Chapter 8
Contact of Thin Inhomogeneous Transversely Isotropic Elastic Layers

Abstract In this chapter we consider contact problems for thin bonded inhomogeneous transversely isotropic elastic layers. In particular, in Sects. 8.1 and 8.2, the deformation problems are studied for the cases of elastic layers with the out-of-plane and thickness-variable inhomogeneous properties, respectively. In Sect. 8.3, the axisymmetric frictionless contact problems for thin incompressible inhomogeneous elastic layers are studied in detail in the framework of the leading-order asymptotic model. Finally, the deformation problem for a transversely isotropic elastic layer bonded to a rigid substrate, and coated with a very thin elastic layer made of another transversely isotropic material is analyzed in Sect. 8.4.

8.1 Deformation of an In-Plane Inhomogeneous Elastic Layer

In the present section, the leading-order asymptotic models for the local indentation of compressible and incompressible elastic layers developed in Chaps. 1 and 2 are generalized for elastic layers with in-plane inhomogeneous material properties.

8.1.1 Deformation Problem Formulation

Recall that the constitutive relationship for a transversely isotropic media based in the Cartesian coordinates (y_1, y_2, z), where the Oy_1y_2 plane coincides with the plane of elastic symmetry, has the following form [9]:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{11} & A_{13} & 0 & 0 & 0 \\ A_{13} & A_{13} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2A_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}. \tag{8.1}$$

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In this section, we assume that the elastic constants A_{kl} are functions of the in-plane coordinates $\mathbf{y} = (y_1, y_2)$, so that the layer possesses an in-plane inhomogeneity.

By substituting the components of strain

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\Big(\frac{\partial v_\alpha}{\partial y_\beta} + \frac{\partial v_\beta}{\partial y_\alpha}\Big), \quad \alpha, \beta = 1, 2,$$

$$\varepsilon_{3\alpha} = \frac{1}{2}\Big(\frac{\partial w}{\partial y_\alpha} + \frac{\partial v_\alpha}{\partial z}\Big), \quad \varepsilon_{33} = \frac{\partial w}{\partial z}$$

into Hooke's law (8.1), we obtain the stress-displacement relations

$$\begin{aligned}
\sigma_{11} &= A_{11}\frac{\partial v_1}{\partial y_1} + A_{12}\frac{\partial v_2}{\partial y_2} + A_{13}\frac{\partial w}{\partial z}, \quad \sigma_{23} = A_{44}\Big(\frac{\partial w}{\partial y_2} + \frac{\partial v_2}{\partial z}\Big),\\
\sigma_{22} &= A_{12}\frac{\partial v_1}{\partial y_1} + A_{11}\frac{\partial v_2}{\partial y_2} + A_{13}\frac{\partial w}{\partial z}, \quad \sigma_{13} = A_{44}\Big(\frac{\partial w}{\partial y_1} + \frac{\partial v_1}{\partial z}\Big),\\
\sigma_{33} &= A_{13}\frac{\partial v_1}{\partial y_1} + A_{13}\frac{\partial v_2}{\partial y_2} + A_{33}\frac{\partial w}{\partial z}, \quad \sigma_{12} = A_{66}\Big(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\Big),
\end{aligned} \tag{8.2}$$

where $A_{12} = A_{11} - 2A_{66}$ due to the in-plane symmetry properties.

The substitution of the above expressions into the equilibrium equations

$$\frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} = 0, \quad i = 1, 2, 3,$$

yields the system of Lamé equations

$$\mathfrak{L}(\nabla_y)\mathbf{v} + \nabla_y\Big(A_{13}\frac{\partial w}{\partial z}\Big) + A_{44}\nabla_y\frac{\partial w}{\partial z} + A_{44}\frac{\partial^2 \mathbf{v}}{\partial z^2} = \mathbf{0}, \tag{8.3}$$

$$\nabla_y \cdot (A_{44}\nabla_y w) + \frac{\partial}{\partial z}\nabla_y \cdot (A_{44}\mathbf{v}) + \frac{\partial}{\partial z}A_{13}\nabla_y \cdot \mathbf{v} + A_{33}\frac{\partial^2 w}{\partial z^2} = 0, \tag{8.4}$$

where $\nabla_y = (\partial/\partial y_1, \partial/\partial y_2)$ is the in-plane Hamilton operator, the scalar product is denoted by a dot, and $\mathfrak{L}(\nabla_y)$ is a 2×2 matrix differential operator such that

$$\mathfrak{L}_{\alpha\alpha}(\nabla_y) = \frac{\partial}{\partial y_\alpha}\Big(A_{11}\frac{\partial}{\partial y_\alpha}\Big) + \frac{\partial}{\partial y_{3-\alpha}}\Big(A_{66}\frac{\partial}{\partial y_{3-\alpha}}\Big),$$

$$\mathfrak{L}_{\alpha\beta}(\nabla_y) = \frac{\partial}{\partial y_\alpha}\Big(A_{12}\frac{\partial}{\partial y_\beta}\Big) + \frac{\partial}{\partial y_\beta}\Big(A_{66}\frac{\partial}{\partial y_\alpha}\Big), \quad \alpha, \beta = 1, 2, \quad \alpha \neq \beta.$$

We consider the deformation of a thin in-plane inhomogeneous elastic layer (with the elastic constants $A_{kl}(\mathbf{y})$) of uniform thickness, h, ideally bonded to a rigid

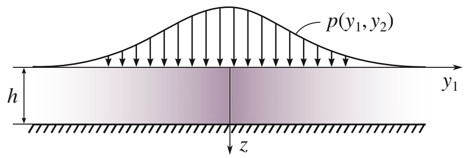

Fig. 8.1 An in-plane inhomogeneous elastic layer of uniform thickness bonded to a rigid substrate and supporting a distributed normal load

substrate (see Fig. 8.1). At the bottom surface of the layer, $z = h$, the following boundary conditions are imposed:

$$\mathbf{v}\big|_{z=h} = \mathbf{0}, \quad w\big|_{z=h} = 0. \tag{8.5}$$

On the upper surface of the elastic layer, we impose the boundary conditions of normal loading with no tangential tractions

$$\sigma_{13}\big|_{z=0} = \sigma_{23}\big|_{z=0} = 0, \quad \sigma_{33}\big|_{z=0} = -p, \tag{8.6}$$

where the normal load $p(\mathbf{y})$ is a given (sufficiently smooth) function of the in-plane coordinates $\mathbf{y} = (y_1, y_2)$.

The problem (8.3)–(8.6) generalizes the deformation problem studied in Chaps. 1 and 2 for the case of transversely isotropic elastic layers with in-plane inhomogeneous properties.

8.1.2 Perturbation Analysis of the Deformation Problem

Assuming that the elastic layer is relatively thin, we require that

$$h = \varepsilon h_*, \tag{8.7}$$

where ε is a small positive parameter, and h_* is a characteristic length, which is assumed to be independent of ε.

By introducing the so-called "stretched" *dimensional* normal coordinate

$$\zeta = \frac{z}{\varepsilon}, \tag{8.8}$$

we transform the Lamé equations (8.3), (8.4) into the following:

$$\frac{1}{\varepsilon^2} A_{44} \frac{\partial^2 \mathbf{v}}{\partial \zeta^2} + \frac{1}{\varepsilon} \frac{\partial}{\partial \zeta} \big(\nabla_y (A_{13} w) + A_{44} \nabla_y w\big) + \mathfrak{L}(\nabla_y)\mathbf{v} = \mathbf{0}, \tag{8.9}$$

$$\frac{1}{\varepsilon^2} A_{33} \frac{\partial^2 w}{\partial \zeta^2} + \frac{1}{\varepsilon} \frac{\partial}{\partial \zeta} \big(\nabla_y \cdot (A_{44}\mathbf{v}) + A_{13} \nabla_y \cdot \mathbf{v}\big) + \nabla_y \cdot (A_{44} \nabla_y w) = 0. \tag{8.10}$$

Correspondingly, the boundary conditions (8.6) become

$$\frac{1}{\varepsilon}\frac{\partial \mathbf{v}}{\partial \zeta} + \nabla_y w\Big|_{\zeta=0} = \mathbf{0}, \tag{8.11}$$

$$\frac{1}{\varepsilon}A_{33}\frac{\partial w}{\partial \zeta} + A_{13}\nabla_y \cdot \mathbf{v}\Big|_{\zeta=0} = -p. \tag{8.12}$$

As before, we apply the perturbation algorithm [14] to construct an approximate solution to the system (8.5), (8.9)–(8.12) in the form of asymptotic expansions

$$\mathbf{v} = \varepsilon \mathbf{v}^0(\mathbf{y}, \zeta) + \varepsilon^2 \mathbf{v}^1(\mathbf{y}, \zeta) + \cdots, \tag{8.13}$$

$$w = \varepsilon w^0(\mathbf{y}, \zeta) + \varepsilon^2 w^1(\mathbf{y}, \zeta) + \cdots, \tag{8.14}$$

where the successive coefficients of the powers of ε are independent of ε.

Following the standard procedure of the perturbation technique, we derive a set of equations for the terms of expansions (8.13) and (8.14). In particular, the leading terms of the asymptotic expansions are determined as solutions of the problems

$$A_{44}\frac{\partial^2 \mathbf{v}^0}{\partial \zeta^2} = \mathbf{0}, \quad \zeta \in (0, h_*), \quad \frac{\partial \mathbf{v}^0}{\partial \zeta}\Big|_{\zeta=0} = \mathbf{0}, \quad \mathbf{v}^0\big|_{\zeta=h_*} = \mathbf{0}; \tag{8.15}$$

$$A_{33}\frac{\partial^2 w^0}{\partial \zeta^2} = 0, \quad \zeta \in (0, h_*), \quad A_{33}\frac{\partial w^0}{\partial \zeta}\Big|_{\zeta=0} = -p, \quad w^0\big|_{\zeta=h_*} = 0, \tag{8.16}$$

from here it follows that

$$\mathbf{v}^0(\mathbf{y}, \zeta) \equiv \mathbf{0}, \quad w^0(\mathbf{y}, \zeta) = \frac{p(\mathbf{y})}{A_{33}}(h_* - \zeta). \tag{8.17}$$

The next two terms of the asymptotic expansions (8.13) and (8.14) satisfy the problems

$$A_{44}\frac{\partial^2 \mathbf{v}^1}{\partial \zeta^2} = -\nabla_y\Big(A_{13}\frac{\partial w^0}{\partial \zeta}\Big) - A_{44}\nabla_y\frac{\partial w^0}{\partial \zeta}, \quad \zeta \in (0, h_*),$$

$$\frac{\partial \mathbf{v}^1}{\partial \zeta}\Big|_{\zeta=0} = -\nabla_y w^0\big|_{\zeta=0}, \quad \mathbf{v}^1\big|_{\zeta=h_*} = \mathbf{0}; \tag{8.18}$$

$$A_{33}\frac{\partial^2 w^1}{\partial \zeta^2} = -\frac{\partial}{\partial \zeta}\nabla_y \cdot (A_{44}\mathbf{v}^0) - A_{13}\frac{\partial}{\partial \zeta}\nabla_y \cdot \mathbf{v}^0, \quad \zeta \in (0, h_*),$$

$$A_{33}\frac{\partial w^1}{\partial \zeta}\Big|_{\zeta=0} = -A_{13}\nabla_y \cdot \mathbf{v}^0\big|_{\zeta=0}, \quad w^1\big|_{\zeta=h_*} = 0. \tag{8.19}$$

In light of $(8.17)_1$, from (8.19), it immediately follows that

$$w^1(\mathbf{y},\zeta) \equiv 0, \tag{8.20}$$

while the substitution of $(8.17)_2$ into Eq. (8.18) leads to the problem

$$\begin{gathered} A_{44}\frac{\partial^2 \mathbf{v}^1}{\partial \zeta^2} = \nabla_y\Bigl(\frac{A_{13}}{A_{33}}p\Bigr) + A_{44}\nabla_y\Bigl(\frac{p}{A_{33}}\Bigr), \quad \zeta \in (0,h_*), \\ \frac{\partial \mathbf{v}^1}{\partial \zeta}\Big|_{\zeta=0} = -h_*\nabla_y\Bigl(\frac{p}{A_{33}}\Bigr), \quad \mathbf{v}^1\big|_{\zeta=h_*} = \mathbf{0}. \end{gathered} \tag{8.21}$$

As the elastic constants and the normal load are functions of the in-plane coordinates only, the right-hand side of Eq. $(8.21)_1$ is independent of ζ. Thus, the double integration of Eq. $(8.21)_1$ with the boundary conditions $(8.21)_2$ and $(8.21)_3$ taken into account yields

$$\mathbf{v}^1(\mathbf{y},\zeta) = -\frac{(h_*^2-\zeta^2)}{2A_{44}}\nabla_y\Bigl(\frac{A_{13}}{A_{33}}p\Bigr) + \frac{(h_*-\zeta)^2}{2}\nabla_y\Bigl(\frac{p}{A_{33}}\Bigr). \tag{8.22}$$

The second non-trivial term of the asymptotic expansion satisfies the problem

$$\begin{gathered} A_{33}\frac{\partial^2 w^2}{\partial \zeta^2} = -\frac{\partial}{\partial \zeta}\bigl[\nabla_y\cdot(A_{44}\mathbf{v}^1) - A_{13}\nabla_y\cdot\mathbf{v}^1\bigr] - \nabla_y\cdot\bigl(A_{44}\nabla_y w^0\bigr), \\ A_{33}\frac{\partial w^2}{\partial \zeta}\Big|_{\zeta=0} = -A_{13}\nabla_y\cdot\mathbf{v}^1\big|_{\zeta=0}, \quad w^2\big|_{\zeta=h_*} = 0. \end{gathered}$$

In light of $(8.17)_2$ and (8.22), the above equations take the form

$$\begin{aligned} A_{33}\frac{\partial^2 w^2}{\partial \zeta^2} &= -\zeta\,\Delta_y\Bigl(\frac{A_{13}}{A_{33}}p\Bigr) - \zeta A_{13}\nabla_y\cdot\biggl(\frac{1}{A_{44}}\nabla_y\Bigl(\frac{A_{13}}{A_{33}}p\Bigr)\biggr) \\ &\quad + (h_*-\zeta)A_{13}\Delta_y\Bigl(\frac{p}{A_{33}}\Bigr), \quad \zeta \in (0,h_*), \end{aligned} \tag{8.23}$$

$$\frac{\partial w^2}{\partial \zeta}\Big|_{\zeta=0} = \frac{h_*^2 A_{13}}{2A_{33}}\biggl[\nabla_y\cdot\biggl(\frac{1}{A_{44}}\nabla_y\Bigl(\frac{A_{13}}{A_{33}}p\Bigr)\biggr) - \Delta_y\Bigl(\frac{p}{A_{33}}\Bigr)\biggr], \quad w^2\big|_{\zeta=h_*} = 0.$$

Integrating Eq. $(8.23)_1$ and taking into account the boundary condition $(8.23)_2$ at $\zeta = 0$, we obtain

$$\begin{aligned} A_{33}\frac{\partial w^2}{\partial \zeta} &= -\frac{\zeta^2}{2}\Delta_y\Bigl(\frac{A_{13}}{A_{33}}p\Bigr) + \frac{(h_*^2-\zeta^2)}{2}A_{13}\nabla_y\cdot\biggl(\frac{1}{A_{44}}\nabla_y\Bigl(\frac{A_{13}}{A_{33}}p\Bigr)\biggr) \\ &\quad + \Bigl(h_*\zeta - \frac{\zeta^2}{2} - \frac{h_*^2}{2}\Bigr)A_{13}\Delta_y\Bigl(\frac{p}{A_{33}}\Bigr), \quad \zeta \in (0,h_*), \end{aligned}$$

By integrating the above equation and taking into account the boundary condition $(8.23)_3$ at $\zeta = h_*$, we arrive at the formula

$$
\begin{aligned}
w^2(\mathbf{y},\zeta) = {} & \frac{1}{6A_{33}}(h_*^3 - \zeta^3)\Delta_y\Big(\frac{A_{13}}{A_{33}}p\Big) \\
& - \frac{A_{13}}{6A_{33}}(h_* - \zeta)^2(\zeta + 2h_*)\nabla_y \cdot \Bigg(\frac{1}{A_{44}}\nabla_y\Big(\frac{A_{13}}{A_{33}}p\Big)\Bigg) \\
& + \frac{A_{13}}{6A_{33}}(h_* - \zeta)^3\Delta_y\Big(\frac{p}{A_{33}}\Big),
\end{aligned}
\tag{8.24}
$$

where for brevity we do not show the argument $\mathbf{y}$ of the functions $p(\mathbf{y})$, $A_{13}(\mathbf{y})$, $A_{33}(\mathbf{y})$, and $A_{44}(\mathbf{y})$.

8.1.3 *Local Indentation of the In-Plane Inhomogeneous Layer: Leading-Order Asymptotics for the Compressible and Incompressible Cases*

Recall that the local indentation of an elastic layer is defined as

$$w_0(\mathbf{y}) \equiv w(\mathbf{y}, 0),$$

where $w(\mathbf{y}, 0)$ is the normal displacement of the layer surface.

In the case of the compressible layer, Eqs. (8.7), (8.8), (8.14), and $(8.17)_2$ yield

$$w_0(\mathbf{y}) \simeq \frac{h}{A_{33}(\mathbf{y})}p(\mathbf{y}), \tag{8.25}$$

so that the deformation response of the elastic layer is analogous to that of a Winkler elastic foundation with the variable modulus

$$k(\mathbf{y}) = \frac{A_{33}(\mathbf{y})}{h}. \tag{8.26}$$

We emphasize that formula (8.26) is valid for a thin bonded compressible transversely isotropic elastic layer with in-plane inhomogeneous properties.

When the material approaches the incompressible limit, the right-hand side of (8.26) increases unboundedly and the first term in the asymptotic expansion (8.14) disappears. Consequently, the ratios A_{44}/A_{33} and A_{13}/A_{33} tend to 0 and 1, respectively.

Therefore, in the limit situation, formula (8.24) reduces to

$$w^2(\mathbf{y},\zeta) = -\frac{1}{6}(h_* - \zeta)^2(\zeta + 2h_*)\nabla_y \cdot \Big(\frac{1}{A_{44}}\nabla_y p\Big). \tag{8.27}$$

In the case of an incompressible bonded elastic layer, formulas (8.7), (8.8), (8.14), and (8.27) give

$$w_0(\mathbf{y}) \simeq -\frac{h^3}{3}\nabla_y \cdot \left(\frac{1}{a_{44}(\mathbf{y})}\nabla_y p(\mathbf{y})\right), \tag{8.28}$$

where $a_{44} = A_{44}$ is the out-of-plane shear modulus.

8.2 Deformation of an Elastic Layer with Thickness-Variable Inhomogeneous Properties

In the present section, the leading-order asymptotic models for the local indentation of thin bonded compressible and incompressible elastic layers developed in Chaps. 1 and 2 are generalized for elastic layers with the so-called thickness-variable inhomogeneous material properties.

8.2.1 Deformation Problem Formulation

Let us consider the deformation of a thin transversely isotropic inhomogeneous elastic layer of uniform thickness, h, with variable properties across the layer thickness (see Fig. 8.2). If the plane of isotropy is parallel to the layer surface, the stress-displacement relations take the form

$$\begin{aligned}
\sigma_{11} &= A_{11}\frac{\partial v_1}{\partial y_1} + A_{12}\frac{\partial v_2}{\partial y_2} + A_{13}\frac{\partial w}{\partial z}, \quad & \sigma_{23} &= A_{44}\Bigl(\frac{\partial w}{\partial y_2} + \frac{\partial v_2}{\partial z}\Bigr),\\
\sigma_{22} &= A_{12}\frac{\partial v_1}{\partial y_1} + A_{11}\frac{\partial v_2}{\partial y_2} + A_{13}\frac{\partial w}{\partial z}, \quad & \sigma_{13} &= A_{44}\Bigl(\frac{\partial w}{\partial y_1} + \frac{\partial v_1}{\partial z}\Bigr),\\
\sigma_{33} &= A_{13}\frac{\partial v_1}{\partial y_1} + A_{13}\frac{\partial v_2}{\partial y_2} + A_{33}\frac{\partial w}{\partial z}, \quad & \sigma_{12} &= A_{66}\Bigl(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\Bigr).
\end{aligned} \tag{8.29}$$

Here, $\mathbf{u} = (\mathbf{u}, w)$ is the displacement vector, $\mathbf{v}$ and w are the in-plane and out-of-plane displacements, respectively, both of which are functions of three-dimensional Cartesian coordinates $(\mathbf{y}, z)$.

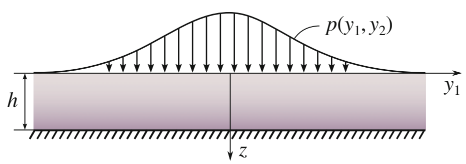

Fig. 8.2 A thickness-variable inhomogeneous elastic layer of uniform thickness bonded to a rigid substrate and supporting a distributed normal load

Let us assume that the elastic constants A_{kl} are represented in the form

$$A_{kl} = \alpha_{kl}\Bigl(\frac{z}{h}\Bigr), \tag{8.30}$$

or, equivalently, the material properties are expressed as functions of the normalized depth coordinate z/h. Note that according to the terminology introduced in [11, 12], we consider a transversely isotropic, transversely homogeneous (TITH) elastic model.

The substitution of the stress-displacement relations (8.29) into the equations of equilibrium

$$\frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} = 0, \quad i = 1, 2, 3,$$

yields the Lamé equations

$$A_{66}\varDelta_y \mathbf{v} + (A_{11} - A_{66})\nabla_y \nabla_y \cdot \mathbf{v} + A_{13}\frac{\partial}{\partial z}\nabla_y w + \frac{\partial}{\partial z}(A_{44}\nabla_y w) + \frac{\partial}{\partial z}\Bigl(A_{44}\frac{\partial \mathbf{v}}{\partial z}\Bigr) = \mathbf{0}, \tag{8.31}$$

$$\frac{\partial}{\partial z}(A_{13}\nabla_y \cdot \mathbf{v}) + A_{44}\frac{\partial}{\partial z}\nabla_y \cdot \mathbf{v} + A_{44}\varDelta_y w + \frac{\partial}{\partial z}\Bigl(A_{33}\frac{\partial w}{\partial z}\Bigr) = 0, \tag{8.32}$$

where $\nabla_y = (\partial/\partial y_1, \partial/\partial y_2)$ and $\varDelta_y = \nabla_y \cdot \nabla_y$ are the in-plane Hamilton and Laplace operators, respectively, and the scalar product is denoted by a dot.

Let us assume that the elastic layer is loaded on the upper surface, $z = 0$, with a normal load, p, without tangential tractions, and that it is perfectly attached to a rigid substrate at the bottom surface, $z = h$, so that the following boundary conditions take place:

$$\sigma_{13}\big|_{z=0} = \sigma_{23}\big|_{z=0} = 0, \quad \sigma_{33}\big|_{z=0} = -p, \tag{8.33}$$

$$\mathbf{v}\big|_{z=h} = \mathbf{0}, \quad w\big|_{z=h} = 0. \tag{8.34}$$

As before, we are interested in the case where the applied load p is specified on the whole upper surface of the layer, and is a sufficiently smooth function of the in-plane coordinates $\mathbf{y} = (y_1, y_2)$.

8.2.2 Perturbation Analysis of the Deformation Problem

We assume that the elastic layer is relatively thin and we set

$$h = \varepsilon h_*, \tag{8.35}$$

where ε is a small positive parameter, and h_* is a characteristic length, which is assumed to be independent of ε.

Let us also introduce the *dimensional* stretched normal coordinate

$$\zeta = \frac{z}{\varepsilon}, \tag{8.36}$$

so that in light of (8.30), (8.35), and (8.36), we obtain

$$A_{kl} = \alpha_{kl}\Big(\frac{\zeta}{h_*}\Big). \tag{8.37}$$

The substitution of the coordinate change (8.36) into the Lamé equations (8.31), (8.4) leads to the system

$$\frac{1}{\varepsilon^2}\frac{\partial}{\partial\zeta}\Big(A_{44}\frac{\partial\mathbf{v}}{\partial\zeta}\Big) + \frac{1}{\varepsilon}\Big(A_{13}\frac{\partial}{\partial\zeta}\nabla_y w + \frac{\partial}{\partial\zeta}(A_{44}\nabla_y w)\Big) + A_{66}\Delta_y\mathbf{v} + (A_{11} - A_{66})\nabla_y\nabla_y\cdot\mathbf{v} = \mathbf{0}, \tag{8.38}$$

$$\frac{1}{\varepsilon^2}\frac{\partial}{\partial\zeta}\Big(A_{33}\frac{\partial w}{\partial\zeta}\Big) + \frac{1}{\varepsilon}\Big(\frac{\partial}{\partial\zeta}(A_{13}\nabla_y\cdot\mathbf{v}) + A_{44}\frac{\partial}{\partial\zeta}\nabla_y\cdot\mathbf{v})\Big) + A_{44}\Delta_y w = 0. \tag{8.39}$$

Correspondingly, the boundary conditions (8.33) on the upper surface of the layer become

$$\frac{1}{\varepsilon}\frac{\partial\mathbf{v}}{\partial\zeta} + \nabla_y w\Big|_{\zeta=0} = \mathbf{0}, \tag{8.40}$$

$$\frac{1}{\varepsilon}A_{33}\frac{\partial w}{\partial\zeta} + A_{13}\nabla_y\cdot\mathbf{v}\Big|_{\zeta=0} = -p. \tag{8.41}$$

The boundary conditions (8.34) on the bottom surface then take the form

$$\mathbf{v}\big|_{\zeta=h_*} = \mathbf{0}, \quad w\big|_{\zeta=h_*} = 0. \tag{8.42}$$

Using the perturbation algorithm [14], we construct an approximate solution to the system (8.38)–(8.42) in the form of asymptotic expansions

$$\mathbf{v} = \varepsilon\mathbf{v}^0(\mathbf{y},\zeta) + \varepsilon^2\mathbf{v}^1(\mathbf{y},\zeta) + \cdots, \tag{8.43}$$

$$w = \varepsilon w^0(\mathbf{y},\zeta) + \varepsilon^2 w^1(\mathbf{y},\zeta) + \cdots, \tag{8.44}$$

where the successive coefficients of the powers of ε are assumed to be independent of the small parameter ε.

The substitution of the asymptotic expansions (8.43) and (8.44) into Eqs. (8.38)–(8.42) produces a set of differential equations that must be satisfied for arbitrary ε.

In particular, the leading terms of the asymptotic expansions (8.43) and (8.44) are determined as solutions of the following two problems:

$$\frac{\partial}{\partial\zeta}\Bigl(A_{44}\frac{\partial\mathbf{v}^0}{\partial\zeta}\Bigr)=\mathbf{0},\ \zeta\in(0,h_*),\quad \frac{\partial\mathbf{v}^0}{\partial\zeta}\Big|_{\zeta=0}=\mathbf{0},\quad \mathbf{v}^0\big|_{\zeta=h_*}=\mathbf{0}; \tag{8.45}$$

$$\frac{\partial}{\partial\zeta}\Bigl(A_{33}\frac{\partial w^0}{\partial\zeta}\Bigr)=0,\ \zeta\in(0,h_*),\quad A_{33}\frac{\partial w^0}{\partial\zeta}\Big|_{\zeta=0}=-p,\quad w^0\big|_{\zeta=h_*}=0. \tag{8.46}$$

From (8.45), it immediately follows that

$$\mathbf{v}^0(\mathbf{y},\zeta)\equiv\mathbf{0}, \tag{8.47}$$

while the non-trivial boundary-value problem (8.46) has the following solution:

$$w^0(\mathbf{y},\zeta)=p(\mathbf{y})\int\limits_{\zeta}^{h_*}\frac{d\zeta'}{A_{33}(\zeta')}. \tag{8.48}$$

As a result of (8.47), it can be easily seen that the problem

$$\frac{\partial}{\partial\zeta}\Bigl(A_{33}\frac{\partial w^1}{\partial\zeta}\Bigr)=-\frac{\partial}{\partial\zeta}(A_{13}\nabla_y\cdot\mathbf{v}^0)-A_{44}\frac{\partial}{\partial\zeta}\nabla_y\cdot\mathbf{v}^0,\quad \zeta\in(0,h_*),$$

$$A_{33}\frac{\partial w^1}{\partial\zeta}\Big|_{\zeta=0}=-A_{13}\nabla_y\cdot\mathbf{v}^0\,\big|_{\zeta=0},\quad w^1\big|_{\zeta=h_*}=0$$

is homogeneous, and therefore its solution is trivial:

$$w^1(\mathbf{y},\zeta)\equiv 0. \tag{8.49}$$

Simultaneously, for the first non-trivial term of the asymptotic expansion (8.43), we have the problem

$$\begin{gathered}\frac{\partial}{\partial\zeta}\Bigl(A_{44}\frac{\partial\mathbf{v}^1}{\partial\zeta}\Bigr)=\frac{\partial A_{13}}{\partial\zeta}\nabla_y w^0-\frac{\partial}{\partial\zeta}\bigl((A_{13}+A_{44})\nabla_y w^0\bigr),\quad \zeta\in(0,h_*),\\ \frac{\partial\mathbf{v}^1}{\partial\zeta}\Big|_{\zeta=0}=-\nabla_y w^0\big|_{\zeta=0},\quad \mathbf{v}^1\big|_{\zeta=h_*}=\mathbf{0}.\end{gathered} \tag{8.50}$$

To solve the above problem, we first rewrite formula (8.48) in the form

$$w^0(\mathbf{y},\zeta)=p(\mathbf{y})\mathscr{W}^0(\zeta), \tag{8.51}$$

where we have introduced the notation

$$\mathscr{W}^0(\zeta) = \int_{\zeta}^{h_*} \frac{d\zeta'}{A_{33}(\zeta')}. \tag{8.52}$$

Observe that we employ the notation $A_{33}(\zeta)$ for brevity, and that as a result of (8.37), Eq. (8.52) can be written as

$$\mathscr{W}^0(\zeta) = \int_{\zeta}^{h_*} \frac{1}{\alpha_{33}(\zeta'/h_*)}\, d\zeta'.$$

Thus, according to (8.51), the solution of the system (8.50) can be represented in the form

$$\mathbf{v}^1(\mathbf{y}, \zeta) = \nabla_y p(\mathbf{y}) \mathscr{V}^1(\zeta), \tag{8.53}$$

where $\mathscr{V}^1(\zeta)$ is a scalar function satisfying the problem

$$\begin{gathered} \frac{d}{d\zeta}\Big(A_{44}\frac{d\mathscr{V}^1}{d\zeta}\Big) = -A_{13}\frac{d\mathscr{W}^0}{d\zeta} - \frac{d}{d\zeta}\big(A_{44}\mathscr{W}^0\big), \quad \zeta \in (0, h_*), \\ \frac{d\mathscr{V}^1}{d\zeta}\Big|_{\zeta=0} = -\mathscr{W}^0(0), \quad \mathscr{V}^1\big|_{\zeta=h_*} = 0. \end{gathered} \tag{8.54}$$

Integrating the differential equation $(8.54)_1$, we obtain

$$\frac{d\mathscr{V}^1}{d\zeta} = \frac{1}{A_{44}(\zeta)} \int_0^{\zeta} \frac{A_{13}(\zeta')}{A_{33}(\zeta')}\, d\zeta' - \mathscr{W}^0(\zeta) + \frac{C_1}{A_{44}(\zeta)},$$

where C_1 is an integration constant. In light of the first boundary condition (8.54), we readily find that $C_1 = 0$.

Upon integration of the above equation, we arrive at the formula

$$\mathscr{V}^1(\zeta) = -\int_{\zeta}^{h_*} \bigg(\frac{1}{A_{44}(\xi)} \int_0^{\xi} \frac{A_{13}(\zeta')}{A_{33}(\zeta')}\, d\zeta' - \mathscr{W}^0(\xi)\bigg) d\xi,$$

which after recalling the definition of the function $\mathscr{W}^0(\zeta)$ (see Eq. (8.52)) can be rewritten as

$$\mathscr{V}^1(\zeta) = -\int\limits_{\zeta}^{h_*} \frac{1}{A_{44}(\xi)} \int\limits_{0}^{\xi} \frac{A_{13}(\zeta')}{A_{33}(\zeta')} \, d\zeta' d\xi + \int\limits_{\zeta}^{h_*} \frac{\xi - \zeta}{A_{33}(\xi)} \, d\xi. \tag{8.55}$$

We now return to the problem for the second non-trivial term of the asymptotic expansion (8.44), that is

$$\begin{gathered}\frac{\partial}{\partial \zeta}\Bigl(A_{33}\frac{\partial w^2}{\partial \zeta}\Bigr) = -\frac{\partial}{\partial \zeta}(A_{13}\nabla_y \cdot \mathbf{v}^1) - A_{44}\frac{\partial}{\partial \zeta}\nabla_y \cdot \mathbf{v}^1 - A_{44}\Delta_y w^0, \\ A_{33}\frac{\partial w^2}{\partial \zeta}\Big|_{\zeta=0} = -A_{13}\nabla_y \cdot \mathbf{v}^1 \Big|_{\zeta=0}, \quad w^2\big|_{\zeta=h_*} = 0.\end{gathered} \tag{8.56}$$

According to (8.51) and (8.53), the solution to the problem (8.56) can be represented in the form

$$w^1(\mathbf{y}, \zeta) = \Delta_y p(\mathbf{y}) \mathscr{W}^2(\zeta), \tag{8.57}$$

where $\mathscr{W}^2(\zeta)$ is a scalar function satisfying the problem

$$\begin{gathered}\frac{d}{d\zeta}\Bigl(A_{33}\frac{d\mathscr{W}^2}{d\zeta}\Bigr) = -\frac{d}{d\zeta}(A_{13}\mathscr{V}^1) - A_{44}\frac{d\mathscr{V}^1}{d\zeta} - A_{44}\mathscr{W}^0, \quad \zeta \in (0, h_*), \\ A_{33}\frac{d\mathscr{W}^2}{d\zeta}\Big|_{\zeta=0} = -A_{13}\mathscr{V}^1 \Big|_{\zeta=0}, \quad \mathscr{W}^2\big|_{\zeta=h_*} = 0.\end{gathered} \tag{8.58}$$

By integrating Eq. $(8.58)_1$ with boundary condition $(8.58)_2$ taken into account, we find

$$\frac{d\mathscr{W}^2}{d\zeta} = -\frac{A_{13}}{A_{33}}\mathscr{V}^1 - \frac{1}{A_{33}} \int\limits_{0}^{\zeta} (\zeta - \zeta') \frac{A_{13}(\zeta')}{A_{33}(\zeta')} \, d\zeta'.$$

A final integration reveals

$$\mathscr{W}^2(\zeta) = \int\limits_{\zeta}^{h_*} \frac{A_{13}(\eta)}{A_{33}(\eta)} \mathscr{V}^1(\eta) \, d\eta + \int\limits_{\zeta}^{h_*} \frac{1}{A_{33}(\xi)} \int\limits_{0}^{\xi} (\xi - \zeta') \frac{A_{13}(\zeta')}{A_{33}(\zeta')} \, d\zeta' d\xi. \tag{8.59}$$

By collecting formulas (8.57) and (8.59), we can write out a closed-form representation for the function $w^2(\mathbf{y}, \zeta)$.

8.2.3 Local Indentation of the Inhomogeneous Layer: Leading-Order Asymptotics for the Compressible and Incompressible Cases

In the case of the compressible layer, formulas (8.44), (8.51), and (8.52) yield for the local indentation

$$w_0(\mathbf{y}) \equiv w(\mathbf{y}, 0)$$

the following leading-order asymptotic approximation:

$$w_0(\mathbf{y}) \simeq \varepsilon p(\mathbf{y}) \int_0^{h_*} \frac{d\zeta}{A_{33}(\zeta)}.$$

Following (8.30), (8.35)–(8.37), the above formula can be transformed to

$$w_0(\mathbf{y}) \simeq p(\mathbf{y}) \int_0^{h} \frac{dz}{A_{33}(z)}. \tag{8.60}$$

In other words, the deformation response of a thin bonded compressible inhomogeneous elastic layer resembles that of a Winkler elastic foundation with the modulus

$$k = \left(\int_0^{h} \frac{dz}{A_{33}(z)} \right)^{-1}. \tag{8.61}$$

It is clear that in the case of a thin compressible homogeneous elastic layer formula (8.61) reduces to (1.65).

When the layer material approaches the incompressibility limit, the ratio A_{44}/A_{33} vanishes, while the ratio A_{13}/A_{33} tends to 1. At the same time, the Winkler foundation modulus k defined by (8.61) tends to infinity. Thus, in the case of the incompressible layer, formula (8.55) results in the following:

$$\mathscr{V}^1(\zeta) = -\int_{\zeta}^{h_*} \frac{\xi \, d\xi}{a_{44}(\xi)}. \tag{8.62}$$

Here, $a_{44} = A_{44}$ is the out-of-plane shear modulus.

Correspondingly, Eq. (8.59) reduces to

$$\mathscr{W}^2(\zeta) = \int_{\zeta}^{h_*} \mathscr{V}^1(\eta)\, d\eta. \tag{8.63}$$

The substitution of (8.62) into (8.63) reveals

$$\begin{aligned} \mathscr{W}^2(0) &= \zeta \mathscr{V}^1(\zeta)\Big|_0^{h_*} - \int_0^{h_*} \zeta \frac{d\mathscr{V}^1}{d\zeta}(\zeta)\, d\zeta \\ &= -\int_0^{h_*} \frac{\zeta^2}{a_{44}(\zeta)}\, d\zeta. \end{aligned} \tag{8.64}$$

Collecting formulas (8.44), (8.57), (8.64), and taking into account Eqs. (8.35)–(8.37), we obtain

$$w_0(\mathbf{y}) \simeq -\Delta_y p(\mathbf{y}) \int_0^h \frac{z^2 dz}{a_{44}(z)}. \tag{8.65}$$

We emphasize that formula (8.65) is derived for a thin bonded incompressible transversely isotropic, transversely homogeneous elastic layer.

8.3 Contact of Thin Bonded Incompressible Inhomogeneous Layers

In this section we briefly consider the axisymmetric frictionless contact problems for thin inhomogeneous transversely isotropic elastic layers bonded to slightly curved rigid substrates. The developed leading-order asymptotic models are validated by comparison with available published results.

8.3.1 Contact Problem Formulation

We consider two thin uniform inhomogeneous elastic layers firmly attached to rigid substrates. In the undeformed configuration the layers are in contact at a single point, O, chosen as the center of the Cartesian coordinate system Oy_2y_2z (see Fig. 3.1). We write the equations of the layer surfaces in the form $z = (-1)^n \varphi_n(\mathbf{y})$ $(n = 1, 2)$, so that the gap between the contacting surfaces is

$$\varphi(\mathbf{y}) = \varphi_1(\mathbf{y}) + \varphi_2(\mathbf{y}). \tag{8.66}$$

Denoting the vertical contact approach of the substrates—as usual—by δ_0, we formulate the linearized unilateral non-penetration condition as follows:

$$\delta_0 - \left(w_0^{(1)}(\mathbf{y}) + w_0^{(2)}(\mathbf{y})\right) \leq \varphi(\mathbf{y}). \tag{8.67}$$

Here, $w_0^{(n)}(\mathbf{y})$ is the local indentation of the nth elastic layer.

Generalizing the results of the previous two sections, we arrive at the approximate formula for the local indentation of the nth layer

$$w_0^{(n)}(\mathbf{y}) = -\nabla_y \cdot \left(\int_0^{h_n} \frac{z^2 dz}{G_n'(\mathbf{y}, z)} \nabla_y p(\mathbf{y}) \right), \tag{8.68}$$

where $p(\mathbf{y})$ is the contact pressure density, h_n is the thickness of the nth layer, and $G_n'(\mathbf{y}, z)$ is the out-of-plane shear modulus as it measured from the layer surface.

The contour Γ of the contact area ω is determined from the condition that the contact pressure is positive inside ω and vanishes at Γ, so that

$$p(\mathbf{y}) > 0, \quad \mathbf{y} \in \omega, \quad p(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{8.69}$$

Moreover, in the case of incompressible layers, we additionally assume a smooth transition of the pressure density $p(\mathbf{y})$ from the contact region ω to the outside region $\mathbf{y} \notin \omega$, where $p(\mathbf{y}) \equiv 0$. Thus, in addition to $(8.69)_2$, we impose the following zero-pressure-gradient boundary condition (see Sect. 2.7.3 and [5, 7, 8]):

$$\frac{\partial p}{\partial n}(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{8.70}$$

Here, $\partial/\partial n$ is the normal derivative directed outward from ω.

By substituting the expressions for the local indentations $w_0^{(1)}(\mathbf{y})$ and $w_0^{(2)}(\mathbf{y})$ provided by formula (8.68) into the contact condition (8.67) and taking into account $(8.69)_1$, we derive the governing differential equation

$$-\nabla_y \cdot \left(\gamma(\mathbf{y}) \nabla_y p(\mathbf{y})\right) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega, \tag{8.71}$$

where we have introduced the notation

$$\gamma(\mathbf{y}) = \int_0^{h_1} \frac{z^2 dz}{G_1'(\mathbf{y}, z)} + \int_0^{h_1} \frac{z^2 dz}{G_1'(\mathbf{y}, z)}. \tag{8.72}$$

Finally, the equilibrium equation for the whole system is

$$\iint_{\omega} p(\mathbf{y})\, d\mathbf{y} = F, \tag{8.73}$$

where F is the external load compressing the elastic layers.

The boundary-value problem (8.69)–(8.71) will be used to find both the contact area ω and the contact pressure $p(\mathbf{y})$, while the equilibrium equation (8.73) will allow determination of the contact approach δ_0, provided that the contact force F is given in advance.

8.3.2 Axisymmetric Unilateral Contact Problem

Introducing the cylindrical coordinate system (r, θ, z), we can write equations of the undeformed layer surfaces in the form $z = (-1)^n \varphi_n(r)$ $(n = 1, 2)$, so that the gap function (8.66) becomes (see Fig. 8.3)

$$\varphi(r) = \varphi_1(r) + \varphi_2(r). \tag{8.74}$$

For the sake of simplicity, we assume that the gap $\varphi(r)$ is a smooth increasing function and thus that the contact area ω is a circle of some radius a.

Due to the chain rule of differentiation, we have

$$\frac{\partial}{\partial y_1} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial y_2} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta},$$

so that the in-plane Hamilton operator is

$$\nabla_y = \Big(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\Big)\mathbf{e}_1 + \Big(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\Big)\mathbf{e}_2,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the basis vectors of the Cartesian coordinate system.

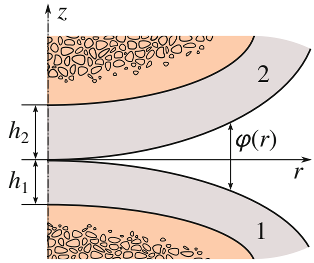

Fig. 8.3 Two inhomogeneous incompressible elastic layers bonded to axisymmetric rigid substrates in the undeformed configuration

For an axisymmetric density $p(r)$, we have

$$\nabla_y p(r) = \cos\theta \frac{dp}{dr}(r)\,\mathbf{e}_1 + \sin\theta \frac{dp}{dr}(r)\,\mathbf{e}_2$$

and correspondingly

$$\begin{aligned}\nabla_y \cdot \big(\gamma(r)\nabla_y p(r)\big) &= \frac{d}{dr}\Big(\gamma(r)\frac{dp}{dr}(r)\Big) + \frac{1}{r}\gamma(r)\frac{dp}{dr}(r)\\ &= \frac{1}{r}\frac{d}{dr}\Big(r\gamma(r)\frac{dp}{dr}(r)\Big),\end{aligned} \tag{8.75}$$

while, as a result of (8.72), the function $\gamma(r)$ is given by

$$\gamma(r) = \int_0^{h_1} \frac{z^2 dz}{G_1'(r,z)} + \int_0^{h_1} \frac{z^2 dz}{G_1'(r,z)}.$$

Further, Eq. (8.71) takes the form

$$-\frac{1}{r}\frac{d}{dr}\Big(r\gamma(r)\frac{dp}{dr}(r)\Big) = \delta_0 - \varphi(r), \quad r \in (0,a), \tag{8.76}$$

whereas the boundary conditions $(8.69)_2$ and (8.70) become

$$p(a) = 0, \quad \frac{dp}{dr}(a) = 0. \tag{8.77}$$

Integrating Eq. (8.76), we find

$$\frac{dp}{dr}(r) = -\frac{\delta_0}{2}\frac{r}{\gamma(r)} + \frac{1}{r\gamma(r)}\int_0^r \varphi(\rho)\rho\,d\rho, \tag{8.78}$$

where the integration constant vanishes due to the regularity condition for the solution of the problem (8.76), (8.77) at the center of the contact area $r = 0$.

By substituting $r = a$ into the above equation and taking into account the boundary condition $(8.77)_2$, we derive the following equation:

$$\delta_0 = \frac{2}{a^2}\int_0^r \varphi(\rho)\rho\,d\rho. \tag{8.79}$$

By integrating Eq. (8.78) and employing the boundary condition $(8.77)_1$, we obtain

$$p(r)=\frac{\delta_0}{2}\int_r^a\frac{\rho\,d\rho}{\gamma(\rho)}-\int_r^a\frac{1}{\rho\gamma(\rho)}\int_0^\rho\varphi(\xi)\xi\,d\xi\,d\rho. \tag{8.80}$$

Formula (8.80) presents the contact pressure in terms of the gap function $\varphi(r)$, given by (8.74), the elastic compliance function $\gamma(r)$, defined by (8.72), and two a priori unknown parameters δ_0 and a, which are related by Eq. (8.79).

An additional equation for determining the contact approach δ_0 and the contact radius a is provided by the equilibrium equation (8.73). Specifically, the substitution of (8.80) into Eq. (8.73) yields

$$F=\frac{\pi}{2}\delta_0\int_0^a\frac{\rho^3 d\rho}{\gamma(\rho)}-\pi\int_0^a\frac{r}{\gamma(r)}\int_0^r\varphi(\rho)\rho\,d\rho dr. \tag{8.81}$$

In the case of in-plane homogeneous elastic layers, where $\gamma(r)$ is constant, formulas (8.80) and (8.81) reduce to the following:

$$p(r)=\frac{\delta_0}{4\gamma}(a^2-r^2)-\frac{1}{\gamma}\Theta(a,r), \tag{8.82}$$

$$F=\frac{\pi}{4\gamma}\int_0^a\varphi(\rho)(2\rho^2-a^2)\rho\,d\rho. \tag{8.83}$$

Here we have introduced the notation

$$\Theta(a,r)=\int_r^a\varphi(\rho)\rho\ln\frac{a}{\rho}\,d\rho-\int_0^r\varphi(\rho)\rho\ln\frac{r}{\rho}\,d\rho. \tag{8.84}$$

Observe that in writing formula (8.83) we have taken into account Eq. (8.79). We also note that the contact radius is determined as a solution of Eq. (8.79).

8.3.3 Contact Problem for a Thin Bonded Non-homogeneous Incompressible Elastic Layer with Fixed Contact Area

Let us now consider contact interaction between a thin elastic layer bonded to a rigid substrate and a punch, under the assumption that the contact area, ω, does not change if the contact load, F, varies. In this case, the contact condition is

$$w_0(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega, \tag{8.85}$$

where δ_0 and $\varphi(\mathbf{y})$ are the punch's normal displacement and the punch shape function, and $w_0(\mathbf{y})$ is the local indentation of the elastic layer.

For a thin incompressible elastic layer, according to the asymptotic analysis performed in Sects. 8.1 and 8.2, we have

$$w_0(\mathbf{y}) = -\nabla_y \cdot \biggl(\int_0^h \frac{z^2 dz}{G'(\mathbf{y}, z)} \nabla_y p(\mathbf{y}) \biggr), \tag{8.86}$$

where $p(\mathbf{y})$ is the contact pressure distribution, h_n and $G'(\mathbf{y}, z)$ are the elastic layer's thickness and out-of-plane shear modulus measured in the z-direction from the layer surface, respectively.

Substituting (8.86) into Eq. (8.85), we arrive at the equation

$$-\nabla_y \cdot \bigl(\gamma(\mathbf{y}) \nabla_y p(\mathbf{y})\bigr) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega, \tag{8.87}$$

where we have introduced the notation

$$\gamma(\mathbf{y}) = \int_0^h \frac{z^2 dz}{G'(\mathbf{y}, z)}. \tag{8.88}$$

The second-order differential equation (8.87) requires some boundary conditions at the contour Γ of the domain ω. As was shown by Aleksandrov [1] (see also Sect. 2.7.2) in the axisymmetric contact problem for a thin incompressible isotropic elastic layer, in order to construct the leading-order inner asymptotic solution for the contact pressure under a flat-ended punch, the differential equation (8.87) should be supplemented with the following boundary condition:

$$p(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{8.89}$$

In the case of a thin bonded homogeneous incompressible elastic layer (when $\gamma(\mathbf{y}) \equiv \text{const}$), Barber [7] has shown that the problem (8.87), (8.89) is formally equivalent to the Saint-Venant torsion problem (see, e.g., [19], Chap. 10), and hence the analytical solutions to many contact problems can be written down.

We emphasize that in the case of an elastic layer indented by a rigid punch with a sharp edge, the contact pressure has a square-root singularity at the contour Γ (see, e.g., [3, 13, 15]). For a thin elastic layer this effect is taken into account by the additional asymptotic solution of the boundary-layer type (see, in particular, [1]).

Finally, the punch's displacement δ_0 and the contact force F are related through the equilibrium equation

$$\iint_{\omega} p(\mathbf{y})\, d\mathbf{y} = F. \tag{8.90}$$

Observe that, from a physical point of view, there exists a constant γ_0 such that the function (8.88) satisfies the condition

$$\gamma(\mathbf{y}) \geq \gamma_0, \quad \mathbf{y} \in \bar{\omega} = \omega \cup \Gamma.$$

According to the weak maximum principle (see, e.g., [10]), if the right-hand side of Eq. (8.87) satisfies the condition

$$\delta_0 - \varphi(\mathbf{y}) \geq 0, \quad \mathbf{y} \in \omega, \tag{8.91}$$

then the minimum of the contact pressure $p(\mathbf{y})$ in $\bar{\omega}$ is achieved on Γ, i.e.,

$$\min_{\mathbf{y}\in\bar{\omega}} p(\mathbf{y}) = \min_{\mathbf{y}\in\Gamma} p(\mathbf{y}). \tag{8.92}$$

Hence, from (8.89) and (8.92), it follows that in the case (8.91), we have throughout the contact area

$$p(\mathbf{y}) \geq 0, \quad \mathbf{y} \in \omega. \tag{8.93}$$

We note that under some assumptions on the domain ω (e.g., for a simply connected domain ω bounded by a smooth contour Γ of class C^2), the strict inequality in (8.91) implies the strict inequality in (8.93). Equivalently, if the local indentation of a thin incompressible elastic layer is positive over the whole contact area, then the contact pressure under the punch is positive, which is generally not the case.

8.3.4 Axisymmetric Contact Problem with Fixed Contact Area

We now consider the following linear contact problem (see Fig. 8.4):

$$-\frac{1}{r}\frac{d}{dr}\Big(r\gamma(r)\frac{dp}{dr}(r)\Big) = \delta_0 - \varphi(r), \quad r \in (0, a), \tag{8.94}$$

$$p(a) = 0, \tag{8.95}$$

$$2\pi \int_0^a p(r) r\, dr = F. \tag{8.96}$$

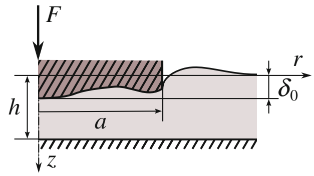

Fig. 8.4 An inhomogeneous incompressible elastic layer bonded to a rigid substrate in full contact with an axisymmetric rigid punch

As a result of (8.88), we have

$$\gamma(r) = \int_0^h \frac{z^2 dz}{G'(r,z)}.$$

The solution to Eqs. (8.94)–(8.96) is given by the formulas (see Sect. 8.3.2):

$$p(r) = \frac{\delta_0}{2} \int_r^a \frac{\rho\, d\rho}{\gamma(\rho)} - \int_r^a \frac{1}{\rho\gamma(\rho)} \int_0^\rho \varphi(\xi)\xi\, d\xi d\rho, \tag{8.97}$$

$$F = \frac{\pi}{2}\delta_0 \int_0^a \frac{\rho^3 d\rho}{\gamma(\rho)} - \pi \int_0^a \frac{r}{\gamma(r)} \int_0^r \varphi(\rho)\rho\, d\rho dr. \tag{8.98}$$

In the case of the in-plane homogeneous elastic layer, where the shear modulus G' does not depend on r and $\gamma(r)$ is constant, Eqs. (8.97) and (8.98) yield simply

$$p(r) = \frac{\delta_0}{4\gamma}(a^2 - r^2) - \frac{1}{\gamma}\Theta(a,r), \tag{8.99}$$

$$F = \frac{\pi}{8\gamma}\delta_0 a^4 - \frac{\pi}{2\gamma} \int_0^a \varphi(\rho)(a^2 - \rho^2)\rho\, d\rho, \tag{8.100}$$

where the factor $\Theta(a,r)$ is given by formula (8.84).

The obtained solution and, in particular, formula (8.99) agree with the so-called degenerate asymptotic solution obtained by Aleksandrov [1] in the isotropic and homogeneous case. The contact problem for an incompressible inhomogeneous isotropic elastic layer bonded to a rigid substrate, and indented without friction by a rigid punch, was studied by Malits [16], who, in particular, constructed the leading-order asymptotic solution in the case of a circular punch of three-dimensional profile, where formula (8.100) takes the following form:

$$F = \frac{\pi}{8\gamma}\delta_0 a^4 - \frac{1}{4\gamma}\int_0^{2\pi} d\theta \int_0^a \varphi(\rho,\theta)(a^2-\rho^2)\rho\, d\rho.$$

For the flat-ended punch, when $\varphi(r) \equiv 0$, Eqs. (8.99) and (8.100) further simplify to

$$p(r) = \frac{\delta_0}{4\gamma}(a^2 - r^2), \tag{8.101}$$

$$F = \frac{\pi}{8\gamma}\delta_0 a^4. \tag{8.102}$$

In the case of the homogeneous isotropic incompressible elastic layer, we have

$$\gamma = \frac{h^3}{3G}, \tag{8.103}$$

where G is the shear modulus, and formula (8.101) coincides with the solution obtained by Aleksandrov [1], while Eq. (8.102), apart from notation, coincides with the corresponding equation obtained by Malits [16].

8.4 Deformation of a Thin Elastic Layer Coated with an Elastic Membrane

In this section we consider the deformation problem for a transversely isotropic elastic layer bonded to a rigid substrate and coated with a very thin elastic layer made of another transversely isotropic material. The leading-order asymptotic model is based on the simplifying assumptions that the generalized plane stress conditions apply to the coating layer, and the flexural stiffness of the coating layer is negligible compared to its tensile stiffness.

8.4.1 Boundary Conditions for a Coated Elastic Layer

We consider a very thin transversely isotropic elastic coating layer (of uniform thickness $\hat{h}$) bonded to an elastic layer (of thickness h) made of another transversely isotropic material (see Fig. 8.5).

Let the five independent elastic constants of the elastic layer and its coating are denoted by A_{11}, A_{12}, A_{13}, A_{33}, A_{44} and $\hat{A}_{11}$, $\hat{A}_{12}$, $\hat{A}_{13}$, $\hat{A}_{33}$, $\hat{A}_{44}$, respectively.

Under the assumption that the two layers are in perfect contact with one another along their common interface, $z = 0$, the following boundary conditions of continuity (interface conditions of perfect bonding) should be satisfied:

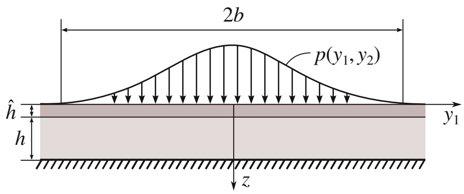

Fig. 8.5 A two-layer elastic system (coated layer and coating layer in perfect bonding) bonded to a rigid substrate and loaded by a normal load

$$\hat{\mathbf{v}}(\mathbf{y},0)=\mathbf{v}(\mathbf{y},0),\quad \hat{w}(\mathbf{y},0)=w(\mathbf{y},0), \tag{8.104}$$

$$\hat{\sigma}_{3j}(\mathbf{y},0)=\sigma_{3j}(\mathbf{y},0),\quad j=1,2,3. \tag{8.105}$$

Here, $(\hat{\mathbf{v}}, \hat{w})$ is the displacement vector of the elastic coating layer, and $\hat{\sigma}_{ij}$ are the corresponding components of stress.

On the upper surface of the two-layer system, $z=-\hat{h}$, we impose the boundary conditions of normal loading with no tangential tractions

$$\hat{\sigma}_{31}(\mathbf{y},-\hat{h})=\sigma_{32}(\mathbf{y},-\hat{h})=0,\quad \hat{\sigma}_{33}(\mathbf{y},-\hat{h})=-p(\mathbf{y}), \tag{8.106}$$

where $p(\mathbf{y})$ is a specified function.

Following Rahman and Newaz [18], we simplify the deformation analysis of the elastic coating layer based on the following two assumptions: (1) the coating layer is assumed to be very thin, so that the generalized plane stress conditions apply; (2) the flexural stiffness of the coating layer in the z-direction is negligible compared to its tensile stiffness.

In the absence of body forces, the equilibrium equations for an infinitesimal element of the coating layer are

$$\frac{\partial\hat{\sigma}_{11}}{\partial y_1}+\frac{\partial\hat{\sigma}_{12}}{\partial y_2}+\frac{\partial\hat{\sigma}_{13}}{\partial z}=0,\quad \frac{\partial\hat{\sigma}_{21}}{\partial y_1}+\frac{\partial\hat{\sigma}_{22}}{\partial y_2}+\frac{\partial\hat{\sigma}_{23}}{\partial z}=0, \tag{8.107}$$

$$\frac{\partial\hat{\sigma}_{31}}{\partial y_1}+\frac{\partial\hat{\sigma}_{32}}{\partial y_2}+\frac{\partial\hat{\sigma}_{33}}{\partial z}=0. \tag{8.108}$$

The stress-strain relationship for the transversely isotropic elastic coating layer is given by

$$\begin{pmatrix}\hat{\sigma}_{11}\\ \hat{\sigma}_{22}\\ \hat{\sigma}_{33}\\ \hat{\sigma}_{23}\\ \hat{\sigma}_{13}\\ \hat{\sigma}_{12}\end{pmatrix}=\begin{bmatrix}\hat{A}_{11}&\hat{A}_{12}&\hat{A}_{13}&0&0&0\\ \hat{A}_{12}&\hat{A}_{11}&\hat{A}_{13}&0&0&0\\ \hat{A}_{13}&\hat{A}_{13}&\hat{A}_{33}&0&0&0\\ 0&0&0&2\hat{A}_{44}&0&0\\ 0&0&0&0&2\hat{A}_{44}&0\\ 0&0&0&0&0&2\hat{A}_{66}\end{bmatrix}\begin{pmatrix}\hat{\varepsilon}_{11}\\ \hat{\varepsilon}_{22}\\ \hat{\varepsilon}_{33}\\ \hat{\varepsilon}_{23}\\ \hat{\varepsilon}_{13}\\ \hat{\varepsilon}_{12}\end{pmatrix}, \tag{8.109}$$

where $2\hat{A}_{66}=\hat{A}_{11}-\hat{A}_{12}$.

Integrating Eqs. (8.107), (8.108) through the thickness of the coating layer and taking into account the interface and boundary conditions (8.105) and (8.106), we find

$$\hat{h}\left(\frac{\partial\hat{\hat{\sigma}}_{11}}{\partial y_1}+\frac{\partial\hat{\hat{\sigma}}_{12}}{\partial y_2}\right)=-\sigma_{13}\big|_{z=0},\quad \hat{h}\left(\frac{\partial\hat{\hat{\sigma}}_{12}}{\partial y_1}+\frac{\partial\hat{\hat{\sigma}}_{22}}{\partial y_2}\right)=-\sigma_{23}\big|_{z=0}, \tag{8.110}$$

$$\hat{h}\left(\frac{\partial\hat{\hat{\sigma}}_{13}}{\partial y_1}+\frac{\partial\hat{\hat{\sigma}}_{23}}{\partial y_2}\right)=-\sigma_{33}\big|_{z=0}-p. \tag{8.111}$$

Here, $\hat{\hat{\sigma}}_{ij}$ are the averaged stresses, i.e.,

$$\hat{\hat{\sigma}}_{ij}(\mathbf{y})=\frac{1}{\hat{h}}\int\limits_{-\hat{h}}^{0}\hat{\sigma}_{ij}(\mathbf{y},z)\,dz.$$

Under the simplifying assumptions made above, we have

$$\hat{\hat{\sigma}}_{13}=\hat{\hat{\sigma}}_{23}=\hat{\hat{\sigma}}_{33}=0. \tag{8.112}$$

Hence, Eq. (8.111) immediately implies that

$$\sigma_{33}\big|_{z=0}=-p. \tag{8.113}$$

Moreover, in light of (8.112), the averaged strain $\hat{\hat{\varepsilon}}_{33}$ must satisfy the equation

$$\hat{A}_{13}\hat{\hat{\varepsilon}}_{11}+\hat{A}_{13}\hat{\hat{\varepsilon}}_{22}+\hat{A}_{33}\hat{\hat{\varepsilon}}_{33}=0,$$

and, therefore, the in-plane averaged stress-strain relationship takes the form

$$\begin{pmatrix}\hat{\hat{\sigma}}_{11}\\ \hat{\hat{\sigma}}_{22}\\ \hat{\hat{\sigma}}_{12}\end{pmatrix}=\begin{bmatrix}\hat{\hat{A}}_{11} & \hat{\hat{A}}_{12} & 0\\ \hat{\hat{A}}_{12} & \hat{\hat{A}}_{11} & 0\\ 0 & 0 & 2\hat{\hat{A}}_{66}\end{bmatrix}\begin{pmatrix}\hat{\hat{\varepsilon}}_{11}\\ \hat{\hat{\varepsilon}}_{22}\\ \hat{\hat{\varepsilon}}_{12}\end{pmatrix}, \tag{8.114}$$

where we have introduced the notation

$$\hat{\hat{A}}_{11}=\hat{A}_{11}-\frac{\hat{A}_{13}^2}{\hat{A}_{33}},\quad \hat{\hat{A}}_{12}=\hat{A}_{12}-\frac{\hat{A}_{13}^2}{\hat{A}_{33}},\quad 2\hat{\hat{A}}_{66}=\hat{\hat{A}}_{11}-\hat{\hat{A}}_{12}. \tag{8.115}$$

On the other hand, in light of the interface conditions (8.104), we have

$$\hat{\hat{\varepsilon}}_{11}=\varepsilon_{11}\big|_{z=0},\quad \hat{\hat{\varepsilon}}_{22}=\varepsilon_{22}\big|_{z=0},\quad \hat{\hat{\varepsilon}}_{12}=\varepsilon_{12}\big|_{z=0}, \tag{8.116}$$

where ε_{11}, ε_{22}, and ε_{12} are the in-plane strains in the coated elastic layer $z \in (0, h)$.

Taking Eqs. (8.114) and (8.116) into account, we transform the boundary conditions (8.110) into the following:

$$-\frac{1}{\hat{h}}\sigma_{31}\big|_{z=0} = \frac{\partial}{\partial y_1}\left(\hat{\hat{A}}_{11}\frac{\partial v_1}{\partial y_1} + \hat{\hat{A}}_{12}\frac{\partial v_2}{\partial y_2}\right) + \hat{\hat{A}}_{66}\frac{\partial}{\partial y_2}\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\right),$$

$$-\frac{1}{\hat{h}}\sigma_{32}\big|_{z=0} = \hat{\hat{A}}_{66}\frac{\partial}{\partial y_1}\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(\hat{\hat{A}}_{12}\frac{\partial v_1}{\partial y_1} + \hat{\hat{A}}_{11}\frac{\partial v_2}{\partial y_2}\right).$$

The above boundary conditions can be rewritten in the matrix form as

$$\sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2\big|_{z=0} = -\hat{\mathfrak{L}}(\nabla_y)\mathbf{v}\big|_{z=0}, \tag{8.117}$$

where $\hat{\mathfrak{L}}(\nabla_y)$ is a 2×2 matrix differential operator such that

$$\begin{aligned}\hat{\mathfrak{L}}_{\alpha\alpha}(\nabla_y) &= \hat{h}\hat{\hat{A}}_{11}\frac{\partial^2}{\partial y_\alpha^2} + \hat{h}\hat{\hat{A}}_{66}\frac{\partial^2}{\partial y_{3-\alpha}^2}\\ \hat{\mathfrak{L}}_{\alpha\beta}(\nabla_y) &= \hat{h}\big(\hat{\hat{A}}_{12} + \hat{\hat{A}}_{66}\big)\frac{\partial^2}{\partial y_\alpha \partial y_\beta}, \quad \alpha, \beta = 1, 2, \quad \alpha \neq \beta.\end{aligned} \tag{8.118}$$

Thus, the deformation problem for an elastic layer coated with a very thin flexible elastic layer is reduced to that for the elastic layer without coating, but subjected to a different set of boundary conditions (8.113) and (8.117) on the surface $z = 0$.

Observe that in the axisymmetric case, as a result of (8.116), we have

$$\hat{\hat{\sigma}}_{rr} = \hat{\hat{A}}_{11}\varepsilon_{rr} + \hat{\hat{A}}_{12}\varepsilon_{\theta\theta}, \quad \hat{\hat{\sigma}}_{\theta\theta} = \hat{\hat{A}}_{12}\varepsilon_{rr} + \hat{\hat{A}}_{11}\varepsilon_{\theta\theta}, \quad \hat{\hat{\sigma}}_{r\theta} = 0,$$

where

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{v_r}{r},$$

and Eqs. (8.110) should be replaced with the following:

$$\frac{\hat{h}}{r}\left(\frac{\partial(r\hat{\hat{\sigma}}_{rr})}{\partial r} - \hat{\hat{\sigma}}_{\theta\theta}\right) = -\sigma_{zr}\big|_{z=0}.$$

Correspondingly, the boundary condition (8.117) takes the following form:

$$\sigma_{zr}\big|_{z=0} = -\hat{h}\hat{\hat{A}}_{11}\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2}\right). \tag{8.119}$$

We note here that the axisymmetric boundary condition (8.119) was previously derived in a number of papers [2, 6, 17, 18].

8.4.2 Deformation Problem Formulation

We now consider a relatively thin transversely isotropic elastic layer of uniform thickness, h, coated with an infinitesimally thin elastic membrane and bonded to a rigid substrate (see Fig. 8.6), so that

$$\mathbf{v}\big|_{z=h} = \mathbf{0}, \quad w\big|_{z=h} = 0. \tag{8.120}$$

In the absence of body forces, the vector $(\mathbf{v}, w)$ of displacements in the elastic layer satisfies the Lamé system

$$A_{66}\Delta_y\mathbf{v} + (A_{11} - A_{66})\nabla_y\nabla_y\cdot\mathbf{v} + A_{44}\frac{\partial^2\mathbf{v}}{\partial z^2} + (A_{13} + A_{44})\frac{\partial}{\partial z}\nabla_y w = \mathbf{0},$$
$$A_{44}\Delta_y w + A_{33}\frac{\partial^2 w}{\partial z^2} + (A_{13} + A_{44})\frac{\partial}{\partial z}\nabla_y\cdot\mathbf{v} = 0. \tag{8.121}$$

Assuming that the coated layer is supporting a normal load and denoting the load density by p, we require that

$$\sigma_{33}\big|_{z=0} = -p. \tag{8.122}$$

Based on the analysis performed in Sect. 8.4.1, the influence of the elastic membrane is represented by the boundary condition

$$\sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2\big|_{z=0} = -\hat{\mathfrak{L}}(\nabla_y)\mathbf{v}\big|_{z=0}, \tag{8.123}$$

where $\hat{\mathfrak{L}}(\nabla_y)$ is the matrix differential operator defined by formulas (8.118).

Taking into account the stress-strain relationship

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0\\ A_{12} & A_{11} & A_{13} & 0 & 0 & 0\\ A_{13} & A_{13} & A_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & 2A_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & 2A_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & 2A_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{pmatrix},$$

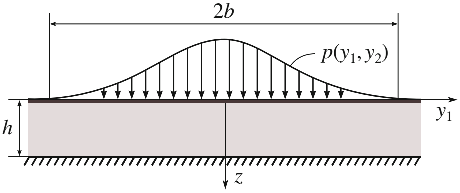

Fig. 8.6 A coated elastic layer of uniform thickness bonded to a rigid substrate and loaded by a normal load

we rewrite Eqs. (8.122), (8.123) as follows:

$$A_{13}\frac{\partial v_1}{\partial y_1} + A_{13}\frac{\partial v_2}{\partial y_2} + A_{33}\frac{\partial w}{\partial z}\bigg|_{z=0} = -p, \tag{8.124}$$

$$A_{44}\Bigl(\nabla_y w + \frac{\partial \mathbf{v}}{\partial z}\Bigr)\Big|_{z=0} = -\hat{\mathfrak{L}}(\nabla_y)\mathbf{v}\big|_{z=0}. \tag{8.125}$$

Equations (8.120), (8.121), (8.124), and (8.125) comprise the deformation problem for the coated transversely isotropic elastic layer.

Here, following Argatov and Mishuris [4], we construct a leading-order asymptotic solution to the deformation problem (8.120)–(8.125).

8.4.3 Asymptotic Analysis of the Deformation Problem

Let h_* be a characteristic length of the external load distribution. Denoting by ε a small positive parameter, we require that

$$h = \varepsilon h_* \tag{8.126}$$

and introduce the stretched dimensionless normal coordinate

$$\zeta = \frac{z}{\varepsilon h_*}.$$

In addition, we non-dimensionalize the in-plane coordinates by the formulas

$$\eta_i = \frac{y_i}{h_*}, \quad i = 1, 2, \quad \boldsymbol{\eta} = (\eta_1, \eta_2),$$

so that

$$\frac{\partial}{\partial z} = \frac{1}{\varepsilon h_*}\frac{\partial}{\partial \zeta}, \quad \nabla_y = \frac{1}{h_*}\nabla_\eta.$$

Moreover, we assume that the tensile stiffness of the coating layer is relatively high, i.e., $\hat{\hat{A}}_{11} \gg A_{11}$, and so on. Continuing, we consider the situation when

$$\hat{\mathfrak{L}}(\nabla_y) = \varepsilon^{-1}\hat{\mathfrak{L}}^*(\nabla_y), \tag{8.127}$$

so that, in particular, the ratio $A_{11}/\hat{\hat{A}}_{11}$ is of the order of ε.

Following the perturbation algorithm [14], the solution to the deformation problem (8.120), (8.121), (8.124), (8.125) is represented as follows:

$$\mathbf{v} = \varepsilon^2\mathbf{v}^1(\boldsymbol{\eta}, \zeta) + \cdots, \tag{8.128}$$

$$w = \varepsilon w^0(\boldsymbol{\eta}, \zeta) + \varepsilon^3 w^2(\boldsymbol{\eta}, \zeta) + \cdots . \tag{8.129}$$

For the sake of brevity, we include here only the non-vanishing terms (for details see Sect. 1.2).

It can be shown that the leading-order term in (8.129) is given by

$$w^0(\boldsymbol{\eta}, \zeta) = \frac{h_* p}{A_{33}}(1 - \zeta), \tag{8.130}$$

whereas the first non-trivial term of the expansion (8.128) satisfies the problem

$$A_{44}\frac{\partial^2 \mathbf{v}^1}{\partial \zeta^2} = -(A_{13} + A_{44})\nabla_\eta \frac{\partial w^0}{\partial \zeta}, \quad \zeta \in (0, 1),$$

$$A_{44}\frac{\partial \mathbf{v}^1}{\partial \zeta} + \frac{1}{h_*}\hat{\mathfrak{L}}^*(\nabla_\eta)\mathbf{v}^1\bigg|_{\zeta=0} = -A_{44}\nabla_\eta w^0\big|_{\zeta=0}, \quad \mathbf{v}^1\big|_{\zeta=1} = \mathbf{0}.$$

Substituting the expansion (8.130) for w^0 into the above equations, we obtain

$$\begin{aligned} A_{44}\frac{\partial^2 \mathbf{v}^1}{\partial \zeta^2} &= \frac{A_{13} + A_{44}}{A_{33}} h_* \nabla_\eta p, \quad \zeta \in (0, 1), \\ A_{44}\frac{\partial \mathbf{v}^1}{\partial \zeta} + \frac{1}{h_*}\hat{\mathfrak{L}}^*(\nabla_\eta)\mathbf{v}^1\bigg|_{\zeta=0} &= -\frac{A_{44}}{A_{33}} h_* \nabla_\eta p, \quad \mathbf{v}^1\big|_{\zeta=1} = \mathbf{0}. \end{aligned} \tag{8.131}$$

The solution to the boundary-value problem (8.131) is represented in the form

$$\mathbf{v}^1 = -\frac{A_{13} + A_{44}}{2A_{33}A_{44}}\zeta(1 - \zeta)h_* \nabla_\eta p + (1 - \zeta)\mathbf{V}^1(\boldsymbol{\eta}), \tag{8.132}$$

where $\mathbf{V}^1(\boldsymbol{\eta})$ satisfies the equation

$$\frac{1}{h_*}\hat{\mathfrak{L}}^*(\nabla_\eta)\mathbf{V}^1 - A_{44}\mathbf{V}^1 = \frac{A_{13} - A_{44}}{2A_{33}} h_* \nabla_\eta p \tag{8.133}$$

on the entire plane $\zeta = 0$.

For the second non-trivial term of the expansion (8.129), we derive the problem

$$A_{33}\frac{\partial^2 w^2}{\partial \zeta^2} = -(A_{13} + A_{44})\nabla_\eta \cdot \frac{\partial \mathbf{v}^1}{\partial \zeta} - A_{44}\Delta_\eta w^0, \quad \zeta \in (0, 1),$$

$$A_{33}\frac{\partial w^2}{\partial \zeta}\bigg|_{\zeta=0} = -A_{13}\nabla_\eta \cdot \mathbf{v}^1\big|_{\zeta=0}, \quad w^2\big|_{\zeta=1} = 0.$$

Substituting the expressions (8.130) and (8.132) for w^0 and $\mathbf{v}^1$, respectively, into the above equations, we arrive at the problem

$$\frac{\partial^2 w^2}{\partial \zeta^2} = -\big[(A_{13}+A_{44})^2(2\zeta-1) + 2A_{44}^2(1-\zeta)\big]\frac{h_* \Delta_\eta p}{2A_{33}^2 A_{44}} + \frac{A_{13}+A_{44}}{A_{33}}\nabla_\eta \cdot \mathbf{V}^1, \quad \zeta \in (0,1), \tag{8.134}$$

$$\left.\frac{\partial w^2}{\partial \zeta}\right|_{\zeta=0} = -\frac{A_{13}}{A_{33}}\nabla_\eta \cdot \mathbf{V}^1, \quad w^2\big|_{\zeta=1} = 0. \tag{8.135}$$

Integrating Eq. (8.134) twice with respect to ζ and taking into account the boundary condition $(8.135)_2$, we obtain

$$w^2 = -\big[(A_{13}+A_{44})^2\big(2\zeta^3 - 3\zeta^2 + 1\big) + 2A_{44}^2(1-\zeta)^3\big]\frac{h_* \Delta_\eta p}{12A_{33}^2 A_{44}} + \frac{A_{13}+A_{44}}{2A_{33}}(1-\zeta)^2\nabla_\eta \cdot \mathbf{V}^1(\boldsymbol{\eta}) + C_2(\boldsymbol{\eta})(1-\zeta), \tag{8.136}$$

where $C_2(\boldsymbol{\eta})$ is an arbitrary function.

The substitution of (8.136) into the boundary condition $(8.135)_1$ yields

$$C_2 = \frac{A_{44}}{2A_{33}^2}h_* \Delta_\eta p - \frac{A_{44}}{A_{33}}\nabla_\eta \cdot \mathbf{V}^1,$$

and thus, in light of this relation, formula (8.136) implies

$$w^2\big|_{\zeta=0} = -\big[(A_{13}+A_{44})^2 - 4A_{44}^2\big]\frac{h_* \Delta_\eta p}{12A_{33}^2 A_{44}} + \frac{A_{13}-A_{44}}{2A_{33}}\nabla_\eta \cdot \mathbf{V}^1, \tag{8.137}$$

where $\mathbf{V}^1$ is the solution of Eq. (8.133).

8.4.4 Local Indentation of the Coated Elastic Layer: Leading-Order Asymptotics for the Compressible and Incompressible Cases

In the case of the compressible layer, Eqs. (8.129) and (8.130) yield

$$w_0(\mathbf{y}) \simeq \frac{h}{A_{33}}p(\mathbf{y}), \tag{8.138}$$

so that the deformation response of the coated elastic layer is analogous to that of a Winkler elastic foundation with the foundation modulus $k = A_{33}/h$. In other words, the deformation of the elastic coating does not contribute substantially to the deformation of a thin compressible layer.

When the material approaches the incompressible limit, the right-hand side of (8.138) decreases to zero and the first term in the asymptotic expansion (8.129) disappears. Hence, the ratios A_{13}/A_{33} and A_{44}/A_{33} tend to 1 and 0, respectively.

Therefore, in the limit situation formula (8.137) reduces to

$$w^2\big|_{\zeta=0} = -\frac{h_*}{12a_{44}}\Delta_\eta p(\boldsymbol{\eta}) + \frac{1}{2}\nabla_\eta \cdot \mathbf{V}^1(\boldsymbol{\eta}), \tag{8.139}$$

where $a_{44} = A_{44}$ is the out-of-plane shear modulus of the elastic layer, and $\mathbf{V}^1(\boldsymbol{\eta})$ satisfies the equation

$$\frac{1}{h_*}\hat{\mathfrak{L}}^*(\nabla_\eta)\mathbf{V}^1(\boldsymbol{\eta}) - a_{44}\mathbf{V}^1(\boldsymbol{\eta}) = \frac{h_*}{2}\nabla_\eta p(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathbb{R}^2. \tag{8.140}$$

Thus, in the case of the incompressible bonded elastic layer, formulas (8.127)–(8.129), (8.139), and (8.140) produce

$$w_0(\mathbf{y}) \simeq -\frac{h^3}{12a_{44}}\Delta_y p(\mathbf{y}) + \frac{h}{2}\nabla_y \cdot \mathbf{v}_0(\mathbf{y}), \tag{8.141}$$

where the vector $\mathbf{v}_0(\mathbf{y})$ satisfies the equation

$$h\hat{\mathfrak{L}}(\nabla_y)\mathbf{v}_0(\mathbf{y}) - a_{44}\mathbf{v}_0(\mathbf{y}) = \frac{h^2}{2}\nabla_y p(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^2. \tag{8.142}$$

Here, $\hat{\mathfrak{L}}(\nabla_y)$ is the matrix differential operator defined by formulas (8.118).

Observe that, as a consequence of (8.132), the vector-function $\mathbf{v}_0(\mathbf{y})$ can be interpreted as the tangential displacement of the surface point $(\mathbf{y}, 0)$ of the elastic layer.

Finally, let us consider two opposite limit situations. First, when the coating is absent and $\hat{\mathfrak{L}}(\nabla_y) \equiv 0$, Eq. (8.142) implies

$$\mathbf{v}_0(\mathbf{y}) = -\frac{h^2}{2a_{44}}\nabla_y p(\mathbf{y}).$$

The substitution of this expression into formula (8.141) leads to

$$w_0(\mathbf{y}) \simeq -\frac{h^3}{3a_{44}}\Delta_y p(\mathbf{y}), \tag{8.143}$$

which agrees completely with the asymptotic model developed in Sect. 2.7.1.

Second, in the case of a very stiff coating we have $\mathbf{v}_0(\mathbf{y}) \equiv \mathbf{0}$, and formula (8.141) reduces to

$$w_0(\mathbf{y}) \simeq -\frac{h^3}{12a_{44}} \Delta_y p(\mathbf{y}). \tag{8.144}$$

In other words, comparing (8.143) and (8.144), we conclude that the inextensible membrane coating attached to the surface of a thin bonded incompressible elastic layer reduces the out-of-plane shear compliance of the layer by a factor of four.

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