Chapter 8
The Near-Horizon Limit

Jiří Daněk

Abstract We present a new analytic coordinate system covering the whole global extension of the ultra-extreme Reissner-Nordström-de Sitter spacetime, analyse radial motions of two particles of different charges and demonstrate our results in an exact Penrose diagram. Further, we use the near-horizon limit to analyse the energy of particles' collisions near the ultra-extreme horizon, which lead to unbound collision energy in the center of mass system, and relate our results to the previously established behavior on simple and extreme horizon of Reissner-Nordström black hole.

8.1 Introduction

In the past black hole horizons enjoyed great attention because of the misunderstanding of coordinate singularities located on them in the original coordinate systems. With the understanding of coordinates' properties and discovery of equivalent nonsingular coordinate systems, the peculiarity of the horizons faded a bit, but black hole horizons are still drawing attention due to their role in the global causal and geometric properties of spacetimes. The near-horizon limit is one way of investigating the direct proximity of an arbitrary horizon and lets us forget about the distant regions we are not interested in. Behaviour of free test particles can also reveal many interesting features of spacetimes and is another tool for investigating the near-horizon regions.

Recently, collision processes leading to unbound collision energies in the center of mass system were found to occur in the direct proximity of the extreme horizon of Kerr black hole [1]. Later it was proven that the extreme horizon of the static charged Reissner-Nordström black hole (RN) can also exhibit unbound collision energy and serve as a particle accelerator as well [7]. Similar investigation was also performed for the single inner horizon [6] of RN or for the single cosmological horizon of Reissner-Nordström-de Sitter spacetime (RNdS) [8].

J. Daněk (✉)
Institute of Theoretical Physics Faculty of Mathematics and Physics,
Charles University in Prague, Prague, Czech Republic
e-mail: jiri.danek@matfyz.cz

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Article is to search for unbound collision energy near the so far neglected ultra-extreme (U-E) horizon of the U-E RNdS (i.e., $9M^2 = 8Q^2 = 2/\Lambda$ [2]), the metric of which can be written as

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{8.1}$$

with the lapse function $f(r) = -\frac{\Lambda}{3r^2}(r-N)(r-r_t)^3$ where $N = -3r_t$, $r_t = (2\Lambda)^{-1/2}$.

This paper is organized as follows: In Sect. 8.2 we will find a new continuous coordinate patch covering the whole global extension of RNdS. The geodesics of radial charged test particles will be investigated in Sect. 8.3. In Sect. 8.4 we will use the near-horizon limit to investigate the collision energy near the U-E horizon. Finally, we will demonstrate our results and collision in an exact Penrose diagram (Fig. 8.1).

8.2 Analytic Coordinates of Ultra-Extreme RNdS

Since the case of the U-E horizon is not sufficiently explored in the literature we suggest a new coordinate system of compact coordinates (U, V) continuous across the triple horizon. We will adapt a method used for Schwarzschild in [4].

Firstly, we need to give the relation for the tortoise coordinate $r_u^\star$. Due to the definition $\frac{\mathrm{d}r_u^\star}{\mathrm{d}r} = \frac{1}{f(r)}$ one gets the following partial fraction decomposition

$$\frac{\mathrm{d}r_u^\star}{\mathrm{d}r} = \frac{A_u}{(r-r_t)^3} + \frac{B_u}{(r-r_t)^2} + \frac{C_u}{r-r_t} + \frac{D_u}{r-N}, \tag{8.2}$$

with constants

$$A_u = -\tfrac{3r_t^3}{2},\ \ B_u = -\tfrac{21r_t^2}{8},\ \ C_u = -\tfrac{27r_t}{32} \text{ and } D_u = +\tfrac{27r_t}{32}. \tag{8.3}$$

Now we can solve the differential equation (8.2) and find a solution

$$r_u^\star = -\frac{1}{2}\frac{A_u}{(r-r_t)^2} - \frac{B_u}{(r-r_t)} + C_u \ln\left|\frac{r}{r_t} - 1\right| + D_u \ln\left|\frac{r}{N} - 1\right|. \tag{8.4}$$

Secondly, we will implicitly define new variables $\left(\hat{U}, \hat{V}\right)$ as

$$\begin{aligned} r_u^\star &= +h(\hat{U}) + h(\hat{V}), \\ t &= -h(\hat{U}) + h(\hat{V}), \end{aligned} \tag{8.5}$$

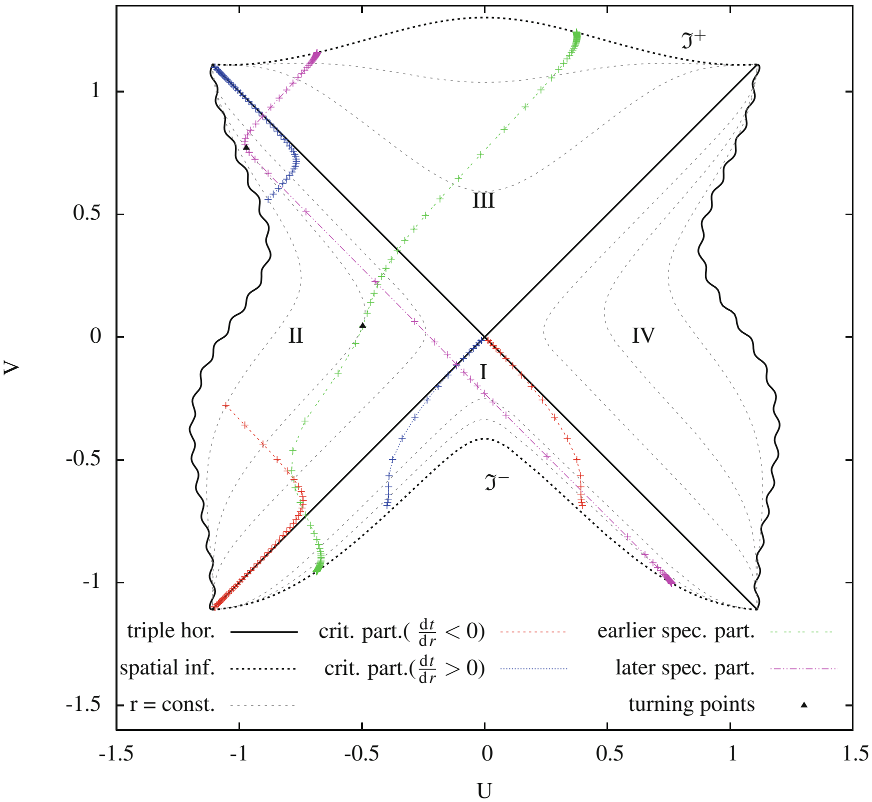

Fig. 8.1 We plotted the paths of four critical and two special particles in the global Penrose diagram of the U-E RNdS with crosses denoting equidistant values of proper time. We distinguish between critical particles with $\frac{dt}{dr} < 0$ and $\frac{dt}{dr} > 0$. As we can see, the points become denser as the critical particles approach the triple horizon, which demonstrates the inability of critical particles to reach the U-E horizon at a finite value of their proper time. On the other hand, special particles cross the horizon with a finite value of their proper time. All special particles have $X_{sp} > 0$ along their paths. In region I collision energy of special particle and critical particle with $\frac{dt}{dr} < 0$ diverges more strongly than in the case $\frac{dt}{dr} > 0$. The stronger divergence does not appear in region II since all future-oriented particles have $X_i > 0$ there and the PS effect is causally prohibited. Two special particles in the diagram differ only in their time of release and the differences in their paths originate from the nontrivial new coordinate path (U, V). Critical particles would reach infinities and the singularity, but we stopped their trajectories at suitable values of the radial coordinate r

where $h(x)$ is a suitable function, which will be discussed later. The new coordinates are directly connected to the classical Eddington-Finkelstein null coordinates $u = t - r_u^\star = -2h(\hat{U})$ and $v = t + r_u^\star = 2h(\hat{V})$.

Thirdly, we rewrite the metric of the U-E case as

$$\mathrm{d}s^2 = 4f(r)\, h'(\hat{U})h'(\hat{V})\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega_2^2, \tag{8.6}$$

where prime denotes derivative with respect to the variable of a function.

At last, we have to determine the form of the function $h(x)$ in order to keep the metric elements smooth and nonzero on the horizon. There is no unique form of $h(x)$ ensuring our requirements, therefore, we define $h(x)$, which is injective on each of the distinct intervals $x \in \left(-\frac{\pi}{2}, 0\right)$ and $x \in \left(0, \frac{\pi}{2}\right)$ separately, to be of the form

$$h(x) = -\frac{1}{2}\frac{A_u}{\tan^2(x)} - \frac{B_u}{\tan(x)} + C_u \ln|\tan(x)| - \tan^2(x). \tag{8.7}$$

Introduction of tangents ensures compactness of the new coordinates $\left(\hat{U}, \hat{V}\right)$.

The new coordinate system covers the whole U-E RNdS and even its global extension despite the usual pathological points $\left(i\frac{\pi}{2}, j\frac{\pi}{2}\right)$, where $i, j \in \{-1, 0, 1\}$.

Definitions $U = \frac{\sqrt{2}}{2}\left(\hat{U} - \hat{V}\right)$ and $V = \frac{\sqrt{2}}{2}\left(\hat{U} + \hat{V}\right)$ only rotate the final Penrose diagram.

8.3 Test Particles in the Ultra-Extreme RNdS

The path of a general radial, massive and charged particle in U-E RNdS spacetime is given by the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(E - \frac{Q\varepsilon}{r}\right) \times \left(f(r)\sqrt{\left(E - \frac{Q\varepsilon}{r}\right)^2 - f(r)}\right)^{-1}, \tag{8.8}$$

where the sign sets the direction of motion, E is the constant of motion connected to the time-like Killing vector and ε is the specific charge of the particle [5].

The equation simplifies enough to be solvable for two special particles listed in the following subsections.

The solutions involve integrals of the form $\mathrm{Int}_n(r, p) = \int\left((r-p)^n \sqrt{r^2 + \mathfrak{B}r + \mathfrak{A}}\right)^{-1} \mathrm{d}r$, the solutions of which are given as

$$\mathrm{Int}_1(r,p) = -\frac{1}{\sqrt{a}} \ln \left| \frac{2a + b(r-p) + 2\sqrt{aR}}{r-p} \right| \quad \text{for } \ a > 0, \tag{8.9}$$

$$\mathrm{Int}_2(r,p) = \frac{\sqrt{R}}{a(r-p)} - \frac{b}{2a}\mathrm{Int}_1(r,p),$$

$$\mathrm{Int}_3(r,p) = \left(\frac{-1}{2a(r-p)^2} + \frac{3b^2}{4a^2(r-p)} \right)\sqrt{R} + \left(\frac{3b^2}{4a^2} - \frac{c}{2a} \right)\mathrm{Int}_1(r,p),$$

with $R = r^2 + \mathfrak{B}r + \mathfrak{A}$, $b = 2p + \mathfrak{B}$ and $a = p^2 + \mathfrak{B}p + \mathfrak{A}$ according to [3].

8.3.1 Special Particle

The special particle is defined by its specific values $E = \frac{M}{Q\varepsilon}$ and $\varepsilon = \pm 1$ where sigh is chosen to satisfy condition $Q\varepsilon > 0$. The trajectory

$$\pm t_u^{sp}(r) = A_u^{sp}\mathrm{Int}_3(r,r_t) + B_u^{sp}\mathrm{Int}_2(r,r_t) + C_u^{sp}\mathrm{Int}_1(r,r_t) + D_u^{sp}\mathrm{Int}_1(r,N) + E_u^{sp}, \tag{8.10}$$

solves (8.8), where E_u^{sp} is one arbitrary constant,

$$A_u^{sp} = -\tfrac{\sqrt{3}r_t^4}{2} \quad B_u^{sp} = -\tfrac{19\sqrt{3}r_t^3}{8}, \quad C_u^{sp} = -\tfrac{45\sqrt{3}r_t^2}{32}, \text{ and } D_u^{sp} = +\tfrac{45\sqrt{3}r_t^2}{32}. \tag{8.11}$$

We also substituted $\mathfrak{A} = -\frac{2r_t^2}{3}$, $\mathfrak{B} = 0$ into the integrals (8.9).

8.3.2 Critical Particle

The critical particle is characterized by a fine-tuned value of its specific charge $\varepsilon = \frac{Er_t}{Q}$, for which (8.8) can be solved as

$$\pm t_u^{cr}(r) = A_u^{cr}\mathrm{Int}_3(r,r_t) + B_u^{cr}\mathrm{Int}_2(r,r_t) + C_u^{cr}\mathrm{Int}_1(r,r_t) + D_u^{cr}\mathrm{Int}_1(r,N) + E_u^{cr}, \tag{8.12}$$

with one arbitrary constant E_u^{cr}, constants $A_u^{cr} = \frac{E}{\sqrt{6}r_t}A_u$, $B_u^{cr} = \frac{E}{\sqrt{6}r_t}B_u$, etc. Expressions $\mathfrak{A} = 3r_t^2(2E^2 - 1)$, $\mathfrak{B} = 2r_t$ must be substituted into (8.9).

8.4 Collision with Unbound Collision Energy in C.M.S

A radial charged particle has a 4-velocity $u_i^\mu = \left(\frac{X_i}{f(r)}, \mp Z_i, 0, 0\right)$, where $X_i = \left(E_i - \frac{Q\varepsilon_i}{r}\right)$ and $Z_i = \sqrt{X_i^2 - f(r)}$. Let us examine the collision energy of two particles in their center of mass system. Let particle $i = 1$ be critical and particle $i = 2$ non-critical (e.g., special).

The total energy in the center of mass frame of the two colliding particles $E_{C.M.}^2 = -\left(m_1 u_{1\mu} + m_2 u_{2\mu}\right)\left(m_1 u_1^\mu + m_2 u_2^\mu\right)$. If we assume both particles to have the same rest mass $m_1 = m_2 = m$, we can write the last formula as $\frac{E_{C.M.}^2}{2m^2} = 1 + \frac{X_1 X_2 - Z_1 Z_2}{f(r)}$.

In the near-horizon limit (i.e., $r \sim r_t$) we can expand these terms as

$$Z_1 \approx \left|1 - \frac{r_t}{r}\right| \left[|E_1| + \frac{\Lambda r^2}{6|E_1|}\left(1 - \frac{N}{r}\right)\left(1 - \frac{r_t}{r}\right)\right], \quad Z_2 \approx |X_{2(H)}| - \frac{f(r)}{2|X_{2(H)}|}, \tag{8.13}$$

with $X_{2(H)} = X_2(r = r_t) \neq 0$. Introducing a new near-horizon coordinate $\varepsilon_t := \frac{r - r_t}{r_t}$ and substituting the above expressions into $\frac{E_{C.M.}^2}{2m^2}$, we are led to the expression

$$\frac{E_{C.M.}^2}{2m^2} \approx 3\frac{\left|E_1 X_{2(H)}\right|(1 + \varepsilon_t)\delta_{s_1, -s_2}}{\mathrm{sign}(\varepsilon_t)\varepsilon_t^2} + \frac{1}{2}\left|\frac{X_{2(H)}}{E_1 \varepsilon_t}\right| + O(\varepsilon_t^0), \tag{8.14}$$

where $s_1 := \mathrm{sign} X_1$, $s_2 := \mathrm{sign} X_{2(H)}$ and $\delta_{i,j}$ is the Kronecker delta. There is a problem for $X_1 X_2 < 0$ on the static side as the location of the collision approaches the horizon (i.e., $\varepsilon_t \to 0^-$), since then $E_{C.M.}^2 \to -\infty$. Fortunately, the so-called PS process with $X_1 X_2 < 0$ is prohibited here by a causality violation. Particle with $X_i > 0$ would move towards the future while particle with $X_j < 0$ would be past-oriented. On the nonstatic side, t is not a time-like coordinate and motions with $X_1 X_2 < 0$ can occur.

8.5 Conclusions

We have examined paths of radial charged particles and discovered that special particles are repulsed by the black hole charge and do not fall into the singularity. They can even reach three regions of globally extended U-E RNdS with a finite value of their proper time. Particle collisions involving the critical particle in the proximity of the U-E horizon exhibit resemblance to the collisions near an extreme horizon and result in infinite collision energies in C.M. The U-E horizon possesses properties of the inner and cosmological non-extreme horizons, since the order of collision energy divergence is not the same on both sides of the triple horizon.

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