Chapter 9
Sensitivity Analysis of Articular Contact Mechanics

Abstract Asymptotic models of articular contact developed in the previous chapters assume, in particular, that the cartilage layers are of uniform thickness and are bonded to rigid substrates shaped like elliptic paraboloids. In this final chapter, treating the term "sensitivity" in a broad sense, we study the effects of deviation of the substrate's shape from the elliptic (Sect. 9.1) and of nonuniform thicknesses of the contacting incompressible layers (Sect. 9.2). It is shown that these effects in multibody dynamics simulations can be minimized if the geometric parameters in question (in particular, the layer thicknesses) are determined in a specific way to minimize the corresponding error in the force-displacement relationship.

9.1 Non-elliptical Contact of Thin Incompressible Viscoelastic Layers: Perturbation Solution

In this section, a more general three-dimensional unilateral contact problem for thin incompressible transversely isotropic viscoelastic layers bonded to rigid substrates, whose shapes are close to those of elliptic paraboloids, is considered and approximately solved by the perturbation technique.

9.1.1 Formulation of the Contact Problem

Consider the frictionless unilateral contact between two thin linear incompressible transversely isotropic viscoelastic layers firmly attached to rigid substrates. Introducing the Cartesian coordinate system (y_1, y_2, z), we write the equations of the layer surfaces $(n = 1, 2)$ in the form $z = (-1)^n \varphi_\varepsilon^{(n)}(\mathbf{y})$, where $\mathbf{y} = (y_1, y_2)$. In the undeformed state, the two layer/substrate systems occupy convex domains $z \leq -\varphi_\varepsilon^{(1)}(\mathbf{y})$ and $z \geq \varphi_\varepsilon^{(2)}(\mathbf{y})$ in contact with the plane $z = 0$ at a single point chosen as the coordinate origin. Let us assume that

$$\varphi_\varepsilon^{(n)}(\mathbf{y}) = \varphi_0^{(n)}(\mathbf{y}) + \varepsilon\phi_n(\mathbf{y}), \tag{9.1}$$

© Springer International Publishing Switzerland 2015
I. Argatov and G. Mishuris, *Contact Mechanics of Articular Cartilage Layers*,
Advanced Structured Materials 50, DOI 10.1007/978-3-319-20083-5_9

where $\varphi_0^{(n)}(\mathbf{y})$ is an elliptic paraboloid, ε is a small positive dimensionless parameter, and the function $\varepsilon\phi_n(\mathbf{y})$ describes a small deviation of the nth substrate surface from the paraboloid shape ($n = 1, 2$).

We denote the normal approach of the substrates by $\delta_\varepsilon(t)$. The linearized unilateral contact condition that the surface points of the viscoelastic layers do not penetrate one into another can then be written as follows:

$$\delta_\varepsilon(t) - w_0^{(1)}(t, \mathbf{y}) - w_0^{(2)}(t, \mathbf{y}) \leq \varphi_\varepsilon^{(1)}(\mathbf{y}) + \varphi_\varepsilon^{(2)}(\mathbf{y}). \tag{9.2}$$

Here, $w_0^{(n)}(t, \mathbf{y})$ is the local indentation (i.e., the normal displacement of the surface points) of the nth layer ($n = 1, 2$).

According to the perturbation analysis performed in Sect. 2.5 (see, in particular, formula (2.152)), the leading-order asymptotic solution for the local indentation of an incompressible viscoelastic layer of thickness h_n is given by

$$w_0^{(n)}(t, \mathbf{y}) = -\frac{h_n^3}{3} \int\limits_{0^-}^{t} J'^{(n)}(t-\tau)\Delta_y \frac{\partial p_\varepsilon}{\partial \tau}(\tau, \mathbf{y})\, d\tau. \tag{9.3}$$

Here, $J'^{(n)}(t)$ is the out-of-plane shear creep compliance of the nth layer ($n = 1, 2$), $p_\varepsilon(t, \mathbf{y})$ is the contact pressure, and $\Delta_y = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ is the Laplace operator.

The equality in relation (9.2) determines the contact region $\omega_\varepsilon(t)$. In other words, the following equation holds within the contact area

$$w_0^{(1)}(t, \mathbf{y}) + w_0^{(2)}(t, \mathbf{y}) = \delta_\varepsilon(t) - \varphi_\varepsilon(\mathbf{y}), \quad \mathbf{y} \in \omega_\varepsilon(t), \tag{9.4}$$

where we have introduced the notation $\varphi_\varepsilon(\mathbf{y}) = \varphi_\varepsilon^{(1)}(\mathbf{y}) + \varphi_\varepsilon^{(2)}(\mathbf{y})$.

According to (9.1), we have

$$\varphi_\varepsilon(\mathbf{y}) = \varphi_0(\mathbf{y}) + \varepsilon\phi(\mathbf{y}), \tag{9.5}$$

where $\varphi_0(\mathbf{y}) = \varphi_0^{(1)}(\mathbf{y}) + \varphi_0^{(2)}(\mathbf{y})$ and $\phi(\mathbf{y}) = \phi_1(\mathbf{y}) + \phi_2(\mathbf{y})$. The function $\varepsilon\phi(\mathbf{y})$ will be called the gap function variation.

Without any loss of generality we may assume that

$$\varphi_0(\mathbf{y}) = \frac{y_1^2}{2R_1} + \frac{y_2^2}{2R_2}, \tag{9.6}$$

where the parameters R_1 and R_2 are positive and can be related to the coefficients of the paraboloids $\varphi_0^{(1)}(\mathbf{y})$ and $\varphi_0^{(2)}(\mathbf{y})$ by known formulas (see Sect. 2.1.1).

Substituting the expressions for displacements $w_0^{(1)}(t, \mathbf{y})$ and $w_0^{(2)}(t, \mathbf{y})$ given by formula (9.3) into Eq. (9.4), we obtain the contact condition in the following form:

$$-\sum_{n=1}^{2} \frac{h_n^3}{3} \int_{0^-}^{t} J'^{(n)}(t-\tau)\Delta_y \frac{\partial p_\varepsilon}{\partial \tau}(\tau, \mathbf{y})\, d\tau = \delta_\varepsilon(t) - \varphi_\varepsilon(\mathbf{y})\mathscr{H}(t). \tag{9.7}$$

Here, we assume that $\mathbf{y} \in \omega_\varepsilon(t)$, and $\mathscr{H}(t)$ is Heaviside's function introduced in a standard way, namely $\mathscr{H}(t) = 0$ for $t < 0$ and $\mathscr{H}(t) = 1$ for $t \geq 0$.

Let $G_0'^{(n)} = 1/J'^{(n)}(0^+)$ be the instantaneous out-of-plane shear elastic modulus of the nth layer. Then, the normalized creep function $\Phi'^{(n)}(t)$ is introduced by

$$J'^{(n)}(t) = \frac{1}{G_0'^{(n)}} \Phi'^{(n)}(t). \tag{9.8}$$

Let us rewrite Eq. (9.7) in the form corresponding to one layer by introducing the compound creep function, $\Phi_\beta(t)$, and the equivalent instantaneous shear elastic modulus, G_0', as follows:

$$\Phi_\beta(t) = \beta_1 \Phi'^{(1)}(t) + \beta_2 \Phi'^{(2)}(t), \tag{9.9}$$

$$G_0' = \frac{(h_1 + h_2)^3 G_0'^{(1)} G_0'^{(2)}}{h_1^3 G_0'^{(2)} + h_2^3 G_0'^{(1)}}, \tag{9.10}$$

$$\beta_1 = \frac{h_1^3 G_0'^{(2)}}{h_1^3 G_0'^{(2)} + h_2^3 G_0'^{(1)}}, \quad \beta_2 = \frac{h_2^3 G_0'^{(1)}}{h_1^3 G_0'^{(2)} + h_2^3 G_0'^{(1)}}. \tag{9.11}$$

Moreover, let us introduce an auxiliary notation

$$m = \frac{3G_0'}{h^3}, \tag{9.12}$$

where $h = h_1 + h_2$ is the joint thickness. Recall that formula (9.10) determines the equivalent modulus in such a way that $\beta_1 + \beta_2 = 1$ and thus, $\Phi_\beta(0) = 1$.

Thus, taking into account (9.8)–(9.12), we rewrite Eq. (9.7) as

$$\int_{0^-}^{t} \Phi_\beta(t-\tau)\Delta_y \frac{\partial p_\varepsilon}{\partial \tau}(\tau, \mathbf{y})\, d\tau = m\big(\varphi_\varepsilon(\mathbf{y})\mathscr{H}(t) - \delta_\varepsilon(t)\big). \tag{9.13}$$

Equation (9.13) will be used to find the contact pressure density $p_\varepsilon(t, \mathbf{y})$. The contour $\Gamma_\varepsilon(t)$ of the contact area $\omega_\varepsilon(t)$ is determined from the condition that the contact pressure is positive and vanishes at the contour of the contact area:

$$p_\varepsilon(t, \mathbf{y}) > 0, \quad \mathbf{y} \in \omega_\varepsilon(t), \tag{9.14}$$

$$p_\varepsilon(t, \mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_\varepsilon(t). \tag{9.15}$$

In the case of the contact problem for an incompressible layer (see, in particular, Sect. 2.7.3), we additionally assume a smooth transition of the surface normal stresses from the contact region $\mathbf{y} \in \omega_\varepsilon(t)$ to the outside region $\mathbf{y} \notin \omega_\varepsilon(t)$. Hence, we impose the following zero-pressure-gradient boundary condition [9, 10, 13, 17]:

$$\frac{\partial p_\varepsilon}{\partial n}(t, \mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_\varepsilon(t). \tag{9.16}$$

Here, $\partial/\partial n$ is the normal derivative directed outward from $\omega_\varepsilon(t)$.

We assume that the density $p_\varepsilon(t, \mathbf{y})$ is defined on the entire plane such that

$$p_\varepsilon(t, \mathbf{y}) = 0, \quad \mathbf{y} \notin \omega_\varepsilon(t). \tag{9.17}$$

From the physical point of view, the contact pressure between the smooth surfaces should satisfy the regularity condition, i.e., in the case (9.5), the function $p_\varepsilon(t, \mathbf{y})$ is assumed to be analytical in the domain $\omega_\varepsilon(t)$.

The equilibrium equation for the whole system is

$$\iint\limits_{\omega_\varepsilon(t)} p_\varepsilon(t, \mathbf{y})\, d\mathbf{y} = F(t), \tag{9.18}$$

where $F(t)$ denotes the external load, which is assumed to be known a priori.

For non-decreasing loads, when $dF(t)/dt \geq 0$, the contact zone should increase. Thus, we assume that the following monotonicity condition holds:

$$\omega_\varepsilon(t_1) \subset \omega_\varepsilon(t_2), \quad t_1 \leq t_2. \tag{9.19}$$

Following Argatov and Mishuris [8], we construct an asymptotic solution for the three-dimensional contact problem formulated by Eq. (9.13), under the monotonicity condition (9.19). We will first consider the problem in its general formulation transforming it to a set of equations more suitable for further analysis.

9.1.2 Equation for the Contact Approach

Integrating Eq. (9.13) over the contact domain $\omega_\varepsilon(t)$, we find

$$\iint\limits_{\omega_\varepsilon(t)} \int\limits_{0^-}^{t} \Phi_\beta(t-\tau) \Delta_y \frac{\partial p_\varepsilon}{\partial \tau}(\tau, \mathbf{y})\, d\tau d\mathbf{y} = m \iint\limits_{\omega_\varepsilon(t)} \big(\varphi_\varepsilon(\mathbf{y})\mathscr{H}(t) - \delta_\varepsilon(t)\big)\, d\mathbf{y}. \tag{9.20}$$

In light of (9.17) and (9.19), we have $\omega_\varepsilon(\tau) \subset \omega_\varepsilon(t)$ for $\tau \in (0, t)$, as well as $p_\varepsilon(\tau, \mathbf{y}) \equiv 0$ for $\mathbf{y} \notin \omega_\varepsilon(\tau)$. Therefore, the integral on the left-hand side of (9.20), which is denoted by $\mathscr{I}(t)$, can be transformed into

$$\mathscr{I}(t) = \iint\limits_{\omega_\varepsilon(t)} \int\limits_{0^-}^{t} \Phi_\beta(t-\tau)\Delta_y \frac{\partial p_\varepsilon}{\partial \tau}(\tau, \mathbf{y})\, d\tau d\mathbf{y}$$
$$= \int\limits_{0^-}^{t} \Phi_\beta(t-\tau) \iint\limits_{\omega_\varepsilon(t)} \Delta_y \frac{\partial p_\varepsilon}{\partial \tau}(\tau, \mathbf{y})\, d\mathbf{y} d\tau$$
$$= \int\limits_{0^-}^{t} \Phi_\beta(t-\tau) \frac{\partial}{\partial \tau} \iint\limits_{\omega_\varepsilon(t)} \Delta_y p_\varepsilon(\tau, \mathbf{y})\, d\mathbf{y} d\tau. \quad (9.21)$$

We note that, as a consequence of (9.15)–(9.17), the density $p_\varepsilon(t, \mathbf{y})$ is a smooth function of the variables y_1 and y_2 on the entire plane.

Further, by employing the second Green's formula

$$\iint\limits_{\omega} \big(u(\mathbf{y})\Delta v(\mathbf{y}) - v(\mathbf{y})\Delta u(\mathbf{y})\big)\, d\mathbf{y} = \int\limits_{\Gamma} \left(u(\mathbf{y})\frac{\partial v}{\partial n}(\mathbf{y}) - v(\mathbf{y})\frac{\partial u}{\partial n}(\mathbf{y})\right) ds, \quad (9.22)$$

where ds is the element of the arc length, we obtain

$$\iint\limits_{\omega_\varepsilon(t)} \Delta_y p_\varepsilon(\tau, \mathbf{y})\, d\mathbf{y} = \int\limits_{\Gamma_\varepsilon(t)} \frac{\partial p_\varepsilon}{\partial n}(\tau, \mathbf{y})\, ds. \quad (9.23)$$

In light of the boundary condition (9.16) and the monotonicity condition (9.19), the right-hand side of Eq. (9.23) vanishes for $t > 0$ and, therefore, Eq. (9.20) reduces to

$$\iint\limits_{\omega_\varepsilon(t)} \big(\varphi_\varepsilon(\mathbf{y})\mathscr{H}(t) - \delta_\varepsilon(t)\big)\, d\mathbf{y} = 0.$$

From here it immediately follows that

$$\delta_\varepsilon(t) = \frac{\mathscr{H}(t)}{A_\varepsilon(t)} \iint\limits_{\omega_\varepsilon(t)} \varphi_\varepsilon(\mathbf{y})\, d\mathbf{y}, \quad (9.24)$$

where $A_\varepsilon(t)$ is the area of $\omega_\varepsilon(t)$ given by the integral

$$A_\varepsilon(t) = \iint\limits_{\omega_\varepsilon(t)} d\mathbf{y}. \quad (9.25)$$

Equation (9.24) connects the unknown contact approach $\delta_\varepsilon(t)$ with some integral characteristics of the contact domain $\omega_\varepsilon(t)$.

9.1.3 Equation for the Integral Characteristics the Contact Area

Substituting the functions $u(\mathbf{y}) = p_\varepsilon(\tau, \mathbf{y})$ and $v(\mathbf{y}) = (y_1^2 + y_2^2)/4$ into Green's formula (9.22) for the domain $\omega_\varepsilon(t)$, assuming that $\tau < t$, and taking into account the boundary conditions (9.15), (9.16) and the monotonicity condition (9.19), we obtain the relation

$$\frac{1}{4}\iint\limits_{\omega_\varepsilon(t)} |\mathbf{y}|^2 \Delta p_\varepsilon(\tau, \mathbf{y})\, d\mathbf{y} = \iint\limits_{\omega_\varepsilon(\tau)} p_\varepsilon(\tau, \mathbf{y})\, d\mathbf{y}. \tag{9.26}$$

Using formula (9.26), we can evaluate the contact load (9.18). Indeed, by multiplying both sides of (9.13) by $(y_1^2 + y_2^2)/4$ and integrating the obtained equation over the contact domain $\omega_\varepsilon(t)$, we obtain

$$\int\limits_{0^-}^{t} \Phi_\beta(t-\tau)\frac{\partial}{\partial\tau}\iint\limits_{\omega_\varepsilon(\tau)} p_\varepsilon(\tau, \mathbf{y})\, d\mathbf{y}d\tau = \frac{m}{4}\iint\limits_{\omega_\varepsilon(t)} |\mathbf{y}|^2\big(\mathscr{H}(t)\varphi_\varepsilon(\mathbf{y}) - \delta_\varepsilon(t)\big)\, d\mathbf{y}. \tag{9.27}$$

Taking into account the notation (9.18) for the contact force, we rewrite (9.27) as

$$\int\limits_{0^-}^{t} \Phi_\beta(t-\tau)\dot{F}(\tau)\, d\tau = \frac{m}{4}\mathscr{H}(t)\iint\limits_{\omega_\varepsilon(t)} |\mathbf{y}|^2\varphi_\varepsilon(\mathbf{y})\, d\mathbf{y} - \delta_\varepsilon(t)\frac{m}{4}\iint\limits_{\omega_\varepsilon(t)} |\mathbf{y}|^2 d\mathbf{y}, \tag{9.28}$$

where the dot denotes the differentiation with respect to time, i.e., $\dot{F}(t) = dF(t)/dt$.

Then, excluding the quantity $\delta_\varepsilon(t)$ from (9.28) by means of Eq. (9.24), we arrive at the following equation:

$$\int\limits_{0^-}^{t} \Phi_\beta(t-\tau)\dot{F}(\tau)\, d\tau = \frac{m}{4}\mathscr{H}(t)\iint\limits_{\omega_\varepsilon(t)} \left(|\mathbf{y}|^2 - \frac{I_\varepsilon(t)}{A_\varepsilon(t)}\right)\varphi_\varepsilon(\mathbf{y})\, d\mathbf{y}. \tag{9.29}$$

Here, $I_\varepsilon(t)$ is the polar moment of inertia of $\omega_\varepsilon(t)$ given by the integral

$$I_\varepsilon(t) = \iint\limits_{\omega_\varepsilon(t)} |\mathbf{y}|^2 d\mathbf{y}. \tag{9.30}$$

Equation (9.29) connects the known contact force $F(t)$ with some integral characteristics of the unknown contact area $\omega_\varepsilon(t)$.

9.1.4 Equation for the Contact Pressure

Let us rewrite Eq. (9.13) in the form

$$\Delta_y P_\varepsilon(t,\mathbf{y}) = m\big(\varphi_\varepsilon(\mathbf{y}) - \delta_\varepsilon(t)\big), \quad \mathbf{y}\in\omega_\varepsilon(t), \tag{9.31}$$

where we have introduced the notation

$$P_\varepsilon(t,\mathbf{y}) = \int\limits_{0^-}^{t} \Phi_\beta(t-\tau)\frac{\partial p_\varepsilon}{\partial\tau}(\tau,\mathbf{y})\,d\tau. \tag{9.32}$$

By denoting the integral operator on the right-hand side of the previous equation by $\mathscr{K}$, we have

$$\mathscr{K} y(\tau) = \int\limits_{0^-}^{t} \Phi_\beta(t-\tau)\dot{y}(\tau)\,d\tau, \tag{9.33}$$

so that formula (9.32) can be represented as

$$P_\varepsilon(t,\mathbf{y}) = \mathscr{K} p_\varepsilon(\tau,\mathbf{y}). \tag{9.34}$$

The inverse operator to $\mathscr{K}$ denoted by $\mathscr{K}^{-1}$ is defined by the formula

$$\mathscr{K}^{-1} Y(\tau) = \int\limits_{0^-}^{t} \Psi_\beta(t-\tau)\dot{Y}(\tau)\,d\tau, \tag{9.35}$$

where $\Psi_\beta(t)$ is the compound relaxation function defined by its Laplace transform

$$\tilde{\Psi}_\beta(s) = \frac{1}{s^2\tilde{\Phi}_\beta(s)}. \tag{9.36}$$

Note that since

$$\tilde{\Phi}_\beta(s) = \beta_1\tilde{\Phi}'^{(1)}(s) + \beta_2\tilde{\Phi}'^{(2)}(s),$$

and

$$\tilde{\Phi}'^{(n)}(s) = \frac{1}{s^2\tilde{\Psi}'^{(n)}(s)},$$

where $\tilde{\Psi}'^{(n)}(s)$ is the Laplace transform of the relaxation function in out-of-plane shear for the nth layer, formula (9.36) can be reduced to the following:

$$\tilde{\Psi}_\beta(s) = \frac{\tilde{\Psi}'^{(1)}(s)\tilde{\Psi}'^{(2)}(s)}{\beta_1\tilde{\Psi}'^{(2)}(s) + \beta_2\tilde{\Psi}'^{(1)}(s)}. \tag{9.37}$$

Recall also that the coefficients β_1 and β_2 are introduced by formulas (9.11) in such a way that $\Psi_\beta(0) = 1$.

As a result of the boundary conditions (9.15) and (9.16), the function $P_\varepsilon(t, \mathbf{y})$ must satisfy the following boundary conditions:

$$P_\varepsilon(t, \mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_\varepsilon(t), \tag{9.38}$$

$$\frac{\partial P_\varepsilon}{\partial n}(t, \mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_\varepsilon(t). \tag{9.39}$$

Thus, in the case of monotonically increasing contact area $\omega_\varepsilon(t)$, the problem (9.31), (9.38), (9.39) allows us to determine the domain $\omega_\varepsilon(t)$ based on the positivity condition for the contact pressure (9.14). Then, from Eq. (9.24) we can determine the contact approach $\delta_\varepsilon(t)$. Finally, by applying the inverse operator (9.35), we obtain a complete solution to the problem.

9.1.5 Limiting Case Problem: Elliptical Contact Area

We now consider the problem for the limiting case $\varepsilon = 0$, when the function $\varphi_0(\mathbf{y})$ represents an elliptic paraboloid. In a special case of the integral operator $\mathscr{K}$, the solution to this problem has previously been presented in [6]. Here we adopt it in the necessary form for the further asymptotic analysis, in order to construct a more general solution to the problem with the slightly perturbed boundary of an arbitrary shape.

In this case the right-hand side of (9.31) takes the form $m\big(\varphi_0(\mathbf{y}) - \delta_0(t)\big)$. This suggests that we assume the domain $\omega_0(t)$ to be elliptical and so we set

$$P_0(t, \mathbf{y}) = Q_0(t)\left(1 - \frac{y_1^2}{a(t)^2} - \frac{y_2^2}{b(t)^2}\right)^2. \tag{9.40}$$

In other words, the contour $\Gamma_0(t)$ is an ellipse with the semi-axises $a(t)$ and $b(t)$. It is simple to verify that the function $P_0(t, \mathbf{y})$ satisfies the boundary conditions (9.38) and (9.39) exactly.

Substituting (9.40) into Eq. (9.31), we obtain after some algebra the following system of algebraic equations:

$$\delta_0(t) = \frac{4Q_0(t)}{m}\left(\frac{1}{a(t)^2} + \frac{1}{b(t)^2}\right), \tag{9.41}$$

$$\frac{1}{R_1} = \frac{8Q_0(t)}{ma(t)^2}\left(\frac{3}{a(t)^2} + \frac{1}{b(t)^2}\right), \quad \frac{1}{R_2} = \frac{8Q_0(t)}{mb(t)^2}\left(\frac{1}{a(t)^2} + \frac{3}{b(t)^2}\right). \tag{9.42}$$

The form of the ellipse $\Gamma_0(t)$ can be characterized by its aspect ratio s defined as

$$s = \frac{b(t)}{a(t)}, \tag{9.43}$$

and from (9.42), it immediately follows that s is constant with time and is determined as a positive root of the equation

$$\frac{R_2}{R_1} = \frac{s^2(3s^2 + 1)}{3 + s^2}. \tag{9.44}$$

In turn, Eq. (9.44) can be reduced to a quadratic equation for s^2, so that

$$s^2 = \sqrt{\left(\frac{R_1 - R_2}{6R_1}\right)^2 + \frac{R_2}{R_1}} - \frac{(R_1 - R_2)}{6R_1}. \tag{9.45}$$

Recall that, along with Eqs. (9.41) and (9.42), we have Eqs. (9.24) and (9.29), which connect the contact approach $\delta_\varepsilon(t)$ and the known contact force $F(t)$ with some integral characteristics of the contact domain $\omega_\varepsilon(t)$.

By taking into account (9.45), we transform Eq. (9.24) into

$$\delta_0(t) = \frac{1}{8}\left(\frac{1}{R_1} + \frac{s^2}{R_2}\right)a(t)^2. \tag{9.46}$$

Then, by excluding the quantity $\delta_0(t)$ from Eqs. (9.41) and (9.46), we obtain

$$Q_0(t) = \frac{m}{32}\frac{s^2}{(s^2 + 1)}\left(\frac{1}{R_1} + \frac{s^2}{R_2}\right)a(t)^4. \tag{9.47}$$

By application of the same method, Eq. (9.29) becomes

$$\mathscr{K}F(\tau) = \frac{\pi m}{384}\left(\frac{3s - s^3}{R_1} + \frac{3s^5 - s^3}{R_2}\right)a(t)^6. \tag{9.48}$$

This allows us to determine the major semi-axis $a(t)$ of the contact area $\omega_0(t)$ as a function of time t in the form

$$a(t) = \left[\frac{\pi m}{384}\left(\frac{3s - s^3}{R_1} + \frac{3s^5 - s^3}{R_2}\right)\right]^{-1/6}\bigl(\mathscr{K}F(\tau)\bigr)^{1/6}. \tag{9.49}$$

As a consequence of (9.49), formulas (9.46) and (9.47) determine the quantities $\delta_0(t)$ and $Q_0(t)$, respectively.

We now turn to evaluating the contact pressure in the case of elliptical contact. In light of (9.33) and (9.40), we obtain the following operator equation for the contact pressure density $p_0(t, \mathbf{y})$:

$$\mathscr{K} p_0(\tau, \mathbf{y}) = Q_0(t)\left(1 - \frac{y_1^2}{a(t)^2} - \frac{y_2^2}{b(t)^2}\right)^2, \quad \mathbf{y} \in \omega_0(t). \tag{9.50}$$

By inverting Eq. (9.50) with the help of (9.35), we obtain

$$p_0(t, \mathbf{y}) = \mathscr{K}^{-1}\left\{ Q_0(\tau)\left(1 - \frac{y_1^2}{a(\tau)^2} - \frac{y_2^2}{b(\tau)^2}\right)^2 \mathscr{H}\left(1 - \frac{y_1^2}{a(\tau)^2} - \frac{y_2^2}{b(\tau)^2}\right)\right\}, \tag{9.51}$$

where the Heaviside factor indicates the contact domain $\omega_0(\tau)$.

Following the notation (9.35), Eq. (9.51) can be transformed into

$$p_0(t, \mathbf{y}) = \int_{0^-}^{t} \Psi_\beta(t - \tau) Q_0(\tau)\left(1 - \frac{y_1^2}{a(\tau)^2} - \frac{y_2^2}{b(\tau)^2}\right)_+^2 d\tau. \tag{9.52}$$

Here the positive part function $(x)_+ = (x + |x|)/2$ is used as an indicator of the current contact area $\omega_0(\tau)$.

9.1.6 Slightly Perturbed Elliptical Contact Area

We now consider the gap function $\varphi_\varepsilon(\mathbf{y})$, given in a general form (9.5) with small $\varepsilon > 0$. The solution corresponding to the limiting case $\varepsilon = 0$ was defined above and is denoted by $p_0(t, \mathbf{y})$ and $\delta_0(t)$, where the contact domain $\omega_0(t)$ is bounded by an ellipse $\Gamma_0(t)$. Then, the function $P_0(t, \mathbf{y}) = \mathscr{K} p_0(\tau, \mathbf{y})$ satisfies the problem

$$\Delta_y p_0(t, \mathbf{y}) = m\big(\varphi_0(\mathbf{y}) - \delta_0(t)\big), \quad \mathbf{y} \in \omega_0(t), \tag{9.53}$$

$$p_0(t, \mathbf{y}) = 0, \quad \frac{\partial p_0}{\partial n}(t, \mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_0(t). \tag{9.54}$$

Recall also that in accordance with (9.24), the contact approach $\delta_0(t)$ is given by

$$\delta_0(t) = \frac{1}{A_0(t)} \iint_{\omega_0(t)} \varphi_0(\mathbf{y})\, d\mathbf{y}, \tag{9.55}$$

where $A_0(t)$ is the area of $\omega_0(t)$. Moreover, in light of (9.29), we have

$$\mathscr{K} F(\tau) = \frac{m}{4} \iint_{\omega_0(t)} B_0(t, \mathbf{y})\varphi_0(\mathbf{y})\, d\mathbf{y}, \tag{9.56}$$

where, with $I_0(t)$ being the polar moment of inertia of $\omega_0(t)$,

$$B_0(t, \mathbf{y}) = |\mathbf{y}|^2 - \frac{I_0(t)}{A_0(t)}. \tag{9.57}$$

We represent the solution to the perturbed auxiliary contact problem (9.31), (9.38), (9.39) as

$$P_\varepsilon(t, \mathbf{y}) = P_0(t, \mathbf{y}) + \varepsilon P_1(t, \mathbf{y}) + O(\varepsilon^2), \tag{9.58}$$

$$\delta_\varepsilon(t) = \delta_0(t) + \varepsilon\delta_1(t) + O(\varepsilon^2), \tag{9.59}$$

and recall that the contact load $F(t)$ is assumed to be specified, while the contact approach $\delta_\varepsilon(t)$ is unknown a priori.

By substituting (9.58) and (9.59) into Eq. (9.31), we arrive at the equation

$$\varDelta_y P_1(t, \mathbf{y}) = m\big(\varphi(\mathbf{y}) - \delta_1(t)\big), \quad \mathbf{y} \in \omega_0(t). \tag{9.60}$$

Let us assume that the unknown boundary $\varGamma_\varepsilon(t)$ of the contact area $\omega_\varepsilon(t)$ (see Fig. 9.1) is described by the equation

$$n = h_\varepsilon(t, \sigma), \quad s \in \varGamma_0(t). \tag{9.61}$$

Here, σ is the arc length along $\varGamma_0(t)$, and n is the distance (taking the sign into account) measured along the outward (with respect to the domain $\omega_0(t)$) normal to the curve $\varGamma_0(t)$. The function $h_\varepsilon(t, \sigma)$ describes the variation of the contact area and should be determined by considering the boundary conditions for Eq. (9.60).

In light of (9.58) and (9.59), we set

$$h_\varepsilon(t, \sigma) = \varepsilon h(t, \sigma), \tag{9.62}$$

where the function $h(t, \sigma)$ is assumed to be independent of ε.

Applying the perturbation technique (see, for example, [20]), we have

$$P_\varepsilon\big|_\varGamma = P_\varepsilon\big|_{\varGamma_0} + h_\varepsilon \frac{\partial P_\varepsilon}{\partial n}\Big|_{\varGamma_0} + O(\varepsilon^2). \tag{9.63}$$

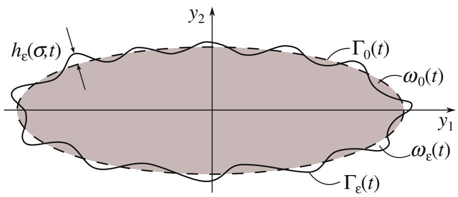

Fig. 9.1 Schematic representation of the contact domain $\omega_\varepsilon(t)$ with the boundary $\varGamma_\varepsilon(t)$, and the limit domain $\omega_0(t)$ with the boundary $\varGamma_0(t)$

Let $\mathbf{n}_\varepsilon$ be the unit outward normal vector to the curve $\Gamma_\varepsilon(t)$. Then, the following formula holds:

$$\mathbf{n}_\varepsilon = \frac{\big(1-\kappa(t,\sigma)h_\varepsilon(t,\sigma)\big)\mathbf{n}_0 - h'_\varepsilon(t,\sigma)\mathbf{t}_0}{\sqrt{\big(1-\kappa(t,\sigma)h_\varepsilon(t,\sigma)\big)^2 + h'_\varepsilon(t,\sigma)^2}}. \tag{9.64}$$

Here, $\mathbf{t}_0$ and $\mathbf{n}_0$ are the unit tangential and outward normal vectors to the curve $\Gamma_0(t)$, $\kappa(t,\sigma)$ is the curvature of $\Gamma_0(t)$, and the prime denotes the derivative with respect to the arc length σ.

Taking into account formula (9.64), we obtain

$$\frac{\partial p_\varepsilon}{\partial n}\Big|_{\Gamma} = \frac{\partial p_\varepsilon}{\partial n}\Big|_{\Gamma_0} - h'_\varepsilon \frac{\partial p_\varepsilon}{\partial \sigma}\Big|_{\Gamma_0} + O(\varepsilon^2), \tag{9.65}$$

and by substituting the expansion (9.58) into Eqs. (9.63), (9.65) and taking into account the boundary conditions (9.54) for the function $P_0(t,\mathbf{y})$, we derive the following boundary conditions at the unperturbed contact boundary:

$$P_1(t,\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_0(t), \tag{9.66}$$

$$\frac{\partial P_1}{\partial n}(t,\mathbf{y}) = -h(t,\sigma)\frac{\partial^2 P_0}{\partial n^2}(t,\mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t). \tag{9.67}$$

Here, in accordance with Eq. (9.53), we have

$$\frac{\partial^2 P_0}{\partial n^2}(t,\mathbf{y}) = m\big(\varphi_0(\mathbf{y}) - \delta_0(t)\big), \quad \mathbf{y} \in \Gamma_0(t). \tag{9.68}$$

Note that the right-hand side of (9.68) is strictly positive. This can be verified by employing the explicit formula obtained in Sect. 9.1.5, or proved by using the maximum principle for harmonic functions.

Finally, Eqs. (9.24) and (9.29) yield

$$A_0(t)\delta_1(t) = \iint\limits_{\omega_0(t)} \phi(\mathbf{y})\, d\mathbf{y} + \int\limits_{\Gamma_0(t)} \varphi_0(\mathbf{y})h(t,\sigma)\, d\sigma_y - A_1(t)\delta_0(t), \tag{9.69}$$

from which it follows that, for $B_0(t,\mathbf{y})$ defined by (9.56),

$$0 = \iint\limits_{\omega_0(t)} B_0(t,\mathbf{y})\phi(\mathbf{y})\, d\mathbf{y} - \delta_0(t)\bigg(I_1(t) - \frac{I_0(t)}{A_0(t)}A_1(t)\bigg) + \int\limits_{\Gamma_0(t)} B_0(t,\mathbf{y})\varphi_0(\mathbf{y})h(t,\sigma)\, d\sigma_y. \tag{9.70}$$

Here, $A_1(t)$ and $I_1(t)$ are the first-order perturbation coefficients of $A_\varepsilon(t)$ and $I_\varepsilon(t)$, respectively, given by

$$A_1(t) = \int\limits_{\Gamma_0(t)} h(t,\sigma)\,d\sigma_y, \tag{9.71}$$

$$I_1(t) = \int\limits_{\Gamma_0(t)} |\mathbf{y}|^2 h(t,\sigma)\,d\sigma_y. \tag{9.72}$$

Equations (9.60), (9.66), (9.67), (9.69) and (9.70) constitute the first-order perturbation problem. By employing the relations (9.68) and (9.71), it is not hard to check that Eq. (9.69) coincides with the solvability condition (see, for instance, (9.23)) of the boundary-value problem (9.60), (9.66), (9.67).

9.1.7 Determination of the Contour of the Contact Area

First, let us express the solution to Eq. (9.60) in the form

$$P_1(t,\mathbf{y}) = m\big(P_1^{[0]}(t,\mathbf{y}) + P_1^{[1]}(t,\mathbf{y})\big), \tag{9.73}$$

where we have introduced the notation

$$P_1^{[0]}(t,\mathbf{y}) = Y_\phi^{[0]}(\mathbf{y}) - \delta_1(t)Y_1^{[0]}(\mathbf{y}), \tag{9.74}$$

$$Y_\phi^{[0]}(\mathbf{y}) = \frac{1}{2\pi}\iint\limits_{\omega_0(t)} \phi(\mathbf{y})\ln|\mathbf{y}-\mathbf{y}|\,d\mathbf{y}, \quad Y_1^{[0]}(\mathbf{y}) = \frac{1}{2\pi}\iint\limits_{\omega_0(t)} \ln|\mathbf{y}-\mathbf{y}|\,d\mathbf{y}. \tag{9.75}$$

Substituting the form (9.73) into Eqs. (9.60), (9.66) and (9.67), we obtain the following boundary value problem for the function $P_1^{[1]}(t,\mathbf{y})$:

$$\Delta_y P_1^{[1]}(t,\mathbf{y}) = 0, \quad \mathbf{y} \in \omega_0(t), \tag{9.76}$$

$$P_1^{[1]}(t,\mathbf{y}) = -Y_\phi^{[0]}(\mathbf{y}) + \delta_1(t)Y_1^{[0]}(\mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t), \tag{9.77}$$

$$\begin{aligned}\frac{\partial P_1^{[1]}}{\partial n}(t,\mathbf{y}) &= -h(t,\sigma)\big(\varphi_0(\mathbf{y}) - \delta_0(t)\big)\\ &\quad - \frac{\partial Y_\phi^{[0]}}{\partial n}(t,\mathbf{y}) + \delta_1(t)\frac{\partial Y_1^{[0]}}{\partial n}(t,\mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t).\end{aligned} \tag{9.78}$$

Here we have used relations (9.68) and (9.74).

Let us denote the first term on the right-hand side of (9.78) by $-\hat{h}(t,\sigma)$, so that

$$h(t,\sigma) = \frac{\hat{h}(t,\sigma)}{\varphi_0(\mathbf{y}) - \delta_0(t)}, \quad \mathbf{y} \in \Gamma_0(t). \tag{9.79}$$

As a result of (9.71), (9.72) and (9.79), Eqs. (9.69), (9.70) and (9.78) respectively take the forms

$$\delta_1(t) = \frac{1}{A_0(t)} \iint\limits_{\omega_0(t)} \phi(\mathbf{y})\, d\mathbf{y} + \frac{1}{A_0(t)} \int\limits_{\Gamma_0(t)} \hat{h}(t,\sigma)\, d\sigma_y, \tag{9.80}$$

$$0 = \iint\limits_{\omega_0(t)} B_0(t,\mathbf{y})\phi(\mathbf{y})\, d\mathbf{y} + \int\limits_{\Gamma_0(t)} B_0(t,\mathbf{y})\hat{h}(t,\sigma)\, d\sigma_y. \tag{9.81}$$

$$\frac{\partial P_1^{[1]}}{\partial n}(t,\mathbf{y}) = -\hat{h}(t,\sigma) - \frac{\partial Y_\phi^{[0]}}{\partial n}(t,\mathbf{y}) + \delta_1(t)\frac{\partial Y_1^{[0]}}{\partial n}(t,\mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t). \tag{9.82}$$

Second, to proceed further with the computation of the contour of the contact area, we now define the Steklov—Poincaré (Dirichlet-to-Neumann) operator $\mathfrak{S} : H^{1/2}(\Gamma_0(t)) \to H^{-1/2}(\Gamma_0(t))$ by

$$\big(\mathfrak{S}g\big)(\mathbf{y}) = \frac{\partial w}{\partial n}(\mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t), \tag{9.83}$$

where $w(\mathbf{y})$ is the unique solution of the Dirichlet problem

$$\Delta_y w(\mathbf{y}) = 0, \quad \mathbf{y} \in \omega_0(t); \qquad w(\mathbf{y}) = g(\mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t). \tag{9.84}$$

The operator $\mathfrak{S}$ for a circular domain is well known, and we will use the above form later in Sect. 9.1.9. To the authors' best knowledge there is no closed form representation for $\mathfrak{S}$ in the case of an elliptic domain. However, the finite element Steklov—Poincaré operator can be computed by standard FEM packages (see for details [15]). In [8], an alternative approach for constructing the operator numerically in terms of conformal mappings was presented.

In terms of the Steklov—Poincaré operator, Eqs. (9.77) and (9.82) yield

$$\begin{aligned}\hat{h}(t,\sigma) &= \big(\mathfrak{S}Y_\phi^{[0]}\big)(\mathbf{y}) - \frac{\partial Y_\phi^{[0]}}{\partial n}(t,\mathbf{y}) \\ &\quad - \delta_1(t)\bigg(\big(\mathfrak{S}Y_1^{[0]}\big)(\mathbf{y}) - \frac{\partial Y_1^{[0]}}{\partial n}(t,\mathbf{y})\bigg), \quad \mathbf{y} \in \Gamma_0(t).\end{aligned} \tag{9.85}$$

Note that the substitution of (9.85) into (9.80) results in an identity, which we check by verifying the following properties:

$$\int\limits_{\Gamma_0(t)} \bigl(\mathfrak{S}g\bigr)(\mathbf{y})\, d\sigma_y = 0, \quad \forall g \in H^{1/2}(\Gamma_0(t)),$$

$$\iint\limits_{\omega_0(t)} \phi(\mathbf{y})\, d\mathbf{y} = \int\limits_{\Gamma_0(t)} \frac{\partial Y_\phi^{[0]}}{\partial n}(t, \mathbf{y})\, d\sigma_y, \quad \forall \varphi \in L^2(\omega_0(t)).$$

Now, excluding the variable $\delta_1(t)$ from (9.85) by means of (9.80), we obtain

$$\hat{h}(t, \sigma) = -\biggl(\frac{1}{A_0(t)} \iint\limits_{\omega_0(t)} \phi(\mathbf{y})\, d\mathbf{y} + \hat{H}_0(t)\biggr)\biggl(\bigl(\mathfrak{S}Y_1^{[0]}\bigr)(\mathbf{y}) - \frac{\partial Y_1^{[0]}}{\partial n}(t, \mathbf{y})\biggr)$$
$$+ \bigl(\mathfrak{S}Y_\phi^{[0]}\bigr)(\mathbf{y}) - \frac{\partial Y_\phi^{[0]}}{\partial n}(t, \mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t), \tag{9.86}$$

where $\hat{H}_0(t)$ is the relative weighted increment of the contact area defined as

$$\hat{H}_0(t) = \frac{1}{A_0(t)} \int\limits_{\Gamma_0(t)} \hat{h}(t, \sigma)\, d\sigma. \tag{9.87}$$

At this point, the function $\hat{h}(t, \sigma)$ is determined by Eq. (9.86) with an accuracy up to its integral characteristics $\hat{H}_0(t)$.

By substitution of the expression (9.86) into Eq. (9.81), we arrive at the following simple equation to determine $\hat{H}_0(t)$:

$$\Xi_1^{[0]}(t)\hat{H}_0(t) = \Xi_\phi^{[0]}(t) + \iint\limits_{\omega_0(t)} \biggl(B_0(t, \mathbf{y}) - \frac{\Xi_1^{[0]}(t)}{A_0(t)}\biggr)\phi(\mathbf{y})\, d\mathbf{y}. \tag{9.88}$$

Here, both functions $\Xi_1^{[0]}(t)$ nd $\Xi_\phi^{[0]}(t)$ are determined by the formula

$$\Xi_{\phi;1}^{[0]}(t) = \int\limits_{\Gamma_0(t)} B_0(t, \mathbf{y})\biggl(\bigl(\mathfrak{S}Y_{\phi;1}^{[0]}\bigr)(\mathbf{y}) - \frac{\partial Y_{\phi;1}^{[0]}}{\partial n}(t, \mathbf{y})\biggr)\, d\mathbf{y}. \tag{9.89}$$

It is clear that the solvability of Eq. (9.86) depends crucially on the property of having fixed sign for $\Xi_1^{[0]}(t)$. In [8], it was proven that $\Xi_1^{[0]}(t) < 0$.

Thus, determining the function $\hat{H}_0(t)$ by dividing both sides of (9.88) by $\Xi_1^{[0]}(t)$ and substituting the obtained result into Eq. (9.86), we find $\hat{h}(t, \sigma)$ and, as a consequence of (9.79), uniquely determine the function $h(t, \sigma)$, which describes the variation of the contact domain $\omega_0(t)$.

9.1.8 Asymptotics of the Contact Pressure

In accordance with (9.34), the contact pressure is given by

$$p_\varepsilon(t, \mathbf{y}) = \mathscr{K}^{-1}\big(P_\varepsilon(\mathbf{y}, \tau)\mathscr{I}_{\omega_\varepsilon(\tau)}(\mathbf{y})\big), \tag{9.90}$$

where the integral operator $\mathscr{K}^{-1}$ is defined by formula (9.35), and $\mathscr{I}_{\omega_\varepsilon(t)}(\mathbf{y})$ is the indicator function of the domain $\omega_\varepsilon(t)$ defined by

$$\mathscr{I}_{\omega_\varepsilon(t)}(\mathbf{y}) = \begin{cases} 1, & \mathbf{y} \in \omega_\varepsilon(t), \\ 0, & \mathbf{y} \notin \omega_\varepsilon(t). \end{cases}$$

In the interior of the contact area $\omega_\varepsilon(t)$, Eqs. (9.58) and (9.90) yield the asymptotic representation

$$p_\varepsilon(t, \mathbf{y}) = p_0(t, \mathbf{y}) + \varepsilon p_1(t, \mathbf{y}) + O(\varepsilon^2), \tag{9.91}$$

where

$$p_i(t, \mathbf{y}) = \mathscr{K}^{-1}\big(P_i(t, \mathbf{y})\mathscr{I}_{\omega_0(t)}(\mathbf{y})\big), \quad i = 0, 1. \tag{9.92}$$

In the boundary-layer region near the contour $\Gamma_\varepsilon(t)$, the so-called outer asymptotic representation (9.91) does not work, and the so-called inner asymptotic representation should be constructed. Here we employ the terminology from the method of matched asymptotic expansions [20].

The inner asymptotic representation

$$p_\varepsilon(t, \mathbf{y}) = \varepsilon^2\mathscr{K}^{-1}\mathscr{P}(\tau, s, \nu) + O(\varepsilon^3) \tag{9.93}$$

will be constructed by making use of the stretched coordinate

$$\nu = \varepsilon^{-1} n. \tag{9.94}$$

In light of (9.73)–(9.75), the function $P_1(t, \mathbf{y})$ will be determined completely as soon as we know the function $P_1^{[1]}(t, \mathbf{y})$ which satisfies the following Dirichlet problem (see (9.76) and (9.77)):

$$\Delta P_1^{[1]}(t, \mathbf{y}) = 0, \quad \mathbf{y} \in \omega_0(t); \quad P_1^{[1]}(t, \mathbf{y}) = g_\phi^{[0]}(t, \mathbf{y}), \quad \mathbf{y} \in \Gamma_0(t).$$

Here we have introduced the following notation (see Eqs. (9.80), (9.87) and (9.88)):

$$g_\phi^{[0]}(t, \mathbf{y}) = -Y_\phi^{[0]}(\mathbf{y}) + \delta_1(t)Y_1^{[0]}(\mathbf{y}), \tag{9.95}$$

and as a consequence of said relations, we have

$$\delta_1(t) = \frac{\Xi_\phi^{[0]}(t)}{\Xi_1^{[0]}(t)} + \frac{1}{\Xi_1^{[0]}(t)} \iint\limits_{\omega_0(t)} B_0(t, \mathbf{y})\phi(\mathbf{y})\, d\mathbf{y}. \tag{9.96}$$

Near the boundary of the domain $\omega_0(t)$, we may use the Taylor expansions

$$P_0(t,\mathbf{y}) = \frac{m}{2}\big[\varphi_0(\mathbf{y}_0(\sigma)) - \delta_0(t)\big]n^2 + O(n^3), \tag{9.97}$$

$$P_1(t,\mathbf{y}) = \frac{m}{2}\big[\phi(\mathbf{y}_0(\sigma)) - \delta_1(t)\big]n^2 - m\hat{h}(t,\sigma)n + O(n^3), \tag{9.98}$$

where $\mathbf{y}_0(\sigma)$ is the point of the curve $\Gamma_0(t)$ with the natural coordinate σ.

Applying the perturbation method developed by Nazarov [18], we construct the auxiliary function of the inner asymptotic representation (9.93) in the form

$$\mathscr{P}(t,\sigma,\nu) = \frac{m}{2}\big[\varphi_0(\mathbf{y}_0(\sigma)) - \delta_0(t)\big](\nu - h(t,\sigma))^2. \tag{9.99}$$

The function (9.99) exactly satisfies the relations (9.15), while the boundary condition (9.16) is satisfied asymptotically. We note that the normals $\mathbf{n}_0$ and $\mathbf{n}_\varepsilon$ to the contours $\Gamma_0(t)$ and $\Gamma_\varepsilon(t)$ are, generally speaking, different (see formula (9.64)).

Finally, taking account of the relations (9.79), (9.94), (9.97)–(9.99), it is not hard to verify verify that the matching asymptotic condition for the outer (9.91) and inner (9.93) asymptotic representations is fulfilled.

9.1.9 Slightly Perturbed Circular Contact Area

Let us assume that a circular domain can be taken as a zero approximation in the form (9.5), where the function $\varphi(\mathbf{y})$ defining the perturbed boundary is given by the polynomials

$$\varphi_0(\mathbf{y}) = \frac{1}{2R}(y_1^2 + y_2^2), \quad \phi(\mathbf{y}) = \sum_{n=0}^{N}\sum_{j=0}^{n} c_{nj}\, y_1^j y_2^{n-j}, \tag{9.100}$$

where c_{nj} are given dimensional coefficients.

In this case, the limit ($\varepsilon = 0$) auxiliary contact problem (9.31), (9.38), (9.39) has the following solution:

$$P_0(t,\mathbf{y}) = Q_0(t)\bigg(1 - \frac{y_1^2 + y_2^2}{a_0(t)^2}\bigg)^2.$$

Correspondingly, Eqs. (9.46), (9.47) and (9.49) take the form

$$\delta_0(t) = \frac{a_0(t)^2}{4R}, \quad Q_0(t) = \frac{m}{32}\frac{a_0(t)^4}{R}, \tag{9.101}$$

$$a_0(t) = \left(\frac{m\pi}{96R}\right)^{-1/6}\left(\int\limits_{0^-}^{t} \Phi_\beta(t-\tau)\dot{F}(\tau)\,d\tau\right)^{1/6}. \tag{9.102}$$

The first-oder perturbation problem (9.60), (9.66), (9.67), (9.69) and (9.70) now can be written as follows:

$$\Delta_y P_1(t,\mathbf{y}) = m\left(\sum_{n=0}^{N}\sum_{j=0}^{n} c_{nj} y_1^j y_2^{n-j} - \delta_1(t)\right), \quad \mathbf{y} \in \omega_0(t), \tag{9.103}$$

$$P_1(t,\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_0(t), \tag{9.104}$$

$$\frac{\partial P_1}{\partial n}(t,\mathbf{y}) = -mh(t,\sigma)\frac{a_0(t)^2}{4R}, \quad \mathbf{y} \in \Gamma_0(t), \tag{9.105}$$

$$\pi a_0(t)^2\delta_1(t) = \sum_{n=0}^{N}\frac{a_0(t)^{n+2}}{n+2}\sum_{j=0}^{n} c_{nj}K_{nj} + \frac{a_0(t)^3}{4R}H_0(t), \tag{9.106}$$

$$0 = \frac{a_0(t)^5}{8R}H_0(t) + \sum_{n=0}^{N}\frac{na_0(t)^{n+4}}{2(n+2)(n+4)}\sum_{j=0}^{n} c_{nj}K_{nj}. \tag{9.107}$$

Here, $\omega_0(t)$ and $\Gamma_0(t)$ are the disc and the circle of the radius $a_0(t)$, $K_{nj} = 0$ for odd n and j, while $K_{nj} = 2B\big((n+1-j)/2, (j+1)/2\big)$ for even values of n and j, and $B(\zeta, \xi)$ is the Beta function defined by

$$B(\zeta,\xi) = \int\limits_0^1 t^{\zeta-1}(1-t)^{\xi-1}dt.$$

Moreover, $H_0(t)$ is an integral characteristics of the contour variation $h(t,\sigma)$ which can be described by the contour variation in polar coordinates as

$$H_0(t) = \int\limits_0^{2\pi} h(\theta,t)\,d\theta.$$

From (9.106) and (9.107), it immediately follows that

$$H_0(t) = -\sum_{n=0}^{N}\frac{4Ra_0(t)^{n-1}}{(n+2)(n+4)}\sum_{j=0}^{n} c_{nj}K_{nj},$$

$$\delta_1(t) = \sum_{n=0}^{N} \frac{(n+3)a_0(t)^n}{\pi(n+2)(n+4)} \sum_{j=0}^{n} c_{nj} K_{nj}. \tag{9.108}$$

Using Green's function $\mathscr{G}(t, \mathbf{y}, \mathbf{y}')$ of the Dirichlet problem for the domain $\omega_0(t)$, we express the solution to the Dirichlet problem (9.103), (9.104) in the form

$$P_1(t, \mathbf{y}) = m \iint\limits_{\omega_0(t)} \phi(\mathbf{y}')\mathscr{G}(t, \mathbf{y}, \mathbf{y}')\, d\mathbf{y}' - \frac{m}{4}\delta_1(t)(|\mathbf{y}|^2 - a_0(t)^2). \tag{9.109}$$

Recall that for a circular domain $\omega_0(t)$ of radius $a_0(t)$, Green's function is

$$\mathscr{G}(t, \mathbf{y}, \mathbf{y}') = \frac{1}{2\pi} \ln \frac{a_0(t)|\mathbf{y}'||\mathbf{y} - \mathbf{y}'|}{\big|\, |\mathbf{y}'|\mathbf{y} - a_0(t)^2\mathbf{x}' \big|},$$

whereas, in polar coordinates, we have

$$\mathscr{G}(t, \mathbf{y}, \mathbf{y}') = \frac{1}{4\pi} \ln \frac{a_0(t)^2\big(r^2 + r'^2 - 2r'r\cos(\theta - \theta')\big)}{a_0(t)^4 + r'^2 r^2 - 2r' r a_0(t)^2 \cos(\theta - \theta')}.$$

Calculating the normal derivative of the function (9.109), we obtain

$$\frac{\partial P_1}{\partial n}(t, \mathbf{y}) = m \iint\limits_{\omega_0(t)} \phi(\mathbf{y}')\frac{\partial \mathscr{G}}{\partial n}(t, \mathbf{y}, \mathbf{y}')\, d\mathbf{y}' - \frac{m}{2}\delta_1(t)|\mathbf{y}|, \tag{9.110}$$

where

$$\frac{\partial}{\partial n}\mathscr{G}(t, \mathbf{y}, \mathbf{y}') = \frac{1}{2\pi} \frac{a_0(t)^2 - r'^2}{a_0(t)\big(a_0(t)^2 + r'^2 - 2r' a_0(t)\cos(\theta - \theta')\big)}.$$

By substitution of the expression (9.110) into the boundary condition (9.105) and taking into account (9.108), we obtain the function $h(t, \theta)$ describing variation of the contact domain in the following form:

$$\begin{aligned} h(t, \theta) = {} & \frac{2R}{\pi a_0(t)} \sum_{n=0}^{N} \frac{(n+3)a_0(t)^n}{(n+2)(n+4)} \sum_{j=0}^{n} c_{nj} K_{nj} \\ & - \frac{2R}{\pi a_0(t)^3} \int_0^{2\pi} d\theta' \int_0^{a_0(t)} \Bigg(\sum_{n=0}^{N} r'^n \sum_{j=0}^{n} c_{nj} \cos^j \theta' \sin^{n-j} \theta' \Bigg) \\ & \times \frac{a_0(t)^2 - r'^2}{a_0(t)^2 + r'^2 - 2r' a_0(t)\cos(\theta - \theta')}\, r'\, dr'. \end{aligned} \tag{9.111}$$

Note that the integral with respect to θ' on the right-hand side of (9.111) can be evaluated with the help of the following relation (see, e.g., [14]):

$$\int_0^\pi \frac{\cos nx\,dx}{1 - 2\rho\cos x + \rho^2} = \frac{\pi\rho^n}{1-\rho^2}, \quad \rho^2 < 1.$$

It was shown [8] that formula (9.111) asymptotically coincides with the exact solution for the ellipse presented in Sect. 9.1.5 in the case of small eccentricity.

9.2 Contact of Two Bonded Thin Transversely Isotropic Elastic Layers with Variable Thicknesses

In this section, a three-dimensional unilateral contact problem for a thin transversely isotropic elastic layer with variable thickness bonded to a rigid substrate is considered. Two cases are studied sequentially: (a) the layer material is compressible; (b) the layer material is incompressible. It is well known that the asymptotic solution for a thin isotropic elastic layer undergoes a dramatic change in the limit as Poisson's ratio ν tends to 0.5, so that the formulas obtained in the case (a) are not applicable when the layer material approaches the incompressible limit. After developing a refined asymptotic model for the deformation of one elastic layer of variable thickness, we apply sensitivity analysis to determine how "sensitive" the mathematical model for contact interactions of two thin uniform layers of thicknesses h_1 and h_2 is to variations in the layer thicknesses. We will consider the term "sensitivity" in a broad sense by allowing variable layer thicknesses $H_1(\mathbf{y})$ and $H_2(\mathbf{y})$, whereas the original model deals with the scalar parameters h_1 and h_2.

9.2.1 Unperturbed Asymptotic Model

As a rule, analytical models of articular contact assume rigid bones and represent cartilage as a thin elastic layer of constant thickness resisting deformation like a Winkler foundation consisting of a series of discrete springs with constant length and stiffness [12]. However, a subject-specific approach to articular contact mechanics requires developing patient-specific models for accurate predictions. A sensitivity analysis of finite element models of hip cartilage mechanics with respect to varying degrees of simplified geometry was performed in [2].

Based on the asymptotic analysis of the frictionless contact problem for a thin elastic layer bonded to a rigid substrate in the thin-layer limit (see Chap. 2), the following asymptotic model for contact interaction of two thin incompressible transversely isotropic layers was established:

$$-\left(\frac{h_1^3}{3G_1'}+\frac{h_2^3}{3G_2'}\right)\Delta_y p(\mathbf{y})=\delta_0-\varphi(\mathbf{y}),\quad \mathbf{y}\in\omega, \tag{9.112}$$

$$p(\mathbf{y})=0,\quad \frac{\partial p}{\partial n}(\mathbf{y})=0,\quad \mathbf{y}\in\Gamma. \tag{9.113}$$

Here, $p(\mathbf{y})$ is the contact pressure density, h_n and G_n' are the thickness and out-of-plane shear modulus of the nth layer material, respectively, $n = 1, 2$, $\Delta_y = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ is the Laplace differential operator, δ_0 is the vertical approach of the rigid substrates to which the layers are bonded, $\varphi(\mathbf{y})$ is the initial gap function defined as the distance between the layer surfaces in the vertical direction, ω is the contact area, and Γ is the contour of ω, $\partial/\partial n$ is the normal derivative.

In the isotropic case, the asymptotic model (9.112), (9.113) was developed in [4, 10]. It was shown [6, 9, 11] that this model describes the instantaneous response of thin biphasic layers to dynamic and impact loading. In [7], the isotropic elastic model (9.112), (9.113) was generalized for a general viscoelastic case.

With respect to articular contact, a case of special interest is when the subchondral bones are approximated by elliptic paraboloids, so that the gap function is given by

$$\varphi(\mathbf{y})=\frac{y_1^2}{2R_1}+\frac{y_2^2}{2R_2} \tag{9.114}$$

with positive curvature radii R_1 and R_2.

In the case (9.114), the exact solution to the problem (9.112), (9.113) has the following form [6, 10]:

$$p(\mathbf{y})=p_0\left(1-\frac{y_1^2}{a_1^2}-\frac{y_2^2}{a_2^2}\right)^2. \tag{9.115}$$

Integration of the contact pressure distribution (9.115) over the elliptical contact area ω with the semi-axes a_1 and a_2 results in the following force-displacement relationship [4]:

$$F=\frac{2\pi}{3}c_F(s)mR_1R_2\delta_0^3. \tag{9.116}$$

Here, $c_F(s)$ is a dimensionless factor depending on the aspect ratio $s = a_2/a_1$ (see Sect. 4.5.6), and the coefficient m is given by

$$m=3\left(\frac{h_1^3}{G_1'}+\frac{h_2^3}{G_2'}\right)^{-1}. \tag{9.117}$$

The asymptotic model (9.112)–(9.114) assumes that the cartilage layers have constant thicknesses, whereas it is well known [1] that articular cartilage has a variable thickness and that the surface of subchondral bone deviates from the ellipsoid shape [19]. A sensitivity analysis of the model (9.112), (9.113) with respect to small perturbations of the gap function (9.114) was performed in [8]. In particular, it has been shown [4] that the influence of the gap function variation on the force-displacement relationship will be negligible if the effective geometrical characteristics R_1 and R_2 are determined by a least square method.

The two-dimensional contact problem for a thin isotropic elastic strip of variable thickness was solved by Vorovich and Peninin [21] using an asymptotic method, under the assumption that the Poisson's ratio of the strip material is not very close to 0.5. A three-dimensional unilateral contact problem for a thin isotropic elastic layer of variable thickness bonded to a rigid substrate was studied in [5]. Here, the results obtained in [5] are generalized for the transversely isotropic case.

9.2.2 Contact Problem for a Thin Transversely Isotropic Elastic Layer with Variable Thickness

We consider (see Fig. 9.2) a homogeneous, isotropic, linearly elastic layer with a planar contact interface, $x_3 = 0$, and a variable thickness, $H(x_1, x_2)$, firmly attached to an uneven rigid surface

$$x_3 = H(x_1, x_2). \tag{9.118}$$

In the absence of body forces, the equilibrium equations and the strain-displacement relations governing small deformations of the elastic layer are

$$\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} = 0, \quad j = 1, 2, 3, \tag{9.119}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3, \tag{9.120}$$

where σ_{ij} is the Cauchy stress tensor, ε_{ij} is the infinitesimal strain tensor, and u_j is the displacement component along the x_j-axis. Here the Cartesian coordinate system (y_1, y_2, z) will be used such that $y_1 = x_1$, $y_2 = x_2$, and $z = x_3$.

Moreover, a transversely isotropic elastic body is characterized by the following stress-strain relationships:

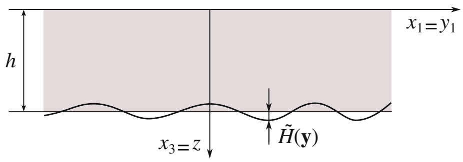

Fig. 9.2 Elastic layer with a variable thickness

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{11} & A_{13} & 0 & 0 & 0 \\ A_{13} & A_{13} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2A_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}. \tag{9.121}$$

Recall that $2A_{66} = A_{11} - A_{12}$, while A_{11}, A_{12}, A_{13}, A_{33}, and A_{44} are five independent elastic constants.

We assume that the elastic layer is indented by a smooth rigid punch in the form of an elliptic paraboloid

$$z = -\varphi(y_1, y_2),$$

where

$$\varphi(\mathbf{y}) = \frac{y_1^2}{2R_1} + \frac{y_2^2}{2R_2}. \tag{9.122}$$

Under the assumption of frictionless contact, we have

$$\sigma_{31}(\mathbf{y}, 0) = 0, \quad \sigma_{32}(\mathbf{y}, 0) = 0, \quad \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2. \tag{9.123}$$

Denoting by δ_0 the indenter's displacement, we formulate the boundary condition on the contact interface as

$$\begin{gathered} u_3(\mathbf{y}, 0) \geq \delta_0 - \varphi(\mathbf{y}), \quad \sigma_{33}(\mathbf{y}, 0) \leq 0, \\ \big(u_3(\mathbf{y}, 0) - \delta_0 + \varphi(\mathbf{y})\big)\sigma_{33}(\mathbf{y}, 0) = 0, \quad \mathbf{y} \in \mathbb{R}^2, \end{gathered} \tag{9.124}$$

while on the rigid substrate surface defined by Eq. (9.118) we have

$$u_j(\mathbf{y}, H(\mathbf{y})) = 0, \quad \mathbf{y} \in \mathbb{R}^2 \quad (j = 1, 2, 3). \tag{9.125}$$

Assuming that the layer is relatively thin in comparison to the characteristic dimensions of ω, we introduce a small dimensionless parameter ε and require that

$$\delta_0 = \varepsilon\delta_0^*, \quad R_1 = \varepsilon^{-1}R_1^*, \quad R_2 = \varepsilon^{-1}R_2^*, \tag{9.126}$$

$$H(\mathbf{y}) = \varepsilon h_*(1 + \varepsilon\psi_*(\mathbf{y})), \tag{9.127}$$

where δ_0^*, R_1^*, and R_2^* are assumed to be comparable with h_*. Without loss of generality we can assume that $|\psi_*(\mathbf{y})| \leq h_*$ for any $\mathbf{y} \in \mathbb{R}^2$.

The problem is to calculate the contact pressure distribution

$$p(x_1, x_2) = -\sigma_{33}(x_1, x_2, 0), \quad (x_1, x_2) \in \omega, \tag{9.128}$$

and the contact force required to indent the punch into the elastic layer

$$F = \iint_{\omega} p(\mathbf{y})\, d\mathbf{y}. \tag{9.129}$$

We introduce the notation

$$h = \varepsilon h_*, \quad \tilde{H}(\mathbf{y}) = \varepsilon^2 \tilde{H}_*(\mathbf{y}), \tag{9.130}$$

where

$$\tilde{H}_*(\mathbf{y}) = h_* \psi_*(\mathbf{y}). \tag{9.131}$$

Hence, the following relation is evident:

$$H(\mathbf{y}) = h + \tilde{H}(\mathbf{y}), \tag{9.132}$$

where h is an average thickness, and $\tilde{H}(\mathbf{y})$ is a small variation such that $\tilde{H}(\mathbf{y}) \ll h$.

We note that as a consequence of (9.128), the unilateral contact condition (9.124) can be rewritten as follows:

$$p(\mathbf{y}) \geq 0, \quad \mathbf{y} \in \mathbb{R}^2,$$

$$p(\mathbf{y}) > 0 \;\;\Rightarrow\;\; u_3(\mathbf{y}, 0) = \delta_0 - \varphi(\mathbf{y}),$$

$$p(\mathbf{y}) = 0 \;\;\Rightarrow\;\; u_3(\mathbf{y}, 0) \geq \delta_0 - \varphi(\mathbf{y}).$$

We pose the following frictionless unilateral contact problem: for a given value of the punch displacement δ_0, find the contact pressure $p(\mathbf{y})$ such that the Signorini boundary condition (9.124) is satisfied both inside the contact area ω and outside (that is $p(\mathbf{y}) = 0$ for $\mathbf{y} \in \mathbb{R}^2 \setminus \bar{\omega}$).

9.2.3 Perturbation Solution

First, we introduce the so-called stretched coordinate

$$\zeta = \varepsilon^{-1} z. \tag{9.133}$$

Substituting (9.121) and (9.120) into Eqs. (9.119) and taking into account the variable transformation (9.133), we arrive at the following Lamé system for the displacement vector $\mathbf{u} = (\mathbf{v}, w)$:

$$\varepsilon^{-2} A_{44} \frac{\partial^2 \mathbf{v}}{\partial \zeta^2} + \varepsilon^{-1} (A_{13} + A_{44}) \nabla_y \frac{\partial w}{\partial \zeta} + A_{66} \Delta_y \mathbf{v} + (A_{11} - A_{66}) \nabla_y \nabla_y \cdot \mathbf{v} = \mathbf{0}, \tag{9.134}$$

$$\varepsilon^{-2}A_{33}\frac{\partial^2 w}{\partial\zeta^2}+\varepsilon^{-1}(A_{13}+A_{44})\nabla_y\cdot\frac{\partial\mathbf{v}}{\partial\zeta}+A_{44}\Delta_y w=0. \tag{9.135}$$

Here, $\nabla_y=(\partial/\partial y_1,\partial/\partial y_2)$ is the Hamilton differential operator, and the dot denotes the scalar product, so that $\nabla_y\cdot\nabla_y=\Delta_y$ is the Laplace operator.

Correspondingly, the boundary condition (9.123) takes the form

$$\varepsilon^{-1}\frac{\partial\mathbf{v}}{\partial\zeta}+\nabla_y w\bigg|_{\zeta=0}=\mathbf{0}. \tag{9.136}$$

In light of (9.124) and (9.126), we have

$$w(\mathbf{y},0)=\varepsilon(\delta_0^*-\varphi^*(\mathbf{y})),\quad \mathbf{y}\in\omega, \tag{9.137}$$

where we have introduced the notation (see Eqs. (9.122) and (9.126))

$$\varphi^*(\mathbf{y})=\frac{y_1^2}{2R_1^*}+\frac{y_2^2}{2R_2^*}. \tag{9.138}$$

Furthermore, by stretching the normal coordinate, Eq. (9.128) is reduced to

$$-p(\mathbf{y})=\varepsilon^{-1}A_{33}\frac{\partial w}{\partial\zeta}+A_{13}\nabla_y\cdot\mathbf{v}\bigg|_{\zeta=0},\quad \mathbf{y}\in\omega. \tag{9.139}$$

The boundary conditions (9.125) on the substrate surface (see Eqs. (9.118) and (9.127))

$$\zeta=h_*(1+\varepsilon\psi_*(\mathbf{y})) \tag{9.140}$$

take the following form:

$$\mathbf{v}\big(\mathbf{y},h_*+\varepsilon h_*\psi_*(\mathbf{y})\big)=\mathbf{0},\quad w\big(\mathbf{y},h_*+\varepsilon h_*\psi_*(\mathbf{y})\big)=0. \tag{9.141}$$

Observe that among Eqs. (9.134)–(9.137), (9.139) and (9.141), there are only two inhomogeneous equations, namely (9.137) and (9.139). The form of (9.137) suggests the asymptotic expansion

$$w(\mathbf{y},\zeta)=\varepsilon w^0(\mathbf{y},\zeta)+\varepsilon^2 w^1(\mathbf{y},\zeta)+\cdots. \tag{9.142}$$

In light of (9.139) and (9.142), we suggest that $p(\mathbf{y})=O(1)$ as $\varepsilon\to 0$. Taking into account the homogeneous conditions (9.136), (9.141), we set

$$\mathbf{v}(\mathbf{y},\zeta)=\varepsilon\mathbf{v}^0(\mathbf{y},\zeta)+\varepsilon^2\mathbf{v}^1(\mathbf{y},\zeta)+\cdots. \tag{9.143}$$

We emphasize that the asymptotic ansatz (9.142), (9.143) is valid only inside the contact region ω. In other words, a plane boundary layer should be constructed near the edge of the contact area. We refer to [3, 13] for more details.

9.2.4 Derivation of Asymptotic Expansions

Substituting (9.142) and (9.143) into Eqs. (9.134) and (9.135), we obtain

$$\varepsilon^{-2}A_{44}\frac{\partial^2\mathbf{v}^0}{\partial\zeta^2}+\varepsilon^{-1}\left((A_{13}+A_{44})\nabla_y\frac{\partial w^0}{\partial\zeta}+A_{44}\frac{\partial^2\mathbf{v}^1}{\partial\zeta^2}\right)+\varepsilon^0\Big\{A_{66}\Delta_y\mathbf{v}^0+(A_{11}+A_{66})\nabla_y\nabla_y\cdot\mathbf{v}^0+(A_{13}+A_{44})\nabla_y\frac{\partial w^1}{\partial\zeta}+A_{44}\frac{\partial^2\mathbf{v}^2}{\partial\zeta^2}\Big\}+\cdots=\mathbf{0}, \quad (9.144)$$

$$\varepsilon^{-2}A_{33}\frac{\partial^2 w^0}{\partial\zeta^2}+\varepsilon^{-1}\left(A_{33}\frac{\partial^2 w^1}{\partial\zeta^2}+(A_{13}+A_{44})\nabla_y\cdot\frac{\partial\mathbf{v}^0}{\partial\zeta}\right)+\varepsilon^0\Big\{A_{33}\frac{\partial^2 w^2}{\partial\zeta^2}+(A_{13}+A_{44})\nabla_y\cdot\frac{\partial\mathbf{v}^1}{\partial\zeta}+A_{44}\Delta_y w^0\Big\}+\cdots=0. \quad (9.145)$$

Further, by substituting (9.142) and (9.143) into the boundary conditions (9.136) and (9.139) at the contact region, we find

$$\varepsilon^{-1}\frac{\partial\mathbf{v}^0}{\partial\zeta}+\varepsilon^0\left(\nabla_y w^0+\frac{\partial\mathbf{v}^1}{\partial\zeta}\right)+\varepsilon\left(\nabla_y w^1+\frac{\partial\mathbf{v}^2}{\partial\zeta}\right)+\cdots\Big|_{\zeta=0}=\mathbf{0}, \quad (9.146)$$

$$A_{33}\frac{\partial w^0}{\partial\zeta}+\varepsilon\left(A_{13}\nabla_y\cdot\mathbf{v}^0+A_{33}\frac{\partial w^1}{\partial\zeta}\right)+\varepsilon^2\left(A_{13}\nabla_y\cdot\mathbf{v}^1+A_{33}\frac{\partial w^2}{\partial\zeta}\right)+\cdots\Big|_{\zeta=0}=-p(\mathbf{y}). \quad (9.147)$$

The substitution of (9.142), (9.143) into Eqs. (9.141) then yields

$$\mathbf{v}^0+\varepsilon\left(\mathbf{v}^1+\tilde{H}_*\frac{\partial\mathbf{v}^0}{\partial\zeta}\right)+\varepsilon^2\left(\mathbf{v}^2+\tilde{H}_*\frac{\partial\mathbf{v}^1}{\partial\zeta}+\frac{\tilde{H}_*^2}{2}\frac{\partial^2\mathbf{v}^0}{\partial\zeta^2}\right)+\cdots\Big|_{\zeta=h_*}=\mathbf{0}, \quad (9.148)$$

$$w^0 + \varepsilon\Big(w^1 + \tilde{H}_* \frac{\partial w^0}{\partial \zeta}\Big) + \varepsilon^2\Big(w^2 + \tilde{H}_* \frac{\partial w^1}{\partial \zeta} + \frac{\tilde{H}_*^2}{2}\frac{\partial^2 w^0}{\partial \zeta^2}\Big) + \cdots \Big|_{\zeta = h_*} = 0, \quad (9.149)$$

where we have used notation (9.131).

Thus, on the basis of Eqs. (9.144)–(9.149), we have arrived at a recurrence system of boundary-value problems for the functions $\mathbf{v}^k$ and w^k ($k = 0, 1, \ldots$). Let us construct the first several terms of the asymptotic series (9.142) and (9.143).

9.2.5 Asymptotic Solution for a Thin Compressible Layer

According to (9.144)–(9.149), the first-order problem takes the form

$$\frac{\partial^2 w^0}{\partial \zeta^2} = 0, \quad \zeta \in (0, h_*), \quad A_{33}\frac{\partial w^0}{\partial \zeta}\Big|_{\zeta=0} = -p(\mathbf{y}), \quad w^0\big|_{\zeta=h_*} = 0; \qquad (9.150)$$

$$\frac{\partial^2 \mathbf{v}^0}{\partial \zeta^2} = \mathbf{0}, \quad \zeta \in (0, h_*), \quad \frac{\partial \mathbf{v}^0}{\partial \zeta}\Big|_{\zeta=0} = \mathbf{0}, \quad \mathbf{v}^0\big|_{\zeta=h_*} = \mathbf{0}. \qquad (9.151)$$

From (9.150) and (9.151), it immediately follows that

$$w^0(\mathbf{y}, \zeta) = \frac{p(\mathbf{y})}{A_{33}}(h_* - \zeta), \qquad (9.152)$$

$$\mathbf{v}^0(\mathbf{y}, \zeta) \equiv \mathbf{0}. \qquad (9.153)$$

As a consequence of (9.153), the second-order problem, derived from (9.144)–(9.149) is

$$\frac{\partial^2 w^1}{\partial \zeta^2} = 0, \quad \zeta \in (0, h_*),$$

$$\frac{\partial w^1}{\partial \zeta}\Big|_{\zeta=0} = 0, \quad w^1\big|_{\zeta=h_*} = -\tilde{H}_*(\mathbf{y})\frac{\partial w^0}{\partial \zeta}\Big|_{\zeta=h_*}; \qquad (9.154)$$

$$A_{44}\frac{\partial^2 \mathbf{v}^1}{\partial \zeta^2} = -(A_{13} + A_{44})\nabla_y \frac{\partial w^0}{\partial \zeta}, \quad \zeta \in (0, h_*),$$

$$\frac{\partial \mathbf{v}^1}{\partial \zeta}\Big|_{\zeta=0} = -\nabla_y w^0\big|_{\zeta=0}, \quad \mathbf{v}^1\big|_{\zeta=h_*} = \mathbf{0}. \qquad (9.155)$$

It is readily seen that the solution of the problem (9.154) is given by

$$w^1(\mathbf{y}, \zeta) = \frac{p(\mathbf{y})}{A_{33}} \tilde{H}_*(\mathbf{y}). \tag{9.156}$$

On the other hand, the solution of the problem (9.155) is

$$\mathbf{v}^1(\mathbf{y}, \zeta) = \Psi(\zeta) \nabla_y p(\mathbf{y}), \tag{9.157}$$

where we have introduced the notation

$$\Psi(\zeta) = \frac{(A_{13} + A_{44})}{2A_{33}A_{44}} (h_* - \zeta)^2 - \frac{h_* A_{13}}{A_{33}A_{44}} (h_* - \zeta). \tag{9.158}$$

Collecting Eqs. (9.142), (9.152) and (9.156), we arrive at the two-term asymptotic approximation for the normal displacement

$$w(\mathbf{y}, \zeta) \simeq \varepsilon \frac{p(\mathbf{y})}{A_{33}} (h_* - \zeta) + \varepsilon^2 \frac{p(\mathbf{y})}{A_{33}} \tilde{H}_*(\mathbf{y}). \tag{9.159}$$

By taking into account the scaling relations (9.130), we rewrite (9.159) in the form

$$u_3(\mathbf{y}, z) \simeq \frac{p(\mathbf{y})}{A_{33}} \big(h + \tilde{H}(\mathbf{y}) - z\big). \tag{9.160}$$

By substituting the expression (9.160) into the contact condition

$$u_3(\mathbf{y}, 0) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega, \tag{9.161}$$

we derive the following equation for the contact pressure density:

$$\frac{h + \tilde{H}(\mathbf{y})}{A_{33}} p(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega. \tag{9.162}$$

In light of the condition $p(\mathbf{y}) > 0$ for $\mathbf{y} \in \omega$, we obtain

$$p(\mathbf{y}) = \frac{A_{33}}{h + \tilde{H}(\mathbf{y})} \big(\delta_0 - \varphi(\mathbf{y})\big)_+, \tag{9.163}$$

where $(x)_+ = \max\{x, 0\}$ is the positive-part function.

Invoking the notation (9.132) for the variable thickness of the elastic layer, we rewrite formula (9.163) as

$$p(\mathbf{y}) = \frac{A_{33}}{H(\mathbf{y})} \big(\delta_0 - \varphi(\mathbf{y})\big)_+. \tag{9.164}$$

Formula (9.164) shows that a thin compressible elastic layer deforms like a Winkler foundation with the variable foundation modulus

$$k(\mathbf{y}) = \frac{A_{33}}{H(\mathbf{y})}. \tag{9.165}$$

Let us recall that the elastic parameter A_{33} is related to Young's moduli, E and E', and Poisson's ratios, ν and ν', by the formulas

$$A_{33} = \frac{E'(1-\nu)}{1-\nu-\dfrac{2E}{E'}\,\nu'^2}, \tag{9.166}$$

Finally, according to [16] (see also Sect. 2.4), the denominator of (9.166) would approach zero if the layer material became more incompressible. It is readily seen from (9.165) that $k(\mathbf{y}) \to \infty$ in the incompressibility limit, implying that the case of an incompressible elastic layer requires special consideration.

9.2.6 Asymptotic Solution for a Thin Incompressible Layer

Let us continue the process of constructing terms of the asymptotic expansions (9.142) and (9.143). In light of (9.156), Eqs. (9.144)–(9.149) yield the following third-order problem:

$$A_{33}\frac{\partial^2 w^2}{\partial \zeta^2} = -(A_{13}+A_{44})\nabla_y \cdot \frac{\partial \mathbf{v}^1}{\partial \zeta} - A_{44}\Delta_y w^0, \quad \zeta \in (0, h_*),$$

$$A_{33}\frac{\partial w^2}{\partial \zeta}\bigg|_{\zeta=0} = -A_{13}\nabla_y \cdot \mathbf{v}^1\big|_{\zeta=0}, \quad w^2\big|_{\zeta=h_*} = 0; \tag{9.167}$$

$$\frac{\partial^2 \mathbf{v}^2}{\partial \zeta^2} = \mathbf{0}, \quad \zeta \in (0, h_*),$$

$$\frac{\partial \mathbf{v}^2}{\partial \zeta}\bigg|_{\zeta=0} = -\nabla_y w^1\big|_{\zeta=0}, \quad \mathbf{v}^2\big|_{\zeta=h_*} = -\tilde{H}_*(\mathbf{y})\frac{\partial \mathbf{v}^1}{\partial \zeta}\bigg|_{\zeta=h_*}. \tag{9.168}$$

Substituting (9.152) and (9.157) into Eqs. (9.167), we derive the problem

$$\frac{\partial^2 w^2}{\partial \zeta^2} = \frac{A_{13}\Delta_y p(\mathbf{y})}{A_{33}^2 A_{44}}\big[(A_{13}+2A_{44})(h_*-\zeta) - (A_{13}+A_{44})h_*\big], \quad \zeta \in (0, h_*),$$

$$\left.\frac{\partial w^2}{\partial \zeta}\right|_{\zeta=0} = -\frac{A_{13}(A_{44}-A_{13})h_*^2}{2A_{33}^2 A_{44}}\Delta_y p(\mathbf{y}), \quad \left.w^2\right|_{\zeta=h_*} = 0. \tag{9.169}$$

It can be verified that the solution to (9.169) condenses into the form

$$w^2(\mathbf{y},\zeta) = \frac{A_{13}\Delta_y p(\mathbf{y})}{A_{33}^2}\left\{\frac{A_{13}+2A_{44}}{6A_{44}}(h_*-\zeta)^3 - \frac{(A_{13}+A_{44})h_*}{2A_{44}}(h_*-\zeta)^2 + \frac{h_*^2}{2}(h_*-\zeta)\right\}. \tag{9.170}$$

Further, in light of (9.156) and (9.157), the problem (9.168) takes the form

$$\frac{\partial^2 \mathbf{v}^2}{\partial \zeta^2} = \mathbf{0}, \quad \zeta \in (0, h_*),$$

$$\left.\frac{\partial \mathbf{v}^2}{\partial \zeta}\right|_{\zeta=0} = -\frac{\nabla_y(p\tilde{H}_*)}{A_{33}}, \quad \left.\mathbf{v}^2\right|_{\zeta=h_*} = -\frac{A_{13}h_*}{A_{33}A_{44}}\tilde{H}_*\nabla_y p, \tag{9.171}$$

where we have omitted the arguments of functions $p(\mathbf{y})$ and $\tilde{H}_*(\mathbf{y})$ for clarity.

It can be easily verified that the solution to (9.171) has the form

$$\mathbf{v}^2(\mathbf{y},\zeta) = \frac{h_*-\zeta}{A_{33}}\nabla_y(p\tilde{H}_*) - \frac{A_{13}h_*}{A_{33}A_{44}}\tilde{H}_*\nabla_y p. \tag{9.172}$$

We emphasize that in contrast to the two-term approximation (9.159), the third term (9.170) does not vanish at the contact interface in the limit as $\nu \to 0.5$. Indeed, formula (9.170) yields

$$w^2(\mathbf{y},0) = -\frac{h_*^3 A_{13}(A_{13}-A_{44})}{3A_{33}^2 A_{44}}\Delta_y p(\mathbf{y}). \tag{9.173}$$

In the incompressibility limit, we have

$$\frac{A_{13}(A_{13}-A_{44})}{A_{33}^2 A_{44}} \to \frac{1}{a_{44}}, \tag{9.174}$$

where $a_{44} = A_{44} = G'$ is the out-of-plane shear modulus.

In order to construct a correction for the leading asymptotic term (9.170), we consider the following problem:

$$A_{33}\frac{\partial^2 w^3}{\partial \zeta^2} = -(A_{13}+A_{44})\nabla_y \cdot \frac{\partial \mathbf{v}^2}{\partial \zeta} - A_{44}\Delta_y w^1, \quad \zeta \in (0, h_*),$$

$$A_{33}\frac{\partial w^3}{\partial \zeta}\bigg|_{\zeta=0} = -A_{13}\nabla_y\cdot \mathbf{v}^2\big|_{\zeta=0}, \quad w^3\big|_{\zeta=h_*} = -\tilde{H}_*(\mathbf{y})\frac{\partial w^2}{\partial \zeta}\bigg|_{\zeta=h_*}. \tag{9.175}$$

By substituting (9.156), (9.170) and (9.172) into Eqs. (9.175), we find

$$\frac{\partial^2 w^3}{\partial \zeta^2} = \frac{A_{13}}{A_{33}^2}\Delta_y(p\tilde{H}_*), \quad \zeta \in (0, h_*), \tag{9.176}$$

$$\frac{\partial w^3}{\partial \zeta}\bigg|_{\zeta=0} = \frac{A_{13}h_*}{A_{33}^2 A_{44}}\big(A_{13}\nabla_y\cdot(\tilde{H}_*\nabla_y p) - A_{44}\Delta_y(p\tilde{H}_*)\big), \tag{9.177}$$

$$w^3\big|_{\zeta=h_*} = \frac{A_{13}h_*^2}{2A_{33}^2}\tilde{H}_*\Delta_y p. \tag{9.178}$$

Integrating Eq. (9.176), we obtain

$$w^3(\mathbf{y}, 0) = \frac{A_{13}}{2A_{33}^2}\Delta_y(p\tilde{H}_*)\zeta^2 + C_1(\mathbf{y})\zeta + C_0(\mathbf{y}), \tag{9.179}$$

where the integration functions $C_1(\mathbf{y})$ and $C_0(\mathbf{y})$ are determined by the boundary conditions (9.177) and (9.178). It can be checked that

$$C_1(\mathbf{y}) = \frac{A_{13}h_*}{A_{33}^2 A_{44}}\big(A_{13}\nabla_y\cdot(\tilde{H}_*\nabla_y p) - A_{44}\Delta_y(p\tilde{H}_*)\big), \tag{9.180}$$

$$C_0(\mathbf{y}) = \frac{A_{13}h_*^2}{2A_{33}^2}[\tilde{H}_*\Delta_y p + \Delta_y(p\tilde{H}_*)] - \frac{A_{13}^2 h_*^2}{A_{33}^2 A_{44}}\nabla_y\cdot(\tilde{H}_*\nabla_y p). \tag{9.181}$$

From (9.179), it immediately follows that

$$w^3(\mathbf{y}, 0) = C_0(\mathbf{y}), \tag{9.182}$$

and it can be shown that in the incompressibility limit (when $A_{13}/A_{33} \to 1$ and $A_{44}/A_{33} \to 0$), we arrive at the following result:

$$C_0(\mathbf{y}) = -\frac{h_*^2}{a_{44}}\nabla_y\cdot(\tilde{H}_*\nabla_y p). \tag{9.183}$$

Thus, collecting Eqs. (9.142), (9.173), (9.182) and (9.183), we obtain the following two-term asymptotic approximation for the normal displacement at the contact interface in the case of the incompressible elastic layer:

$$w(\mathbf{y}, 0) \simeq -\varepsilon^3\frac{h_*^3}{3a_{44}}\Delta_y p(\mathbf{y}) - \varepsilon^4\frac{h_*^2}{a_{44}}\nabla_y\cdot(\tilde{H}_*(\mathbf{y})\nabla_y p(\mathbf{y})). \tag{9.184}$$

Recalling the scaling relations (9.130) and the notation $a_{44} = G'$, we rewrite (9.184) in the form

$$u_3(\mathbf{y}, 0) \simeq -\frac{h^3}{3G'}\Delta_y p(\mathbf{y}) - \frac{h^2}{G'}\nabla_y \cdot (\tilde{H}(\mathbf{y})\nabla_y p(\mathbf{y})). \tag{9.185}$$

Now, substituting the expression (9.185) into the contact condition (9.161), we arrive at a partial differential equation in the domain ω with respect to the function $p(\mathbf{y})$. According to the asymptotic analysis [13] (see also Sect. 2.7.3), at the contour Γ of ω, we impose the following boundary conditions:

$$p(\mathbf{y}) = 0, \quad \frac{\partial p}{\partial n}(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{9.186}$$

Here, $\partial/\partial n$ is the normal derivative. We underline that the location of the contour Γ must be determined as part of the solution. For analytical evaluation of the contour Γ, a perturbation-based method was developed in [8].

9.2.7 Perturbation of the Contact Pressure in the Compressible Case

Collecting Eqs. (9.142), (9.152), (9.156), (9.170) and (9.179), we obtain

$$\begin{aligned} w(\mathbf{y}, 0) \simeq{}& \varepsilon\frac{h_*}{A_{33}}p(\mathbf{y}) + \varepsilon^2\frac{\tilde{H}_*(\mathbf{y})}{A_{33}}p(\mathbf{y}) - \varepsilon^3\frac{h_*^3 A_{13}(A_{13} - A_{44})}{3A_{33}^2 A_{44}}\Delta_y p(\mathbf{y}) \\ &+ \varepsilon^4\bigg\{\frac{h_*^2 A_{13}}{2A_{33}^2}\big[\tilde{H}_*(\mathbf{y})\Delta_y p(\mathbf{y}) + \Delta_y\big(p(\mathbf{y})\tilde{H}_*(\mathbf{y})\big)\big] \\ &- \frac{h_*^2 A_{13}^2}{A_{33}^2 A_{44}}\nabla_y \cdot \big(\tilde{H}_*(\mathbf{y})\nabla_y p(\mathbf{y})\big)\bigg\}. \end{aligned} \tag{9.187}$$

By substitution of the asymptotic expansion (9.187) into the contact condition (9.137) and using the notation (9.131), we arrive at the equation

$$\begin{aligned} p(\mathbf{y}) &+ \varepsilon\psi_*(\mathbf{y})p(\mathbf{y}) - \varepsilon^2\frac{h_*^3 A_{13}(A_{13} - A_{44})}{3A_{33}A_{44}}\Delta_y p(\mathbf{y}) \\ &+ \varepsilon^3\frac{A_{13}h_*^2}{2A_{33}A_{44}}\bigg\{A_{44}\big[\psi_*(\mathbf{y})\Delta_y p(\mathbf{y}) + \Delta_y\big(p(\mathbf{y})\psi_*(\mathbf{y})\big)\big] \\ &- 2A_{13}\nabla_y \cdot \big(\psi_*(\mathbf{y})\nabla_y p(\mathbf{y})\big)\bigg\} = \frac{A_{33}}{h_*}f^*(\mathbf{y}), \end{aligned} \tag{9.188}$$

where we have introduced the shorthand notation

$$f^*(\mathbf{y}) = \delta_0^* - \varphi^*(\mathbf{y}). \tag{9.189}$$

It should be noted that Eq. (9.188) is applied in the case of compressible materials when its right-hand side makes sense.

By employing a perturbation method, the third-order asymptotic solution to Eq. (9.188) is expressed in the form

$$p(\mathbf{y}) \simeq \frac{A_{33}}{h_*}\big(\sigma_0(\mathbf{y}) + \varepsilon\sigma_1(\mathbf{y}) + \varepsilon^2\sigma_2(\mathbf{y}) + \varepsilon^3\sigma_3(\mathbf{y})\big). \quad (9.190)$$

Upon substitution of (9.195) into (9.188), we straightforwardly obtain

$$\sigma_0 = f^*, \quad \sigma_1 = -\psi_* f^*, \quad \sigma_2 = \psi_*^2 f^* + \frac{h_*^2 A_{13}(A_{13} - A_{44})}{3A_{33}A_{44}}\Delta_y f^*, \quad (9.191)$$

$$\sigma_3 = -\psi_*^3 f^* - \frac{h_*^2 A_{13}(A_{13} - A_{44})}{3A_{33}A_{44}}\big(\psi_*\Delta_y f^* + \Delta_y(\psi_* f^*)\big)$$
$$- \frac{h_*^2 A_{13}}{2A_{33}A_{44}}\big\{A_{44}\big[\psi_*\Delta_y f^* + \Delta_y(f^*\psi_*)\big] - 2A_{13}\nabla_y\cdot(\psi_*\nabla_y f^*)\big\}, \quad (9.192)$$

where, for the sake of brevity, the argument $\mathbf{y}$ is omitted.

Further, by making use of the differential identities

$$\nabla\cdot(\psi\nabla f) = \nabla\psi\cdot\nabla f + \psi\Delta f,$$

$$\Delta(f\psi) = \psi\Delta f + f\Delta\psi + 2\nabla f\cdot\nabla\psi,$$

we simplify formula (9.192) as follows:

$$\sigma_3 = -\psi_*^3 f^* - \frac{h_*^2 A_{13}(2A_{13} + A_{44})}{6A_{33}A_{44}} f^*\Delta_y\psi_*$$
$$+ \frac{h_*^2 A_{13}(A_{13} - A_{44})}{3A_{33}A_{44}}\big(\nabla_y f^*\cdot\nabla_y\psi_* + \psi_*\Delta_y f^*\big). \quad (9.193)$$

Observe also that formula (9.181) can be transformed to

$$C_0(\mathbf{y}) = -\frac{h_*^2 A_{13}(A_{13} - A_{44})}{A_{33}^2 A_{44}}\big[\nabla_y\tilde{H}_*\cdot\nabla_y p + \tilde{H}_*\Delta_y p\big] + \frac{h_*^2 A_{13}}{2A_{33}^2} p\Delta_y\tilde{H}_*, \quad (9.194)$$

where the expression in the brackets in (9.194) is equal to $\nabla_y\cdot(\tilde{H}_*\nabla_y p)$.

In the isotropic case, Eqs. (9.190)–(9.193) reduce to the following [5]:

$$p(\mathbf{y}) \simeq \frac{2\mu + \lambda}{h_*}\big(\sigma_0(\mathbf{y}) + \varepsilon\sigma_1(\mathbf{y}) + \varepsilon^2\sigma_2(\mathbf{y}) + \varepsilon^3\sigma_3(\mathbf{y})\big), \quad (9.195)$$

$$\sigma_0 = f^*, \quad \sigma_1 = -\psi_* f^*, \quad \sigma_2 = \psi_*^2 f^* + \frac{h_*^2 \lambda(\lambda - \mu)}{3\mu(2\mu + \lambda)} \Delta_y f^*, \tag{9.196}$$

$$\sigma_3 = -\psi_*^3 f^* - \frac{h_*^2 \lambda(2\lambda + \mu)}{6\mu(2\mu + \lambda)} f^* \Delta_y \psi_* + \frac{h_*^2 \lambda(\lambda - \mu)}{3\mu(2\mu + \lambda)} \big(\nabla_y f^* \cdot \nabla_y \psi_* + \psi_* \Delta_y f^*\big). \tag{9.197}$$

It can be easily checked that the four-term asymptotic expansion (9.195), with the coefficients given by (9.196) and (9.197) in the 2D case, recovers the corresponding solution obtained in [21], where the next asymptotic term for the contact pressure in (9.195) was given explicitly.

9.2.8 Application to Sensitivity Analysis of the Contact Interaction Between Two Thin Incompressible Layers

According to (9.185) and (9.186), the refined asymptotic model for contact interaction of thin incompressible layers bonded to rigid substrates takes the form

$$-m^{-1} \Delta_y p(\mathbf{y}) - \sum_{n=1}^{2} \frac{h_n^2}{G_n'} \nabla_y \cdot (\tilde{H}_n(\mathbf{y}) \nabla_y p(\mathbf{y})) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \tilde{\omega}, \tag{9.198}$$

$$p(\mathbf{y}) = 0, \quad \frac{\partial p}{\partial n}(\mathbf{y}) = 0, \quad \mathbf{y} \in \tilde{\Gamma}, \tag{9.199}$$

where $\tilde{\Gamma}$ is the contour of the contact region $\tilde{\omega}$, and m is given in (9.117).

Let us define

$$p(\mathbf{y}) = \bar{p}(\mathbf{y}) + \tilde{p}(\mathbf{y}), \tag{9.200}$$

where $\bar{p}(\mathbf{y})$ is the solution to the original asymptotic model (9.112), (9.113), and $\tilde{p}(\mathbf{y})$ represents a perturbation due to the variability of the layer thickness.

Then, under the assumption that the thickness variation functions $\tilde{H}_1(\mathbf{y})$ and $\tilde{H}_2(\mathbf{y})$ introduce a small variation into the elliptical contact region ω corresponding to the density $\bar{p}(\mathbf{y})$, we derive from (9.198)–(9.200) the following limit problem for the variation of the contact pressure density:

$$-m^{-1} \Delta_y \tilde{p}(\mathbf{y}) = \sum_{n=1}^{2} \frac{h_n^2}{G_n'} \nabla_y \cdot (\tilde{H}_n(\mathbf{y}) \nabla_y \bar{p}(\mathbf{y})), \quad \mathbf{y} \in \omega, \tag{9.201}$$

$$\tilde{p}(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{9.202}$$

Here, Γ is the contour corresponding to the contact pressure (9.115).

Moreover, the thickness variation functions $\tilde{H}_1(\mathbf{y})$ and $\tilde{H}_2(\mathbf{y})$ will not greatly influence the resulting force-displacement relationship, if

$$\iint\limits_{\omega} \tilde{p}(\mathbf{y})\, d\mathbf{y} = 0. \tag{9.203}$$

In this case, the contact force P is related to the displacement δ_0 by the same relation as that derived in framework of the original asymptotic model (9.112), (9.113).

Let us derive the conditions for $\tilde{H}_1(\mathbf{y})$ and $\tilde{H}_2(\mathbf{y})$ under which the equality (9.203) holds true. With this aim we consider an auxiliary problem

$$\Delta_y \Theta(\mathbf{y}) = 1, \quad \mathbf{y} \in \omega, \quad \Theta(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma \tag{9.204}$$

with the solution

$$\Theta(\mathbf{y}) = -\frac{a_1^2 a_2^2}{2(a_1^2 + a_2^2)}\theta(\mathbf{y}),$$

where

$$\theta(\mathbf{y}) = 1 - \frac{y_1^2}{a_1^2} - \frac{y_2^2}{a_2^2}. \tag{9.205}$$

As a result of (9.204), we rewrite Eq. (9.203) as

$$\iint\limits_{\omega} \tilde{p}(\mathbf{y}) \Delta_y \Theta(\mathbf{y})\, d\mathbf{y} = 0. \tag{9.206}$$

Applying the second Green's formula and taking into account Eqs. (9.201), (9.202) and (9.204), we reduce Eq. (9.206) to

$$\iint\limits_{\omega} \theta(\mathbf{y}) \sum_{n=1}^{2} \frac{h_n^2}{G_n'} \nabla_y \cdot (\tilde{H}_n(\mathbf{y}) \nabla_y \bar{p}(\mathbf{y}))\, d\mathbf{y} = 0. \tag{9.207}$$

After rewriting Eq. (9.207) in the form

$$\iint\limits_{\omega} \theta(\mathbf{y}) \sum_{k=1}^{2} \frac{\partial}{\partial y_k} \left(\frac{\partial \bar{p}}{\partial y_k}(\mathbf{y}) \sum_{n=1}^{2} \frac{h_n^2}{G_n'} \tilde{H}_n(\mathbf{y}) \right) d\mathbf{y} = 0$$

and integrating by parts with (9.205) taken into account, we find

$$-\iint_{\omega} \sum_{k=1}^{2} \frac{2y_k}{a_k^2} \frac{\partial \bar{p}}{\partial y_k}(\mathbf{y}) \sum_{n=1}^{2} \frac{h_n^2}{G_n'} \tilde{H}_n(\mathbf{y})\, d\mathbf{y}$$
$$+ \int_{\Gamma} \theta(\mathbf{y}) \sum_{k=1}^{2} \cos(n, y_k) \frac{\partial \bar{p}}{\partial y_k}(\mathbf{y}) \sum_{n=1}^{2} \frac{h_n^2}{G_n'} \tilde{H}_n(\mathbf{y})\, ds_y = 0. \tag{9.208}$$

It is clear that the line integral in (9.208) vanishes due to the boundary condition (9.205). Hence, taking into account the exact expression (9.115) for $\bar{p}(\mathbf{y})$, we finally transform Eq. (9.208) into

$$\sum_{n=1}^{2} \frac{h_n^2}{G_n'} \iint_{\omega} \tilde{H}_n(\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} = 0, \tag{9.209}$$

where we have introduced the notation

$$\rho(\mathbf{y}) = \left(\frac{s y_1^2}{a_1^2} + \frac{y_2^2}{s a_2^2}\right)\left(1 - \frac{y_1^2}{a_1^2} - \frac{y_2^2}{a_2^2}\right). \tag{9.210}$$

Based on the derived Eq. (9.209), we suggest the following optimization criterion for determining the average thicknesses h_1 and h_2:

$$\min_{h_n} \iint_{\omega_*} (H_n(\mathbf{y}) - h_n)^2 \rho_*(\mathbf{y})\, d\mathbf{y}. \tag{9.211}$$

Here, ω_* is a characteristic elliptic domain with semi-axes a_1^* and a_2^*. In particular, in the capacity of ω_* we can take the average elliptic contact area for a class of admissible contact loadings, while $\rho_*(\mathbf{y})$ is given by

$$\rho_*(\mathbf{y}) = \left(\frac{s^* y_1^2}{a_1^{*2}} + \frac{y_2^2}{s^* a_2^{*2}}\right)\left(1 - \frac{y_1^2}{a_1^{*2}} - \frac{y_2^2}{a_2^{*2}}\right), \tag{9.212}$$

where $s^* = a_2^*/a_1^*$ is the aspect ratio of ω_*.

It is clear that the necessary optimality condition for (9.211) has the form

$$\iint_{\omega_*} (H_n(\mathbf{y}) - h_n)\rho_*(\mathbf{y})\, d\mathbf{y} = 0, \tag{9.213}$$

from which it follows that

$$h_n = \frac{1}{R_*} \iint\limits_{\omega_*} H_n(\mathbf{y}) \rho_*(\mathbf{y}) \, d\mathbf{y}, \tag{9.214}$$

where

$$R_* = \iint\limits_{\omega_*} \rho_*(\mathbf{y}) \, d\mathbf{y} = \frac{\pi}{12} \big(a_1^{*2} + a_2^{*2} \big).$$

It remains to show that Eq. (9.209) follows from (9.213) if ω_* coincides with ω. Indeed, as a consequence of (9.132), Eq. (9.213) is equivalent to the following:

$$\iint\limits_{\omega_*} \tilde{H}_n(\mathbf{y}) \rho_*(\mathbf{y}) \, d\mathbf{y} = 0, \quad n = 1, 2. \tag{9.215}$$

By adding the two equations above, multiplied by h_n^2/G_n', $n = 1, 2$, respectively, we arrive at Eq. (9.209).

It is interesting to observe that Eq. (9.214) indicates that, in order to obtain the optimal average thickness h_n, the corresponding variable thickness $H_n(\mathbf{y})$ has been averaged with the weight function $\rho_*(\mathbf{y})$ given by (9.212).

To conclude, we note that in the case of compressible layers, the optimal value of the average thickness h_n coincides with the simple average of $H_n(\mathbf{y})$.

References

1. Akiyama, K., Sakai, T., Sugimoto, N., Yoshikawa, H., Sugamoto, K.: Three-dimensional distribution of articular cartilage thickness in the elderly talus and calcaneus analyzing the subchondral bone plate density. Osteoarthritis Cartilage **20**, 296–304 (2012)
2. Anderson, A.E., Ellis, B.J., Maas, S.A., Weiss, J.A.: Effects of idealized joint geometry on finite element predictions of cartilage contact stresses in the hip. J. Biomech. **43**, 1351–1357 (2010)
3. Argatov, I.I.: Pressure of a punch in the form of an elliptic paraboloid on a thin elastic layer. Acta Mech. **180**, 221–232 (2005)
4. Argatov, I.: Development of an asymptotic modeling methodology for tibio-femoral contact in multibody dynamic simulations of the human knee joint. Multibody Syst. Dyn. **28**, 3–20 (2012)
5. Argatov, I.: Contact problem for a thin elastic layer with variable thickness: Application to sensitivity analysis of articular contact mechanics. Appl. Math. Model. **37**, 8383–8393 (2013)
6. Argatov, I., Mishuris, G.: Elliptical contact of thin biphasic cartilage layers: Exact solution for monotonic loading. J. Biomech. **44**, 759–761 (2011)
7. Argatov, I., Mishuris, G.: Frictionless elliptical contact of thin viscoelastic layers bonded to rigid substrates. Appl. Math. Model. **35**, 3201–3212 (2011)
8. Argatov, I.I., Mishuris, G.S.: Contact problem for thin biphasic cartilage layers: perturbation solution. Quart. J. Mech. Appl. Math. **64**, 297–318 (2011)
9. Ateshian, G.A., Lai, W.M., Zhu, W.B., Mow, V.C.: An asymptotic solution for the contact of two biphasic cartilage layers. J. Biomech. **27**, 1347–1360 (1994)

10. Barber, J.R.: Contact problems for the thin elastic layer. Int. J. Mech. Sci. **32**, 129–132 (1990)
11. Barry, S.I., Holmes, M.: Asymptotic behaviour of thin poroelastic layers. IMA J. Appl. Math. **66**, 175–194 (2001)
12. Bei, Y., Fregly, B.J.: Multibody dynamic simulation of knee contact mechanics. Med. Eng. Phys. **26**, 777–789 (2004)
13. Chadwick, R.S.: Axisymmetric indentation of a thin incompressible elastic layer. SIAM J. Appl. Math. **62**, 1520–1530 (2002)
14. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products. Academic, New York (1980)
15. Hsiao, G.C., Steinbach, O., Wendland, W.L.: Domain decomposition methods via boundary integral equations. J. Comput. Appl. Math. **125**, 521–537 (2000)
16. Itskov, M., Aksel, N.: Elastic constants and their admissible values for incompressible and slightly compressible anisotropic materials. Acta Mech. **157**, 81–96 (2002)
17. Jaffar, M.J.: Asymptotic behaviour of thin elastic layers bonded and unbonded to a rigid foundation. Int. J. Mech. Sci. **31**, 229–235 (1989)
18. Nazarov, S.A.: Perturbations of solutions of the Signorini problem for a second-order scalar equation. Math. Notes **47**, 115–126 (1990)
19. Siu, D., Rudan, J., Wevers, H.W., Griffiths, P.: Femoral articular shape and geometry: A three-dimensional computerized analysis of the knee. J. Arthroplasty **11**, 166–173 (1996)
20. Van Dyke, M.D.: Perturbation Methods in Fluid Mechanics. Academic Press, New York (1964)
21. Vorovich, I.I., Penin, O.M.: Contact problem for an infinite strip of variable height [in Russian]. Solid Mech. **5**, 112–121 (1971)