Chapter 6
Contact of Thin Biphasic Layers

Abstract In Sect. 6.1, a three-dimensional deformation problem for an articular cartilage layer is studied in the framework of the linear biphasic model. The articular cartilage bonded to subchondral bone is modeled as a transversely isotropic biphasic material consisting of a solid phase and a fluid phase. In Sect. 6.2, the same problem is reconsidered with the effect of inherent viscoelasticity of the solid matrix taken into account. The frictionless unilateral contact problem for the articular cartilage layers is considered in Sect. 6.3. It is assumed that the subchondral bones are rigid and shaped like elliptic paraboloids. The obtained short-time leading-order asymptotic solution is valid for monotonically increasing loading conditions.

6.1 Deformation of a Thin Bonded Biphasic Layer

In this section, the short-time leading-order asymptotic solution of the deformation problem for a thin transversely isotropic biphasic layer bonded to a rigid impermeable substrate and subjected to a normal load is constructed. Also, the long-term response of the biphasic layer under constant load is briefly discussed.

6.1.1 Deformation Problem Formulation

Let us consider a thin transversely isotropic biphasic layer of uniform thickness, h, ideally bonded to a rigid impermeable substrate and loaded by a normal time dependent load, q, (see Fig. 6.1). In the following, the two-dimensional Cartesian coordinate system (x_1, x_2) in the plane of the biphasic layer will be denoted by $\mathbf{y} = (y_1, y_2)$, so that $\mathbf{x} = (\mathbf{y}, z)$, where z is the normal coordinate. Also, the displacement vector of the solid matrix is represented as $\mathbf{u} = (\mathbf{v}, w)$, where $\mathbf{v}$ and w are the in-plane displacement vector and the normal displacement, respectively.

The system of governing differential equations (5.16)–(5.19) for a biphasic medium can now be rewritten as

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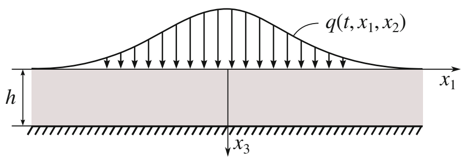

Fig. 6.1 A biphasic layer bonded to a rigid impermeable substrate and supporting a time-dependent normal load

$$A^s_{66}\Delta_y \mathbf{v} + (A^s_{11} - A^s_{66})\nabla_y \nabla_y \cdot \mathbf{v} + A^s_{44}\frac{\partial^2 \mathbf{v}}{\partial z^2} + (A^s_{13} + A^s_{44})\frac{\partial}{\partial z}\nabla_y w = \nabla_y p,$$

$$A^s_{44}\Delta_y w + A^s_{33}\frac{\partial^2 w}{\partial z^2} + (A^s_{13} + A^s_{44})\frac{\partial}{\partial z}\nabla_y \cdot \mathbf{v} = \frac{\partial p}{\partial z}, \quad (6.1)$$

$$\frac{\partial}{\partial t}\left(\nabla_y \cdot \mathbf{v} + \frac{\partial w}{\partial z}\right) = k_1 \Delta_y p + k_3 \frac{\partial^2 p}{\partial z^2}, \quad (6.2)$$

$$\mathbf{w}^f = -k_1 \nabla_y p - k_3 \frac{\partial p}{\partial z}\mathbf{e}_3. \quad (6.3)$$

Here, $\nabla_y = (\partial/\partial y_1)\mathbf{e}_1 + (\partial/\partial y_2)\mathbf{e}_2$ and $\Delta_y = \nabla_y \cdot \nabla_y$ are the in-plane Hamilton and Laplace operators, respectively, while the scalar product is denoted by a dot.

At the bottom surface of the biphasic layer, $z = h$, the boundary conditions (5.20) and (5.21) now take the form

$$\mathbf{v}\big|_{z=h} = \mathbf{0}, \quad w\big|_{z=h} = 0, \quad \frac{\partial p}{\partial z}\bigg|_{z=h} = 0.$$

As on the upper surface, $z = 0$, the layer is assumed to be loaded only by a variable distributed normal load q, the traction boundary conditions

$$\sigma_{33}\big|_{z=0} = -q, \quad \sigma_{13}\big|_{z=0} = \sigma_{23}\big|_{z=0} = 0$$

can be rewritten as follows (see Eqs. (5.24) and (5.25)):

$$-p + A^s_{13}\nabla_y \cdot \mathbf{v} + A^s_{33}\frac{\partial w}{\partial z}\bigg|_{z=0} = -q, \quad (6.4)$$

$$\nabla_y w + \frac{\partial \mathbf{v}}{\partial z}\bigg|_{z=0} = \mathbf{0}.$$

Moreover, assuming that the normal load q is transferred from an impermeable punch, we require that

$$\frac{\partial p}{\partial z}\bigg|_{z=0} = 0,$$

that is no fluid flow takes place across the contact interface.

Equations (6.1)–(6.3) along with the above boundary conditions and the zero initial conditions (see Eq. (5.30))

$$\mathbf{v} = \mathbf{0}, \quad w = 0, \quad p = 0, \quad \mathbf{w}^{\mathrm{f}} = \mathbf{0}, \quad -\infty < t < 0,$$

constitute the deformation problem for a thin biphasic layer.

6.1.2 Perturbation Analysis of the Deformation Problem: Short-Time Asymptotic Solution

Assuming that the biphasic layer is relatively thin, we set

$$h = \varepsilon h_*, \tag{6.5}$$

where ε is a small positive parameter, h_* is independent of ε and has the order of magnitude of a characteristic length in the plane of the layer.

Now, we introduce the dimensionless in-plane coordinates

$$\boldsymbol{\eta} = (\eta_1, \eta_2), \quad \eta_i = \frac{y_i}{h_*}, \quad i = 1, 2, \tag{6.6}$$

and the stretched dimensionless normal coordinate

$$\zeta = \varepsilon^{-1}\frac{z}{h_*}. \tag{6.7}$$

Also, following Ateshian et al. [7], the governing equations are non-dimensionalized using non-dimensional variables

$$\tau = \frac{k_3 A_{33}^{\mathrm{s}}}{h^2}t, \quad \mathbf{V} = \frac{\mathbf{v}}{h}, \quad W = \frac{w}{h}, \quad P = \frac{p}{A_{44}^{\mathrm{s}}}, \quad Q = \frac{q}{A_{44}^{\mathrm{s}}}. \tag{6.8}$$

Observe that as a consequence of (6.5), the first formula above can be simply rewritten as $\tau = \varepsilon^{-2}(k_3 A_{33}^{\mathrm{s}}/h_*^2)t$, so that a finite interval for the fast variable τ corresponds to a very short interval for the time variable t. Correspondingly, the approximate solution obtained below represents the short-time asymptotics.

Therefore, the system of differential equations (6.1)–(6.3) with the corresponding boundary and initial conditions takes the form

$$\frac{\partial^2\mathbf{V}}{\partial\zeta^2} + \varepsilon\Big((1+\beta_{13})\nabla_\eta\frac{\partial W}{\partial\zeta} - \nabla_\eta P\Big)$$
$$+ \varepsilon^2\big(\beta_{66}\Delta_\eta\mathbf{V} + (\beta_{11}-\beta_{66})\nabla_\eta\nabla_\eta\cdot\mathbf{V}\big) = \mathbf{0},$$

$$\beta_{33}\frac{\partial^2 W}{\partial \zeta^2} - \frac{\partial P}{\partial \zeta} + \varepsilon(1+\beta_{13})\nabla_\eta \cdot \frac{\partial \mathbf{V}}{\partial \zeta} + \varepsilon^2 \Delta_\eta W = 0, \tag{6.9}$$
$$\beta_{33}\frac{\partial^2 W}{\partial \tau \partial \zeta} - \frac{\partial^2 P}{\partial \zeta^2} + \varepsilon\beta_{33}\frac{\partial}{\partial \tau}\nabla_\eta \cdot \mathbf{V} - \varepsilon^2 \kappa_1 \Delta_\eta P = 0,$$

$$\mathbf{V}\big|_{\zeta=1} = \mathbf{0}, \quad W\big|_{\zeta=1} = 0, \quad \frac{\partial P}{\partial \zeta}\bigg|_{\zeta=1} = 0,$$

$$Q - P + \varepsilon\beta_{13}\nabla_\eta \cdot \mathbf{V} + \beta_{33}\frac{\partial W}{\partial \zeta}\bigg|_{\zeta=0} = 0, \tag{6.10}$$

$$\frac{\partial \mathbf{V}}{\partial \zeta} + \varepsilon \nabla_\eta W\bigg|_{\zeta=0} = \mathbf{0}, \quad \frac{\partial P}{\partial \zeta}\bigg|_{\zeta=0} = 0,$$

$$\mathbf{V} = \mathbf{0}, \quad W = 0, \quad P = 0, \quad -\infty < \tau < 0.$$

Here we have introduced the notation

$$\beta_{11} = \frac{A^s_{11}}{A^s_{44}}, \quad \beta_{13} = \frac{A^s_{13}}{A^s_{44}}, \quad \beta_{33} = \frac{A^s_{33}}{A^s_{44}}, \quad \beta_{66} = \frac{A^s_{66}}{A^s_{44}}, \quad \kappa_1 = \frac{k_1}{k_3}. \tag{6.11}$$

Following Ateshian et al. [7], the asymptotic ansatz for the solution to the system (6.9) and (6.10) is represented in the form

$$\begin{aligned} P &= P^0 + \varepsilon^2 P^1 + \cdots, \\ \mathbf{V} &= \varepsilon \mathbf{V}^0 + \cdots, \\ W &= \varepsilon^2 W^0 + \cdots, \end{aligned} \tag{6.12}$$

where only non-vanishing leading asymptotic terms are included. Note that this asymptotic ansatz is particularly motivated by the only nonhomogeneous equation (6.4) in the deformation problem under consideration.

Substituting the asymptotic expressions (6.12) into Eq. (6.9) and the boundary conditions (6.10), after collecting terms of like order, we obtain

$$P^0 \equiv Q \tag{6.13}$$

and arrive at the following problems:

$$\frac{\partial^2 \mathbf{V}^0}{\partial \zeta^2} = \nabla_\eta Q, \quad \zeta \in (0,1), \quad \frac{\partial \mathbf{V}^0}{\partial \zeta}\bigg|_{\zeta=0} = \mathbf{0}, \quad \mathbf{V}^0\big|_{\zeta=1} = \mathbf{0}; \tag{6.14}$$

$$\beta_{33}\frac{\partial^2 W^0}{\partial\zeta^2}-\frac{\partial P^1}{\partial\zeta}=-(1+\beta_{13})\nabla_\eta\cdot\frac{\partial \mathbf{V}^0}{\partial\zeta},\quad \zeta\in(0,1),$$
$$\beta_{33}\frac{\partial^2 W^0}{\partial\tau\partial\zeta}-\frac{\partial^2 P^1}{\partial\zeta^2}=\kappa_1\Delta_\eta Q-\beta_{33}\frac{\partial}{\partial\tau}\nabla_\eta\cdot\mathbf{V}^0,\quad \zeta\in(0,1), \tag{6.15}$$

$$W^0\big|_{\zeta=1}=0,\quad \frac{\partial P^1}{\partial\zeta}\bigg|_{\zeta=1}=0,$$
$$-P^1+\beta_{33}\frac{\partial W^0}{\partial\zeta}\bigg|_{\zeta=0}=-\beta_{13}\nabla_\eta\cdot\mathbf{V}^0\big|_{\zeta=0},\quad \frac{\partial P^1}{\partial\zeta}\bigg|_{\zeta=0}=0. \tag{6.16}$$

By direct integration of the ordinary boundary-value problem (6.14), we find

$$\mathbf{V}^0=-\frac{1}{2}(1-\zeta^2)\nabla_\eta Q. \tag{6.17}$$

The substitution of (6.17) into (6.15) and (6.16) yields

$$\beta_{33}\frac{\partial^2 W^0}{\partial\zeta^2}-\frac{\partial P^1}{\partial\zeta}=-(1+\beta_{13})\zeta\Delta_\eta Q,\quad \zeta\in(0,1),$$
$$\beta_{33}\frac{\partial^2 W^0}{\partial\tau\partial\zeta}-\frac{\partial^2 P^1}{\partial\zeta^2}=\kappa_1\Delta_\eta Q+\frac{\beta_{33}}{2}(1-\zeta^2)\frac{\partial}{\partial\tau}\Delta_\eta Q,\quad \zeta\in(0,1), \tag{6.18}$$

$$W^0\big|_{\zeta=1}=0,\quad \frac{\partial P^1}{\partial\zeta}\bigg|_{\zeta=1}=0,$$
$$\beta_{33}\frac{\partial W^0}{\partial\zeta}-P^1\bigg|_{\zeta=0}=\frac{\beta_{13}}{2}\Delta_\eta Q,\quad \frac{\partial P^1}{\partial\zeta}\bigg|_{\zeta=0}=0. \tag{6.19}$$

The exact solution of this resulting problem (6.18) and (6.19) can be determined via the Laplace transform method (see, e.g., [19, 20]).

6.1.3 Solution of the Resulting Ordinary Boundary-Value Problem

We proceed by first remarking that, along with the coordinate system $(\mathbf{y}, z)$ where the coordinate center is placed at the contact interface and the z axis is directed into the layer, another coordinate system is commonly used with its coordinate center placed at the bottom surface of the layer (see Fig. 6.2). In this case we have

$$\bar{z}=h-z,\quad \bar{\mathbf{y}}=-\mathbf{y}, \tag{6.20}$$

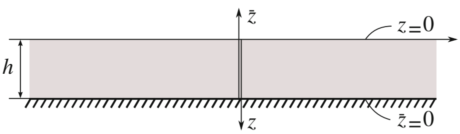

Fig. 6.2 A biphasic layer of uniform thickness bonded to a rigid impermeable substrate: Two systems of coordinates

where $\bar{z}$ and $\bar{\mathbf{y}}$ are the new normal and in-plane coordinates.

Moreover, since the normal axis has changed direction to its opposite, the normal displacements should be related by

$$\bar{W}(t, \bar{\mathbf{y}}, \bar{z}) = -W(t, \mathbf{y}, z). \tag{6.21}$$

The coordinate transformation (6.20) and (6.21) must be taken into account when comparing the obtained results with other studies.

Making use of (6.20) and (6.21), we transform the problem (6.18) and (6.19) to

$$\begin{aligned} &\beta_{33}\frac{\partial^2 \bar{W}^0}{\partial \bar{\zeta}^2} - \frac{\partial P^1}{\partial \bar{\zeta}} = (1+\beta_{13})(1-\bar{\zeta})\Delta_\eta Q, \quad \bar{\zeta} \in (0,1), \\ &\beta_{33}\frac{\partial^2 \bar{W}^0}{\partial \tau \partial \bar{\zeta}} - \frac{\partial^2 P^1}{\partial \bar{\zeta}^2} = \kappa_1 \Delta_\eta Q + \frac{\beta_{33}}{2}\bar{\zeta}(2-\bar{\zeta})\frac{\partial}{\partial \tau}\Delta_\eta Q, \quad \bar{\zeta} \in (0,1), \end{aligned} \tag{6.22}$$

$$\begin{aligned} &\bar{W}^0\big|_{\bar{\zeta}=0} = 0, \quad \frac{\partial P^1}{\partial \bar{\zeta}}\bigg|_{\bar{\zeta}=0} = 0, \\ &\beta_{33}\frac{\partial \bar{W}^0}{\partial \bar{\zeta}} - P^1\bigg|_{\bar{\zeta}=1} = \frac{\beta_{13}}{2}\Delta_\eta Q, \quad \frac{\partial P^1}{\partial \bar{\zeta}}\bigg|_{\bar{\zeta}=1} = 0. \end{aligned} \tag{6.23}$$

Now, let $\tilde{\bar{W}}^0$, $\tilde{P}^1$, and $\tilde{Q}$ denote the Laplace transforms of $\bar{W}^0$, P^1, and Q, respectively, with respect to the dimensionless time variable τ, and s be the Laplace transform parameter.

Taking into account the zero initial conditions, the Laplace transformation of Eqs. (6.22) and (6.23) leads to the system

$$\begin{aligned} &\beta_{33}\frac{\partial^2 \tilde{\bar{W}}^0}{\partial \bar{\zeta}^2} - \frac{\partial \tilde{P}^1}{\partial \bar{\zeta}} = (1+\beta_{13})(1-\bar{\zeta})\Delta_\eta \tilde{Q}, \quad \bar{\zeta} \in (0,1), \\ &s\beta_{33}\frac{\partial \tilde{\bar{W}}^0}{\partial \bar{\zeta}} - \frac{\partial^2 \tilde{P}^1}{\partial \bar{\zeta}^2} = \kappa_1 \Delta_\eta \tilde{Q} + s\beta_{33}\frac{\bar{\zeta}}{2}(2-\bar{\zeta})\Delta_\eta \tilde{Q}, \quad \bar{\zeta} \in (0,1), \end{aligned} \tag{6.24}$$

$$\begin{gathered}
\tilde{\bar{W}}^0\big|_{\bar{\zeta}=0} = 0, \quad \frac{\partial \tilde{P}^1}{\partial \bar{\zeta}}\bigg|_{\bar{\zeta}=0} = 0, \\
\beta_{33}\frac{\partial \tilde{\bar{W}}^0}{\partial \bar{\zeta}} - \tilde{P}^1\bigg|_{\bar{\zeta}=1} = \frac{\beta_{13}}{2}\Delta_\eta \tilde{Q}, \quad \frac{\partial \tilde{P}^1}{\partial \bar{\zeta}}\bigg|_{\bar{\zeta}=1} = 0.
\end{gathered} \tag{6.25}$$

The homogeneous differential system corresponding to Eq. (6.24) has the characteristic equation $\lambda^4 - s\lambda^2 = 0$, with three roots $\lambda_{1,2} = 0$, $\lambda_{3,4} = \pm\sqrt{s}$, and its general solution is given by

$$\begin{aligned}
\tilde{\bar{W}}_0^0 &= C_0 + C_1 \cosh\sqrt{s}\bar{\zeta} + C_2 \sinh\sqrt{s}\bar{\zeta}, \\
\tilde{P}_0^1 &= C_3 + \beta_{33}\sqrt{s}\big(C_1 \sinh\sqrt{s}\bar{\zeta} + C_2 \cosh\sqrt{s}\bar{\zeta}\big),
\end{aligned}$$

where $C_0, \ldots, C_3$ are arbitrary functions of the Laplace transform parameter s.

A particular solution of the system (6.24), which does not necessarily satisfy the boundary conditions (6.25), can be found by the method of undetermined coefficients in the form

$$\begin{aligned}
\tilde{\bar{W}}_1^0 &= \left(\frac{1}{s\beta_{33}}\big(1 + \kappa_1 + \beta_{13} - \beta_{33}\big)\bar{\zeta} + \frac{\bar{\zeta}^2}{2} - \frac{\bar{\zeta}^3}{6}\right)\Delta_\eta \tilde{Q}, \\
\tilde{P}_1^1 &= (\beta_{33} - 1 - \beta_{13})\left(\bar{\zeta} - \frac{\bar{\zeta}^2}{2}\right)\Delta_\eta \tilde{Q}.
\end{aligned}$$

Now, substituting the expressions

$$\tilde{\bar{W}}^0 = \tilde{\bar{W}}_0^0 + \tilde{\bar{W}}_1^0, \quad \tilde{P}^1 = \tilde{P}_0^1 + \tilde{P}_1^1$$

into the system of boundary conditions (6.25), we derive a system of four linear algebraic equations for determining C_0, C_1, C_2, and C_3 as follows:

$$\begin{gathered}
C_0 = -C_1 = -\frac{1}{\beta_{33}s}\big(1 + \beta_{13} - \beta_{33}\big)\Delta_\eta \tilde{Q}, \quad C_2 = \frac{\cosh\sqrt{s}}{\sinh\sqrt{s}}C_0, \\
C_3 = \left(\frac{1}{2} + \frac{1}{s}\big(1 + \kappa_1 + \beta_{13} - \beta_{33}\big)\right)\Delta_\eta \tilde{Q}.
\end{gathered}$$

Collecting the above formulas, we thus obtain

$$\begin{aligned}
\tilde{\bar{W}}^0 &= \Delta_\eta \tilde{Q}\left\{\frac{\delta_1}{\beta_{13}s}\left(1 - \frac{\sinh\sqrt{s}(1-\bar{\zeta})}{\sinh\sqrt{s}}\right) + \frac{\bar{\zeta}^2}{2}\left(1 - \frac{\bar{\zeta}}{3}\right) + \frac{\delta_0}{\beta_{13}s}\bar{\zeta}\right\}, \\
\tilde{P}^1 &= \Delta_\eta \tilde{Q}\left\{\delta_1\left(\frac{\cosh\sqrt{s}(1-\bar{\zeta})}{\sqrt{s}\sinh\sqrt{s}} + \bar{\zeta}\left(1 - \frac{\bar{\zeta}}{2}\right)\right) + \frac{1}{2} + \frac{\delta_0}{s}\right\},
\end{aligned}$$

where, for simplicity, we have introduced the auxiliary notation

$$\delta_0 = 1 + \kappa_1 + \beta_{13} - \beta_{33}, \quad \delta_1 = \beta_{33} - 1 - \beta_{13}. \tag{6.26}$$

By performing the inverse Laplace transform using the residue theorem, we get

$$\begin{aligned}\bar{W}^0 = {} & \Delta_\eta Q(\tau)\frac{\bar{\zeta}^2}{2}\Big(1 - \frac{\bar{\zeta}}{3}\Big) + \frac{(\delta_1 + \delta_0\bar{\zeta})}{\beta_{33}}\int_0^\tau \Delta_\eta Q(\tau')\,d\tau' \\ & - \frac{\delta_1}{\beta_{33}}\int_0^\tau \Delta_\eta Q(\tau')\bigg\{1 - \bar{\zeta} \\ & + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{\sin \pi n(1-\bar{\zeta})}{n} e^{-\pi^2 n^2(\tau - \tau')}\bigg\}\,d\tau',\end{aligned} \tag{6.27}$$

$$\begin{aligned}P^1 = {} & \Delta_\eta Q(\tau)\bigg(\frac{1}{2} + \delta_1\bar{\zeta}\Big(1 - \frac{\bar{\zeta}}{2}\Big)\bigg) + \delta_0\int_0^\tau \Delta_\eta Q(\tau')\,d\tau' \\ & + \delta_1\int_0^\tau \Delta_\eta Q(\tau')\bigg\{1 + 2\sum_{n=1}^{\infty}(-1)^n \cos \pi n(1-\bar{\zeta})e^{-\pi^2 n^2(\tau-\tau')}\bigg\}\,d\tau'.\end{aligned} \tag{6.28}$$

Note that in the isotropic case we have $\delta_0 = 0$ and $\delta_1 = 1$, i.e.,

$$1 + \kappa_1 + \beta_{13} - \beta_{33} = 0, \quad \beta_{33} - 1 - \beta_{13} = 1,$$

and formulas (6.27) and (6.28) agree with the leading-order asymptotic solution originally obtained by Ateshian et al. [7].

6.1.4 Displacements of the Solid Matrix

By recovering the dimensional variables (see, in particular, Eqs. (6.6)–(6.8), (6.11), (6.12), (6.17), (6.20), (6.21), and (6.27)), we arrive at the following leading-order asymptotic approximations for the in-plane (tangential) and out-of-plane (normal) displacements:

$$\mathbf{v} \simeq -\frac{h^2}{2A_{44}^{\mathrm{s}}}\Big(1 - \frac{z^2}{h^2}\Big)\nabla_y q(t, \mathbf{y}),$$

$$w \simeq -\frac{h^3}{3A_{44}^{\mathrm{s}}}\Delta_y q(t,\mathbf{y})\Big(1-\frac{z}{h}\Big)^2\Big(1+\frac{z}{2h}\Big)$$
$$-\,hk_1\int_0^t \Delta_y q(t',\mathbf{y})\,dt'\Bigg(1-\frac{\big[(k_1+k_3)A_{44}^{\mathrm{s}}+k_3\big(A_{13}^{\mathrm{s}}-A_{33}^{\mathrm{s}}\big)\big]}{k_1A_{44}^{\mathrm{s}}}\frac{z}{h}\Bigg)$$
$$+\,\frac{hk_3\big(A_{33}^{\mathrm{s}}-A_{44}^{\mathrm{s}}-A_{13}^{\mathrm{s}}\big)}{A_{44}^{\mathrm{s}}}\Bigg\{\frac{z}{h}\int_0^t \Delta_y q(t',\mathbf{y})\,dt'$$
$$+\,\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\pi n\frac{z}{h}\int_0^t \Delta_y q(t',\mathbf{y})\exp\Big(-\pi^2n^2\frac{k_3A_{33}^{\mathrm{s}}}{h^2}(t-t')\Big)dt'\Bigg\}.$$

According to the derived solution, the displacements of the surface points of the bonded thin biphasic layer are

$$\mathbf{v}\big|_{z=0} \simeq -\frac{h^2}{2A_{44}^{\mathrm{s}}}\nabla_y q(t,\mathbf{y}), \tag{6.29}$$

$$w\big|_{z=0} \simeq -\frac{h^3}{3A_{44}^{\mathrm{s}}}\Delta_y q(t,\mathbf{y}) - hk_1\int_0^t \Delta_y q(\tau,\mathbf{y})\,d\tau. \tag{6.30}$$

The leading-order asymptotic relations (6.29) and (6.30) derived for the so-called local indentation will be used to formulate asymptotic models for the unilateral frictionless contact interaction between thin bonded biphasic layers.

6.1.5 Interstitial Fluid Pressure and Relative Fluid Flux

In light of (6.6)–(6.8), $(6.12)_1$, (6.13), and (6.28), we obtain

$$p \simeq q(t,\mathbf{y}) + \frac{h^2\big(A_{44}^{\mathrm{s}}+2A_{33}^{\mathrm{s}}-2A_{13}^{\mathrm{s}}\big)}{6A_{44}^{\mathrm{s}}}\Delta_y q(t,\mathbf{y}) + k_1A_{33}^{\mathrm{s}}\int_0^t \Delta_y q(t',\mathbf{y})\,dt'$$
$$-\,\frac{2h^2}{\pi^2}\frac{\big(A_{33}^{\mathrm{s}}-A_{44}^{\mathrm{s}}-A_{13}^{\mathrm{s}}\big)}{A_{44}^{\mathrm{s}}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\pi n\frac{z}{h} \tag{6.31}$$
$$\times\int_{0^-}^t \Delta_y \dot{q}(t',\mathbf{y})\exp\Big(-\pi^2n^2\frac{k_3A_{33}^{\mathrm{s}}}{h^2}(t-t')\Big)dt'.$$

Recall that the lower integration limit 0^- in the last integral in (6.31) allows consideration of the load discontinuity at time zero.

We now introduce the dimensionless variables (6.6)–(6.8) into Eq. (6.3), and obtain

$$\mathbf{w}^{\mathrm{f}} = -\frac{k_1 A_{44}^{\mathrm{s}}}{h_*} \nabla_\eta P - \frac{k_3 A_{44}^{\mathrm{s}}}{\varepsilon h_*} \frac{\partial P}{\partial \zeta} \mathbf{e}_3.$$

As a result of $(6.12)_1$, we state the following asymptotic formulas

$$w_1^{\mathrm{f}} \mathbf{e}_1 + w_2^{\mathrm{f}} \mathbf{e}_2 \simeq -\frac{k_1 A_{44}^{\mathrm{s}}}{h_*} \big(\nabla_\eta Q + \varepsilon^2 \nabla_\eta P^1 \big),$$

$$w_3^{\mathrm{f}} \simeq -\varepsilon \frac{k_3 A_{44}^{\mathrm{s}}}{h_*} \frac{\partial P^1}{\partial \zeta}.$$

The in-plane and out-of-plane components of the relative fluid flux can be evaluated by differentiating the asymptotic expansion (6.31).

6.1.6 Stresses in the Solid and Fluid Phases

As a consequence of (6.8) and (6.12), the above asymptotic analysis yields the following leading-order asymptotic formulas for the solid matrix strains:

$$\varepsilon_{11} \simeq \varepsilon^2 \frac{\partial V_1^0}{\partial \eta_1}, \quad \varepsilon_{22} \simeq \varepsilon^2 \frac{\partial V_2^0}{\partial \eta_2}, \quad \varepsilon_{33} \simeq \varepsilon^2 \frac{\partial W^0}{\partial \zeta},$$
$$\varepsilon_{12} \simeq \frac{\varepsilon^2}{2} \left(\frac{\partial V_1^0}{\partial \eta_2} + \frac{\partial V_2^0}{\partial \eta_1} \right), \quad \varepsilon_{13} \simeq \frac{\varepsilon}{2} \frac{\partial V_1^0}{\partial \zeta}, \quad \varepsilon_{23} \simeq \frac{\varepsilon}{2} \frac{\partial V_2^0}{\partial \zeta}. \tag{6.32}$$

Substituting these asymptotic approximations into Eq. (5.13), we can evaluate the effective stresses σ_{ij}^{e} in the solid matrix. After that, in light of (5.1) and (5.7), the stresses in the solid matrix can be approximately evaluated by the formula

$$\boldsymbol{\sigma}^{\mathrm{s}} = -(1 - \phi_{\mathrm{f}}) p \mathbf{I} + \boldsymbol{\sigma}^{\mathrm{e}},$$

where ϕ_{f} is the porosity of the solid matrix (fluid volume fraction), and the stresses in the fluid phase are defined by the following formula (see Eq. (5.4)):

$$\boldsymbol{\sigma}^{\mathrm{f}} = -\phi_{\mathrm{f}} p \mathbf{I}.$$

In particular, according to (5.13), (6.17), and (6.32), we obtain

$$\sigma_{31}^{\mathrm{e}} \mathbf{e}_1 + \sigma_{32}^{\mathrm{e}} \mathbf{e}_2 \simeq z \nabla_y q(t, \mathbf{y}), \tag{6.33}$$

from which it follows that the maximum shear stress in a thin bonded biphasic layer under distributed normal loading is achieved at the bonding interface, $z = h$, at the location of maximum gradient $|q(t, \mathbf{y})|$.

Observe [7] that, as the dominant terms in the deformations and stresses are the lowest-order quantities, the normal strains and effective stresses as well as the in-plane shear strain and effective stress, ε_{12} and σ^{e}_{12}, are $O(\varepsilon^2)$, while the out-of-plane strains and effective shear stresses are $O(\varepsilon)$ (see Eq. (6.32)).

In light of $(6.12)_1$, the hydrostatic pressure is $O(1)$ and is significantly larger than the effective normal stresses. Also, as its lowest-order term is $O(\varepsilon)$, the shear stresses σ^{s}_{3i} $(i = 1, 2)$ in the solid phase, which are equal to the effective shear stresses σ^{e}_{3i} $(i = 1, 2)$, are one order of magnitude greater than the effective normal stresses.

We emphasize (see, e.g., [7]) that this order of magnitude analysis has major implications on how a thin layer of biphasic tissue (e.g., articular cartilage) supports distributed compressive load under the bonding condition.

Finally, let us now introduce the in-plane typical length scale, L, such that $\varepsilon = h/L$. (Note that, as a consequence of (6.5), we simply have $L = h_*$.) In contact problem [7], L refers to the characteristic length of the contact area. Then, the first formula (6.8) can be rewritten as

$$\tau = \frac{t}{\mathcal{T}_3},$$

where $\mathcal{T}_3 = h^2/(k_3 A^{s}_{33})$ is the typical vertical diffusion time within the biphasic layer [8]. On the other hand, formula $(6.8)_1$ can be rewritten as

$$\tau = \varepsilon^{-2} \frac{k_3 A^{s}_{33}}{L^2} t,$$

which shows that the dimensionless time variable τ is introduced by stretching the dimensionless time variable $(k_3 A^{s}_{33}/L^2)t$.

We thus underline that formulas (6.29) and (6.30) present the leading-order asymptotic solution, which is valid for short times only.

6.1.7 Long-Term (Equilibrium) Response of a Thin Bonded Biphasic Layer Under Constant Loading

To begin, we assume that the normal load distribution q has a finite support and does not depend on the time variable t. Following [7], we consider the equilibrium (long-term, when $t/\mathcal{T}_3 \gg 1$) response of a thin bonded layer of biphasic material after the relative motion of the interstitial fluid has ceased and the fluid pressure has vanished. In this case the system of governing differential equations (6.1) and (6.2) reduces to that for a single phase *compressible* elastic layer with material properties coinciding with those of the solid matrix.

The corresponding asymptotic solution was derived in Sect. 1.2 and the leading asymptotic terms are summarized below (see Eqs. (1.22) and (1.23))

$$\mathbf{v} \simeq \frac{h^2}{A_{33}^{\mathrm{s}}}\left\{\frac{1}{2}\left(1+\frac{A_{13}^{\mathrm{s}}}{A_{44}^{\mathrm{s}}}\right)\left(1-\frac{z}{h}\right)^2-\frac{A_{13}^{\mathrm{s}}}{A_{44}^{\mathrm{s}}}\left(1-\frac{z}{h}\right)\right\}\nabla_y q(\mathbf{y}),$$
$$w \simeq \frac{h}{A_{33}^{\mathrm{s}}}\left(1-\frac{z}{h}\right)q(\mathbf{y}).$$

Finally, we observe [7] that during the biphasic creep process, the short-time asymptotic solution gradually evolves into the equilibrium (long-term) asymptotic solution, which has drastically different characteristics.

6.2 Deformation of a Thin Transversely Isotropic Biphasic Poroelastic Layer Bonded to a Rigid Impermeable Substrate

In this section, the short-time leading-order asymptotic solution of the deformation problem for a thin biphasic poroelastic (BPVE) layer is constructed. The main result of the section (see Sect. 6.2.3) is an approximate formula for the local indentation of a thin bonded BPVE layer.

6.2.1 Deformation Problem Formulation

In this section, we follow the problem formulation given in detail in Sect. 6.1, with the sole difference being that the total stresses within a thin biphasic poroviscoelastic (BPVE) layer are determined by the constitutive relations

$$\begin{aligned}
\sigma_{11} &= -p + B_{11}^{\mathrm{s}} * \varepsilon_{11} + B_{12}^{\mathrm{s}} * \varepsilon_{22} + B_{13}^{\mathrm{s}} * \varepsilon_{33}, \quad & \sigma_{23} &= 2B_{44}^{\mathrm{s}} * \varepsilon_{23},\\
\sigma_{22} &= -p + B_{12}^{\mathrm{s}} * \varepsilon_{11} + B_{11}^{\mathrm{s}} * \varepsilon_{22} + B_{13}^{\mathrm{s}} * \varepsilon_{33}, \quad & \sigma_{13} &= 2B_{44}^{\mathrm{s}} * \varepsilon_{13},\\
\sigma_{33} &= -p + B_{13}^{\mathrm{s}} * \varepsilon_{11} + B_{13}^{\mathrm{s}} * \varepsilon_{22} + B_{33}^{\mathrm{s}} * \varepsilon_{33}, \quad & \sigma_{12} &= 2B_{66}^{\mathrm{s}} * \varepsilon_{12},
\end{aligned} \tag{6.34}$$

where p is the pressure in the fluid phase, $B_{11}^{\mathrm{s}}(t)$, $B_{12}^{\mathrm{s}}(t)$, $B_{13}^{\mathrm{s}}(t)$, $B_{33}^{\mathrm{s}}(t)$, and $B_{44}^{\mathrm{s}}(t)$ are independent stress-relaxation functions of the solid phase, $B_{66}^{\mathrm{s}}(t) = \big(B_{11}^{\mathrm{s}}(t) - B_{12}^{\mathrm{s}}(t)\big)/2$, and the symbol $*$ denotes the Stieltjes integral, i.e.,

$$B_{kl}^{\mathrm{s}} * \varepsilon_{ij} = \int_{-\infty}^{t} B_{kl}^{\mathrm{s}}(t-\tau)\, d\varepsilon_{ij}(\tau).$$

Correspondingly, the equilibrium equations of the solid matrix take the form

$$B^{\mathrm{s}}_{66} * \Delta_y \mathbf{v} + (B^{\mathrm{s}}_{11} - B^{\mathrm{s}}_{66}) * \nabla_y \nabla_y \cdot \mathbf{v} + B^{\mathrm{s}}_{44} * \frac{\partial^2 \mathbf{v}}{\partial z^2} + (B^{\mathrm{s}}_{13} + B^{\mathrm{s}}_{44}) * \frac{\partial}{\partial z} \nabla_y w = \nabla_y p, \quad (6.35)$$

$$B^{\mathrm{s}}_{44} * \Delta_y w + B^{\mathrm{s}}_{33} * \frac{\partial^2 w}{\partial z^2} + (B^{\mathrm{s}}_{13} + B^{\mathrm{s}}_{44}) * \frac{\partial}{\partial z} \nabla_y \cdot \mathbf{v} = \frac{\partial p}{\partial z}, \quad (6.36)$$

where $\mathbf{v}$ and w are the in-plane displacement vector and the normal displacement of the solid matrix, respectively.

The continuity equation for the BPVE medium has the same form as for biphasic mixtures, i.e.,

$$\frac{\partial}{\partial t}\left(\nabla_y \cdot \mathbf{v} + \frac{\partial w}{\partial z}\right) = k_1 \Delta_y p + k_3 \frac{\partial^2 p}{\partial z^2}. \quad (6.37)$$

The boundary conditions at the bottom surface of the layer, $z = h$, and at the top surface, $z = 0$, can be written as follows:

$$\mathbf{v}\big|_{z=h} = \mathbf{0}, \quad w\big|_{z=h} = 0, \quad \frac{\partial p}{\partial z}\bigg|_{z=h} = 0;$$

$$-p + B^{\mathrm{s}}_{13} * \nabla_y \cdot \mathbf{v} + B^{\mathrm{s}}_{33} * \frac{\partial w}{\partial z}\bigg|_{z=0} = -q, \quad (6.38)$$

$$B^{\mathrm{s}}_{44} * \left(\nabla_y w + \frac{\partial \mathbf{v}}{\partial z}\right)\bigg|_{z=0} = \mathbf{0}, \quad \frac{\partial p}{\partial z}\bigg|_{z=0} = 0.$$

Equations (6.35)–(6.37) with the given above boundary conditions and the initial conditions

$$\mathbf{v} = \mathbf{0}, \quad w = 0, \quad p = 0, \quad -\infty < t < 0,$$

constitute the deformation problem for a bonded BPVE layer.

Here, following Argatov and Mishuris [4], we construct a leading-order asymptotic solution to the deformation problem (6.35)–(6.37).

6.2.2 *Short-Time Asymptotic Analysis of the Deformation Problem*

Introducing a characteristic length, h_*, and a small parameter, ε, we require that

$$h = \varepsilon h_*.$$

Moreover, as usual, we introduce the dimensionless in-plane coordinates

$$\boldsymbol{\eta} = (\eta_1, \eta_2), \quad \eta_i = \frac{y_i}{h_*}, \quad i = 1, 2,$$

and stretch the normal coordinate as follows:

$$\zeta = \varepsilon^{-1} \frac{z}{h_*}.$$

The governing equations will be non-dimensionalized using the following non-dimensional variables (cf. Eq. (6.8)):

$$\tau = \frac{k_3 B_{44}^{s0}}{h^2} t, \quad \mathbf{V} = \frac{\mathbf{v}}{h}, \quad W = \frac{w}{h}, \quad P = \frac{p}{B_{44}^{s0}}, \quad Q = \frac{q}{B_{44}^{s0}}. \tag{6.39}$$

Here, $B_{44}^{s0} = B_{44}^{s}(0)$ is the instantaneous shear modulus.

Following non-dimensionalisation, we apply the Laplace transformation to the obtained system and arrive at the following problem:

$$\bar{b}_{44}^{s} \frac{\partial^2 \tilde{\mathbf{V}}}{\partial \zeta^2} + \varepsilon (\bar{b}_{13}^{s} + \bar{b}_{44}^{s}) \nabla_\eta \frac{\partial \tilde{W}}{\partial \zeta} + \varepsilon^2 \big(\bar{b}_{66}^{s} \Delta_\eta \tilde{\mathbf{V}} + (\bar{b}_{11}^{s} - \bar{b}_{66}^{s}) \nabla_\eta \nabla_\eta \cdot \tilde{\mathbf{V}} \big) = \varepsilon \nabla_\eta \tilde{P}, \tag{6.40}$$

$$\bar{b}_{33}^{s} \frac{\partial^2 \tilde{W}}{\partial \zeta^2} + \varepsilon (\bar{b}_{13}^{s} + \bar{b}_{44}^{s}) \nabla_\eta \cdot \frac{\partial \tilde{\mathbf{V}}}{\partial \zeta} + \varepsilon^2 \bar{b}_{44}^{s} \Delta_\eta \tilde{W} = \frac{\partial \tilde{P}}{\partial \zeta}, \tag{6.41}$$

$$s \Big(\frac{\partial \tilde{W}}{\partial \zeta} + \varepsilon \nabla_\eta \cdot \tilde{\mathbf{V}} \Big) = \frac{\partial^2 \tilde{P}}{\partial \zeta^2} + \varepsilon^2 \kappa_1 \Delta_\eta \tilde{P}, \tag{6.42}$$

$$\tilde{\mathbf{V}}\big|_{\zeta=1} = \mathbf{0}, \quad \tilde{W}\big|_{\zeta=1} = 0, \quad \frac{\partial \tilde{P}}{\partial \zeta}\bigg|_{\zeta=1} = 0,$$

$$-\tilde{P} + \bar{b}_{13}^{s} \nabla_\eta \cdot \tilde{\mathbf{V}} + \bar{b}_{33}^{s} \frac{\partial \tilde{W}}{\partial \zeta}\bigg|_{\zeta=0} = -\tilde{Q}, \tag{6.43}$$

$$\nabla_\eta \tilde{W} + \frac{\partial \tilde{\mathbf{V}}}{\partial \zeta}\bigg|_{\zeta=0} = \mathbf{0}, \quad \frac{\partial \tilde{P}}{\partial \zeta}\bigg|_{\zeta=0} = 0.$$

The Laplace transforms are denoted by a tilde, $\kappa_1 = k_1/k_3$, and $\bar{b}_{kl}^{s} = s\tilde{B}_{kl}^{s}/B_{44}^{s0}$, where $\tilde{B}_{kl}^{s}$ is the Laplace transform of $B_{kl}^{s}\big(h^2\tau/(B_{44}^{s0}k_3)\big)$ with respect to the dimensionless time variable τ.

Following Ateshian et al. [7], we represent the asymptotic ansatz for the solution to the system (6.40)–(6.43) in the form

$$\tilde{P} \simeq \tilde{Q} + \varepsilon^2 \tilde{P}^1, \quad \tilde{\mathbf{V}} \simeq \varepsilon \tilde{\mathbf{V}}^0, \quad \tilde{W} \simeq \varepsilon^2 \tilde{W}^0. \tag{6.44}$$

Substituting the asymptotic expressions into Eqs. (6.40)–(6.42) and the boundary conditions (6.43), we find after some simple calculations

$$\tilde{\mathbf{V}}^0 = -\frac{1}{2\bar{b}^{\mathrm{s}}_{44}}(1-\zeta^2)\nabla_\eta \tilde{Q}, \tag{6.45}$$

where the pair $\tilde{W}^0$ and $\tilde{P}^1$ should be determined as the solution of the problem

$$\begin{aligned} \bar{b}^{\mathrm{s}}_{33}\frac{\partial^2 \tilde{W}^0}{\partial\zeta^2} - \frac{\partial \tilde{P}^1}{\partial\zeta} &= -(\bar{b}^{\mathrm{s}}_{44} + \bar{b}^{\mathrm{s}}_{13})\nabla_\eta \cdot \frac{\partial \tilde{\mathbf{V}}^0}{\partial\zeta}, \\ s\frac{\partial \tilde{W}^0}{\partial\zeta} - \frac{\partial^2 \tilde{P}^1}{\partial\zeta^2} &= \kappa_1 \Delta_\eta \tilde{Q} - s\nabla_\eta \cdot \tilde{\mathbf{V}}^0, \end{aligned} \tag{6.46}$$

$$\begin{aligned} \tilde{W}^0\big|_{\zeta=1} &= 0, \quad \frac{\partial \tilde{P}^1}{\partial\zeta}\bigg|_{\zeta=1} = 0, \\ -\tilde{P}^1 + \bar{b}^{\mathrm{s}}_{33}\frac{\partial \tilde{W}^0}{\partial\zeta}\bigg|_{\zeta=0} &= -\bar{b}^{\mathrm{s}}_{13}\nabla_\eta \cdot \tilde{\mathbf{V}}^0\big|_{\zeta=0}, \quad \frac{\partial \tilde{P}^1}{\partial\zeta}\bigg|_{\zeta=0} = 0. \end{aligned} \tag{6.47}$$

The general solution of the homogeneous differential system corresponding to Eq. (6.46) is given by

$$\begin{aligned} \tilde{W}^0_0 &= C_0 + C_1 \cosh\sqrt{f(s)}\zeta + C_2 \sinh\sqrt{f(s)}\zeta, \\ \tilde{P}^1_0 &= C_3 + \frac{s}{\sqrt{f(s)}}\big(C_1 \sinh\sqrt{f(s)}\zeta + C_2 \cosh\sqrt{f(s)}\zeta\big), \end{aligned}$$

where we have introduced the notation

$$f(s) = \frac{s}{\bar{b}^{\mathrm{s}}_{33}}.$$

It can be shown that, in light of (6.45), the following pair represents a particular solution of the system (6.46):

$$\begin{aligned} \tilde{W}^0_1 &= \left(\frac{2}{s}\big[\bar{b}^{\mathrm{s}}_{44} + \bar{b}^{\mathrm{s}}_{13} - \bar{b}^{\mathrm{s}}_{33} + \kappa_1 \bar{b}^{\mathrm{s}}_{44}\big] + 1 - \frac{\zeta^2}{6}\right)\frac{\zeta}{2\bar{b}^{\mathrm{s}}_{44}}\Delta_\eta \tilde{Q}, \\ \tilde{P}^1_1 &= \frac{(\bar{b}^{\mathrm{s}}_{44} + \bar{b}^{\mathrm{s}}_{13} - \bar{b}^{\mathrm{s}}_{33})}{2\bar{b}^{\mathrm{s}}_{44}}\zeta^2 \Delta_\eta \tilde{Q}. \end{aligned}$$

Substituting the expressions

$$\tilde{W}^0 = \tilde{W}_0^0 + \tilde{W}_1^0, \quad \tilde{P}^1 = \tilde{P}_0^1 + \tilde{P}_1^1 \tag{6.48}$$

into the system of boundary conditions (6.47) and taking into account Eq. (6.45), we evaluate the integration constants C_0, C_1, C_2, and C_3 as follows:

$$C_0 = -\Big(\frac{1}{3\bar{b}_{44}^{\rm s}} + \frac{\kappa_1}{s}\Big)\Delta_\eta \tilde{Q}, \quad C_1 = 0, \quad C_2 = -\frac{(\bar{b}_{44}^{\rm s} + \bar{b}_{13}^{\rm s} - \bar{b}_{33}^{\rm s})}{s\bar{b}_{44}^{\rm s}} \frac{\Delta_\eta \tilde{Q}}{\sinh\sqrt{f(s)}},$$

$$C_3 = \frac{\bar{b}_{33}^{\rm s}}{2\bar{b}_{44}^{\rm s}}\bigg(\frac{2}{s}(\bar{b}_{44}^{\rm s} + \bar{b}_{13}^{\rm s} - \bar{b}_{33}^{\rm s} + \kappa_1\bar{b}_{44}^{\rm s}) + 1 - \frac{\bar{b}_{13}^{\rm s}}{\bar{b}_{33}^{\rm s}}\bigg)\Delta_\eta \tilde{Q}.$$

The functions W, $\mathbf{V}$, and P can thus be obtained by performing the inverse Laplace transform.

6.2.3 Local Indentation of a Thin BPVE Layer

Recall that $B_{44}^{\rm s}(t)$ represents the out-of-plane relaxation modulus in shear so that, in light of the zero initial conditions, Eqs. $(6.34)_2$ and $(6.34)_4$ take the form

$$\sigma_{3i}(t) = 2\int_{0^-}^{t} B_{44}^{\rm s}(t-\tau)\dot{\varepsilon}_{3i}(\tau)\,d\tau, \quad i = 1, 2. \tag{6.49}$$

Let us introduce the out-of-plane creep compliance in shear of the solid matrix, $J_{44}^{\rm s}(t)$, which governs the deformation response of the solid phase under application of a step out-of-plane shear stress of unit magnitude. Hence, the inverse relations for (6.49) are given by

$$2\varepsilon_{3i}(t) = \int_{0^-}^{t} J_{44}^{\rm s}(t-\tau)\dot{\sigma}_{3i}(\tau)\,d\tau, \quad i = 1, 2.$$

For a given relaxation modulus $B_{44}^{\rm s}(t)$ and its Laplace transform $\tilde{B}_{44}^{\rm s}(s)$ (with respect to the time variable t), the corresponding creep compliance can be evaluated through its Laplace transform

$$\tilde{J}_{44}^{\rm s}(s) = \frac{1}{s^2\tilde{B}_{44}^{\rm s}(s)}. \tag{6.50}$$

Thus, collecting formulas (6.39), (6.44), (6.45), (6.48) and taking account of (6.50), we obtain the following asymptotic representations for the displacements of the surface points of the thin bonded BPVE layer:

$$\mathbf{v}\big|_{z=0} \simeq -\frac{h^2}{2} \int\limits_{0^-}^{t} J_{44}^{\mathrm{s}}(t-\tau) \frac{\partial}{\partial \tau} \nabla_y q(\tau, \mathbf{y})\, d\tau, \tag{6.51}$$

$$w\big|_{z=0} \simeq -\frac{h^3}{3} \int\limits_{0^-}^{t} J_{44}^{\mathrm{s}}(t-\tau) \frac{\partial}{\partial \tau} \Delta_y q(\tau, \mathbf{y})\, d\tau - hk_1 \int\limits_{0}^{t} \Delta_y q(\tau, \mathbf{y})\, d\tau. \tag{6.52}$$

Observe that the derived asymptotic formula (6.52) reflects two types of mechanisms, which are responsible for time-dependent effects in articular cartilage: the flow independent and the flow dependent, characterized by the first and second terms on the right-hand side of (6.52), respectively.

The asymptotic relations (6.51) and (6.52) will be used to formulate asymptotic models for the frictionless contact interaction between thin bonded BPVE layers.

6.2.4 Reduced Relaxation and Creep Function for the Fung Model

Recall (see Sect. 5.4.1) that the so-called reduced stress-relaxation function, $\psi(t)$, is defined by

$$B_{44}^{\mathrm{s}}(t) = B_{44}^{\mathrm{s}\infty}\psi(t),$$

where $B_{44}^{\mathrm{s}\infty} = B_{44}^{\mathrm{s}}(+\infty)$ is the equilibrium modulus.

Let us now consider the reduced creep function, $\varphi(t)$, defined by the formula

$$J_{44}^{\mathrm{s}}(t) = J_{44}^{\mathrm{s}\infty}\varphi(t).$$

Here, $J_{44}^{\mathrm{s}\infty} = J_{44}^{\mathrm{s}}(+\infty)$ is the equilibrium compliance such that $J_{44}^{\mathrm{s}\infty} = 1/B_{44}^{\mathrm{s}\infty}$.

The following normalization conditions then hold:

$$\psi(+\infty) = 1, \quad \varphi(+\infty) = 1.$$

The reduced relaxation function can be represented in terms of a relaxation spectrum $S(\tau)$ as follows:

$$\psi(t) = 1 + \int\limits_{0}^{\infty} S(\tau)e^{-t/\tau}\, d\tau.$$

According to Fung [13], in order to account for the weakly frequency dependent behavior of soft biological tissues, the relaxation spectrum is taken in the form

$$S(\tau) = \begin{cases} \dfrac{c}{\tau}, & \tau_1 \le \tau \le \tau_2, \\ 0, & \tau < \tau_1, \quad \tau > \tau_2, \end{cases}$$

where τ_1 and τ_2 have the dimension of time, and c is dimensionless.

The Fung reduced relaxation function can be evaluated as

$$\psi(t) = 1 + c\Big[E_1\Big(\frac{t}{\tau_2}\Big) - E_1\Big(\frac{t}{\tau_1}\Big)\Big], \tag{6.53}$$

where $E_1(x) = \int_x^\infty e^{-\xi}/\xi \, d\xi$ is the exponential integral function.

The Fung reduced creep function $\varphi(t)$ corresponding to the reduced relaxation function $\psi(t)$ given by Eq. (6.53) can be obtained by employment of the Laplace transform and Eq. (6.50), that is

$$\tilde{\psi}(s)\tilde{\varphi}(s) = \frac{1}{s^2},$$

where the Laplace transform $\tilde{\psi}(s)$ is given by formula (5.197). Note also that the above relation immediately implies that

$$\int_0^t \psi(t-\tau)\varphi(\tau)\, d\tau = t.$$

According to Dortmans et al. [10], the following formula holds:

$$\varphi(t) = 1 - \frac{(\tau_c - \tau_2)(\tau_c - \tau_1)}{c\tau_c(\tau_2 - \tau_1)} e^{-t/\tau_c} - c\int_{\tau_1}^{\tau_2} e^{-t/\tau} \frac{1}{\tau} \frac{1}{\Big(1 + c \ln \dfrac{\tau_2 - \tau}{\tau - \tau_1}\Big)^2 + \pi^2 c^2}\, d\tau, \tag{6.54}$$

where we have used the notation

$$\tau_c = \frac{\tau_2 e^{1/c} - \tau_1}{e^{1/c} - 1}.$$

From (6.53) and (6.54), we find

$$\psi(0) = 1 + c \ln \frac{\tau_2}{\tau_1},$$

$$\varphi(0) = 1 - \frac{(\tau_c - \tau_2)(\tau_c - \tau_1)}{c\tau_c(\tau_2 - \tau_1)} - c\int_{\tau_1}^{\tau_2} \frac{1}{\tau} \frac{1}{\left(1 + c\ln\dfrac{\tau_2 - \tau}{\tau - \tau_1}\right)^2 + \pi^2 c^2}\, d\tau.$$

It can be numerically verified that $\psi(0)\varphi(0) = 1$. We conclude this case by noting that the creep spectrum corresponding to (6.54) was discussed in [10].

6.3 Contact of Thin Bonded Transversely Isotropic BPVE Layers

In this section, the leading-order asymptotic models have been developed for the short-time frictionless contact interaction between thin biphasic poroviscoelastic layers bonded to rigid impermeable substrates shaped like elliptic paraboloids.

6.3.1 Contact Problem Formulation for BPVE Cartilage Layers

When studying contact problems for real joint geometries, a numerical analysis, such as the finite element method, is necessary [6, 14, 27], since exact analytical solutions were obtained only for two-dimensional [5, 16], or axisymmetric and simple geometries [11, 12, 21]. In particular, the two-dimensional contact creep problem between two cylindrical biphasic layers bonded to rigid impermeable substrates was solved by Kelkar and Ateshian [18] for all times and arbitrary layer thicknesses using the integral transform method. The frictionless rolling contact problem for cylindrical biphasic layers was analytically studied by Ateshian and Wang [5].

An asymptotic solution for the contact problem of two identical isotropic biphasic cartilage layers attached to two rigid impermeable spherical bones of equal radii modeled as elliptic paraboloids was obtained by Ateshian et al. [7]. This solution was extended by Wu et al. [24] to a more general model by combining the assumption of the kinetic relationship from classical contact mechanics [17] with the joint contact model for the contact of two biphasic cartilage layers [7]. An improved solution for the contact of two biphasic cartilage layers which can be used for dynamic loading was obtained by Wu et al. [25]. These solutions have been widely used as the theoretical background in modeling articular contact mechanics.

Later, Mishuris and Argatov [1, 22] refined the analysis of [7, 24] by formulating the contact condition which takes into account the tangential displacements at the contact interface. Finally, the axisymmetric model of articular contact mechanics originally developed in [7, 24] was generalized in [2] in the case of elliptical contact.

In this section, the asymptotic model of articular contact for isotropic biphasic layers [2, 7, 24] is extended for the transversely isotropic BPVE case.

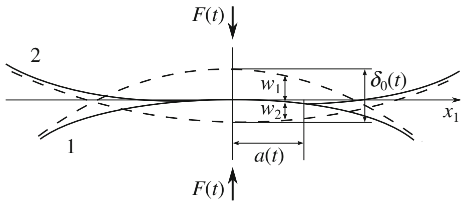

Fig. 6.3 Schematic diagram of the contact of articular cartilage surfaces *1* and *2* under the external load $F(t)$. The *dashed lines* imply the surfaces' profiles in the undeformed state

Consider two thin articular cartilage layers of uniform thicknesses h_1 and h_2 firmly attached to subchondral bones. Let $w_0^{(1)}(t, \mathbf{y})$ and $w_0^{(2)}(t, \mathbf{y})$ be the absolute values of the vertical displacements of the boundary points of the cartilage layers (see Fig. 6.3). Let also $\delta_0(t)$ denote the contact (vertical) approach of the rigid subchondral bones under a specified external vertical load, $F(t)$, which is assumed to be a function of the time variable t.

Further, let $\varphi(\mathbf{y})$ denote the gap between the layer surfaces before deformation. Here, following Argatov and Mishuris [2, 3], we consider a special case of the gap function represented by an elliptic paraboloid

$$\varphi(\mathbf{y}) = \frac{y_1^2}{2R_1} + \frac{y_2^2}{2R_2}, \tag{6.55}$$

where R_1 and R_2 are positive constants having dimensions of length.

Then, the linearized contact condition in the contact area $\omega(t)$ can be written as

$$w_0^{(1)}(t, \mathbf{y}) + w_0^{(2)}(t, \mathbf{y}) = \delta_0(t) - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega(t). \tag{6.56}$$

In the case of unilateral contact, the contact pressure between the cartilage layers, $p(\mathbf{y})$, is assumed to be positive inside the contact area $\omega(t)$ and satisfies the following boundary conditions [7] (see also [15, 23]):

$$p(t, \mathbf{y}) = 0, \quad \frac{\partial p}{\partial n}(t, \mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma(t).$$

Here, $\partial/\partial n$ is the normal derivative at the contour $\Gamma(t)$ of the domain $\omega(t)$.

Moreover, the following equilibrium equation holds:

$$F(t) = \iint\limits_{\omega(t)} p(t, \mathbf{y})\, d\mathbf{y}.$$

Applying the leading-order asymptotic model (6.52) for the short-time deformation of a thin bonded biphasic poroviscoelastic (BPVE) layer, we approximate the vertical displacement of the surface points of the nth cartilage layer by the formula

$$w_0^{(n)}(t, \mathbf{y}) = -\frac{h_n^3}{3} \int\limits_{0^-}^{t} J_{44}^{s(n)}(t-\tau) \frac{\partial}{\partial \tau} \Delta_y p(\tau, \mathbf{y})\, d\tau - h_n k_1^{(n)} \int\limits_{0}^{t} \Delta_y p(\tau, \mathbf{y})\, d\tau, \tag{6.57}$$

where $J_{44}^{s(n)}(t)$ and $k_1^{(n)}$ are the out-of-plane creep compliance in shear of the solid matrix and the transverse (in-plane) permeability coefficient of the nth cartilage layer, respectively.

Let us simplify the mathematical formalism. First, by letting

$$G_0^{\prime s(n)} = \frac{1}{J_{44}^{s(n)}(0^+)}$$

we introduce the instantaneous out-of-plane shear elastic modulus of the solid matrix of the nth layer.

Second, using the integration by parts formula, we represent the second integral in (6.57) as follows:

$$\int\limits_{0}^{t} \Delta_y p(\tau, \mathbf{y})\, d\tau = \int\limits_{0^-}^{t} (t-\tau) \frac{\partial}{\partial \tau} \Delta_y p(\tau, \mathbf{y})\, d\tau.$$

Third, combining the fluid flow-independent viscoelasticity and the fluid flow-dependent viscous effects in articular cartilage, we introduce the following generalized normalized creep function of the nth thin BPVE layer:

$$\Phi^{(n)}(t) = G_0^{\prime s(n)} J_{44}^{s(n)}(t) + \frac{3G_0^{\prime s(n)}}{h_n^2} k_1^{(n)} t. \tag{6.58}$$

Finally, the compound creep function, $\Phi_\beta(t)$, and the equivalent instantaneous shear elastic modulus, G_0', can be defined as follows (cf. Eqs. (4.102)–(4.105)):

$$\Phi_\beta(t) = \beta_1 \Phi'^{(1)}(t) + \beta_2 \Phi'^{(2)}(t),$$

$$G_0' = \frac{(h_1+h_2)^3 G_0^{\prime s(1)} G_0^{\prime s(2)}}{h_1^3 G_0^{\prime s(2)} + h_2^3 G_0^{\prime s(1)}}, \tag{6.59}$$

$$\beta_1 = \frac{h_1^3 G_0^{\prime s(2)}}{h_1^3 G_0^{\prime s(2)} + h_2^3 G_0^{\prime s(1)}}, \quad \beta_2 = \frac{h_2^3 G_0^{\prime s(1)}}{h_1^3 G_0^{\prime s(2)} + h_2^3 G_0^{\prime s(1)}}.$$

Thus, following the above relations, and after the substitution of the asymptotic approximations (6.57) into Eq. (6.56), we arrive at the governing integro-differential equation

$$-\int_{0^-}^{t} \Phi_\beta(t-\tau)\Delta_y \frac{\partial p}{\partial \tau}(\tau,\mathbf{y})\,d\tau = m\big(\delta_0(t) - \varphi(\mathbf{y})\mathscr{H}(t)\big), \quad \mathbf{y} \in \omega(t). \tag{6.60}$$

where the Heaviside function factor $\mathscr{H}(t)$ takes into account the zero initial conditions for $t < 0$, and m is given by (with $h = h_1 + h_2$ being the joint thickness)

$$m = \frac{3G_0'}{h^3}.$$

Equation (6.60) can be used to determine the contact pressure distribution $p(t,\mathbf{y})$ between BPVE cartilage layers under the monotonicity condition $\omega(t_1) \subset \omega(t_2)$ for $t_1 \leq t_2$. The monotonicity condition that the contact zone increases for non-decreasing loads, when $dF(t)/dt \geq 0$, should be checked a posteriori.

6.3.2 Exact Solution for Monotonic Loading

As in the Hertz theory of elliptic contact between two elastic bodies, the contact area $\omega(t)$ between the cartilage layers with the initial gap determined by Eq. (6.55) is elliptic with the semi-axes $a(t)$ and $b(t)$ changing with time. The form of the ellipse $\Gamma(t)$ can be characterized by its aspect ratio $s = b(t)/a(t)$. Assuming, as usual, that $R_1 \geq R_2$, we obtain $a(t) \geq b(t)$, and, generally, $0 < s \leq 1$ where the value $s = 1$ corresponds to a circular contact area. We emphasize that the parameter s is constant during loading and depends only on the ratio R_2/R_1 via the following relation (see Sect. 4.5):

$$s = \sqrt{\sqrt{\left(\frac{R_1 - R_2}{6R_1}\right)^2 + \frac{R_2}{R_1}} - \frac{(R_1 - R_2)}{6R_1}}.$$

The evolution of the major semi-axis of the contact area is governed by formula

$$a(t) = \left(\frac{96\sqrt{R_1 R_2}}{\pi m}\right)^{1/6} c_a(s)\left(\int_{0-}^{t} \Phi_\beta(t-\tau)\frac{dF(\tau)}{d\tau}\,d\tau\right)^{1/6}. \tag{6.61}$$

Here, $\sqrt{R_1 R_2}$ is a geometric mean of the radii R_1 and R_2, while $c_a(s)$ is a dimensionless factor given by (see Fig. 6.4)

$$c_a(s) = \left(\frac{\sqrt{(3s^2+1)(s^2+3)}}{4s^4}\right)^{1/6}.$$

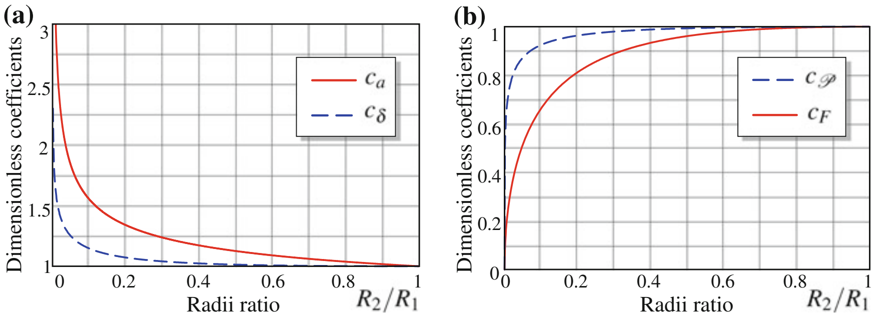

Fig. 6.4 Dimensionless scaling factors: **a** Coefficients c_a and c_δ; **b** Coefficients $c_{\mathscr{P}}$ and c_F. It is interesting that this behavior for the different scaling factors is overall substantially similar.

The contact approach between the subchondral bones is given by

$$\delta_0(t) = \left(\frac{3}{2\pi m R_1 R_2}\right)^{1/3} c_\delta(s)\left(\int\limits_{0-}^{t} \Phi_\beta(t-\tau)\frac{dF(\tau)}{d\tau}\,d\tau\right)^{1/3}, \tag{6.62}$$

where we have introduced the notation

$$c_\delta(s) = \left(\frac{2(s^2+1)^3}{s(3s^2+1)(s^2+3)}\right)^{1/3}.$$

Now, if the contact load $F(t)$ is known, then Eqs. (6.61) and (6.62) allow us to determine the quantities $a(t)$ and $\delta_0(t)$, respectively.

The contact pressure is calculated by means of the formula

$$p(t,\mathbf{y}) = \mathscr{P}_0(t)\left(1 - \frac{y_1^2}{a(t)^2} - \frac{y_2^2}{s^2 a(t)^2}\right)^2 - \int\limits_{t_*(\mathbf{y})}^{t} \frac{\partial \Psi_\beta}{\partial \tau}(t-\tau)\mathscr{P}_0(\tau)\left(1 - \frac{y_1^2}{a(\tau)^2} - \frac{y_2^2}{s^2 a(\tau)^2}\right)^2 d\tau. \tag{6.63}$$

Here, $\Psi_\beta(t)$ is the corresponding generalized normalized relaxation function determined by its Laplace transform $\tilde{\Psi}_\beta(s) = 1/[s^2\tilde{\Phi}_\beta(s)]$, s is the Laplace transform parameter, and $\mathscr{P}_0(t)$ is an auxiliary function given by

$$\mathscr{P}_0(t) = \left(\frac{27m}{96\pi^2\sqrt{R_1R_2}}\right)^{1/3} c_{\mathscr{P}}(s)\left(\int\limits_{0-}^{t} \Phi_\beta(t-\tau)\frac{dF(\tau)}{d\tau}\,d\tau\right)^{2/3},$$

where we have introduced the notation

$$c_{\mathscr{P}}(s) = \left(\frac{4s}{\sqrt{(3s^2+1)(s^2+3)}} \right)^{1/3}.$$

The quantity $t_*(\mathbf{y})$, which enters the lower limit in the integral (6.63), is the time when the contour of the contact area $\omega(t)$ first reaches the point $\mathbf{y}$. If, however, the point under consideration lies inside the instantaneous contact area, i.e., $\mathbf{y} \in \omega(0^+)$, then $t_*(\mathbf{y}) \equiv 0$. The quantity $t_*(\mathbf{y})$ is called the time-to-contact for the point $\mathbf{y}$. When the point $\mathbf{y}$ is located outside of $\omega(0^+)$, the nonzero quantity of $t_*(\mathbf{y})$ is determined by the equation $a(t_*)^2 = y_1^2 + y_2^2/s^2$, or in accordance with Eq. (6.61) by the following:

$$\int\limits_{0-}^{t_*} \Phi_\beta(t_* - \tau) \frac{dF(\tau)}{d\tau}\, d\tau = \frac{\pi m}{96\sqrt{R_1 R_2} c_a(s)^6} \left(y_1^2 + \frac{y_2^2}{s^2} \right)^3.$$

In the case of a stepwise loading, we have $F(t) = F_0 \mathscr{H}(t)$, and the above equation reduces to

$$\Phi_\beta(t_*) = \frac{1}{a_0^6} \left(y_1^2 + \frac{y_2^2}{s^2} \right)^3, \tag{6.64}$$

where $a_0 = a(0^+)$ is the instantaneous value of the major semi-axis, which is given by

$$a_0 = \left(\frac{96\sqrt{R_1 R_2}}{\pi m} \right)^{1/6} c_a(s) F_0^{1/6}.$$

We note that the asymptotic model (6.57), $n = 1, 2$, holds true only when the cartilage thicknesses are small compared with the characteristic size of the contact area, i.e., $\max\{h_1, h_2\} \ll a_0$. For this reason the contact force F_0, and $F(0^+)$ generally, should not take too small values.

Table 6.1 Isotropic and transversely isotropic biphasic material properties of human articular cartilage. The highlighted values are used in the asymptotic models [9]

Material property	Isotropic	Transversely isotropic
Young's modulus E_3^s (MPa)	0.69	0.46
Young's modulus E_1^s (MPa)	0.69	5.8
Poisson's ratio ν_{12}^s	0.0	0.0
Poisson's ratio ν_{31}^s	0.0	0.0
Shear modulus G_{13}^s (MPa)	0.345	0.37
Shear modulus $G_{12}^s = E_1^s/[2(1+\nu_{12}^s)]$ (MPa)	0.345	0.23
Permeability $k_1 = k_3$ ($\times 10^{-15}$m^4/Ns)	3.0	5.1
Solid volume fraction ϕ^s	0.25	0.25

Thus, in the case of a stepwise loading, formula (6.63), where quantity $t_*(\mathbf{y})$ is determined by Eq. (6.64), represents the sought for solution of Eq. (6.60). Note that in the case of the biphasic layers the derived expression for the contact pressure coincides with the result obtained previously in [2].

Table 6.1 shows some typical values for biphasic material properties of human articular cartilage in the isotropic and transversely isotropic cases. It is known that the mechanical properties of cartilage may change with disease. In particular, the early stages of osteoarthritis are characterized [26] by increased permeability, increased thickness of the cartilage layers, reduced shear modulus, increased Poisson's ratio, and/or a combination of these effects.

To conclude, we emphasize (see, e.g., [7, 25]) that specifically in regard to articular cartilage, the constructed asymptotic model, which is based on the short-time asymptotic solution of the deformation problem for a thin BPVE layer and assumes that most of the contact load is carried by the interstitial fluid, can be used for time periods of several thousand seconds, when the articular joint is biologically functional, and becomes invalid for time $t \to \infty$, when the interstitial fluid is pushed out of the cartilage layer underlying the contact area and the total contact pressure is carried only by the solid phase of the cartilage.

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