $G(n, \hat{p})$ with $\hat{p} \sim \rho^2 m$, in particular there exists a threshold function for the appearance of the giant component in the evolution of $G_{n,m,\rho}$.

In the proof of Theorem 4 we generate relations of the random presentation in two stages. The relations generated in the first stage are used to introduce an auxiliary random intersection graph H whose vertices are elements from $S \cup S^{-1}$

and two vertices a and b are adjacent if the corresponding elements from $S \cup S^{-1}$ are equal in Γ due to the presence of both relations acd and bcd in the presentation for some $c, d \in S \cup S^{-1}$. It turns out that for p large enough H a.a.s. contains a giant component; i.e. a large set of generators collapse to a single element. We then use relations generated in the second stage to show that a.a.s. all the elements from $S \cup S^{-1}$ collapse to the identity.

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Generalized Involution Models of Projective Reflection Groups

Fabrizio Caselli and Eric Marberg

Abstract The main motivation of this work was to investigate the generalized involution models of the projective reflection groups $G(r,p,q,n)$. This family of groups parametrizes all quotients of the complex reflection groups $G(r,p,n)$ by scalar subgroups. Our classification is ultimately incomplete, but we provide several necessary and sufficient conditions for generalized involution models to exist in various cases. In the process we have been led to consider and solve several intermediate problems concerning the structure of projective reflection groups. We derive a simple criterion for determining whether two groups $G(r,p,q,n)$ and $G(r,p',q',n)$ are isomorphic. We also describe explicitly the form of all automorphisms of $G(r,p,q,n)$, outside a finite list of exceptional cases. Building on prior work, this allows us to prove that $G(r,p,1,n)$ has a generalized involution model if and only if $G(r,p,1,n) \cong G(r,1,p,n)$. We also classify which groups $G(r,p,q,n)$ have generalized involution models when $n = 2$, or q is odd, or n is odd.

The details for this work are provided in our paper (F. Caselli and E. Marberg, Isomorphisms, automorphisms, and generalized involution models of projective reflection groups, Israel J. Math., 50 pp., in press).

A *model* for a finite group G is a set $\{\lambda_i : H_i \to \mathbb{C}\}$ of linear characters of subgroups of G, such that the sum of induced characters $\sum_i \mathrm{Ind}_{H_i}^G(\lambda_i)$ is equal to the multiplicity-free sum of all irreducible characters $\sum_{\psi \in \mathrm{Irr}(G)} \psi$. Models are interesting because they lead to interesting representations in which the irreducible representations of G live. This is especially the case when the subgroups H_i are taken to be the stabilizers of the orbits of some natural G-action.

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Example 1 Let $G = G(r,n)$ be the group of complex $n \times n$ matrices with exactly one nonzero entry, given by an rth root of unity, in each row and column. Assume r is odd. Then G acts on its symmetric elements by $g : X \mapsto gXg^T$, and the distinct orbits of this action are represented by the block diagonal matrices of the form

$$X_i \stackrel{\text{def}}{=} \begin{pmatrix} J_{2i} & 0 \\ 0 & I_{n-2i} \end{pmatrix},$$

where J_n denotes the $n \times n$ matrix with ones on the anti-diagonal and zeros elsewhere. Write H_i for the stabilizer of X_i in G. The elements of H_i preserve the standard copy of $\mathbb{C}^{2i}$ in $\mathbb{C}^n$, inducing a map $\pi_i : H_i \to \mathrm{GL}_{2i}(\mathbb{C})$. If $\lambda_i \stackrel{\text{def}}{=} \det \circ \pi_i$ then $\{\lambda_i : H_i \to \mathbb{C}\}$ is a model for $G(r,n)$ [2, Theorem 1.2].

The following definition of Bump and Ginzburg [5] captures the salient features of this example. Let ν be an automorphism of G with $\nu^2 = 1$. Then G acts on the set of *generalized involutions*

$$\mathscr{I}_{G,\nu} \stackrel{\text{def}}{=} \{\omega \in G : \omega^{-1} = \nu(\omega)\}$$

by the twisted conjugation $g : \omega \mapsto g \cdot \omega \cdot \nu(g)^{-1}$. We write

$$C_{G,\nu}(\omega) \stackrel{\text{def}}{=} \{g \in G : g \cdot \omega \cdot \nu(g)^{-1} = \omega\}$$

to denote the stabilizer of $\omega \in \mathscr{I}_{G,\nu}$ under this action, and say that a model $\{\lambda_i : H_i \to \mathbb{C}\}$ is a *generalized involution model* (or *GIM* for short) with respect to ν if each H_i is the stabilizer $C_{G,\nu}(\omega)$ of a generalized involution $\omega \in \mathscr{I}_{G,\nu}$, with each twisted conjugacy class in $\mathscr{I}_{G,\nu}$ contributing exactly one subgroup. The model in Example 1 is a GIM with respect to the inverse transpose automorphism of $G(r,n)$.

In [13, 14], the second author classified which finite complex reflection groups have GIMs. Subsequently, the first author discovered an interesting reformulation of this classification, which suggests that these results are most naturally interpreted in the broader context of *projective reflection groups*. These groups were introduced in [7] and further studied, for example, in [4, 6, 8]. They include as an important special case an infinite series of groups $G(r,p,q,n)$ defined as follows.

For positive integers r,p,n with p dividing r, let $G(r,p,n)$ denote the subgroup of $G(r,n)$ consisting of the matrices whose nonzero entries, multiplied together, form an (r/p)th root of unity. Apart from 34 exceptions, the irreducible finite complex reflection groups are all groups $G(r,p,n)$ of this kind. The projective reflection group $G(r,p,q,n)$ is defined as the quotient

$$G(r,p,q,n) \stackrel{\text{def}}{=} G(r,p,n)/C_q$$

where C_q is the cyclic subgroup of scalar $n \times n$ matrices of order q. Note that for this quotient to be well-defined we must have $C_q \subset G(r,p,n)$, which occurs precisely when q divides r and pq divides rn. Observe also that $G(r,n) = G(r,1,n)$ and $G(r,p,n) = G(r,p,1,n)$.

There is an interesting notion of duality for projective reflection groups; by definition, the projective reflection group *dual* to $G = G(r,p,q,n)$ is $G^* \stackrel{\text{def}}{=} G(r,q,p,n)$. This notion of duality has been crucial in the study of some aspects of the invariant theory of these groups in [7] and in the construction of other type of models in [6, 8]. The starting point of the present collaboration is now the following theorem which reformulates the main result of [13].

Theorem 1 *The complex reflection group $G = G(r,p,1,n)$ has a GIM if and only if $G \cong G^*$; i.e., if and only if $G(r,p,1,n) \cong G(r,1,p,n)$.*

Remark 1 Explicitly, G has a GIM if and only if (i) $n \neq 2$ and $\mathrm{GCD}(p,n) = 1$ or (ii) $n = 2$ and either p or r/p is odd; this is the statement of [13, Theorem 5.2].

Deducing this theorem from [13, Theorem 5.2] is straightforward, given our next main result. Let r, n be positive integers and let p, p', q, q' be positive divisors of r such that $pq = p'q'$ divides rn. The following result simplifies and extends [7, Theorem 4.4].

Theorem 2 *The projective reflection groups $G(r,p,q,n)$ and $G(r,p',q',n)$ are isomorphic if and only if either (i) $\mathrm{GCD}(p,n) = \mathrm{GCD}(p',n)$ and $\mathrm{GCD}(q,n) = \mathrm{GCD}(q',n)$ or (ii) $n = 2$ and the numbers $p + p'$ and $q + q'$ and $\frac{r}{pq}$ are all odd integers.*

As a corollary, we can say precisely when the group $G(r,p,q,n)$ is "self-dual" as in Theorem 1.

Corollary 1 *The projective reflection group $G = G(r,p,q,n)$ is isomorphic to its dual $G^* = G(r,q,p,n)$ if and only if either (i) $\mathrm{GCD}(p,n) = \mathrm{GCD}(q,n)$ or (ii) $n = 2$ and $\frac{r}{pq}$ is an odd integer.*

On seeing Theorem 1 one naturally asks whether for arbitrary projective reflection groups the property of having a GIM is equivalent to self-duality. Theorem 2 allows us to attack this question directly; its answer turns out to be false, and the rest of our results are devoted to clarifying which groups $G(r,p,q,n)$ have GIMs. The following theorem completely solves this problem in the often pathological case $n = 2$.

Theorem 3 *The projective reflection group $G(r,p,q,2)$ has a GIM if and only if $(r,p,q) = (4,1,2)$ or $G(r,p,q,2) \cong G(r,q,p,2)$.*

Remark 2 By Theorem 2, the condition $G(r,p,q,2) \cong G(r,q,p,2)$ holds if and only if (i) p and q have the same parity or (ii) $\frac{r}{pq}$ is an odd integer.

A few notable differences between complex reflection groups and projective reflection groups complicates the task of determining the existence of GIMs, and in the case $n \neq 2$ our classification is incomplete. For example, the groups $G(r,p,q,n)$ occasionally can have conjugacy class-preserving outer automorphisms. The fact that the groups $G(r,p,n)$ never have such automorphisms [15, Proposition 3.1] was the source of a significant reduction in the proof of [13, Theorem 5.1] which is no longer available in many cases of interest. Nevertheless, by carrying out a detailed analysis of the conjugacy classes and automorphisms of $G(r,p,q,n)$, we are able to prove the following theorem.

Theorem 4 *Let $G = G(r,p,q,n)$ and assume $n \neq 2$.*

(1) If $\mathrm{GCD}(p,n) = 1$ *then G has a GIM if q or n is odd.*
(2) If $\mathrm{GCD}(p,n) = 2$ *then G has a GIM only if q is even.*
(3) If $\mathrm{GCD}(p,n) = 3$ *then G has a GIM if and only if* (r,p,q,n) *is*

$$(3,3,3,3) \textit{ or } (6,3,3,3) \textit{ or } (6,6,3,3) \textit{ or } (6,3,6,3).$$

(4) If $\mathrm{GCD}(p,n) = 4$ *then G has a GIM only if* $r \equiv p \equiv q \equiv n \equiv 4 \pmod 8$.
(5) If $\mathrm{GCD}(p,n) \geq 5$ *then G does not have a GIM.*

In arriving at this result, we prove a useful criterion for determining conjugacy in $G(r,p,n)$ and give an explicit description of the automorphism group of $G(r,p,q,n)$. We note as a corollary that the theorem provides a complete classification when q or n is odd. This shows that projective reflection groups which are not self-dual may still possess GIMs.

Corollary 2 *Let $G = G(r,p,q,n)$ and assume $n \neq 2$ and (r,p,q,n) is not one of the four exceptions* $(3,3,3,3)$ *or* $(6,3,3,3)$ *or* $(6,6,3,3)$ *or* $(6,3,6,3)$. *If q or n is odd, then G has a GIM if and only if* $\mathrm{GCD}(p,n) = 1$.

Combining Theorems 2 and 4 shows that to completely determine which projective reflection groups $G(r,p,q,n)$ have GIMs, it remains only to consider groups of the form

$$G(2r,1,2q,2n) \quad \text{or} \quad G(2r,2,2q,2n) \quad \text{or} \quad G(8r+4,4,8q+4,8n+4).$$

(Of course we only need to consider the first two types when $2n > 2$.) We also have some conjectures concerning which of these groups should have GIMs.

This research continues a line of inquiry taken up by a number of people in the past few decades. Researchers originally considered *involution models*, which are simply GIMs defined with respect to the identity automorphism. Inglis et al. described an elegant involution model for the symmetric group in [9] (which is precisely the model in Example 1 when $r = 1$). In his doctoral thesis, Baddeley [3] classified which finite Weyl groups have involution models. Vinroot [16] extended this classification to show that the finite Coxeter groups with involution models are precisely those of type A_n, BC_n, D_{2n+1}, F_4, H_3, and $I_2(m)$. In extending this classification to reflection groups, it is natural to consider generalized involution models, since only groups whose representations are all realizable over the real numbers can possess involution models. Adin et al. [2] constructed a GIM for

$G(r,n)$ extending Inglis, Richardson, and Saxl's original model for S_n, which provides the starting point of [13, 14].

As mentioned at the outset, these sorts of classifications are interesting because they lead to interesting representations. We close our contribution with some recent evidence of this phenomenon. The model in Example 1 with $r = 1$ gives rise via induction to a representation of S_n on the vector space spanned by its involutions. This representation turns out to have a simple combinatorial definition [1, Sect. 1.1], which surprisingly makes sense *mutatis mutandis* for any Coxeter group. The generic Coxeter group representation we get in this way corresponds to an involution model (in the finite cases) in precisely types A_n, H_3, and $I_2(2m+1)$. What's more, recent work of Lusztig and Vogan [11, 12] and Vogan [10] indicates that this representation is the specialization of a Hecke algebra representation which for Weyl groups is expected to have deep connections to the unitary representations of real reductive groups.

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Universal Gröbner Bases for Maximal Minors of Matrices of Linear Forms

Aldo Conca

Abstract Bernstein, Sturmfels and Zelevinsky proved in 1993 that the maximal minors of a matrix of variables form a universal Gröbner basis. We present a very short proof of this result, along with a broad generalization to matrices with multi-homogeneous structures. Our main tool is a rigidity statement for radical Borel-fixed ideals in multigraded polynomial rings. For a more detailed exposition of the matter of this chapter we refer to the paper "Universal Gröbner bases for maximal minors" arXiv:1302.4461 written with Emanuela De Negri and Elisa Gorla.

1 From Algebra to Combinatorics and Back

Gröbner and Sagbi bases allow us to associate combinatorial objects (posets, matroids, simplical complexes, graphs, polytopes) to algebraic objects (ideals, modules, subalgebras) via term orders and deformations. Let us recall how.

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K, and let $>$ be a term order, that is, a total order on the set of monomials of S that is compatible with the semigroup structure of the set of monomials and such that 1 is the smallest monomial.

For every $0 \neq F \in S$ we set

$$\text{in}_>(F)\text{=the largest monomial in } F.$$

Then for every K-vector subspace $V \subseteq S$ we set

$$\text{in}_>(V) = K\text{-span}\{\text{in}_>(F) : F \in V, F \neq 0\}.$$

Since $>$ is compatible with the semigroup structure of the set of monomials of S one has $\text{in}_>(FG) = \text{in}_>(F)\,\text{in}_>(G)$ and hence $\text{in}_>(V)$ is an ideal of S if V is an ideal of S. Similarly $\text{in}_>(V)$ is a K-subalgebra of S if V is a K-subalgebra of S.

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By construction, $\text{in}_>(V)$ is generated by monomials and hence it is combinatorial object.

If I is an ideal, then polynomials $F_1, \dots, F_s \in I$ form a Gröbner basis of I with respect to $>$ if

$$\text{in}_>(I) = (\text{in}_>(F_1), \dots, \text{in}_>(F_s)).$$

Similarly, if A is a K-subalgebra of S, then polynomials $F_1, \dots, F_s \in A$ form a Sagbi basis of A with respect to $>$ if

$$\text{in}_>(A) = K[\text{in}_>(F_1), \dots, \text{in}_>(F_s)].$$

A set of polynomials $F_1, \dots, F_s \in I$ is a universal Gröbner basis of I if they are a Gröbner basis with respect to every term order $>$. Similarly one defines universal Sagbi bases. Hilbert's basis theorem ensures that every ideal of S posses a finite Gröbner basis. Unfortunately there is no Hilbert's basis theorem for subalgebras of S and there are finitely generated K-without a finite Sagbi basis, no matter what the term order is.

The study of invariants and properties of an ideal or K-subalgebra V of S can be done via deformation to monomial ideals/algebras in three steps:

(Step 1) Compute $\text{in}_>(V)$.

(Step 2) Prove that $\text{in}_>(V)$ has certain properties or invariants by looking at the associated combinatorial object.

(Step 3) Transfer back the information to the original ideal or algebra.

Usually (Step 1) and (Step 2) involve a lot of combinatorics. The typical example is Hodge straightening law for the ideal of Plücker relations defining the Grassmannian $G(m, n)$. It can be reformulated as a statement about the Plücker relations being a Gröbner basis. The associated initial ideal is attached to a distributive lattice as the Stanley-Reisner ideals of the corresponding order complex. On the other hand, for (Step 3) one knows that most of the homological properties and numerical invariants behave well user such a deformation.

Presenting the Grassmannian $G(m, n)$ as the subalgebra of the polynomial ring generated by the maximal minors of the generic matrix, one can apply the deformation strategy to it using Sagbi basis as well. In this way, the deformed object is a toric algebra, known as the Hibi ring, associated to the underlying distributive lattice.

2 Determinantal Ideals

Let $X = (x_{ij})$ be a $m \times n$ matrix with x_{ij} distinct variables over K. Assume $m \le n$. Set $S = K[x_{ij}]$. For $t \in \mathbb{N}$ the (generic) determinantal ideal is the ideal $I_t(X)$ generated by all the $t \times t$ subminors of X. These ideals appear in various contexts, e.g. classical invariant theory.

Theorem 1 (Sturmfels [5]) *The t-minors of X form a Gröbner basis of $I_t(X)$ with respect to a suitable term order (called diagonal term). The corresponding initial ideal* $\mathrm{in}_>(I_t(X))$ *is associated to a shellable simplicial complex.*

The main tools used are: the Knuth-Robinson-Schensted correspondence, the standard monomial theory of Rota-Doubilet-Stein and results of Björner on multiple chains complexes associated to planar distributive lattices.

What about universal Gröbner basis for these ideals? The ideal $I_2(X)$ of 2-minors defines Segre embedding of

$$\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n+m-1}.$$

Its universal Gröbner basis is given by the following:

Theorem 2 (Sturmfels [4] and Villarreal [7]) *A universal Gröbner basis of the ideal of 2-minos $I_2(X)$ is the set of the all binomials associated to cycles of the complete bipartite graph $K_{m,n}$.*

For example, the cycle of $K_{3,4}$ depicted in Fig. 1 corresponds to the binomial $x_{11}x_{24}x_{32} - x_{21}x_{34}x_{12}$. It turns out that, since $I_2(X)$ defines a toric algebra, one even has:

Proposition 3 *Every initial ideal of $I_2(X)$ defines a Cohen-Macaulay ring. More precisely, it is associated to a shellable simplicial complex.*

For maximal minors one has:

Theorem 4 (Sturmfels-Zelevinsky [6] and Bernstein-Zelevinsky [1]) *The set of the m-minors of X is a universal Gröbner basis of $I_m(X)$.*

Theorem 5 (Boocher [2]) *Every initial ideal of $I_m(X)$ defines a Cohen-Macaulay ring.*

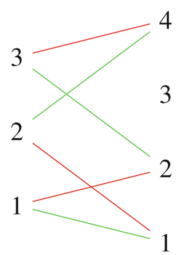

Fig. 1 A cycle of $K_{3,4}$

See also [3] for Theorem 4. Our first contribution is a simple specialization argument that explains both Theorems 4 and 5. The argument is based on the following

Lemma 6 *Let R be a standard graded K-algebra, let M, N, T be finitely generated graded modules R-modules, and $J = (y_1, \dots, y_s) \subseteq R$ be a homogeneous ideal. Suppose that:*

(1) There exists a surjective graded R-homomorphism $f : T \to N$.

(2) The modules M and N have the same Hilbert series.

(3) The modules M/JM and T/JT have the same Hilbert series.

(4) $y_1, \dots, y_s$ is M-regular sequence.

Then f is an isomorphism and $y_1, \dots, y_s$ is a T-regular sequence.

The idea is to use Lemma 6 in combination with the fact that $I_m(X)$ and every ideal generated by the initial terms of the maximal minors of X degenerate to the ideal of the square-free monomials of degree m of a set of n variables.

Our second contribution is the generalization of Theorems 4 and 5 to matrices of linear forms that are either column graded or row graded. We consider first the graded structure on $K[x_{ij}]$ induced by the column grading, i.e. $\deg x_{ij} = e_j \in \mathbb{Z}^n$ for all i, j. Then a matrix $L = (L_{ij})$ is said to be column graded if L_{ij} has degree e_j. Similarly for the row grading. For example,

$$L = \begin{pmatrix} x_{11} & 0 & x_{13} - 2x_{23} & -x_{24} \\ 0 & x_{12} + x_{22} & x_{23} & -x_{24} \end{pmatrix}$$

is column graded while

$$L = \begin{pmatrix} x_{11} & x_{11} + x_{12} & x_{11} - x_{12} & x_{14} \\ 0 & x_{21} & x_{21} + 4x_{24} & x_{24} \end{pmatrix}$$

is row graded. We prove the following

Theorem 7 *Assume $L = (L_{ij})$ is column-graded. Then:*

(a) The maximal minors of L form a universal Gröbner basis of $I_m(L)$.

(b) The ideal $I_m(L)$ is radical and it has a linear resolution.

(c) Every initial ideal J of $I_m(L)$ is radical and $\beta_{ij}(I_m(L)) = \beta_{ij}(J)$ for all i, j.

(d) The projective dimension of $I_m(L)$ (and hence of all its initial ideals) is $n - m$ unless $I_m(L) = 0$ or a column of $I_m(L)$ is identically 0.

For the row-graded case we have a weaker statement:

Theorem 8 *Assume $L = (L_{ij})$ is row-graded and assume that the codimension of $I_m(L)$ is $n - m + 1$ (the largest possible value). Then:*

(a) The ideal $I_m(L)$ has a universal Gröbner basis of elements of (total) degree m.

(b) The ideal $I_m(L)$ is radical.

(c) Every initial ideal J of $I_m(L)$ is radical, it has a linear resolution and defines a Cohen-Macaulay ring. In particular, $\beta_{ij}(I_m(L)) = \beta_{ij}(J)$ for all i, j.

In (a) of Theorem 8 we cannot expect that the maximal minors are a universal Gröbner basis since they might have all the same initial term. Experiments suggest that assumption on the codimension in Theorem 8 might be superfluous.

The main idea in the proof of Theorems 7 and 8 is the use of multigraded generic initial ideals and their "rigidity" property. These are ideals in a multigraded polynomial ring that are invariant under the action of the Borel subgroup of the upper triangular block matrices. They are obtained as Gröbner degeneration of any multigraded ideal in generic coordinates. They have a strong rigidity behaviour expressed by the following

Theorem 9 *Suppose that I and J are Borel fixed multigraded ideals with the same multigraded Hilbert function. If I is radical then $I = J$.*

Let us conclude by observing that no universal Gröbner basis of $I_t(X)$ is known for $2 < t < m$. But simple examples show that $I_t(X)$ has in general non-radical initial ideals and also non-Cohen-Macaulay initial ideals. For example,

Example 10 For the 3-minors of the generic 4×4 matrix consider the lexicographic term order $>$ associated to the total order:

$$x_{11}, x_{22}, x_{33}, x_{44}, x_{12}, x_{23}, x_{34}, x_{21}, x_{32}, x_{43}, x_{13}, x_{24}, x_{31}, x_{42}, x_{14}, x_{41}.$$

Then $\text{in}_>(I_3(X))$ has $x_{12}x_{23}x_{31}x_{44}^2$ among its minimal generators.

Example 11 For the 3-minors of the generic 4×5 matrix consider the lexicographic term order $>$ associated to the total order:

$$x_{11}, x_{22}, x_{33}, x_{44}, x_{12}, x_{23}, x_{34}, x_{45}, x_{21}, x_{32}, x_{43}, x_{13}, x_{24}, x_{35}, x_{31}, x_{42}, x_{14}, x_{25}, x_{41}, x_{15}.$$

Then $\text{in}_>(I_3(X))$ has $x_{12}x_{23}x_{31}x_{45}^2$ and $x_{12}x_{23}x_{31}x_{44}^2$ among its minimal generators and it does not define a Cohen-Macaulay ring.

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Torsion in the Homology of Milnor Fibers of Hyperplane Arrangements

Graham Denham and Alexander I. Suciu

Abstract As is well-known, the homology groups of the complement of a complex hyperplane arrangement are torsion-free. Nevertheless, as we showed in a recent paper (Denham and Suciu, Proc. Lond. Math. Soc. **108**(6), 1435–1470, 2014), the homology groups of the Milnor fiber of such an arrangement can have non-trivial integer torsion. We give here a brief account of the techniques that go into proving this result, outline some of its applications, and indicate some further questions that it brings to light.

This is a report on the main results of [2]. We give an outline of our approach and a summary of our conclusions. Our main result gives a construction of a family of projective hypersurfaces for which the Milnor fiber has torsion in homology. The hypersurfaces we use are hyperplane arrangements, for which techniques are available to examine the homology of finite cyclic covers quite explicitly, by reducing to rank 1 local systems.

The parameter spaces for rank 1 local systems with non-vanishing homology are known as characteristic varieties. In the special case of complex hyperplane arrangement complements, the combinatorial theory of multinets largely elucidates their structure, at least in degree 1. We make use of an iterated parallel connection construction to build arrangements with suitable characteristic varieties, then vary the characteristic of the field of definition in order to construct finite cyclic covers with torsion in first homology. These covers include the Milnor fiber. We now give some detail about each step.

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1 The Milnor Fibration

A classical construction due to J. Milnor associates to every homogeneous polynomial $f \in \mathbb{C}[z_1, \dots, z_\ell]$ a fiber bundle, with base space $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, total space the complement in $\mathbb{C}^\ell$ to the hypersurface defined by f, and projection map $f\colon \mathbb{C}^\ell \setminus f^{-1}(0) \to \mathbb{C}^*$.

The Milnor fiber $F = f^{-1}(1)$ has the homotopy type of a finite, $(\ell - 1)$-dimensional CW-complex, while the monodromy of the fibration, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n} z$, where n is the degree of f. If f has an isolated singularity at the origin, then F is homotopic to a bouquet of $(\ell - 1)$-spheres, whose number can be determined by algebraic means. In general, though, it is a rather hard problem to compute the homology groups of the Milnor fiber, even in the case when f completely factors into distinct linear forms: that is, when the hypersurface $\{f = 0\}$ is a hyperplane arrangement.

Building on our previous work with D. Cohen [1], we show there exist projective hypersurfaces (indeed, hyperplane arrangements) whose complements have torsion-free homology, but whose Milnor fibers have torsion in homology. Our main result can be summarized as follows.

Theorem 1 *For every prime $p \geq 2$, there is a hyperplane arrangement whose Milnor fiber has non-trivial p-torsion in homology.*

This resolves a problem posed by Randell [7, Problem 7], who conjectured that Milnor fibers of hyperplane arrangements have torsion-free homology. Our examples also give a refined answer to a question posed by Dimca and Némethi [3, Question 3.10]: torsion in homology may appear even if the hypersurface is defined by a reduced equation. We note the following consequence:

Corollary 2 *There are hyperplane arrangements whose Milnor fibers do not have a minimal cell structure.*

This stands in contrast with arrangement complements, which always admit perfect Morse functions. Our method also allows us to compute the homomorphism induced in homology by the monodromy, with coefficients in a field of characteristic not dividing the order of the monodromy.

It should be noted that our approach produces only examples of arrangements $\mathscr{A}$ for which the Milnor fiber $F(\mathscr{A})$ has torsion in qth homology, for some $q > 1$. This leaves open the following question.

Question 3 *Is there an arrangement $\mathscr{A}$ such that $H_1(F(\mathscr{A}), \mathbb{Z})$ has non-zero torsion?*

Since our methods rely on complete reducibility, it is also natural to ask: do there exist projective hypersurfaces of degree n for which the Milnor fiber has homology

p-torsion, where p divides n? If so, is there a hyperplane arrangement with this property?

A much-studied question in the subject is whether the Betti numbers of the Milnor fiber of an arrangement $\mathscr{A}$ are determined by the intersection lattice, $L(\mathscr{A})$. While we do not address this question directly, our result raises a related, and arguably even more subtle problem.

Question 4 *Is the torsion in the homology of the Milnor fiber of a hyperplane arrangement combinatorially determined?*

As a preliminary question, one may also ask: can one predict the existence of torsion in the homology of $F(\mathscr{A})$ simply by looking at $L(\mathscr{A})$? As it turns out, under fairly general assumptions, the answer is yes: if $L(\mathscr{A})$ satisfies certain very precise conditions, then automatically $H_*(F(\mathscr{A}), \mathbb{Z})$ will have non-zero torsion, in a combinatorially determined degree.

2 Hyperplane Arrangements

Let $\mathscr{A}$ be a (central) arrangement of n hyperplanes in $\mathbb{C}^\ell$, defined by a polynomial $Q(\mathscr{A}) = \prod_{H \in \mathscr{A}} f_H$, where each f_H is a linear form whose kernel is H. The starting point of our study is the well-known observation that the Milnor fiber of the arrangement, $F(\mathscr{A})$, is a cyclic, n-fold regular cover of the projectivized complement, $U(\mathscr{A})$; this cover is defined by the homomorphism $\delta\colon \pi_1(U(\mathscr{A})) \twoheadrightarrow \mathbb{Z}_n$, taking each meridian generator x_H to 1.

Now, if $\Bbbk$ is an algebraically closed field whose characteristic does not divide n, then $H_q(F(\mathscr{A}), \Bbbk)$ decomposes as a direct sum, $\bigoplus_\rho H_q(U(\mathscr{A}), \Bbbk_\rho)$, where the rank 1 local systems $\Bbbk_\rho$ are indexed by characters $\rho\colon \pi_1(U(\mathscr{A})) \to \Bbbk^*$ that factor through δ. Thus, if there is such a character ρ for which $H_q(U(\mathscr{A}), \Bbbk_\rho) \neq 0$, but there is no corresponding character in characteristic 0, then the group $H_q(F(\mathscr{A}), \mathbb{Z})$ will have non-trivial p-torsion.

To find such characters, we first consider multi-arrangements $(\mathscr{A}, m)$, with positive integer weights m_H attached to each hyperplane $H \in \mathscr{A}$. The corresponding Milnor fiber, $F(\mathscr{A}, m)$, is defined by the homomorphism $\delta_m\colon \pi_1(U(\mathscr{A})) \twoheadrightarrow \mathbb{Z}_N$, $x_H \mapsto m_H$, where N denotes the sum of the weights. Fix a prime p. Starting with an arrangement $\mathscr{A}$ supporting a suitable multinet, we find a deletion $\mathscr{A}' = \mathscr{A} \setminus \{H\}$, and a choice of multiplicities m' on $\mathscr{A}'$ such that $H_1(F(\mathscr{A}', m'), \mathbb{Z})$ has p-torsion. Finally, we construct a "polarized" arrangement $\mathscr{B} = \mathscr{A}' \| m'$, and show that $H_*(F(\mathscr{B}), \mathbb{Z})$ has p-torsion.

3 Characteristic Varieties

Our arguments depend on properties of the jump loci of rank 1 local systems. The *characteristic varieties* of a connected, finite CW-complex X are the subvarieties $\mathcal{V}^q_d(X, \Bbbk)$ of the character group $\widehat{G} = \mathrm{Hom}(G, \Bbbk^*)$, consisting of those characters ρ for which $H_q(X, \Bbbk_\rho)$ had dimension at least d.

Suppose $X^\chi \to X$ is a regular cover, defined by an epimorphism $\chi\colon G \to A$ to a finite abelian group, and if $\Bbbk$ is an algebraically closed field of characteristic p, where $p \nmid |\mathscr{A}|$, then $\dim H_q(X^\chi, \Bbbk) = \sum_{d \geq 1} \left|\mathrm{im}(\hat{\chi}_\Bbbk) \cap \mathcal{V}^q_d(X, \Bbbk)\right|$, where $\hat{\chi}_\Bbbk\colon \hat{A} \to \hat{G}$ is the induced morphism between character groups.

Theorem 5 *Let $X^\chi \to X$ be a regular, finite cyclic cover. Suppose that* $\mathrm{im}(\hat{\chi}_\mathbb{C}) \not\subseteq \mathcal{V}^q_1(X, \mathbb{C})$*, but* $\mathrm{im}(\hat{\chi}_\Bbbk) \subseteq \mathcal{V}^q_1(X, \Bbbk)$*, for some field $\Bbbk$ of characteristic p not dividing the order of the cover. Then $H_q(X^\chi, \mathbb{Z})$ has non-zero p-torsion.*

4 Multinets

In the case when $X = M(\mathscr{A})$ is the complement of a hyperplane arrangement, the positive-dimensional components of the characteristic variety $\mathcal{V}^1_1(X, \mathbb{C})$ have a combinatorial description, for which we refer in particular to work of Falk and Yuzvinsky in [5].

A *multinet* consists of a partition of $\mathscr{A}$ into at least three subsets $\mathscr{A}_1, \ldots, \mathscr{A}_k$, together with an assignment of multiplicities, $m\colon \mathscr{A} \to \mathbb{N}$, and a subset $\mathscr{X}$ of the rank 2 flats, such that any two hyperplanes from different parts intersect at a flat in $\mathscr{X}$, and several technical conditions are satisfied: for instance, the sum of the multiplicities over each part $\mathscr{A}_i$ is constant, and for each flat $Z \in \mathscr{X}$, the sum $n_Z := \sum_{H \in \mathscr{A}_i : H \supset Z} m_H$ is independent of i. Each multinet gives rise to an orbifold fibration $X \to \mathbb{P}^1 \setminus \{k \text{ points}\}$; in turn, such a map yields by pullback an irreducible component of $\mathcal{V}^1_1(X, \mathbb{C})$.

We say that a multinet on $\mathscr{A}$ is *pointed* if for some hyperplane H, we have $m_H > 1$ and $m_H \mid n_Z$ for each flat $Z \subset H$ in $\mathscr{X}$. We show that the complement of the deletion $\mathscr{A}' := \mathscr{A} \setminus \{H\}$ supports an orbifold fibration with base $\mathbb{C}^*$ and at least one multiple fiber. Consequently, for any prime $p \mid m_H$, and any sufficiently large integer r not divisible by p, there exists a regular, r-fold cyclic cover $Y \to U(\mathscr{A}')$ such that $H_1(Y, \mathbb{Z})$ has p-torsion.

Furthermore, we also show that any finite cyclic cover of an arrangement complement is dominated by a Milnor fiber corresponding to a suitable choice of multiplicities. Putting things together, we obtain the following result.

Theorem 6 *Suppose $\mathscr{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity vector m. Let p be a prime dividing m_H. There is then a choice of multiplicity vector m' on the deletion $\mathscr{A}' = \mathscr{A} \setminus \{H\}$ such that $H_1(F(\mathscr{A}', m'), \mathbb{Z})$ has non-zero p-torsion.*

For instance, if $\mathscr{A}$ is the reflection arrangement of type B_3, defined by the polynomial $Q = xyz(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$, then $\mathscr{A}$ satisfies the conditions of the theorem, for $m = (2, 2, 2, 1, 1, 1, 1, 1, 1)$ and $H = \{z = 0\}$. Choosing then multiplicities $m' = (2, 1, 3, 3, 2, 2, 1, 1)$ on $\mathscr{A}'$ shows that $H_1(F(\mathscr{A}', m'), \mathbb{Z})$ has non-zero 2-torsion.

Similarly, for primes $p > 2$, we use the fact that the reflection arrangement of the full monomial complex reflection group, $\mathscr{A}(p, 1, 3)$, admits a pointed multinet. This yields p-torsion in the first homology of the Milnor fiber of a suitable multi-arrangement on the deletion.

5 Parallel Connections and Polarizations

The last step of our construction replaces multi-arrangements with simple arrangements. We add more hyperplanes and increase the rank by means of suitable iterated parallel connections. The complement of the parallel connection of two arrangements is diffeomorphic to the product of the respective complements, by a result of Falk and Proudfoot [4]. Then the characteristic varieties of the result are given by a formula due to Papadima and Suciu [6].

We organize the process by noting that parallel connection of matroids has an operad structure, and we analyze a special case which we call the *polarization* of a multi-arrangement $(\mathscr{A}, m)$. By analogy with a construction involving monomial ideals, we use parallel connection to attach to each hyperplane H the central arrangement of m_H lines in $\mathbb{C}^2$, to obtain a simple arrangement we denote by $\mathscr{A} \| m$. A crucial point here is the connection between the respective Milnor fibers: the pullback of the cover $F(\mathscr{A} \| m) \to U(\mathscr{A} \| m)$ along the canonical inclusion $U(\mathscr{A}) \to U(\mathscr{A} \| m)$ is equivalent to the cover $F(\mathscr{A}, m) \to U(\mathscr{A})$. Using this fact, we prove the following.

Theorem 7 *Suppose $\mathscr{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathscr{A}' = \mathscr{A} \setminus \{H\}$ such that the Milnor fiber of the polarization $\mathscr{A}' \| m'$ has p-torsion in homology, in degree* $1 + \left|\{K \in \mathscr{A}' : m'_K \geq 3\}\right|$.

For instance, if $\mathscr{A}'$ is the deleted B_3 arrangement as above, then choosing $m' = (8, 1, 3, 3, 5, 5, 1, 1)$ produces an arrangement $\mathscr{B} = \mathscr{A}' \| m'$ of 27 hyperplanes in $\mathbb{C}^8$, such that $H_6(F(\mathscr{B}), \mathbb{Z})$ has 2-torsion of rank 108.

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Decompositions of Betti Diagrams of Powers of Monomial Ideals: A Stability Conjecture

Alexander Engström

Abstract For any fixed monomial ideals the resolution of high enough powers are predictable. To actually gain explicit information about the stable behavior of projective resolutions of high powers is rather non-trivial if the ideals aren't particularly well behaved. We describe how the asymptotic decomposition of Betti tables of high powers can be conjecturally described using polytopes as a new invariant for the stable regime.

1 The Conjecture

We state a conjecture on the stability of Betti diagrams of powers of monomial ideals. Boij and Söderberg [1] conjectured that Betti diagrams can be decomposed into pure diagrams, and that was proved by Eisenbud and Schreyer [2]. We don't cover the basics of Boij-Söderberg theory here, see Fløystad [4] for a survey.

Up to scaling a pure diagram is determined by its non-zero positions. In our setting the top left corner is always non-zero and we normalise by assigning it the value one. For higher powers of ideals we need taller pure diagrams in a sequential way. A *translation of a pure diagram* for $k = 0, 1, 2, \ldots$ is a sequence of pure diagrams on the form

$$\pi(k) = \begin{array}{r|llll} & 0 & 1 & 2 & \cdots \\ \hline 0 & 1 & & & \\ \vdots & & & & \\ l(k) & & \multicolumn{3}{l}{\text{A fixed shape for}} \\ & & \multicolumn{3}{l}{\text{non-zero entries}} \end{array}$$

where $l(k)$ is a linear function.

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According to Boij-Söderberg theory there is for every ideal I in S a decomposition of the Betti diagram $\beta(S/I) = w_1\pi_1 + \cdots + w_m\pi_m$ where each w_i is a non-negative real number and each π_i is a pure diagram. Usually there are many choices of weights and when considering algorithms to find decompositions there is a point to finding a particular one. But the amount of choices is also a measure of the complexity of the Betti table, and one might notice that for ideals that we know the invariants of for large powers, the complexity in this sense is quite low. For example, if all powers are linear, then there are no choices at all.

For any $\beta(S/I)$ there is a finite set of pure diagrams that can be included in a decomposition with a positive weights. We call the set of possible weight vectors for $\beta(S/I)$ the *polytope of Betti diagram decompositions*.

We conjecture that for high powers the polytope of Betti diagram decompositions stabilises.

Conjecture 1 *Let I be a monomial ideal in S with all generators of the same degree. Then there is a k_0 such that for all $k > k_0$,*

1. *For some translations of pure diagrams $\pi_1(k), \ldots, \pi_m(k)$ any decomposition of $\beta(S/I^k)$ is a weighted sum of the form $w_1\pi_1(k) + \cdots + w_m\pi_m(k)$. Denote the polytope of Betti diagram decompositions of $\beta(S/I^k)$ in $\mathbb{R}^m$ by P_k.*
2. *All P_k are of the same combinatorial type as a polytope P_I. For any vertex v of P_I there is a function $h_v(k) \in \mathbb{R}^m$, which is rational in each coordinate, such that the vertex corresponding to v in P_k is $h_v(k)$.*

The conjecture is true for ideals whose large enough powers are all linear: The polytope is a point. This follows from the fact that the column sums in Betti diagrams stabilise to polynomials for large powers according to Kodiyalam [5], and from the procedure to derive the unique decomposition of linear diagrams. The conjecture holds for many small examples that the author has calculated. There is unfortunately no abundance of ideals in the literature for which the Betti diagrams of all powers are given explicitly, since these concepts are fairly new. But there are many interesting tools accessible, for example from algebraic and topological combinatorics, that should make serious attempts to derive them fruitful.

2 An Example

In this section we give an example of an ideal satisfying the conjecture. Engström and Norén [3] constructed explicit cellular minimal resolutions of S/I^k for all k and n, where

$$S = \mathbf{k}[x_1, x_2, \ldots, x_n] \text{ and } I = \langle x_1x_2, x_2x_3, \ldots, x_{n-1}x_n \rangle,$$

and calculated the Betti numbers:

$$\beta_{i,j}(S/I^k) = \binom{n+3k-j-2}{2j-3i-3k+3}\binom{n+4k+2i-2j-4}{2k+2i-j-2}\binom{j-i-k}{k-1}.$$

The Betti diagram of $\mathbf{k}[x_1,x_2,x_3,x_4,x_5,x_6]/\langle x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_6\rangle^k$ is

	0	1	2	3	4	5
0	1					
$\vdots$						
$2k-1$		$\binom{k+4}{4}$	$4\binom{k+3}{4}$	$6\binom{k+2}{4}$	$4\binom{k+1}{4}$	$\binom{k}{4}$
			$k(k+2)$	$2k(k+1)$	k^2	

The translations of pure diagrams:

$\pi_1(k) =$

	0	1	2	3
0	1			
$\vdots$				
$2k-1$		$*$	$*$	$*$

$\pi_2(k) =$

	0	1	2	3
0	1			
$\vdots$				
$2k-1$		$*$	$*$	
				$*$

$\pi_3(k) =$

	0	1	2	3
0	1			
$\vdots$				
$2k-1$		$*$		
			$*$	$*$

$\pi_4(k) =$

	0	1	2	3	4
0	1				
$\vdots$					
$2k-1$		$*$	$*$	$*$	$*$

$\pi_5(k) =$

	0	1	2	3	4
0	1				
$\vdots$					
$2k-1$		$*$	$*$	$*$	
					$*$

$\pi_6(k) =$

	0	1	2	3	4
0	1				
$\vdots$					
$2k-1$		$*$	$*$		
				$*$	$*$

$\pi_7(k) =$

	0	1	2	3	4
0	1				
$\vdots$					
$2k-1$		$*$			
			$*$	$*$	$*$

$\pi_8(k) =$

	0	1	2	3	4	5
0	1					
$\vdots$						
$2k-1$		$*$	$*$	$*$	$*$	$*$

The polytope P_k is a triangle whose vertices have the coordinates $h_1(k)$, $h_2(k)$ and $h_3(k)$.

$$h_1(k) = \left(0, 0, \frac{k+2}{2k+3}, w_4(k), \frac{(2k+5)(k-1)}{(2k+1)(k+2)(k+1)}, \frac{(4k+5)(k+1)}{(2k+3)(2k+1)(k+2)}, 0, w_8(k)\right)$$

$$h_2(k) = \left(0, \frac{2(k+2)(k+2)}{(2k+3)(2k+1)}, 0, w_4(k), \frac{(2k+5)(k-1)}{(2k+1)(k+2)(k+1)}, \frac{(k+1)(k-1)}{(2k+3)(2k+1)(k+2)}, \frac{1}{2k+3}, w_8(k)\right)$$

$$h_3(k) = \left(\frac{k+2}{2k+1}, 0, 0, w_4(k), \frac{k^2-7}{(2k+1)(k+2)(k+1)}, \frac{k+1}{(2k+3)(2k+1)(k+2)}, \frac{1}{2k+3}, w_8(k)\right)$$

$$w_4(k) = \frac{(7k+5)(k-1)(k-2)}{4(2k+3)(2k+1)(k+1)}$$

$$w_8(k) = \frac{(k-1)(k-2)(k-3)}{4(2k+3)(2k+1)(k+1)}.$$

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Matroids Over a Ring

Alex Fink and Luca Moci

Abstract We introduce the notion of a matroid M over a commutative ring R, assigning to every subset of the ground set an R-module according to some axioms. When R is a field, we recover matroids. When $R = \mathbb{Z}$, and when R is a DVR, we get (structures which contain all the data of) quasi-arithmetic matroids, and valuated matroids i.e. tropical linear spaces, respectively.

More generally, whenever R is a Dedekind domain, we extend all the usual properties and operations holding for matroids (e.g., duality), and we explicitly describe the structure of the matroids over R. Furthermore, we compute the Tutte-Grothendieck ring of matroids over R. We also show that the Tutte quasi-polynomial of a matroid over $\mathbb{Z}$ can be obtained as an evaluation of the class of the matroid in the Tutte-Grothendieck ring.

1 Introduction

Matroid theory has proved to be a versatile language to deal with many problems on the interfaces of combinatorics and algebra. Following the introduction of matroids, a number of enriched variants thereof have arisen, among them oriented matroids [2], valuated matroids [7], complex matroids [1], and (quasi-)arithmetic matroids [6, 10]. Each of these structures retains some information about a vector configuration, or an equivalent object, which is richer than the purely linear algebraic information that matroids retain.

It is natural to ask how well these generalizations of matroids can be unified under one framework. One such framework was proposed in [7]. In this paper we suggest a different approach, defining the notion of a *matroid M over a commutative ring R* (Definition 1). We find this definition to have multiple agreeable features. For one, by building on the well-studied setting of modules over commutative rings, we

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get a theory where the considerable power and development of commutative algebra can be easily brought to bear. For another, unlike arithmetic and valuated matroids, a matroid over R is not defined as a matroid decorated with extra data; there are only two axioms, and we suggest that they are comparably simple to the matroid axioms themselves. In particular, a *representable* matroid over R is precisely a vector configuration in a finitely generated R-module.

When R is a field, a matroid M over R is nothing but a matroid: the data $M(A)$ is a vector space, which contains only the information of its dimension, and this directly encodes the rank function of M. When $R = \mathbb{Z}$, every module $M(A)$ is an abelian group, and by extracting its torsion subgroup we get a quasi-arithmetic matroid. When R is a discrete valuation ring (DVR), we may similarly extract a valuated matroid, which corresponds to a tropical linear space. More generally, whenever R is a Dedekind domain, we can extend the usual properties and operations holding for matroids, such as duality.

One of the most-loved invariants of matroids is their Tutte polynomial $\mathbf{T}_M(x, y)$. It thus comes as no surprise that the Tutte polynomial has been considered for generalizations of matroids as well. A quasi-arithmetic matroid $\hat{M}$ has an associated arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x, y)$; this is a specialization of the *Tutte quasi-polynomial* of a quasi-arithmetic matroid defined in [3].

Among its properties, the Tutte polynomial of a classical matroid is the universal deletion-contraction invariant. In more algebraic language, following [4], the class of a matroid in the Tutte-Grothendieck group for deletion-contraction relations is exactly its Tutte polynomial. Our generalization of the Tutte polynomial for matroids over a Dedekind ring R is also the class in the Tutte-Grothendieck group, so it retains the universality of the usual Tutte polynomial, and we obtain the two generalizations of Tutte just mentioned as evaluations of it.

This paper is a survey on the theory developed in [8], to which the interested reader is encouraged to refer for material that has been omitted here.

2 Matroids Over a Ring

Throughout, we let R be a Dedekind domain. Though not strictly necessary for the theory, this assumption has many useful structural consequences, reviewed in [8, Sect. 3], and is a hypothesis of all our main results.

By R-Mod we mean the category of finitely generated R-modules over R. We write "f.g." for "finitely generated".

Definition 1 A *matroid over R* on the ground set E is a function M assigning to each subset $A \subseteq E$ a f.g. R-module $M(A)$ satisfying the following axioms:

(M1) For any $A \subseteq E$ and $b \in E \setminus A$, there exists a surjection $M(A) \twoheadrightarrow M(A \cup \{b\})$ whose kernel is a cyclic submodule of $M(A)$.

(M2) For any $A \subseteq E$ and $b \neq c \in E \setminus A$, there exists a pushout

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{b\}) \\ \downarrow & \lrcorner & \downarrow \\ M(A \cup \{c\}) & \longrightarrow & M(A \cup \{b, c\}) \end{array}$$

where all four morphisms are surjections with cyclic kernel.

Clearly, Axiom (M1) is redundant if $|E| \geq 2$, and the choice of the modules $M(A)$ is only relevant up to isomorphism.

The fundamental way of producing matroids over R is from vector configurations in an R-module: these are the *representable* matroids. Given a f.g. R-module N and a list $X = x_1, \ldots, x_n$ of elements of N, the matroid M_X of X associates to the sublist A of X the quotient

$$M_X(A) = N \Big/ \left(\sum_{x \in A} Rx \right). \tag{1}$$

For each $x \in X$ there is a map $M_X(A) \xrightarrow{/x} M_X(A \cup \{x\})$, which quotients out by the image of Rx in $M_X(A)$; these maps satisfy axioms (M1) and (M2).

In other words, a represented matroid over R is a certain kind of functor from the Boolean poset $\mathscr{B}(E)$ to R-Mod. In contrast an arbitrary matroid over R is a map merely of the objects of $\mathscr{B}(E)$, not the morphisms, satisfying analogous conditions.

Minors and direct sums are defined in the obvious fashion. Let M and M' be matroids over R on respective ground sets E and E'. We define their *direct sum* $M \oplus M'$ on the ground set $E \cup E'$ by $(M \oplus M')(A \cup A') = M(A) \oplus M'(A')$. If i is an element of E, we define two matroids over R on the ground set $E \setminus \{i\}$: the *deletion* of i in M, denoted $M \setminus i$, by $(M \setminus i)(A) = M(A)$ and the *contraction* of i in M, denoted $M \setminus i$, by $(M/i)(A) = M(A \cup \{i\})$.

Matroids in the classical sense are matroids over fields, essentially since dimension is a complete isomorphism invariant of modules over fields. There is one hitch in the equivalence, corresponding to the ability to choose a vector configuration that does not span its ambient space. Accordingly, let us say that a matroid M over R is *full-rank* if no nontrivial projective module is a direct summand of $M(E)$. Very little is lost in restricting to full-rank matroids: any non-full-rank matroid splits as a direct sum involving a matroid on zero elements.

Proposition 2 *Let $\mathbb{K}$ be a field. Full-rank matroids M over $\mathbb{K}$ are equivalent to (classical) matroids. If M is a full-rank matroid over $\mathbb{K}$, then* $\dim M(A)$ *is the corank of A in the corresponding classical matroid. Furthermore, a matroid over $\mathbb{K}$ is representable if and only if, as a classical matroid, it is representable over $\mathbb{K}$.*

Given a map $R \to S$ of rings, applying $— \otimes_R S$ gives a function from matroids over R to matroids over S. For instance, matroids over R can be *localized* at primes by tensoring along $R \to R_{\mathfrak{m}}$; the *generic matroid*, a classical matroid, is obtained by tensoring along $R \to \mathrm{FracR}$.

3 Duality for Matroids Over Dedekind Domains

Let M be a matroid over R, on ground set E. Fix a free module F that surjects on $M(\emptyset)$. For any $A \subseteq E$ and maximal flag of subsets $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{|A|} = A$, we obtain a composite surjection

$$F \to M(\emptyset) \to M(A_1) \to \cdots \to M(A).$$

Using the horseshoe lemma, we may assemble minimal projective resolutions of each step of this composition into a projective resolution of $F/M(A)$, yielding a projective resolution

$$P(A)_\bullet : \quad 0 \to P_2(A) \to P_1(A) \stackrel{d_1}{\to} F \to M(A) \to 0$$

of $M(A)$. As usual, we write $^\vee$ for the contravariant functor $\mathrm{Hom}(—, R)$.

Definition 3 Define the module $M^*(E \setminus A)$ as the cokernel of the map dual to d_1 in $P(A)_\bullet$, that is

$$M^*(E \setminus A) \doteq \mathrm{coker}\big(\mathrm{F}^\vee \xrightarrow{\mathrm{d}_1^\vee} \mathrm{P}_1(\mathrm{A})^\vee\big).$$

This is well-defined ([8, Lemma 4.2]). We define M^*, the *dual* matroid over R to M, to be the collection of these modules $M^*(E \setminus A)$.

Theorem 4 *If R is a Dedekind domain, and M is a matroid over R, then its dual M^* is a full-rank matroid over R. Furthermore, M is the direct sum of M^{**} and the projective empty matroid for $M(E)_{\mathrm{proj}}$; in particular, if M is full-rank, $M^{**} = M$.*
If M is representable, also M^* is.

The last statement above gives a generalization of the classical Gale duality of vector configurations. Furthermore, duality of matroids over rings is well-behaved with respect to deletion, contraction, direct sums, and tensor products, as shown in [8, Proposition 4.9].

4 Structure of Matroids Over a DVR

Let R be a DVR with maximal ideal $\mathfrak{m}$. If N is a f.g. R-module and $i \geq 1$ is an integer, define $d_{\leq i}(N) \doteq \sum_{j=1}^{i} d_j(N)$. Then isomorphism classes of modules N are in bijection with infinite nonincreasing sequences $d_{\leq \bullet}(N)$ of nonnegative integers. Proposition 5.4 of [8] records necessary and sufficient conditions on the $d_{\leq \bullet}(M(A))$ in a matroid over R: briefly, (M1) becomes the Pieri rule (interpreted as in the Hall algebra), while (M2) becomes submodularity of the functions $A \mapsto -d_{\leq n}(M(A))$ with equality demanded in the submodular inequality under some conditions.

These matroids turn out to have a tropical-geometric character. Fink and Moci[8, Proposition 5.5] implies that the *tropicalizations* of the relations

$$p_{Ab}p_{Acd} - p_{Ac}p_{Abd} + p_{Ad}p_{Abc} = 0 \tag{2}$$

hold of the family of numbers $d_{\leq n}(M(—))$, where we abbreviate $A \cup \{b, c\}$ as Abc and similarly. These relations (2) are Plücker relations for the full flag variety (of type A), which has a tropical analogue, the *tropical flag Dressian* [9]. In fact:

Theorem 5 *Let M be a matroid over $(R, \mathfrak{m})$. The function $A \mapsto \dim_{R/\mathfrak{m}} M(A)$ produces a valuated matroid, in the sense of Dress and Wenzel [7]. That is, M determines a point of the tropical flag Dressian.*

5 Global Structure of Matroids Over a Dedekind Domain

The structure of matroids over arbitrary Dedekind domains R is the subject of [8, Propositions 6.1, 6.2]. In brief, a system of R-modules forms a matroid over R only if all its localizations at primes do; and given such a system of localizations there is at most one matroid over R, that is the projective summands are uniquely determined if they exist.

If M is a matroid over $\mathbb{Z}$, then we define a corank function for M as the corank function of the generic matroid $M \otimes_{\mathbb{Z}} \mathbb{Q}$, that is cork(A) is the rank over $\mathbb{Z}$ of the projective part of $M(A)$. We also define $m(A)$ to be the cardinality of the torsion subgroup of A.

Corollary 6 *The triple $(E, \text{cork}, \text{m})$ is a quasi-arithmetic matroid, as defined in [8, Remark 6.4].*

Thus matroids over $\mathbb{Z}$ recover many of the essential features of the second author's theory of *arithmetic matroids* from [6], but are richer in that the torsion subgroups carry more information than merely m.

6 The Tutte-Grothendieck Group

All matroids over R in this section are full-rank. Essentially following Brylawski [4], define the *Tutte-Grothendieck group* $K(R\text{-Mat})$ of matroids over R to be the abelian group generated by a symbol $\mathbf{T}_M$ for each unlabelled full-rank matroid M over R, modulo the relations

$$\mathbf{T}_M = \mathbf{T}_{M\setminus a} + \mathbf{T}_{M/a}$$

whenever a is not a loop or coloop of the generic matroid (so that $M \setminus a$ and M/a remain full-rank).

Define $\mathbb{Z}[R\text{-Mod}]$ to be the monoid ring of R-modules under direct sum, i.e. the ring with a $\mathbb{Z}$-linear basis $\{u^N\}$ with an element u^N for each f.g. R-module N up to isomorphism, and product given by $u^N u^{N'} = u^{N\oplus N'}$.

Theorem 7 *The Tutte-Grothendieck group $K(R\text{-Mat})$ is a ring without unity, with product given by $\mathbf{T}_M \cdot \mathbf{T}_{M'} = \mathbf{T}_{M\oplus M'}$. As a ring it injects into $\mathbb{Z}[R\text{-Mod}]\otimes\mathbb{Z}[R\text{-Mod}]$, and under this injection, the class of M maps to*

$$\mathbf{T}_M = \sum_{A\subseteq E} X^{M(A)} Y^{M^*(E\setminus A)}, \tag{3}$$

where $\{X^N\}$ and $\{Y^N\}$ are the respective bases of the two tensor factors $\mathbb{Z}[R\text{-Mod}]$. The ring $\mathbb{K}(R\text{-Mat})$ is the subring of $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ generated by X^P and Y^P as P ranges over rank 1 projective modules, and $(XY)^N$ as N ranges over torsion modules.

If R is a field, Theorem 7 gives the classical Tutte polynomial in the bivariate polynomial ring $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ on setting $X = x-1$ and $Y = y-1$.

6.1 Arithmetic Tutte Polynomial and Quasi-Polynomial

If M is a matroid over $\mathbb{Z}$, the arithmetic Tutte polynomial of its associated quasi-arithmetic matroid $\hat{M}$, and its Tutte quasi-polynomial, are each images of $\mathbf{T}_M$ under ring homomorphisms. When $R = \mathbb{Z}$, the Picard group is trivial, and

$$\mathbf{T}_M = \sum_{A\subseteq E} (X^R)^{\mathrm{cork_M(A)}} (Y^R)^{\mathrm{nullity}_M(A)} (XY)^{M(A)_{\mathrm{tors}}},$$

where we use the notation $\mathrm{nullity}_M(A) = \mathrm{cork_{M^*}(E \setminus A)} = \mathrm{\dim M^*(E \setminus A)}$.

The arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x, y)$ of the quasi-arithmetic matroid $\hat{M}$ defined by M, which is

$$\mathbf{M}_{\hat{M}}(x, y) = \sum_{A \subseteq E} m(A)(x-1)^{\mathrm{rk(E)-rk(A)}}(y-1)^{|A|-\mathrm{rk(A)}},$$

is a specialization of $\mathbf{T}_M$ by specializing X^R to $(x-1)$, Y^R to $(y-1)$, and $(XY)^N$ to the cardinality of N for each torsion module N. (See [5, 6, 10] for applications of $\mathbf{M}_{\hat{M}}$.)

In [3] a *Tutte quasi-polynomial* $\mathbf{Q}_M(x, y)$ was defined, specializing both to $\mathbf{T}_{\tilde{M}}(x, y)$ and $\mathbf{M}_{\hat{M}}(x, y)$, and in a sense interpolating between them. In fact $\mathbf{Q}_M(x, y)$ is an invariant of the matroid over $\mathbb{Z}$, though not of $\hat{M}$. We show explicitly how to compute it from the universal invariant.

For every positive integer q, let us define a function V_q as $V_q((XY)^{\mathbb{Z}/p^k}) = 1$ if p^k divides q, while $V_q((XY)^{\mathbb{Z}/p^k}) = p^{k-j}$ if p^k does not divide q and $j \geq 0$ is the largest integer such that p^j divides q. We will extend this to define $V_q((XY)^N)$ multiplicatively for any torsion abelian group N. Then we define a specialization of $\mathbf{T}_M$ to the ring of quasipolynomials by specializing X^R to $(x-1)$, Y^R to $(y-1)$, and $(XY)^N$ to $V_{(x-1)(y-1)}((XY)^N)$. This gives

$$\mathbf{Q}_M(x, y) = \sum_{A \subseteq E} \frac{|M(A)_{\mathrm{tors}}|}{|(x-1)(y-1)M(A)_{\mathrm{tors}}|}(x-1)^{\mathrm{rk(E)-rk(A)}}(y-1)^{|A|-\mathrm{rkrk(A)}}.$$

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A Minimal Irreducible Triangulation of $\mathbb{S}^3$

Florian Frick

Abstract We present a very symmetric triangulation of the 3-sphere, where every edge is in at most five facets but which is not the boundary of a polytope. This shows that not every triangulation of a sphere, where angles around faces of codimension two are less than 2π in the metric pieced together by regular Euclidean simplices, is polytopal. The counterexample presented here is the smallest triangulation of $\mathbb{S}^3$ where every edge is contained in an empty triangle. Moreover, it shows that a triangulation of $\mathbb{S}^3$ that is embeddable into $\mathbb{R}^4$ with straight faces is not necessarily weakly vertex-decomposable.

1 Construction

We will construct a triangulation of $\mathbb{S}^3$ that is positively curved but not polytopal and geometrically embeddable into $\mathbb{R}^4$ but not weakly vertex-decomposable. These notions will be defined later.

By a triangulation we will always mean a simplicial complex. The triangulation $\mathcal{T}$ presented here is uniquely determined by the property that the link of every vertex is an octahedron in which two opposite triangles have been stacked. This triangulates $\mathbb{S}^3$ on ten vertices.

We will explicitly construct $\mathcal{T}$ as a subcomplex of the five-dimensional cross-polytope. Label the vertices of the five-dimensional cross-polytope by $a_0, \dots, a_4$, $b_0, \dots, b_4$, where a_i is opposite to b_i. The five triangles of the form (a_i, a_{i+1}, a_{i+2}) are a triangulated Möbius strip, as are the triangles (b_i, b_{i+2}, b_{i+4}), where the indices are always modulo 5. These Möbius strips have five boundary edges and five interior edges. A tetrahedron σ in the five-dimensional cross-polytope belongs to the subcomplex $\mathcal{T}$ if and only if either a triangle of σ is contained in one of the Möbius strips and the other vertex of σ is in the other Möbius strip or σ has two edges on the two opposite Möbius strips, such that either both are interior edges or both are boundary edges.

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The resulting complex is homeomorphic to $\mathbb{S}^3$, and every edge of the 5-dimensional cross-polytope is in $\mathscr{T}$. The f-vector of this subcomplex is $(10, 40, 60, 30)$. There is dihedral symmetry along the Möbius strips and the two strips can be interchanged. In particular, the automorphism group of $\mathscr{T}$ acts transitively on the vertices.

2 Irreducibility

In a simplicial 2-sphere one can successively contract edges to obtain the tetrahedron. It is not possible to contract an arbitrary 3-sphere to the 4-simplex. An edge is called *contractible* if it is not contained in any empty face, i.e. a non-face all of whose subfaces are part of the complex. The process of contracting an edge (v, w) in a simplicial complex consists of identifying the vertices v and w in every face. A simplicial complex is *irreducible* if no edge is contractible.

Every surface has a finite number of irreducible triangulations [1]. This means that for each surface there is a finite list of triangulations, such that any other triangulation of this surface can be obtained from a member of this list by repeatedly splitting vertices. This is false in higher dimensions. This is just since the connected sum of two irreducible complexes different from the boundary of the simplex is again irreducible, where a connected sum of two triangulations is obtained by deleting a facet in each and gluing along the resulting boundary. An infinite family of irreducible triangulations is obtained by repeatedly taking the connected sum with an irreducible triangulation of the sphere, which does not change the topology. Here we present the smallest irreducible triangulation of the 3-sphere other than the boundary of the 4-simplex.

Theorem 1 *Every triangulation of the* 3*-sphere with fewer than ten vertices, apart from the boundary of the* 4*-simplex, has contractible edges. Among the triangulations of* $\mathbb{S}^3$ *on ten vertices the smallest instances of irreducible* 3*-spheres have* f*-vector* $(10, 40, 60, 30)$ *and there are six such triangulations.*

This is proved by a computer enumeration using data obtained by Lutz [7]. The triangulation $\mathscr{T}$ presented here is one of these six minimal irreducible triangulation of $\mathbb{S}^3$.

3 Positive Curvature

The dihedral angles of a regular tetrahedron are $\arccos(\frac{1}{3})$, which is slightly less than $\frac{2\pi}{5}$. There is a regular spherical tetrahedron with all dihedral angles equal to $\frac{2\pi}{5}$, which can be obtained by projecting the 600-cell radially onto the unit sphere. Inducing the metric of this regular spherical tetrahedron of edge length $\frac{\pi}{5}$ on every facet of a three-dimensional triangulation introduces an angle defect (or surplus) around any edge e, depending on the *valence* of e, that is, the number of facets e is contained in. The angle around an edge is at most 2π if and only if its valence is at most five. A three-dimensional triangulation where valences are bounded by five is called *positively curved*, since the metric above is an Alexandrov space of positive curvature.

Such a positive curvature bound from below yields a volume bound. The largest possible positively curved triangulation is the 600-cell. There are 4787 such triangulations—enumerated by Frank H. Lutz and John M. Sullivan—and 4761 of those triangulate $\mathbb{S}^3$. In addition, there are four different quotients of $\mathbb{S}^3$, which can be obtained as topological types of positively curved triangulations [4].

The triangulation $\mathscr{T}$ is one of these 4761 examples, since the vertex degrees in its vertex links are bounded by five. Notice that a positively curved triangulation with the 1-skeleton of the five-dimensional cross-polytope is necessarily irreducible, since each edge is contained in six 3-cycles, but at most in five triangles. Thus, every edge is contained in at least one empty triangle.

4 Non-polytopality

Every triangulation of $\mathbb{S}^2$ is polytopal, that is, is combinatorially equivalent to the boundary of a 3-polytope. This is false in higher dimensions. In fact, most triangulations of $\mathbb{S}^d$ for $d \geq 3$ are not the boundary of any $(d+1)$-polytope. This was shown by Kalai for $d \geq 4$ [5] and Pfeifle and Ziegler for $d = 3$ [8]. While an algorithm exists to decide whether a given triangulation of a sphere is polytopal, it is far from efficient. Much effort has been invested in finding large classes of polytopal triangulations, especially in finding operations which preserve polytopality, and—on the other hand—investigating which properties constitute obstructions to a triangulation being the boundary complex of a polytope.

Every knot can be realized by just three edges in some triangulation of $\mathbb{S}^3$. If this knot is non-trivial the triangulation cannot be polytopal. It turns out that some polytopes fail to be reducible, vertex-decomposable [6], and even weakly vertex-decomposable [3]. However, it is still an open question whether there are simplicial 4-polytopes that are not weakly vertex-decomposable.

Closely related to the question of polytopality is deciding whether a simplicial complex admits a geometric embedding into d-dimensional Euclidean space, where geometric means that every k-face is contained in a k-dimensional affine subspace. The triangulation $\mathscr{T}$ shows that a simplicial d-sphere which admits a geometric embedding into $\mathbb{R}^{d+1}$ cannot necessarily be perturbed to give a strictly convex, and thus polytopal, embedding.

Bokowski and Schuchert found four coherently oriented matroids for the dual complex of $\mathscr{T}$, but showed these are not realizable [2]. Since the primal is polytopal if and only if the dual is polytopal, $\mathscr{T}$ is not the boundary complex of a 4-polytope.

While it might be natural to assume that all positively curved triangulations are boundaries of polytopes, this is not true as $\mathscr{T}$ shows. However, $\mathscr{T}$ is embeddable into $\mathbb{R}^4$ with straight faces. Since $\mathscr{T}$ is a subcomplex of the five-dimensional cross-polytope, it can be realized in a Schlegel diagram of the cross-polytope in $\mathbb{R}^4$.

5 Further Properties

Besides being a minimal irreducible, positively curved, and non-polytopal triangulation of $\mathbb{S}^3$, $\mathscr{T}$ is not weakly vertex-decomposable and vertex-transitive. Here a pure, i.e. all facets have the same dimension, simplicial complex is *weakly vertex-decomposable* if it is a simplex or there is a vertex, such that deleting this vertex results in a weakly vertex-decomposable simplicial complex. In dimensions five and higher there are simplicial polytopes whose boundary complexes are not weakly vertex-decomposable [3]. It is unknown whether simplicial 4-polytopes exist that are not weakly vertex-decomposable. However, the triangulation $\mathscr{T}$ is geometrically embeddable into $\mathbb{R}^4$ but not weakly vertex-decomposable.

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Tropical Oriented Matroids

Silke Horn

Abstract Tropical oriented matroids (as defined by Ardila and Develin) are a tropical analogue of classical oriented matroids in the sense that they encode the properties of the types of points in an arrangement of tropical hyperplanes—in much the same way as the covectors of (classical) oriented matroids describe the types in arrangements of linear hyperplanes.

They are in correspondence with other objects of interest in discrete geometry: subdivisions of products of simplices and mixed subdivisions of simplices.

The details for this work are provided in Horn (A topological representation theorem for tropical oriented matroids: part I, 2012. arXiv:12120714 [math.CO]; part II, 2012. arXiv:12122080 [math.CO]); see also Horn (DMTCS Proc. **01**, 135–146, 2012).

1 Introduction

Oriented matroids abstract the combinatorial properties of arrangements of real hyperplanes and are ubiquitous in combinatorics. In fact, an arrangement of n (oriented) real hyperplanes in $\mathbb{R}^d$ induces a regular cell decomposition of $\mathbb{R}^d$. Then the covectors of the associated oriented matroid encode the position of the points of $\mathbb{R}^d$ (respectively, the cells in the subdivision) relative to the each of the hyperplanes in the arrangement. Conversely, the famous Topological Representation Theorem by Folkman and Lawrence [5] (see also [2]), states that every oriented matroid can be realised as an arrangement of PL-*pseudo*hyperplanes.

This paper is about the *tropical* analogue of oriented matroids.

Tropical geometry is concerned with the algebraic geometry over the tropical semiring $(\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \oplus, \otimes)$, where $\oplus : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}} : a \oplus b := \min\{a, b\}$ and $\otimes : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}} : a \otimes b := a + b$ are the tropical addition and multiplication. A *tropical hyperplane* is the vanishing locus of a linear tropical polynomial. From the combinatorial point of view though a tropical hyperplane in the tropical torus

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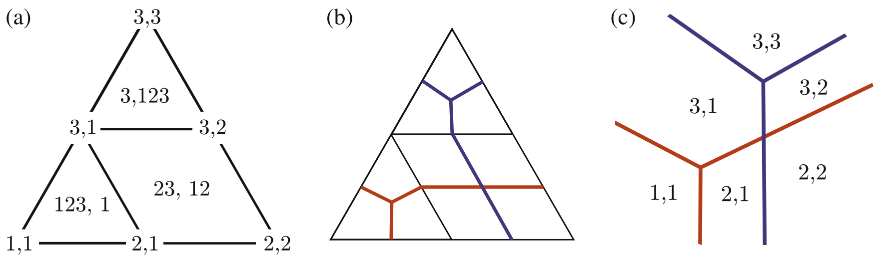

Fig. 1 The correspondence between mixed subdivisions and tropical pseudohyperplane arrangements. (**a**) A (regular) mixed subdivision of $2\triangle^2$. (**b**) The Poincaré dual of Fig. 1a. (**c**) An arrangement of tropical hyperplanes

$\mathbb{T}^{d-1} \cong \mathbb{R}^{d-1}$ is just the (codimension-1-skeleton of the) polar fan of the $(d-1)$-dimensional simplex $\triangle^{d-1}$. For a $(d-2)$-dimensional tropical hyperplane H the d connected components of $\mathbb{T}^{d-1} \setminus H$ are called the *(open) sectors* of H.

An arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$ induces a cell decomposition of $\mathbb{T}^{d-1}$ and each cell can be assigned a *type* that describes its position relative to each of the tropical hyperplanes. See Fig. 1c for an illustration in dimension 2.

It turns out that tropical curves—and as such in particular arrangements of tropical hyperplanes—have relationships to other interesting objects. By Develin and Sturmfels [4] *regular* subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ are dual to arrangements of n tropical hyperplanes in $\mathbb{T}^{d-1}$. See Fig. 1 for an illustration.

2 Tropical Oriented Matroids

A central concept is that of an (n,d)-type. For $n, d \geq 1$ an (n,d)*-type* is an n-tuple $(A_1, \ldots, A_n)$ of non-empty subsets of $[d]$.

For convenience we will write sets like $\{1,2,3\}$ as 123 throughout this chapter.

An (n,d)-type A can be represented as a subgraph K_A of the complete bipartite graph $K_{n,d}$: Denote the vertices of $K_{n,d}$ by $N_1, \ldots, N_n, D_1, \ldots, D_d$. Then the edges of K_A are $\{\{N_i, D_j\} \mid j \in A_i\}$.

Besides tropical hyperplane arrangements there are other objects that share the notion of an (n,d)-type:

Products of Simplices and Mixed Subdivisions If we label the vertices of $\triangle^{n-1}$ by $1, \ldots, n$, the vertices of the polytope $\triangle^{n-1} \times \triangle^{d-1}$ are in canonical bijection with the edges of the complete bipartite graph $K_{n,d}$. Then a cell C in a subdivision of $\triangle^{n-1} \times \triangle^{d-1}$ is assigned the type corresponding to the subgraph of $K_{n,d}$ containing all edges that mark vertices of C. See e.g. De Loera et al. [3] for a thorough treatment of triangulations and other subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$.

A polytopal subdivision of $n\triangle^{d-1}$ is *mixed* if every cell is a Minkowski sum of n faces of $\triangle^{d-1}$. By identifying the faces of $\triangle^{d-1}$ with the subsets of $[d]$, the faces of such a mixed subdivision again correspond to (n, d)-types. See Fig. 1a for an example.

By the *Cayley Trick* (cf. Huber et al. [10]) subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ are in bijection with mixed subdivisions of $n\triangle^{d-1}$.

Tropical Oriented Matroids Tropical oriented matroids as defined by Ardila and Develin [1] via a set of covector axioms generalise tropical hyperplane arrangements. In fact, by Ardila and Develin [1] the types in an arrangement of tropical hyperplanes always form a tropical oriented matroid. But there are *non-realisable* tropical oriented matroids which are *not realised* by any arrangement of tropical hyperplanes.

It turns out that the three concepts of subdivisions of products of two simplices, mixed subdivisions of dilated simplices and tropical oriented matroids are in fact equivalent:

By Ardila and Develin [1, Theorem 6.3], the types of a tropical oriented matroid with parameters (n, d) yield a subdivision of $\triangle^{n-1} \times \triangle^{d-1}$. Conversely, by Ardila and Develin [1, Proposition 6.4], the types of the cells in a mixed subdivision of $n\triangle^{d-1}$ satisfy all but one of the tropical oriented matroid axioms. In Oh and Yoo [11] it is proven that *fine* mixed subdivisions satisfy the fourth axiom, the *elimination axiom*.

In Sect. 3 we introduce a Topological Representation Theorem (in analogy to the one by Folkman and Lawrence [5] for classical oriented matroids) which states that any tropical oriented matroid can be represented as an arrangement of tropical *pseudo*hyperplanes. The equivalence of tropical oriented matroids and mixed subdivisions of $n\triangle^{d-1}$ is then a corollary of this theorem (see Sect. 4).

For quick reference, the general picture is depicted in Fig. 2.

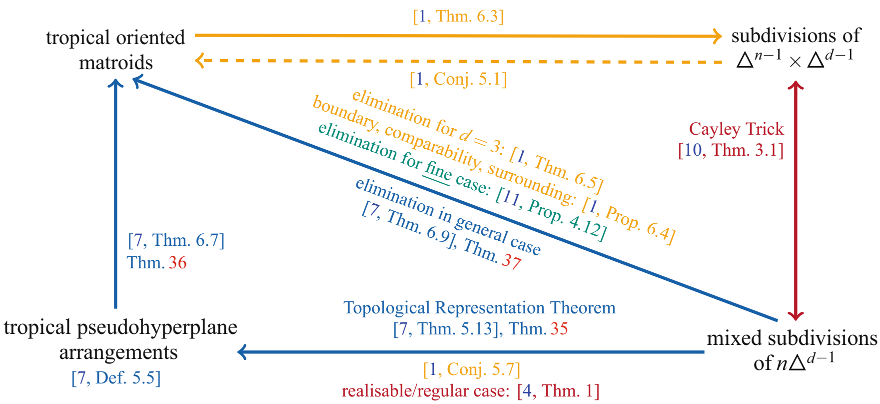

Fig. 2 The correspondences between the four concepts of tropical oriented matroids, mixed subdivisions of $n\triangle^{d-1}$, subdivisions of a product of two simplices and tropical pseudohyperplane arrangements

3 The Topological Representation Theorem

A *tropical pseudohyperplane* is basically a set which is PL-homeomorphic to a tropical hyperplane (see also [7, Definition 5.1]). Moreover, by Horn [6, Theorem 4.2] the *Poincaré dual* of a mixed subdivisions of $n\triangle^{d-1}$ yields a family of tropical pseudohyperplanes. (See Fig. 1b for an illustration.)

In order to define arrangements of these we have to impose restrictions on the intersections of the pseudohyperplanes. To this end, we borrow from the classical arrangements of (linear) pseudohyperplanes (see e.g. [2, Definition 5.1.3]):

A family of tropical pseudohyperplanes is an *arrangement* if any set of tropical pseudohalfspace boundaries either has empty intersection or forms an arrangement of linear pseudohyperplanes.

With this definition we obtain the Topological Representation Theorem:

Theorem 1 (Topological Representation Theorem [7, Theorem 5.13]) *Every tropical oriented matroid (in general position) can be realised by an arrangement of tropical pseudohyperplanes.*

4 Convexity in Tropical Oriented Matroids

The *elimination axiom* of tropical oriented matroids states that for any two types A, B in a tropical oriented matroid M (with parameters (n, d)) and any $k \in [n]$ there is a type $C \in M$ such that $C_k = A_k \cup B_k$ and $C_i \in \{A_i, B_i, A_i \cup B_i\}$ for each $i \in [n]$.

We define the *convex hull* of two types $A, B \in M$ as the set

$$M_{AB} := \{C \mid C_i \in \{A_i, B_i, A_i \cup B_i\} \text{ for all } i \in [n]\}.$$

It then turns out that the types of a mixed subdivision S of $n\triangle^{d-1}$ satisfy the elimination axiom if and only if S_{AB} is path connected for any two cells/types in S.

We can approximate this convex hull by an intersection of tropical pseudohalfspaces as illustrated in Fig. 3. See also [7, Sect. 6] for more details.

We then obtain the following corollaries of Theorem 1:

Theorem 2 (Cf. [7, Theorem 6.7]) *Arrangements of tropical pseudohyperplanes satisfy the elimination axiom.*

Theorem 3 (Cf. [1, Conjecture 5.1], [7, Theorem 6.9]) *Tropical oriented matroids with parameters (n, d) are in bijection with subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ and mixed subdivisions of $n\triangle^{d-1}$.*

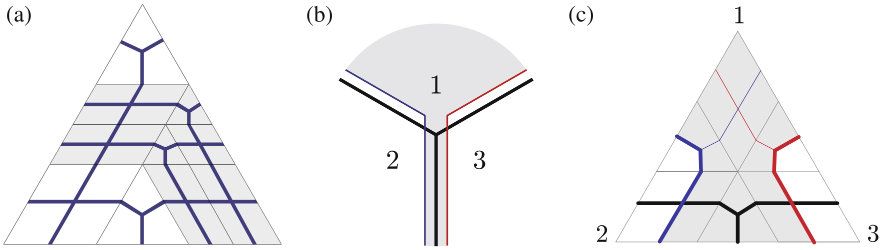

Fig. 3 The blow-up operation subdivides one simplex cell and extends this throughout the subdivision. Dually we add a copy of a tropical pseudohyperplane close to the original one. This can be used to approximate convex hulls of cells. (**a**) A *blow-up* of one tropical pseudohyperplane in a mixed subdivision of $3\triangle^2$. (**b**) An approximation of the convex hull of $A = (1)$ and $B = (23)$. (**c**) A combinatorial version of Fig. 3b using blow-ups

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Rota's Conjecture, the Missing Axiom, and Prime Cycles in Toric Varieties

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Abstract Rota's conjecture predicts that the coefficients of the characteristic polynomial of a matroid form a log-concave sequence. I will outline a proof for representable matroids using Milnor numbers and the Bergman fan. The same approach to the conjecture in the general case (for possibly non-representable matroids) leads to several intriguing questions on higher codimension algebraic cycles in toric varieties.

Attempting to solve the four color problem, Birkhoff introduced a polynomial associated to a graph which coherently encodes the answers to the analogous q-color problem for all natural numbers q [1]. This polynomial, called the chromatic polynomial, is a fundamental invariant of graphs. Any other numerical invariant of a simple graph which can be recursively computed by deletion and contraction of edges is a specialization of the chromatic polynomial.

In previous work [5] it is proved that the coefficients of the chromatic polynomial form a log-concave sequence for any graph, thus resolving a conjecture of Ronald Read [5]. An important step in the proof was to construct a complex algebraic variety associated to a graph and ask a more general question on the characteristic class of the algebraic variety. It turned out that the property of the characteristic class responsible for log-concavity is that it is *realizable*, meaning that the homology class is the class of an irreducible subvariety.

Proposition 1 ([5]) *Let ξ be a homology class in a product of projective spaces*

$$\xi = \sum_i d_i [\mathbb{P}^{k-i} \times \mathbb{P}^i] \in H_{2k}(\mathbb{P}^m \times \mathbb{P}^n; \mathbb{Z}).$$

Then some positive multiple of ξ is realizable if and only if $\{d_i\}$ form a log-concave sequence of nonnegative integers with no internal zeros.

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In general, for any complex projective variety X, one may define the space of prime cycles of X as a closed subset of $H_{2k}(X;\mathbb{R})$ which consists of limits of homology classes of irreducible k-dimensional subvarieties up to a positive multiple:

$$\mathcal{P}_k(X) := \overline{\{\xi \mid n \cdot \xi = [V] \text{ for some } n \in \mathbb{N} \text{ and } V \subseteq X\}} \subseteq H_{2k}(X;\mathbb{R}).$$

This subset showing asymptotic distribution of primes in the homology of X played a key role in the solution to the graph theory problem.

A motivating observation for further investigation is that, even for very simple toric varieties such as the one above, the orderly structure of the space of prime cycles becomes visible only after allowing positive multiples of homology classes. For example, there is no irreducible subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ which has the homology class

$$1[\mathbb{P}^5 \times \mathbb{P}^0] + 2[\mathbb{P}^4 \times \mathbb{P}^1] + 3[\mathbb{P}^3 \times \mathbb{P}^2] + 4[\mathbb{P}^2 \times \mathbb{P}^3] + 2[\mathbb{P}^1 \times \mathbb{P}^4] + 1[\mathbb{P}^0 \times \mathbb{P}^5],$$

although $(1, 2, 3, 4, 2, 1)$ is a log-concave sequence with no internal zeros [6]. One may expect that the same holds for the space of prime cycles of any smooth projective toric variety.

Conjecture 2 For any smooth projective toric variety X, the space of prime cycles $\mathcal{P}_k(X)$ is a closed semialgebraic subset of $H_{2k}(X;\mathbb{R})$.

Read's conjecture on graphs was later extended by Gian-Carlo Rota to combinatorial geometries, also called matroids, whose defining axioms are modeled on the relation of linear independence in a vector space. The above mentioned algebro-geometric proof does not work in this more general setting for one very interesting reason: not every matroid is realizable as a configuration of vectors in a vector space. Mathematicians since David Hilbert, who found a finite projective plane which is not coordinatizable over any field, have been interested in this tension between the axioms of combinatorial geometry and algebraic geometry. After numerous unsuccessful quests for the "missing axiom" which guarantees realizability, logicians found that one cannot add finitely many new axioms to matroid theory to resolve the tension [8, 9]. On the other hand, computer experiments revealed that numerical invariants of small matroids behave as if they were realizable, confirming Rota's conjecture in particular for all matroids within the range of our computational capabilities.

I believe that the discrepancy can be properly explained in the framework of algebraic geometry through a better understanding of the space of prime cycles. Here a matroid M of rank $k + 1$ on $n + 1$ elements can be viewed as a k-dimensional integral homology class Δ_M in the toric variety X_n constructed from the n-dimensional permutohedron. This homology class is the Bergman fan of M, and the realizability of M translates to the statement that the Bergman of M is realizable as an *integral* homology class of X_n. The toric variety X_n is equipped with a natural map

$$X_n \longrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

and the chromatic (characteristic) polynomials appear through the pushforward [7]

$$\Delta_M \longmapsto (\text{coefficients of the chromatic polynomial of } M).$$

Under this translation, Rota's log-concavity conjecture and its numerical evidences suggest an intriguing possibility that *any* matroid is realizable as a *real* homology class.

Conjecture 3 For any matroid M of rank $k+1$ on $n+1$ elements, we have

$$\Delta_M \in \mathcal{P}_k(X_n).$$

If true, this will not only prove the log-concavity conjecture but also explain the subtle discrepancy between combinatorial geometry and algebraic geometry.

Little is known on the space of prime cycles, even for very simple toric varieties such as X_n. What is needed as a first step in understanding the space of prime cycles is a systematic study of positivity of homology classes in toric varieties. More precisely, one needs to understand relations between the nef cone, the pseudoeffective cone, and the cone of movable cycles in a smooth projective toric variety X.

Question 4 (Nef Implies Effective) Is it true that the nef cone of k-dimensional cycles in X is contained in the pseudoeffective cone of k-dimensional cycles in X?

Question 5 (Toric Moving Lemma) Is it true that some positive multiple of a nef and effective cycle in X is homologous to an effective cycle intersecting all torus orbits of X properly?

Using the fact that the Bergman fan of a matroid spans an extremal ray of the nef cone of the toric variety X_n, one can show that affirmative answers to both questions for X_n implies Conjecture 3 for that n. An affirmative answer to the first question can be deduced from results of [3, 4]. The first question has a negative answer when X is not necessarily a toric variety [2].

In the special case when the dimension or the codimension is one, the above questions have an affirmative answer, and these statements form a basic part of the well-established theory relating the nef cone, the pseudoeffective cone, and the movable cone of a toric variety. Concrete results on cycles of higher dimension and codimension in toric varieties will guide the development of an analogous theory on positivity of algebraic cycles in more general algebraic varieties.

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A Combinatorial Classification of Buchsbaum Simplicial Posets

Jonathan Browder and Steven Klee

Abstract The family of Buchsbaum simplicial posets over a field **K** provides an algebraic abstraction of the family of (**K**-homology) manifold triangulations. In 2008, Novik and Swartz established lower bounds on the face numbers of a Buchsbaum simplicial poset as a function of its dimension and its topological Betti numbers over **K**. They conjectured that these lower bounds are sufficient to classify face numbers of Buchsbaum simplicial posets with prescribed Betti numbers. We prove this conjecture by using methods from the theory of (pseudo)manifold crystallizations to construct simplicial posets with prescribed face numbers and Betti numbers.

A *simplical poset* is a poset with a unique minimal element $\hat{0}$ in which each interval $[\hat{0}, \sigma]$ is isomorphic to a Boolean lattice. A simplicial poset P is naturally graded by declaring that $\mathrm{rk}(\sigma) = k$ if $[\hat{0}, \sigma]$ is isomorphic to a Boolean lattice of rank k. To any simplicial poset P, we associate a regular CW complex $|P|$ called the *geometric realization* of P which contains a $(k-1)$-dimensional simplex for each element $\sigma \in P$ of rank k. As such, simplicial posets are also called *simplicial cell complexes*, and we refer to the elements of P as *faces*. By way of comparison, a *simplicial complex* is a simplicial poset in which each pair of faces has a unique greatest lower bound. Geometrically, two faces in a simplicial complex intersect along a unique (possibly empty) face, whereas two faces in a simplicial poset can intersect along a subcomplex that is common to each of their boundaries.

The most natural combinatorial invariant of a finite $(d-1)$-dimensional simplicial poset is its *f-vector*, $f(P) := (f_{-1}(P), f_0(P), \ldots, f_{d-1}(P))$, where the *f-numbers* $f_i(P)$ count the number of i-dimensional faces in P. Often it is more natural to study a certain integer transformation of the f-vector called the *h-vector*, $h(P) :=$

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$(h_0(P), h_1(P), \ldots, h_d(P))$ whose entries, the *h-numbers* of P, are defined by the formula

$$h_j(P) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}(P).$$

For any $(d-1)$-dimensional simplicial poset P, $h_0(P) = 1$ and $h_d(P) = (-1)^{d-1}\tilde{\chi}(P)$, where $\tilde{\chi}(P)$ denotes the reduced Euler characteristic of P. Since the Euler characteristic of P is inherently related to both the combinatorial and topological structure of P, we will also be interested in studying the *(reduced) Betti numbers* of P (over a field $\mathbf{k}$), which are defined as $\beta_i(P) = \beta_i(P; \mathbf{k}) := \dim_{\mathbf{k}} \tilde{H}_i(P; \mathbf{k})$.

The primary reason for studying h-numbers instead of f-numbers is that they arise naturally when studying the *face ring* of a simplicial poset. We will not define the face ring here since the properties we are interested in studying can be defined equivalently in terms of topological information. We refer to Stanley's book [12] for further information on the algebraic properties of face rings.

We will be interested in studying two families of simplicial posets known as *Cohen-Macaulay* simplicial posets and *Buchsbaum* simplicial posets. These conditions are defined in terms of algebraic conditions on the face ring, but Reisner [7] and Schenzel [8] showed that these are in fact topological properties of a simplicial poset. We summarize these results in the following proposition, which we will use as our definitions of Cohen-Macaulay and Buchsbaum simplicial posets.

Proposition 1 *A $(d-1)$-dimensional simplicial poset P is Cohen-Macaulay (over a field $\mathbf{k}$) if and only if*

$$\tilde{H}_i(\mathrm{lk}_P(\tau); \mathbf{k}) = 0,$$

for all faces $\tau \in P$ (including $\tau = \emptyset$) and all $i < \dim(\mathrm{lk}_P(\tau))$. The simplicial poset P is Buchsbaum (over $\mathbf{k}$) if and only if it is pure and the link of each of its atoms is Cohen-Macaulay (over $\mathbf{k}$).

As a consequence, any simplicial cell decomposition of a sphere or ball is Cohen-Macaulay, and any simplicial cell decomposition of a manifold is Buchsbaum.

The h-numbers of a Cohen-Macaulay simplicial poset are nonnegative [12] because they arise naturally as the dimensions of certain graded vector spaces associated to the face ring. If P is a Buchsbaum simplicial poset, the dimensions of the analogous vector spaces are given by the *h'-numbers* of P [6], which are defined as

$$h'_j(P) = h_j(P) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(P),$$

for all $0 \le j \le d$.

Note that as $h_d(P) = (-1)^{d-1}\tilde{\chi}(P)$, it follows that $h'_d(P) = \beta_{d-1}(P)$. Further, a Cohen-Macaulay simplicial poset may only have non-vanishing reduced homology in top degree, in which case the h- and h'-numbers coincide.

1 The Problem

One of the main problems in the study of simplicial complexes and simplicial posets is to characterize the h-vectors of certain families of simplicial posets. As we will see below, these classifications for simplicial posets are often less restrictive than their counterparts for simplicial complexes.

Stanley [9] showed that a vector $\mathbf{h} = (h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}_{\geq 0}$ is the h-vector of a Cohen-Macaulay simplicial complex of dimension $d-1$ if and only if (1) $h_0 = 1$ and (2) $\mathbf{h}$ is an M-vector. In contrast, $\mathbf{h}$ is the h-vector of a Cohen-Macaulay simplicial poset of dimension $d-1$ if and only if $h_0 = 1$ and $h_j \geq 0$ for all j [11].

Similarly, Stanley [9] and Masuda [4] showed that a vector $\mathbf{h} = (h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}_{\geq 0}$ is the h-vector of a simplicial poset decomposition of $\mathbb{S}^{d-1}$ if and only if (1) $h_0 = 1$, (2) $h_j = h_{d-j}$ for all j, and (3) $\sum_{j=0}^{d} h_j$ is even if $h_j = 0$ for some j. In contrast, it is still unknown whether the more technical conditions of the g-theorem [1, 10] continue to hold for simplicial complexes that decompose $\mathbb{S}^{d-1}$.

Novik and Swartz gave necessary conditions on the h'-vectors of Buchsbaum posets.

Theorem 2 ([6, Theorem 6.4]) *Let P a Buchsbaum simplicial poset of rank d. Then $h'_j(P) \geq \binom{d}{j}\beta_{j-1}(P)$ for $j = 1, 2, \ldots, d-1$.*

In the case that P is Cohen-Macaulay, this reduces to the condition that the h-numbers are non-negative and $h_0 = 1$. Thus it is natural to ask [6, Question 7.4] if the lower bound of Theorem 2 is sufficient to characterize h'-vectors of Buchsbaum simplicial posets. The main result our work [2] is to answer this question in the affirmative.

Theorem 3 *Let $\beta_0, \beta_1, \ldots, \beta_{d-1}, h'_0, h'_1, \ldots, h'_d$ be non-negative integers. Then there is a Buchsbaum simplicial poset P of rank d with $h'_i(P) = h_i$ and $\beta_i(P) = \beta_i$ if and only if $h'_0 = 1$, $h'_d = \beta_{d-1}$ and for $j = 1, 2, \ldots, d-1$, $h'_j \geq \binom{d}{j}\beta_{j-1}$.*

To prove Theorem 3, it suffices to construct, for any β_i and h'_i satisfying the conditions of the theorem, a Buchsbaum simplicial poset having those Betti numbers and h'-numbers. Novik and Swartz showed it is sufficient to construct a family of Buchsbaum simplicial posets $X(k, d)$ for all d and all $0 \leq k \leq d-1$ such that

$$\beta_i(X(k,d)) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k, \end{cases} \quad \text{and} \quad h'_j(X(k,d)) = \begin{cases} \binom{d}{j}, & \text{if } j = 0, k+1 \\ 0, & \text{otherwise.} \end{cases}$$

For $X(0, d)$ we may take the disjoint union of two $(d-1)$-dimensional simplices, and for $X(d-1, d)$ we may take two $(d-1)$-simplices identified along their boundaries. Novik and Swartz also gave constructions for $X(1, d)$ [6, Lemma 7.6] and $X(d-2, d)$ [6, Lemma 7.7] for all d, along with an ad-hoc construction for $X(2, 5)$.

In [2], we provide a unified construction of $X(k, d)$ for all d and all $0 < k < d$.

Theorem 4 *For all $d \geq 2$ and all $0 \leq k \leq d-1$ there exists a Buchsbaum simplicial poset $X(k, d)$ with the following properties.*

1. *For all $0 \leq i \leq d-1$ and all $0 \leq j \leq d$,*

$$\beta_i(X(k,d)) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k, \end{cases} \quad \text{and} \quad h'_j(X(k,d)) = \begin{cases} \binom{d}{j}, & \text{if } j = 0, k+1 \\ 0, & \text{otherwise}. \end{cases}$$

2. *The link of each atom of $X(k, d)$ is shellable, and hence Cohen-Macaulay.*
3. *For each atom v of $X(k, d)$,*

$$h_j(\mathrm{lk}_{X(k,d)}(v)) = \begin{cases} \binom{d-1}{j}, & \text{if } j = 0, k \\ 0, & \text{otherwise.} \end{cases}$$

Condition (2) guarantees that the complex $X(k, d)$ is Buchsbaum. The significance of condition (3) above is that if P is a simplicial poset of rank d with $f_0(P) = d$ for which condition (3) is satisfied, then P has the h'-numbers and Betti numbers that are required for $X(k, d)$ [2, Lemma 3.2].

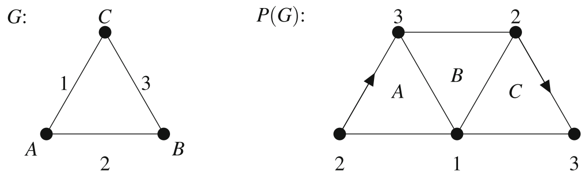
2 Constructing Buchsbaum Simplicial Posets

To construct our simplicial posets $X(k, d)$ we will adopt a graph-theoretic approach, related to the method of crystallizations of manifolds ([3] is a good reference) and the graphical posets of [5]. However we will allow less restricted classes of graphs, as we do not require that our simplicial cell complexes are even pseudomanifolds.

Definition 5 Let G be a finite connected multigraph whose edges are labeled by colors in $[d]$. For any $S \subseteq [d]$, let G_S be the restriction of G to the edges whose label belongs to S (and keeping all vertices of G). We define a poset $P(G)$ as follows: the elements of $P(G)$ are pairs (H, S), where $S \subset [d]$ and H is a connected component of G_S, ordered by $(H, S) \leq (H', S')$ if $S' \subseteq S$ and H' is a subgraph of H.

The geometric connection between the simplicial poset $P(G)$ and the graph G is very natural. The vertices of G correspond to the facets of $P(G)$. Each facet is a $(d-1)$-simplex, and hence has d vertices which we color as $1, 2, \ldots, d$. If F and F' are facets of $P(G)$ whose corresponding vertices in G are connected by an edge of color i, then F and F' intersect along their corresponding codimension-one faces

Fig. 1 A graph G (*left*) and the simplicial poset $P(G)$ (*right*)



opposite their respective vertices of color i. Thus the graph G is an edge-labeled dual graph to the cell complex $P(G)$.

Example 6 We illustrate this construction with the graph G and its corresponding simplicial poset $P(G)$ shown in Fig. 1.

The simplicial poset $X(k, d)$ of Theorem 4 is defined in terms of its dual graph $G(k, d)$ as follows.

Definition 7 Let k and d be integers with $0 \leq k < d$. Let W_d be the set of words in the alphabet $\{0, 1\}$ of length d whose first letter is 1, and let $W_d(k)$ denote the set of all such words with exactly $k + 1$ blocks of zeros and ones.

We define $G(k, d)$ as the edge-labeled multigraph on vertex set $W_d(k) \cup \{\alpha\}$ with the following edges:

- $G(k, d)$ has an edge labeled j connecting $\mathbf{w}, \mathbf{v} \in W_d(k)$ if and only if $\mathbf{w}$ and $\mathbf{v}$ differ only in position j, and
- $G(k, d)$ has an edge labeled j connecting $\mathbf{w} \in W_d(k)$ to α if and only if w_j is contained in a block of size one (alternatively, if "flipping" the bit at position j in $\mathbf{w}$ decreases the number of blocks).

Example 8 The graph G from Example 6 realizes $G(1, 3)$ under the identification of vertex A with the word 100, vertex B with the word 110, and vertex C with α.

As a larger example, we also include the graph $G(2, 5)$ (Fig. 2). The vertices in this case are labeled as

$$v_{12} = 10111 \qquad v_{23} = 11011$$
$$v_{13} = 10011 \qquad v_{24} = 11001$$
$$v_{14} = 10001 \qquad v_{34} = 11101.$$

We have drawn the vertex α three times to make the drawing planar, and we have made this vertex white in order to distinguish it from the other vertices.

Since the vertex α in this example is connected to vertices v_{12}, v_{13}, and v_{14} by an edge of color 1, the corresponding facet has a codimension-one face that is contained in four facets. Thus $X(2, 5)$ is not a manifold. In fact, aside from the families $X(0, d)$, $X(1, d)$, and $X(d-1, d)$, none of the simplicial posets we have constructed can be

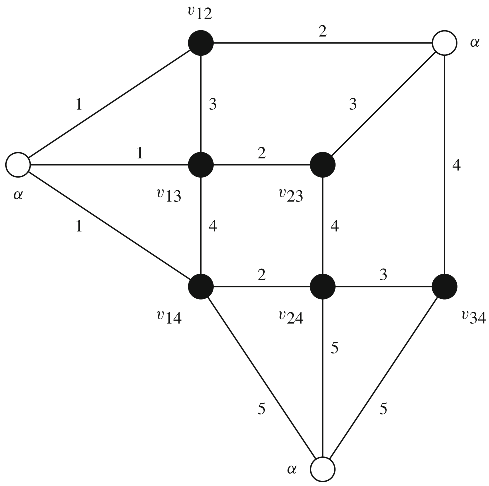


Fig. 2 The graph $G(2,5)$

realized as manifolds for this same reason. It remains an interesting question to determine whether or not it is possible to realize simplicial cell decompositions of manifolds that have the same face numbers and Betti numbers as the complexes $X(k,d)$.

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Dimensional Differences Between Faces of the Cones of Nonnegative Polynomials and Sums of Squares

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Abstract We study dimensions of the faces of the cone of nonnegative polynomials and the cone of sums of squares; we show that there are dimensional differences between corresponding faces of these cones. These dimensional gaps occur in all cases where there exist nonnegative polynomials that are not sums of squares. The gaps occur generically, they are not the product of selecting special faces of the cones. For ternary forms and quaternary quartics, we completely characterize when these differences are observed. Moreover, we provide an explicit description for these differences in the two smallest cases, in which the cone of nonnegative polynomials and the cone of sums of squares are different.

1 Introduction

Let $H_{n,2d}$ denote the set of homogeneous polynomials (forms) in n variables of degree $2d$ over $\mathbb{R}$ and let $\mathbb{RP}^{n-1}$ resp. $\mathbb{CP}^{n-1}$ denote the $(n-1)$-dimensional real resp. complex projective space. For a fixed number of variables n and degree $2d$, nonnegative polynomials and sums of squares form closed convex cones in $H_{n,2d}$. We call these cones $P_{n,2d}$ and $\Sigma_{n,2d}$, respectively, i. e.,

$$P_{n,2d} = \left\{p \in H_{n,2d} \;\middle|\; p(x) \geq 0 \;\text{ for all }\; x \in \mathbb{RP}^{n-1}\right\},$$

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$$\Sigma_{n,2d} = \left\{ p \in P_{n,2d} \;\mid\; p(x) = \sum q_i^2 \;\text{ for some }\; q_i \in H_{n,d} \right\}.$$

The relationship between the cone of nonnegative polynomials and the cone of sums of squares has been studied since Hilbert's seminal paper in 1888 [11]. Since every polynomial that is a sum of squares is nonnegative, one always has the containment $\Sigma_{n,2d} \subseteq P_{n,2d}$. In the aforementioned article [11], Hilbert completely characterized, when also the reverse inclusion is true, i. e., when, indeed the two cones $P_{n,2d}$ and $\Sigma_{n,2d}$ are equal. More precisely, he showed that a nonnegative form in n variables of even degree $2d$ has to be a sum of squares only in the following cases: the form is bivariate, i. e., $n = 2$, the form is quadratic, i. e., $2d = 2$, or, the form is a ternary quartic, i. e., $n = 3$ and $2d = 4$. In all other cases, he proved existence of nonnegative polynomials that are not sums of squares. It is remarkable that Hilbert's proof was existential and not constructive, and the first explicit nonnegative polynomial not being a sum of squares was found only 70 years later by Motzkin [15, 16]. The example, he provided, also referred to as *Motzkin polynomial*, was the following degree 6 homogeneous polynomial in three variables

$$M(x, y, z) := \frac{1}{3}x^2y^4 + \frac{1}{3}x^4y^2 + \frac{1}{3}z^6 - x^2y^2z^2.$$

He showed that polynomial is nonnegative but cannot be written as a sum of squares. A related question, known as Hilbert's 17th Problem, see e.g., [12], was the following

Question 1 Is it true that every nonnegative polynomial f can be written as a sum of squares of rational functions: $f = \sum_i \left(\frac{g_i}{h_i}\right)^2$?

In 1925, Artin was able to answer this question in the affirmative [2]. The Motzkin polynomial, for instance, can be written as a sum of squares of rational functions with denominator $(x^2 + y^2 + z^2)$. However, in general, the degree of the multiplier might be very large and a current line of research is concerned with the construction of multipliers and with bounding their degree. Recently, in small dimensions, several aspects of the differences between the two cones, $P_{n,2d}$ and $\Sigma_{n,2d}$, the structure of the dual cones as well as the algebraic boundaries of these cones were investigated (see [4–6]). Understanding the precise relationship between these cones is interesting from the point of view of computational complexity in polynomial optimization and also for practical testing for nonnegativity (see, e. g., [13]). Indeed, while testing whether a polynomial is nonnegative is NP-hard already in degree 4 [9], testing whether a polynomial is a sum of squares can be reduced to a semidefinite programming problem, which can be solved efficiently [14]. Unfortunately, except for the cases of $n = 2$, the univariate case for nonhomogeneous polynomials, the case $2d = 2$ (see [3, Sects. II.11 and II.12]), and, to some extent, the case of ternary quartics, neither the structure of these cones nor their precise relationship with each other is very well understood.

In this chapter, we focus on the study of faces of the cones $P_{n,2d}$ and $\Sigma_{n,2d}$. We will identify certain exposed faces of these cones, derive estimates for the dimensions of those faces and establish dimensional differences between corresponding exposed faces in many cases. The results of this chapter can be found in more detail in the recent article [7]. We also refer to [7] for further background and explanations and for most of the proofs.

2 Exposed Faces and *d*-Independence

A face F of a convex set K is called *exposed* if there exists a supporting hyperplane H such that $F = H \cap K$. It is easy to see that the boundary of the cone $P_{n,2d}$ consists of all the forms with at least one zero, whereas its interior consists of all strictly positive forms. In particular, a maximal proper face of $P_{n,2d}$ consists of all forms with exactly one prescribed zero [8, Chap. 4]. Given a set Γ of distinct points in $\mathbb{RP}^{n-1}$, the forms in $P_{n,2d}$ vanishing at all points of Γ form an exposed face of $P_{n,2d}$, which we call $P_{n,2d}(\Gamma)$:

$$P_{n,2d}(\Gamma) = \{p \in P_{n,2d} \ \mid \ p(s) = 0 \ \text{ for all } \ s \in \Gamma\}.$$

Similarly, we let $\Sigma_{n,2d}(\Gamma)$ be the exposed face of $\Sigma_{n,2d}$ consisting of forms that vanish at all points of Γ:

$$\Sigma_{n,2d}(\Gamma) = \{p \in \Sigma_{n,2d} \ \mid \ p(s) = 0 \ \text{ for all } \ s \in \Gamma\}.$$

Moreover, any exposed face of $P_{n,2d}$ has a description of the above form, and the set Γ can be chosen to be finite [8, Chap. 4]. We note that, despite this simple description of exposed faces, the full facial structure of $P_{n,2d}$ "should" be very difficult to fully describe since—as already mentioned—the problem of testing for nonnegativity is known to be NP-hard.

We pose the questions, which conditions need to be imposed on the set $\Gamma \subseteq \mathbb{RP}^{n-1}$, such that $P_{n,2d}(\Gamma) = \Sigma_{n,2d}(\Gamma)$. So as to answer this question we study the weaker question, for which sets $\Gamma \subseteq \mathbb{RP}^{n-1}$ the exposed faces $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ have the same dimension.

2.1 Exposed Faces of $\Sigma_{n,2d}$

The goal of this section is to compute the dimension of the exposed face $\Sigma_{n,2d}(\Gamma)$ for a finite set $\Gamma \subseteq \mathbb{RP}^{n-1}$.
In the following, let $I(\Gamma)$ be the vanishing ideal of Γ and let

$$I_d(\Gamma) = \{f \in H_{n,d} \ : \ f(s) = 0 \text{ for all } s \in \Gamma\}$$

be its degree d part. Moreover, let

$$(I_d(\Gamma))^2 = \{f \in H_{n,2d} \ : \ f = \sum_i \alpha_i q_i^2 \text{ for some } q_i \in I_d(\Gamma) \text{ and } \alpha_i \in \mathbb{R}\}.$$

Since any polynomial $p \in \Sigma_{n,2d}(\Gamma)$ belongs to $(I_d(\Gamma))^2$, we have the inclusion $\Sigma_{n,2d}(\Gamma) \subseteq (I_d(\Gamma))^2$. Moreover, this inclusion is full-dimensional, since one can choose a basis of $(I_d(\Gamma))^2$ consisting of squares and any nonnegative linear combination of these squares lies in $\Sigma_{n,2d}(\Gamma)$.

Proposition 2 *Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a finite set. Then $\Sigma_{n,2d}(\Gamma)$ is a full-dimensional convex cone in the vector space of all forms of degree $2d$ in $(I_d(\Gamma))^2$, i. e.,* $\dim \Sigma_{n,2d}(\Gamma) = \dim(I_d(\Gamma))^2$.

2.2 Exposed Faces of $P_{n,2d}$

The goal of this section consists of determining the dimension of an exposed face $P_{n,2d}(\Gamma)$ for a finite set $\Gamma \subseteq \mathbb{RP}^{n-1}$. For this aim, we consider the *second symbolic power* $I^{(2)}(\Gamma)$ of $I(\Gamma)$, i. e., the ideal of all forms in $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ vanishing at every point of Γ to order at least 2:

$$I^{(2)}(\Gamma) = \{p \in \mathbb{R}[x] \mid \nabla p(s) = 0 \ \text{ for all } \ s \in \Gamma\}.$$

Since every nonnegative form that is zero on $s \in \Gamma$ must vanish to order 2 on s, it follows that the face $P_{n,2d}(\Gamma)$ is contained in the degree $2d$ part of $I^{(2)}(\Gamma)$, i. e., $P_{n,2d}(\Gamma) \subseteq I^{(2)}_{2d}(\Gamma)$. We pose the question, under which assumptions $P_{n,2d}(\Gamma)$ is a full-dimensional subcone of $I^{(2)}_{2d}(\Gamma)$. In order to answer this question, the following crucial definition is required.

Definition 3 Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a finite set of distinct points. We call $\Gamma \subset \mathbb{RP}^{n-1}$ *d-independent* if Γ satisfies the following two conditions:

(i) The forms in $I_d(\Gamma)$ share no common zeroes in $\mathbb{CP}^{n-1}$ outside of Γ.
(ii) For any $s \in \Gamma$ the forms that vanish to order 2 on s and vanish at the rest of Γ to order 1 form a vector space of codimension $|\Gamma| + n - 1$ in $H_{n,d}$.

The second condition in the above definition simply states that the constraints of vanishing at Γ and additionally double vanishing at any point $s \in \Gamma$ are all linearly independent. The next proposition provides an answer to the previously posed question.

Proposition 4 *Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a d-independent set. Then $P_{n,2d}(\Gamma)$ is a full-dimensional convex cone in $I^{(2)}_{2d}(\Gamma)$:* $\dim P_{n,2d}(\Gamma) = \dim I^{(2)}_{2d}(\Gamma)$.

Full-dimensionality of $P_{n,2d}(\Gamma)$ in $I^{(2)}_{2d}(\Gamma)$ is established by finding a form $p \in P_{n,2d}(\Gamma)$ that after adding a suitably small multiple of any double vanishing form remains nonnegative:

$$p + \epsilon q \in P_{n,2d}(\Gamma) \text{ for some sufficiently small } \epsilon \text{ and any } q \in I^{(2)}_{2d}(\Gamma).$$

The form p can be viewed as a certificate of full-dimensionality of $P_{n,2d}(\Gamma)$ in $I^{(2)}_{2d}(\Gamma)$. The important point is that p can be any form, in particular, we will focus on finding such p that is a sum of squares. This approach follows that of [16] and, indeed, it can be traced to the original proof of Hilbert [11].

For a form p, let its Hessian H_p be the matrix of second derivatives of p, i. e., $H_p = \left(\frac{\partial^2 p}{\partial x_i \partial x_j}\right)$. We note that if a form p vanishes at a point s, then, by homogeneity, p needs to vanish at a line through s. Therefore, s lies in the kernel of the Hessian of p at s: $H_p(s)s = 0$. If a form p is nonnegative, then its Hessian at any zero s is positive semidefinite since 0 is a minimum for p. We call a nonnegative form p *round* at a zero $s \in \mathbb{RP}^{n-1}$ if $H_p(s)$ is positive definite on the subspace $s^{\perp}$ of vectors perpendicular to s, i. e., if $y^T H_p(s) y > 0$ for all $y \in s^{\perp}$.

For a form p, we let $Z(p)$ denote the real projective variety of p.

The next corollary, which follows from Lemma 3.1 in [16], will be crucial for the proof of Proposition 4.

Corollary 5 *Let Γ be a finite set in $\mathbb{RP}^{n-1}$. Suppose that there exists a nonnegative form p in $P_{n,2d}(\Gamma)$ such that $Z(p) = \Gamma$ and p is round at every point $s \in \Gamma$. Then the face $P_{n,2d}(\Gamma)$ is full-dimensional in the vector space $I^{(2)}_{2d}(\Gamma)$.*

Proof of Proposition 4 Let $q_1, \dots, q_k$ be a basis of $I_d(\Gamma)$. We claim that $p = \sum_{i=1}^k q_i^2$ has the properties of Corollary 5, which implies that $P_{n,2d}(\Gamma)$ is full-dimensional in $I^{(2)}_{2d}(\Gamma)$.

Since Γ forces no additional zeroes and since $q_1, \dots, q_k$ is a basis of $I_d(\Gamma)$, it follows that the forms q_i have no common zeroes outside of Γ and thus $Z(p) = \Gamma$.

Now choose $s \in \Gamma$. It remains to show that p is round at s. Since the forms in $I_d(\Gamma)$ that double vanish at s form a vector space of codimension $n-1$ in $I_d(\Gamma)$, we see that for $1 \le i \le k$ the gradients of q_i at s span a vector space of dimension $n-1$. Since, by Euler's identity (see, e. g., [10, Lemma 11.4]), $\langle \nabla q_i, s\rangle = 0$ for all i, this implies that the gradients actually span $s^{\perp}$.

Note that the Hessian of p is the sum of the Hessians of q_i^2, i. e., $H_p = \sum_{i=1}^k H_{q_i^2}$. Since $q_i(s) = 0$ for all i and $s \in \Gamma$, we conclude that

$$\frac{\partial^2 q_i^2}{\partial x_l \partial x_j}(s) = 2\frac{\partial q_i}{\partial x_l}(s)\frac{\partial q_i}{\partial x_j}(s).$$

Therefore, we see that the Hessian of q_i^2 at any $s \in \Gamma$ is actually double the tensor of the gradient of q_i at s with itself: $H_{q_i^2}(s) = 2\nabla q_i \otimes \nabla q_i(s)$. It is now straightforward to verify that $\nabla q_j(s)^T H_{q_i^2}(s) \nabla q_j(s) > 0$ for all $1 \le i, j \le k$ and $s \in \Gamma$, which shows the claim.

We want to remark that Proposition 4 allows to actually determine the dimension of $P_{n,2d}(\Gamma)$ since by the Alexander-Hirschowitz Theorem [1] the dimension of $I^{(2)}(\Gamma)$ is known.

2.3 d-Independence

Given the results from the previous section, we are able to determine the dimension of an exposed face $P_{n,2d}(\Gamma)$, if Γ is a d-independent set in $\mathbb{RP}^{n-1}$. The obvious question that arises however is how restrictive this condition of being d-independent is. We are able to show the following:

Proposition 6 *Let Γ be a generic collection of points in $\mathbb{RP}^{n-1}$ such that $|\Gamma| \leq \binom{n+d-1}{d} - n$. Then Γ is d-independent.*

The proof of this result precedes in two steps. The first step is to show that the set of d-independent configurations of k points in $\mathbb{RP}^{n-1}$ is a Zariski open subset of $(\mathbb{RP}^{n-1})^k$. The second step consists of identifying a subset of $\mathbb{RP}^{n-1}$ of cardinality $\binom{n+d-1}{d} - n$ that is d-independent. More precisely, one can show that the set

$$S_{n,d} := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \ : \ 0 \leq \alpha_i \leq d-1,\ \alpha_i \in \mathbb{N},\ \sum_{i=1}^{n} \alpha_i = d\}$$

has this property.

In view of Propositions 2, 4 and 6 our original question of finding a dimensional difference between the faces $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ can be reduced to the following:

Question 7 Let $\Gamma \subseteq \mathbb{RP}^{n-1}$ (or equivalently $\mathbb{CP}^{n-1}$) be a generic set of points such that $|\Gamma| \leq \binom{n+d-1}{d} - n$, and let $I(\Gamma)$ be the vanishing ideal of Γ. For what values of $|\Gamma|$ does equality

$$I_{2d}^{(2)}(\Gamma) = (I_d(\Gamma))^2$$

hold?

We will study this problem in more detail in the next section for $n = 3$ and $n = 4$.

3 The Cases $(n, 2d) \in \{(3, 2d), (4, 4)\}$

Though we cannot provide an answer to Question 7 in full generality, we are able to do so for ternary forms and quaternary quartics. Thereby, we also obtain a complete characterization of when dimensional differences between corresponding exposed faces of the cones of nonnegative polynomials and sums of squares are observed in these cases. For ternary forms we show the following:

Theorem 8 *Let Γ be a d-independent set of points in $\mathbb{RP}^2$ such that $|\Gamma| \leq \binom{d+1}{2}$. Then*

$$\dim I_{2d}^{(2)}(\Gamma) = \dim(I_d(\Gamma))^2.$$

Moreover, if $\binom{d+1}{2} + 1 \leq |\Gamma| \leq \binom{d+1}{2} + (d-2)$, then $\dim I_{2d}^{(2)}(\Gamma) > \dim(I_d(\Gamma))^2$.

As already explained, the next corollary is an immediate consequence of Theorem 8.

Corollary 9 *Let $\Gamma \subseteq \mathbb{RP}^2$ be d-independent with $|\Gamma| \leq \binom{d+1}{2}$. Then*

$$\dim P_{3,2d}(\Gamma) = \dim \Sigma_{3,2d}(\Gamma).$$

Furthermore, for $\binom{d+1}{2} + 1 \leq |\Gamma| \leq \binom{d+1}{2} + (d-2)$ we have

$$\dim P_{3,2d}(\Gamma) > \dim \Sigma_{3,2d}(\Gamma).$$

Similarly, for $n = 4$, we derive the minimal size of a 2-independent set Γ such that the dimensions of $(I_2(\Gamma))^2$ and $I_4^{(2)}(\Gamma)$ are distinct.

Theorem 10 *Let $\Gamma \subseteq \mathbb{RP}^3$ be a finite set in general linear position. Then the following hold:*

(i) If $|\Gamma| = 6$, then $\dim(I_2(\Gamma))^2 = 10 < 11 = \dim I_4^{(2)}(\Gamma)$.
(ii) If $|\Gamma| \leq 5$, then $\dim(I_2(\Gamma))^2 = \dim I_4^{(2)}(\Gamma)$.

As a direct consequence, we obtain the following corollary.

Corollary 11 *Let $\Gamma \subset \mathbb{RP}^3$ be a finite set in general linear position. Then the following hold:*

(i) If $|\Gamma| = 6$, then $\dim \Sigma_{4,4}(\Gamma) = 10 < 11 = \dim P_{4,4}(\Gamma)$.
(ii) If $|\Gamma| \leq 5$, then $\dim \Sigma_{4,4}(\Gamma) = \dim P_{4,4}(\Gamma)$.

We are not only able to establish dimensional differences but also to construct nonnegative forms, belonging to a certain exposed face that are not sums of squares. The next example illustrates this for the case $(n, 2d) = (4, 4)$ and also explains the general strategy for this construction.

Example 12 Let $\Gamma = \{s_1, \ldots, s_6\}$ with

$$s_1 = (0,0,1,1), s_2 = (0,1,0,1), s_3 = (0,1,1,0), s_4 = (1,0,0,1),$$
$$s_5 = (1,0,1,0), s_6 = (1,1,0,0).$$

In the first step of the construction, we choose a basis $f_1, \ldots, f_r$ of $I_2(\Gamma)$. For our specific set Γ, one can show that the polynomials $f_1(x_1, x_2, x_3, x_4) = x_1(x_1 - x_2 - x_3 - x_4)$, $f_2(x_1, x_2, x_3, x_4) = x_2(x_2 - x_1 - x_3 - x_4)$, $f_3(x_1, x_2, x_3, x_4) = x_3(x_3 - x_1 - x_2 - x_4)$ and $f_4(x_1, x_2, x_3, x_4) = x_4(x_4 - x_1 - x_2 - x_3)$ form a basis for $I_2(\Gamma)$.

In the second step of the construction, one needs to find a form $q \in I_4^{(2)}(\Gamma) \setminus (I_2(\Gamma))^2$. Note that such a form exists whenever there is a dimensional difference between the corresponding faces. The key point of the construction is that for sufficiently small $\epsilon > 0$ one can show that

$$\sum_{i=1}^{r} f_i^2 + \epsilon q \in P_{4,4}(\Gamma) \setminus \Sigma_{4,4}(\Gamma).$$

In our specific example, the polynomial $q(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$ lies in $I_4^{(2)}(\Gamma)$ but not in $(I_4(\Gamma))^2$ and one can verify that

$$f_1^2 + f_2^2 + f_3^2 + f_4^2 + q \in P_{4,4}(\Gamma) \setminus \Sigma_{4,4}(\Gamma).$$

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On a Conjecture of Holtz and Ron Concerning Interpolation, Box Splines, and Zonotopes

Matthias Lenz

Abstract Let X be a list of vectors that is unimodular and let B_X be the box spline defined by X. We discuss the proof of the following conjecture by Holtz and Ron: every real-valued function on the set of interior lattice points of the zonotope defined by X can be extended to a function on the whole zonotope of the form $p(D)B_X$ in a unique way, where $p(D)$ is a differential operator that is contained in the so-called internal $\mathcal{P}$-space. We construct an explicit solution to this interpolation problem in terms of truncations of the Todd operator. As a corollary we obtain a slight generalisation of the Khovanskii-Pukhlikov formula that relates the volume and the number of lattice points in a smooth lattice polytope.

1 Introduction

Box splines and multivariate splines measure the volume of certain variable polytopes. Since vector partition functions measure the number of integral points in polytopes, they can be seen as discrete versions of these spline functions. Splines and vector partition functions have recently received a lot of attention by researchers in various fields including approximation theory, algebra, combinatorics, and representation theory. A standard reference from the approximation theory point of view is the book [7] by de Boor et al. The combinatorial and algebraic aspects are stressed in the book [8] by De Concini and Procesi.

Given a set $\Theta = \{u_1, \ldots, u_k\}$ of k distinct points on the real line and a function $f : \Theta \to \mathbb{R}$, it is well-known that there exists a unique polynomial p_f in the space of univariate polynomials of degree at most $k-1$ such that $p_f(u_i) = f(u_i)$ for $i = 1, \ldots, k$. If Θ is contained in $\mathbb{R}^d$ for an integer $d \geq 2$, the situation becomes more difficult. Not all of the properties of the univariate case can be preserved simultaneously. In particular, for fixed Θ it is difficult to find a space of polynomials that contains a unique interpolant for every function $f : \Theta \to \mathbb{R}$. One of our results states that if Θ is the set of interior lattice points of a zonotope, such a

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space is obtained by applying certain differential operators to the box spline in the unimodular case. This solves a conjecture by Holtz and Ron [12].

Khovanskii and Pukhlikov proved a remarkable formula that relates the volume and the number of integer points in a smooth polytope [13]. The connection is made via Todd operators, i.e. differential operators of type $\frac{\partial_x}{1-e^{\partial_x}}$. The formula is closely related to the Hirzebruch-Riemann-Roch Theorem for smooth projective toric varieties (see [5, Chap. 13]).

It will turn out that truncations of certain shifted Todd operators provide explicit solutions to the Holtz-Ron conjecture. We will be able to deduce slight generalisations of a deconvolution formula of De Concini et al. [9] and of the Khovanskii-Pukhlikov formula.

Proofs of our results and more details can be found in [16–18].

2 Notation and Mathematical Background

Our notation is similar to the one used in [8]. We fix a d-dimensional real vector space U and a lattice $\Lambda \subseteq U$. Let $X = (x_1, \ldots, x_N) \subseteq \Lambda$ be a finite list of vectors that spans U. We assume that X is unimodular with respect to Λ, i.e. every basis for U that can be selected from X is also a lattice basis for Λ. Note that X can be identified with a linear map $X : \mathbb{R}^N \to U$. Let $u \in U$. We define the variable polytopes

$$\Pi_X(u) := \{w \in \mathbb{R}^N_{\geq 0} : Xw = u\} \quad \text{and} \quad \Pi^1_X(u) := \Pi_X(u) \cap [0,1]^N. \tag{1}$$

Note that any convex polytope can be written in the form $\Pi_X(u)$ for suitable X and u. The dimension of these two polytopes is at most $N - d$. We define the

$$\text{vector partition function } \mathcal{T}_X(u) := \left|\Pi_X(u) \cap \mathbb{Z}^N\right|, \tag{2}$$

$$\text{the box spline } B_X(u) := \det(XX^T)^{-1/2} \operatorname{vol}_{N-d} \Pi^1_X(u), \tag{3}$$

$$\text{and the multivariate spline } T_X(u) := \det(XX^T)^{-1/2} \operatorname{vol}_{N-d} \Pi_X(u). \tag{4}$$

Note that we have to assume that 0 is not contained in the convex hull of X in order for T_X and $\mathcal{T}_X$ to be well-defined. Otherwise, $\Pi_X(u)$ may be unbounded. The *zonotope* $Z(X)$ is defined as

$$Z(X) := \left\{ \sum_{i=1}^{N} \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\} = X \cdot [0,1]^N. \tag{5}$$

We denote its set of interior lattice points by $\mathcal{Z}_-(X) := \operatorname{int}(Z(X)) \cap \Lambda$.

The symmetric algebra over U is denoted by $\operatorname{Sym}(U)$. We fix a basis $s_1, \ldots, s_d$ for the lattice Λ. This makes it possible to identify Λ with $\mathbb{Z}^d$, U with $\mathbb{R}^d$, $\operatorname{Sym}(U)$ with the polynomial ring $\mathbb{R}[s_1, \ldots, s_d]$, and X with a $(d \times N)$-matrix. Then X is unimodular if and only if every non-singular square submatrix of this matrix has determinant 1 or -1.

We denote the dual vector space by $V = U^*$ and we fix a basis $t_1, \dots, t_d$ that is dual to the basis for U. An element of $\mathrm{Sym}(U)$ can be seen as a differential operator on $\mathrm{Sym}(V)$, i.e. $\mathrm{Sym}(U) \cong \mathbb{R}[s_1, \dots, s_d] \cong \mathbb{R}[\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_d}]$. For $f \in \mathrm{Sym}(U)$ and $p \in \mathrm{Sym}(V)$ we write $f(D)p$ to denote the polynomial in $\mathrm{Sym}(V)$ that is obtained when f acts on p as a differential operator. It is known that the box spline is piecewise polynomial and its local pieces are contained in $\mathrm{Sym}(V)$. We will mostly use elements of $\mathrm{Sym}(U)$ as differential operators on its local pieces.

Note that a vector $u \in U$ defines a linear form $u \in \mathrm{Sym}(U)$. For a sublist $Y \subseteq X$, we define $p_Y := \prod_{y \in Y} y$. For example, if $Y = ((1,0),(1,2))$, then $p_Y = s_1(s_1 + 2s_2)$. Furthermore, $p_\emptyset := 1$. We define the *rank* of a sublist $Y \subseteq X$ as the dimension of the vector space spanned by Y. We denote it by $\mathrm{rk}(Y)$. Now we define the

$$\textit{central } \mathcal{P}\textit{-space } \mathcal{P}(X) := \mathrm{span}\{p_Y : \mathrm{rk}(X \setminus Y) = \mathrm{rk}(X)\} \tag{6}$$

$$\text{and the } \textit{internal } \mathcal{P}\textit{-space } \mathcal{P}_-(X) := \bigcap_{x \in X} \mathcal{P}(X \setminus x). \tag{7}$$

The space $\mathcal{P}_-(X)$ was introduced in [12] where it was also shown that the dimension of this space is equal to $|\mathcal{Z}_-(X)|$. The space $\mathcal{P}(X)$ first appeared in approximation theory [1, 6, 10].

3 Results

The internal $\mathcal{P}$-space can be characterised in terms of box splines:

Theorem 1 *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is unimodular and spans U. Then*

$$\mathcal{P}_-(X) = \{f \in \mathcal{P}(X) : f(D)B_X \textit{ is a continuous function}\}. \tag{8}$$

Theorem 1 ensures that the derivatives of B_X that appear in the following theorem exist.

Theorem 2 (Holtz-Ron Conjecture [12, Conjecture 1.8]) *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is unimodular and spans U. Let f be a real valued function on $\mathcal{Z}_-(X)$, the set of interior lattice points of the zonotope defined by X.*

Then there exists a unique polynomial $p \in \mathcal{P}_-(X) \subseteq \mathbb{R}[s_1, \dots, s_d]$, s.t. $p(D)B_X|_{\mathcal{Z}_-(X)} = f$.

Let $z \in U$. As usual, the exponential is defined as $e^z := \sum_{k \geq 0} \frac{z^k}{k!} \in \mathbb{R}[[s_1, \dots, s_d]]$. We define the ($z$-shifted) *Todd operator*

$$\mathrm{Todd}(X, z) := e^{-z} \prod_{x \in X} \frac{x}{1 - e^{-x}} \in \mathbb{R}[[s_1, \dots, s_d]]. \tag{9}$$

The Todd operator was introduced by Hirzebruch in the 1950s [11] and plays a fundamental role in the Hirzebruch-Riemann-Roch theorem for complex algebraic varieties. It can be expressed in terms of the *Bernoulli numbers* $B_0 = 1, B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}, \ldots$ that are defined by the equation $\frac{s}{e^s-1} = \sum_{k\geq 0} \frac{B_k}{k!} s^k$. One should note that $e^z \frac{z}{e^z-1} = \frac{z}{1-e^{-z}} = \sum_{k\geq 0} \frac{B_k}{k!}(-z)^k$. For $z \in \mathcal{Z}_-(X)$ we can fix a list $S \subseteq X$ s. t. $z = \sum_{x\in S} x$. Let $T := X \setminus S$. Then we can write the Todd operator as $\operatorname{Todd}(X, z) = \prod_{x\in S} \frac{x}{e^x-1} \prod_{x\in T} \frac{x}{1-e^{-x}}$.

A sublist $C \subseteq X$ is called a *cocircuit* if $\operatorname{rk}(X \setminus C) < \operatorname{rk}(X)$ and C is inclusion minimal with this property. We consider the *cocircuit ideal* $\mathcal{J}(X) := \operatorname{ideal}\{p_C : C \text{ cocircuit}\} \subseteq \operatorname{Sym}(U)$. It is known [10, 12] that $\operatorname{Sym}(U) = \mathcal{P}(X) \oplus \mathcal{J}(X)$. Let

$$\psi_X : \mathcal{P}(X) \oplus \mathcal{J}(X) \to \mathcal{P}(X) \tag{10}$$

denote the projection. Note that this is a graded linear map and that ψ_X maps to zero any homogeneous polynomial whose degree is at least $N - d + 1$. This implies that one can easily extend it to a function $\psi_X : \mathbb{R}[[s_1, \ldots, s_d]] \to \mathcal{P}(X)$.

Let

$$f_z = f_z^X := \psi_X(\operatorname{Todd}(X, z)). \tag{11}$$

Theorem 3 *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is unimodular and spans U. Let z be a lattice point in the interior of the zonotope $Z(X)$.*

Then $f_z \in \mathcal{P}_-(X)$ and $f_z(D)B_X|_\Lambda = \delta_z$.

Dahmen and Micchelli observed that

$$T_X = B_X *_d \mathcal{T}_X := \sum_{\lambda\in\Lambda} B_X(\cdot - \lambda)\mathcal{T}_X(\lambda) \tag{12}$$

(cf. [8, Proposition 17.17]). Using this result, the following variant of the Khovanskii-Pukhlikov formula [13] (see also [19] and [4, Chap. 10]) follows immediately.

Corollary 4 *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is unimodular and spans U, $u \in \Lambda$ and $z \in \mathcal{Z}_-(X)$. Then*

$$|\Pi_X(u - z) \cap \Lambda| = \mathcal{T}_X(u - z) = f_z(D)T_X(u). \tag{13}$$

The central $\mathcal{P}$-space and various other generalised $\mathcal{P}$-spaces have a canonical basis [12, 15]. Up to now, no general construction for a basis of the internal space $\mathcal{P}_-(X)$ was known (cf. [3, 12, 14]). The polynomials f_z form such a basis.

Corollary 5 *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is unimodular and spans U. Then $\{f_z : z \in \mathcal{Z}_-(X)\}$ is a basis for $\mathcal{P}_-(X)$.*

We also obtain a new basis for the central space $\mathcal{P}(X)$. Let $w \in U$ be a *short affine regular vector*, i.e. a vector whose Euclidian length is close to zero that is not

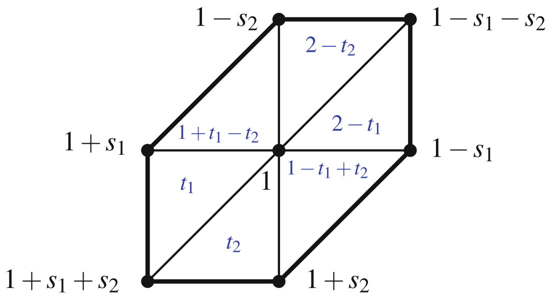

Fig. 1 The local pieces of the box spline (*blue*) and the polynomials f_z (*black*) corresponding to Example 7

contained in any hyperplane generated by sublists of X. Let $\mathcal{Z}(X, w) := (Z(X) - w) \cap \Lambda$. It is known that $\dim \mathcal{P}(X) = |\mathcal{Z}(X, w)| = \text{vol}(Z(X))$ [12].

Corollary 6 *Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is unimodular and spans U. Then $\{f_z : z \in \mathcal{Z}(X, w)\}$ is a basis for $\mathcal{P}(X)$.*

Example 7 Let $X = ((1, 0), (0, 1), (1, 1)) \subseteq \mathbb{Z}^2$. Then $\mathcal{P}_-(X) = \mathbb{R}$, $\mathcal{P}(X) = \text{span}\{1, s_1, s_2\}$, $\mathcal{Z}_-(X) = \{(1, 1)\}$, and $f_{(1,1)} = 1$. $\Pi_X(u_1, u_2) \cong [0, \min(u_1, u_2)] \subseteq \mathbb{R}^1$. The multivariate spline and the vector partition function are:

$$T_X(u_1, u_2) = \begin{cases} u_2 & \text{for } 0 \le u_2 \le u_1 \\ u_1 & \text{for } 0 \le u_1 \le u_2 \end{cases} \quad \text{and } \mathcal{T}_X(u_1, u_2) = \begin{cases} u_2{+}1 & \text{for } 0 \le u_2{\le}u_1 \\ u_1{+}1 & \text{for } 0 \le u_1{\le}u_2 \end{cases}.$$

Corollary 4 correctly predicts that $T_X(u)|_{\mathbb{Z}^2} = \mathcal{T}_X(u-(1, 1))$. Figure 1 shows the six non-zero local pieces of the piecewise linear function B_X and the seven polynomials f_z attached to the lattice points of the zonotope $Z(X)$.

4 Deletion and Contraction

A crucial part of the proofs of our results are deletion-contraction arguments. Let $x \in X$. We call the list $X \setminus x$ the *deletion* of x. The image of $X \setminus x$ under the canonical projection $\pi_x : U \to U/\,\text{span}(x) =: U/x$ is called the *contraction* of x. It is denoted by X/x. Note that since X is unimodular, $\Lambda/x \subseteq U/x$ is a lattice for every $x \in X$ and X/x is unimodular with respect to this lattice. All structures studied in this chapter behave nicely under deletion and contraction. Namely:

1. The box spline satisfies $D_x B_X = \nabla_x B_{X\setminus x}$ and $B_{X/x}(\bar{u}) = \sum_{\lambda \in \mathbb{Z}} B_X(u + \lambda x)$.
2. There is a canonical bijection $\mathcal{Z}_-(X) \setminus \mathcal{Z}_-(X \setminus x) \to \mathcal{Z}_-(X/x)$.
3. The equation $x f_z^{X\setminus x} = f_z^X - f_{z+x}^X$ holds.
4. The following sequence is exact:

$$0 \to \mathcal{P}_-(X \setminus x) \xrightarrow{\cdot p_x} \mathcal{P}_-(X) \xrightarrow{\pi_x} \mathcal{P}_-(X/x) \to 0. \tag{14}$$

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Root Polytopes of Crystallographic Root Systems

Mario Marietti

Abstract Let Φ be a finite (reduced) irreducible crystallographic root system. We give a case-free explicit description of the convex hull of all roots in Φ, that we denote by $\mathcal{P}_\Phi$ and call the root polytope of Φ. This description is attained by considering a set of distinguished faces, indexed by the subsets of a fixed root basis Π, which is a complete set of representatives of the orbits of the faces under the action of the Weyl group W. The description reveals a rich combinatorial structure of the root polytope $\mathcal{P}_\Phi$ and gives as by-products some results on root systems which may be interesting on their own. Even if the proofs (which also are case-free) are clearly omitted, the results are presented in the order they are proved. This is a report on Cellini (Int. Math. Res. Not. **12**, 4392–4420 (2015); J. Algebr. Comb. **39**(3), 607–645 (2014)).

1 Coordinate Faces

Let us fix an arbitrary finite (reduced) irreducible crystallographic root system Φ in an n-dimensional Euclidean space $\mathbb{E}$, and denote by $\mathcal{P}_\Phi$ the root polytope of Φ, which is the convex hull of all roots in Φ.

The choice of a root basis $\Pi = \{\alpha_1, \dots, \alpha_n\}$ of Φ provides a special set of faces of $\mathcal{P}_\Phi$. If $\breve{\Omega} = \{\breve{\omega}_1, \dots, \breve{\omega}_n\}$ is the dual basis of Π in $\mathbb{E}$, i.e., the corresponding set of fundamental coweights, Φ^+ is the corresponding set of positive roots, and $\theta = \sum_{i=1}^{n} m_i \alpha_i \in \Phi^+$ is the corresponding highest root of Φ, then each hyperplane $\{x \in \mathbb{E} : (x, \breve{\omega}_i) = m_i\}$, for $i = 1, \dots, n$, supports a face F_i of $\mathcal{P}_\Phi$ containing θ. As in [2], we call F_i the ith coordinate face.

Coordinate faces may or may not be facets of $\mathcal{P}_\Phi$. For example, in type A_2, both coordinate faces are facets while, in type C_2, the coordinate face F_2 is a facet properly containing the coordinate face F_1 (the simple roots are numbered according to [1]).

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In the following proposition, we collect some of the properties of coordinate faces. We recall that the root poset is the set of positive roots Φ^+ partially ordered by the relation: $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of simple roots. We denote by $c_k(x)$ the k-th coordinate of x with respect to the root basis Π.

Proposition 1.1 *The coordinate faces F_i, $i \in \{1, 2, \dots, n\}$, satisfy the following properties:*

1. *$F_i \neq F_j$ if $i \neq j$.*
2. *The sum of two roots in F_i is never a root.*
3. *$F_i \cap \Phi$ is an interval in the root poset, i.e., there exists $\eta_i \in \Phi^+$ such that*

$$F_i \cap \Phi = [\eta_i, \theta].$$

4. *The dimension of F_i equals the cardinality of the set $\{k \mid c_k(\eta_i) \neq c_k(\theta)\}$.*
5. *The barycenter of F_i is parallel to $\breve{\omega}_i$.*
6. *Two coordinate faces are never in the same W-orbit.*

The following proposition shows that the poset of the coordinate faces with respect to inclusion can be directly detected from the extended (i.e. affine) Dynkin diagram of Φ. We denote by α_0 the affine simple root.

Proposition 1.2 *Let $i, j \in \{1, 2, \dots, n\}$. Then:*

- *F_i is a facet if and only if F_i is maximal among the coordinate faces.*
- *$F_i \subseteq F_j$ if and only if every path from α_j to α_0 in the extended Dynkin diagram of Φ contains α_i.*

Note that, by definition, $F_i \subseteq F_j$ means that every root in Φ^+ with ith coordinate (with respect to Π) equal to m_i (i.e. as greater as possible) has also jth coordinate equal to m_j.

2 Standard Parabolic Faces

For each $I \subseteq \{1, \dots, n\}$, the intersection $F_I := \cap_{i \in I} F_i$ is a face of $\mathcal{P}_\Phi$. As in [2], we call the F_I, for all $I \subseteq \{1, \dots, n\}$, the *standard parabolic faces*.

Many of the properties of Proposition 1.1 hold also for the standard parabolic faces.

Proposition 2.1 *Let $I \subseteq \{1, 2, \dots, n\}$. Then:*

1. *The sum of two roots in F_I is never a root.*
2. *$F_I \cap \Phi$ is an interval in the root poset, i.e., there exists $\eta_I \in \Phi^+$ such that*

$$F_I \cap \Phi = [\eta_I, \theta].$$

3. *The dimension of F_I equals the cardinality of the set $\{k \mid c_k(\eta_I) \neq c_k(\theta)\}$.*

4. *The barycenter of* F_I *is in the cone generated by* $\check{\omega}_i$, $i \in I$.
5. *Two standard parabolic faces are never in the same* W*-orbit.*

Differently from the case of the coordinate faces, we have that $F_I = F_J$ does not imply $I = J$. In the characterization of those subsets $J \subseteq \{1, 2, \ldots, n\}$ such that F_J equals a prescribed F_I, the extended Dynkin diagram again comes out.

We let $\hat{\Pi} := \Pi \cup \{\alpha_0\}$ and $\Pi_I := \{\alpha_i \mid i \in I\}$ for each $I \subseteq [n]$, and we denote by $(\hat{\Pi} \setminus \Pi_I)_0$ the connected component of α_0 in the Dynkin graph of $\hat{\Pi} \setminus \Pi_I$. We define the closure $\bar{I}$ and the border ∂I of I by

- $\bar{I} := \{k \mid \alpha_k \notin (\hat{\Pi} \setminus \Pi_I)_{\alpha_0}\}$.
- $\partial I := \{k \mid \alpha_k \notin (\hat{\Pi} \setminus \Pi_I)_{\alpha_0},$ and α_k is adjacent to $(\hat{\Pi} \setminus \Pi_I)_{\alpha_0}\}$.

Clearly $\partial I \subseteq I \subseteq \bar{I}$. The subsets J such that F_J equals a prescribed F_I form an interval in the Boolean lattice whose top and bottom elements are, respectively, the closure and the border of I; furthermore, the first gives the dimension of F_I, while the second gives its stabilizer.

Theorem 2.2 *Let* $I \subseteq \{1, 2, \ldots, n\}$. *Then:*

1. $F_J = F_I$ *if and only if* $\partial I \subseteq J \subseteq \bar{I}$.
2. *The dimension of* F_I *is equal to* $n - |\bar{I}|$ *(the cardinality of* $\bar{I}$*).*
3. *The stabilizer of* F_I *in* W *is* $W_{\Pi \setminus \Pi_{\partial I}}$*, the parabolic subgroup of* W *generated by the reflections through the roots not in* $\Pi_{\partial I}$.

As a consequence of Theorem 2.2, we have the following result.

Corollary 2.3 *There is an inclusion preserving bijection* σ *between the standard parabolic faces and the connected subdiagrams of the extended Dynkin diagram containing* α_0*; the map* σ *sends* F_I *to* $(\hat{\Pi} \setminus \Pi_I)_{\alpha_0}$*,and the dimension of* F_I *equals the number of vertices of* $\sigma(F_I) - 1$.

3 Global Properties of the Root Polytope $\mathcal{P}_\Phi$

The standard parabolic faces are the unique faces up to the action of the Weyl group.

Theorem 3.1 *The standard parabolic faces form a complete set of representatives of the orbits of the action of the Weyl group* W *on the set of faces (of all dimensions).*

Thanks to Theorem 3.1 and the results in the previous section, we obtain some global properties of the root polytope $\mathcal{P}_\Phi$. We collect them in the following result.

Corollary 3.2

1. *The* W*-orbits are parametrized by the connected subdiagrams of the extended Dynkin diagram which contain the affine simple root* α_0.

2. *The half-space representation of the root polytope is the following:*

$$\mathcal{P}_\Phi = \{x \mid (w\breve{\omega}_i, x) \leq c_i(\theta)\}$$

where i varies over all the integer in $\{1, \dots, n\}$ such that $\hat{\Pi} \setminus \{\alpha_i\}$ is connected, and w varies over W^{α_i} (the minimal coset representatives of the parabolic subgroup $W_{\Phi\setminus\{\alpha_i\}}$). Moreover, the above one is the minimal set of linear inequalities that defines $\mathcal{P}_\Phi$ as an intersection of half-spaces.

3. *The f-polynomial of the root polytope is:*

$$\sum_{\Gamma} [W : W_{\Pi\setminus\Pi_{\partial(\Pi\setminus\Gamma)}}] t^{|\Gamma|}$$

where the sum is over $\Gamma \subseteq \Pi$ such that $\Gamma \cup \{\alpha_0\}$ is connected.

The first assertion of Corollary 3.2 could also be obtained as a consequence of some results by Vinberg (see [6]).

Remark 3.3

1. Each standard parabolic face has a natural algebraic interpretation. Let $\mathfrak{g}$ be a complex simple Lie algebra having root system Φ with respect to a Cartan subalgebra $\mathfrak{h}$. For all $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root space corresponding to α, and let $\mathfrak{b} = \mathfrak{h} + \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ be the standard Borel subalgebra of $\mathfrak{g}$ corresponding to Φ^+.
   An Abelian ideal of $\mathfrak{b}$ is a subspace $\mathfrak{i} \subseteq \mathfrak{b}$ such that $[\mathfrak{b}, \mathfrak{i}] \subseteq \mathfrak{i}$ and $[\mathfrak{i}, \mathfrak{i}] = \{0\}$. These are of the form $\mathfrak{i} = \sum_{\alpha\in\Gamma} \mathfrak{g}_\alpha$ for $\Gamma \subseteq \Phi^+$ such that $(\Phi^+ + \Gamma) \cap \Phi^+ \subseteq \Gamma$ and $(\Gamma + \Gamma) \cap \Phi^+ = \emptyset$. Properties (1) and (2) of Proposition 2.1 imply that

$$\sum_{\alpha\in F_I} \mathfrak{g}_\alpha$$

   is a principal Abelian ideal of $\mathfrak{b}$ (generated by any non-zero vector in $\mathfrak{g}_{\eta_I}$).
2. Our results have also a direct interesting application in the study of partition functions. More precisely, for all γ in the root lattice, let $|\gamma|$ be the minimum number of roots needed to express γ as a sum of roots. In [5], Chirivì uses the results on the root polytope $\mathcal{P}_\Phi$ to prove several properties of the map $\gamma \mapsto |\gamma|$; in particular, the map is piecewise quasi-linear with the cones over the facets of $\mathcal{P}_\Phi$ as quasi-linearity domains.
3. Types A and C root polytopes have many further apparently unrelated special properties. We refer the reader to [3, 4].

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On Product Formulas for Volumes of Flow Polytopes

Karola Mészáros

Abstract We outline the construction of a family of polytopes $\mathcal{P}_{m,n}$, indexed by $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 2}$, whose volumes are given by the product

$$\prod_{i=m+1}^{m+n-2} \frac{1}{2i+1} \binom{m+n+i}{2i}.$$

The Chan-Robbins-Yuen polytope CRY_n, whose volume is $\prod_{i=1}^{n-2} C_i$, coincides with $\mathcal{P}_{0,n-1}$. Our construction of the polytopes $\mathcal{P}_{m,n}$ is an application of a systematic method we develop for expressing volumes of a class of flow polytopes as the number of certain triangular arrays. The latter is also the constant term of a formal Laurent series.

1 Flow Polytopes and the Chan-Robbins-Yuen Polytope

This contribution is based on the Mészáros work in [4], where we develop an encoding of triangulations for a large class of flow polytopes. Using this encoding, we prove volume formulas for a family of flow polytopes, of which, the Chan-Robbins-Yuen (CRY_n) polytope is a special case. In this section we define flow polytopes, CRY_n, and explain how they relate to each other. We also construct a family of flow polytopes generalizing CRY_n. In the next section we define Kostant partition functions and connect them to flow polytopes. Finally, in Sect. 3 we give general results about volumes of flow polytopes and explain how certain conjectures of Chan, Robins and Yuen about triangular arrays can be seen as conjectures about volumes of flow polytopes. Note that throughout this note we are working with normalized volumes, but for brevity we omit the word normalized.

Chan et al. defined CRY_n as the convex hull of the set of $n \times n$ permutation matrices π with $\pi_{ij} = 0$ if $j \geq i + 2$ [3], which can be shown to equal the flow polytope of K_{n+1}. The volume of CRY_n is the product of the first $n - 2$ Catalan

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numbers, which Zeilberger [7] proved analytically. The lack of a combinatorial understanding of vol(CRY_n) has captivated combinatorialists since the CRY_n polytope was introduced. We prove that CRY_n is a member of a larger family of polytopes which have nice product formulas as their volumes. A combinatorial proof for these volume formulas, including vol(CRY_n), is yet to be found.

Theorem 1.1 ([4, Theorem 8]) *The volume of the flow polytope* $\mathcal{P}_{m,n} = \mathcal{F}_{\tilde{G}_{m,n}}$ *is*

$$\prod_{i=m+1}^{m+n-1} \frac{1}{2i+1}\binom{m+n+i+1}{2i}.$$

The Chan-Robins-Yuen polytope CRY_n *can be identified with* $\mathcal{P}_{0,n-2}$.

Before defining graphs $G_{m,n}$, we include a definition of flow polytopes and explain how to see CRY_n as one. Given a loopless graph G on the vertex set $[n+1]$ with all edges directed from their smallest to their largest vertex, let in(e) denote the smallest (initial) vertex of edge e and fin(e) the biggest (final) vertex of edge e. Let $E(G) = \{\{e_1, \ldots, e_l\}\}$ be the multiset of edges of G. A *flow* f of size one on G is a function $f : E \to \mathbb{R}_{\geq 0}$ from the edge set E of G to the set of nonnegative real numbers such that

$$1 = \sum_{e \in E, \mathrm{in}(e)=1} f(e) = \sum_{e \in E, \mathrm{fin}(e)=n+1} f(e),$$

and for $2 \leq i \leq n$

$$\sum_{e \in E, \mathrm{fin}(e)=i} f(e) = \sum_{e \in E, \mathrm{in}(e)=i} f(e).$$

The *flow polytope* $\mathcal{F}_G$ associated to the graph $G = (V, E)$ is the set of all flows $f : E \to \mathbb{R}_{\geq 0}$ of size one on G. The polytope $\mathcal{F}_G$ is a convex polytope in the Euclidean space $\mathbb{R}^{|E|}$ of all functions $f : E \to \mathbb{R}$. Its dimension is $\dim(\mathcal{F}_G) = |E| - |V| + 1$ [2]. Figure 1 shows the equations of $\mathcal{F}_{K_5}$ and explains why this polytope is the same as

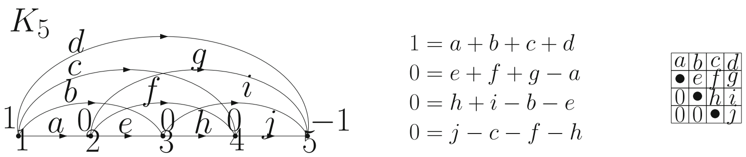

Fig. 1 Graph K_5 is given with *arrows* on its edges suggestive of the direction of the flows. The flow variables on the edges are $a, b, c, d, e, f, g, h, i, j$. Vertices $1, 2, 3, 4, 5$ are written below the graph, while the net flow vector $(1, 0, 0, 0, -1)$ is above the vertices. The equations defining the flow polytope corresponding to K_5 are in the *middle*. Note that these same equations define CRY_4 as can be seen from the matrix on the *left*, where we denoted by *filled circle* entries that are determined by the variables $a, b, c, d, e, f, g, h, i, j$

CRY_4. The arguments can be generalized to show that the the flow polytope $\mathcal{F}_{K_{n+1}}$ coincides with the Chan et al. polytope CRY_n [3]. CRY_n can also be thought of as $\mathcal{F}_{\widetilde{K_{n-1}}}$, which is how it is in Theorem 1.1, since $\tilde{G}$ is constructed from G by adding two extra vertices (one smaller than all the vertices of G, and one bigger) which connect to all the other vertices.

The graphs $G_{m,n}$, $m \in \mathbb{Z}_{\geq 0}$ from Theorem 1.1, are on the vertex set $[n+1]$ and their multiset of edges contain all edges of the complete graph, where the edges incident to 1 have multiplicity $m+1$. In Sect. 3 we explain how this theorem can be proved.

2 Flow Polytopes and Kostant Partition Functions

Flow polytopes are inherently tied to representation theory as the followings explain. Postnikov (2010, personal communication) and Stanley [6] discovered a remarkable connection between the volume of the flow polytope and the Kostant partition function K_G. Namely, they proved that given a loopless graph G on the vertex set $[n+1]$, the normalized volume $\mathrm{vol}(\mathcal{F}_G)$ of the flow polytope associated to graph G is

$$\mathrm{vol}(\mathcal{F}_G) = K_G(0, d_2, \ldots, d_n, -\sum_{i=2}^{n} d_i), \tag{1}$$

where $d_i = indeg_G(i) - 1$ for $i \in \{2, \ldots, n\}$, and $K_G(\mathrm{v})$ denotes the *Kostant partition function*, which is the number of ways to write the vector v as a nonnegative linear combination of the positive type A_n roots corresponding to the edges of G, without regard to order. To the edge (i, j), $i < j$, of G corresponds the positive type A_n root $e_i - e_j$, where e_i is the ith standard basis vector in $\mathbb{R}^{n+1}$. It is not hard to see based on the definitions of flow polytopes and Kostant partition functions that the Ehrhart polynomial of a flow polytope is a Kostant partition function, however, the result in (1) is far from obvious.

Along with Postnikov (2010, personal communication) and Stanley [6], Baldoni and Vergne [1, 2] also studied type A_n flow polytopes extensively with residue techniques. Mészáros and Morales [5] worked on flow polytopes of other types using combinatorial techniques.

We use the above connection between flow polytopes and Kostant partition functions to evaluate the latter at special vectors. These evaluations are of interest in representation theory, and the general computation of Kostant partition functions is #P-hard.

Proposition 2.1 ([4, Corollary 9])

$$K_{A_n^+}(m+1, m+2, \ldots, m+n, -nm - \binom{n+1}{2}) = \prod_{i=m+1}^{m+n-1} \frac{1}{2i+1} \binom{m+n+i+1}{2i}.$$

The above proposition was previously proved by Baldoni and Vergne [2, Sect. 3] using residue techniques.

3 Formulas for Volumes of Flow Polytopes

The main tool for establishing Theorem 1.1 is a systematic subdivision procedure we explain in [4], the most general consequences of which are the following (equivalent) results:

Theorem 3.1 ([4, Theorem 5]) *The volume* $\mathrm{vol}(\mathcal{F}_{\tilde{G}})$*, where G is a graph on the vertex set* $[n+1]$*, is equal to the number of triangular arrays* $(b_{i,j})_{i>j}$*,* $j \in [n-1]$*,* $i \in \{j+1, \ldots, n\}$*, with the constraints*

$$\sum_{j=i+1}^{n} b_{j,i} \leq indeg(i+1) + \sum_{k=1}^{i-1} b_{ik}, \text{ for all } i \in [n-1]$$

and constraints $b_{j,i} = 0$ *if* $(i+1, j+1) \notin E(G)$*.*

Theorem 3.2 ([4, Theorem 6]) *The volume* $\mathrm{vol}(\mathcal{F}_{\tilde{G}})$*, where G is a graph on the vertex set* $[n+1]$*, is equal to the constant term of*

$$\prod_{i=2}^{n}(1-x_i)^{-1} \prod_{(i,n+1)\in E(G)\,:\,2\leq i} (1-x_i)^{-1} \prod_{i=2}^{n} x_i^{-indeg(i)} \prod_{(i,j)\in E(G)\,:\,j\leq n} \left(1-\frac{x_i}{x_j}\right)^{-1}.$$

In [4] we use Theorems 3.1 and 3.2 in several instances. Namely, we show that several conjectures of Chan, Robins and Yuen, stated purely in the form of triangular arrays, refer to volumes of polytopes or sums of volumes of polytopes. For example, Chan et al. [3, Conjecture 2] describe a set of triangular arrays whose cardinality is $N(n,k) \times \prod_{i=1}^{n-1} C_i$. Using this description and the above theorems we construct the polytopes $P_i^{n,k}$, $i \in C_{n,k}$ in [4, Sect. 5] satisfying the following:

Theorem 3.3 ([4, Theorem 13]) *Fix* $n, k \in \mathbb{Z}$ *such that* $1 \leq k \leq n$*. Then, the sum of the volumes of the polytopes* $P_i^{n,k}$*,* $i \in C_{n,k}$*, is equal to*

$$N(n,k) \times \prod_{i=1}^{n-1} C_i.$$

Just as Theorem 3.3 corresponds to [3, Conjecture 2], so does Theorem 1.1 to [3, Conjecture 3]. For a detailed explanation see [4].

Theorem 3.1 can also be used to construct polytopes with combinatorial volumes. As an illustration of this in [4, Sect. 6] we construct polytopes whose volumes equal the number of r-ary trees on n internal nodes, $\frac{1}{(r-1)n+1}\binom{rn}{n}$.

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On the Topology of the Cambrian Semilattices

Myrto Kallipoliti and Henri Mühle

Abstract For an arbitrary Coxeter group W and a Coxeter element $\gamma \in W$, Reading and Speyer defined the Cambrian semilattice $\mathcal{C}_\gamma$ as the sub-semilattice of the weak order on W induced by so-called γ-sortable elements. In this note, we define an edge-labeling of $\mathcal{C}_\gamma$, and show that this is an EL-labeling for every closed interval of $\mathcal{C}_\gamma$. In addition, we use our labeling to show that every finite open interval in a Cambrian semilattice is either contractible or spherical, and we characterize the spherical intervals, generalizing a result by Reading.

1 Introduction

For every Coxeter group W, Reading and Speyer defined a family of subsemilattices of the weak order semilattice of W, indexed by the Coxeter elements of W, the so-called *Cambrian semilattice of* W, and realized these semilattices in terms of *sortable elements*, see [4].

This note investigates the topological properties of the order complex of the proper part of closed intervals in a Cambrian semilattice. Recall that a closed interval $[x, y]$ in a lattice is called *nuclear* if y is the join of atoms of $[x, y]$. Our main results are the following.

Theorem 1.1 *Every closed interval in* $\mathcal{C}_\gamma$ *is EL-shellable for every (possibly infinite) Coxeter group W and every Coxeter element* $\gamma \in W$.

Theorem 1.2 *Let W be a (possibly infinite) Coxeter group and let* $\gamma \in W$ *be a Coxeter element. Every finite open interval in the Cambrian semilattice* $\mathcal{C}_\gamma$ *is either contractible or spherical. Furthermore, a finite open interval* $(x, y)_\gamma$ *is spherical if and only if the corresponding closed interval* $[x, y]_\gamma$ *is nuclear.*

We remark that for finite crystallographic Coxeter groups Theorem 1.1 is implied by Ingalls and Thomas [1, Theorem 4.17] and on the other hand, for finite Coxeter groups, Theorem 1.2 is implied by concatenating [3, Theorem 1.1] and

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[3, Propositions 5.6 and 5.7]. However the methods used in [1, 3] cannot be applied to infinite Coxeter groups. On the contrary, our proofs are obtained completely within the framework of Coxeter-sortable elements, and thus have the advantage that they are uniform and direct. For a more detailed exposition of our results, we refer to [2].

2 Coxeter-Sortable Elements

Let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element, and define the half-infinite word

$$\gamma^{\infty} = s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | \cdots .$$

The vertical bars in the representation of γ^{∞} are "dividers", which have no influence on the structure of the word, but shall serve for a better readability. Clearly, every reduced word for $w \in W$ can be considered as a subword of γ^{∞}. Among all reduced words for w, there is a unique reduced word, which is lexicographically first considered as a subword of γ^{∞}. This reduced word is called the *γ-sorting word of w*.

Example 2.1 Consider the Coxeter group $W = \mathfrak{S}_5$, generated by $S = \{s_1, s_2, s_3, s_4\}$, where s_i corresponds to the transposition $(i, i+1)$ for all $i \in \{1, 2, 3, 4\}$ and let $\gamma = s_1 s_2 s_3 s_4$. Clearly, s_1 and s_4 commute. Hence, $w_1 = s_1 s_2 | s_1 s_4$ and $w_2 = s_1 s_2 s_4 | s_1$ are reduced words for the same element $w \in W$. Considering w_1 and w_2 as subwords of γ^{∞}, we find that w_2 is a lexicographically smaller subword of γ^{∞} than w_1. There are six other reduced words for w, namely

$$w_3 = s_1 s_4 | s_2 | s_1, \qquad w_4 = s_4 | s_1 s_2 | s_1, \qquad w_5 = s_4 | s_2 | s_1 s_2,$$
$$w_6 = s_2 s_4 | s_1 s_2, \qquad w_7 = s_2 | s_1 s_4 | s_2, \qquad w_8 = s_2 | s_1 s_2 s_4.$$

It is easy to see that among these w_2 is the lexicographically first subword of γ^{∞}, and hence w_2 is the γ-sorting word of w.

In the following, we consider only γ-sorting words, and we write

$$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} \mid s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} \mid \cdots \mid s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}}, \tag{1}$$

where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$. For each $i \in \{1, 2, \ldots, l\}$, we say that

$$b_i = \{s_j \mid \delta_{i,j} = 1\} \subseteq S$$

is the *ith block of w*. Then, w is called *γ-sortable* if and only if $b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l$, and we denote the set of γ-sortable elements of W by C_{γ}. The *γ-Cambrian* *semilattice of* W is then the semilattice $\mathcal{C}_\gamma = (C_\gamma, \leq_\gamma)$, where $\leq_\gamma$ denotes the restriction of the weak order on W to C_γ.

Example 2.2 Let us continue the previous example. We have seen that $w_2 = s_1s_2s_4|s_1$ is a γ-sorting word in W, and $b_1 = \{s_1, s_2, s_4\}$, and $b_2 = \{s_1\}$. Since $b_2 \subseteq b_1$, we see that w_2 is indeed γ-sortable.

3 EL-Shellability and Topology of the Closed Intervals in $\mathcal{C}_\gamma$

Now we define an edge-labeling of $\mathcal{C}_\gamma$ and sketch the proofs of Theorems 1.1 and 1.2.

3.1 El-Shellability

Define for every $w \in W$ the *set of positions* of the γ-sorting word of w as

$$\alpha_\gamma(w) = \big\{(i-1)\cdot n + j \mid \delta_{i,j} = 1\big\} \subseteq \mathbb{N},$$

where the $\delta_{i,j}$'s are the exponents from (1). We remark that the set of positions of w depends not only on the choice of the Coxeter element γ, but also on the choice of the reduced word of γ.

Example 3.1 Let $W = \mathfrak{S}_4$, $\gamma = s_1s_2s_3$ and consider $u = s_1s_2s_3|s_2$, and $v = s_2s_3|s_2|s_1$. Then, $\alpha_\gamma(u) = \{1, 2, 3, 5\}$, and $\alpha_\gamma(v) = \{2, 3, 5, 7\}$, where $u \in C_\gamma$, while $v \notin C_\gamma$.

Denote by $\mathcal{E}(\mathcal{C}_\gamma)$ the set of covering relations of $\mathcal{C}_\gamma$, and define an edge-labeling of $\mathcal{C}_\gamma$ by

$$\lambda_\gamma : \mathcal{E}(\mathcal{C}_\gamma) \to \mathbb{N}, \quad (u, v) \mapsto \min\big\{i \mid i \in \alpha_\gamma(v) \setminus \alpha_\gamma(u)\big\}. \tag{2}$$

Figure 1 shows the Hasse diagram of a part of a Cambrian semilattice associated with the affine Coxeter group $\tilde{A}_2$, together with the labels defined by the map λ_γ.

We prove Theorem 1.1 by showing that the map λ_γ defined in (2) is an EL-labeling for every closed interval in $\mathcal{C}_\gamma$. For this, we need the following lemma, which uses many of the deep results on Cambrian semilattices developed in [4].

Lemma 3.2 *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$ and let s be initial in γ. If $s \not\leq_\gamma u$ and $s \leq_\gamma v$, then the join $s \vee_\gamma u$ covers u in $\mathcal{C}_\gamma$.*

Now the proof of Theorem 1.1 is straightforward by using induction on rank and length, as well as Lemma 3.2.

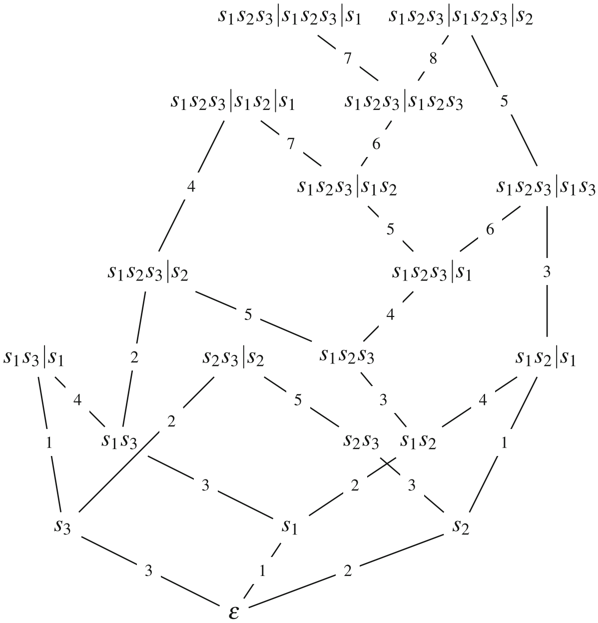

Fig. 1 The first seven ranks of an $\tilde{A}_2$-Cambrian semilattice, with the edge-labeling from (2)

3.2 Topology

We prove the following theorem by counting the falling maximal chains with respect to the labeling defined in (2).

Theorem 3.3 *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$ and let k denote the number of atoms of the interval $[u, v]_\gamma$. Then, $\mu(u, v) = (-1)^k$ if and only if $[u, v]_\gamma$ is nuclear. Otherwise, $\mu(u, v) = 0$.*

Again, the proof is straightforward using induction on rank and length and the following technical lemma.

Lemma 3.4 *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$ and let s be initial in γ. If $s \not\leq_\gamma u$ and $s \leq_\gamma v$, then the following are equivalent.*

1. *The interval $[u, v]_\gamma$ is nuclear.*
2. *There exists an element $v' \in [u, v]_\gamma$ satisfying $s \not\leq_\gamma v' \lessdot_\gamma v$ such that the interval $[u, v']_\gamma$ is nuclear.*

4 Example

Consider the affine Coxeter group $\tilde{A}_2$, which is generated by the set $\{s_1, s_2, s_3\}$ satisfying $(s_1s_2)^3 = (s_1s_3)^3 = (s_2s_3)^3 = \varepsilon$, as well as $s_1^2 = s_2^2 = s_3^2 = \varepsilon$ and let $\gamma = s_1s_2s_3$. Figure 1 shows the sub-semilattice of the Cambrian semilattice $\mathcal{C}_\gamma$ consisting of all γ-sortable elements of $\tilde{A}_2$ of length ≤ 7.

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cd-Index for CW-Posets

Satoshi Murai

Abstract The flag f-vector is a basic combinatorial invariant of graded posets that counts the number of chains. For an Eulerian poset, its flag f-vector is efficiently encoded by a certain non-commutative polynomial, called the **cd**-index. In this note, we give an extensions of the **cd**-index which can be defined for all CW-posets that are not necessary Eulerian. The details for this work are provided in our paper (Murai and Yanagawa, Squarefree P-modules and the **cd**-index, Adv. Math. **265**, 241–279 (2014).).

First of all, we quickly recall the definition of the **cd**-index. We refer the readers to [5] for basics on the theory of partially ordered sets. Let P be a finite graded poset having rank n with the minimal element $\hat{0}$. For a subset $S \subset [n] = \{1, 2, \ldots, n\}$, an S-*chain* of P is a chain $\hat{0} < \sigma_1 < \cdots < \sigma_k$ in P with $\{\text{rank}\,\sigma_1, \ldots, \text{rank}\,\sigma_k\} = S$. Let $f_S(P)$ be the number of S-chains of P. Define $h_S(P)$ by

$$h_S(P) = \sum_{T \subset S} (-1)^{|S|-|T|} f_T(P)$$

for all $S \subset [n]$. The vectors $(f_S : S \subset [n])$ and $(h_S(P) : S \subset [n])$ are called the *flag* f-*vector* and the flag h-*vector* of P, respectively. The flag h-vector of P is often expressed as a homogeneous non-commutative polynomial in variables $\mathbf{a}$ and $\mathbf{b}$, called the **ab**-*index* of P, defined by

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{S \subset [n]} h_S(P) w_S \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle,$$

where $w_S = w_1 w_2 \cdots w_n$ is defined by $w_i = \mathbf{a}$ if $i \notin S$ and $w_i = \mathbf{b}$ if $i \in S$ and where $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is the non-commutative polynomial ring with variables $\mathbf{a}$ and $\mathbf{b}$ over $\mathbb{Z}$.

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A finite poset P with the minimal element $\hat{0}$ and the maximal element $\hat{1}$ is called *Eulerian* if $\mu_P(\sigma,\tau) = (-1)^{\operatorname{rank}\tau-\operatorname{rank}\sigma}$ for all $\sigma < \tau$ in P, where $\mu_P(-,-)$ is the Möbius function of P. It was proved by Fine that if P is Eulerian, then $\Psi_{P-\{\hat{1}\}}(\mathbf{a},\mathbf{b})$ can be written as a polynomial in $\mathbf{a}+\mathbf{b}$ and $\mathbf{ab}+\mathbf{ba}$, that is, there is a polynomial $\Phi_P(\mathbf{c},\mathbf{d}) \in \mathbb{Z}\langle \mathbf{c},\mathbf{d}\rangle$ such that $\Psi_{P-\{\hat{1}\}}(\mathbf{a},\mathbf{b}) = \Phi_P(\mathbf{a}+\mathbf{b},\mathbf{ab}+\mathbf{ba})$. See [5, Theorem 3.17.1]. The polynomial $\Phi_P(\mathbf{c},\mathbf{d})$ is called the **cd**-*index* of P. The **cd**-index has a nice property if P has a nice structure. Indeed, it was shown in [2] that the **cd**-index of Gorenstein* posets (this class contains face posets of regular CW-decompositions of spheres if we ignore $\hat{1}$) is non-negative. In this note, we give a generalization of the **cd**-index.

We regard polynomials in $\mathbb{Z}\langle \mathbf{c},\mathbf{d}\rangle$ as polynomials in $\mathbb{Z}\langle \mathbf{a},\mathbf{b}\rangle$ by the identifications $\mathbf{c}=\mathbf{a}+\mathbf{b}$ and $\mathbf{d}=\mathbf{ab}+\mathbf{ba}$. For a poset P and $\sigma,\tau \in P$, let $[\sigma,\tau]=\{\rho \in P : \sigma \le \rho \le \tau\}$ and $(\sigma,\tau)=\{\rho\in P : \sigma<\rho<\tau\}$. We say that P is *Cohen–Macaulay* if the order complex

$$\Delta(P) = \big\{\{\sigma_1,\dots,\sigma_k\} \subset P : \sigma_1 < \cdots < \sigma_k\big\}$$

is a Cohen–Macaulay simplicial complex [4, Chap. II]. The following is the general version of the main result.

Theorem 1 *Let P be a finite graded poset with the minimal element $\hat{0}$. If $[\hat{0},\sigma]$ is Eulerian for all $\sigma \in P$, then there are unique* **cd**-*polynomials $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}, \Phi^{\mathbf{b}}$ such that*

$$\Psi_P(\mathbf{a},\mathbf{b}) = \Phi^{\mathbf{d}}\cdot\mathbf{d} + \Phi^{\mathbf{a}}\cdot\mathbf{a} + \Phi^{\mathbf{b}}\cdot\mathbf{b}.$$

Moreover, if P is Cohen–Macaulay then all the coefficients of $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}, \Phi^{\mathbf{b}}$ are non-negative.

The above theorem gives a generalization of the **cd**-index of Eulerian posets since if P is an Eulerian poset minus $\hat{1}$ then it clearly satisfies the assumption of the theorem and $\Psi_P(\mathbf{a},\mathbf{b}) = \Psi_P(\mathbf{b},\mathbf{a})$ (see [5, Corollary 3.16.6]), which imply that $\Phi^{\mathbf{a}} = \Phi^{\mathbf{b}}$ and $\Phi^{\mathbf{d}}\cdot\mathbf{d} + \Phi^{\mathbf{a}}\cdot\mathbf{c}$ is the ordinal **cd**-index.

A finite poset P with the minimal element $\hat{0}$ is said to be a *CW-poset* if the order complex of $(\hat{0},\sigma)$ is homeomorphic to a sphere for all $\sigma \in P-\{\hat{0}\}$. The name of CW-posets comes from the result of Björner [1] who proved that a poset P is a CW-poset if and only if it is the face poset of a regular CW-complex. It is straightforward that a CW-poset satisfies the assumption of Theorem 1. The following special case of Theorem 1 gives a generalization of the **cd**-index for regular CW-complexes.

Corollary 2 *Let P be a CW-poset. There are unique* **cd**-*polynomials $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}, \Phi^{\mathbf{b}}$ such that*

$$\Psi_P(\mathbf{a},\mathbf{b}) = \Phi^{\mathbf{d}}\cdot\mathbf{d} + \Phi^{\mathbf{a}}\cdot\mathbf{a} + \Phi^{\mathbf{b}}\cdot\mathbf{b}. \tag{1}$$

Moreover, if P is Cohen–Macaulay then all the coefficients of $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}, \Phi^{\mathbf{b}}$ are non-negative.

The above corollary gives an efficient way to express flag f-vectors of CW-posets. Indeed, the existence of the expression (1) describes all linear equations satisfied by the flag f-vectors of CW-posets. See [3, Proposition 5.2]. Also, the non-negativity statement for Cohen–Macaulay CW-posets implies the following consequence on ordinal h-vectors. Recall that if P is a CW-poset having rank n, then the order complex of $P - \{\hat{0}\}$ is the barycentric subdivision of a regular CW-complex corresponding to P, and its h-vector $(h_0, h_1, \ldots, h_n)$ is given by $h_i = \sum_{S \subset [n], |S|=i} h_S(P)$.

Corollary 3 *If P is a Cohen–Macaulay CW-poset then the h-vector of the order complex of $P - \{\hat{0}\}$ is unimodal. In other words, the h-vector of the barycentric subdivision of a Cohen–Macaulay finite regular CW-complex is unimodal.*

In the rest of this note, we prove the existence part of Theorem 1 and Corollary 3. A full proof of Theorem 1 can be found in [3].

Proof of the Existence Part of Theorem 1

Suppose rank$P = n$. We first claim that there are polynomials $\Omega, \Upsilon \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ such that

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \Omega + \Upsilon \cdot \mathbf{b}. \tag{2}$$

For $\sigma \in P$, let $\partial\sigma = [\hat{0}, \sigma] - \{\sigma\}$. It follows from [5, p. 317, Eq. (3.81)] that

$$\Psi_P(\mathbf{a}, \mathbf{b}) = (\mathbf{a} - \mathbf{b})^n + \sum_{\sigma \in P - \{\hat{0}\}} \Psi_{\partial\sigma} \cdot \mathbf{b}(\mathbf{a} - \mathbf{b})^{n - \operatorname{rank} \sigma}.$$

Since $\Psi_{\partial\sigma} \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ for any $\sigma \in P$ by the assumption, to prove the existence of (2), it suffices to prove that, for any $\Phi, \Psi \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$, there are $\Phi', \Psi' \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ such that $(\Phi + \Psi\mathbf{b})(\mathbf{a} - \mathbf{b}) = \Phi' + \Psi'\mathbf{b}$. Then, the next computation proves the claim.

$$(\Phi + \Psi\mathbf{b})(\mathbf{a} - \mathbf{b}) = \Phi \cdot (\mathbf{a} - \mathbf{b}) + \Psi \cdot (\mathbf{ba} - \mathbf{b}^2) = \Phi \cdot (\mathbf{c} - 2\mathbf{b}) + \Psi \cdot (\mathbf{d} - \mathbf{cb}).$$

Consider the expression (2). Write $\Omega = \Omega^{\mathbf{c}} \cdot \mathbf{c} + \Omega^{\mathbf{d}} \cdot \mathbf{d}$. Then we have

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \Omega + \Upsilon \cdot \mathbf{b} = \Omega^{\mathbf{d}} \cdot \mathbf{d} + \Omega^{\mathbf{c}} \cdot \mathbf{a} + (\Omega^{\mathbf{c}} + \Upsilon) \cdot \mathbf{b},$$

which gives the desired formula. □

Proof of Corollary 3

Let $\Psi_P(\mathbf{a},\mathbf{b}) = \Phi^{\mathbf{d}} \cdot \mathbf{d} + \Phi^{\mathbf{a}} \cdot \mathbf{a} + \Phi^{\mathbf{b}} \cdot \mathbf{b}$ be the expression (1). By substituting $\mathbf{a} = 1$, one obtains

$$\Psi_P(1,\mathbf{b}) = \Phi^{\mathbf{d}}(1+\mathbf{b}, 2\mathbf{b}) \cdot 2\mathbf{b} + \Phi^{\mathbf{a}}(1+\mathbf{b}, 2\mathbf{b}) + \Phi^{\mathbf{b}}(1+\mathbf{b}, 2\mathbf{b}) \cdot \mathbf{b}. \tag{3}$$

On the other hand, by the definition of the h-vector, one has

$$\Psi_P(1,\mathbf{b}) = h_0 + h_1\mathbf{b} + \cdots + h_n\mathbf{b}^n. \tag{4}$$

For any homogeneous **cd**-polynomial $\Upsilon(\mathbf{c},\mathbf{d}) \in \mathbb{Z}\langle \mathbf{c},\mathbf{d}\rangle$ of degree k, where $\deg \mathbf{c} = 1$ and $\deg \mathbf{d} = 2$, whose coefficients are non-negative, $\Upsilon(1+\mathbf{b}, 2\mathbf{b})$ can be written as

$$\Upsilon(1+\mathbf{b}, 2\mathbf{b}) = \alpha_0(1+\mathbf{b})^k + \alpha_1\mathbf{b}(1+\mathbf{b})^{k-2} + \alpha_2\mathbf{b}^2(1+\mathbf{b})^{k-4} + \cdots$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots$ are non-negative integers. Thus if we write

$$\Upsilon(1+\mathbf{b}, 2\mathbf{b}) = \gamma_0 + \gamma_1\mathbf{b} + \cdots + \gamma_k\mathbf{b}^k,$$

then $(\gamma_0, \gamma_1, \ldots, \gamma_k)$ satisfies $\gamma_0 \le \cdots \le \gamma_{\frac{k}{2}} \ge \cdots \ge \gamma_k$ when k is even and $\gamma_0 \le \cdots \le \gamma_{\frac{k-1}{2}} = \gamma_{\frac{k+1}{2}} \ge \cdots \ge \gamma_k$ when k is odd. The desired statement follows by applying these facts to (3) and (4). ☐

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Bipartite Rigidity

Eran Nevo

Abstract We develop a bipartite rigidity theory for bipartite graphs parallel to the classical rigidity theory for general graphs. This theory coincides with the study of Babson–Novik's balanced shifting restricted to graphs. We establish bipartite analogs of the cone, contraction, deletion, and gluing lemmas, and apply these results to derive a bipartite analog of the rigidity criterion for planar graphs. Our result asserts that a bipartite graph is planar only if its balanced shifting does not contain $K_{3,3}$. We also discuss potential applications of this theory to Jockusch's cubical lower bound conjecture and to upper bound conjectures for embedded simplicial complexes.

Motivating Results and Conjectures First we recall an important *rigidity* criterion for planarity of graphs. An embedding of a graph G into $\mathbb{R}^d$ is a map assigning a vector $\phi(v) \in \mathbb{R}^d$ to every vertex v. The embedding is *stress-free* if there is no way to assign weights w_{uv} to edges, so that not all weights are equal to zero and every vertex is "in equilibrium:"

$$\sum_{v\,:\,uv \in E(G)} w_{uv}(\phi(u) - \phi(v)) = 0 \quad \text{for all } u. \tag{1}$$

The embedding is *infinitesimally rigid* if every assignment of velocity vectors $V(u) \in \mathbb{R}^d$ to vertices of G that satisfies

$$\langle V(v) - V(u), \phi(v) - \phi(u) \rangle = 0 \tag{2}$$

for every $uv \in E(G)$, must satisfy relation (2) for *every pair* of vertices.[1]

Author Eran Nevo reflecting a joint work in progress with Gil Kalai and Isabella Novik.

[1]Relation (2) asserts that the velocities respect (infinitesimally) the distance along an embedded edge. If these relations apply to all pairs of vertices the velocities necessarily come from a rigid motion of the entire space.

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Proposition 1 (Gluck, Dehn, Alexandrov, Cauchy) *A generic embedding of a simple planar graph in* $\mathbb{R}^3$ *is stress free. A generic embedding of a maximal simple planar graph in* $\mathbb{R}^3$ *is also infinitesimally rigid.*

This result of Gluck [10] is closely related to Cauchy's rigidity theorem for polytopes of dimension three, and is easily derived from its infinitesimal counterpart by Dehn and Alexandrov. It implies the following, which also follows from Euler's formula.

Proposition 2 (Euler, Descartes) *A simple planar graph with* $n \geq 3$ *vertices has at most* $3n - 6$ *edges.*

In higher dimensions, the following Euler-type upper bound inequality is conjectured. Let $\mathbb{S}^k$ denote the k-sphere.

Conjecture 1 For any nonnegative integer d, there is a constant $c(d)$ such that an arbitrary d-dimensional simplicial complex K that embeds in $\mathbb{S}^{2d}$ satisfies $f_d(K) \leq c(d)f_{d-1}(K)$.

This conjecture with the sharp constant $c(d) = d + 2$ was raised by Kalai, Sarkaria, and perhaps others.

Algebraic (symmetric) shifting is an operation introduced by Kalai [14, 15] that replaces a simplicial complex K with a "shifted" simplicial complex K^s. For graphs, symmetric shifting is closely related to infinitesimal rigidity. The shifting operation preserves various properties of the complex, and, in particular, the numbers of faces of every dimension. In dimension one, shifted graphs are known as *threshold graphs*.[2]

The following result (see [15, 18]) is closely related to Gluck's theorem, and clearly implies Euler's inequality of Proposition 2:

Proposition 3 *If* G *is a planar graph then the symmetric algebraic shifting of* G*,* G^s*, does not contain* K_5 *as a subgraph. Equivalently,* G^s *does not contain the edge* $\{4, 5\}$. *More generally, the same conclusion holds for any graph* G *that does not contain* K_5 *as a minor.*

Similarly, the following conjecture implies Conjecture 1 with $c(d) = d + 2$ [15]:

Conjecture 2 (Kalai, Sarkaria) If K is a d-dimensional complex embeddable in $\mathbb{S}^{2d}$, then K^s does not contain the Flores complex $\binom{[2d+3]}{\leq d+1}$.

One drawback of Proposition 3 is that G^s may contain $K_{3,3}$ and hence the planarity property is lost under shifting. This will be fixed using balanced shifting!

Preliminaries on Bipartite Graphs and Balanced Shifting A d-dimensional simplicial complex is called *balanced* if its vertices are colored with $d + 1$ colors

[2]In higher dimensions the class of shifted complexes is much reacher than the class of threshold complexes.

such that any edge is bicolored; in particular the $d + 1$ colors for the vertices of every d-simplex are all different. Thus a balanced graph is simply a bipartite graph.

Babson and Novik [4] defined a notion of balanced shifting, and associated with every balanced simplicial complex K a *balanced shifted* complex K^b. The following properties of K^b are important to us: it is a balanced complex, with the same face numbers as K, and is balanced shifted, namely, w.r.t the fixed total order on the vertices of each color, if $i \in F \in K^b$ and $j < i$ are vertices of the same color then $F \cup j \setminus i \in K^b$.

Let $G = (V = A \uplus B, E)$ be a bipartite graph. We usually identify A and B with the ordered sets $\{1 < 2 < \ldots < n\} := [n]$ and $\{1' < 2' < \ldots < m'\} := [m']$, respectively, and denote the edge connecting vertices i and j' by ij'. Given a pair of two fixed integers $k \leq n$ and $l \leq m$, we say that a total order on $V = A \uplus B$ is (k, l)-*admissible* if (i) it extends the natural orders on A and B, and (ii) the set $[k] \cup [l']$ forms an initial segment of V w.r.t. $<$. Instead of recalling the definition of $G^b = G^{b,<}$ (when computed w.r.t. a (k, l)-admissible order $<$), we will phrase properties of G^b in terms of (k, l)-*rigidity*, introduced below.

For G bipartite and u, v two vertices from the *same* part, the *contraction* of u with v is the graph G' on the vertex set $V - \{u\}$ obtained from G by identifying u with v and deleting the extra copy from each double edge that was created. Observe that G' is also bipartite.

(k, l)-Rigidity The goal here is to develop a rigidity theory for bipartite graphs, paralleling the one for general graphs [1, 2, 22, 24]. We recall from [15], see also [16, 17], that a (non-bipartite) graph G on the vertex set $[n]$ is *generically d-stress free* if and only if the pair $(d+1)(d+2)$ is *not* an edge of G^s, and that G is *generically d-rigid* if and only if the pair *dn is* an edge of G^s. Motivated by these results, we make the following definition:

Definition 1 Let $G = (A \uplus B, E)$ be a bipartite graph, let $k \leq n$ and $l \leq m$ be two fixed integers, and let $<$ be a (k, l)-admissible order on $A \uplus B$. We call G (generically) (k, l)-*stress free* if the pair $(k + 1)(l + 1)'$ is not an edge of $G^{b,<}$. We say that G is (generically) (k, l)-*rigid* if all pairs $ij' \in A \times B$ such that $i \leq k$ or $j' \leq l'$ are edges of $G^{b,<}$.

We now turn to two equivalent formulations, paralleling ones from classical rigidity.

Definition 2 Let $G = (A \uplus B, E)$ be a bipartite graph and let $\Theta \in \mathrm{GL}_n(\mathbb{R}) \times GL_m(\mathbb{R})$ be a block-generic matrix. Let $R^{(k,l)}(G)$ be an $|E| \times (l|A| + k|B|)$ matrix whose rows are labeled by the edges of G, whose columns occur in blocks of size l for each vertex in A and blocks of size k for each vertex in B, and whose block corresponding to $v \in V$ and $ab' \in E$ is given by

$$\begin{cases} (\theta_{i'b'} : \ 1 \leq i \leq l) & \text{if } v = a, \\ (\theta_{ia} : \ 1 \leq i \leq k) & \text{if } v = b, \\ 0 & \text{if } v \notin \{a, b\}. \end{cases}$$

The matrix $R^{(k,l)}(G)$ is called the *bipartite (k, l)-rigidity matrix of G*.

Proposition 4 *Let $G = (A \uplus B, E)$ be a bipartite graph. Then G is (k,l)-stress free if and only if the rows of $R^{(k,l)}(G)$ are linearly independent, and G is (k,l)-rigid if and only if* $\text{rank}(R^{(k,l)}(G)) = l|A| + k|B| - kl$. *The later happens if and only if the row spans satisfy* $\text{row}(R^{(k,l)}(G)) = \text{row}(R^{(k,l)}(K_{A,B}))$.

In the case of $k = l$, G is (k,k)-stress free if and only if it is k-acyclic in the sense of [12]. The case $k = 1$ can be traced to Whiteley [23].

The notions of (k,l)-stress freeness and (k,l)-rigidity have simple geometric interpretations analogous to Eqs. (1) and (2). To derive such interpretations, define a (k,l)-*embedding* of a bipartite graph $G = (A \uplus B, E)$ to be a map $\phi : A \uplus B \to \mathbb{R}^k \times \mathbb{R}^l$ that assigns to every $a \in A$ a vector $\phi(a) \in \mathbb{R}^k \times (0)$, and to every $b \in B$ a vector $\phi(b) \in (0) \times \mathbb{R}^l$. For instance, letting $\phi(a) = (\theta_{ia} \ : i \in [k]) \times (0)$ and $\phi(b) = (0) \times (\theta_{j'b} \ : j \in [l])$ defines a *generic* (k,l)-embedding.

Remark 1 A bipartite graph $G = (A \uplus B, E)$ is (k,l)-stress free if and only if for a *generic* (k,l)-embedding ϕ, there is no way to assign weights w_{ab} to edges so that not all weights are equal to zero and every vertex u satisfies:

$$\sum_{v \,:\, uv \in E} w_{uv}\phi(v) = 0.$$

A bipartite graph G is (k,l)-rigid if and only if for a generic (k,l)-embedding ϕ, every assignment of velocity vectors $V(a) \in (0) \times \mathbb{R}^l$ for $a \in A$ and $V(b) \in \mathbb{R}^k \times (0)$ for $b \in B$ that satisfies

$$\langle V(a), \phi(b) \rangle + \langle V(a), \phi(b) \rangle = 0 \tag{3}$$

for all $ab \in E$, must satisfy Eq. (3) for all $ab \in A \times B$.

We now establish bipartite analogs of the deletion, contraction, gluing and cone lemmas in classical rigidity.

Lemma 1 (Deletion Lemma) *Let G be a bipartite graph, v a vertex of G of degree d, and $G' = G - v$ the graph obtained from G by deleting v.*

1. *If G' is (k,l)-stress free and $d \leq \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-stress free.*
2. *If G' is (k,l)–rigid and $d \geq \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-rigid.*

Lemma 2 (Contraction Lemma) *Let $G = (V,E)$ be a bipartite graph, v and w two vertices of G that belong to the same part, C the set of common neighbors of*

v and w, and $G' = (V - \{v\}, E')$ the graph obtained from G by contracting v with w.

1. *If G' is (k,l)-stress free and $|C| \leq \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-stress free.*
2. *If G' is (k,l)-rigid and $|C| \geq \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-rigid.*

Lemma 3 (Gluing Lemma) *Let $G = (A \uplus B, E)$ be a bipartite graph written as the union $G = G_1 \cup G_2$ of two bipartite graphs $G_1 = (A_1 \uplus B_1, E_1)$ and $G_2 = (A_2 \uplus B_2, E_2)$.*

1. *If G_1 and G_2 are (k,l)-rigid, $|A_1 \cap A_2| \geq k$, and $|B_1 \cap B_2| \geq l$, then G is (k,l)-rigid.*
2. *If G_1 and G_2 are (k,l)-stress free, and $G_1 \cap G_2$ is (k,l)-rigid, then G is (k,l)-stress free.*

Definition 3 Let $G = (A \uplus B, E)$ be a bipartite graph, where $A = [n]$ and $B = [m']$. Let $A^* := A \cup \{0\}$ and $B^* := B \cup \{0'\}$. The *left-side cone over G*, $C^L G$, is the bipartite graph with the vertex set $A^* \uplus B$ and the edge set $E \cup \{0b' : b' \in B\}$. The *right-side cone over G*, $C^R G$, is the bipartite graph with the vertex set $A \uplus B^*$ and the edge set $E \cup \{a0' : a \in A\}$.

To compute the balanced shifting of $C^L G$, we extend our order $<$ on V to an order $<_0$ on $A^* \cup B$ by requiring that 0 is the smallest vertex. Similarly, to work with $C^R G$, we extend $<$ to an order $<_{0'}$ on $A \cup B^*$ by requiring that $0'$ as the smallest vertex. Note that if $<$ is (k,l)-admissible, then $<_0$ is $(k+1,l)$-admissible and $<_{0'}$ is $(k,l+1)$-admissible.

Lemma 4 (Cone Lemma) *The operations of coning and shifting commute, that is,*

$$(C^L G)^{b,<_0} = C^L(G^{b,<}) \quad \text{and} \quad (C^R G)^{b,<_{0'}} = C^R(G^{b,<}).$$

Thus, G is (k,l)-rigid if and only if $C^L G$ is $(k+1,l)$-rigid (equivalently, if and only if $C^R G$ is $(k,l+1)$-rigid), and G is (k,l)-stress free if and only if $C^L G$ is $(k+1,l)$-stress free (equivalently, if and only if $C^R G$ is $(k,l+1)$-stress free).

Bipartite Planar Graphs The following is a bipartite analogue of Proposition 3.

Theorem 1 *If G is a planar bipartite graph and $<$ is a $(2,2)$-admissible order, then $K_{3,3}$ is not a subgraph of $G^{b,<}$. Equivalently, planar bipartite graphs are $(2,2)$-stress free.*

In particular, the inequality $E \leq 2V - 4$ follows. Note that for G planar, G^b is also planar, unlike the case of G^s; see Proposition 7 for a higher dimensional analog.

Our proof of Theorem 1 can be considered as a bipartite analog of Whiteley's proof [24] of Gluck's result. It relies on the Contraction Lemma as well as on some combinatorial properties of bipartite planar graphs.

Theorem 1 along with the "minor"-part of Proposition 3 lead to the following problem: find an interesting notion of a minor for bipartite graphs, denoted $<_b$, for which $K_{3,3} <_b G$ would imply that G is not planar, and $K_{3,3} \subseteq G^b$ would imply that $K_{3,3} <_b G$. In a recent work [7] we suggest a definition for $<_b$ which yields an analog of Wagner's theorem [21]; specifically, we prove:

if G is bipartite then G is planar if and only if $K_{3,3} \not<_b G$.

Graphs of Cubical Polytopes We now discuss potential applications of bipartite rigidity, á la Kalai [13], to lower bound conjectures on face numbers of cell complexes with a bipartite 1-skeleton. Recall that by a result of Blind and Blind [5], if P is a cubical d-polytope with $d > 2$, then the graph of P, $G(P)$, is bipartite. Moreover, if $d > 2$ is even, then the two sides of $G(P)$ have the same number of vertices. (These results were generalized to arbitrary cubical spheres by Babson and Chan [3].) We are interested in the following conjecture of Jockusch [11].

Conjecture 3 (Jockusch) If K is a cubical polytope of dimension $d \geq 3$ with $f_0(K)$ vertices and $f_1(K)$ edges, then $f_1(K) \geq \frac{d+1}{2} f_0(K) - 2^{d-1}$.

Note that if G is $(2, d-1)$-rigid, and has the same number of vertices on each side, then G has at least $\frac{d+1}{2} f_0(G) - 2(d-1)$ edges. The graph $G(P)$ of a stacked cubical polytope P is bipartite, has the same number of vertices on each side, but has only $\frac{d+1}{2} f_0(P) - 2^{d-1}$ edges. (Recall that a stacked cubical polytope is a polytope obtained starting with a cube and repeatedly gluing (combinatorial) cubes onto facets.)

Proposition 5 *If P is a stacked cubical d-polytope,* $d \geq 3$*, then*

(i) It is possible to add to $G(P)$ *exactly* $2^{d-1} - 2(d-1)$ *edges, all in one facet of P, such that the resulting graph is* $(2, d-1)$*-rigid and stress free.*
(ii) It is possible to add to $G(P)$ *exactly* $2^{d-1} - d$ *edges, all in one facet of P, such that the resulting graph is* $(1, d)$*-rigid and stress free.*

This yields the following approach to Jockusch's conjecture; specifically, a positive answer to the following problem will imply Conjecture 3 for all *even* $d > 2$:

Problem 1 *Let G be the graph of a cubical d-polytope, where* $d > 2$ *is even. Is it possible to add* $2^{d-1} - 2(d-1)$ *edges to G to obtain a* $(2, d-1)$*-rigid graph? Is it possible to add* $2^{d-1} - d$ *edges to G to obtain a* $(1, d)$*-rigid graph?*

A similar reasoning shows that a positive answer to the following problem will imply Conjecture 3 for an arbitrary d:

Problem 2 *Let G be the graph of a cubical d-polytope, where* $d > 2$*. Is it possible to add* $2^{d-1} - \lfloor \frac{d+1}{2} \rfloor \lceil \frac{d+1}{2} \rceil$ *edges to G to obtain a* $(\lfloor \frac{d+1}{2} \rfloor, \lceil \frac{d+1}{2}) \rceil$*-rigid graph?*

The proof of Proposition 5(i) uses the Contraction, Gluing and Cone lemmas. The proof of part (ii) uses a Laman-type condition proved by Whiteley [23].

Balanced Complexes and Euler-Type Upper Bounds We posit the following bipartite analog of Conjecture 2:

Conjecture 4 Let K be a d-dimensional balanced complex that is topologically embeddable in $\mathbb{S}^{2d}$, and let $<$ be a $(2, 2, \ldots, 2)$-admissible order. Then $K^{b,<}$ does not contain the van Kampen complex $[3]^{*(d+1)}$, i.e., the $(d+1)$-fold join of 3 points.

In particular, $f_d(K) \leq 2f_{d-1}(K)$.

As with the Kalai–Sarkaria conjecture, Conjecture 4 is known so far only for $d = 0, 1$: the case $d = 0$ is obvious and the case $d = 1$ is Theorem 1. Using a random coloring and basic properties of balanced shifting, we prove that:

Proposition 6 *Conjecture 4 implies Conjecture 1 with* $c(d) = \frac{2(d+1)^{d+2}}{(d+1)!}$.

We now turn to rephrasing Conjecture 4 in terms of embeddability of K^b, a formulation that is *not* available for Conjecture 2: indeed, while shifted graphs not containing K_5 may be nonplanar (for instance, G^s where G is the graph of the octahedron), *balanced-shifted* bipartite graphs not containing $K_{3,3}$ are necessarily planar. This statement extends to higher dimensions, as the following proposition shows.

Proposition 7 *Let K be a d-dimensional balanced-shifted simplicial complex not containing* $[3]^{*(d+1)}$ *as a subcomplex. Then K is PL embeddable in* $\mathbb{S}^{2d}$.

Combining Proposition 7 with the well-known fact that the complex $[3]^{*(d+1)}$ is not PL embeddable in $\mathbb{S}^{2d}$ [8, 20], we obtain that Conjecture 4 is equivalent to the following:

Conjecture 5 If K is a d-dimensional balanced complex that is topologically embeddable in $\mathbb{S}^{2d}$, and $<$ is a $(2, \ldots, 2)$-admissible order, then $K^{b,<}$ is PL embeddable in $\mathbb{S}^{2d}$.

Let $\mathfrak{o}(K)$ denote the van Kampen obstruction to PL embeddability of a d-dimensional complex K in $\mathbb{S}^{2d}$, computed with coefficients in $\mathbb{Z}$. (One may also use other coefficients, e.g., $\mathbb{Z}/2\mathbb{Z}$). Recall that if K is PL embeddable in $\mathbb{S}^{2d}$, then $\mathfrak{o}(K) = 0$ (and the converse also holds provided $d \neq 2$), see [9, 19, 25]. As $\mathfrak{o}([3]^{*(d+1)}) \neq 0$ (even with $\mathbb{Z}_2$ coefficients) and as, according to [6], for $d \geq 3$ the topological embeddability of a d-dimensional complex K in $\mathbb{S}^{2d}$ is equivalent to the PL embeddability, we obtain that for $d \geq 3$, the following conjecture implies Conjecture 5, even when considered with $\mathbb{Z}_2$ coefficients.

Conjecture 6 Let K be a d-dimensional balanced complex. If $\mathfrak{o}(K) = 0$, then $\mathfrak{o}(K^b) = 0$.

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Balanced Manifolds and Pseudomanifolds

Isabella Novik

Abstract A simplicial $(d-1)$-dimensional complex K is called balanced if the graph of K (i. e., the 1-dimensional skeleton) is d-colorable. Here we discuss some recent results as well as several open questions on face numbers of balanced manifolds and pseudomanifolds; we also present constructions of balanced manifolds (with and without boundary) with few vertices. This work is joint with Steve Klee.

1 Basics of Simplicial Complexes

Let K be a $(d-1)$-dimensional simplicial complex. The main object of our study is the *f-vector* of K, $f(K) := (f_{-1}(K), f_0(K), \ldots, f_{d-1}(K))$, where $f_i = f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. We will be mainly concerned with the f-vectors of simplicial spheres, manifolds, and pseudomanifolds, that is, simplicial complexes whose geometric realizations form a topological sphere, manifold, and pseudomanifold, respectively. All terminology that is undefined here such as Stanley–Reisner rings, Cohen–Macaulay complexes, etc., can be found in [18].

A $(d-1)$-dimensional simplicial complex K is called balanced if the graph of K is d-colorable. (Thus, balanced 1-dimensional complexes are simply bipartite graphs.) The class of balanced complexes is a fascinating class of objects that arise often in combinatorics, algebra, and topology: for instance, barycentric subdivisions of *all* regular CW complexes are balanced; therefore, *every* triangulable space has a balanced triangulation! Coxeter complexes and Tits buildings form another large family of balanced complexes. It is also worth mentioning that the class of balanced complexes is closed under taking links. In addition, rank selected subcomplexes of a balanced complex are smaller-dimensional balanced complexes that inherit some of the properties of the original complex such as Cohen-Macaulayness (or Buchsbaumness) [17].

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What can we say about the face numbers of balanced simplicial spheres, manifolds, and pseudomanifolds? Do the results known for the class of *all* simplicial spheres, manifolds, and pseudomanifolds have balanced analogs? This is one of the main questions we discuss here.

It turns out that to study this question, for a $(d-1)$-dimensional simplicial complex K, it is sometimes more convenient to work not with the f-vector of K, but with the h-vector, $h(K) = (h_0(K), \ldots, h_d(K))$, defined by $\sum_{i=0}^{d} h_i(K)x^{d-i} = \sum_{i=0}^{d} f_{i-1}(K)(x-1)^{d-i}$. Thus, the h-vector is obtained from the f-vector by an invertible linear transformation, and hence contains exactly the same information as the f-vector. In contrast with f-numbers, h-numbers can, in general, be negative. However, by a result of [16] the h-numbers of simplicial spheres are always positive.

2 The Lower Bound Theorem

The Lower Bound Theorem (LBT, for short), due originally to Barnette [2], provides tight lower bounds on the number of i-dimensional faces of a (connected) simplicial manifold K in terms of the number of vertices of K and the dimension of K. An alternative proof of the LBT via the rigidity theory of frameworks was given in the seminal work of Kalai [9]. Kalai's proof was extended by Fogelsanger [5] to all simplicial complexes whose geometric realization is a normal pseudomanifold.

In the language of h-numbers, the LBT for spheres, manifolds, and normal pseudomanifolds translates to a single inequality: $h_2 \geq h_1$. The strengthening of this inequality proved in [14] asserts that if $d \geq 4$ and K is a simplicial $(d-1)$-dimensional orientable manifold, then

$$h_2(K) \geq h_1(K) + \binom{d+1}{2}\beta_1(K), \tag{2.1}$$

where $\beta_1(K) = \dim \tilde{H}_1(K)$ is the first Betti number of K. The cases of equality were characterized in [9] and [14], respectively: if K is a simplicial $(d-1)$-dimensional manifold and $d \geq 4$, then $h_2(K) = h_1(K)$ holds if and only if K is a stacked sphere, and if $d \geq 5$, equality holds in (2.1) if and only if K is in the Walkup class $\mathcal{H}^d$ introduced in [20]. Stacked spheres are those spheres that can be obtained by repeatedly taking connected sums of boundaries of simplices, and a $(d-1)$-dimensional simplicial manifold belongs to $\mathcal{H}^d$ if it is obtained from a stacked sphere through iterated handle addition.

As stacked spheres are very far from being balanced, one immediate question is if there are sharper versions of the LBT for balanced manifolds and pseudomanifolds. Such a balanced LBT was established in [6] for balanced spheres, and in [3] for balanced manifolds. Very recently we extended this result to balanced normal pseudomanifolds as well as treated the case of equality:

Theorem 2.1

1. *Let K be a balanced* $(d-1)$*-dimensional connected normal pseudomanifold, where* $d \geq 3$*. Then* $2h_2(K) \geq (d-1)h_1(K)$*.*
2. *Let K be a balanced* $(d-1)$*-dimensional connected manifold, where* $d \geq 4$*. Then* $2h_2(K) = (d-1)h_1(K)$ *if and only if K is a stacked cross-polytopal sphere.*

Stacked cross-polytopal spheres are obtained by forming connected sums of the boundaries of cross-polytopes (where, to preserve balanced-ness, we identify vertices of the same color). As such their number of vertices is a multiple of d. Hence, the above theorem implies that for $d \geq 4$, balanced spheres whose number of vertices n is not divisible by d satisfy the strict inequality $2h_2 > (d-1)h_1$. Can this inequality be further sharpened for such values of n? Does, for every n not divisible by d, the class of balanced $(d-1)$-spheres with n vertices contain a sphere that simultaneously minimizes all the f-numbers?

Mimicking the argument of [19, Theorem 4.3], it is not hard to show that if $d \geq 3$ and K is a balanced connected simplicial $(d-1)$-manifold with $\beta_1(K;\mathbb{Q}) \neq 0$, then $2h_2 - (d-1)h_1 \geq 4\binom{d}{2}$. This leads us to the following

Conjecture 2.2 If $d \geq 4$ and K is a connected balanced simplicial $(d-1)$-manifold (orientable over some field $\mathbb{F}$), then

$$2h_2(K) - (d-1)h_1(K) \geq 4\binom{d}{2}\beta_1(K;\mathbb{F}).$$

If Conjecture 2.2 is true, then the proposed inequality is sharp: the equality is achieved by the elements of $\mathcal{BH}^d$ — a natural balanced analog of Walkup's class $\mathcal{H}^d$. Moreover, we conjecture that for $d \geq 5$, the equality is achieved only if K belongs to $\mathcal{BH}^d$.

3 Constructions of Balanced Manifolds

What is the minimum number of vertices needed to triangulate a given triangulable manifold? Is there a triangulation of a given manifold that simultaneously minimizes all the f-numbers? While for 2-dimensional manifolds the answers are known from late 60s-early 70s (they follow from the inequalities of [7] and the constructions given in [15] and [8]), despite a tremendous amount of effort that went into studying this question in the last two decades (see for instance [12] and the upcoming book [13]) there are not too many higher-dimensional manifolds for which there is even a conjectural answer. The main difficulty appears to be a lack of standard constructions to produce triangulations with few vertices.

For a $(d-1)$-dimensional sphere, the minimal triangulation is given by the boundary of a simplex and requires d vertices. The minimal balanced triangulation of a $(d-1)$-dimensional sphere is given by the boundary of a cross-polytope and requires $2d$ vertices. Hence the next family of manifolds to study is the product of two spheres. If one of these spheres is a circle, then the minimum number of vertices needed was established in [1] and independently in [4]. What is the minimum number of vertices needed for a balanced triangulation? It is easy to see that one needs at least $3d$ vertices. Are $3d$ vertices enough? Very recently we established the following partial answers:

Theorem 3.1

1. *There is a construction of a balanced simplicial complex with $3d$ vertices whose geometric realization is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ if d is odd and to the non-orientable $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ if d is even.*
2. *For every $n \geq 3d + 2$, there is a construction of a balanced triangulation of both an orientable and a non-orientable $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ with n vertices.*
3. *The above mentioned $3d$-vertex triangulation simultaneously minimizes all the face numbers among all balanced triangulations of closed $(d-1)$-dimensional manifolds with a non-vanishing β_1 (computed with rational coefficients).*

This raises the following questions:

Problem 3.2 Are there balanced triangulations of spherical bundles over a circle with $3d + 1$ vertices? Are there balanced triangulations with $3d$ vertices for the non-orientable bundle when d is odd and the orientable bundle when d is even?

We end by discussing similar-type questions for manifolds with boundary. It follows from the results of [14] that if M is a triangulable $(d-1)$-dimensional manifold with boundary such that $\beta_i(M) \neq 0$ for some $0 \leq i \leq d-2$, then one needs at least $2d-i$ vertices to triangulate M. On the other hand, it is easy to see that to triangulate M in a balanced way one needs at least $2d$ vertices. The bound $2d-i$ in the non-balanced case is achieved by (i) disjoint unions of two $(d-1)$-simplices (here $i = 0$ and $f_0 = 2d$), and (ii) Kühnel's solid tori (if d is odd) and twisted tori (if d is even) [11] — in this case $i = 1$ and $f_0 = 2d - 1$. To the best of our knowledge these are the only known infinite families of examples. Surprisingly, in the balanced case there is a construction with $2d$ vertices for *every* pair $0 \leq i \leq d-2$, see [10]:

Theorem 3.3 *For every $0 \leq i \leq d-2$, there is a $(d-1)$-dimensional balanced complex $B_{i,d}$ such that $\|B_{i,d}\|$ is a manifold with boundary, $f_0(B_{i,d}) = 2d$, $\beta_i(B_{i,d}) = 1$, and $\beta_j(B_{i,d}) = 0$ for all $j \neq i$.*

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Some Combinatorial Constructions and Relations with Artin Groups

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Abstract Motivated by the computation of local system cohomology of Artin groups (and their configuration spaces) we introduce some very special class of sheaves over posets, called *weighted sheaves* over posets, which seem to be interesting by themselves. We relate them with some *graph complexes* arising from Artin groups. In particular, we find an interesting connection between some local cohomology of the braid groups, localized at the d-cyclotomic polynomial, and the cohomology of what we call *d-independent graph complexes*.

Let $(\mathbf{W}, S)$ be a Coxeter system, $|S|$ finite. Recall that one can associate:

- A **hyperplane arrangement** in $\mathbb{R}^n$

$$\mathcal{A} := \{H : H \text{ is the fixed point set of some reflection in } \mathbf{W}\}.$$

- **Configuration spaces**

$$\mathbf{Y} := [int(U) + i\mathbb{R}^n] \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \quad \text{and} \quad \mathbf{Y_W} := \mathbf{Y}/\mathbf{W}$$

  where U is the *Tits cone*

$$U := \mathbf{W}.\text{closure}(C_0) \qquad (C_0 \text{ is a fixed } \textit{chamber}) .$$

- A *finite* CW-complex $\mathbf{X_W} \subseteq \mathbf{Y_W}$, constructed from a union of convex polyhedra, one for each finite parabolic subgroup of $\mathbf{W}$, by explicit identifications on their faces, such that $\mathbf{Y_W}$ deformation retracts onto $\mathbf{X_W}$ (see [11, 12]).

Of special interest to us is the simplicial complex $K := K_{\mathbf{W}}$ of all subsets $\Gamma \subseteq S$ such that the parabolic subgroup $\mathbf{W}_\Gamma$ is finite.

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The **Artin group** of type $\mathbf{W}$ is $G_{\mathbf{W}} := \pi_1(\mathbf{Y_W})$. It has a presentation

$$G_{\mathbf{W}} =< g_s,\ s \in S\ :\ g_s g_{s'} g_s g_{s'} \ldots\ =\ g_{s'} g_s g_{s'} g_s \ldots\ (\ m(s,s')\ \text{factors}) >$$

where $m(s,s')$ are the coefficients of the Coxeter matrix. The main example is when $\mathbf{W}$ is the symmetric group Σ_n with $S = \{(i, i+1),\ i = 1, \ldots, n-1\}$: we get the classical configuration space of n different ordered points in $\mathbb{C}$ given by $\mathbf{Y} := \mathbb{C}^n \setminus \cup_{i,j} H_{ij}$, where $H_{ij} := \{z_i = z_j\}$ and the **orbit space** $\mathbf{Y}_{\Sigma_n} := \left(\mathbb{C}^n \setminus \cup_{i,j} H_{ij}\right) / \Sigma_n$, which is the configuration space of n unordered points in $\mathbb{C}$. The *braid group* Br_n is given by $\pi_1(\mathbf{Y}_{\Sigma_n})$.

Classical (and easy) fact is that $\mathbf{Y}_{\Sigma_n}$ is a space of type $K(\pi, 1)$ (while the analog result for the configuration spaces is known for some Artin groups but only conjectured in general). Recall also that for a discrete group G we have $H_*(G; M) := H_*(K(G,1); \mathcal{L}_M)$ for any G-module M, where $\mathcal{L}_M$ is the associated local system of coefficients. Therefore the homology (and cohomology) of $\mathbf{Y}_{\Sigma_n}$, with any twisted coefficients, equals the homology (and cohomology) of the braid group Br_n.

Particularly significant for geometrical reasons is the abelian representation which takes any generator g_s of the Artin group $G_{\mathbf{W}}$ into the $(-q)$-multiplication in the ring $R := \mathbb{Q}[q^{\pm 1}]$ of Laurent polynomials (minus sign is only for technical reasons). For many Artin groups, including finite type and some affine type Artin groups, several computations were done (see some references at the end).

We recall [12]: The homology groups $H_*(\mathbf{X_W}; R_q)$ are computed by the algebraic complex

$$C_k\ :=\ \oplus_{J \subseteq S,\ |J| = k,\ \mathbf{W}_J\ \text{finite}}\quad R\, e_J$$

with

$$\partial(e_J)\ =\ \sum_{I \subseteq J,\ |I| = k-1,\ \mathbf{W}_I\ \text{finite}} [I : J]\, \frac{\mathbf{W}_J(q)}{\mathbf{W}_I(q)}\, e_I. \tag{1}$$

Here $\mathbf{W}_J(q)$ is the Poincaré polynomial of $\mathbf{W}_J$ and $[I : J]$ is an incidence number.

Recall also (see [9] and [5] (different method) for case A_n, [6] for all other non-exceptional finite type cases over $\mathbb{Q}[q^{\pm 1}]$, [4] for the exceptional cases of finite type, and for case A_n over $\mathbb{Z}[q^{\pm 1}]$ see [1]):

Theorem 1

$$H_*(Br_n; R_q)_{(\varphi_d)} = \begin{cases} R/(\varphi_d) & \text{if } n \equiv 0 \text{ or } 1 \ (\text{mod } d) \\ 0 & \text{otherwise} \end{cases}$$

where the non vanishing term is in degree $(d-2)k$ *if* $n = dk$ *or* $n = dk + 1$. *On the left we have the* φ_d*-primary component of the homology,* φ_d *being the* d*-th cyclotomic polynomial.*

Let G be a graph on the set of vertices $\{1, \dots, n\}$. A *(vertex) property* $\mathcal{P}(G)$ of G is a collection of full vertex-subgraphs of G, closed with respect to inclusion: $\Gamma \in \mathcal{P}(G),\ \Gamma \supset \Gamma' \ \Rightarrow\ \Gamma' \in \mathcal{P}(G)$. Every property of G defines a simplicial scheme $K := K(\mathcal{P}(G))$ on $\{1, \dots, n\}$. The main example here is given by the *k-independent graph complexes*

$Ind_k(G) := \{$subgraphs $\Gamma \subseteq G$ such that each connected component of Γ has at most k vertices$\}$.

We consider here $Ind_k(A_n)$, where A_n is the linear graph with n-vertices. We have:

Theorem 2 (Sal.)

$$H_*(Br_{n+1}; R)_{(\varphi_d)} \ = \ \tilde{H}_{*+(d-1)}(Ind_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)})$$

(trivial coefficients in the right side).

Therefore Theorem 1 gives the homology of the independent graph complexes:

$$\tilde{H}_*(Ind_{(d-2)}(A_n)) \ = \ \begin{cases} \tilde{H}_*(S^{dk-2k-1}) & \text{for } n = dk \text{ or } n = dk-1. \\ 0 & \text{otherwise} \end{cases}$$

Conversely, one can compute directly (similar to [10] where the case of Ind_1 is considered) the homology of the independent graph complexes, and deduce the twisted cohomology of the braid group.

Let $(P, \prec)$ be a poset and let $(R, |)$ be the above ring endowed with the divisibility relation. Any morphism $\psi : (P, \prec) \to (R, |) : x \longrightarrow \psi(x) = w_x \in R$ defines a *sheaf of rings* over P (or a *diagram of rings*) by the collections

$$\{\ R/(w_x)\ ,\ x \in P\}, \quad \{\ i_{x,y} :\ R/(w_y) \ \to\ R/(w_x)\ ,\ x \preceq y\}$$

where $i_{x,y}$ is induced by the identity of R.

Definition We call the triple (P, R, ψ) a ***weighted sheaf*** over P and the coefficients w_x the ***weights*** of the sheaf.

The main example for us is when the poset is given by a simplicial complex (or a simplicial scheme) K with partial ordering $\sigma \prec \tau \ \Leftrightarrow\ \sigma \subseteq \tau$: then a weighted sheaf over K is given by assigning to each simplex $\sigma \in K$ a weight $w_\sigma \in R$, with $\sigma \prec \tau \Rightarrow w_\sigma | w_\tau$.

Let K be a simplicial scheme defined over $I_n := \{1, \dots, n\}$, and let $(C^0_*(K), \partial^0)$ be the standard algebraic complex of the simplicial homology of K.

Definition The **weighted complex** associated to the weighted sheaf (K, R, ψ) is the algebraic complex $(L_*(K), \partial)$ defined by

$$L_k(K) \ := \bigoplus_{|\sigma|=k} \frac{R}{(w_\sigma)}\, \bar{e}_\sigma$$

and boundary $\partial : L_k \rightarrow L_{k-1}$ induced by ∂^0 :

$$\partial(a_\sigma\, \bar{e}_\sigma) \;=\; \sum_{\tau \prec \sigma} [\tau : \sigma]\, i_{\tau,\sigma}(a_\sigma)\, \bar{e}_\tau.$$

Given any irreducible $\varphi \in R$, the φ-primary component of the weighted sheaf is obtained by using the weight $w_{(\varphi)} : K \rightarrow R$, defined as $w_{(\varphi)}(\sigma) := \varphi^{v_\varphi(\sigma)},\ \sigma \in K,$ where $v_\varphi(\sigma) :=$ max power of φ dividing w_σ. Then $S_{(\varphi)} := (K, R, w_{(\varphi)})$ is a weighted sheaf with associated complex $L_{(\varphi)}$. There is a natural increasing filtration into subcomplexes of $L_{(\varphi)}$ by the powers of φ_d:

$$F^s(L_{(\varphi)}) \;:=\; \bigoplus_{v_\varphi(\sigma) \leq s} \frac{R}{(\varphi^{v_\varphi(\sigma)})}\, \bar{e}_\sigma$$

and an associated increasing filtration of the simplicial complex K into subcomplexes:

$$K_{(\varphi),s} \;:=\; \{\sigma \in K \mid v_\varphi(\sigma) \leq s\}.$$

Then $F^s(L_{(\varphi)})$ is the weighted complex associated to the weighted sheaf $(K_{(\varphi),s}, R, w_{(\varphi)}|_{K_{(\varphi),s}})$.

Theorem 3 *Let (K, R, w) be a weighted sheaf, with associated weighted complex L_*. For any irreducible $\varphi \in R$, there exists a spectral sequence*

$$E^0_{p,q} \quad \Rightarrow \quad H_*(L_{(\varphi)})$$

that abuts to the homology of the φ-primary component of the associated algebraic complex L_*. Moreover the $E^1_{p,q}$-term is isomorphic to the relative $(p+q)$-homology with trivial coefficients *of the pair $(K_{(\varphi),p}, K_{(\varphi),p-1})$.*

A *variation* of standard methods from **Discrete Morse Theory** [7, 8] fits very well here.

Definition A *weighted acyclic matching* on a weighted sheaf (P, R, w) over P is an acyclic matching $\mathcal{M}$ on P such that $x \lhd y \in \mathcal{M} \Rightarrow w(x) = w(y)$.

To a weighted acyclic matching on $\mathcal{S} := (K, R, w)$ one can associate a Morse complex $(C^{\mathcal{M}}_*,\ \partial^{\mathcal{M}})$, which is a torsion complex generated by the critical cells. As in the standard case we still have:

Theorem 4 *One has an isomorphism $H_*(L_*(K), \partial) \cong H_*(C^{\mathcal{M}}_*, \partial^{\mathcal{M}})$.*

Given an irreducible φ, the associated filtration on the weighted complex induces a filtration $F^p C_*^{\mathcal{M}}$ onto the Morse complex. Let

$$\mathcal{F}^p C_*^{\mathcal{M}} \ := \ F^p C_*^{\mathcal{M}} \ / \ F^{p-1} C_*^{\mathcal{M}} \ \cong \ \oplus_{\sigma \text{ critical}, \ v_\varphi(\sigma) = p} \ \frac{R}{(\varphi^p)} \ \bar{e}_\sigma$$

be the quotient complex.

Theorem 5 *Let $\mathcal{M}$ be an acyclic matching for the weighted complex $\mathcal{S}_{(\varphi)}$, where φ is any irreducible. Then for the above spectral sequence one has $E^1_{p,q} \cong H_{p+q}(\mathcal{F}^p C_*^{\mathcal{M}})$. The differential $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ is induced by the boundary of the Morse complex and it is computed by using alternating paths in $\mathcal{M}$.*

To any Coxeter system $(\mathbf{W}, S)$ we can associate, for any d, a weighted sheaf (K, R, ψ_d) over the previous defined $K = K_{\mathbf{W}}$ by considering $\psi_d(J)$ as the maximum power of φ_d which divides $\mathbf{W}_J(q)$. The homology of the associated weighted complex is strictly related (by formula (1)) to the homology of the Artin group (actually, of the associated configuration space).

Coming back to the case of braid groups, for any d-th cyclotomic polynomial φ_d, a weighted sheaf (K, R, ψ_d) over $K(A_n)$ is defined by the weights

$$\psi_d(J) \ = \ \varphi_d^{\sum_{i=1}^m \lfloor \frac{n_i+1}{d} \rfloor}$$

where n_i is the cardinality of the i-th connected component of the subgraph of A_n generated by $J \subseteq \{1, \ldots, n\}$. The proof of Theorem 2 requires a suitable weighted acyclic matching on these weighted sheaves.

Further computations in some other not yet known cases were done, namely for some affine type Artin groups. By using the above described theory and suitable matchings on the associated weighted sheaves, we get [13] a full description of the twisted homology for all exceptional affine cases (some infinite families of affine type where already computed by different methods in [2, 3]).

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Deterministic Abelian Sandpile and Square-Triangle Tilings

Sergio Caracciolo, Guglielmo Paoletti, and Andrea Sportiello

Abstract The Abelian Sandpile Model, seen as a deterministic lattice automaton, on two-dimensional periodic graphs, generates complex regular patterns displaying (fractal) self-similarity. In particular, on a variety of lattices and initial conditions, at all sizes, there appears what we call an *exact Sierpinski structure*: the volume is filled with periodic patterns, glued together along straight lines, with the topology of a triangular Sierpinski gasket. Various lattices (square, hexagonal, kagome, ...), initial conditions, and toppling rules show Sierpinski structures which are apparently unrelated and involve different mechanisms. As will be shown elsewhere, all these structures fall under one roof, and are in fact different projections of a unique mechanism pertinent to a family of deterministic surfaces in a four-dimensional lattice. This short note gives a description of this surface, and of the combinatorics associated to its construction.

1 Introduction

Let Λ be the lattice in dimension 4, tensor product of two copies of the triangular lattice, $\Lambda = \langle e_1, e_2, e_3, e_4, e_5, e_6 \mid \sum_{1\leq i\leq 3} e_i = \sum_{4\leq i\leq 6} e_i = 0\rangle_{\mathbb{Z}}$. Consider the two-dimensional cell complex containing all the vertices and edges of Λ, and, as (oriented) faces, the triangles of the two lattices and the parallelograms spanned by pairs (e_1, e_4), (e_2, e_5) and (e_3, e_6). We choose the orientation such that the cycles (e_1, e_2, e_3), $(-e_1, -e_2, -e_3)$ and $(e_1, -e_4, -e_1, e_4)$ are upward faces, and similarly with $(123) \to (456)$ and $(14) \to (25), (36)$. We call a *surface* a connected and simply-connected collection of faces in the cell complex above, with all upward faces.

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Embeddings of $e_1, \dots, e_6$ in $\mathbb{R}^2$ satisfying the forementioned orientation constraints correspond to projections of the four-dimensional cell complex on a two-dimensional real space, such that surfaces are mapped injectively. An example of such an embedding is $(e_1, \dots, e_6) = (\omega^3, \omega^{11}, \omega^7, \omega^0, \omega^8, \omega^4)$, where $\omega^k = (\cos\frac{k\pi}{6}, \sin\frac{k\pi}{6})$. In this case, surfaces correspond to tilings of regions of the plane, composed only of squares and triangles of unit sides, and along directions multiple of $\pi/6$. These tilings are called *square-triangle tilings* in the literature.

Any other projection is topologically equivalent, provided that $e_1 + e_2 + e_3 = e_4 + e_5 + e_6 = 0$ and the orientation of the faces is preserved. We call *valid* such a projection. The set of valid projections is an open portion of an algebraic projective variety. We call *degenerate projections* those on the boundary of this open set. Under degenerate projections, the image of some faces is a segment or a point.

A seminal work of de Brujin for Penrose–Ammann lozenge tilings [2] has first illustrated the possibility that projections of deterministic surfaces from a high-dimensional periodic cell-complex could explain features of two-dimensional aperiodic incommensurable tilings. The square-triangle case discussed here shows a similar phenomenon.

Square-triangle tilings have also distinguished properties, among which is a relation with Algebraic Geometry, generalising the well-known connection between lozenge tilings and Schur functions (see e.g. [1]). The algebra of Schur functions has ubiquitous three-index structure constants $c^{\bar{\nu}}_{\lambda,\mu}$, called *Littlewood–Richardson* (LR) coefficients [16]. When the Young diagrams λ, μ, $\bar{\nu}$ are boxed in a rectangle $(d-n) \times n$ (as is the case, e.g., when they label cells of the Schubert variety), there exists a relation (*Poincaré duality*) which acts as complementation at the level of diagrams, $\nu \leftrightarrow \bar{\nu}$, and the LR coefficients are symmetric in all three indices if the upper one is complemented, $c^{\bar{\nu}}_{\lambda,\mu} =: c_{\lambda,\mu,\nu}$. As shown by P. Zinn-Justin [21] and Purbhoo [20], the LR coefficients correspond to the enumerations of square-triangle tilings over triangoloids whose three sides are built from λ, μ and ν, respectively. Two degenerate projections of these surfaces reduce to portions of the square and of the triangular lattice. As degenerate projections transform some faces into segments or points, the bijective correspondence is preserved only if extra integer labelings, encoding the disappeared faces, are added to the resulting structures. These limiting tilings, together with the auxiliary labelings, correspond to the original Littlewood–Richardson rule [16] in the square case, and to the Knutson–Tao (discrete) honeycombs [11, 12] in the triangular case.

2 ASM and Square-Triangle Tilings

The purpose of this paper is to illustrate another unsuspected feature specific of square-triangle tilings, namely of encoding the *exact Sierpinski structures* that arise in the Deterministic Abelian Sandpile Model. These structures have been identified on various regular two-dimensional lattices, under various abelian toppling rules, initial conditions and deterministic evolution protocols, and square-triangle tilings describe them in a unified way.

The first occurrences of such structures have been presented, by the authors, in [3, 19], while the observation of approximated versions of these structures (reproduced at a coarse-grained scale, but locally deformed by some one-dimensional defects) is much older [18], and has first been made, only on the square lattice, for the two most natural deterministic protocols: the evaluation of the identity configuration in simple geometries [6, 13, 17], and the relaxation of a large amount of sand put at the origin, in the (elsewhere empty) infinite lattice [7, 8, 14, 15].

The 'universal role' of the square-triangle tiling, in different ASM realisations, should sound surprising, as the generic projection gives incommensurable parallelogram-triangle tilings and does not live on a discrete two-dimensional lattice, as is instead the case for the sandpile models we consider. What comes out is that, in a remarkable analogy with the mechanism discussed above for the combinatorics of the Littlewood–Richardson rule and Knutson–Tao honeycombs, different lattice ASM realisations occur at different "rational" points in the set of valid projections (and its boundary, of degenerate projections).

As this short paper is within a series, we do not give here an introduction to the Abelian Sandpile Model. The interested reader can consult the beautiful review by Deepak Dhar [5], who first established a large part of the theory. For aspects of the model more strictly related to the features discussed here, the reader can refer to the PhD thesis of one of the authors [19], or the shorter papers [3] and [4]. Here we will only concentrate on the aspects concerning the surfaces in the square-triangle tiling corresponding to the exact Sierpinski structures in the ASM.

The sandpile configurations are height vectors $\vec{z} = \{z_i\}$, with variables $z_i \in \mathbb{N}$ associated to vertices i of a graph $G = (V, E)$. There exists a notion of *stable* configuration, and a more restrictive notion of *recurrent* one. *Transient* is a synonimous for non-recurrent. There exists a notion of *forbidden sub-configuration* (FSC), and a stable configuration is recurrent iff it has no FSC. More generally, a configuration is *recurrent over* $W \subseteq V(G)$ if it has no FSC contained within W, thus making recurrency a local notion (like instability). Local recurrency and instability are dual notions, if we set in the wider frame of multitoppling ASM, as first shown in [4]. The *toppling matrix* Δ encodes the dynamics of the sandpile, and determines a subdivision of $\mathbb{Z}^{V(G)}$ into equivalence classes. There exists exactly one stable recurrent configuration within each class. Unstable configurations $\vec{z}$ can be *relaxed* to stable ones, $\vec{w} = \mathcal{R}\vec{z}$. Stable transient configurations can be *projected* to the unique recurrent representative in the class, $\vec{w} = \mathcal{P}\vec{z}$. The operators $\mathcal{R}$ and $\mathcal{P}$ correspond to find the fixed point of iterated maps, $\mathcal{R}_0$ and $\mathcal{P}_0$, corresponding to "rounds" of the procedure.[1]

[1]In $\mathcal{R}_0$ one can perform at most one toppling per site, in $\mathcal{P}_0$ one adds a single frame identity, and then relax.

A number of structures and operations on square-triangle tilings can be introduced, that will reproduce, under the various projection procedures, the forementioned counterparts in the various ASM realisations. We dub all these features of the square-triangle setting with the "axiomatic" attribute, as the reason for their names emerges only when the projection procedure is explicitated. Note that we are not able to reproduce *all* the relevant features of the sandpile model. In particular, we are not able to reproduce the a_i operators (nor their counterparts $a_i^\dagger$ defined in [4]). The main things we are able to reproduce are summarised by the following list:

- The notion of (ASM-)equivalence of configurations is trivialised at the axiomatic level: two tilings are equivalent if they have the same boundary.
- The axiomatic notion of FSC correspond to cycles in the tiling satisfying certain local rules.
- We have an axiomatic notion of $\mathcal{P}_0$, consisting in a local deformation along the cycles of maximal FSC's (w.r.t. inclusion).
- Similarly, we can certify that regions encircled by certain cycles will undergo a round of relaxation. This gives an axiomatic local notion of unstable subconfiguration (USC).[2]
- We have an axiomatic notion of $\mathcal{R}_0$, consisting in a local deformation along the cycles of maximal USC's (w.r.t. inclusion).
- We have a recursive description of the Sierpinski structures at the axiomatic level. As these structures in the ASM determine the classification of patches and propagators in certain backgrounds [19], this induces a corresponding classification of axiomatic patches and propagators.
- A choice of vectors $e_1, \ldots, e_6 \in \mathbb{R}^2$, and of "masses" $\{m_{123}, m_{456}, m_{14}, m_{25}, m_{36}\}$ for the five types of tiles, induces a notion of density for the patches. This allows to state an axiomatic version of the Dhar–Sadhu–Chandra incidence formula, first introduced, for the ASM, in [7].

3 Sierpinski Structures

Let $\mathbf{s} = (\mathbf{s_k}, \ldots, \mathbf{s_1}, \mathbf{s_0})$ be a finite string of positive integers, and $n(\mathbf{s}) = \sum_i 3^i s_i$. A Sierpinski structure is labeled by a string $\mathbf{s}$, and $n(\mathbf{s})$ is its size. Structures of the same size are equivalent.

An abstract Sierpinski gasket of index k is defined as follows. At index 0, it is just a dark upward triangle. At index $k+1$, it is obtained from the gasket at index k by subdividing all dark upward triangles into three dark upward and one light downward triangles, all of half the side. Light triangles which are there at index k, will remain unchanged at all $k' > k$. A light triangle has index k if it first appeared

[2] Corresponding to the *waves of topplings* [9, 10].

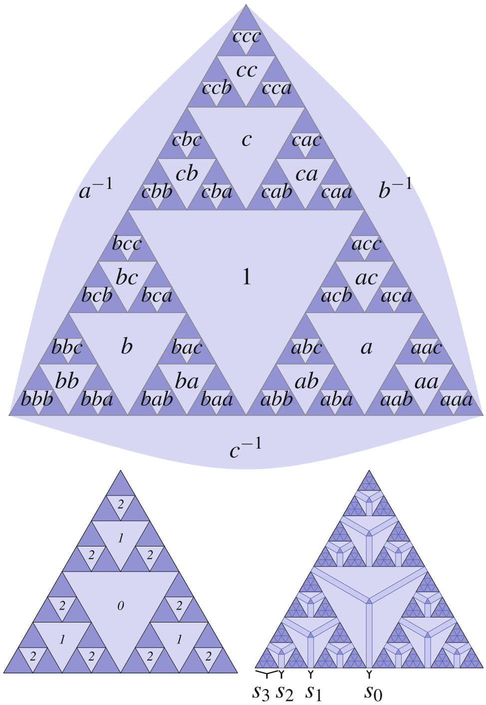

Fig. 1 *Top*: The labeling of the *light triangles*; *Bottom left*: the Sierpinski gasket of index $k = 3$; *Bottom right*: The structure of the patches, and the role of the parameter $\mathbf{s}$

in a gasket at index $k + 1$. A gasket of index k has 3^k dark triangles, and 3^h light triangles of index h, for $0 \leq h < k$. See Fig. 1.

In the sandpile setting, the triangles of the gasket will determine polygonal regions filled with a biperiodic patterns, called *patches* [18]. Patches may be recurrent, transient or marginal, depending on their behaviour under the burning test (see [3]).

In a Sierpinski structure identified by $\mathbf{s}$, all the dark triangles correspond to transient patches, of triangular shape, with a side of s_k unit tiles. Light triangles of index h correspond to polygonal regions filled with recurrent patches. These regions have the aspect of triangoloids with concave sides, the sides being polygonal lines composed of $2^{k-h} - 1$ segments. The packing of unit tiles depends in a certain fixed way on the integer $k - h$ and the variables $s_{h'}$ for $h' > h$, and has no extra freedom, with an exception: starting at the vertices of the triangoloids, we can have a band of a patch with marginal density, of width $s_h - 1$.[3] The three bands meet at a triangular transient patch.

[3]This corresponds to $s_h - 1$ parallel *type-I propagators*, w.r.t. the definitions in [3, 19].

A transient patch contains a FSC only if "sufficiently large", namely if it contains at least 7 unit tiles, packed in a shape . Thus, a triangle of side up to 3 units filled with a transient patch, i.e. the shape , may still be part of an overall recurrent configuration. This has a consequence on our Sierpinski structures: a structure with label $\mathbf{s}$ is recurrent if and only if $1 \leq s_h \leq 3$ for all $h \leq k$. These are the structures ultimately appearing in sandpile protocols.

Each region of the Sierpinski structure is filled with a periodic pattern. The geometry of every region, including the number and location of the unit tiles, is determined through a recursive procedure. Also the shape of the unit tiles, and their content in terms of elementary squares and triangles, are determined recursively. At this aim it is useful to introduce a labeling of the regions of the Sierpinki gasket. We label the dark upward triangles with words in the alphabet $\{a, b, c\}$, and the light downward triangles with the same word as the dark triangle that originated them. When a triangle of label w is split, the three new triangles, in the three directions, have labels wa, wb and wc. We also give labels to the three external regions of the triangles, as a^{-1}, b^{-1} and c^{-1}. See Fig. 1.

A triangle with label w has three larger adjacent light triangles, in the three directions, that have labels $\alpha(w)$, $\beta(w)$ and $\gamma(w)$. These three functions can be defined as follows. Let α_w, β_w and γ_w the rightmost position along w such that, at its right, there are no more a, b or c, respectively; let us call $w|_\ell$ the truncation of w to its first ℓ letters; let us understand that $aa^{-1} = bb^{-1} = cc^{-1} = 1$. Then $\alpha(w) = w|_{\alpha_w} a^{-1}$, and so on.

Complex tiles arise from the superposition of more elementary ones. Only three tiles are indecomposable, and must be given as input. These tiles correspond to the three square orientations in our square-triangle tilings. The corresponding tilings appear outside the triangle, at the three sides. The unit tile of label w is composed of the superposition of two copies of the tiles of labels $\alpha(w)$, $\beta(w)$ and $\gamma(w)$. Unless $w = 1$, one of these three words has higher degree than the other (say $\alpha(w)$). In this case, the six tiles do not overlap, with the only exception that the two $\alpha(w)$ tiles do overlap exactly on a $\alpha(\alpha(w))$ tile. If $w = 1$, no tiles overlap. Each tile has 12 special positions along its boundary, which determine the translation vectors of the recurrent, transient and marginal tilings involving it, and the new tile inheritates its owns positions from those of the three subtiles. This mechanism is illustrated in Fig. 2.

4 Dual Tiles

Our construction in terms of the vectors $e_1, \dots, e_6$ has a number of covariances that allow to shorten our description

C_3**-covariance** ($2\pi/3$ rotations):
$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_2, e_3, e_1, e_5, e_6, e_4)$;
exchange (123) $\leftrightarrow$ **(456)** ($\pi/2$ rotations):
$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_4, e_5, e_6, -e_1, -e_2, -e_3)$;

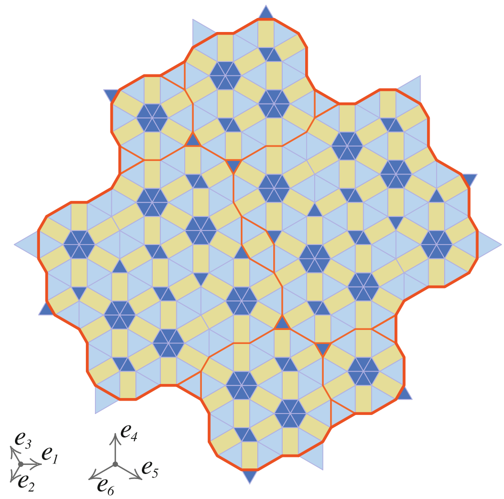

Fig. 2 The tile associated to $w = cba$. The interior *orange lines* describe the decomposition into $\alpha(w)$, $\beta(w)$ and $\gamma(w)$ tiles. The overlap, composed of a $\alpha(\alpha(w))$ tile and two *light triangles*, is in the *middle*. The *triangles* outside the tile denote the 12 special positions. Here we have $u(P) = \big((\underline{6}1), (\underline{2}\underline{6}1\underline{6}), (434\underline{2}\underline{6}1\underline{6}\underline{2}), (34), (\underline{5}343), (\underline{1}6\underline{1}\underline{5}343\underline{5})\big)$

central symmetry (π rotations):

$$(e_1, e_2, \cdots, e_6) \to (-e_1, -e_2, \cdots, -e_6).$$

We call *polygon* a closed curve that is the boundary of some square-triangle tiling. A polygon P is determined by a cyclic sequence in the alphabet $\{1, \ldots, 6, \underline{1}, \ldots, \underline{6}\}$, where 1, $\underline{1}$ stand for $+e_1$, $-e_1$, and so on. We use the shortcuts $\blacktriangle$, $\blacktriangledown$, $\triangleright$ and $\triangleleft$ for the polygons (123), $(\underline{1}\underline{2}\underline{3})$, (456) and $(\underline{4}\underline{5}\underline{6})$, respectively.

A centrally symmetric polygon P is determined by a sequence of the form $P = (i_1 i_2 \ldots i_k \underline{i_1}\, \underline{i_2} \ldots \underline{i_k})$, where $\underline{\underline{i}} = i$. We use the shortcut $(i_1 i_2 \ldots i_k \|)$ in such a case.

A polygon P is a *dual tile* if both the triple of polygons $(P, \blacktriangle, \blacktriangledown)$ and the triple $(P, \triangleright, \triangleleft)$ (in these proportions) tile periodically the plane. We call a *transient/recurrent hex tiling* a tiling of the two forms above, respectively.

The three fundamental parallelogram tiles are dual tiles. The dodecagon, $(1\underline{6}\underline{2}43\underline{5}\|)$, is another example. All the tilings associated to dual tiles, except those deriving from the fundamental parallelograms, have the topology of a *hexagonal tiling*: each polygon P is neighbour to other 6 P's. The fundamental triangles are at the 6 triple points, with alternating orientations cyclically along each P.

To each word w as in the previous section can be associated a dual tile $P(w)$, which is centrally symmetric. The three fundamental parallelograms are $(41\|) = P(a^{-1})$ and so on. The dodecagon is $(1\underline{6}\underline{2}43\underline{5}\|) = P(1)$.

A pair of polygons (P, Q) is a *dual pair* if the sextuplet $(P, Q, \blacktriangle, \blacktriangledown, \triangleright, \triangleleft)$ (in these proportions) tiles the plane. We call a *sq-oc tiling* a tiling obtained as above. Neglecting triangles (e.g., replacing them with Y-shapes), such a tiling has the square-octagon topology: any Q tile is neighbour to 4 P ones, and any P tile is neighbour to 4 P's and 4 Q's, alternating.[4] The fundamental triangles are at the triple points of the square-octagon topology. Each P and Q tile is adjacent to 8 and 4 triangles, respectively, alternating dark / light, and, within dark and light ones, of opposite orientations.

For each w, the pairs of tiles $\big(P(\alpha(w)), P(w)\big)$, $\big(P(\beta(w)), P(w)\big)$ and $\big(P(\gamma(w)), P(w)\big)$ are dual pairs. For example, the dodecagon and any of the fundamental parallelograms form a dual pair.

Exceptionally, and analogously to what happens for hex tilings, also all pairs of fundamental parallelograms are dual pairs, although with a different topology, and with no ordering.

As a consequence, each tile $P = P(w)$ appears in two hex tilings, three sq-oc tilings as 'octagon', and infinitely many sq-oc tilings as 'square'. The union of the positions of triple points among all these tilings has cardinality 12. These 12 special positions break the perimeter of the tile into open paths, related by the central symmetry (see again Fig. 2). Thus, a list of 6 paths, $u(P) = (u_1, \dots, u_6)$, determines simultaneously the perimeter and the special positions, and $P = (1u_1\underline{6}u_2\underline{2}u_3 4u_4 3u_5\underline{5}u_6\|)$.

The recursive construction, at the level of these paths, leads to the formulas (completed by C_3-covariance)

$$(u_1\underline{6}u_2)_w = (u_1\underline{6}u_2\underline{2}u_3)_{\beta(w)}\ \underline{6}\ (\underline{u_6}1u_1\underline{6}u_2)_{\gamma(w)}$$

$$\begin{cases} (u_1)_w = (u_1)_{\alpha(w)} & |\alpha(w)| > |\beta(w)| \\ (u_2)_w = (u_2)_{\beta(w)} & |\alpha(w)| < |\beta(w)| \\ (u_1)_w = (u_2)_w = \varnothing & \alpha(w) = a^{-1}, \beta(w) = b^{-1}. \end{cases}$$

The geometry of these paths is such that:

- The sq-oct patches based on a (P, Q) dual pair may be adjacent to both recurrent and transient hex patches, based both on P and on Q, although with a restriction on the direction of the (straight) boundary.
- The hex transient tiling based on $P(w)$ can be adjacent to the hex recurrent tiling based on $P(w')$, if w' is a prefix of w.

[4]This fixes who's who among P and Q.

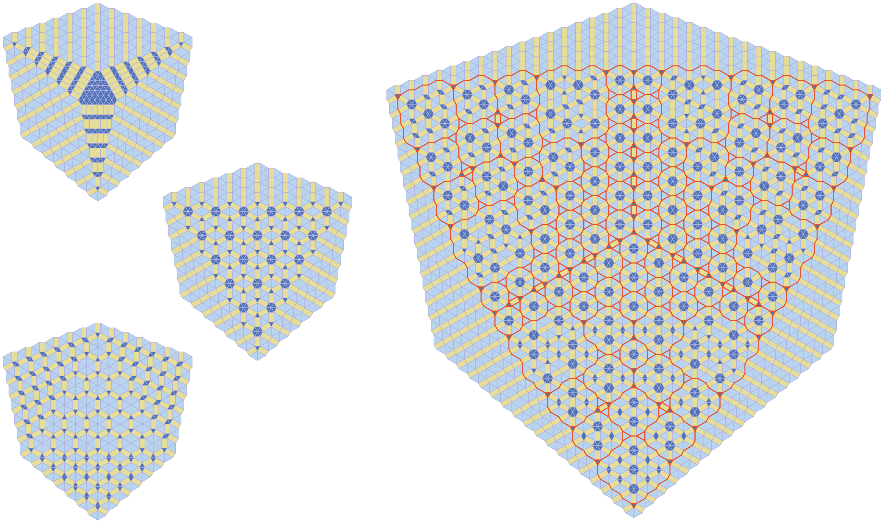

Fig. 3 Four classes of equivalent configurations: *Left*: three deterministic configurations, of size $n = 6$. The two *on top* are stable but transient, and the one on the *bottom* is recurrent but unstable. Applying $\mathcal{P}$ and $\mathcal{R}$, respectively, we obtain our axiomatic Sierpinski structure, (*on the right* at $\mathbf{s} = (1, 2, 2)$, thus $n(\mathbf{s}) = 17$). The patch structure is highlighted by the *orange* construction lines, showing the same topology of the Sierpinski gasket in Fig. 1

This ultimately leads to the consistency of the construction of the Sierpinski structures (see Fig. 3).

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A Special Feature of Quadratic Monomial Ideals

Matteo Varbaro

Abstract We will see the proof of the lemma that allowed Caviglia, Constantinescu and myself in (Isr. J. Math. **204**, 469–475 (2014)) to show a version of a conjecture by Kalai. We will also discuss possible consequences of the lemma on the study of dual graphs of flag complexes.

Our aim is to show a property peculiar of quadratic monomial ideals, namely the Main Lemma below, proved in collaboration with Giulio Caviglia and Alexandru Constantinescu in [4]. The original motivation for looking up such property was to use it in the study of a conjecture by Gil Kalai, for which indeed it has been extremely helpful. During the talk, I mentioned how this lemma could be exploited in other situations. First of all let us recall it:

Main Lemma *Let k be an infinite field, and $I \subseteq k[x_1, \ldots, x_n] =: S$ an ideal of height c. If I is monomial and generated in degree 2, then there exist linear forms $\ell_{i,j}$ for $i \in \{1, 2\}$ and $j \in \{1, \ldots, c\}$ such that $\ell_{1,1}\ell_{2,1}, \ldots, \ell_{1,c}\ell_{2,c}$ is an S-regular sequence contained in I.*

In general, if R is a standard graded Cohen-Macaulay k-algebra, where k is an infinite field, and $J \subseteq R$ is a height c homogeneous ideal generated in a single degree d, then we can always find an R-regular sequence $g_1, \ldots, g_c$ of degree d elements inside J. So, with the notation of the Main Lemma, since the polynomial ring S is Cohen-Macaulay, we already know that there is an S-regular sequence $f_1, \ldots, f_c$ consisting of quadratic polynomials inside I. The point of the result is that we can choose each f_i being a product of two linear forms.

Before sketching the (elementary) proof of the Main Lemma, let us see that the analog property fails for nonquadratic monomial ideals.

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Example Let J be the height 2 monomial ideal $(x^2y, y^2z, xz^2) \subseteq k[x, y, z]$. It is straightforward to check that the only products of linear forms $\ell_1\ell_2\ell_3$ in J are (scalar multiples of) the monomials x^2y, y^2z and z^2x. Clearly, no combination of 2 such monomials form a $k[x, y, z]$-regular sequence.

Proof of the Main Lemma The proof goes like follows: Take a height c minimal prime $\mathfrak{p}$ of I

$$\mathfrak{p} = (x_1, \dots, x_c)$$

(this of course can be done after relabeling the variables). We decompose the k-vector space generated by the quadratic monomials of I as:

$$I_2 = \bigoplus_{i=1}^{c} x_iV_i,$$

where $V_i = \langle x_j : x_ix_j \in I \text{ and } j \geq i\rangle$. We aim to find $\ell_i \in V_i$ such that

$$\dim_k(\langle x_i : i \in A\rangle + \langle \ell_i : i \in [c] \setminus A\rangle) = c \quad \forall\, A \subseteq [c] = \{1, \dots, c\}.$$

Indeed, one can easily check that the condition above characterizes the fact that $x_1\ell_1, \dots, x_c\ell_c$ is an S-regular sequence. The trick to find such linear forms ℓ_i is to construct a family of bipartite graphs G_A, for all $A \subseteq [c]$, in the following way:

(i) $V(G_A) = [c] \cup \{x_1, \dots, x_n\}$.
(ii) $E(G_A) = \{\{i, x_i\} : i \in [c] \setminus A\} \cup \{\{i, x_j\} : i \in A \text{ and } x_j \in V_i\}$.

Claim There is a matching of G_A containing all the vertices of $[c]$. To show this, we will appeal to the Marriage Theorem, by showing that, for a subset $B \subseteq [c]$, the set $N(B)$ of vertices adjacent to some vertex in B has cardinality not smaller than that of B. To this purpose, note that:

$$|N(B)| = \dim_k\Bigg(\sum_{i\in A\cap B} V_i + \sum_{i\in([c]\setminus A)\cap B} \langle x_i\rangle\Bigg) =$$
$$\dim_k\Bigg(\sum_{i\in A\cap B} V_i + \sum_{i\in[c]\setminus(A\cap B)} \langle x_i\rangle\Bigg) - \dim_k\Bigg(\sum_{i\in[c]\setminus B} \langle x_i\rangle\Bigg) \geq |B|,$$

where the inequality follows from the fact that the ideal $\sum_{i\in A\cap B}(V_i) + (x_i : i \in [c] \setminus (A \cap B))$, containing I, must have height at least c. Therefore, we get by the Marriage Theorem the existence of

$$j(1, A),\ j(2, A), \dots,\ j(c, A)$$

such that $\{1, x_{j(1,A)}\}, \ldots, \{c, x_{j(c,A)}\}$ is a matching of G_A, which implies that

$$\dim_k \langle x_{j(1,A)}, \ldots, x_{j(c,A)} \rangle = c.$$

Now it is enough to put

$$\ell_i = \sum_{A \subseteq [c]} \lambda(A) x_{j(i,A)},$$

where $\lambda(A)$ are general elements of k. ☐

As already mentioned, we were motivated in proving the Main Lemma to study the following conjecture of Kalai:

Conjecture (Kalai) *The f-vector of a Cohen-Macaulay flag simplicial complex is the f-vector of a Cohen-Macaulay balanced simplicial complex.*

It is convenient to recall here that the f-vector $(f_{-1}, \ldots, f_{d-1})$ of a $(d-1)$-dimensional simplicial complex Δ is defined as:

$$f_j = |\{j\text{-dimensional faces of } \Delta\}| .$$

As a consequence of the Main Lemma, we get the following:

Theorem *The h-vector of a Cohen-Macaulay flag simplicial complex is the h-vector of a Cohen-Macaulay balanced simplicial complex.*

To show how the theorem above follows form the Main Lemma, we remind that, if $(h_0, \ldots, h_d)$ and $(f_{-1}, \ldots, f_{d-1})$ are, respectively, the h- and f-vector of a given $(d-1)$-dimensional simplicial complex, then the following equations hold true:

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1} \quad \text{and} \quad f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i .$$

From these formulas one sees that the theorem above is very close to answering positively the conjecture of Kalai: The only defect is that h- and f-vector determine each other only if the dimension of the simplicial complex is known, and the h-vector does not provide such an information (in fact it may happen that $h_d = 0$).

For the convenience of the reader, let us recall the definitions of the objects occurring in the conjecture of Kalai:

(i) A simplicial complex Δ is said to be Cohen-Macaulay (over k) if its Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay.
(ii) A simplicial complex is said to be flag if all its minimal nonfaces have cardinality 2.
(iii) A $(d-1)$-dimensional simplicial complex is said to be balanced if its 1-skeleton is a d-colorable graph.

Why were we able to prove the "h-version" of the conjecture of Kalai and not the original one? The point is that the entries of the h-vector of a simplicial complex Δ are the coefficients of the h-polynomial of $k[\Delta]$. So, by combining the Main Lemma with the main result obtained by Abedelfatah in [1], we get the following:

Theorem (A1) *The h-vector of a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex on $n+d$ vertices equals to the Hilbert function of S/J where J is an ideal of S containing $(x_1^2, \ldots, x_n^2)$.*

Notice that the ideal J of Theorem A1 can be chosen monomial (just passing to the initial ideal). In this case, the Hilbert function of S/J is just the f-vector of the simplicial complex Γ on n vertices whose faces are $\{i_1, \ldots, i_r\}$ where $x_{i_1} \cdots x_{i_r} \notin J$. Therefore Theorem A1 can be re-stated as:

Theorem (A2) *The h-vector of a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex Δ on $n+d$ vertices is the f-vector of some simplicial complex Γ on n vertices.*

On the other hand, Theorem A2 is in turn equivalent to:

Theorem (A3) *The h-vector of a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex Δ on $n+d$ vertices is the h-vector of an $(n-1)$-dimensional Cohen-Macaulay balanced simplicial complex Ω on $2n$ vertices.*

Proof By Theorem A2 we know that there is a simplicial complex Γ on n vertices with f-vector equal to the h-vector of Δ. Set:

$$J = I_\Gamma + (x_1^2, \ldots, x_n^2) \subseteq S$$

and consider the polarization J' of J in $S[y_1, \ldots, y_n]$. Then $J' = I_\Omega$ where, since the polarization preserves the minimal graded free resolution, the h-vector of Ω is the Hilbert function of S/J (which is the f-vector of Γ) and Ω is Cohen-Macaulay.

Now, notice that Ω is $(n-1)$-dimensional, and the coloring $\mathrm{col}(x_i) = \mathrm{col}(y_i) = i$ provides an n-coloring of the 1-skeleton of Ω, so that Ω is balanced.

Remark To show that Theorem A3 $\implies$ Theorem A1, one has to use that the Stanley-Reisner ring of a pure $(d-1)$-dimensional balanced simplicial complex Ω has a special system of parameters, namely:

$$\ell_i = \sum_{\mathrm{col}(x_j)=i} x_j \quad \forall\, i = 1, \ldots, d.$$

If Ω is on m vertices, then $k[\Omega]/(\ell_1, \ldots, \ell_d) \cong k[x_1, \ldots, x_{m-d}]/J$, where J is an ideal containing the squares of the variables. Furthermore, if Ω is Cohen-Macaulay, then $\ell_1, \ldots, \ell_d$ is a $k[\Omega]$-regular sequence, so the h-polynomial of $k[\Omega]/(\ell_1, \ldots, \ell_d)$ is the same as the h-polynomial of $k[\Omega]$.

Remark Theorem A1 can be seen as the solution of a particular case of a general conjecture of Eisenbud-Green-Harris, namely the *quadratic monomial case*. For the precise statement of the conjecture see [5, 6]. During the conference in Cortona, Abedelfatah posted on the arxiv a solution of the EGH conjecture in the *monomial case* (any degree), see [2]. His proof is not an obvious extension of ours, essentially because the Example given in the first page.

At the end of the talk, I discussed another aspect in which the Main Lemma might be helpful: The dual graph $G(\Delta)$ of a pure $(d-1)$-dimensional simplicial complex Δ is defined as:

(i) $V(G(\Delta)) = \{\text{facets of } \Delta\}$.
(ii) $E(G(\Delta)) = \{\{F, G\} : |F \cap G| = d-1\}$.

Recently, Adiprasito and Benedetti proved in [3] that, if Δ is a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex on n vertices, then

$$\operatorname{diam}(G(\Delta)) \leq n - d.$$

The Main Lemma might be helpful to get further understanding of $G(\Delta)$ for Δ flag as follows: Recall that

$$I_\Delta = \bigcap_{F \text{ facet of } \Delta} (x_i : i \notin F),$$

so the graph $G(\Delta)$ may be thought also as the graph of minimal primes of I_Δ; i.e. the graph whose vertices are the minimal primes of I_Δ, and two minimal primes are connected by an edge if and only if the height of their sum is one more than the height of I_Δ. The Main Lemma says that there exist linear forms $\ell_{i,j}$ such that $\ell_{1,1}\ell_{2,1}, \ldots, \ell_{1,n-d}\ell_{2,n-d}$ is an S-regular sequence contained in I_Δ. This implies that the graph $G(\Delta)$ is an induced subgraph of the graph of minimal primes of the ideal

$$(\ell_{1,1}\ell_{2,1}, \ldots, \ell_{1,n-d}\ell_{2,n-d}).$$

Such a graph is quite simple to describe: It is obtained by contracting some edges (which ones depends on the geometry of the matroid given by $\ell_{1,1}, \ell_{2,1}, \ldots, \ell_{1,n-d}, \ell_{2,n-d}$) of the graph $\mathbb{G}$ defined as:

1. $V(\mathbb{G}) = 2^{\{1,\ldots,n-d\}}$.
2. $\{A, B\} \in E(\mathbb{G})$ if and only if $|A \cup B| - |A \cap B| = 1$.

This should give strong restrictions on the structure of $G(\Delta)$ for Δ flag.

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Resonant Bands, Local Systems and Milnor Fibers of Real Line Arrangements

Masahiko Yoshinaga

Abstract This is a short note on the study of cohomology groups of rank one local systems of real line arrangements via resonant bands. Results on Milnor fibers and several conjectures are also stated.

1 Local Systems

Let $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{C}^2$. We can identify $\mathbb{C}^2$ with $\mathbb{CP}^2 \setminus \overline{H}_0$, where $\overline{H}_0$ is the line at infinity. We define $c\mathcal{A} = \{\overline{H}_0, \overline{H}_1, \dots, \overline{H}_n\}$, where $\overline{H}_i$ is the closure of H_i in $\mathbb{CP}^2$. The complement of lines is denoted by $M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^n H_i = \mathbb{CP}^2 \setminus \bigcup_{i=0}^n \overline{H}_i$.

We define the character torus by $\mathbb{T}(\mathcal{A}) = \mathrm{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*)$. Since the fundamental group $\pi_1(M(\mathcal{A}))$ is generated by meridians γ_i of H_i $(i = 0, \dots, n)$, $\rho \in \mathrm{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*)$ is specified by the images $(\rho(\gamma_0), \rho(\gamma_1), \dots, \rho(\gamma_n)) \in (\mathbb{C}^*)^{n+1}$. By this correspondence, we have the following isomorphism

$$\mathbb{T}(\mathcal{A}) \simeq \{(q_0, q_1, \dots, q_n) \in (\mathbb{C}^*)^{n+1} \mid q_0 q_1 \cdots q_n = 1\}.$$

The character torus $\mathbb{T}(\mathcal{A})$ can also be identified with the moduli space of complex rank one local systems. For a given $q = (q_0, q_1, \dots, q_n)$ with $\prod q_i = 1$, we denote by $\mathcal{L}_q$ the associated local system, i.e., the local system which has the monodromy $q_i \in \mathbb{C}^*$ around the line H_i.

The twisted cohomology $H^1(M(\mathcal{A}), \mathcal{L}_q)$ is related to many other problems about the topology of $M(\mathcal{A})$ [6, 7]. One of the central problem is combinatorial decidability of $H^1(M(\mathcal{A}), \mathcal{L}_q)$.

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2 Chambers and Bands

From now, we assume that each line $H \in \mathcal{A}$ is defined over the real number field $\mathbb{R}$. Our purpose is to describe $H^1(M(\mathcal{A}), \mathcal{L}_q)$ in terms of real structure.

A connected component C of $\mathbb{R}^2 \setminus \bigcup_{i=1}^n H_i$ is called a *chamber*. The set of all chambers is denoted by $\mathrm{ch}(\mathcal{A})$. Let $C, C' \in \mathrm{ch}(\mathcal{A})$. A line $H \in \mathcal{A}$ is said to separate C and C' if C and C' belong to opposite half spaces defined by $H \subset \mathbb{R}^2$.

Definition 1 $\mathrm{Sep}(C, C') := \{H \in \mathcal{A} \mid H \text{ separates } C \text{ and } C'\}$.

Definition 2 We call the number of separating lines $d(C, C') := |\mathrm{Sep}(C, C')|$ the *distance* of C from C'.

The following object is useful to compute $H^1(M(\mathcal{A}), \mathcal{L}_q)$. See [8, 9] for more details and examples.

Definition 3 A *band* is a region bounded by a pair of consecutive parallel lines.

Let B be a band. Then there are exactly two unbounded chambers in B. We call them $U_1(B)$ and $U_2(B) \in \mathrm{ch}(\mathcal{A})$. The distance $d(U_1(B), U_2(B))$ is called the *length* of the band B, denoted by $|B|$.

Definition 4 Let B be a band bounded by two parallel lines H and H'. The closures $\overline{H}, \overline{H}' \subset \mathbb{RP}^2$ intersects on the line at infinity $\overline{H}_0$. The intersection is denoted by $X(B) := \overline{H} \cap \overline{H}' \in \overline{H}_0$. We also have $X(B) = \overline{B} \cap \overline{H}_0$, where $\overline{B}$ is the closure of B in $\mathbb{RP}^2$.

3 Resonant Bands

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a line arrangement define over $\mathbb{R}$ as in the previous section. Let $q_1, \dots, q_n \in \mathbb{C}^*$ be nonzero complex numbers. We set $q_0 := (q_1 q_2 \cdots q_n)^{-1}$. For each q_i, we fix $t_i \in \mathbb{C}^*$ such that $t_i^2 = q_i$, $i = 0, 1, \dots, n$.

Definition 5 Let $C, C' \in \mathrm{ch}(\mathcal{A})$. Define

$$\Delta_q(C, C') := \prod_{H_i \in \mathrm{Sep}(C,C')} t_i - \prod_{H_i \in \mathrm{Sep}(C,C')} t_i^{-1}.$$

The following proposition is straightforward.

Proposition 6 $\Delta_q(C, C') = 0$ *if and only if* $\prod_{H_i \in \mathrm{Sep}(C,C')} q_i = 1$.

Definition 7 A band B is said to be $\mathcal{L}_q$*-resonant* if $\Delta_q(U_1(B), U_2(B)) = 0$.

Let B be a band. Note that each line $H \in \mathcal{A}$ is either parallel to B or across B. Hence we have

$$\mathrm{ch}(\mathcal{A}) = (c\mathcal{A})_{X(B)} \sqcup \mathrm{Sep}(U_1(B), U_2(B)), \tag{1}$$

where $(c\mathcal{A})_{X(B)}$ is the set of lines passing through $X(B)$. Using the relation $q_0 q_1 \cdots q_n = 1$, we have the following.

Proposition 8 *A band B is $\mathcal{L}_q$-resonant if and only if* $q_{X(B)} := \prod_{H_i \in (c\mathcal{A})_{X(B)}} q_i = 1$.

Definition 9 Denote by $\mathrm{RB}_{\mathcal{L}_q}(\mathcal{A})$ the set of all $\mathcal{L}_q$-resonant bands.

Next we define a linear map

$$\nabla : \mathbb{C}[\mathrm{RB}_{\mathcal{L}_q}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathrm{ch}(\mathcal{A})] \tag{2}$$

from the vector space spanned by the $\mathcal{L}_q$-resonant bands to the vector space spanned by the chambers.

Definition 10 Let $B \in \mathrm{RB}_{\mathcal{L}_q}(\mathcal{A})$. Define $\nabla(B) \in \mathbb{C}[\mathrm{ch}(\mathcal{A})]$ by the following formula.

$$\nabla(B) := \sum_{C \subset B} \Delta_q(U_1(B), C) \cdot [C].$$

Theorem 11 *Assume that $q_0 \neq 1$. Then*

$$\mathrm{Ker}(\nabla : \mathbb{C}[\mathrm{RB}_{\mathcal{L}_q}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathrm{ch}(\mathcal{A})]) \simeq H^1(M(\mathcal{A}), \mathcal{L}_q).$$

See [9] for proofs and applications. From Theorem 11 we also have the following vanishing result.

Theorem 12 *Assume that $q_0 \neq 1$.*

(i) Suppose that there does not exist point $X \in \overline{H}_0$ such that $|(c\mathcal{A})_X| \geq 3$ and $q_X = 1$. Then $H^1(M(\mathcal{A}), \mathcal{L}_q) = 0$.
(ii) Suppose that there exists unique $X \in \overline{H}_0$ such that $|(c\mathcal{A})_X| \geq 3$ and $q_X = 1$. Then

$$\dim H^1(M(\mathcal{A}), \mathcal{L}_q) = \begin{cases} 0, & \text{if there is } i \text{ with } X \notin \overline{H}_i \text{ and } q_i \neq 1, \\ |(\mathcal{A})_X| - 2, & \text{if, for every } i,\ X \notin \overline{H}_i \text{ implies } q_i = 1. \end{cases}$$

Remark 13 By a result by Cohen et al. [1], Theorem 12 (i) is true for any complex arrangement.

In general, two lines H, H' on the real projective plane $\mathbb{RP}^2$ divide the space into two regions. A pair of lines $\overline{H}_i, \overline{H}_j \in c\mathcal{A} = \{\overline{H}_0, \overline{H}_1, \dots, \overline{H}_n\}$ is called *sharp pair* if one of two regions does not contain any intersections of $c\mathcal{A}$ in its interior.

The existence of sharp pairs gives an upper bound of the dimension of the twisted cohomology groups.

Theorem 14 *Suspose that there exists a sharp pair* $\overline{H}_i, \overline{H}_j \in c\mathcal{A}$ *such that* $q_i \neq 1$ *and* $q_j \neq 1$*. Then* $\dim H^1(M(\mathcal{A}), \mathcal{L}_q) \leq 1$.

4 Milnor Fibers

The Milnor fiber $F(\mathcal{A})$ of the cone of $c\mathcal{A}$ is a $\mathbb{Z}/(n+1)\mathbb{Z}$ cyclic covering space of $M(\mathcal{A})$. One of the open problems is the combinatorial description of the Betti numbers of $F(\mathcal{A})$, especially $b_1(F(\mathcal{A}))$.

There is a natural automorphism $\rho : F(\mathcal{A}) \longrightarrow F(\mathcal{A})$, called the monodromy automorphism. Since ρ is order $n+1$, the cohomology group decomposes into the sum of eigen spaces

$$H^k(F(\mathcal{A}), \mathbb{C}) = \bigoplus_{\lambda^{n+1}=1} H^K(F(\mathcal{A}), \mathbb{C})_\lambda, \tag{3}$$

where the sum runs over all complex numbers satisfying $\lambda^{n+1} = 1$ and $H^k(F(\mathcal{A}), \mathbb{C})_\lambda$ is the λ-eigenspace.

Let λ be a complex number satisfying $\lambda^{n+1} = 1$. Let us denote by $\mathcal{L}_\lambda$ the local system corresponding to $(\lambda, \lambda, \dots, \lambda) \in \mathbb{T}(\mathcal{A})$. It is known [2] that the λ-eigenspace is isomorphic to the twisted cohomology group of $M(\mathcal{A})$, namely, $H^k(F(\mathcal{A}), \mathbb{C}) \simeq H^k(M(\mathcal{A}), \mathcal{L}_\lambda)$. To compute this, we can apply the result in the previous section. Note that $\Delta_\lambda(C, C') = \lambda^{d(C,C')} - \lambda^{-d(C,C')}$.

Now we fix a complex number $\lambda \in \mathbb{C}^*$ of order $k > 1$ such that $k|(n+1)$.

Proposition 15 *A band* B *is* $\mathcal{L}_\lambda$*-resonant if and only if* $k|d(U_1(B), U_2(B))$*. Equivalently,* $\lambda^{|(c\mathcal{A})_{X(B)}|} = 1$.

Let B be a $\mathcal{L}_\lambda$-resonant band. Then

$$\nabla(B) = \sum_{C \subset B} (\lambda^{d(U_1(B),C)} - \lambda^{-d(U_1(B),C)}) \cdot [C]. \tag{4}$$

Theorem 16 $H^1(F(\mathcal{A}), \mathbb{C})_\lambda \simeq \mathrm{Ker}(\nabla : \mathbb{C}[\mathrm{RB}_{\mathcal{L}_\lambda}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathrm{ch}(\mathcal{A})])$.

Using the above theorem, we can prove some vanishing results.

Definition 17 A point $p \in \overline{H}_0$ is said to be a $\mathcal{L}_\lambda$*-resonant edge* if $|(c\mathcal{A})_p| \geq 3$ and $|(c\mathcal{A})_p|$ is divisible by k.

Theorem 18 *If there are no* $\mathcal{L}_\lambda$*-resonant edges, then* $H^1(F(\mathcal{A}))_\lambda = 0$.

The proof of this theorem is now an easy one. First, we have $\mathrm{RB}_{\mathcal{L}_\lambda}(\mathcal{A}) = \emptyset$ by the assumption. Then obviously $\mathrm{Ker}\,\nabla = 0$. Theorem 18 is due to Libgober [5]. We should note that Libgober's result is more general than Theorem 18, for he proved it for any complex arrangements.

We call the affine line arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ in $\mathbb{R}^2$ *essential* if there is at least one nontrivial intersection. This assumption is not a strong restriction. Indeed it avoids only the case "$H_1, \dots, H_n$ are parallel". Under the essentiality hypothesis, we can strengthen the previous result.

Theorem 19 *Suppose $\mathcal{A}$ is essential. If there exists at most one $\mathcal{L}_\lambda$-resonant edge on $\overline{H}_0$, then $H^1(F(\mathcal{A}))_\lambda = 0$.*

The above theorem says that (if $\mathcal{A}$ is essential) $H^1(F(\mathcal{A}))_\lambda \neq 0$ implies that every line $\overline{H}_i \in c\mathcal{A}$ has at least two points of $\mathcal{L}_\lambda$-resonant edges. It seems natural to ask what happens if there are exactly two $\mathcal{L}_\lambda$-resonant edges on $\overline{H}_0$. The following result gives an answer.

Theorem 20 *Suppose that there exist two $\mathcal{L}_\lambda$-resonant edges. If $H^1(F(\mathcal{A}))_\lambda \neq 0$, then $c\mathcal{A}$ is projectively equivalent to the so called A_3-arrangement defined by the equation $xyz(x-z)(y-z)(x-y) = 0$.*

Corollary 21 *Suppose $|c\mathcal{A}| \geq 7$ and $H^1(F(\mathcal{A}))_\lambda \neq 0$. Then each line $\overline{H}_i \in c\mathcal{A}$ has at least three $\mathcal{L}_\lambda$-resonant edges on it.*

Theorem 22 *If $c\mathcal{A}$ has a sharp pair of lines, then* $\dim H^1(F(\mathcal{A}))_\lambda \leq 1$.

5 Conjectures

Conjecture 23 Theorems 19 and 20 hold for any complex line arrangements.

Conjecture 24 For a real arrangement $c\mathcal{A}$, $\dim H^1(F(\mathcal{A}))_\lambda \leq 1$ for any $\lambda \neq 1$. Furthermore, if $\lambda^3 \neq 1$, then $H^1(F(\mathcal{A}))_\lambda = 0$.

For simplicial arrangements [4], we have a more precise conjecture.

Conjecture 25 Let $c\mathcal{A}$ be a simplicial arrangement on $\mathbb{RP}^2$. Then the following are equivalent.

(1) $H^1(F(\mathcal{A}))_{\neq 1} \neq 0$.
(2) $\dim H^1(F(\mathcal{A}))_{\exp(2\pi\sqrt{-1}/3)} = 1$.
(3) $c\mathcal{A}$ has 3-multinet structure (of multiplicity 1) [3].
(4) $c\mathcal{A}$ is of type $A(6m, 1)$ [4].

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On Highly Regular Embeddings

Pavle V.M. Blagojević, Wolfgang Lück, and Günter M. Ziegler

Abstract A continuous map $\mathbb{R}^d \to \mathbb{R}^N$ is *k-regular* if it maps any k pairwise distinct points to k linearly independent vectors. Our main result on k-regular maps is the following lower bound for the existence of such maps between Euclidean spaces, in which $\alpha(k)$ denotes the number of ones in the dyadic expansion of k:

For $d \geq 1$ and $k \geq 1$ there is no k-regular map $\mathbb{R}^d \to \mathbb{R}^N$ for

$$N < d(k - \alpha(k)) + \alpha(k).$$

This reproduces a result of Chisholm from 1979 for the case of d being a power of 2; for the other values of d our bounds are in general better than Karasev's [13], who had only recently gone beyond Chisholm's special case. In particular, our lower bound turns out to be tight for $k \leq 3$.

The framework of Cohen and Handel (1979) relates the existence of a k-regular map to the existence of a specific inverse of an appropriate vector bundle. Thus non-existence of regular maps into $\mathbb{R}^N$ for small N follows from the non-vanishing of specific dual Stiefel–Whitney classes. This we prove using the general Borsuk–Ulam–Bourgin–Yang theorem combined with a key observation by Hung [12] about the cohomology algebras of unordered configuration spaces.

Our study produces similar topological lower bound results also for the existence of *ℓ-skew embeddings* $\mathbb{R}^d \to \mathbb{R}^N$ for which we require that the images of the tangent spaces of any ℓ distinct points are skew affine subspaces. This extends work by Ghomi and Tabachnikov [8] for $\ell = 2$.

The details for this work are provided in our paper *On highly regular embeddings*, Transactions of American Mathematical Society, Published electronically: May 6, 2015, http://dx.doi.org/10.1090/tran/6559.

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1 Introduction

Let $d \geq 1$ and $k \geq 1$ be integers. A map $\mathbb{R}^d \to \mathbb{R}^N$ is *k-regular* if it maps any k pairwise distinct points to k linearly independent vectors. Such a map only exists if N is large enough. How large does N have to be?

The study of the existence of k-regular maps was initiated by Borsuk [5] in 1957 and latter attracted additional attention due to its connection to approximation theory, via the Haar–Kolmogorov–Rubinstein theorem (see [7]). The problem and its extensions were extensively studied by Chisholm, F. Cohen, Handel, and others in the 1970s and 1980s [6, 7, 9, 10], and then again by Handel and Vassiliev in the 1990s [11, 14, 15].

Some basic examples and results are as follows, where $N(d,k)$ denotes the smallest dimension for which a k-regular map $\mathbb{R}^d \to \mathbb{R}^N$ exists:

- $N(d,k) \geq k$ is trivial.
- $N(d,1) = 1$.
- $N(d,2) = d+1$.
- $N(1,k) = k$ follows from the existence of the real *moment curve*

$$\gamma : \mathbb{R} \to \mathbb{R}^k, \quad t \mapsto (1, t, \ldots, t^{k-1}).$$

- $N(2,k) \leq 2k-1$ follows from the *complex moment curve*

$$\gamma_{\mathbb{C}} : \mathbb{C} \to \mathbb{R} \times \mathbb{C}^{k-1}, \quad t \mapsto (1, t, \ldots, t^{k-1}).$$

- $N(d,3) \leq d+2$ is obtained from embeddings

$$\mathbb{R}^d \hookrightarrow S^d \hookrightarrow \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \times \{1\} \hookrightarrow \mathbb{R}^{d+2}.$$

- $N(d,k) \geq (d+1)\lfloor \frac{k}{2} \rfloor$ was proven by Boltjanskiĭ et al. [4].
- $N(d,k) \leq (d+1)k$ may be obtained from a general position smooth embedding, see Boltjanski [3] and Handel [11].

2 Main Result

Our main result on k-regular maps—see [2] for details—is the following lower bound for the existence of such maps between Euclidean spaces:

Theorem 1 *For any $d \geq 1$ and any $k \geq 1$ there is no k-regular map $\mathbb{R}^d \to \mathbb{R}^N$ for*

$$N < d(k - \alpha(k)) + \alpha(k),$$

where $\alpha(k)$ denotes the number of ones in the dyadic expansion of k.

This reproduces a result of Chisholm [6] from 1979 for the case when d is a power of 2; for the other values of d our bounds are in general better than Karasev's [13], who had only recently gone beyond Chisholm's special case. In particular, our lower bound turns out to be tight for $k = 3$ (see also Handel [9]).

3 Methods

Any k-regular map $f : \mathbb{R}^d \to \mathbb{R}^N$ yields an $\mathfrak{S}_k$-equivariant map

$$F(\mathbb{R}^d, k) \longrightarrow_{\mathfrak{S}_k} V_k(\mathbb{R}^N)$$

from the configuration space of (ordered) k-tuples of distinct points in $\mathbb{R}^d$ to the Stiefel manifold of ordered k-frames in $\mathbb{R}^N$. Cohen and Handel [7] showed that the existence of such an equivariant map is equivalent to the existence of an $(N-k)$-dimensional inverse of the vector bundle

$$\xi_{d,k} : \mathbb{R}^k \longrightarrow F(\mathbb{R}^d, k) \times_{\mathfrak{S}_k} \mathbb{R}^k \longrightarrow F(\mathbb{R}^d, k)/\mathfrak{S}_k,$$

that is, to the existence of an embedding into a trivial bundle of rank N over the unordered configuration space $F(\mathbb{R}^d, k)/\mathfrak{S}_k$.

Thus if the k-regular map $f : \mathbb{R}^d \to \mathbb{R}^N$ exists, then the dual Stiefel–Whitney class $\overline{w}_{N-k+1}(\xi_{d,k})$ vanishes. Hence Theorem 1 is a consequence of the following, which is our main technical result.

Theorem 2 *For any $d \geq 1$ and $k \geq 1$,*

$$\overline{w}_{(d-1)(k-\alpha(k))}(\xi_{d,k}) \neq 0.$$

Chisholm has proved this for the case $d = 2^e$ in 1979 [6].

Proof Our proof proceeds in five steps. The first four of them treat the special case $k = 2^m$.

In the first step we pass to the image of the cohomology of $\mathfrak{S}_k$ under the restriction to the cohomology of the subgroup $E_m := (\mathbb{Z}/2)^m$. The image of the restriction is known to be generated by the Dickson invariants $q_{m,i}$ of degree $2^m - 2^i$.

In the second step, we identify $q_{m,0}^{d-1}$ with $w_{k-1}(\xi_{d,k})^{d-1}$, which is non-zero according to our previous work in [1]. All other monomials in the classes w_i, $i < k-1$, of degree at least $(d-1)(k-1)$, vanish according to a key observation by Hung [12] together with the structure of our model of the configuration space $F(\mathbb{R}^d, n)$ from [1].

In the third step, we proceed by induction on d, based on the general Borsuk–Ulam–Bourgin–Yang theorem.

In the fourth step, we study the monomial expansion of the dual Stiefel–Whitney invariants,

$$\overline{w}_{(d-1)(k-\alpha(k))}(\xi_{d,k}) = \sum_{\substack{j_1,\dots,j_{k-1}\geq 0 \\ j_1+2j_2+\dots+(k-1)j_{k-1}=(d-1)(k-1)}} \binom{j_1+\dots+j_{k-1}}{j_1,j_2,\dots,j_{k-1}} w_1^{j_1}\cdots w_{k-1}^{j_{k-1}}$$
$$= \quad w_{k-1}^{d-1} \quad + \quad \text{other terms},$$

where Chisholm [6] had exploited that for $d = 2^e$ some of the relevant multinomial coefficients vanish mod 2, while we need and get that all the "other terms" are zero.

In the fifth and last step we extend the result to general k.

We refer to [2] for the details.

4 Further Work

Our study [2] produces similar topological lower bound results also for the existence of *ℓ-skew embeddings* $\mathbb{R}^d \to \mathbb{R}^N$, for which we require that the images of the tangent spaces of any ℓ distinct points are skew affine subspaces. This extends work by Ghomi and Tabachnikov [8] for $\ell = 2$.

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Part II

Positive Sum Systems

Anders Björner

Abstract Let $x_1, x_2, \ldots, x_n$ be real numbers summing to zero, and let $\mathcal{P}^+$ be the family of all subsets $J \subseteq [n] := \{1, 2, \ldots, n\}$ such that $\sum_{j \in J} x_j > 0$. Subset families arising in this way are the objects of study here.

We prove that the order complex of $\mathcal{P}^+$, viewed as a poset under set containment, triangulates a shellable ball whose f-vector does not depend on the choice of x, and whose h-polynomial is the classical Eulerian polynomial. Then we study various components of the flag f-vector of $\mathcal{P}^+$ and derive some inequalities satisfied by them.

It has been conjectured by Manickam, Miklós and Singhi in 1986 that $\binom{n-1}{k-1}$ is a lower bound for the number of k-element subsets in $\mathcal{P}^+$, unless n/k is too small. We discuss some related results that arise from applying the order complex and flag f-vector point of view.

Some remarks at the end include brief discussions of related extensions and questions. For instance, we mention positive sum set systems arising in matroids whose elements are weighted by real numbers.

1 Introduction

Let $x_1, x_2, \ldots, x_n$ be real numbers summing to zero,[1] and let $\mathcal{P}^+ = \mathcal{P}^+(x)$ be the family of all subsets $J \subseteq [n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ such that $x(J) \stackrel{\text{def}}{=} \sum_{j \in J} x_j > 0$. Such a set family $\mathcal{P}^+(x)$ will here be called a *Positive Sum System—PSS* for short. A standing

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[1] The zero sum condition can in several cases be relaxed to nonnegative sum, with only slight modification of arguments. We leave this without further mention.

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assumption is that the *weight* vector x is *generic*, by which is meant that $x(J) \neq 0$, for all proper subsets J.

This same concept comes up in several applied areas, such as Physics, Game Theory, Economics, and Computer Science, typically under a multitude of different names such as “linear threshold hypergraphs”, “maximal unbalanced families”, etc.[2] See the paper [2] for many references to applications.

Is there some recognizable structure to such a set system $\mathcal{P}^+$? Interesting enumerative questions have been discussed in the literature, some of which are mentioned here, see Sect. 7. Approaching PSS systems from another direction, we prove that the order complex of $\mathcal{P}^+$, viewed as a poset under set containment, triangulates a shellable ball whose h-polynomial is the classical Eulerian polynomial. One surprising consequence is that the h-vector of $\mathcal{P}^+(x)$, and hence also its f-vector, does not depend on the weight vector x.

Example 1.1 Let $x = (4, 1, -2, -3)$. Then

$$\mathcal{P}^+(x) = \{1, 2, 12, 13, 14, 123, 124\},$$

where for notational ease set brackets are omitted. The set system $\mathcal{P}^+(x)$ is shown as a poset in Fig. 1, where to distinguish from different uses of the numerals we use the labeling $(1, 2, 3, 4) = (a, b, c, d)$.

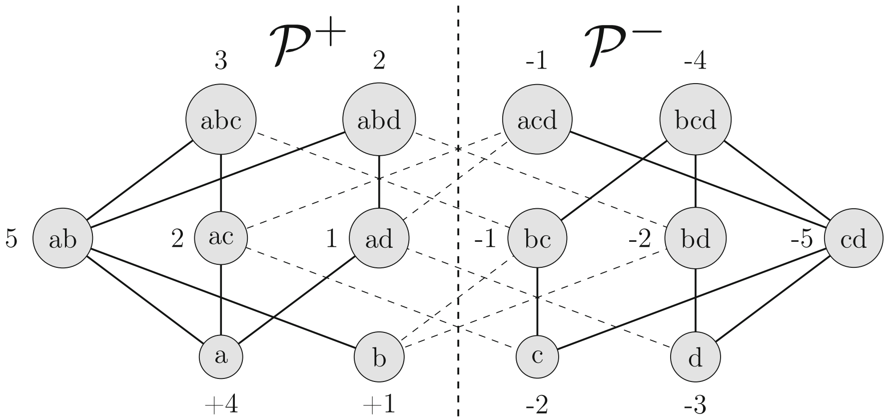

Fig. 1 A PSS system and its dual with set inclusion edges

[2]Opposite to H. Poincaré’s aphorism “Mathematics is the art of giving the same name to different things”.

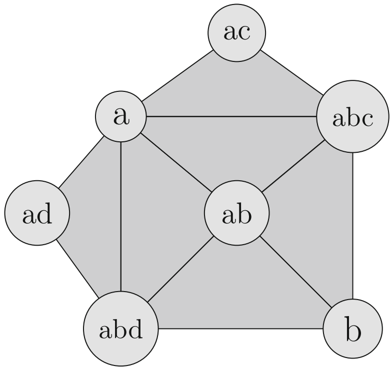

Fig. 2 The order complex, a ball

This PSS has six maximal chains, each of cardinality three. Its order complex consists of six triangles (2-simplices) glued together this way to form a ball (Fig. 2):

We assume some familiarity with the combinatorics of simplicial complexes and their face numbers. As a general reference, see [4].

The ball $\Delta(\mathcal{P}^+(x))$ is balanced, so we may talk also about its *flag f-vector*, $(f_J \mid J \subseteq [n])$. This number array turns out to depend on x, however certain linear combinations of the flag f-numbers are invariant. Several equalities and inequalities for the flag f-vectors of PSS systems come from the fact that $\Delta(\mathcal{P}^+)$ triangulates a ball, but there are also other ones.

How many k-element subsets must there be in $\mathcal{P}^+$? Let $\mathcal{P}_k^+ := \{J \in \mathcal{P}^+ : |J| = k\}$. It has been conjectured by Manickam, Miklós and Singhi in 1986 that $|\mathcal{P}_k^+| \geq \binom{n-1}{k-1}$ for $n \geq 4k$. Despite much work this conjecture remains open. We discuss some related results that naturally arise from taking a "flag f-vector" point of view.

2 Some Basic Properties

The following summarizes some basic properties of the PSS poset $\mathcal{P}^+ = \mathcal{P}^+(x)$. A standard assumption is this: $x_1 \geq \cdots \geq x_n$. Also, recall that we demand that $x(J) = 0$ if and only if $J = \emptyset$ or $J = [n]$.

A family Γ of subsets of $[n]$ is said to be *shifted* if $A \in \Gamma$, $1 \leq y < z \leq n$, $y \notin A$, $z \in A$ implies that $(A \setminus z) \cup \{y\} \in \Gamma$. We let $\mathcal{B}_n$ denote the Boolean poset of all proper subsets of $[n]$, ordered by inclusion.

Proposition 2.1 *Let $\mathcal{P}^+$ be a PSS. Then,*

(1) $|\mathcal{P}^+| = 2^{n-1} - 1$.
(2) *$\mathcal{P}^+$ is a shifted set family.*
(3) *Given $x \in \mathbb{R}^n$, there exists $y \in \mathbb{Z}^n$, such that $P^+(x) = P^+(y)$, and for non-empty $I, J \subseteq [n]$: $I \neq J \Rightarrow y(I) \neq y(J)$.*

(4) *If J covers $H_1, \ldots, H_k$ in $\mathcal{B}_n$, then*

$$x(J) = \frac{x(H_1) + \cdots + x(H_k)}{k-1}$$

(5) *If J is covered by $G_1, \ldots, G_{n-k}$ in $\mathcal{B}_n$, then*

$$x(J) = \frac{x(G_1) + \cdots + x(G_{n-k})}{n-k-1}$$

(6) *All maximal chains of $\mathcal{P}^+$ have the same length $n-2$.*

Proof In each pair of a set J and its complement $J^c \stackrel{\text{def}}{=} [n] \setminus J$, one set is positive and the other negative, so $|\mathcal{P}^+| = \frac{|\mathcal{B}_n|}{2} = 2^{n-1} - 1$.

Part (2) is immediate from the definition.

Statement (3) is seen by perturbing the points x_i to sufficiently nearby generic rational points, followed by dilation to integer points.

In the sum $x(G_1) + \cdots + x(G_{n-k})$, each element of J is counted $n-k$ times, and each element not in J is counted once. Therefore,

$$x(G_1) + \cdots + x(G_{n-k}) = (n-k)x(J) + x(J^c) = (n-k-1)x(J)$$

which proves (5). Part (4) is handled similarly.

Part (4) shows that if J has positive weight, then at least one of the covering sets $G_1, \ldots, G_{n-k}$ must be positive. It is a direct consequence that all maximal chains of $\mathcal{P}^+$ have full length. □

Corollary 2.2

$$x(J) = \frac{\sum x(H_i) + \sum x(G_j)}{n-2}$$

If we forget the order structure of $\mathcal{B}_n$ for a moment and view the situation on the edge graph of the n-dimensional cube, then the corollary expresses a kind of "subharmonic" property of the function $x(\cdot)$: its value at any vertex of the cube is, up to a $\frac{n-2}{n}$-factor, the average of its values at the neighbors of that vertex.

Parts (4) and (5) allow the following immediate generalization. Suppose that $|J| = k$, and that $p < k < q$. Let $H_1, \ldots, H_{\binom{k}{p}}$ be the p-element subsets of J. Then

$$x(J) = \frac{x(H_1) + \cdots + x(H_{\binom{k}{p}})}{p}$$

Similarly, if $G_1, \ldots, G_{\binom{n-k}{q-k}}$ are the q-element supersets of J, then

$$x(J) = \frac{x(G_1) + \cdots + x(G_{\binom{n-k}{q-k}})}{n-q}$$

3 PSS Are Balls

Theorem 3.1 *The order complex $\Delta(\mathcal{P}^+)$ is a shellable $(n-2)$-dimensional ball.*

Proof The complex $\Delta(\mathcal{P}^+)$ is pure and $(n-2)$-dimensional. Furthermore it has the property that every $(n-3)$-dimensional face borders at most two facets, and some border only one facet. It is well known that under these conditions shellability implies being homeomorphic to a ball.

We use the method of lexicographic shellability [6]. The proof that $\mathcal{P}^+$ is EL-shellable is straight-forward. Label each covering edge in the Hasse diagram of $\mathcal{P}^+$ by the new element x_i that is added when going up along that edge. Then order the labels in reverse magnitude order: $x_1 < x_2 < \cdots < x_n$. One easily checks that in each interval in $\mathcal{P}$ there is a unique increasing chain and it comes lexicographically first. □

The triangulated balls occurring as PSS balls have some very particular structure. For instance, as a consequence of Theorem 5.3 they have precisely one internal vertex, namely the set of all positive elements.

In the case $n = 3$ every PSS ball consists of two edges joined at a common vertex. For $n = 4$ there are exactly two isomorphism types of PSS balls, as shown in Fig. 3. In this case, if the number of positive elements is two then the ball will be of type Δ, and if the number of positive elements is one or three it will be of type Γ.

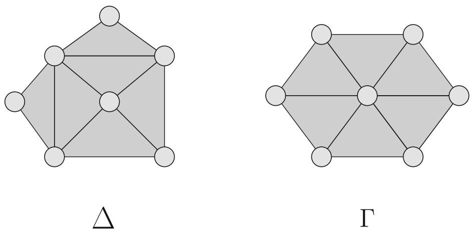

Fig. 3 The two types of PSS balls obtainable when $n = 4$

4 Rotation of Compositions

We now discuss some combinatorial properties of compositions, or ordered partitions. There are two kinds.

Definition 4.1

(i) (*Set composition*) A composition with k parts of a set E, an (E,k)-*composition*, is an ordered sequence of nonempty and pairwise disjoint subsets $< D_1, \ldots, D_k >$ such that

$$D_1 \cup D_2 \cup \cdots \cup D_k = E.$$

(ii) (*Number composition*) A composition with k parts of the number n, an (n,k)-*composition*, is an ordered sequence of positive integers $< c_1, \ldots, c_k >$ such that

$$c_1 + c_2 + \cdots + c_k = n.$$

The set of all (E,k)-compositions is permuted by the *rotation* operator:

$$\mathbf{Rot}_k < D_1, D_2, \cdots, D_{k-1}, D_k > \stackrel{\text{def}}{=} < D_2, D_3, \cdots, D_k, D_1 >$$

Similarly, for the set of (n,k)-compositions:

$$\mathbf{rot}_k < c_1, c_2, \cdots, c_{k-1}, c_k > \stackrel{\text{def}}{=} < c_2, c_3, \cdots, c_k, c_1 >$$

There is an obvious forgetful map φ, mapping set compositions to number compositions by replacing the subsets by their cardinalities. This map commutes with rotation:

$$\varphi \circ \mathbf{Rot}_k = \mathbf{rot}_k \circ \varphi$$

Note that, whereas all orbits of the $\mathbf{Rot}_k$ operator are of size k, all we can say about the orbits of $\mathbf{rot}_k$ is that their sizes divide k.

We need two facts about the orbits of the rotation maps, expressed in the following lemmas.

Lemma 4.2 *Let c be a (n,k)-composition whose* **rot**-*orbit has size d. Then the set*

$$T(c) \stackrel{\text{def}}{=} \varphi^{-1}(c) \cup \varphi^{-1}(\mathbf{rot}(c)) \cup \varphi^{-1}(\mathbf{rot}^2(c)) \cup \cdots \cup \varphi^{-1}(\mathbf{rot}^{d-1}(c))$$

is a union of **Rot**-*orbits, and the number of these orbits is* $\frac{d}{k}\binom{n}{c_1,\ldots,c_k}$.

Proof We have that

$$|\varphi^{-1}(\mathbf{rot}^j(c))| = \binom{n}{c_1, \ldots, c_k}$$

for all j. Hence, $|T(c)| = d\binom{n}{c_1,\ldots,c_k}$, and the claim follows. □

Now, suppose that we have real numbers $x_1, x_2, \ldots, x_n$ summing to zero. We assume genericity, as previously defined. By the *charge* of a set composition $< D_1, \ldots, D_k >$ we mean

$$\text{charge}(D) \stackrel{\text{def}}{=} \text{the number of } j \text{ such that } x(D_1 \cup D_2 \cup \cdots \cup D_j) < 0$$

Thus, $0 \leq \text{charge}(D) \leq k - 1$.

Lemma 4.3 *Each* $\mathbf{Rot}_k$*-orbit contains precisely one composition of charge s, for each* $s = 0, 1, 2, \ldots k - 1$.

Proof Let j be such that $x(D_1 \cup D_2 \cup \cdots \cup D_j)$ is minimal. This requirement uniquely determines j, due to the genericity assumption. Then $\text{charge}(\mathbf{Rot}^j(D)) = 0$. Next, let j be such that $x(D_1 \cup D_2 \cup \cdots \cup D_j)$ is second smallest. Then $\text{charge}(\mathbf{Rot}^j(D)) = 1$, and so on. □

5 *f*-Vectors and Eulerian Numbers

We now return to the discussion of the order complex $\Delta(\mathcal{P}^+)$ of a PSS system $\mathcal{P}^+(x)$. Let f_k denote the number of k-dimensional faces. Our concern is to study these face numbers, as recorded in the *f-vector* $(f_{-1}, f_0, \ldots, f_{n-2})$ and the *h-vector* $(h_0, h_1, \ldots, h_{n-1})$, related via

$$\sum_{i=0}^{n-1} f_{i-1} t^{n-1-i} = \sum_{i=0}^{n-1} h_i (t+1)^{n-1-i}$$

Consider our running example, the PSS system defined in Example 1.1. Its order complex is a two-dimensional ball Δ with f and h-vectors $f = (1, 7, 12, 6)$ and $h = (1, 4, 1, 0)$. Note that the other two-dimensional PSS ball Γ shown in Fig. 3 has the same face numbers.

We know that, in general, the order complex $\Delta(\mathcal{P}^+)$ is an $(n-2)$-ball embedded in the $(n-2)$-sphere $\Delta(\mathcal{B}_n)$. The following relationship between the order complex f-vectors of $\mathcal{P}^+$ and of $\mathcal{B}_n$ (the barycentric subdivision of the boundary of an $(n-1)$-simplex) is a key observation:

Lemma 5.1

$$f_j(\Delta(\mathcal{P}^+)) = \frac{1}{j+2} f_j\,(\Delta(\mathcal{B}_n))$$

Proof The j-faces of $\Delta(\mathcal{B}_n)$ are chains of $j+1$ proper subsets of $[n]$. By taking successive difference sets, such chains can be equivalently identified as $([n], j+2)$-compositions.

The set of all $([n], j+2)$-compositions is partitioned into **Rot**-orbits, all of size $j+2$. Lemma 4.3 tells that in each orbit there is exactly one composition of zero charge. A composition has charge $=0$ if and only if the corresponding chain is in $\mathcal{P}^+$, so we are done. □

Theorem 5.2 *Let* $\mathcal{P}^+ = \mathcal{P}^+(x)$.

(1) The f-polynomial (and hence also the h-polynomial) of $\Delta(\mathcal{P}^+)$ *is an invariant, the same for all weight vectors x.*
(2) The h-polynomial of $\Delta(\mathcal{P}^+)$ *is the classical Eulerian polynomial. That is,*

$$h_i(\Delta(\mathcal{P}^+)) = \text{card}\,\{\textit{permutations} \in S_n \textit{ with } i \textit{ descents}\}$$

Proof Part (1) is a direct consequence of Lemma 5.1. Because of it we are free to work with any weight vector x.

The vector $x = \{1, 1, \ldots 1, 1-n\}$ is a good choice. Then $\Delta(\mathcal{P}^+)$ is the order complex of a half-open interval $(\emptyset, [n-1]]$ in $\mathcal{B}_n$. This is a cone over the barycentric subdivision of the boundary of an $(n-2)$-simplex (equivalently, the Coxeter complex of the symmetric group S_{n-1}), and the description of its h-vector is well known. For instance, it is easy to derive via the method of R-labeling, see [6, 11]. □

See [11] for information about Eulerian numbers and polynomials. As an example, $h = (1, 11, 11, 1, 0)$ and $h_{\Delta(\mathcal{P}^+)}(t) = t + 11t^2 + 11t^3 + t^4$ for every PSS with $n = 5$.

It is well known that the h-vector of a ball determines not only the face numbers of the ball, but also the face numbers of the bounding sphere and the face numbers of the ball's interior, see e.g. [4, Thm. 18.3.6].

Thus, relying on the easily computable f-numbers of the barycentric subdivision of the boundary of a simplex, we derive expressions like the following.

Theorem 5.3 *We have for the* $(n-2)$*-ball* $\mathcal{P}^+(x)$*:*

(i) $f_0(\mathcal{P}^+) = \frac{1}{2} f_0(\mathcal{B}_n) = 2^{n-1} - 1$
(ii) $f_1(\mathcal{P}^+) = \frac{1}{3} f_1(\mathcal{B}_n) = 3^{n-1} - 2^n + 1$
(iii) $f_{n-3}(\mathcal{P}^+) = \frac{1}{n-1} f_{n-3}(\mathcal{B}_n) = \frac{n!}{2}$
(iv) $f_{n-2}(\mathcal{P}^+) = \frac{1}{n} f_{n-2}(\mathcal{B}_n) = (n-1)!$

and for its boundary $\partial\mathcal{P}^+(x)$*, an* $(n-3)$*-sphere:*

(v) $f_0(\partial\mathcal{P}^+(x)) = 2^{n-1} - 2$
(vi) $f_1(\partial\mathcal{P}^+(x)) = 3(3^{n-2} - 2^{n-1} + 1)$

(vii) $f_{n-4}(\partial\mathcal{P}^+(x)) = (n-1)!\frac{n-2}{2}$
(viii) $f_{n-3}(\partial\mathcal{P}^+(x)) = (n-1)!$

Part (iv) states the useful fact that the number of totally positive maximal chains in the Boolean poset $\mathcal{B}_n$ is $(n-1)!$. Parts (i) and (v) together imply that the number of internal vertices is one.

6 Flag *f*-Vectors

Flag f- and h-vectors are defined in the following way.

Definition 6.1 For $Z \subseteq \mathcal{B}_n$ and $J = \{i_1 < i_2 < \cdots < i_k\} \subseteq [n-1]$, let

(i) $f_J = f_J(Z) = \#\{chains\ z_1 < \cdots < z_k\ in\ Z\ such\ that\ |z_j| = i_j\}$,

(ii) $h_J = h_J(Z) = \sum_{I \subseteq J}(-1)^{||J \setminus I||} f_J$.

There is an obvious bijective correspondence ϕ between number (n, k)-compositions and $(k-1)$-subsets of $[n-1]$:

$$c_1 + c_2 + \cdots + c_k \longleftrightarrow \{c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1}\}$$

Hence, instead of indexing flag f-vectors by subsets J, we can index by number compositions $c =< c_1 + c_2 + \cdots + c_k >$. Also, the rotation operator can be applied to subsets as well as to compositions:

$$rot\ (J) \overset{\text{def}}{=} \phi^{-1} \circ \mathbf{rot} \circ \phi\ (J)$$

The flag f-vector of $\mathcal{P}^+(x)$ refines the information given by the ordinary f-vector of the previous section. It is therefore natural to ask if also the flag f-vector $f_J(\mathcal{P}^+(x))$ is independent of x. Let us take a look at the two $n = 4$ complexes Δ and Γ shown in Fig. 3. Here are their flag f-vectors:

J	∅	**1**	2	**3**	1, 2	1, 3	2, 3	1, 2, 3
Δ	1	**2**	3	**2**	4	4	4	6
Γ	1	**3**	3	**1**	6	3	3	6

And these are the flag h-vectors:

J	∅	**1**	2	**3**	1, 2	1, 3	2, 3	1, 2, 3
Δ	1	**1**	2	**1**	0	1	0	0
Γ	1	**2**	2	**0**	1	0	0	0

These tables show that the flag f-vector is not an invariant. However, there are certain relations that do not depend on x.

Theorem 6.2 *Let J be a $(k-1)$-element subset of $[n-1]$ corresponding to the composition $c_1 + c_2 + \cdots + c_k = n$, whose rotation orbit has size d. Then, writing for simplicity $f_J = f_J(\mathcal{P}^+(x))$, we have*

$$f_J + f_{rot(J)} + f_{rot^2(J)} + \cdots + f_{rot^{d-1}(J)} = \frac{d}{k}\binom{n}{c_1, \ldots, c_k}$$

Proof This follows from Lemmas 4.2 and 4.3. □

Example 6.3 Here are the rotation relations for flag f-vectors, $n = 4$:

(i) $f_{\{1\}} + f_{\{3\}} = \binom{4}{1,3} = 4$
(ii) $f_{\{2\}} = \frac{1}{2}\binom{4}{2,2} = 3$
(iii) $f_{\{1,2\}} + f_{\{2,3\}} + f_{\{1,3\}} = \binom{4}{1,2,1} = 6$

In the tables above for our running example, the relation (i) is the sum of the boldface entries, while relation (iii) is the sum of the underlined entries.

The rotation relations for flag f-vectors, $n = 5$, are as follows:

(i) $f_{\{1\}} + f_{\{4\}} = 5$
(ii) $f_{\{2\}} + f_{\{3\}} = 10$
(iii) $f_{\{1,2\}} + f_{\{3,4\}} + f_{\{1,4\}} = 20$
(iv) $f_{\{1,3\}} + f_{\{2,3\}} + f_{\{2,4\}} = 30$
(v) $f_{\{1,2,3\}} + f_{\{2,3,4\}} + f_{\{1,3,4\}} + f_{\{1,2,4\}} = 30$

7 Positive k-Sets, Edges and Matchings

In this section we discuss inequalities for two particularly interesting components of the flag f-vector: $f_{\{k\}}(\mathcal{P}^+)$ and $f_{\{k-1,k\}}(\mathcal{P}^+)$. Also, we prove the existence of matchings of positive $(k-1)$-sets to positive k-sets under certain conditions.

How many k-element subsets must $\mathcal{P}^+$ contain? An intriguing conjecture by Manickam, Miklós and Singhi from 1986 states that[3]

$$f_{\{k\}}(\mathcal{P}^+) \geq \binom{n-1}{k-1} \quad \text{for } n \geq 4k. \tag{7.1}$$

[3]Note the distinction between $f_k(\mathcal{P}^+)$ (defined in Sect. 5) and $f_{\{k\}}(\mathcal{P}^+)$ (a component of the flag f-vector).

This lower bound, if true, is tight, as shown by PSS systems governed by weight vectors x with only one positive entry.

Example 7.1 Let $x = (3, 3, 3, 3, 3, 3, 3, -7, -7, -7)$. This determines a PSS with parameters $(n, k) = (10, 3)$. Here $|\mathcal{P}_3^+| = \binom{7}{3} = 35$, but the inequality (7.1) promises $|\mathcal{P}_3^+| \geq \binom{9}{2} = 36$. This shows that some restriction such as $n \geq 4k$ is needed in the MMS conjecture

A lot of work has been done on the MMS conjecture. Manickam, Miklós and Singhi showed that if n is sufficiently large with respect to k, then the inequality $f_{\{k\}}(\mathcal{P}^+) \geq \binom{n-1}{k-1}$ holds. It is natural to wonder "how large"?

For a long time the best lower bounds known for n were exponential in k. A breakthrough came in 2011 when Alon et al. [2] proved the MMS inequality (7.1) to hold for $n \geq 33k^2$. Finally, a linear bound with huge constant ($n \geq 10^{46}k$) was given in 2013 by Pokrovskiy [9]. The inequality is also known to be true when k divides n, as a consequence of the hypergraph theorem of Baranyai [3]. See [2] for details about these and other references.

The following result show relaxations of the MMS inequality, valid for all $n > k$. The second part generalizes the Baranyai bound.

Theorem 7.2 *For all $n > k$,*

(i)

$$f_{\{k-1,k\}}(\mathcal{P}^+) \geq \binom{n-1}{k-1}.$$

(ii)

$$f_{\{k\}}(\mathcal{P}^+) \geq \frac{\gcd(n,k)}{k}\binom{n-1}{k-1}$$

(iii)

$$f_{\{k\}}(\mathcal{P}^+) \geq \frac{1}{p}\binom{n-1}{k-1},$$

where p is the average number of positive elements in a positive k-subset.

Proof There are $(n-1)!$ totally positive maximal chains in $\mathcal{B}_n$, each of them must pass through a positive $\{k-1, k\}$ edge, and each such edge is contained in $(k-1)!(n-k)!$ maximal chains. This implies part (i).

Let $d = \gcd(n, k)$ and consider the family $\mathcal{C}_{n,d}$ of chains of type $< d, d, \ldots, d >$. There are $\binom{n}{d,d,\ldots,d}$ such chains, and the $\mathbf{Rot}_{n/d}$ operator acts on $\mathcal{C}_{n,d}$ with orbits of size n/d. In each orbit there is, by Lemma 4.3, exactly one totally positive chain.

Hence, there are

$$\frac{1}{n/d}\binom{n}{d,d,\ldots,d} = \binom{n-1}{d-1}\binom{n-d}{d,d,\ldots}$$

totally positive chains in $\mathcal{C}_{n,d}$. Each such chain must contain some $z \in \mathcal{P}_k^+$, and each $z \in \mathcal{P}_k^+$ is contained in exactly $\binom{k}{d,d,\ldots}\binom{n-k}{d,d,\ldots}$ chains.

Hence,

$$\binom{n-1}{d-1}\binom{n-d}{d,d,\ldots} \leq f_{\{k\}}(\mathcal{P}^+)\binom{k}{d,d,\ldots}\binom{n-k}{d,d,\ldots}$$

which simplifies to part (ii):

$$\frac{\gcd(n,k)}{k}\binom{n-1}{k-1} \leq f_{\{k\}}(\mathcal{P}^+).$$

It is a consequence of part (ii) that $f_{\{k\}}(\mathcal{P}^+) \geq \frac{1}{k}\binom{n-1}{k-1}$. From this, part (iii) is easily proved by averaging. We omit the details. □

The set $\mathcal{P}^+_{\{k-1,k\}}$ is the collection of edges of a bipartite graph whose vertices are $\mathcal{P}^+_{\{k-1\}}$ and $\mathcal{P}^+_{\{k\}}$. We have seen some lower bounds, proven and conjectured, to the sizes of the two vertex sets and to $f_{\{k-1,k\}}(\mathcal{P}^+)$. We end with a structural result concerning matchings.

Let $\mathrm{pos}(x)$ denote the number of positive entries in the weight vector $x = (x_1,\ldots,x_n)$, and let $\mathcal{P}_k^+(x)$ be the family of all k-element sets in $\mathcal{P}^+(x)$. A *matching* is an injective mapping $\phi : \mathcal{P}^+_{k-1}(x) \to \mathcal{P}^+_k(x)$, such that $T < \phi(T)$ for all T.

Theorem 7.3 *Suppose that pos*$(x) \geq 2k-1$. *Then there exists a matching* $\mathcal{P}^+_{k-1}(x) \to \mathcal{P}^+_k(x)$.

Proof The existence of matchings between adjacent cardinality levels in the full Boolean lattice is well known. The construction has been described by various authors.

We refer to the description in [12, §2.2]. The injective mapping $\phi : \binom{[n]}{k-1} \to \binom{[n]}{k}$ constructed on pages 36–37 of [12] has the property that if $\phi(T) = T \cup \{t\}$, then $t \leq 2k-1$. Therefore, $x_t \geq x_{2k-1} > 0$, and if T is positive, so is also $\phi(T)$. □

Corollary 7.4 *Suppose that pos*$(x) \geq 2k-1$. *Then,*

$$f_{\{1\}}(\mathcal{P}^+) \leq f_{\{2\}}(\mathcal{P}^+) \leq \cdots \leq f_{\{k-1\}}(\mathcal{P}^+) \leq f_{\{k\}}(\mathcal{P}^+)$$

Thus, if the number of positive entries and of negative entries in x are roughly equal, then this increasing sequence will run until circa $k = n/4$. Can we hope for more? Are there inequalities of the type $f_{\{j-1\}}(\mathcal{P}^+) \leq f_{\{j\}}(\mathcal{P}^+)$ for other j? This seems likely, but we leave this open and end with a small example, namely the PSS defined in Example 7.1, for which

$$(f_{\{1\}}(\mathcal{P}^+), \dots, f_{\{9\}}(\mathcal{P}^+) = (7, 21, 35, 140, 126, 70, 85, 24, 3)$$

8 Remarks

8.1 A Flag MMS Conjecture

The MMS conjecture says that a PSS system $\mathcal{P}^+ = \mathcal{P}^+(x)$ minimizes $f_{\{k\}}(\mathcal{P}^+)$ (for $k \leq n/4$) if its weight vector x has only one positive entry. The same intuition leads to the following generalization.

Conjecture 8.1 *Let $\mathcal{P}^+ = \mathcal{P}^+(x)$ be a PSS system, and let J be a subset of $[n-1]$ corresponding to a composition $k_1 + k_2 + \cdots + k_t = n$, such that $k_t \geq 3n/4$. Then,*

$$f_J(\mathcal{P}^+) \geq \binom{n-1}{k_1 - 1, k_2 - k_1, \dots, k_t - k_{t-1}}$$

Using the same strategy as in the proof of Theorem 7.2, we easily obtain this weaker approximation to that lower bound.

$$f_J(\mathcal{P}^+) \geq \frac{1}{k}\binom{n-1}{k_1 - 1, k_2 - k_1, \dots, k_t - k_{t-1}}, \text{ for all } n.$$

8.2 Counting PSS Systems of Given Size

How many distinct positive sum set systems can arise as one varies the vector $x_1, x_2, \dots, x_n$? Let E_n be this number. It has been computed up to $n = 9$ by quantum theory physicists, for whom these numbers have a meaning in connection with thermal field theory. Here are the beginning values [13]:

n	2	3	4	5	6	7	8
E_n	2	6	32	370	11 292	1 066 044	347 326 352

No exact formula for E_n is known. Its asymptotic growth has been studied, and this estimate is known:

$$E_n = 2^{n^2 + o(n^2)}, \quad \text{or more precisely, } 2^{\binom{n-1}{2}} < E_n < 2^{(n-1)^2}$$

See the papers [5, 14] for these results and further information.

8.3 PSS Systems in Matroids

The definition of a PSS can be immediately adapted to any finite set family. For instance, weighting the elements of a matroid by real numbers, one gets the systems of positive bases, positive circuits, positive flats, and so on. In this direction there is the following recent result of Adiprasito and the author [1] concerning the system of positive flats in a matroid.

Let $\mathcal{L}$ be a geometric lattice of rank r, with set of atoms A. Let $x : A \to \mathbb{R}$ be a mapping such that $\sum_{a\in A} x(a) = 0$, and let $\mathcal{L}^+(x) = \{z \in \mathcal{L} \mid \sum_{a\in z} x(a) > 0\}$.

Theorem 8.2 ([1]) *The order complex of $\mathcal{L}^+(x)$ is topologically $(r-3)$-connected. In fact, it is homotopy Cohen-Macaulay.*

This result had been conjectured by G. Mikhalkin and G.M. Ziegler (The positive part of a geometric lattice, 2008, personal communication), motivated by its relevance for questions concerning Lefschetz hyperplane theorems in tropical geometry.

The wider context of matroid theory offers new perspectives and possibilities. Let M be a connected matroid of rank k on n elements, and suppose given a weight vector x summing to zero but otherwise generic, as previously defined.

Let $\tilde{\mu}(M)$ be the Möbius number of M, i.e., the absolute value of $\mu(\hat{0}, \hat{1})$ computed over the lattice of flats of M. It is known [7, Thm. 7.1] that the matroid M has at least $\frac{n}{k}\tilde{\mu}(M)$ bases. How many of these must be of positive weight?

Conjecture 8.3 *M has at least $\tilde{\mu}(M)$ positive bases, if $n \geq 4k$.*

For the special case when M is the k-uniform matroid we have that $\tilde{\mu}(M) = \binom{n-1}{k-1}$. Hence, Conjecture 8.3 contains the MMS conjecture as a special case.

8.4 Scarf Complexes

Motivated by questions in integer programming, H.E. Scarf has introduced a class of simplicial complexes with vertices in the integer point lattice and faces determined by movable linear constraints. See [10] for details and references. It can be shown that the PSS complexes $\Delta(\mathcal{P}^+)$ studied here are Scarf complexes (called "Top complexes" by him).

In [8] the author studied the f-vectors of Scarf complexes. The points of view in the two papers are quite different, but it should be mentioned that some of the results here are proved in greater generality there. For instance, Lemma 5.1 is a special case of [8, Proposition 11(i)], and Sect. 4 of [8] gives a more detailed discussion of its consequences than what is offered here. On the other hand, flag f-vectors are not considered in [8], since Scarf complexes are in general not order complexes of posets or otherwise balanced.

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The S_{n+1} Action on Spherical Models and Supermaximal Models of Tipe A_{n-1}

Filippo Callegaro and Giovanni Gaiffi

Abstract In this paper we recall the construction of the De Concini–Procesi wonderful models of the braid arrangement: these models, in the case of the braid arrangement of type A_{n-1}, are equipped with a natural S_n action, but only the minimal model admits an 'hidden' symmetry, i.e. an action of S_{n+1} that comes from its moduli space interpretation. We show that this hidden action can be lifted to the face poset of the corresponding minimal real spherical model and we compute the number of its orbits. Then we provide a spherical version of the construction of the supermaximal model (see Callegaro, Gaiffi, On models of the braid arrangement and their hidden symmetries. Int. Math. Res. Not. (published online 2015). doi: 10.1093/imrn/rnv009), i.e. the smallest model that can be projected onto the maximal model and again admits the extended S_{n+1} action.

1 Introduction

The first part of this paper is a short survey on the construction of De Concini–Procesi (real and complex) wonderful models of subspace arrangements (see [5, 6]) and on the similar construction of real spherical models (see [20]). After describing the main properties of these spaces, we recall that they are particular cases of some more general constructions that, starting from a 'good' stratified variety, produce models by blowing up a suitable subset of strata (see [30, 31] and also [10] for further references).

We then consider, as an example, the case of the braid arrangement, i.e., the root arrangement of type A. Here there is a natural action of the symmetric group: S_n acts on the root arrangement of type A_{n-1}, and therefore on all the above mentioned wonderful models.

It turns out that the minimal projective (real or complex) De Concini–Procesi model associated with this root arrangement is isomorphic to the moduli space $\overline{M}_{0,n+1}$ of $n+1$-pointed stable curves of genus 0, therefore it carries an 'hidden'

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extended action of S_{n+1} that has been studied by several authors (see for instance [13, 25, 36]).

The second part of this paper is devoted to this symmetric group action in the spherical case. First we show (in Sect. 3) that the extended S_{n+1} action can be lifted to the face poset of the minimal real spherical model of type A_{n-1} (this model is the disjoin union of $n!$ Stasheff's associahedra) and we give formulas that count the number of orbits (see Sect. 4). It turns out that this problem of orbit counting is strictly related with the classical problem of counting the dissections of a convex polygon with $n+1$ edges modulo the action of the rotations. Our formulas involve a power series in two variables and can be compared with the formulas in the literature (see [1, 11, 32, 38, 39]).

In the third part of this paper we recall the construction of supermaximal models appeared in [2]. As a difference from [2] we mainly focus on the spherical case. We start in Sect. 5 by addressing the following problem: there are several De Concini–Procesi models associated with the arrangement of type A_{n-1}; among these spaces there are a minimal and a maximal one. We observe that the S_{n+1} action cannot be extended to the non-minimal models (we show this by an example). Why does this happen?

We answer to this question by showing in Sect. 7 that the maximal model is, in some sense, "too small". This takes two steps, that we express here from the point of view of spherical models:

1. We identify in a natural way its strata with a subset $\mathcal{T}$ of 1-codimensional strata of a "spherical supermaximal" model on which the S_{n+1} action is defined. This spherical supermaximal model is obtained by blowing up some strata in the spherical maximal model, but it also belongs to the family $\mathcal{L}$ of models obtained by blowing up 'building sets' of strata in the spherical minimal model; in fact it is the model obtained by blowing up all the strata of the spherical minimal model.
2. We show that the closure of $\mathcal{T}$ under the S_{n+1} action is the set of all the strata of the spherical supermaximal model. This means that the spherical supermaximal model is the minimal model in $\mathcal{L}$ that admits a projection onto the spherical maximal model and is equipped with the S_{n+1} action.

The second step is the content of Theroem 36.

We conclude by counting the orbits of the action of S_{n+1} on the spherical supermaximal model and quoting a theorem (proved in [2]) that describes a basis for the cohomology of a complex supermaximal model.

2 Wonderful Models

2.1 The Geometric Definition of Building Sets and Nested Sets

In this section we recall from [5, 6] the definitions of nested set and building set of subspaces in an euclidean space V.

Let $\mathcal{A}$ be a central subspace arrangement in an Euclidean space V. We denote by $\mathcal{C}_{\mathcal{A}}$ the closure under the sum of $\mathcal{A}$ and by $\mathcal{A}^{\perp}$ the arrangement

$$\mathcal{A}^{\perp} = \{A^{\perp} \mid A \in \mathcal{A}\}.$$

In particular we notice that if $\{O\} \notin \mathcal{A}$ then also $\{O\} \notin \mathcal{C}_{\mathcal{A}}$.

Definition 1 The collection of subspaces $\mathcal{G} \subset \mathcal{C}_{\mathcal{A}}$ is called *building set associated to* $\mathcal{A}$ if $\mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\mathcal{G}}$ and every element C of $\mathcal{C}_{\mathcal{A}}$ is the direct sum $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ of the maximal elements $G_1, G_2, \ldots, G_k$ of $\mathcal{G}$ contained in C (this is called the $\mathcal{G}$-decomposition of C).

Given a subspace arrangement $\mathcal{A}$, there are several building sets associated to it. Among these there always are a maximum and a minimum (with respect to inclusion). The maximum is $\mathcal{C}_{\mathcal{A}}$, the minimum is the building set of *irreducibles* that is defined as follows.

Definition 2 Given a subspace $U \in \mathcal{C}_{\mathcal{A}}$, a **decomposition of** U in $\mathcal{C}_{\mathcal{A}}$ is a collection $\{U_1, \cdots, U_k\}$ $(k > 1)$ of non zero subspaces in $\mathcal{C}_{\mathcal{A}}$ such that

1. $U = U_1 \oplus \cdots \oplus U_k$.
2. For every subspace $A \subset U$, $A \in \mathcal{C}_{\mathcal{A}}$, we have $A \cap U_1, \cdots, A \cap U_k \in \mathcal{C}_{\mathcal{A}}$ and $A = (A \cap U_1) \oplus \cdots \oplus (A \cap U_k)$.

Definition 3 A non zero subspace $F \in \mathcal{C}_{\mathcal{A}}$ which does not admit a decomposition is called **irreducible** and the set of irreducible subspaces is denoted by $\mathcal{F}_{\mathcal{A}}$.

Remark 4 In the case of a root arrangement (i.e. when $\mathcal{A}^{\perp}$ is the hyperplane arrangement provided by the hyperplanes orthogonal to the roots of a root system Φ) the building set of irreducibles is the set of subspaces spanned by the irreducible root subsystems of Φ (see [41]).

Definition 5 Let $\mathcal{G}$ be a building set associated to $\mathcal{A}$. A subset $\mathcal{S} \subset \mathcal{G}$ is called ($\mathcal{G}$-) nested, if given a subset $\{U_1, \ldots, U_h\}$ (with $h > 1$) of pairwise non comparable elements in $\mathcal{S}$, then these elements are in direct sum and $U_1 \oplus \cdots \oplus U_h \notin \mathcal{G}$.

2.2 The Example of the Root System A_{n-1}

Let us consider the root arrangement of type A_{n-1}. We think of it as an *essential* arrangement, i.e. we consider the hyperplanes with equation $x_i - x_j = 0$ in the quotient space $\mathbb{R}^n / < (1, 1, \ldots, 1) >$.

Let us denote by $\mathcal{F}_{A_{n-1}}$ the building set of irreducibles associated to this arrangement. According to Remark 4, it is made by all the subspaces spanned by the irreducible root subsystems. Therefore there is a bijective correspondence between the elements of $\mathcal{F}_{A_{n-1}}$ and the subsets of $\{1, \cdots, n\}$ of cardinality at least two: if the annihilator of $A \in \mathcal{F}_{A_{n-1}}$ is the subspace described by the equation $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ then we represent A by the set $\{i_1, i_2, \ldots, i_k\}$. As a consequence a $\mathcal{F}_{A_{n-1}}$- nested set $\mathcal{S}$ is a set (which we still call $\mathcal{S}$) of subsets of $\{1, \cdots, n\}$ with the property that any of its elements has cardinality ≥ 2 and if I and J belong to $\mathcal{S}$ than either $I \cap J = \emptyset$ or one of the two sets is included into the other.

Let us now denote by $\mathcal{F}_{A_{n-1 fund}}$ the subset of $\mathcal{F}_{A_{n-1}}$ made by the subspaces that can be spanned by a set of simple roots. It coincides with the building set of irreducibles associated to the arrangement given by the hyperplanes orthogonal to the simple roots. With regards to the bijection mentioned above, the subspaces in $\mathcal{F}_{A_{n-1 fund}}$ correspond to subsets of the form $\{i, i+1, \ldots, j\}$, where $1 \leq i < j \leq n$.

When we associate to a nested set of $\mathcal{F}_{A_{n-1 fund}}$ its set of subsets of $\{1, \cdots, n\}$ we realize that this is equivalent to giving a parenthesization of the list $1, 2, 3, \ldots, n$. For instance,

$$\{1, 2\}, \{3, \{4, 5, 6\}, 7\}, 8, 9$$

represents the of $\mathcal{F}_{A_{8 fund}}$- nested set made by the three sets $\{1, 2\}$, $\{4, 5, 6\}$, $\{3, 4, 5, 6, 7\}$, i.e. by the three subspaces in $\mathbb{R}^9 / < (1, 1, \ldots, 1) >$ described by the equations $x_1 = x_2$, $x_4 = x_5 = x_6$ and $x_3 = x_4 = x_5 = x_6 = x_7$.

Let us now focus on the maximal building set $\mathcal{C}_{A_{n-1}}$ associated with the root arrangement of type A_{n-1}. It is made by all the subspaces that can be obtained as span of roots. Using the same notation as before, these subspaces can be put in bijective correspondence with the partitions of $\{1, \cdots, n\}$ such that at least one part has cardinality ≥ 2. Each part with cardinality ≥ 2 correspond to a subspace, and all these subspaces are in direct sum; the parts of cardinality 1 don't correspond to subspaces.

For instance,

$$\{1, 3\}, \{2\}, \{4, 5, 9\}, \{7\}, \{6, 8\}$$

corresponds to the subspace described by the equations $x_1 = x_3$, $x_4 = x_5 = x_9$ and $x_6 = x_8$.

Remark 6 In the sequel we will not write the parts with only one element, i.e. we will omit the parentheses $\{i\}$.

The $\mathcal{C}_{A_{n-1}}$-nested sets are given by chains of subspaces in $\mathcal{C}_{A_{n-1}}$ (with respect to inclusion). In terms of partitions, this corresponds to give chains of the above described partitions of $\{1, \cdots, n\}$ (with respect to the refinement relation).

As before, we denote by $\mathcal{C}_{A_{n-1 fund}}$ the subset of $\mathcal{C}_{A_{n-1}}$ made by the subspaces that can be spanned by a set of simple roots. Its subspaces are described by the partitions of the list $1, 2, \ldots, n$ such that at least one part has cardinality ≥ 2. The $\mathcal{C}_{A_{n-1 fund}}$-nested sets are described by chains of these partitions of $1, 2, \ldots, n$.

One can find in [23] a description of the maximal model $Y_{\mathcal{C}_{A_{n-1}}}$ and in [24] a description of all the S_n invariant building sets associated with the root system A_{n-1}.

2.3 The Construction of Wonderful Models

In this section we recall the construction of De Concini–Procesi models from [6].

The interest in these models was at first motivated by an approach to Drinfeld's construction of special solutions for Khniznik–Zamolodchikov equation (see [12]). Moreover, in [6] it was shown, using the cohomology description of these models, that the rational homotopy type of the complement of a complex subspace arrangement depend only on the intersection lattice.

Then real and complex De Concini–Procesi models turned out to play a key role in several fields of mathematical research: subspace and toric arrangements, toric varieties (see for instance [8, 17, 37]), tropical geometry (see [16]), moduli spaces and configuration spaces (see for instance [13, 29]), box splines, vector partition functions and index theory (see [3, 7]), discrete geometry (see [14]).

Let us recall how they are defined. Let $\mathcal{A}$ be a subspace arrangement in an euclidean space V and let $\mathcal{M}(\mathcal{A}^{\perp})$ be the complement of the arrangement $\mathcal{A}^{\perp}$. Let $\mathcal{G}$ be a building set associated to $\mathcal{A}$ (we can suppose that it contains V), Then one considers the map

$$i : \mathbb{P}(\mathcal{M}(\mathcal{A}^{\perp})) \to \mathbb{P}(V) \times \prod_{D \in \mathcal{G} - \{V\}} \mathbb{P}(V/D^{\perp})$$

where in the first coordinate we have the inclusion and the map from $\mathcal{M}(\mathcal{A}^{\perp})$ to $\mathbb{P}(V/D^{\perp})$ is the restriction of the canonical projection $(V - D^{\perp}) \to \mathbb{P}(V/D^{\perp})$.

Definition 7 The (compact) wonderful model $Y_{\mathcal{G}}$ is obtained by taking the closure of the image of i.

De Concini and Procesi in [6] proved that the complement $\mathcal{D}$ of $\mathbb{P}(\mathcal{M}(\mathcal{A}^{\perp}))$ in $Y_{\mathcal{G}}$ is a divisor with normal crossings whose irreducible components are in bijective correspondence with $\mathcal{G} - \{V\}$ and are denoted by $\mathcal{D}_G$ ($G \in \mathcal{G} - \{V\}$).

More precisely, let us introduce the following notation:

Definition 8 Given a subspace $C \subset V$, we define the following two (possibly empty) subspace arrangements.

1. $\mathcal{A}_C = \{H \in \mathcal{A} \mid C \subset H\}$.
2. $\mathcal{A}^C = \{B \cap C \mid B \in \mathcal{A} - \mathcal{A}_C\}$.

If we now denote by π the projection onto the first component $\mathbb{P}(V)$, $\mathcal{D}_G$ is equal to the closure of

$$\pi^{-1}\left(\mathbb{P}(G^{\perp}) - \bigcup_{L^{\perp}\in(\mathcal{A}^{\perp})^{G^{\perp}}} \mathbb{P}(L^{\perp})\right).$$

It can also be characterized as the unique irreducible component such that $\pi(\mathcal{D}_G) = \mathbb{P}(G^{\perp})$.

A complete characterization of the boundary is then provided by the observation that, if we consider a collection $\mathcal{T}$ of subspaces in $\mathcal{G}$ containing V, then

$$\mathcal{D}_{\mathcal{T}} = \bigcap_{A\in\mathcal{T}-\{V\}} \mathcal{D}_A$$

is non empty if and only if $\mathcal{T}$ is $\mathcal{G}$-nested, and in this case $\mathcal{D}_{\mathcal{T}}$ is a smooth irreducible subvariety.

Remark 9 The same construction can be done also in the complex case, providing complex wonderful models (up to now we have always assumed V to be euclidean, but the definitions of building sets and nested sets can be easily extended, endowing the complex vector space with a non-degenerate Hermitian product).

Let us recall a similar construction, that, in the euclidean case, produces "spherical models" (see [20, 21]).

Let us denote by $S(V)$ the $n-1$-th dimensional unit sphere in V (n being the dimension of V), let $\hat{\mathcal{M}}(\mathcal{A}^{\perp}) = \mathcal{M}(\mathcal{A}^{\perp}) \cap S(V)$ and, for every subspace $A \subset V$, let $S(A) = A \cap S(V)$. Let $\mathcal{G}$ be a building set associated to $\mathcal{A}$ that contains V, and let us consider the compact manifold

$$K = S(V) \times \prod_{A\in\mathcal{G}-\{V\}} S(A).$$

There is an open embedding $\varphi : \hat{\mathcal{M}}(\mathcal{A}^{\perp}) \longrightarrow K$ which is the inclusion on the first factor and on the following factors it is given by projection and normalization.

Definition 10 We denote by $CY_{\mathcal{G}}$ the closure in K of $\varphi(\hat{\mathcal{M}}(\mathcal{A}^{\perp}))$.

It turns out (see [20]) that $CY_{\mathcal{G}}$ is a smooth manifold with corners. It is a differentiable model for $\hat{\mathcal{M}}(\mathcal{A}^{\perp})$ in the following sense: if we denote by $c\pi$ the projection onto the first component $S(V)$, then $c\pi$ is surjective and it is an isomorphism on the preimage of $\hat{\mathcal{M}}(\mathcal{A}^{\perp})$. Furthermore, $c\pi$ establishes a bijective correspondence between the (closures of) codimension 1 open strata (that can be not connected) in the boundary of $CY_{\mathcal{G}}$ and the elements of $\mathcal{G} - \{V\}$.

More precisely, if $A \in \mathcal{G} - \{V\}$, its associated boundary component is

$$C\mathcal{D}_A = c\pi^{-1}\overline{\left(S(A^{\perp}) - \bigcup_{B^{\perp} \in (\mathcal{A}^{\perp})^{A^{\perp}}} S(B^{\perp}) \right)}.$$

We notice that the combinatorial structure of the boundary mimicks the one of De Concini–Procesi models (see [20]):

Theorem 11 *Let $\mathcal{T}$ be a subset of $\mathcal{G}$ which includes V; then*

$$C\mathcal{D}_{\mathcal{T}} = \bigcap_{B \in \mathcal{T} - \{V\}} C\mathcal{D}_B$$

is not empty if and only if $\mathcal{T}$ is $\mathcal{G}$-nested.

Remark 12 We notice that, when $\mathcal{G}$ is the building set associated with a root arrangement $\mathcal{A}$, $CY_{\mathcal{G}}$ has as many connected components as the number of chambers of $\mathcal{M}(\mathcal{A}^{\perp})$. Moreover in [22] these connected components have been linearly realized as polytopes (nestohedra) inside the chambers.

The relations between the projective and the spherical construction of models have been pointed out in [20] by describing the combinatorial properties of a surjective map $F : CY_{\mathcal{G}} \to Y_{\mathcal{G}}$.

Let us recall the definition of F: the model $CY_{\mathcal{G}}$ is embedded in

$$K = S(V) \times \prod_{A \in \mathcal{G} - \{V\}} S(A)$$

while $Y_{\mathcal{G}}$ is embedded inside

$$K' = \mathbb{P}(V) \times \prod_{D \in \mathcal{G} - \{V\}} \mathbb{P}(V/D^{\perp}).$$

Now, given any $A \in \mathcal{G}$, we can consider the natural isomorphism between A and $V/A^{\perp}$ provided by the projection.

As a consequence of this identification, there is a map F' from K to K' whose restriction to each factor $S(A)$ is the $2 \mapsto 1$ projection $S(A) \mapsto \mathbb{P}(V/A^{\perp})$ (in particular this means that on the first factor we are considering the projection $S(V) \mapsto \mathbb{P}(V)$).

Theorem 13 (see [20]) *If we restrict F' to $CY_{\mathcal{G}}$ we obtain a surjective map*

$$F : CY_{\mathcal{G}} \to Y_{\mathcal{G}}.$$

Let $\mathcal{S}$ be a $\mathcal{A}$-nested set which contains V. Then F restricted to the internal points of $C\mathcal{D}_{\mathcal{S}}$ is a $2^{|\mathcal{S}|}$-sheeted covering of the open part of the boundary component $\mathcal{D}_{\mathcal{S}}$ in $Y_{\mathcal{A}}$.

2.4 A More General Construction

The construction of De Concini–Procesi models and of spherical models that we recalled in the preceding section is a special case of other more general constructions that, starting from a 'good' stratified variety, produce models by blowing up a suitable subset of strata. We have in mind for instance the models described by MacPherson and Procesi in [31] and by Li in [30] in the algebro-geometric case and the ones described in [20] in the case of manifolds with corners.[1] We recall here the main definitions of Li's construction in the algebro-geometric case.

Definition 14 A simple arrangement of subvarieties (or 'simple stratification') of a nonsingular variety Y is a finite set $\Lambda = \{\Lambda_i\}$ of nonsingular closed subvarieties Λ_i properly contained in Y satisfying the following conditions:

(i) The intersection of Λ_i and Λ_j is nonsingular and the tangent bundles satisfy $T(\Lambda_i \cap \Lambda_j) = T(\Lambda_i)_{|(\Lambda_i\cap\Lambda_j)} \cap T(\Lambda_j)_{|(\Lambda_i\cap\Lambda_j)}$.
(ii) $\Lambda_i \cap \Lambda_j$ either is equal to some stratum in Λ or is empty.

Definition 15 Let Λ be an arrangement of subvarieties of Y. A subset $\mathcal{G}' \subseteq \Lambda$ is called a building set of Λ if $\forall \Lambda_i \in \Lambda - \mathcal{G}'$ the minimal elements in $\{G \in \mathcal{G}' \ : \ G \supseteq \Lambda_i\}$ intersect transversally and the intersection is Λ_i.

Then, if one has an arrangement Λ of a nonsingular variety Y and a building set $\mathcal{G}'$, one can construct a wonderful model $Y_{\mathcal{G}'}$ considering (by analogy with [6]) the closure of the image of the locally closed embedding

$$(Y - \cup_{\Lambda_i\in\Lambda}\Lambda_i) \to \prod_{G\in\mathcal{G}'} Bl_G Y$$

where $Bl_G Y$ is the blowup of Y along G.

It turns out that

Theorem 16 (See Theorem 1.3 in [30]) *If one arranges the elements $G_1, G_2, \ldots,$ G_N of $\mathcal{G}'$ in such a way that for every $1 \le i \le N$ the set $\{G_1, G_2, \ldots, G_i\}$ is building,*

[1] In Li's paper one can find a short and useful comparison among several constructions of wonderful compactifications by Fulton–Machpherson [18], Ulyanov [40], Kuperberg–Thurston [28], Hu [27]. A further interesting survey including tropical compactifications can be found in Denham's paper [10].

then $Y_{\mathcal{G}'}$ is isomorphic to the variety

$$Bl_{\tilde{G}_N} Bl_{\tilde{G}_{N-1}} \cdots Bl_{\tilde{G}_2} Bl_{G_1} Y$$

where $\tilde{G}_i$ denotes the dominant transform of G_i in $Bl_{\tilde{G}_{i-1}} \cdots Bl_{\tilde{G}_2} Bl_{G_1} Y$.

We will not recall here the theorems that describe the boundary of these more general wonderful models, but we show two examples that will be crucial in the following sections.

Example 17 In the case of real or complex subspace arrangements of a vector space V, the constructions described in the preceding sections and the above construction produce the same models (we only have to pay attention to the fact that in the preceding sections a building set $\mathcal{G}$ was described in a dual way so the building set of subvarieties $\mathcal{G}'$ is made by the orthogonals or annihilators of the subspaces in $\mathcal{G}$).

The same remark holds for spherical models, comparing the definition of $CY_{\mathcal{G}}$ with the more general construction in [20].

Example 18 Given a building set of subspace arrangements $\mathcal{G}$, the boundary strata of the De Concini–Procesi wonderful model $Y_{\mathcal{G}}$ give rise to a simple arrangement of subvarieties, and the set of all strata is a building set. So it is possible to obtain a "model of the model $Y_{\mathcal{G}}$". The same remark holds for the spherical model $CY_{\mathcal{G}}$. The boundary strata of these "models of models" are indexed by the nested sets of the building set of all strata. According to the definition given in Sect. 4 of [31], a nested set $\mathcal{S}$ in this sense is a collection of $\mathcal{G}$-nested sets linearly ordered by inclusion (we will come back to this in Sect. 6).

In the case of the root arrangements A_n, these (very big) models will play a role in the sequel, motivated by the search of geometrical representations of the symmetric group.

3 Action on the Face Poset of $CY_{\mathcal{F}_{A_{n-1}}}$

3.1 Action in the Language of Lists and Dissections

We recall that there is a well know 'extended' S_{n+1} action on the De Concini–Procesi model $Y_{\mathcal{F}_{A_{n-1}}}$, that is a quotient of $CY_{\mathcal{F}_{A_{n-1}}}$: it comes from the isomorphism with the moduli space $\overline{M}_{0,n+1}$ and the character of the resulting representation on cohomology has been computed in [36].

In this section we show how this action can be lifted to the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$ (as usual the partial order on the faces of $CY_{\mathcal{F}_{A_{n-1}}}$ is given by reverse inclusion). As mentioned in Remark 12, a spherical model associated with a root system is a union of nestohedra: more precisely, in this case the strata are indexed by the parenthesizations of the list $1, 2, .., n$ whose parentheses contain at least two numbers (see Sect. 2.2) and $CY_{\mathcal{F}_{A_{n-1}}}$ is a union of $n!$ Stasheff's associahedra of dimension $n-2$.

Therefore, according to the notation in [22], the elements of this face poset are couples $(\gamma, \mathcal{S})$ where $\gamma \in S_n$ and $\mathcal{S}$ is a $\mathcal{F}_{A_{n-1 fund}}$-nested set containing V, i.e. a parenthesization of the list $1, 2, .., n$ whose parentheses contain at least two numbers: the nested set $\mathcal{S}$ represents a face F of the associahedron $P_{\mathcal{G}_{fund}}$ which lies in the fundamental chamber $\mathcal{C}$, and the element γ tells us that we are considering the face γF that lies in the chamber $\gamma\mathcal{C}$). Another way to represent the element $(\gamma, \mathcal{S})$ is by a parenthesization of the list $\gamma(1), \gamma(2), \gamma(3), \ldots, \gamma(n)$.

Example 19 Let $(e, \{\{1, 2\}, \{4, 5, 6\}\})$ be an element of the face poset of $CY_{\mathcal{F}_{A_{6 fund}}}$. The element e on the left tells us that it corresponds to a face o the associahedron that lies in the fundamental chamber. This element can also be represented as

$$\{1, 2\}, 3, \{4, 5, 6\}, 7$$

while

$$1, 2, 3, 4, 5, 6, 7$$

represents the full associahedron in the fundamental chamber (we always suppose that the nested set $\mathcal{S}$ contains V but we omit the corresponding parentheses, in this case $\{1, 2, 3, 4, 5, 6, 7\}$).

Example 20 Let γ be the cyclic permutation $(2, 3, 4)$. Then the face

$$(\gamma, \{\{1, 2\}, \{4, 5, 6\}\})$$

of $CY_{\mathcal{F}_{A_{n-1}}}$ can be represented as

$$\{1, 3\}, 4, \{2, 5, 6\}, 7$$

which is

$$\{\gamma(1), \gamma(2)\}, \gamma(3), \{\gamma(4), \gamma(5), \gamma(6)\}, \gamma(7).$$

Let us now describe in the language of lists an S_{n+1} action on the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$, compatible with the action on the quotient space $Y_{\mathcal{F}_{A_{n-1}}}$.

First we add 0 to all the parenthesized lists that, as mentioned above, describe the face poset. For instance, the face

$$\{1, 2\}, 3, \{4, 5, 6\}, 7$$

is now indicated as

$$0, \{1, 2\}, 3, \{4, 5, 6\}, 7.$$

Then we identify S_{n+1} with the group which permutes the numbers $0, 1, \ldots, n$: in order to explain how S_{n+1} acts it is sufficient to explain how the transposition $(0, 1)$ acts.

In our example, we want to compute

$$(0,1)\,(0, \{1,2\}, 3, \{4,5,6\}, 7)\,.$$

First we let $(0, 1)$ act by permuting 0 and 1 in the list:

$$(0,1)\,(0, \{1,2\}, 3, \{4,5,6\}, 7) = 1, \{0,2\}, 3, \{4,5,6\}, 7.$$

The list on the right is not a valid list since it does not start with 0. Then we read it cyclically starting from 0, and obtain

$$\{0,2\}, 3, \{4,5,6\}, 7, 1.$$

We notice that there is a couple of parentheses which includes 0, so this still does not represent a face of $CY_{\mathcal{F}_{A_6}}$. As our last step, we substitute the parentheses $\{0, 2\}$ that contain 0 with their "complement", i.e. with the parentheses $\{3, 4, 5, 6, 7, 1\}$ that contain all the other numbers in the list. At the end we obtain

$$(0,1)\,(0, \{1,2\}, 3, \{4,5,6\}, 7) = 0, 2, \{3, \{4,5,6\}, 7, 1\}.$$

The list on the right represents a face of $CY_{\mathcal{F}_{A_6}}$ (which does not lie in the fundamental chamber).

Example 21 Here it is another example of this process, in the face poset of $CY_{\mathcal{F}_{A_4}}$:

$$\begin{aligned}(0,1)\,(0, \{1,2,3\}, \{4,5\}) &= 1, \{0,2,3\}, \{4,5\}\\ &= \{0,2,3\}, \{4,5\}, 1\\ &= 0, 2, 3, \{\{4,5\}, 1\}.\end{aligned}$$

In this way, as one can easily check, one defines a S_{n+1} action.[2] Here we introduce an interpretation of the face poset $CY_{\mathcal{F}_{A_{n-1}}}$ and of the action described above in terms of polygon dissections. We start by observing that the parenthesized lists of numbers $1, 2, \ldots, n$ are in bijective correspondence with the dissections of a polygon with $n + 1$ labelled edges. This well known correspondence is illustrated by Fig. 1.

The face poset of $CY_{\mathcal{F}_{A_{n-1}}}$, as a union of abstract simplicial complexes, is isomorphic to the set of dissections of $(n + 1)$-gons with sides labelled $0, 1, \ldots, n$ in some order, which is a union of abstract simplicial complexes on the set of

[2]Notice that our description of the action could also be given in terms of planted labelled trees.

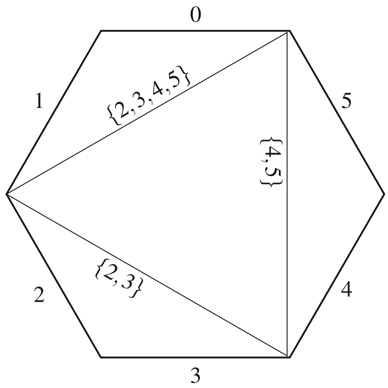


Fig. 1 This dissection of the hexagon produces the parenthesized list $1, \{\{2, 3\}, \{4, 5\}\}$

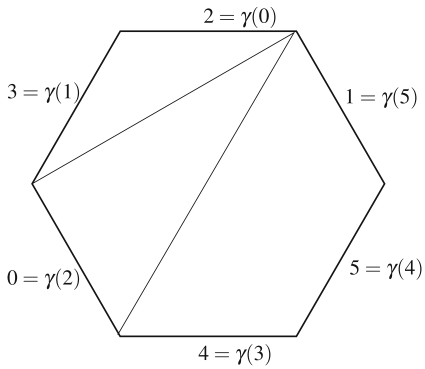


Fig. 2 The dissection of the hexagon associated to the facet $(\gamma, \mathcal{S})$ for $\gamma = (02)(1345)$ and $\mathcal{S} = (1, \{2, \{3, 4, 5\}\})$

diagonals of each such labelled n-gon. Under this isomorphism the facet of the form $(\gamma, \mathcal{S})$, where $\mathcal{S} = \{V, A_1, \ldots, A_k\}$ is a nested set in $\mathcal{F}_{A_{n-1 fund}}$, corresponds to the polygon with sides labelled cyclically $\gamma(0), \gamma(1), \gamma(2), \ldots, \gamma(n)$ with the following k diagonals: if A_s is the interval $[i_s, j_s]$, it is associated to the diagonal that separates the sides $\gamma(i_s), \gamma(i_s + 1), \ldots, \gamma(j_s)$ from the other sides (see the example in Fig. 2).

3.2 Action in the Language of Root Systems and Nested Sets

As an exercise we want to describe the action of Sect. 3.1 in a different way, in the language of root systems and nested sets.

Let $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be a basis for the root system of type A_n (we added to a basis of A_{n-1} the extra root α_0) and let $\tilde{\Delta} = \{\tilde{\alpha}, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be the set of roots that appear in the affine diagram. We identify in the standard way S_{n+1} with the group which permutes $\{0, 1, \ldots, n\}$ and s_{α_0} with the transposition $(0, 1)$. Therefore S_n, the subgroup generated by $\{s_{\alpha_1}, \ldots, s_{\alpha_{n-1}}\}$, is identified with the subgroup which permutes $\{1, \ldots, n\}$.

Let $\mathcal{S} = \{V, A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_s\}$ be a nested set in $\mathcal{F}_{A_{n-1}fund}$ and let $\sigma \in S_{n+1}$. Let us then denote by C the cyclic subgroup generated by $(0, 1, 2, 3, 4, 5, \ldots, n)$ and by $w = \sigma(0, 1, 2, 3, 4, 5, \ldots, n)^r$ the only element in the coset σC which fixes 0 (therefore w belongs to S_n).

Moreover, let us suppose that, for every subspace A_j, some of the roots contained in σA_j have α_0 in their support (when they are written with respect to the basis Δ), while this doesn't happen for the subspaces σB_t. Then for every j we denote by $\overline{A_j}$ the subspace generated by all the roots of $\tilde{\Delta}$ which are orthogonal to A_j.

As a first step in the description of the S_{n+1} action, we put

$$\sigma \cdot (e, \mathcal{S}) = (w, \{V, .., w^{-1}\sigma\overline{A_j}, .., w^{-1}\sigma B_s, ..\}).$$

As one can quickly check, this can be extended to an S_{n+1} action on the full face poset of $CY_{\mathcal{F}_{A_{n-1}}}$ by imposing that σ maps the face $(\gamma, \mathcal{S})$, where $\gamma \in S_n$, to the face $\sigma\gamma \cdot (e, \mathcal{S})$. This action coincides with the one described before in the language of lists.

Example 22 Let us re-consider in this different language the Example 21. Let $\mathcal{S}$ be the nested set of $\mathcal{F}_{A_4fund}$ made by V, $A =< \alpha_1, \alpha_2 >$ and $B =< \alpha_4 >$. The group S_6 is generated by the reflections $s_{\alpha_0}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}$ and we identify S_5 with the subgroup generated by $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}$. Now we compute $s_{\alpha_0}(e, \{V, A, B\})$, which in the language of lists is

$$(0, 1)\,(0, \{1, 2, 3\}, \{4, 5\})\,.$$

We notice that the root $s_{\alpha_0}\alpha_1$ contains α_0 in its support (when it is written with respect to the basis Δ). We then denote by $\overline{A}$ the subspace generated by all the roots of $\tilde{\Delta}$ which are orthogonal to A: $\overline{A} =< \tilde{\alpha}, \alpha_4 >$.

Let $w = (0, 1)(0, 1, 2, 3, 4, 5)$ i.e. the representative of the coset $(0, 1)C$ in S_6 which leaves 0 fixed. Then $s_{\alpha_0} = (0, 1)$ maps the face $(e, \{V, A, B\})$ to the face

$$(w, \{V, w^{-1}s_{\alpha_0}\overline{A}, w^{-1}s_{\alpha_0}B\}) = (w, \{V, < \alpha_3, \alpha_4 >, < \alpha_3 >\}).$$

This face, in the language of lists, is

$$0, 2, 3, \{\{4, 5\}, 1\}$$

as expected.

Let us make another example, this time with the help of some pictures. This will show why this extra action is typical of the root system of type A.

Let us consider in $CY_{\mathcal{F}_{A_6}}$ the face

$$(e, \{V, A\})$$

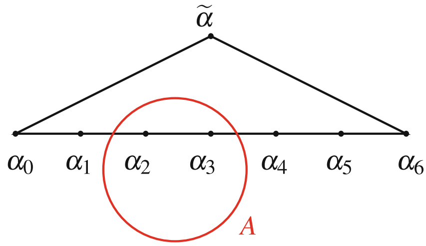


Fig. 3 The subspace A

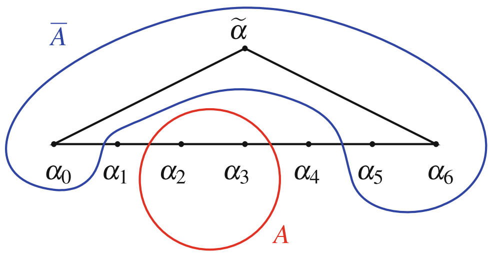


Fig. 4 The subspace $\overline{A}$

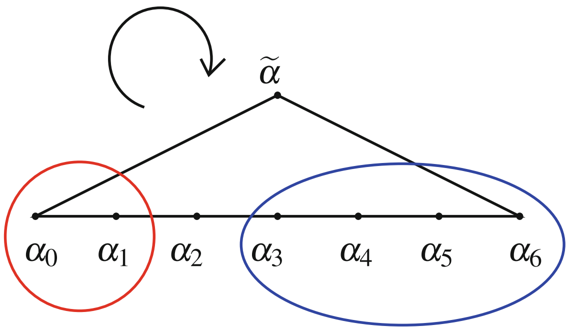


Fig. 5 The rotation maps $\overline{A}$ to $< \alpha_3, \alpha_4, \alpha_5, \alpha_6 >$

where $A = < \alpha_2, \alpha_3 >$ (see Fig. 3) and let us compute $(0, 1, 2)(e, \{V, A\})$. Since $(0, 1, 2)\alpha_2 = \alpha_0 + \alpha_1 + \alpha_2$ we have to consider the subspace $\overline{A}$ (see Fig. 4) and then we 'rotate' our subspaces clockwise (so that α_2 goes to α_0, Fig. 5). This rotation (that makes sense only in the case of the root system of type A_n, which is the only one whose affine diagram contains a cycle) maps $\overline{A}$ to $< \alpha_3, \alpha_4, \alpha_5, \alpha_6 >$. Then we have

$$(0, 1, 2)(e, \{V, A\}) = ((0, 1, 2)(0, 1, 2, 3, 4, 5, 6, 7)^2, \{V, < \alpha_3, \alpha_4, \alpha_5, \alpha_6 >\}).$$

4 Orbits of the Action in the Spherical Case

4.1 A First Remark on Stabilizers

The action of $\sigma \in S_{n+1}$ on the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$ described in the preceding section maps (the face poset of) the associahedron which lies in the fundamental chamber onto (the face poset of) an associahedron that lies in a chamber which may be different from the fundamental one.

It is easy to provide examples where two associahedra that lie in two adjacent chambers are sent to two associahedra whose chambers are not adjacent. This shows that this lifted action is not induced by an isometry.

Anyway it provides geometrical realizations of all the representations $Ind_G^{S_{n+1}} Id$, where G is any subgroup of a conjugate of the cyclic group C generated by $(0, 1, \dots, n)$ and Id is its trivial representation (the case $G = \{e\}$, i.e. the regular representation of S_{n+1}, occurs only if $n \geq 4$).

In fact the stabilizer of an element of the face poset is by construction a subgroup of a conjugate of the cyclic group C.

We want to show that all the above mentioned representations appear. Up to conjugation it is sufficient to consider only the subgroups of C.

As for the regular representation of S_{n+1}, we notice that, for instance, if $n \geq 4$ the stabilizer of $(e, \{V, < \alpha_1, \dots, \alpha_{n-2} >\})$ is the trivial subgroup.

Let then $d < n + 1$ be a divisor of $n + 1$. We will exhibit an element of the face poset whose stabilizer is generated by $(0, 1, 2, 3, 4, 5, \dots, n)^d$.

If $d > 2$ and $dk = n + 1$ we consider for instance the face $(e, \{V, < \alpha_1, \dots, \alpha_{d-2} >, < \alpha_{d+1}, \dots, \alpha_{2d-2} >, \dots, < \alpha_{(k-1)d+1}, \dots, \alpha_{kd-2} >\})$: its stabilizer is generated by $(0, 1, 2, 3, 4, 5, \dots, n)^d$.

If $d = 2$ and $2k = n + 1$ we consider $(e, \{V, < \alpha_1, \alpha_2, \dots, \alpha_{n-2} >, < \alpha_1 >, < \alpha_3 >, \dots, < \alpha_{n-2} >\})$. Its stabilizer is generated by $(0, 1, 2, 3, 4, 5, \dots, n)^2$.

If $d = 1$ we consider $(e, \{V\})$. Its stabilizer is, by definition of the S_{n+1} action, the full cyclic group C.

Remark 23 In terms of polygon dissections we can also describe the previous groups as stabilizers of inscribed regular $\frac{n+1}{d}$-gons.

4.2 A Formula for the Orbits and Polygon Dissections

Our plan in this section is the following:

- We will compute, given $d | n + 1$, the number $\mathcal{O}_{d,k}$ of the faces of the fundamental associahedron that have codimension k and are fixed by the subgroup of C generated by $(0, 1, 2, 3, 4, 5, \dots, n)^d$.

  Going back to the language of lists, this is equivalent to counting all the parenthesized lists fixed by the permutation $(0, 1, 2, 3, 4, 5, \dots, n)^d$.

- We will then obtain formulas for the number $\overline{\mathcal{O}}_{d,k}$ of the faces of the fundamental associahedron that have codimension k and whose stabilizer coincides with the subgroup $< (0, 1, 2, 3, 4, 5, \ldots, n)^d >$. Then the number of orbits of cardinality d with respect to the action of C on the k-codimensional faces of the fundamental associahedron is $\frac{\overline{\mathcal{O}}_{d,k}}{d}$. This is also the number of orbits of cardinality $n!d$ with respect to the action of S_{n+1} on the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$.

We observe that this problem of counting the faces of the fundamental associahedron fixed by $(0, 1, 2, 3, \ldots, n)^d$, in view of the observations at the end of Sect. 3.1, is closely related to the well known problem of counting polygon dissections modulo the action of the group of rotations. In the language of polygon dissections, the action of the cyclic group C corresponds to a (counterclockwise) cyclic permutation of the labels of the edges of the $n+1$-gon, so if we sum up the numbers $\frac{\overline{\mathcal{O}}_{d,k}}{d}$ as d ranges among the divisors of $n+1$ we find the number of different polygon dissections with k diagonals modulo the action of rotations.

The formulas we compute in this survey (see Propositions 26 and 28) can therefore be compared with the formulas for this number that can be found in the literature (see for instance [32] for the complete dissections, [26, 38] from the point of view of planar trees, [39] from the point of view of the cyclic sieving phenomenon, [11] and the more recent [1] which we refer to for further bibliography).

Let $D_{\lambda,\mu}$ be the Cayley number that counts the dissections of a convex polygon with λ edges and μ diagonals (see [4]):

$$D_{\lambda,\mu} = \frac{1}{\mu+1}\binom{\lambda-3}{\mu}\binom{\lambda+\mu-1}{\mu}.$$

We notice that we can define the numbers $D_{\lambda,\mu}$ for every $\lambda, \mu \in \mathbb{Z}$ by putting $D_{\lambda,\mu} = 0$ if $\lambda \leq 2$ or, if $\lambda > 2$ and $\mu < 0$ of if $\lambda > 2$ and $\mu > \lambda - 3$. This also follows from Cayley's formula if we adopt the usual conventions on the binomials (the only case where Cayley's formula can't be considered is when $\mu = -1$). Then let us define the series

$$\Phi(x, q) = \sum_{\lambda,\mu\in\mathbb{Z}} D_{\lambda,\mu} x^{\lambda-1} q^{\mu}.$$

Proposition 24 *Let $d|(n+1)$, $0 < d < \frac{n+1}{2}$. The number $\mathcal{O}_{d,k}$ is equal to 0 if $k \neq \frac{n+1}{d}t$ with $0 \leq t \leq d-1$. The number $\mathcal{O}_{d,\frac{n+1}{d}t}$ coincides with the coefficient of $x^d q^t$ in the following series:*

$$x^1 + x^2 + qx(1+x)\frac{d}{dx}\Phi(x,q) + \frac{1}{2}qx(1+q)\frac{d}{dx}\Phi(x,q)^2 + (1+q)x\Phi(x,q). \quad (1)$$

Proof Let us consider a dissection of the $n+1$-gon that is invariant with respect to the counterclockwise rotation of $\frac{2\pi d}{n+1}$, i.e. with respect to the action of $(0, 1, 2, 3, 4, 5, \ldots, n)^d$.

We denote by C_0 the maximal diagonal (in the dissection) that separates the edge (labelled by) 0 from the center of the polygon, that is the diagonal closest to the center of the polygon, among the diagonals that separate the center from 0, or eventually the edge 0 itself, if there is no such diagonal in the dissection; we write $C_0 = [v_i, v_{n-r}]$ to indicate its vertices, with $i \geq 0, r \geq 0$ (we number the vertices counterclockwise starting from the edge 0, as in Fig. 6). We observe that $i + r + 1 \leq d$ by invariance and that if $i = r = 0$ we have a degenerate case, where C_0 is the edge 0 (which therefore lies in the connected component that contains also the center).

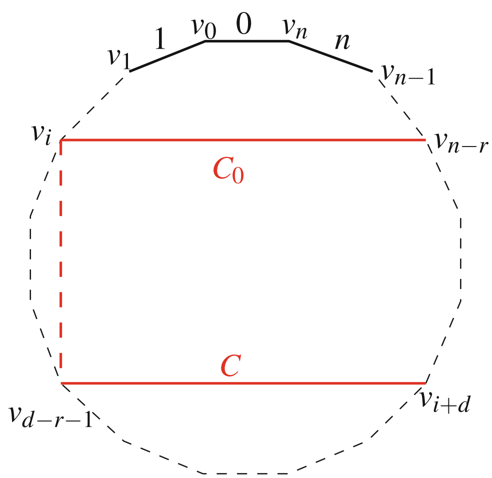
Then we observe that in the dissection there is also the diagonal $C = [v_{d-r-1}, v_{i+d}]$ by invariance and that, once we determine the diagonals that lie in the portion of polygon delimited by the vertices v_i, v_{i+d}, our invariant dissection is fully determined.

Now we can consider the polygon P_1 delimited by the $r+i+1$ edges $d-r, \ldots .d+i$ and by C (see Fig. 6). Our dissection can contain any dissection of this polygon, i.e.

$$\sum_{s=0}^{i+r-1} D_{i+r+2,s}$$

possible dissections. At the same way let us consider the polygon P_2 delimited by the $d-r-i-1$ edges $i+1, \ldots, d-r-1$ and by the diagonal $[v_i, v_{d-r-1}]$. We don't know if this diagonal is in our dissection (this is why in Fig. 6 it is represented by a dotted segment); we have then to take into account two cases, depending on if $[v_i, v_{d-r-1}]$ belongs or doesn't belong to the dissection. In any case, our dissection

Fig. 6 The edges are numbered from 0 to n counterclockwise, and the vertices are indexed from v_0 to v_n counterclockwise. The diagonal C_0 is in *red* and has vertices v_i and v_{n-r}



can contain any internal dissection of the polygon P_2, i.e.

$$\sum_{u=0}^{d-r-i-3} D_{d-r-i,u}$$

dissections.

Let us fix the value $i + r + 1$ and let us suppose that the polygons P_1 and P_2 are not degenerate, i.e. they have at least three edges. Then let us consider all the invariant dissections that have a picture like the one described above: this means that we are considering the $i + r + 1$ ways to place the diagonal C_0.

The sum, over all these dissections, of the dissections of the portion of the polygon delimited by v_i, v_{i+d} that have t diagonals is counted by the coefficient of $x^d q^t$ in the following series:

$$\frac{1}{2} qx(1 + q)\frac{d}{dx}\Phi(x, q)^2.$$

To explain this more in detail, we notice that this series is equal to

$$qx(1 + q)\Phi(x, q)\frac{d}{dx}\Phi(x, q)$$

i.e.

$$\left[(1 + q) \sum_{\lambda,\mu\in\mathbb{Z}} D_{\lambda,\mu}x^{\lambda-1}q^{\mu}\right]\left[qx \sum_{\lambda,\mu\in\mathbb{Z}} D_{\lambda,\mu}(\lambda - 1)x^{\lambda-2}q^{\mu}\right]$$

where:

- The series in the left brackets counts the contribution of the dissections of the polygon P_2, and $(1 + q)$ takes into account the two cases (presence or absence of the diagonal $[v_i, v_{d-r-1}]$).
- The series in the right brackets counts the contribution of the polygon P_1: the derivative produces the addendum $(i+r+1)D_{i+r+2,s}x^{i+r}q^s$ that takes into account the $i + r + 1$ ways to place the diagonal C_0, the factor q takes into account the presence of the diagonal C and the factor x has been added for technical reasons, so that the relevant coefficient, after the product of the two series in the brackets, is the coefficient of $x^d q^t$.

The other terms in the series (1) allow us to include all the cases where one (or both) of the polygons P_1, P_2 are degenerate. In particular:

- The term x corresponds to the case $d = 1$.
- The term x^2 has been added to complete the case $d = 2$.
- The term $qx(1 + x)\frac{d}{dx}\Phi(x, q)$ corresponds to the cases $d - r - i = 2$ and $d - r - i = 1$.

- The term $(1+q)x\Phi(x,q)$ corresponds to the case $i=r=0$. □

Proposition 25 *If* $d=\frac{n+1}{2}|(n+1)$ *the number* $\mathcal{O}_{\frac{n+1}{2},2t}$ *with* $0\leq t\leq \frac{n+1}{2}-1$ *coincides with the coefficient of* $x^{\frac{n+1}{2}}q^t$ *in the following series:*

$$x^1+x^2+qx^2\frac{d}{dx}\Phi(x,q)+\frac{1}{2}qx(1+q)\frac{d}{dx}\Phi(x,q)^2+(1+q)x\Phi(x,q)$$

while the number $\mathcal{O}_{\frac{n+1}{2},2t-1}$ *with* $1\leq t\leq\frac{n+1}{2}-1$ *coincides with the coefficient of* $x^{\frac{n+1}{2}}q^t$ *in the following series:*

$$qx\frac{d}{dx}\Phi(x,q).$$

Proof The proof if analogous to the one in the preceding proposition, but we have to pay attention: if $i+r+1$ is maximum, i.e. equal to $\frac{n+1}{2}$, the diagonal C coincides with the diagonal C_0. □

The following propositions are easy consequences of Propositions 24 and 25.

Proposition 26 *Let* $d|(n+1)$, $0<d\leq\frac{n+1}{2}$. *The number* $\overline{\mathcal{O}}_{d,k}$ *of the faces of the fundamental associahedron that have codimension k and whose stabilizer coincides with the subgroup of C generated by* $(0,1,2,3,4,5,\ldots,n)^d$ *is given by the following formula:*

$$\overline{\mathcal{O}}_{d,k}=\sum_{c|d}\mu\left(\frac{d}{c}\right)\mathcal{O}_{c,k}$$

where $\mu(\,)$ *is the Möebius function.*

Remark 27 It follows in particular that if $d|(n+1)$, $d\neq\frac{n+1}{2}$, the number $\overline{\mathcal{O}}_{d,k}$ is 0 if $k\neq\frac{n+1}{d}t$ with $0\leq t\leq d-1$.

Proposition 28 *Let* $d|(n+1)$, $0<d\leq\frac{n+1}{2}$. *The number of orbits of the* S_{n+1} *action on the face poset of* $CY_{\mathcal{F}_{A_{n-1}}}$ *that are made by k-codimensional faces and have cardinality exactly equal to* $n!d$ *is therefore* $\frac{\overline{\mathcal{O}}_{d,k}}{d}$. *The number of orbits that have cardinality* $(n+1)!$ *that are made by k-codimensional faces can be computed as* $\frac{\overline{\mathcal{O}}_{n+1,k}}{n+1}$ *where*

$$\overline{\mathcal{O}}_{n+1,k}=D_{n+1,k}-\sum_{d|(n+1),\ d\leq\frac{n+1}{2}}\overline{\mathcal{O}}_{d,k}.$$

5 The Extended Action on Bigger Models: The Example of A_3

5.1 A Problem: There Is Not an Extended Action on $CY_{maxA_{n-1}}$

From now on the minimal and the maximal models associated with the root system A_{n-1} will play a special role in this paper. Hence it is convenient to single out them by a new notation: we will denote by $CY_{minA_{n-1}}$ the minimal model $CY_{\mathcal{F}_{A_{n-1}}}$ and by $CY_{maxA_{n-1}}$ the maximal model $CY_{\mathcal{C}_{A_{n-1}}}$. If one tries to extend the S_{n+1} action from the face poset of $CY_{minA_{n-1}}$ to the face poset of $CY_{maxA_{n-1}}$ one realizes that this is not possible. Let us show this by an example in the case CY_{maxA_3}.

Example 29 In the case A_3, the subspace $< \alpha_1, \alpha_3 >$ belongs to $\mathcal{C}_{A3_{fund}}$, so it corresponds to an edge (denoted by $(e, \{V, < \alpha_1, \alpha_3 >\})$) of the hexagon which lies inside the fundamental chamber (the maximal model $CY_{maxA_{n-1}}$ is a union of $n!$ permutohedra, and the two dimensional permutohedron is an hexagon).

This edge has a vertex in common with the edge corresponding to $< \alpha_1 >$ (this edge is denoted by $(e, \{V, < \alpha_1 >\})$ and their common vertex is $(e, \{V, < \alpha_1 >, < \alpha_1, \alpha_3 >\}))$ and another vertex in common with the edge corresponding to $< \alpha_3 >$ (this edge is denoted by $(e, \{V, < \alpha_3 >\})$ and their common vertex is $(e, \{V, < \alpha_3 >, < \alpha_1, \alpha_3 >\}))$.

Now the edges $(e, \{V, < \alpha_1 >\})$ and $(e, \{V, < \alpha_3 >\})$ also appear in the face poset of CY_{minA_3}. If we apply the permutation $(01) = s_{\alpha_0}$ to these two edges, and ask that it acts as it acts in CY_{minA_3}, we get

$$s_{\alpha_0}(e, \{V, < \alpha_1 >\}) = (w, \{V, w^{-1} s_{\alpha_0} \overline{< \alpha_1 >}\}) = (w, \{V, < \alpha_2, \alpha_3 >\})$$

and

$$s_{\alpha_0}(e, \{V, < \alpha_3 >\}) = (w, \{V, w^{-1} s_{\alpha_0} < \alpha_3 >\}) = (w, \{V, < \alpha_2 >\})$$

where $w = (0,1)(0,1,2,3,4)$. Now we notice that the intersection of these two edges is the vertex denoted by $(w, \{V, < \alpha_2, \alpha_3 >, < \alpha_2 >\})$ in CY_{minA_3} while in CY_{maxA_3} the image of $(e, \{V, < \alpha_1, \alpha_3 >\})$ via s_{α_0} should be an edge adjacent to $(w, \{V, < \alpha_2, \alpha_3 >\})$ and to $(w, \{V, < \alpha_2 >\})$. Since an edge with these properties does not exist, we have shown that the S_5 action on the face poset of CY_{minA_3} cannot be extended to the face poset of CY_{maxA_3}.

5.2 The Search for a 'Supermaximal' Model: The Example for A_3

We want to construct a model which is 'bigger' than $CY_{maxA_{n-1}}$ (i.e., it is obtained from $CY_{maxA_{n-1}}$ by a series of blowups) and is equipped with a S_{n+1} action. We will call 'supermaximal' a model which is minimal among the models that have these properties.

Example 30 In the case discussed in the Example 29 one has to blowup the vertex $(w, \{V, < \alpha_2 >, < \alpha_2, \alpha_3 >\})$ in the maximal model of type A_3 in order to obtain an edge that is the image of the edge $(e, \{V, < \alpha_1, \alpha_3 >\})$ under s_{α_0}.

As a consequence, because of the S_4 symmetry, one has to blowup all the vertices $(\gamma, \{V, < \alpha_i >, < \alpha_j, \alpha_{j+1} >\})$ with $\gamma \in S_4, j = 1, 2$ and $i = j$ or $i = j + 1$.

Let us denote by $CY_{supermaxA_3}$ the model obtained as a result of all these blowups: one can immediately check that the S_5 action on the face poset of CY_{minA_3} described in Sect. 3 extends to the face poset of $CY_{supermaxA_3}$.

In the next section we will introduce the necessary combinatorial notation to extend the construction of a supermaximal model from A_3 to the general case A_{n-1}.

6 Combinatorial Building Sets

After De Concini and Procesi's paper [6], nested sets and building sets appeared in the literature, connected with several combinatorial problems. In [15] building sets and nested sets were defined in the general context of meet-semilattices, and in [9] their connection with Dowling lattices was investigated. Other purely combinatorial definitions were used to give rise to the polytopes that were named *nestohedra* in [35].

Here we recall the combinatorial definitions of building sets and nested sets of a power set in the spirit of [34, 35] (one can refer to Sect. 2 of [33] for a short comparison among various definitions and notations in the literature).

Definition 31 A building set of the power set $\mathcal{P}(\{1, 2, \ldots, n\})$ is a subset $\mathcal{B}$ of $\mathcal{P}(\{1, 2, \ldots, n\})$ such that:

a) If $A, B \in \mathcal{B}$ have nonempty intersection, then $A \cup B \in \mathcal{B}$.
b) The set $\{i\}$ belongs to $\mathcal{B}$ for every $i \in \{1, 2, \ldots, n\}$.

Definition 32 A (nonempty) subset $\mathcal{S}$ of a building set $\mathcal{B}$ is a $\mathcal{B}$-nested set (or just nested set if the context is understood) if and only if the following two conditions hold:

a) For any $I, J \in \mathcal{S}$ we have that either $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$.
b) Given elements $\{J_1, \ldots, J_k\}$ $(k \geq 2)$ of $\mathcal{S}$ pairwise not comparable with respect to inclusion, their union is not in $\mathcal{B}$.

The nested set complex $\mathcal{N}(\mathcal{B})$ is the poset of all the nested sets of $\mathcal{B}$ ordered by inclusion (in fact it is a simplicial complex).

In particular, let us denote by $\mathcal{N}(minA_{n-1})$ the simplicial complex given by the nested sets in $\mathcal{F}_{A_{n-1}}$ that contain $\{V\}$. In fact, as observed in Sect. 2.3, the strata of $Y_{minA_{n-1}}$ are indexed by the nested sets of $\mathcal{F}_{A_{n-1}}$ containing $\{V\}$.

We observe that an element in $\mathcal{N}(minA_{n-1})$ can be obtained by the union of $\{V\}$ with an element of

$$\mathcal{P}(\mathcal{A}'_1) \cup \mathcal{P}(\mathcal{A}'_2) \cup \ldots \cup \mathcal{P}(\mathcal{A}'_s)$$

where $\mathcal{A}'_j = \mathcal{A}_j - \{V\}$ and $\mathcal{A}_j$ are all the maximal nested sets associated with the building set $\mathcal{F}_{A_{n-1}}$ (and $\mathcal{P}(\ )$, as in the preceding section, denotes the power set). Given a simplicial complex $\mathcal{C}$ which is based on some sets $\mathcal{A}'_1, .., \mathcal{A}'_s$ (i.e., it is equal to $\mathcal{P}(\mathcal{A}'_1) \cup \mathcal{P}(\mathcal{A}'_2) \cup \ldots \cup \mathcal{P}(\mathcal{A}'_s)$), Feichtner and Kozlov's definition of building set of a meet semilattice (see Sect. 2 of [15]) can be expressed in the following way: $\mathcal{B} \subseteq \mathcal{C}$ is a building set of $\mathcal{C}$ if and only if for every $j = 1, 2, \ldots, s$ the set $\mathcal{B} \cap \mathcal{P}(\mathcal{A}'_j)$ is a building set of $\mathcal{P}(\mathcal{A}'_j)$ in the sense of Definition 31.

Again, according to Feichtner and Kozlov, given a building set $\mathcal{B}$ of $\mathcal{C}$ as before, a $\mathcal{B}$-nested set is a subset $\mathcal{S}$ of $\mathcal{B}$ such that, for every antichain (with respect to inclusion) $\{X_1, X_2, \ldots, X_l\} \subseteq \mathcal{S}$, the union $X_1 \cup X_2 \cup .. \cup X_l$ belongs to $\mathcal{C} - \mathcal{B}$.

Now, starting from the simplicial complex $\mathcal{N}(minA_{n-1})$, we choose its maximal building set, i.e. we choose $\mathcal{B}(n-1) = \mathcal{N}(minA_{n-1})$.

Remark 33 We can therefore form the nested set complex of $\mathcal{B}(n-1)$, i.e. $\mathcal{N}(\mathcal{B}(n-1)) = \mathcal{N}(\mathcal{N}(minA_{n-1}))$, and then we can continue… As one can see, the complexity of this construction grows fastly. For a convenient notation and an interesting method to handle this complexity see [33].

In an analogous way, we define the simplicial complex $\mathcal{N}(minA_{n-1_{fund}})$ and we choose its maximal building set, i.e. we choose $\mathcal{B}(n-1)_{fund} = \mathcal{N}(minA_{n-1_{fund}})$.

According to Feichtner and Kozlov's definition, an arbitrary $\mathcal{B}(n-1)$-nested set (resp. $\mathcal{B}(n-1)_{fund}$-nested set) is a totally ordered by inclusion set of elements of $\mathcal{N}(minA_{n-1})$ (resp. $\mathcal{N}(minA_{n-1_{fund}})$).

The complex $\mathcal{B}(n-1)$ describes the strata in the boundary of the variety $CY_{minA_{n-1}}$, (and $Y_{minA_{n-1}}$ both in the real and complex case): as remarked in Sect. 2.4, the set of all these strata is a building set in the sense of [30, 31] and [20], therefore we can construct the corresponding variety $CY_{\mathcal{B}(n-1)}$ (and $Y_{\mathcal{B}(n-1)}$, real or complex).

Translating into combinatorial terms the definition given in Sect. 4 of [31], the strata in the boundary of the real or complex variety $Y_{\mathcal{B}(n-1)}$ and of $CY_{\mathcal{B}(n-1)}$ are indexed by the nested sets of $\mathcal{B}(n-1) - \{V\}$. As a convention we will add $\{V\}$ to these nested sets: a stratum of codimension r is indexed by $\{\{V\}, \mathcal{T}_1, \ldots, \mathcal{T}_r\}$ where each $\mathcal{T}_i$ belongs to $\mathcal{N}(minA_{n-1})$ and

$$\{V\} \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_r.$$

7 The S_{n+1} Action on $CY_{\mathcal{B}(n-1)}$

The face poset of the model $CY_{\mathcal{B}(n-1)}$ can be described in the following way: it is made by the couples $(w, \mathcal{S})$ where $w \in S_n$ and $\mathcal{S}$ is a nested set of the building set $\mathcal{B}(n-1)_{fund}$ containing $\{V\}$ (we remark that every element of $\mathcal{S}$ is a nested set of $\mathcal{F}_{A_{n-1fund}}$ containing V and that the elements in $\mathcal{S}$ are linearly ordered by inclusion).

The standard action of S_n on the face poset of $CY_{\mathcal{B}(n-1)}$ can be extended to an S_{n+1} action:

Proposition 34 *The action of S_{n+1} on the face poset of $CY_{minA_{n-1}}$ induces a S_{n+1} action on the face poset of $CY_{\mathcal{B}(n-1)}$.*

Proof Let $(w, \{\{V\}, \mathcal{T}_1, \ldots, \mathcal{T}_r\})$ be an element of the face poset of $CY_{\mathcal{B}(n-1)}$. The action of $\sigma \in S_{n+1}$ maps this element to $(w', \{\{V\}, \mathcal{T}'_1, \ldots, \mathcal{T}'_r\})$ where w' and $\mathcal{T}'_i$ are defined by the action on $CY_{minA_{n-1}}$: $\sigma(w, \mathcal{T}_i) = (w', \mathcal{T}'_i)$. From the inclusions $\{V\} \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_r$ it immediately follows that $\{V\} \subsetneq \mathcal{T}'_1 \subsetneq \cdots \subsetneq \mathcal{T}'_r$. □

Proposition 35 *There is a graded poset embedding φ of the face poset of $CY_{maxA_{n-1}}$ into the face poset of $CY_{\mathcal{B}(n-1)}$.*

Proof Let $(w, \mathcal{T})$ be an element in the face poset of $CY_{maxA_{n-1}}$. Then $\mathcal{T} = \{B_0 = V, B_1, B_2, \ldots, B_r\}$ is a nested set of the building set $\mathcal{C}_{A_{n-1fund}}$ containing V. This means that its elements are linearly ordered by inclusion: $V \supset B_1 \supset \cdots \supset B_r$. Now we can express every B_i as the direct sum of some irreducible subspaces A_{ij}, i.e. elements of $\mathcal{F}_{A_{n-1fund}}$ $(j = 1, \ldots, k_i)$. We notice that, for every $i = 1, \ldots, r$, the sets $\mathcal{T}'_i = \{A_{sj}\} \cup \{V\}$ (with $s > r - i$ and, for every $s, j = 1, \ldots, k_s$) is nested in $\mathcal{F}_{A_{n-1fund}}$. The map φ defined by

$$\varphi((w, \mathcal{T})) = (w, \{\{V\}, \mathcal{T}'_1, \ldots, \mathcal{T}'_r\})$$

if $r \geq 1$, otherwise

$$\varphi((w, \mathcal{T})) = (w, \{\{V\}\})$$

is easily seen to be a poset embedding (notice that $(w, \{\{V\}\})$ represents the connected component of $CY_{\mathcal{B}(n-1)}$ that lies in the chamber associated to w). □

Given a disjoint union P of (combinatorial) polytopes, we will denote by $F(P)$ its face poset and by $F^k(P)$ the subset made by the elements that correspond to faces of codimension k (for instance, $F^1(P)$ is made by the elements that correspond to the facets of P).

The restriction of φ to $F^1(CY_{maxA_{n-1}})$ is an embedding of $F^1(CY_{maxA_{n-1}})$ into $F^1(CY_{\mathcal{B}(n-1)})$. Now $F^1(CY_{\mathcal{B}(n-1)})$ can be identified with $F(CY_{minA_{n-1}})$ (the identification maps $(w, \{\{V\}, \mathcal{S}\})$, with $\mathcal{S}$ a nested set of $\mathcal{F}_{A_{n-1fund}}$ that strictly contains V, to $(w, \mathcal{S})$) and we still call φ the embedding from $F^1(CY_{maxA_{n-1}})$ to $F(CY_{minA_{n-1}})$. More explicitly, if $B_1 \in \mathcal{C}_{A_{n-1fund}}$ is a subspace which is the direct sum of the irreducible subspaces $A_{11}, \ldots, A_{1k_1}$ then

$$\varphi((w, \{V, B_1\})) = (w, \{V, A_{11}, \ldots, A_{1k_1}\}).$$

Theorem 36 *The minimal building subset of $F(CY_{minA_{n-1}})$ which contains the image $\varphi(F^1(CY_{maxA_{n-1}}))$ and is closed under the S_{n+1} action is $F(CY_{minA_{n-1}})$ itself. A similar statement holds in the non spherical case.*

Proof We will prove the theorem in the spherical case (the proof is of combinatorial nature so it can be applied to the non spherical case with minor formal changes).

Let us denote by π_2 the projection of $F(CY_{minA_{n-1}})$ onto $\mathcal{B}(n-1)_{fund}$ given by $(w, \mathcal{S}) \mapsto \mathcal{S}$ and let us denote by $C(n-1)$ the image of the restriction of π_2 to $\varphi(F^1(CY_{maxA_{n-1}}))$.

Let us consider a subset $\tilde{\Gamma}$ of $F(CY_{minA_{n-1}})$ that is building and closed under the S_{n+1} action, and let us denote by Γ the set of $\pi_2(\tilde{\Gamma}) \subseteq \mathcal{B}(n-1)_{fund}$. We will prove the claim by showing that $\Gamma = \mathcal{B}(n-1)_{fund}$, since this implies that $\tilde{\Gamma} = F(CY_{minA_{n-1}})$.

This can be done by induction on the depth of a nested set, which is defined in the following way: let $\mathcal{T}$ be a $\mathcal{F}_{A_{n-1}}$-nested set that contains V and consider the levelled graph associated to $\mathcal{T}$ (it is a tree: it coincides with the Hasse diagram of the poset induced by the inclusion relation, where the leaves are the minimal subspaces of $\mathcal{S}$ and the root, in level 0, is V). We say that $\mathcal{T}$ has depth k if k is the highest level of this tree.

Now we prove by induction that every element in $\mathcal{B}(n-1)_{fund}$ with depth k belongs to Γ.

When $k = 0, 1$ this is immediate: given $B \neq V \in \mathcal{C}_{A_{n-1 fund}}$, the right member of $\varphi((w, \{V, B\}))$ is the nested set of depth 1 whose elements are V and the maximal elements of $\mathcal{F}_{A_{n-1 fund}}$ contained in B. In this way one can show that all the nested sets of $\mathcal{F}_{A_{n-1 fund}}$ with depth 1 that contain V belong to Γ.

Let us check the case $k = 2$. One first observes that, in view of the definition of the S_{n+1} action, every nested set of depth 2 of the form $\{V, B, B_1\}$, where $B_1 \subset B$, belongs to Γ since it can be obtained as $\pi_2(\sigma(e, \mathcal{S}))$ for a suitable choice of $\sigma \in S_{n+1}$ and of a the nested set of depth 1 $\mathcal{S} \in \mathcal{B}(n-1)_{fund}$. Now we show that also all the nested sets of depth 2 of the form $\{V, B, B_1, \ldots, B_j\}$, with $j \geq 2$ and $B_i \subset B$ for every i, belong to Γ. In fact we can obtain $\{V, B, B_1, \ldots, B_j\}$ as a union, for every i, of the nested sets $\{V, B, B_i\}$ that belong to Γ as remarked above. Since all these sets have a nontrivial intersection $\{V, B\}$ and Γ is building in the Feichtner-Kozlov sense (see Sect. 6), this shows that $\{V, B, B_1, \ldots, B_j\}$ belongs to Γ.

Then let us consider a nested set of depth 2 $\{V, B, B_1, \ldots, B_j\}$ ($j \geq 2$), where

i) At least one of $B_1, \ldots, B_{j-1}$ (say B_1) is included in B.
ii) B_j is not included in B.
iii) All the B_i in level 2 are included in B.

This nested set is in Γ since it can be obtained as a union of the nested sets (with depth 1) $\mathcal{S}_1 = \{V, B_1, \ldots, B_j\}$ and $\mathcal{S}_2$, where $\mathcal{S}_2$ is any nested subset of

$\{V, B, B_1, \ldots, B_j\}$ with depth 2. We notice that $\mathcal{S}_1$ and $\mathcal{S}_2$ are in Γ, and have nonempty intersection, therefore their union belongs to Γ.

Now we can show that in Γ there are all the nested sets of depth 2: if the set $\{V, C_1, \ldots C_s, B_1, \ldots, B_j\}$ has depth 2, where $s \geq 2$ and each of the subspaces $C_1, \ldots, C_s$ contains some of the B_i's, while the B_i's are the leaves of the tree, we can obtain $\{V, C_1, \ldots C_s, B_1, \ldots, B_j\}$ as the union of the nested sets $\{C_i, B_1, \ldots, B_j\}$ (for every $i = 1, .., s$) that have pairwise nonempty intersection and belong to Γ, as we have already shown.

Let us now suppose that every nested set in $\mathcal{B}(n-1)_{fund}$ with depth $\leq k$ (with $k \geq 2$) belongs to Γ and let $\mathcal{T}$ be a nested set of depth $k+1$. Let us denote by $\mathcal{T}_k$ the nested set obtained removing from $\mathcal{T}$ the $k+1$-th level: it belongs to Γ by the inductive hypothesis. Then we consider the nested set $\mathcal{T}'$ obtained removing from $\mathcal{T}$ the levels $2, \ldots, k$: since $\mathcal{T}'$ has depth 2 it belongs to Γ again by the inductive hypothesis. We observe that $\mathcal{T}_k$ and $\mathcal{T}'$ have nonempty intersection, therefore their union $\mathcal{T}$ must belong to Γ. □

8 Final Remarks

8.1 *The Models $CY_{\mathcal{B}(n-1)}$ and $Y_{\mathcal{B}(n-1)}$ Are Supermaximal Models*

Theorem 36 proved in the preceding section shows that $CY_{\mathcal{B}(n-1)}$ and $Y_{\mathcal{B}(n-1)}$ are supermaximal models, so we can answer to the question, raised in Sect. 5.2, about how to construct a model that is 'bigger' than the maximal model, admits the extended S_{n+1} action and is minimal with these properties.

Let us state this in a more formal way, in the case of $Y_{\mathcal{B}(n-1)}$ (the spherical case is analogue). Let us consider the simplicial complex $\mathcal{B}(n-1)$ that indicizes the strata of the minimal model $Y_{minA_{n-1}}$, and let us denote by $\mathcal{L}$ the family of spherical models obtained by blowing up all the building subsets of these strata. We observe that $\mathcal{L}$ has a natural poset structure given by the relation $Y_{\mathcal{G}_1} \leq Y_{\mathcal{G}_2}$ if and only if $\mathcal{G}_1 \subseteq \mathcal{G}_2$ (by Li's definition, this also means that there is a birational projection of $Y_{\mathcal{G}_2}$ onto $Y_{\mathcal{G}_1}$).

Let us denote by $\mathcal{T}$ the set of nested sets of $\mathcal{F}_{A_{n-1}}$ with depth 1. From Theorem 16 it follows that if we blowup in $Y_{minA_{n-1}}$ the strata that correspond to the elements of $\mathcal{T}$ (in a suitable order, i.e. first the strata with bigger codimension) we obtain the model $Y_{maxA_{n-1}}$.

Now the supermaximal model associated with the root arrangement A_{n-1} can be defined as the minimal model $Y_{\mathcal{K}}$ in the poset $\mathcal{L}$ that admits the S_{n+1} action and such that $\mathcal{K} \supseteq \mathcal{T}$ (this last property means that it admits a birational projection onto $Y_{maxA_{n-1}}$).

Then Theorem 36 shows that $Y_{\mathcal{B}(n-1)}$ is the only model in $\mathcal{L}$ that satisfies these properties so it is the supermaximal model.

8.2 Orbits in the Supermaximal Case

We can compute the orbits of the S_{n+1} action on the supermaximal spherical model $CY_{\mathcal{B}(n-1)}$. The strata of the connected component that lies in the fundamental chamber (combinatorially equivalent to a nestohedron) are indexed by the $\mathcal{B}(n-1)_{fund}$-nested sets that contain $\{V\}$: a stratum of codimension r is indexed by $\{\{V\}, \mathcal{T}_1, \ldots, \mathcal{T}_r\}$ where each $\mathcal{T}_i$ is a $\mathcal{F}_{A_{n-1}fund}$-nested set and

$$\{V\} \subsetneq \mathcal{T}_1 \subsetneq \quad \subsetneq \mathcal{T}_r.$$

As in the minimal case, the stabilizer of a stratum is a subgroup of the cyclic group $C =< (0, 1, \ldots, n) >$. Let us consider d with $0 < d < \frac{n+1}{2}$ and $d|(n+1)$. Then, the number of strata of codimension r (with $d - 1 \geq r \geq 1$) whose stabilizer is $< (0, 1, \ldots, n)^d >$ is given by the formula:

$$\sum_{k \ s.t. \ d-1 \geq k \geq r} r! S(k, r) \overline{\mathcal{O}}_{d, \frac{n+1}{d} k} \tag{2}$$

where $\overline{\mathcal{O}}_{d, \frac{n+1}{d} k}$ is the number (computed in Sect. 4) of the dissections of a $n+1$-gon with $\frac{n+1}{d} k$ diagonals and with stabilizer $< (0, 1, \ldots, n)^d >$ and $S(k, r)$ is the Stirling number of the second kind. In fact, given a dissection of a $n+1$-gon with $\frac{n+1}{d} k$ diagonals and with stabilizer $< (0, 1, \ldots, n)^d >$, it determines a $\mathcal{F}_{A_{n-1 fund}}$-nested set with $\frac{n+1}{d} k + 1$ subspaces (including V). This can be viewed as the bigger nested set $\mathcal{T}_r$ in the list

$$\{V\} \subsetneq \mathcal{T}_1 \subsetneq \quad \subsetneq \mathcal{T}_r.$$

Once $\mathcal{T}_r$ is fixed, there are $r! S(k, r)$ ways to choose the list of smaller nested sets $\mathcal{T}_1 \subsetneq \quad \subsetneq \mathcal{T}_{r-1}$ (if $r = 1$ this gives 1, as expected).

Remark 37 Since all the nested sets $\mathcal{T}_1, \ldots, \mathcal{T}_r$ have to be $< (0, 1, \ldots, n)^d >$-invariant, the codimension of an invariant stratum is $\leq d - 1$, so the above formula covers all the interesting cases. If $d = \frac{n+1}{2}$ the formula (2) has to be modified in the following way:

$$\sum_{k \ s.t. \ \frac{n+1}{2} - 1 \geq k \geq r} r! S(k, r) [\overline{\mathcal{O}}_{\frac{n+1}{2}, 2k} + \overline{\mathcal{O}}_{\frac{n+1}{2}, 2k-1}]. \tag{3}$$

8.3 The Cohomology of a Complex Supermaximal Model

The discussion in the preceding sections points out the interest of the supermaximal models and of the corresponding symmetric group actions. In the paper [2] the construction of the supermaximal models was generalized to the case of any subspace arrangement and the following description of a basis of the integer cohomology was given.

Let $\mathcal{F}$ be the building set of irreducible subspaces associated with a subspace arrangement in a complex vector space V of dimension n. Let $\mathcal{B}(\mathcal{F})$ be the corresponding supermaximal building set (so $Y_{\mathcal{B}(\mathcal{F})}$ is the model obtained by blowing up all the strata of the minimal model $Y_{\mathcal{F}}$). Let us denote by $\pi_{\mathcal{F}}^{\mathcal{B}(\mathcal{F})}$ the projection from $Y_{\mathcal{B}(\mathcal{F})}$ onto $Y_{\mathcal{F}}$.

Theorem 38 *A basis of the integer cohomology of the complex model $Y_{\mathcal{B}(\mathcal{F})}$ is given by the following monomials:*

$$\eta \, c_{\mathcal{S}_1}^{\delta_1} c_{\mathcal{S}_2}^{\delta_2} \cdots c_{\mathcal{S}_k}^{\delta_k}$$

where

1. *$\mathcal{S}_1 \subsetneq \mathcal{S}_2 \subsetneq \cdots \subsetneq \mathcal{S}_k$ is a chain of $\mathcal{F}$-nested sets (possibly empty, i.e. $k = 0$), with $\{V\} \subsetneq \mathcal{S}_1$.*
2. *The exponents δ_i, for $i = 1, \ldots, k$, satisfy the following inequalities: $1 \leq \delta_i \leq |\mathcal{S}_i| - |\mathcal{S}_{i-1}| - 1$, where we put $\mathcal{S}_0 = \{V\}$.*
3. *η is the image via $(\pi_{\mathcal{F}}^{\mathcal{B}(\mathcal{F})})^*$ of a monomial in a basis[3] of $H^*(D_{\mathcal{S}_1})$ if $k \geq 1$, and it is the image via $(\pi_{\mathcal{F}}^{\mathcal{B}(\mathcal{F})})^*$ of a monomial in a basis[3] of $H^*(Y_{\mathcal{F}})$ if $k = 0$.*
4. *The element $c_{\mathcal{S}_i}$ is the Chern class of the normal bundle of $L_{\mathcal{S}_i}$ (the proper transform of $D_{\mathcal{S}_i}$) in $Y_{\mathcal{B}(\mathcal{F})}$.*

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h-Vectors of Matroid Complexes

Alexandru Constantinescu and Matteo Varbaro

Abstract In this paper we partition in classes the set of matroids of fixed dimension on a fixed vertex set. In each class we identify two special matroids, respectively with minimal and maximal h-vector in that class. Such extremal matroids also satisfy a long-standing conjecture of Stanley. As a byproduct of this theory we establish Stanley's conjecture in various cases, for example the case of Cohen-Macaulay type less than or equal to 3.

1 Introduction

In 1977 Stanley conjectured that the h-vectors of matroids are pure O-sequences [19, p.59], that is they are h-vectors of Artinian monomial level algebras or, equivalently, f-vectors of pure order ideals. Ever since, the h-vectors of matroids have been in the focus of many researchers (see [4, 8, 9, 11, 18, 23]). Pure O-sequences themselves have attracted a lot of attention as well, quite a few conjectures being made regarding their shape [2, gives an overview of the topic]. Although several researchers have approached Stanley's conjecture, to our knowledge only very specific cases have been proven. The case of cographic matroids was proven in [4, 12], that of lattice path matroids in [17] and more generally the one of cotransversal matroids in [15]. Low rank and degree situations were recently investigated in [6, 21, 22].

In the present paper we prove Stanley's conjecture in several cases, which appear in every rank and codimension. As a particular case, we obtain the conjecture for all matroid complexes of Cohen–Macaulay type less than or equal to 3. For any positive integers n and d, we divide the $(d-1)$-dimensional matroids on n vertices in different classes, which are indexed by the partitions of n with length at least d. For each class we build the set of all possible h-vectors of the duals of the matroids

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in the respective class. We then identify two special matroids whose duals have minimal, respectively maximal h-vectors in that set. For all these extremal matroids we prove in a constructive way that Stanley's conjecture holds.

Our approach passes via an equivalent phrasing of Stanley's conjecture. The h-vector of a matroid Δ is defined as the h-vector of the corresponding Stanley–Reisner ring and we will denote it by h_Δ. To a simplicial complex in general, apart from the Stanley–Reisner ideal I_Δ, one can associate its vertex cover ideal $J(\Delta)$. We will denote the h-vector of the quotient ring of $J(\Delta)$ by h^Δ. If we denote by Δ^c the dual of Δ (that is the simplicial complex generated by the complements of the facets in the vertex set), we have that

$$J(\Delta) = I_{\Delta^c} \qquad \text{and} \qquad h^\Delta = h_{\Delta^c}.$$

A classical theorem of matroid theory says that Δ is a matroid if and only if Δ^c is a matroid. This implies the following equivalent formulation of Stanley's conjecture:

Conjecture (Stanley) *For any matroid Δ, the vector* h^Δ *is a pure O-sequence.*

Let us summarize the contents of the paper. Section 2 is mainly devoted to preliminary results and establishing the notation. The recursive formula (1) for the behavior of h^Δ under deletion and contraction of vertices will be a crucial tool throughout the paper. Such a formula depends heavily on the matroid structure and fails for simplicial complexes in general. In Remark 7 we also present a counterexample to the Interval Conjecture for Pure O-sequences formulated by Boij et al. in [2].

In Sect. 3, we first provide some structural results for matroid complexes. We show that the 1-skeleton of a matroid is a complete p-partite graph. The division of the matroids into classes will be done in correspondence with these partitions of the vertex set. In each class we then define $d-1$ matroids: $\Delta_t(d, p, \mathbf{a})$, for $t = 0, \ldots, d-2$, where $\mathbf{a}$ is the partition of n. All these matroids are representable over fields with "enough" elements, and in most cases they are neither graphic nor transversal. We will call $\Delta_0(d, p, \mathbf{a})$ complete p-partite matroids. These are a simultaneous generalization of both uniform and partition matroids.

Later on in this section, we attach to each matroid Δ another matroid ${}^{\mathrm{si}}\Delta$, named simplified matroid, of the same dimension but on less vertices. The simplified matroid reflects many properties of the original matroid. For example, the total Betti numbers of $J(\Delta)$ and $J({}^{\mathrm{si}}\Delta)$ are the same (Proposition 15). In Proposition 17 we provide a formula which computes h^Δ for one-dimensional matroids. It turns out that the set of h-vectors of matroid complexes of the type $(1, 2, h_2, \ldots, h_s)$ coincides with the set of pure O-sequences of the form $(1, 2, h_2, \ldots, h_s)$.

In Sect. 4 we prove the conjecture of Stanley in various instances. In Theorem 25 we show that h^Δ is a pure O-sequence whenever Δ is a $(d-1)$-dimensional complete p-partite matroid for some $p \geq d$. Using Theorem 25, we prove the more general statement that $\mathrm{h}^{\Delta_t(d,p,\mathbf{a})}$ is a pure O-sequence for all $t = 0, \ldots, d-2$ (Theorem 26).

In Sect. 5, for any partition $\mathbf{a}$ of n with $p \geq d$ parts, we denote by $\mathcal{M}(d, p, \mathbf{a})$ the set of $(d-1)$-matroids on n vertices, whose 1-skeleton is p-partite and the cardinalities of the partition sets correspond to **a**. By the results of Sect. 3, every matroid belongs to exactly one of these sets. In Theorems 32 and 35, we show that

$$\mathrm{h}^{\Delta_{d-2}(d,p,\mathbf{a})} \leq \mathrm{h}^{\Delta} \leq \mathrm{h}^{\Delta_0(d,p,\mathbf{a})}, \ \ \forall\, \Delta \in \mathcal{M}(d,p,\mathbf{a}).$$

A priori, the existence of a matroid in $\mathcal{M}(d,p,\mathbf{a})$ with minimal h-vector is not clear at all. Indeed, a striking consequence is the validity of Stanley's conjecture whenever the Cohen–Macaulay type of S/I_Δ is less than or equal to three. In other words, we establish Stanley conjecture for all the h-vectors of type $(h_0, h_1, \ldots, h_s)$ with $h_s \leq 3$.

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2 Preliminaries

In this section we will recall most of the algebraic and combinatorial notions that we will use throughout the paper. For general aspects on the topics presented below we refer the reader to the books of Stanley [20], of Bruns and Herzog [3] and of Oxley [16].

For a positive integer n denote by $[n]$ the set $\{1,\ldots,n\}$. A simplicial complex Δ on $[n]$ is a collection of subsets of $[n]$ such that $F \in \Delta$ and $F' \subseteq F$ imply $F' \in \Delta$. Notice that we are not requiring that $\bigcup_{F\in\Delta} F = [n]$, therefore Δ can be viewed as a simplicial complex on any overset of $\bigcup_{F\in\Delta} F$. Each element $F \in \Delta$ is called a *face* of Δ. The dimension of a face F is $|F| - 1$ and the *dimension of* Δ is $\max\{\dim F \ : \ F \in \Delta\}$. A maximal face of Δ with respect to inclusion is called a *facet* and we will denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. A simplicial complex is called *pure* if all facets have the same cardinality. We call a vertex v a *cone point* of Δ if $v \in F$ for any $F \in \mathcal{F}(\Delta)$. If $F_1,\ldots,F_m$ are subsets of $[n]$, then we denote by $\langle F_1,\ldots,F_m\rangle$ the smallest simplicial complex on $[n]$ containing them. Explicitly:

$$\langle F_1,\ldots,F_m\rangle = \{F \subseteq [n] \ : \ \exists\, i \in \{1,\ldots,m\} : F \subseteq F_i\}.$$

We say that $F_1,\ldots,F_m$ generate the simplicial complex $\langle F_1,\ldots,F_m\rangle$. Clearly every simplicial complex is generated by its set of facets. For any face F the *link* of F in Δ is the following simplicial complex:

$$\mathrm{link}_\Delta F = \{F' \in \Delta \ : \ F' \cup F \in \Delta \text{ and } F' \cap F = \emptyset\}.$$

For a set of vertices $W \subseteq [n]$, the *restriction* of Δ to W is the following subcomplex of Δ:

$$\Delta|_W = \{F \in \Delta \ : \ F \subseteq W\}.$$

The subcomplex $\Delta|_W$ is also called the subcomplex of Δ *induced by* the vertex set W. If F is a face of Δ, then the *face deletion* of F in Δ is $\Delta \setminus F = \{F' \in \Delta \,:\, F \not\subseteq F'\}$. Whenever F is a zero-dimensional face $\{v\}$ we will just write $\Delta \setminus v$ for the face deletion of $\{v\}$ and $\operatorname{link}_\Delta v$ for the link of $\{v\}$. Notice that $\Delta \setminus v = \Delta|_{[n]\setminus\{v\}}$ for all $v \in [n]$. The *dual complex* of Δ is the simplicial complex Δ^c on $[n]$ with facets

$$\mathcal{F}(\Delta^c) = \{[n] \setminus F \,:\, F \in \mathcal{F}(\Delta)\}.$$

For any integer $0 \leq k \leq \dim \Delta$, the *k-skeleton* of Δ is defined as the simplicial complex with facet set $\{F \in \Delta \,:\, \dim F = k\}$.

We will now associate to a simplicial complex two square-free monomial ideals. We will then see how these ideals are related via the dual complex. Denote by $S = \Bbbk[x_1, \dots, x_n]$ the polynomial ring in n variables over a field $\Bbbk$. For each subset $F \subseteq [n]$ define the monomial x_F and the prime ideal $\mathcal{P}_F$ as follows:

$$\mathrm{x}_F = \prod_{i \in F} x_i,$$

$$\mathcal{P}_F = (x_i \,:\, i \in F).$$

The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of S generated by the square-free monomials x_F, with $F \notin \Delta$. In particular we have

$$I_\Delta = (\mathrm{x}_F \,:\, F \text{ is a minimal nonface of } \Delta).$$

The second square-free monomial ideal we can associate to Δ is the *cover ideal* of Δ, namely

$$J(\Delta) = \bigcap_{F \in \mathcal{F}(\Delta)} \mathcal{P}_F.$$

The name "cover ideal" comes from the following fact. A collection of vertices $A \subseteq [n]$ is called a *vertex cover* of Δ if $A \cap F \neq \emptyset$ for any $F \in \mathcal{F}(\Delta)$. A vertex cover A is called *basic* if no proper subset of A is again a vertex cover. It is easy to check that we have

$$J(\Delta) = (\mathrm{x}_A \,:\, A \text{ is a basic vertex cover of } \Delta).$$

It is a well known fact that the prime decomposition of the Stanley–Reisner ideal is

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} \mathcal{P}_{[n]\setminus F}.$$

The following equality, which follows directly from the definition, will be very important for the approach of this paper:

$$J(\Delta) = I_{\Delta^c}.$$

We denote by $\Bbbk[\Delta] = S/I_\Delta$ the *Stanley–Reisner ring* of Δ. Let $h_{\Bbbk[\Delta]} = (h_0, h_1, \ldots, h_s)$ be its h-vector. If $\mathrm{HS}_{\Bbbk[\Delta]}(t)$ is the Hilbert series of $\Bbbk[\Delta]$, then we have

$$\mathrm{HS}_{\Bbbk[\Delta]}(t) = \frac{h_0 + h_1 t + \ldots + h_s t^s}{(1-t)^d},$$

where $h_s \neq 0$ and $d = \dim \Bbbk[\Delta] = \dim \Delta + 1$.

In the classical terminology, the h-vector of a simplicial complex is the h-vector of its Stanley–Reisner ring. As we will mainly deal with cover ideals, in order to avoid the over-use of the word dual, we will fix the following notation and terminology.

Notation Let Δ be any simplicial complex.

1. We denote the h-vector of $\Bbbk[\Delta]$ by h_Δ.
2. We denote the h-vector of $S/J(\Delta)$ by h^Δ.
3. We will refer throughout this paper to h^Δ as *the h-vector of* Δ.

Remark 1 As the cover ideal of the matroid is the Stanley–Reisner ideal of the dual matroid, with the above notation we have

$$h^\Delta = h_{\Delta^c}.$$

For a $(d-1)$-dimensional simplicial complex Δ on $[n]$ such that $S/J(\Delta)$ is Cohen–Macaulay, we denote by type(Δ) the last total Betti number in the minimal free resolution of $S/J(\Delta)$, namely

$$\mathrm{type}(\Delta) = \beta_d(S/J(\Delta)) = \dim_\Bbbk \mathrm{Tor}_d^S(S/J(\Delta), \Bbbk).$$

Matroid theory was born out of the need to study the concept of dependence in an abstract way. In this paper we will view matroids as simplicial complexes whose faces correspond to the independent sets. A characteristic of matroids is that they admit many different but equivalent definitions (see [16] and [20, Chap. III.3]). We present here three of them.

Definition 2 A simplicial complex Δ is called a *matroid complex* (or just matroid) if one of the following equivalent properties hold:

1. *The augmentation axiom:* For any two faces $F, G \in \Delta$ with $|F| < |G|$ there exists $i \in G$ such that $F \cup \{i\} \in \Delta$.

2. *The exchange property:* For any two facets $F, G \in \mathcal{F}(\Delta)$ and for any $i \in F$ there exists a $j \in G$ such that $(F \setminus \{i\}) \cup \{j\} \in$ d.
3. For any subset $W \subseteq [n]$ the restriction $\Delta|_W$ is pure.

A basic result in matroid theory that we will use intensively is the following:

Theorem 3 ([16, Theorem 2.1.1]) *A simplicial complex Δ on $[n]$ is a matroid if and only if Δ^c is a matroid.*

An algebraic characterization of matroid complexes has been given in [14] and [24], namely a simplicial complex Δ is a matroid if and only if all the symbolic powers of I_Δ are Cohen–Macaulay. Another algebraic property that will be important for us (even if it does not characterize matroids) is the following: the Stanley–Reisner ring of a matroid is level [20, Chap. III, Theorem 3.4]. This means that $\mathbb{k}[\Delta]$ is Cohen–Macaulay and the socle of its Artinian reduction lies in exactly one degree. A prototype of level algebras are the Gorenstein algebras, which correspond to socle dimension 1. An important consequence of $S/J(\Delta)$ being level is that the type can be expressed only in terms of the last entry of the h-vector, namely

$$\text{type}(\Delta) = \text{h}^{\Delta}(s) \quad \text{where } s = \max\{i \ : \ \text{h}^{\Delta}(i) \neq 0\}.$$

A crucial ingredient in many of our proofs is a recursive formula for the h-vector of $S/J(\Delta)$. For the following well known statements regarding the Tutte polynomial of a matroid we refer the reader to [1]. If $T_\Delta(x, y)$ is the Tutte polynomial of Δ, then the Tutte polynomial of the dual matroid is $T_{\Delta^c}(x, y) = T_\Delta(y, x)$. The coefficients of x^k in $T_{\Delta^c}(x, 1)$ are exactly $h^\Delta(k)$, for every $k = 0, \ldots, s$. Using the recurrence relation of the Tutte polynomial under deletion and contraction of a vertex $v \in \Delta$, which is not a cone point, one immediately obtains that

$$\text{h}^{\Delta}(k) = \text{h}^{\Delta \setminus v}(k-1) + \text{h}^{\text{link}_\Delta v}(k) \quad \forall\, k \in \mathbb{Z}. \tag{1}$$

Remark 4 The h-vector of the Stanley–Reisner ideal of any simplicial complex satisfies a similar recursive formula. However, the h-vector of the cover ideal of simplicial complex in general fails to satisfy (1). Already a path of length 4, viewed as a simplicial complex, is such an example.

An *order ideal* is a finite collection Γ of monomials of some standard graded polynomial ring, such that $M \in \Gamma$ and N divides M imply $N \in \Gamma$. The partial order given by the divisibility of monomials gives Γ a poset structure. An order ideal is called *pure* if all maximal monomials have the same degree. To every order ideal Γ we associate its f-vector $f(\Gamma) = (f_0(\Gamma), \ldots, f_s(\Gamma))$, where for every $i = 0, \ldots, d$ we have

$$f_i(\Gamma) = |\{M \in \Gamma \ : \ \deg(M) = i\}|.$$

A *pure O-sequence* is a vector $h = (h_0, \dots, h_s)$ that can be obtained as the f-vector of some pure order ideal.

Remark 5 Pure O-sequences can also be presented as the h-vectors of Artinian monomial level algebras, i.e. Artinian level algebras A which are isomorphic to R/I for some polynomial ring R and some monomial ideal $I \subseteq R$. It is very easy to see that, in this situation, if A is Gorenstein then I is forced to be a complete intersection. So the pure O-sequences of type $(h_0, h_1, \dots, h_{s-1}, 1)$ are well understood: they are h-vectors of complete intersections. In particular, it emerges that pure O-sequences are much more special than h-vectors of level algebras in general.

However, already a characterization of pure O-sequences of the type $(h_0, h_1, \dots, h_{s-1}, 2)$, i.e. when the Artinian monomial level algebra A has Cohen-Macaulay type 2, is not known (see [2]).

In [19] Stanley phrased his conjecture in terms of the h-vector of the Stanley–Reisner ring. By Theorem 3 an equivalent statement is the following:

Conjecture 6 (Stanley) If Δ is a matroid, then the h-vector of $S/J(\Delta)$ is a pure O-sequence.

Conjecture 6 is known for some families; we list here the most general of them.

1. When $S/J(\Delta)$ is Gorenstein, see [21, Theorem 4.4.10].
2. When $\mathrm{h}^{\Delta} = (1, h_1, h_2, h_3)$, see [21] and [6].
3. When Δ is a graphic matroid, see [12].
4. When Δ is a transversal matroid, see [15].
5. When the dual of Δ is a paving matroid, see [13]. This corresponds to the case in which $h_i = \binom{v+i-1}{i}$ for all $i < s$, where v is the number of zero-dimensional faces of Δ^c.
6. When $\mathrm{h}^{\Delta} = (1, 2, h_2, \dots, h_s)$. Indeed, one can see by the Hilbert–Burch theorem that, in the height 2 case, pure O-sequences coincide with h-vectors of level algebras (see [2, Proposition 4.5] for the precise proof), so one can deduce the validity of the conjecture in this case by [20, Chap. III, Theorem 3.4].

Computational experiments using the computer algebra system CoCoa [5] were an important part in the preparation of this work. In our investigation, we found a counterexample to the Interval Conjecture for Pure O-sequences (see [2]):

Remark 7 One can check that the vectors $(1, 4, 10, 13, 12, 9, 3)$ and $(1, 4, 10, 13, 14, 9, 3)$ are pure O-sequences. Indeed, the order ideals are generated by $\{x^3y^2z,\ x^3yt^s2,\ x^3z^2t\}$, respectively by $\{x^4y^2,\ x^3yzt,\ x^2z^2t^2\}$. Looking at all possible choices of three monomials of degree 6 in 4 variables, it is possible to compute all the pure O-sequences of the form $(1, 4, h_2, \dots,\ h_5, 3)$. Checking the obtained list, one can realize that $(1, 4, 10, 13, 13, 9, 3)$ does not appear among the pure O-sequences, a contradiction to the above-mentioned conjecture.

3 The Structure of Matroids

In this paper we will stratify the set of matroids of fixed dimension and on a fixed vertex set in terms of partitions of the vertex set. To this aim, in this section we will prove some technical facts. Most of these are well known facts for matroid theory specialists, however we consider it convenient to provide proofs as well. We will then present the simplified matroid associated to any given matroid. This matroid has only trivial parallel classes, but important information, such as the total Betti numbers $\beta_i(S/J(\Delta))$, is preserved. We conclude the section presenting a formula that computes the h-vector of a codimension two Stanley–Reisner ring of a matroid.

From now on, unless otherwise stated, we will consider simplicial complexes Δ on $[n]$ with the property that $v \in \Delta$ for all $v \in [n]$. Notice that the number of vertices not belonging to Δ does not influence h^Δ, so this is no restriction in terms of our goals. This assumption can be also expressed as $[n] = \bigcup_{F\in\Delta} F$ and if Δ is a $(d-1)$-dimensional matroid on $[n]$, a remark in [20, p. 94] implies that

$$n - d = \max\{i \ : \ \mathrm{h}^\Delta(i) \neq 0\}.$$

The following easy remark is the starting point for many of the following technical results.

Remark 8 If Δ is a one-dimensional simplicial complex on $[n]$, then Δ is a matroid if and only if for any $v, w \in [n]$ with $\{v, w\} \notin \Delta$ we have that $\mathrm{link}_\Delta(v) = \mathrm{link}_\Delta(w)$.

One-dimensional simplicial complexes can be viewed as graphs on the same vertex set; the edges are the faces of dimension one. For this reason we will switch between *graph* and *simplicial complex* whenever we find ourselves in this case. Let us recall that a graph is called a *complete p-partite* graph if and only if its vertex set can be partitioned into p disjoint nonempty sets $A_1, \ldots, A_p$ such that $\{v, w\}$ is an edge if and only if v and w lie in different sets of the partition. The following proposition shows that one-dimensional matroids and complete p-partite graphs are actually the same thing.

Proposition 9 *If Δ is a one-dimensional matroid, then Δ is a complete p-partite graph, for some integer $p \geq 2$.*

Proof We will proceed by induction on n, the number of vertices. Assume that $n \geq 2$, choose v a vertex of Δ and consider the set $A_v = \{w \in \Delta \ : \ \{v, w\} \notin \Delta\}$. As Δ is a matroid, we have by Remark 8 that $\mathrm{link}_\Delta v = \mathrm{link}_\Delta w$ for any $w \in A_v$. This implies that A_v is an independent set of vertices. Clearly $\mathrm{link}_\Delta v$ is a zero-dimensional simplicial complex whose faces correspond to the elements in $[n] \setminus A_v$. Moreover, as one can check by definition, the restriction $\Delta|_{[n]\setminus A_v}$ is also a matroid.

If $\dim \Delta|_{[n]\setminus A_v} = 1$, we have by induction that $\Delta|_{[n]\setminus A_v}$ is a complete p-partite graph, with p-partition of the vertex set $A_1 \cup \ldots \cup A_p$. In this case it follows that Δ is a complete $(p+1)$-partite graph with partition $[n] = A_v \cup A_1 \cup \ldots \cup A_p$.

If $\dim \Delta|_{[n]\setminus A_v} = 0$ then $[n] \setminus A_v$ is an independent set of vertices, so Δ is a complete bipartite graph with bipartition $[n] = A_v \cup ([n] \setminus A_v)$.

The next corollary gives a stratification of the set of all $(d-1)$-dimensional matroids on $[n]$ that will be crucial throughout this work. Clearly, the k-skeleton of a matroid is again a matroid, so we have the following.

Corollary 10 *Let Δ be a simplicial complex. If Δ is a matroid, then there exists a positive integer $p \geq 2$ such that the 1-skeleton of Δ is a complete p-partite graph.*

Before showing the next technical lemmas let us fix more notation. From now on, using Corollary 10, Δ will be a $(d-1)$-dimensional matroid on $[n]$, with p-partition of its 1-skeleton $A_1, \dots, A_p$. We will call the sets of independent vertices given by the p-partition *parallel classes*. Whenever necessary we will denote the vertices of a given parallel class as follows

$$A_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,a_i}\}.$$

For any integer $r \in \{1, \dots, p\}$ and any indices $1 \leq i_1 < \dots < i_r \leq p$ we denote by $\Delta_{i_1,\dots,i_r}$ the restriction of Δ to the vertex set $A_{i_1} \cup \dots \cup A_{i_r}$. We call $\Delta_{i_1,\dots,i_r}$ the restriction of Δ to the parallel classes $A_{i_1}, \dots, A_{i_r}$.

Lemma 11 *If for $r \leq d$ parallel classes $A_{i_1}, \dots, A_{i_r}$, with $1 \leq i_1 < \dots < i_r \leq p$, there exist r vertices $v_{i_j} \in A_{i_j}$ such that $\{v_{i_1}, \dots, v_{i_r}\} \in \Delta$, then for any r vertices $u_{i_j} \in A_{i_j}$ we have that $\{u_{i_1}, \dots, u_{i_r}\} \in \Delta$.*

Proof We may assume without loss of generality that $i_j = j$, for $j = 1, \dots, r$. Choose now r vertices $u_j \in A_j$ and assume that $\{u_1, \dots, u_r\} \notin \Delta$. Let $s < r$ be the maximum size of a subset of $\{u_1, \dots, u_r\}$ that belongs to Δ. Again we may assume that actually $\{u_1, \dots, u_s\} \in \Delta$. The simplicial complex $\Delta_{1,\dots,r}$ is a matroid. Since $\{v_1, \dots, v_r\} \in \Delta_{1,\dots,r}$ and the 1-skeleton of $\Delta_{1,\dots,r}$ is complete r-partite, we have $\dim \Delta_{1,\dots,r} = r - 1$. As a matroid is pure, we have that u_{s+1} belongs to some $(r-1)$-dimensional facet F of $\Delta_{1,\dots,r}$. Notice that, by the r-partition of $\Delta_{1,\dots,r}$'s 1-skeleton, the facet F has to contain exactly one vertex from each parallel class. By the augmentation axiom, we know that there exist $r - s$ vertices $w_1, \dots, w_{r-s} \in F$ such that $G = \{u_1, \dots, u_s, w_1, \dots, w_{r-s}\} \in \mathrm{d}_{1,\dots,r}$. As $\mathrm{d}_{1,\dots,r}$ is r-partite, G has to contain a vertex from each parallel class. In particular it has to contain one from $F \cap A_{s+1} = \{u_{s+1}\}$. In particular $\{u_1, \dots, u_s, u_{s+1}\} \in \mathrm{d}$, a contradiction to the maximality of s.

Lemma 12 *Let A_i be one of the parallel classes of Δ and let $v, w \in A_i$. Then*

$$\mathrm{link}_\Delta v = \mathrm{link}_\Delta w.$$

Proof Choose $\{a_1, \dots, a_{d-1}\} \in \mathrm{link}_\Delta v$. The restriction $\Delta|_{\{v,w,a_1,\dots,a_{d-1}\}}$ is a $(d-1)$-dimensional pure complex. As $\{v, w\} \notin \Delta$ we obtain that $\{a_1, \dots, a_{d-1}, w\} \in \Delta|_{\{v,w,a_1,\dots,a_{d-1}\}}$ and thus $\{a_1, \dots, a_{d-1}\} \in \mathrm{link}_\Delta w$.

Exploiting the results of Lemmas 11 and 12 we will simplify notation in the following way. We will write $A_{i_1} \dots A_{i_r} \in \Delta$ if there exist vertices $v_{i_j} \in A_{i_j}$ for all $j = 1, \dots, r$ such that $\{v_{i_1}, \dots, v_{i_r}\} \in \Delta$. By Lemma 11 this holds for any choice of r vertices, one in each parallel class. As by Lemma 12 the link of all the vertices in one parallel class is the same, we will denote by $\mathrm{link}_\Delta A_i$ the link of some vertex $v \in A_i$. These two lemmas lead us to the following definition (see [16, p. 49] for the classical matroid-theoretical definition).

Definition 13 Let Δ be a simplicial complex with complete p-partite 1-skeleton, satisfying Lemma 11. Let $A_1 \cup \dots \cup A_p$ be the p-partition and choose for each $i = 1, \dots, p$ a vertex $v_{i,1} \in A_i$. We define the associated *simplified* complex as

$$^{\mathrm{si}}\Delta = \Delta|_{\{v_{1,1},\dots,v_{p,1}\}}.$$

We will call a parallel class of Δ a *cone class* if the corresponding vertex in $^{\mathrm{si}}\Delta$ is a cone point of $^{\mathrm{si}}\Delta$. This is clearly equivalent to every facet of Δ containing a vertex of that parallel class.

Remark 14 Let Δ be a simplicial complex with complete p-partite 1-skeleton. Then, using Lemma 11, we have

$$\Delta \text{ is a matroid} \iff {}^{\mathrm{si}}\Delta \text{ is a matroid.}$$

The next proposition shows the close relation between a matroid Δ and $^{\mathrm{si}}\Delta$.

Proposition 15 *Given a matroid Δ on $[n]$, we have $\beta_i(S/J(\Delta)) = \beta_i(S/J(^{\mathrm{si}}\Delta))$ for all i. In particular,* $\mathrm{type}(\Delta) = \mathrm{type}(^{\mathrm{si}}\Delta)$.

Proof Set $R = \Bbbk[y_1, \dots, y_p]$, and consider the $\Bbbk$-algebra homomorphism

$$\begin{aligned} \phi : R &\longrightarrow S \\ y_i &\mapsto \textstyle\prod_{j \in A_i} x_j = m_i. \end{aligned}$$

One can check that $\phi(J(^{\mathrm{si}}\Delta))S = J(\Delta)$. Moreover it is obvious that $m_1, \dots, m_p$ form a regular sequence of S, so by a theorem of Hartshorne [7, Proposition 1] S is a flat R-module via ϕ. So, if $F_\bullet$ is a minimal free resolution of $R/J(^{\mathrm{si}}\Delta)$ over R, then it follows that $F_\bullet \otimes_R S$ is a minimal free resolution of $S/J(\Delta)$ over S. Therefore we may conclude.

Remark 16 With the notation of Proposition 15, notice that ϕ allows also to recover the *graded* Betti numbers of $J(\Delta)$ from those of $J(^{\mathrm{si}}\Delta)$. Provided that the partition of the 1-skeleton of Δ is known, it is enough to consider the natural $\mathbb{Z}^p$-grading both on R and on S. The $\mathbb{Z}^p$-grading on S is given by the p-partition.

We will conclude this section with a first application of Eq. (1). We will find a formula for the h-vectors h^Δ where Δ is a one-dimensional matroid. By Theorem 3, this is equivalent to describing the h-vectors of $\mathbb{k}[\Delta]$, where Δ is a matroid such that its Stanley–Reisner ideal has height 2.

By Proposition 9, a one-dimensional matroid Δ is actually a complete p-partite graph on n vertices. For all $k = 1, \ldots, n-1$, let us set

$$c_k(\Delta) = |\{i \in \{1, \ldots, p\} \ : \ |A_i| \geq k\}| - 1.$$

Proposition 17 *Let Δ be a one-dimensional matroid on $[n]$. For all $k = 0, \ldots, n-2$, we have*

$$\mathrm{h}^{\Delta}(k) = \sum_{i=1}^{n-k-1} c_i(\Delta).$$

Proof Let us choose a vertex $v \in A_p$. Clearly, the cover ideal of the link of v is the principal ideal

$$J(\mathrm{link}_{\Delta} v) = \Bigg(\prod_{i \in [n] \setminus A_p} x_i\Bigg).$$

In particular, we have

$$\mathrm{h}^{\mathrm{link}_{\Delta} v}(i) = \begin{cases} 1 \text{ if } 0 \leq i < n - |A_p|, \\ 0 \text{ otherwise.} \end{cases}$$

The partition sets of the matroid $\Delta \setminus v$ are $A_1, A_2, \ldots, A_{p-1}, A_p \setminus \{v\}$, so we have

$$c_k(\Delta \setminus v) = \begin{cases} c_k(\Delta) & \text{if } k \neq |A_p|, \\ c_k(\Delta) - 1 & \text{if } k = |A_p|. \end{cases}$$

By induction we have

$$\mathrm{h}^{\Delta \setminus v}(k) = \sum_{i=1}^{n-k-2} c_i(\Delta \setminus v),$$

for all $k = 0, \ldots, n-3$. On the other side, by (1) we have

$$\mathrm{h}^{\Delta}(k) = \mathrm{h}^{\Delta \setminus v}(k-1) + \mathrm{h}^{\mathrm{link}_{\Delta} v}(k), \quad \forall\, k = 0, \ldots, n-1.$$

Therefore,

$$\mathrm{h}^{\Delta}(k) = \begin{cases} \sum_{i=1}^{n-k-1} c_i(\Delta) - 1 + \mathrm{h}^{\mathrm{link}_\Delta v}(k) = \sum_{i=1}^{n-k-1} c_i(\Delta) \text{ if } k \le n - |A_p| - 1, \\ \\ \sum_{i=1}^{n-k-1} c_i(\Delta) + \mathrm{h}^{\mathrm{link}_\Delta v}(k) \qquad = \sum_{i=1}^{n-k-1} c_i(\Delta) \text{ otherwise.} \end{cases}$$

Corollary 18 *For a sequence* $h = (1, 2, h_3, \dots, h_s)$*, the following are equivalent:*

(i) There is a matroid Δ *such that* h *is the h-vector of* $\Bbbk[\Delta]$.
(ii) There is a matroid Δ *such that* h *is the h-vector of* $S/J(\Delta)$.
(iii) h *is a pure O-sequence.*
(iv) h *is the h-vector of a level algebra.*
(v) $h_{i+1} \le 2h_i + h_{i-1}$ *for all* $i = 1, \dots, s$.

Proof The equivalence between (i) and (ii) follows by Theorem 3, whereas (iii) is equivalent to (iv) by the Hilbert–Burch theorem. The equivalence between (iv) and (v) was shown by Iarrobino in [10]. As $S/J(\Delta)$ is level, and thus (ii) implies (iv), we just need to prove that (v) implies (ii) and this follows easily from Proposition 17.

Corollary 19 *If* Δ *is a one-dimensional matroid, then* $\mathrm{type}(\Delta) = p - 1$*, where* Δ *is p-partite.*

4 Stanley's Conjecture

The main result of this section is Theorem 26, in which we prove that Stanley's conjecture holds for certain matroids which we identify in a natural way. The first discussion of this section and Theorem 25 are particular cases of the main result. They are the starting point of the inductive procedure in the proof of Theorem 26. For a better understanding of the construction which we present here, we will start with a closer look at an already known case of Stanley's Conjecture 6, namely the codimension two case. In this first part we will concentrate on examples which hopefully provide the necessary intuition for the more technical proofs.

We start by presenting a more general recursive formula, which will be used in many of the proofs in this section.

Remark 20 Let Δ be a p-partite, rank d matroid and fix a parallel A_i of cardinality a_i. Using the fact that all the vertices in a parallel class have the same link (Lemma 12), by applying the recursive formula (1) consecutively for every vertex $v \in A_i$ we obtain:

$$h^{\Delta}(k) = h^{\Delta \setminus A_i}(k - a_i) + \sum_{j=0}^{a_i - 1} h^{\mathrm{link}_\Delta A_i}(k - j). \tag{2}$$

Consider a one-dimensional matroid Δ on $[n]$, thus by Proposition 9 it is a complete p-partite graph. Recall that we denote the partition sets of the graph by A_i and for $i = 1, \dots, p$ we have $a_i = |A_i|$. For simplicity, we assume for the moment that $a_1 \leq \dots \leq a_p$. We will now present an inductive method to compute the h-vector of Δ.

When we restrict to the first layer A_1, we obtain a zero-dimensional matroid on a_1 vertices. It is clear that in this case $J(\Delta_1) = (x_1 \cdots x_{a_1})$ so the h-vector of Δ_1 is the vector of length a_1: $(1, 1, \dots, 1)$. Let $v_{2,1}$ be the first vertex of the parallel class A_2. This vertex will be a cone-point of the one-dimensional matroid $\Delta|_{A_1 \cup \{v_{2,1}\}}$, so the h-vector will be the same as the one of Δ_1. We will now use the recursive formula (1) to compute the h-vector of $\Delta_{1,2}$. By Lemma 12 we have

$$\text{link}_{\Delta_{1,2}} v_{2,i} = A_1, \quad \forall\ v_{2,i} \in A_2.$$

So the h-vector of $\Delta|_{A_1 \cup \{v_{2,1}, v_{2,2}\}}$ is computed as follows:

$$\begin{array}{r} 1\ 1 \dots 1\ 1 \\ 1\ 1\ 1 \dots 1\ 0 \\ \hline 1\ 2\ 2 \dots 2\ 1 \end{array}$$

where the first row represents the h-vector of $\Delta|_{A_1 \cup \{v_{2,1}\}}$, the second row represents the h-vector of the link, i.e. of Δ_1. The last row is the h-vector of $\Delta|_{A_1 \cup \{v_{2,1}, v_{2,2}\}}$. To compute the h-vector of $\Delta|_{A_1 \cup \{v_{2,1}, v_{2,2}, v_{2,3}\}}$ we proceed in the same way. All together we have to apply this procedure $a_2 - 1$ times. This can be done also directly in the following way:

$$\begin{array}{r} 1\ \dots 1\ 1\ 1 \\ 1\ \ 1\ \dots 1\ 1\ 0 \\ 1\ \ 1\ \ 1\ \dots 1\ 0\ 0 \\ \\ \dots \\ \\ 1\ 1\ 1 \dots\ 1\ \ 0\ \dots 0\ 0\ 0 \\ \hline 1\ 2\ 3 \dots a_1\ a_1 \dots 3\ 2\ 1 \end{array}$$

The h-vector of $\Delta_{1,2,3}$ of is computed in a similar way. The only difference is that $v_{3,1}$ will no longer be a cone point. Thus the first row will be $h^{\Delta_{1,2}}$ and the number of shifted rows will be a_3. Repeating this procedure, we can imagine that h^{Δ} is computed summing the columns of the staircase in Fig. 1. Notice that the last nonzero entry of h^{Δ} is $p - 1$. In Fig. 2 we can see one example of how the corresponding order ideal is constructed in the case when Δ is the one-dimensional matroid on 15 vertices, with 4-partition (3,3,4,5). Notice that the columns contain monomials of the same degree and that the exponent of x is constant on the rows.

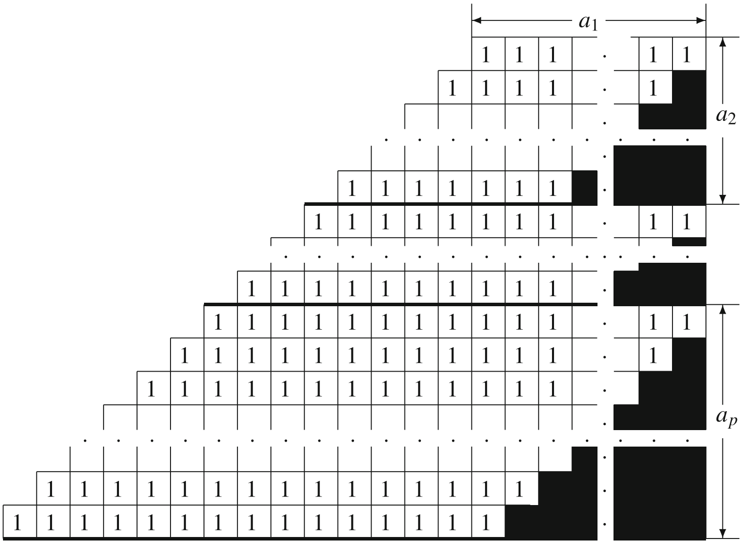

Fig. 1 Computing h^{Δ} for $d = 2$

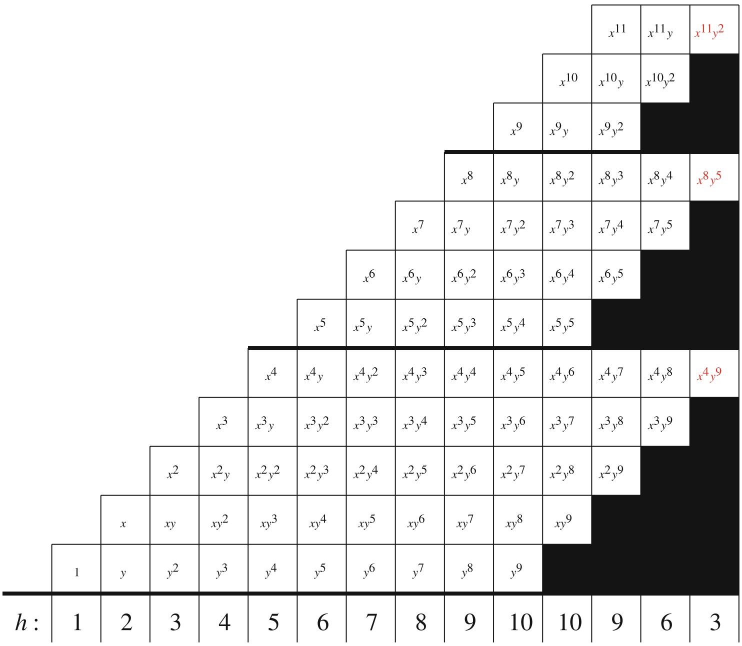

Fig. 2 The order ideal corresponding to the 4-partition $(3, 3, 4, 5)$

Depending on the order of the parallel classes we can build a total of 12 different staircases, each one producing an order ideal. Eliminating the symmetry given by exchanging x and y, we are left with six different order ideals with the right f-vector. For example ordering the partition as $(4, 3, 3, 5)$ we obtain the order ideal generated by $\{x^4y^9, x^7y^6, x^{10}y^3\}$.

In higher dimensions the picture becomes more complicated. One can either imagine d-dimensional staircases, where each cube has value one, or two-dimensional staircases, where each row is the h-vector of the link of a parallel class. As we already saw, the order of the a_i's plays no role in the computation of h^Δ, providing us with several ways to construct an order ideal with the same f-vector. A complicated example in dimension 2, with 6-partite 1-skeleton shows that unfortunately with this method there is no "canonical" choice. By canonical we understand a construction that should be independent of the values of the a_i's.

There is one case in which the choice of the order ideal is unique, namely the case when $d = p$. As we will see in Remark 33, this is equivalent to $J(\Delta)$ being Gorenstein.

Lemma 21 *If Δ is a $(d-1)$-dimensional, d-partite matroid (so $d = p$) with partition $(a_1, \ldots, a_d)$, then*

$$h^\Delta = f(\langle y_1^{a_1-1} \cdots y_d^{a_d-1} \rangle).$$

Proof The minimal generators of $J(\Delta)$ are the monomials corresponding to the basic covers of Δ. In this situation, $A_1, \ldots, A_d$ are the unique basic covers of Δ, so $J(\Delta)$ is a complete intersection with d generators of degrees $a_1, \ldots, a_d$. The conclusion follows because the h-vector of a complete intersection depends only on the degree of its minimal generators.

We will now define a class of matroids and prove that the Stanley conjecture holds for this class. When one fixes the dimension and the p-partition of the vertex set, these matroids will have all the admissible faces, thus they are in a sense a generalization of the Gorenstein matroids.

Definition 22 Let Δ be a $d-1$-dimensional matroid on $[n]$ with p-partite 1-skeleton. We say that Δ is a *complete p-partite matroid* if

$$A_{i_1} \ldots A_{i_d} \in \Delta, \quad \text{for any subset } \{i_1, \ldots, i_d\} \subseteq \{1, \ldots, p\}.$$

Whenever p is clear from the context, we will just call Δ *complete*. Notice that a complete matroid is uniquely determined by the cardinalities of the parallel classes $a_1, \ldots, a_p$ and by d. It is also clear that a matroid is complete if and only if its simplification ${}^{\mathrm{si}}\Delta$ is the *uniform matroid* $U_{d,p}$ (see [16, p. 17]). Complete matroids also generalize *partition matroids* (see [16, p. 18]), which correspond to the case $p = d$. In Proposition 9 we proved that for $d = 2$ all matroids are complete. For $d > 2$ this is no longer true, as the following easy example shows.

Example 23 Let $n = 4$ and $\Delta = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$. It is clear that Δ is a matroid. The 1-skeleton of Δ is

$$\Delta^1 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\} = K_4,$$

so it is a complete 4-partite graph. This means that $a_1 = a_2 = a_3 = a_4 = 1$. Clearly this matroid is not complete, as the face $\{2,3,4\}$ is missing. The complete two-dimensional matroid corresponding to the above a_i's is $\Delta' = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.

Remark 24 Let Δ be a complete p-partite matroid. We have

(i) For any subset of vertices $M \subseteq [n]$ the restriction of Δ to M is also a complete matroid.
(ii) For any parallel class A_i, the link in Δ of any of its vertices $\mathrm{link}_\Delta A_i$ is also a complete matroid.

Theorem 25 *Let Δ be a complete, $(d-1)$-dimensional matroid with p-partition of the 1-skeleton $A_1,\ldots,A_p$. For $i = 1,\ldots,p$ we denote by $a_i = |A_i|$. Let Γ be the pure multi-complex on $\{y_1,\ldots,y_d\}$ with facets*

$$\mathcal{F}(\Gamma) = \{y_1^{(\sum_{i=l_0}^{l_1-1} a_i)-1} y_2^{(\sum_{i=l_1}^{l_2-1} a_i)-1} \cdots y_d^{(\sum_{i=l_{d-1}}^{p} a_i)-1}$$
$$: \forall\ 1 = l_0 < l_1 < l_2 < \ldots < l_{d-1} \le p\}.$$

Then we have that

$$\mathrm{h}^\Delta = f(\Gamma),$$

where h^Δ *is the h-vector of the algebra* $S/J(\Delta)$.

Before we start the proof, let us make a few easy remarks and introduce some notation. For each $i \in \{d,\ldots,p\}$, we denote the link of the i-parallel class in the restriction of Δ to the first i parallel classes by

$$L_i = \mathrm{link}_{\Delta_{1,\ldots,i}} A_i.$$

Notice that L_i is the $(d-2)$-skeleton of $\Delta_{1,\ldots,i-1}$. We will write $r(i)$ for the length of the h-vector of L_i. As the number of vertices of L_i is $a_1 + \ldots + a_{i-1}$ and its dimension is $d-2$, we have that

$$r(i) = 2 - d + \sum_{j=1}^{i-1} a_j.$$

Proof We will prove this theorem by simultaneous induction on d and $p - d$. The case $d = 1$ is trivially true and by Lemma 21 we know that the theorem is true for $p = d$.

For each i, denote by Γ^{L_i} the pure multi-complex corresponding to L_i which is given by the inductive hypothesis.

We assume now that $p > d > 1$ and that $\mathrm{h}^{\Delta_{1,\dots,p-1}} = f(\Gamma_{p-1})$, where

$$\Gamma_{p-1} = \langle y_1^{(\sum_{i=1}^{l_1-1} a_i)-1} y_2^{(\sum_{i=l_1}^{l_2-1} a_i)-1} \cdots y_d^{(\sum_{i=l_{d-2}}^{p} a_i)-1} \\ : \forall\ 1 < l_1 < l_2 < \ldots < l_{d-1} \leq p-1 \rangle.$$

We will use $\mathrm{h}^{\Delta_{1,\dots,p-1}}$ and h^{L_p} to compute h^{Δ} via the formula given in (2), namely

$$\mathrm{h}^{\Delta}(j) = \mathrm{h}^{\Delta_{1,\dots,p-1}}(j - a_p) + \sum_{k=0}^{a_p-1} \mathrm{h}^{L_p}(j-k), \tag{3}$$

for all $0 \leq j \leq 1 - d + \sum_{k=1}^{p} a_k$. To conclude we just need to check that the f-vectors of Γ, Γ_{p-1} and Γ^{L_p} satisfy the same formula. To this purpose, for any $j \in \mathbb{Z}$, let us denote $F_j = \{M \in \Gamma \ : \ \deg M = j\}$, $G_j = \{M \in \Gamma^{L_p} \ : \ \deg M = j\}$ and $H_j = \{M \in \Gamma_{p-1} \ : \ \deg M = j\}$. Let us furthermore partition F_j as

$$F_j = F_{j,\geq a_p} \bigcup \left(\bigcup_{k=0}^{a_p-1} F_{j,k} \right),$$

where $F_{j,\geq a_p} = \{M \in F_j \ : \ y_d^{a_p} \mid M\}$ and $F_{j,k} = \{M \in F_j \ : \ y_d^k \mid M \text{ and } y_d^{k+1} \nmid M\}$. It is easy to check the bijections of sets

$$G_{j-a_p} \xrightarrow{\cong} F_{j,\geq a_p}$$
$$M \mapsto M \cdot y_d^{a_p}$$

and, for all $k = 0, \dots, a_p - 1$,

$$H_{j-k} \xrightarrow{\cong} F_{j,k}$$
$$M \mapsto M \cdot y_d^k.$$

Therefore we get the formula

$$f_j(\Gamma) = f_{j-a_p}(\Gamma_{p-1}) + \sum_{k=0}^{a_p-1} f_{j-k}(\Gamma^{L_p}) \quad \forall\, j \in \mathbb{Z},$$

which, together with (3), yields the conclusion by induction.

Fixing two positive integers d and n and a vector $\mathbf{a} = (a_1, \ldots, a_p) \in (\mathbb{Z}_+)^p$ such that $p \geq d$, $a_1 + \ldots + a_p = n$, we introduce the class

$$\mathcal{M}(d, p, \mathbf{a}),$$

consisting of all $(d-1)$-dimensional matroids with p-partite 1-skeleton, where the partition sets A_i have cardinality a_i for all $i = 1, \ldots, p$. Note that the classes $\mathcal{M}(d, p, \mathbf{a})$ depend only on the set $\{a_1, \ldots, a_p\}$. That is, $\mathcal{M}(d, p, \mathbf{a})$ coincides with $\mathcal{M}(d, p, \mathbf{a}^\sigma)$ for any permutation σ of p elements ($\mathbf{a}^\sigma$ means $(a_{\sigma(1)}, \ldots, a_{\sigma(p)})$). Furthermore notice that, if $d = 2$ or $p = d$, $\mathcal{M}(d, p, \mathbf{a})$ consists of a single matroid, but this happens only in these cases. To see this, it is enough to consider for $t = 0, \ldots, d-2$, the following simplicial complexes

$$\Delta_t(d, p, \mathbf{a}) = \langle \{v_1, v_2, \ldots, v_t, v_{i_1}, \ldots, v_{i_{d-t}}\} \ : \ t < i_1 < \ldots < i_{d-t} \leq p \text{ where } v_i \in A_i \rangle. \tag{4}$$

It is easy to see that $\Delta_t(d, p, \mathbf{a})$ are elements of $\mathcal{M}(d, p, \mathbf{a})$. Moreover, one can show that, if $p > d$, they are not isomorphic pairwise—the easiest way to show this is to notice that they have a different number of facets. The matroid $\Delta_0(d, p, \mathbf{a})$ is just the complete p-partite matroid whose partition sets $A_1, \ldots, A_p$ satisfy $|A_i| = a_i$ for all $i = 1, \ldots, p$. Notice that, apart from the case $t = 0$, the matroid $\Delta_t(d, p, \mathbf{a})$ depends on the vector $\mathbf{a}$, not just on the set of its entries.

Remark 26 For every t, d, n and $\mathbf{a}$ as above the matroids $\Delta_t(d, p, \mathbf{a})$ are representable. To see this it is enough to notice that their simplification satisfies

$${}^{\mathrm{si}}\Delta_t(d, p, \mathbf{a}) = \{v_1, \ldots, v_t\} * U_{d-t, p-t} = \langle \{v_1, \ldots, v_t\} \cup F \ : \ F \in \mathcal{F}(U_{d-t, p-t}) \rangle,$$

where the v_i's are fixed vertices and $U_{d-t,p-t}$ is the uniform matroid of rank $d - t$ on $p - t$ vertices. Thus, a representation of $\Delta_t(d, p, \mathbf{a})$ is obtained by taking a_i copies of the ith column ($i = 1, \ldots, p$) in a representation of $\{v_1, \ldots, v_t\} * U_{d-t,p-t}$. Furthermore, it is easy to check that in order to obtain a representation over a field $\mathbb{F}$, its cardinality has to be "large enough".

As a first thing, we want to show that Stanley's conjecture holds true for all $\Delta_t(d, p, \mathbf{a})$.

Theorem 26 *Let $d, p \in \mathbb{N}$ be such that $p \geq d \geq 1$ and $\mathbf{a} = (a_1, \ldots, a_p) \in (\mathbb{Z}_+)^p$. Then $h^{\Delta_t(d,p,\mathbf{a})}$ is a pure O-sequence for all $t = 0, \ldots, d-2$.*

Proof The case $t = 0$ has already been treated in Theorem 25. So, we will use induction on t, assuming that $t \geq 1$. Let us write Δ_t for $\Delta_t(d, p, \mathbf{a})$. The restricted simplicial complex $\Delta_t' = (\Delta_t)_{2,3,\ldots,p}$ is just $\Delta_{t-1}(d-1, p-1, \tilde{\mathbf{a}})$, where $\tilde{\mathbf{a}} = (a_2, \ldots, a_p)$. Therefore, we know by induction that $h^{\Delta_t'}$ is a pure O-sequence. Set $A_1 = \{v_{1,1}, \ldots, v_{1,a_1}\}$ and $\Delta_t^i \subseteq \Delta_t$ the sub-complex induced by the vertices $A_2 \cup \ldots \cup A_p \cup \{v_{1,1}, \ldots, v_{1,i}\}$ for all $i = 1, \ldots, a_1$. We have $h^{\Delta_t^1} = h^{\Delta_t' * a_{1,1}} = h^{\Delta_t'}$.

Moreover, for all $i \geq 2$ and $k \in \mathbb{Z}$, we have

$$h^{\Delta_t^i}(k) = h^{\Delta_t^{i-1}}(k-1) + h^{\Delta_t'}(k).$$

Particularly, since $\Delta_t = \Delta_t^{a_1}$, we get

$$h^{\Delta_t}(k) = \sum_{j=0}^{a_1-1} h^{\Delta_t'}(k-j) \quad \forall\, k \in \mathbb{Z}. \tag{5}$$

We know that $h^{\Delta_t'}$ is a pure O-sequence, so let Γ' be the order ideal such that $f_{\Gamma'} = h^{\Delta_t'}$. Let us suppose that the set of maximal degree monomials of Γ' is

$$\mathcal{F}_{\Gamma'} = \{u_1, \dots, u_s \,:\, u_i \in \Bbbk[y_2, \dots, y_d] \text{ and } \deg(u_i) = a_2 + \dots + a_p - d + 1\}.$$

Let Γ be the pure order ideal with the following set of maximal monomials:

$$\mathcal{F}(\Gamma) = \{u_1 y_1^{a_1-1}, \dots, u_s y_1^{a_1-1}\}.$$

One can easily see that

$$f_\Gamma(k) = \sum_{j=0}^{a_1-1} f_{\Gamma'}(k-j), \qquad \forall\, k \in \mathbb{Z},$$

so (5) yields the conclusion.

Putting together Theorem 25 and the proof of Theorem 26 we obtain an explicit construction for an order ideal with the f-vector we are looking for. Namely, we obtain the following corollary.

Corollary 27 *If we denote by* $\Gamma_t(d, p, \mathbf{a})$ *the following order ideal:*

$$\langle y_1^{a_1-1} \cdots y_t^{a_t-1} y_{t+1}^{(\sum_{i=t+1}^{l_1-1} a_i)-1} \cdots y_d^{(\sum_{i=l_{d-t-1}}^{p} a_i)-1} \,:\, \forall\, t{+}1 < l_1 < l_2 < \dots < l_{d-t-1} \leq p\rangle,$$

we have that

$$h^{\Delta_t(d,p,\mathbf{a})} = f(\Gamma_t(d, p, \mathbf{a})).$$

In particular,

$$\operatorname{type}(S/J(\Delta_t(d, p, \mathbf{a}))) = \binom{p-t-1}{d-t-1}. \tag{6}$$

A consequence of Theorem 26 is the following interesting fact:

Corollary 28 *Let $d \geq 1$. For all $\mathbf{a} \in (\mathbb{Z}_+)^{d+1}$ and $\Delta \in \mathcal{M}(d, d+1, \mathbf{a})$, h^Δ is a pure O-sequence.*

Proof We want to show that Δ actually is $\Delta_t(d, p, \mathbf{a})$ for some $t = 0, \ldots, d-2$, so that Theorem 26 would give the thesis. Passing to ${}^{\mathrm{si}}\Delta$, a proof in the case $\mathbf{a} = \mathbf{1} = (1, 1, \ldots, 1) \in (\mathbb{Z})_+^{d+1}$ is enough. Notice that any $(d-1)$-dimensional pure simplicial complex on the vertex set $\{1, \ldots, d+1\}$ is a matroid. In order to have the complete graph on $d+1$ vertices as 1-skeleton, ${}^{\mathrm{si}}\Delta$ must have $m \geq 3$ facets. Moreover, if Δ is a $(d-1)$-simplicial complex on $d+1$ vertices with $m \geq 3$ facets, then it is easy to prove that Δ is isomorphic to the matroid $\Delta_{d-m+1}(d, d+1, \mathbf{1})$.

5 Minimal and Maximal *h*-Vectors

Among the matroids described in (4), two play a fundamental role:

$$\begin{aligned} \Delta_{\max}(d, p, \mathbf{a}) &= \Delta_0(d, p, \mathbf{a}), \\ \Delta_{\min}(d, p, \mathbf{a}) &= \Delta_{d-2}(d, p, \mathbf{a}^\sigma), \end{aligned} \tag{7}$$

where σ is a permutation of p elements such that $a_{\sigma(1)} \leq \ldots \leq a_{\sigma(p)}$. In this section we will see that, for any $\Delta \in \mathcal{M}(d, p, \mathbf{a})$, we have

$$h^{\Delta_{\min}(d,p,\mathbf{a})} \leq h^\Delta \leq h^{\Delta_{\max}(d,p,\mathbf{a})}$$

component-wise.

Given a matroid Δ with parallel classes $A_1, \ldots, A_p$, we need to consider in the following lemma the matroid $\Delta_{r\leftrightarrow s}$, where the parallel classes A_r and A_s are switched. Let us give a more rigorous definition: The matroid $\Delta_{r\leftrightarrow s}$ has as facets the subsets $F = \{v_{i_1}, \ldots, v_{i_d}\}$ of $[n]$ such that one of the following happens:

(i) $|F \cap (A_r \cup A_s)| \in \{0, 2\}$ and $F \in \mathcal{F}(\Delta)$.
(ii) $v_{i_j} \in A_r, F \cap A_s = \emptyset$ and there exists $v \in A_s$ such that $(F \setminus \{v_{i_j}\}) \cup \{v\} \in \mathcal{F}(\Delta)$.
(iii) $v_{i_k} \in A_s, F \cap A_r = \emptyset$ and there exists $u \in A_r$ such that $(F \setminus \{v_{i_k}\}) \cup \{u\} \in \mathcal{F}(\Delta)$.

Example 29 Let Δ be the rank 3, 5-partite matroid, with parallel classes $A_1, \ldots, A_5$ and facets:

$$A_1A_2A_5,\ A_1A_3A_5,\ A_1A_4A_5,\ A_2A_3A_5,\ A_2A_4A_5,\ A_3A_4A_5.$$

To compute $\Delta_{3\leftrightarrow 5}$ we just have to switch the indices 3 and 5 in the above list. Notice that, while ${}^{\mathrm{si}}\Delta$ and ${}^{\mathrm{si}}\Delta_{3\leftrightarrow 5}$ are isomorphic, Δ and $\Delta_{3\leftrightarrow 5}$ are isomorphic if and only if $|A_3| = |A_5|$. The h-vectors h^Δ and $h^{\Delta_{3\leftrightarrow 5}}$ are computed in Example 36

Lemma 30 *Let $p > d$ and $\mathbf{a} = (a_1, \dots, a_p) \in (\mathbb{Z}_+)^p$ be a vector such that $a_1 \leq \dots \leq a_p$. Let $\Delta \in \mathcal{M}(d, p, \mathbf{a})$ be a matroid such that A_p is a cone class for Δ (i.e. it corresponds to a cone point in ${}^{\mathrm{si}}\Delta$). Pick $\ell \in \{1, \dots, p-1\}$ such that A_ℓ is not a cone class for Δ (it exists because $p > d$). Then*

$$h^{\Delta_{\ell \leftrightarrow p}} \leq h^{\Delta}.$$

Proof Set $L_p = \mathrm{link}_\Delta A_p$ and $\overline{L}_\ell = \mathrm{link}_{\Delta_{\ell\leftrightarrow p}} A_\ell$. Furthermore, let $L'_p = L_p \setminus A_\ell$ and $\overline{L'}_\ell = \overline{L}_\ell \setminus A_p$. Notice that $L'_p \cong \overline{L'}_\ell$ and that $T = \mathrm{link}_{L_p} A_\ell \cong \mathrm{link}_{\overline{L'}_\ell} A_p = U$. When applying the recursive formula (2) twice for Δ, respectively $\Delta_{\ell\leftrightarrow p}$, we obtain for all $k \in \mathbb{Z}$:

$$h^{\Delta}(k) = h^{\Delta \setminus A_p \setminus A_\ell}(k - a_p - a_\ell) + \sum_{i=0}^{a_p - 1} h^{L'_p}(k - a_\ell - i) + \sum_{i=0}^{a_p-1} \sum_{j=1}^{a_\ell} h^T(k - a_\ell - i + j) \tag{8}$$

and

$$h^{\Delta_{\ell\leftrightarrow p}}(k) = h^{\Delta_{\ell\leftrightarrow p} \setminus A_\ell \setminus A_p}(k - a_\ell - a_p) + \sum_{i=0}^{a_\ell - 1} h^{\overline{L'}_\ell}(k - a_p - i) + \sum_{i=0}^{a_\ell - 1} \sum_{j=1}^{a_p} h^U(_{k - a_p - i + j}). \tag{9}$$

Clearly, as $\Delta \setminus A_p \setminus A_\ell = \Delta_{\ell\leftrightarrow p} \setminus A_\ell \setminus A_p$, we have equality for the frist summands in the two equations above. From the above discussion we have $h^{L'_p}(r) = h^{\overline{L'}_\ell}(r) =: h'_r$ and $h^T(r) = h^U(r) =: h''_r$ for all $r \in \mathbb{Z}$. Let us set

$$M_1 = \sum_{i=0}^{a_p - 1} h'_{k - a_\ell - i}, \quad M_2 = \sum_{i=0}^{a_p-1} \sum_{j=1}^{a_\ell} h''_{k - a_\ell - i + j}$$

and

$$N_1 = \sum_{i=0}^{a_\ell - 1} h'_{k - a_p - i}, \quad N_2 = \sum_{i=0}^{a_\ell-1} \sum_{j=1}^{a_p} h''_{k - a_p - i + j}.$$

Because $a_\ell \leq a_p$, obviously $N_1 \leq M_1$. Moreover we claim that $N_2 = M_2$. To see this, it is enough to notice that

$$h''_{k - a_p - i + j} = h''_{k - a_\ell - (a_p - j) + (a_\ell - i)}.$$

So, we get that (9) is less than or equal to (8).

We need one more technical lemma.

Lemma 31 *Let $A_1, \dots, A_p$ and $B_1, \dots, B_q$ be partitions of $\{1, \dots, n\}$ of cardinality $|A_i| = a_i$ and $|B_j| = b_j$, where $p \geq d$ and $q \geq d$, such that*

(i) $a_1 \leq \dots \leq a_p$.
(ii) $b_1 \leq \dots \leq b_q$.
(iii) $B_j = \bigcup_{k=1}^{r_j} A_{i_{j,k}}$.
(iv) $\bigcup_{i=1}^{d} A_i \subseteq \bigcup_{i=1}^{d} B_i$.

Set $\tilde{\mathbf{a}} = (a_1, a_2, \dots, a_{d-1}, a_d + \dots + a_p)$ and $\mathbf{b} = (b_1, \dots, b_q)$. If Γ is the only $(d-1)$-dimensional matroid in $\mathcal{M}(d, d, \tilde{\mathbf{a}})$, then

$$h^{\Gamma} \leq h^{\Delta} \quad \forall\, \Delta \in \mathcal{M}(d, q, \mathbf{b}).$$

Proof If $q = d$, then the assertion can be deduced by inspection on the h-vectors of Δ and Γ, described in Theorem 25. In fact, using this theorem, one can show a more general statement: Let $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}_+)^d$ and $\beta = (\beta_1, \dots, \beta_d) \in (\mathbb{Z}_+)^d$ be vectors such that $\alpha_1 \leq \dots \leq \alpha_d$, $\beta_1 \leq \dots \leq \beta_d$, $\sum_{i=1}^{d} \alpha_i = \sum_{i=1}^{d} \beta_i$ and $\alpha_i \leq \beta_i$ for all $i = 1, \dots, d-1$. Then, the h-vector of the only matroid in $\mathcal{M}(d, d, \alpha)$ is less than or equal to the h-vector of the only matroid in $\mathcal{M}(d, d, \beta)$. We leave the easy proof of this fact to the reader.

We will use induction on p. Notice that, as we always have $d \leq q \leq p$, the case $p = d$, implies $q = d$, so we are done by the above discussion.

If $p > d$ and $q > d$, then $i_{q,k} > d$ for all $k = 1, \dots, r_q$. Consider the sub-complex $\Gamma' \subseteq \Gamma$ induced by the vertices not in B_q and set $L = \mathrm{link}_{\Gamma} B_q$. As B_q is a subset of a parallel class in Γ, L is well defined and for all $k \in \mathbb{Z}$ we have

$$h^{\Gamma}(k) = h^{\Gamma'}(k - b_q) + \sum_{i=1}^{b_q} h^{L}(k - b_q + i).$$

In the same vein, we can consider the sub-complex $\Delta' \subseteq \Delta$ induced by all the vertices of Δ not in B_q and we set $K = \mathrm{link}_{\Delta} B_q$. Once again we have, for all $k \in \mathbb{Z}$,

$$h^{\Delta}(k) = h^{\Delta'}(k - b_q) + \sum_{i=1}^{b_q} h^{K}(k - b_q + i).$$

By Lemma 30 we can assume that B_q is not a cone class of Δ, so that Δ' has dimension $d-1$. Therefore by the induction on p we immediately get $h^{\Gamma'} \leq h^{\Delta'}$.

On the other hand, L is the unique $(d-1)$-partite $(d-2)$-dimensional matroid on the partition $(a_1, \dots, a_{d-1})$, whereas K is a $(d-2)$-dimensional matroid on a certain partition $C_1, \dots, C_r$. For sure $r \geq d-1$ and, provided that $|C_1| \leq \dots \leq |C_r|$, we get also that $a_i \leq b_i \leq |C_i|$ for all $i = 1, \dots, d-1$. Take a facet $\{v_{i_1}, \dots, v_{i_{d-1}}\}$ of K and suppose that each $v_{i_k} \in C_{i_k}$. Then the sub-complex $K' \subseteq K$ induced by the vertices of $C_{i_1} \cup \dots \cup C_{i_{d-1}}$ is a complete $(d-1)$-partite $(d-2)$-dimensional matroid. We can assume $i_1 < \dots < i_d$, so that $a_k \leq |C_{i_k}|$ for all $k = 1, \dots, d-1$. So we can choose a_k vertices in each one of the C_{i_k}s. It turns out that L is isomorphic to the sub-complex of K' induced by these vertices. Therefore L is isomorphic to an induced sub-complex of K, which implies $h^L \leq h^K$. So we can conclude.

Theorem 32 *If* $d \geq 1$ *and* $\mathbf{a} = (a_1, \ldots, a_p) \in (\mathbb{Z}_+)^p$ *with* $p \geq d$*, then*

$$h^{\Delta_{\min}(d,p,\mathbf{a})} \leq h^{\Delta} \quad \forall\ \Delta \in \mathcal{M}(d, p, \mathbf{a}).$$

Proof Since neither the matroid $\Delta_{\min}(d, p, \mathbf{a})$ nor the set $\mathcal{M}(d, p, \mathbf{a})$ depend on the order of $a_1, \ldots, a_p$, we are allowed to assume that $a_1 \leq \ldots \leq a_p$. We induct on p. If $p = d$, then the theorem is trivial, since $\mathcal{M}(d, d, \mathbf{a})$ consists of only one matroid, namely $\Delta_{\min}(d, d, \mathbf{a})$. If $p > d$, let us set $\Delta_{\min}(d, p, \mathbf{a})' \subseteq \Delta_{\min}(d, p, \mathbf{a})$ and $\Delta' \subseteq \Delta$ the sub-complexes induced by the vertices of $A_1 \cup \ldots \cup A_{p-1}$. Furthermore set $\mathbf{a}' = (a_1, \ldots, a_{p-1})$. We have that $\Delta_{\min}(d, p, \mathbf{a})' = \Delta_{\min}(d, p-1, \mathbf{a}')$. Exploiting Lemma 30, we can assume that $\dim \mathrm{d}' = d - 1$, so that $\mathrm{d}' \in \mathcal{M}(d, p-1, \mathbf{a}')$. So, by induction, we get

$$h^{\Delta_{\min}(d,p,\mathbf{a})'} \leq h^{\Delta'}.$$

Now set $L = \mathrm{link}_{\Delta_{\min}(d,p,\mathbf{a})} A_p$ and $K = \mathrm{link}_{\Delta} A_p$. It turns out that L is the unique $(d-1)$-partite $(d-2)$-dimensional matroid on the partition $(a_1, \ldots, a_{d-2}, a_{d-1} + \ldots + a_{p-1})$. Instead K will be a $(d-2)$-dimensional matroid on a certain partition $(b_1, \ldots, b_q)$. Such partitions satisfy the hypotheses of Lemma 31, so we get

$$h^L \leq h^K.$$

This yields the conclusion, since for all $k \in \mathbb{Z}$

$$\begin{aligned} h^{\Delta_{\min}(d,p,\mathbf{a})}(k) &= h^{\Delta_{\min}(d,p,\mathbf{a})'}(k - a_p) + \textstyle\sum_{i=1}^{a_p} h^L(k - a_p + i) \quad \text{and} \\ h^{\Delta}(k) &= h^{\Delta'}(k - a_p) + \textstyle\sum_{i=1}^{a_p} h^K(k - a_p + i). \end{aligned}$$

Remark 33 By Theorem 32 and (6) one has that, for all $\Delta \in \mathcal{M}(d, p, \mathbf{a})$,

$$\mathrm{type}(S/J(\Delta)) \geq p - d + 1. \tag{10}$$

This implies that, for any matroid Δ, S/I_Δ is Gorenstein if and only if I_Δ is a complete intersection if and only if $p = d$. So we recover [21, Theorem 4.4.10].

Equation (10) allows us to prove the following:

Theorem 34 *If* Δ *is a matroid on* $\{1, \ldots, n\}$ *such that* $\mathrm{type}(S/I_\Delta) \leq 3$*, then* $h(\Delta) = h(\Bbbk[\Delta])$ *is a pure O-sequence. Equivalently, if the h-vector of a matroid has the shape* $(1, h_1, \ldots, h_s)$ *with* $h_s \leq 3$*, then it is a pure O-sequence.*

Proof First of all we replace I_Δ for $J(\Delta)$. If $\mathrm{type}(S/J(\Delta)) \leq 2$, then Δ has to belong or to $\mathcal{M}(d, d, \mathbf{a})$ or to $\mathcal{M}(d, d+1, \mathbf{a})$ thanks to Eq. (10), so in these cases the statement follows at once by Lemma 21 and Corollary 28. Thus we have only to care of the case type$(S/J(\Delta)) = 3$. Again using Eq. (10), Lemma 21 and Corollary 28, we can assume that Δ is in $\mathcal{M}(d, d+2, \mathbf{a})$. The Cohen–Macaulay type of $S/J(\Delta)$ is the same as the one of $S/J({}^{\mathrm{si}}\Delta)$ by Proposition 15. But the dual of ${}^{\mathrm{si}}\Delta$ is a rank 2 matroid (possibly on less than $d+2$ vertices), so it is some complete p-partite graph G (see Proposition 9). Furthermore $h^{{}^{\mathrm{si}}\Delta} = h_G = (1, h_1, h_2)$, where $h_2 = e - v + 1$ (v denotes the number of vertices of G and e the number of its edges). But we want $h_2 = 3$, that is $e = v + 2$. It is easy to check that the only complete p-partite graph on v vertices with $v + 2$ edges is the complete graph on 4 vertices. This means that ${}^{\mathrm{si}}\Delta = \Delta_{d-2}(d, d+2, \mathbf{1})$, so $\Delta = \Delta_{d-2}(d, d+2, \mathbf{a})$. Now the conclusion follows from Theorem 26.

To show that $\Delta_{\max}(d, p, \mathbf{a})$ has maximal h-vector among the matroids $\Delta \in \mathcal{M}(d, p, \mathbf{a})$ is much easier.

Theorem 35 *If $d \geq 1$ and $\mathbf{a} = (a_1, \ldots, a_p) \in (\mathbb{Z}_+)^p$ with $p \geq d$, then*

$$h^{\Delta} \leq h^{\Delta_{\max}(d,p,\mathbf{a})} \quad \forall\ \Delta \in \mathcal{M}(d, p, \mathbf{a}).$$

Proof It is harmless to assume that $\Bbbk$ is infinite; otherwise we can tensor with its algebraic closure. Looking at the respective vertex covers, it is clear that $J(\Delta_{\max}(d, p, \mathbf{a})) \subseteq J(\Delta)$ for all $\Delta \in \mathcal{M}(d, p, \mathbf{a})$. Since both $S/J(\Delta_{\max}(d, p, \mathbf{a}))$ and $S/J(\Delta)$ are $(n-d)$-dimensional Cohen–Macaulay rings, we can choose $n-d$ linear forms which are both $S/J(\Delta_{\max}(d, p, \mathbf{a}))$- and $S/J(\Delta)$- regular (the generic ones have this property). Passing to the Artinian reduction, the inclusion is preserved, so we infer the desired inequality.

Example 36 We will compute now the upper and lower bounds for the h-vectors of the two matroids Δ and $\Delta_{3\leftrightarrow 5}$, of Example 29, when the cardinalities of the parallel classes are $\mathbf{a} = (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (2, 2, 3, 3, 4)$. The facets of $\Delta_{\max}(3, 5, \mathbf{a})$ and $\Delta_{\min}(3, 5, \mathbf{a})$, which produce the maximal (respectively minimal) h-vectors in $\mathcal{M}(3, 5, \mathbf{a})$, are

$$\begin{aligned}\mathcal{F}(\Delta_{\max}(3,5,\mathbf{a})) = \{&A_1A_2A_3,\ A_1A_2A_4,\ A_1A_2A_5,\ A_1A_3A_4,\ A_1A_3A_5,\\ &A_1A_4A_5,\ A_2A_3A_4,\ A_2A_3A_5,\ A_2A_4A_5,\ A_3A_4A_5\},\\ \mathcal{F}(\Delta_{\min}(3,5,\mathbf{a})) = \{&A_1A_2A_3,\ A_1A_2A_4,\ A_1A_2A_5,\ A_1A_3A_4,\ A_1A_3A_5,\ A_1A_4A_5\}.\end{aligned}$$

The corresponding h-vectors are:

$$\begin{aligned} h^{\Delta_{\max}(3,5,\mathbf{a})} &= (1,3,6,10,15,21,27,30,27,18,6)\\ h^{\Delta} &= (1,3,6,10,14,18,21,20,15,\ \ 9,3)\\ h^{\Delta_{3\leftrightarrow 5}} &= (1,3,6,\ \ 9,12,15,17,18,15,\ \ 9,3)\\ h^{\Delta_{\min}(3,5,\mathbf{a})} &= (1,3,5,\ \ 7,\ \ 9,11,13,14,13,\ \ 9,3). \end{aligned}$$

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