Covering the Recursive Sets

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Abstract. We give solutions to two of the questions in a paper by Brendle, Brooke-Taylor, Ng and Nies. Our examples derive from a 2014 construction by Khan and Miller as well as new direct constructions using martingales.

At the same time, we introduce the concept of i.o. subuniformity and relate this concept to recursive measure theory. We prove that there are classes closed downwards under Turing reducibility that have recursive measure zero and that are not i.o. subuniform. This shows that there are examples of classes that cannot be covered with methods other than probabilistic ones. It is easily seen that every set of hyperimmune degree can cover the recursive sets. We prove that there are both examples of hyperimmune-free degree that can and that cannot compute such a cover.

1 Introduction

An important theme in set theory has been the study of cardinal characteristics. As it turns out, in the study of these there are certain analogies with recursion theory, where the recursive sets correspond to sets in the ground model. In a recent paper by Brendle, Brooke-Taylor, Ng and Nies [1], the authors point out analogies between cardinal characteristics and the study of algorithmic randomness. We address two questions raised in this paper that are connected to computing covers for the recursive sets.

In the following, we will assume that the reader is familiar with various notions from computable measure theory, in particular, with the notions of

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Martin-Löf null, Schnorr null and Kurtz null set. For background on these notions we refer the reader to the books of Downey and Hirschfeldt [4], Li and Vitányi [12] and Nies [14].

Our notation from recursion theory is mostly standard. The natural numbers are denoted by ω, and 2^ω denotes the Cantor space and $2^{<\omega}$ the set of all finite binary sequences. We denote the concatenation of strings σ and τ by $\sigma\tau$. The notation $\sigma \sqsubseteq \tau$ denotes that the finite string σ is an initial segment of the (finite or infinite) string τ. We identify sets $A \subseteq \omega$ with their characteristic sequences, and $A{\upharpoonright}n$ denotes the initial segment $A(0)\ldots A(n-1)$. We use λ to denote the empty string. Throughout, μ denotes the Lebesgue measure on 2^ω.

Definition 1. A function $M : 2^{<\omega} \to \mathbb{R}^{\geqslant 0}$ is a *martingale* if for every $x \in 2^{<\omega}$, M satisfies the averaging condition

$$2M(\sigma) = M(\sigma 0) + M(\sigma 1), \tag{1}$$

A martingale M *succeeds on* a set A if

$$\limsup_{n\to\infty} M(A{\upharpoonright}n) = \infty.$$

The class of all sets on which M succeeds is denoted by $S[M]$.

The following definition is taken from Rupprecht [17].

Definition 2. An oracle A is *Schnorr covering* if the union of all Schnorr null sets is Schnorr null relative to A. An oracle A is *weakly Schnorr covering* if the set of recursive reals is Schnorr null relative to A.

Definition 3. A *Kurtz test* relative to A is an A-recursive sequence of closed-open sets G_i such that each G_i has measure at most 2^{-i}; these closed-open sets are given by explicit finite lists of strings and they consist of all members of $\{0,1\}^\omega$ extending one of the strings. Note that $i \to \mu(G_i)$ can be computed relative to A. The intersection of a Kurtz test (relative to A) is called a Kurtz null set (relative to A). An oracle A is *Kurtz covering* if there is an A-recursive array $G_{i,j}$ of closed-open sets such that each i-th component is a Kurtz test relative to A and every unrelativized Kurtz test describes a null-set contained in $\cap_j G_{i,j}$ for some i; A is *weakly Kurtz covering* if there is such an array and each recursive sequence is contained in some A-recursive Kurtz null set $\cap_j G_{i,j}$.

Brendle, Brook-Taylor, Ng and Nies [1] called the notion of (weakly) Schnorr covering in their paper (weakly) Schnorr *engulfing*. In this paper, we will use the original terminology of Rupprecht [17]. We have analogous notions for the other notions of effective null sets. For example, a set A is weakly Kurtz covering if the set of recursive reals is Kurtz null relative to A. We also have Baire category analogues of these notions of covering: A set A is *weakly meager covering* if it computes a meager set that contains all recursive reals. Recall that a set A is diagonally nonrecursive (DNR) if there is a function $f \leqslant_T A$ such that, for all x, if $\varphi_x(x)$ is defined then $\varphi_x(x) \neq f(x)$. A set A has hyperimmune-free Turing degree if for every $f \leqslant_T A$ there is a recursive function g with $\forall x\, [f(x) \leqslant g(x)]$.

2 Solutions to Open Problems

In [1, Question 4.1], Brendle, Brooke-Taylor, Ng and Nies posed three questions, (7), (8) and (9). In this section, we will provide the answers to the questions (7) and (9). For this we note that by [1, Theorem 3] and [9, Theorem 5.1] we have the following result.

Theorem 4. *A set A is weakly meager covering iff it is high or of* DNR *degree.*

We recall the following well-known definitions and results.

Definition 5. A function ψ, written $e \mapsto (n \mapsto \psi_e(n))$, is a recursive numbering if the function $(e, n) \mapsto \psi_e(n)$ is partial recursive. For a given recursive numbering ψ and a function h, we say that f is DNR_h^ψ if for all n, $f(n) \neq \psi_n(n)$ and $f(n) \leq h(n)$. An *order function* is a recursive, nondecreasing, unbounded function.

Theorem 6 (Khan and Miller [8, Theorem 4.3]). *For each recursive numbering ψ and for each order function h, there is an $f \in \mathrm{DNR}_h^\psi$ such that f computes no Kurtz random real.*

Wang (cf. [4, Theorem 7.2.13]) gave a martingale characterization of Kurtz randomness. While it is obvious that weakly Kurtz covering implies weakly Schnorr covering for the martingale notions, some proof is needed in the case that one uses tests (as done here).

Proposition 7. *If A is weakly Kurtz covering then A is weakly Schnorr covering.*

Proof. Suppose A is weakly Kurtz covering, as witnessed by the A-recursive array of closed-open sets $G_{i,j}$. Then the sets $F_j = \cup_i G_{i,i+j+1}$ form an A-recursive Schnorr test, as each F_j has at most the measure $\sum_j 2^{-i-j-2} = 2^{-i-1}$ and the measures of the F_j is uniformly A-recursive as one can relative to A compute the measure of each $G_{i,i+j+1}$ and their sum is fast converging. As for each recursive set there is an i such that all $G_{i,i+j+1}$ contain the set, each recursive set is covered by the Schnorr test. □

Theorem 8. *There is a recursive numbering ψ and an order function h such that for each set A, if A computes a function f that is DNR_h^ψ then A is weakly Kurtz covering.*

Proof. Fix a correspondence between strings and natural numbers $\mathrm{num} : 2^{<\omega} \to \omega$ such that

$$2^{|\sigma|} - 1 \leq \mathrm{num}(\sigma) \leq 2^{|\sigma|+1} - 2.$$

For instance, $\mathrm{num}(\sigma)$ could be the position of σ in the length-lexicographically lexicographic ordering of all strings as proposed by Li and Vitányi [12]. Let $\mathrm{str}(n) = \mathrm{num}^{-1}(n)$ be the string representation of the number n. Thus

$$2^{|\mathrm{str}(n)|} - 1 \leq \mathrm{num}(\mathrm{str}(n)) = n \leq 2^{|\mathrm{str}(n)|+1} - 2.$$

Let φ be any fixed recursive numbering, let

$$\langle a,b\rangle = \mathrm{num}(1^{|\mathrm{str}(a)|}0\mathrm{str}(a)\mathrm{str}(b))$$

in concatenative notation. Let $\psi_{2\langle e,n\rangle}(x) = \varphi_e(n)$ for any x and $\psi_{2y+1} = \varphi_y$. Note that ψ is an acceptable numbering. Let $s(e,n) = 2\langle e,n\rangle$. Then if f is DNR with respect to ψ then f has the following property with respect to φ:

$$f(s(e,n)) \neq \varphi_e(n).$$

Indeed,

$$f(s(e,n)) = f(2\langle e,n\rangle) \neq \psi_{2\langle e,n\rangle}(2\langle e,n\rangle) = \varphi_e(n).$$

Moreover,

$$\begin{aligned} s(a,b) = 2\langle a,b\rangle &\leq 2(2^{|1^{|\mathrm{str}(a)|}0\mathrm{str}(a)\mathrm{str}(b)|}) = 4(2^{|1^{|\mathrm{str}(a)|}|}2^{|\mathrm{str}(a)|}2^{|\mathrm{str}(b)|}) \\ &= 4(2^{|\mathrm{str}(a)|}2^{|\mathrm{str}(a)|}2^{|\mathrm{str}(b)|}) \leq 4(a+1)^2(b+1). \end{aligned}$$

Consider a partition of ω into intervals I_m such that $|I_m|$ is $2+\log(m+1)$ rounded down, and let $h(m) = |I_m|$. If f is DNR_h^ψ then we have

$$\forall \varphi_e\, \forall n\, (f(s(e,n)) \in \{0,1\}^{I_{s(e,n)}} \text{ and } f(s(e,n)) \neq \varphi_e(n)).$$

Given a recursive set R, there is, by the fixed-point theorem, an index e such that, for all n, $\varphi_e(n) = R \restriction I_{s(e,n)}$ and $f(s(e,n)) \neq R \restriction I_{s(e,n)}$. Note that for every fixed e,

$$\prod_{n=0}^{\infty}(1-2^{-|I_{s(e,n)}|}) \leqslant \prod_{n=e+2}^{\infty}(1-2^{-(2+\log(4(e+1)^2(n+1)+1))}) \leqslant$$

$$\prod_{n=e+2}^{\infty}(1-2^{-(3+\log(4(e+1)^2(n+1)))}) = \prod_{n=e+2}^{\infty}(1-2^{-(5+2\log(e+1)+\log(n+1))}).$$

The last product in this formula is 0, as the sum

$$\sum_{n=e+2}^{\infty} 2^{-(5+2\log(e+1)+\log(n+1))} = 1/32 \cdot (e+1)^{-2} \cdot \sum_{n=e+2}^{\infty} 1/(n+1)$$

diverges. Thus

$$\mu(\{B : \exists e \forall n\, [B \restriction I_{s(e,n)} \neq f(s(e,n))]\}) \leqslant \sum_e \prod_{n=0}^{\infty}(1-2^{-|I_{s(e,n)}|}) = 0$$

So if f is computable from A then we have a $\Sigma_2^0(A)$ null set that contains all recursive sets, as desired. □

Theorem 9 (Affirmative answer to Brendle, Brooke-Taylor, Ng and Nies [1, Question 4.1(7)]). *There exists a set A such that*

1. *A is weakly meager covering,*
2. *A does not compute any Schnorr random set,*
3. *A is of hyperimmune-free degree,*
4. *A is weakly Schnorr covering.*

Proof. Let h and ψ as in Theorem 8. By Theorem 6, there is an $f \in \mathrm{DNR}_h^\psi$ such that f computes no Kurtz random real. Let A be a set Turing equivalent to f.

1. By Theorem 4, A is weakly meager covering. Alternatively, one could use the fact that every weakly Kurtz covering oracle is also weakly meager covering and derive the item 1 from the proof of item 4.
2. Since each Schnorr random real is Kurtz random, A does not compute any Schnorr random real.
3. Since A does not compute any Kurtz random real, A is of hyperimmune-free degree.
4. By Theorem 8, A is weakly Kurtz covering. In particular, by Proposition 7, A is weakly Schnorr covering.

This completes the proof. □

Franklin and Stephan [6] characterised that a set A is *Schnorr trivial* iff for every $f \leqslant_{tt} A$ there is a recursive function g such that, for all n, $f(n) \in \{g(n,0), g(n,1), \ldots, g(n,n)\}$; this characterisation serves here as a definition.

Theorem 10 (Affirmative answer to Brendle, Brooke-Taylor, Ng and Nies [1, Question 4.1 (9)]). *There is a hyperimmune-free oracle A which is not DNR (and thus low for weak* 1*-genericity) and which is not Schnorr trivial and which does not Schnorr cover all recursive sets.*

Remark 11. The reader may object that the original question in [1] asked for a set that was *not low for Schnorr tests* rather than *not Schnorr trivial*. However, we can recall the following facts:

– Kjos-Hanssen, Nies and Stephan [10] showed that if A is low for Schnorr tests then A is low for Schnorr randomness;
– Franklin [5] showed that if A is low for Schnorr randomness then A is Schnorr trivial.

3 Infinitely Often Subuniformity and Covering

Let $\langle .,. \rangle$ denote a standard recursive bijection from $\omega \times \omega$ to ω. For a function $P : \omega \to \omega$ define

$$P_n(m) = P(\langle n, m \rangle)$$

and say that P *parametrizes* the class of functions $\{P_n : n \in \omega\}$. We identify sets of natural numbers with their characteristic functions. A class $\mathcal{A}$ is *(recursively) uniform* if there is a recursive function P such that $\mathcal{A} = \{P_n : n \in \omega\}$, and *(recursively) subuniform* if $\mathcal{A} \subseteq \{P_n : n \in \omega\}$. These notions relativize to any oracle A to yield the notions of *A-uniform* and *A-subuniform*.

It is an elementary fact of recursion theory that the recursive sets are not uniformly recursive. The following theorem, as cited in Soare's book [18, p. 255], quantifies exactly how difficult it is to do this:

Theorem 12 (Jockusch). *The following conditions are equivalent:*

(i) *A is high, that is, $A' \geqslant_T \emptyset''$,*
(ii) *the recursive functions are A-uniform,*
(iii) *the recursive functions are A-subuniform,*
(iv) *the recursive sets are A-uniform.*
If A has r.e. degree then (i)–(iv) *are each equivalent to:*
(v) *the recursive sets are A-subuniform.*

In the following we study infinitely often parametrizations and the relation to computing covers for the recursive sets.

3.1 Infinitely Often Subuniformity

Definition 13. We say that a set X *covers* a class $\mathcal{A}$ if there is an X-recursive martingale M such that $\mathcal{A} \subseteq S[M]$.

Note that for X recursive this is just the definition of recursive measure zero. For basics about computable martingales see [4, p. 207].

Definition 14. A class $\mathcal{A} \subseteq 2^\omega$ is called *infinitely often subuniform* (i.o. subuniform for short) if there is a recursive function $P \in \{0,1,2\}^\omega$ such that

$$\forall A \in \mathcal{A}\, \exists n \left[\exists^\infty x\big(P_n(x) \neq 2\big) \wedge \forall x\big(P_n(x) \neq 2 \rightarrow P_n(x) = A(x)\big)\right]. \qquad (2)$$

That is, for every $A \in \mathcal{A}$ there is a row of P that computes infinitely many elements of A without making mistakes. Again, we can relativize this definition to an arbitrary set X: A class $\mathcal{A}$ is i.o. X-subuniform if P as above is X-recursive.

Let REC denote the class of recursive sets. Recall that A is a PA-complete set if A can compute a total extension of every $\{0,1\}$-valued partial recursive function. Note that if a set A is PA-complete then REC is A-subuniform (cf. Proposition 15 below).

For every recursive set A there is a recursive set $\hat{A}$ such that A can be reconstructed from any infinite subset of $\hat{A}$. Namely, let $\hat{A}(x) = 1$ precisely when x codes an initial segment of A. So it might seem that any i.o. sub-parametrization of REC can be converted into a subparametrization in which every recursive set is completely represented. However, we cannot do this uniformly (since we cannot get rid of the rows that have $P_n(x) = 2$ a.e.) and indeed the implication does not hold.

Proposition 15. *We have the following picture of implications:*

$$\begin{array}{ccccc} A \text{ is PA-complete} \Rightarrow & \text{REC is } A\text{-subuniform} & \Rightarrow & \text{REC is i.o. } A\text{-subuniform} \\ & \Uparrow & & \Uparrow \\ & A \text{ is high} & \Rightarrow & A \text{ has hyperimmune degree} \end{array}$$

No other implications hold than the ones indicated.

Proposition 16. *Every i.o. subuniform class has recursive measure zero. This relativizes to: If $\mathcal{A}$ is i.o. X-subuniform then X covers $\mathcal{A}$.*

Proof. The ability to compute infinitely many bits from a set clearly suffices to define a martingale succeeding on it. The uniformity is just what is needed to make the usual sum argument work. □

Proposition 17. *There exists a class of recursive sets that has recursive measure zero and that is not i.o. subuniform.*

Proof. The class of all recursive sets A satisfying $\forall x\,[A(2x) = A(2x+1)]$ has recursive measure 0 but is not i.o. subuniform: If P would witness this class to be i.o. subuniform then Q defined as $Q_i(x) = \min\{P_i(2x), P_i(2x+1)\}$ would witness REC to be i.o. subuniform, a contradiction. □

Above the recursive sets, the 1-generic sets are a natural example of such a class that has measure zero but that is not i.o. subuniform: It is easy to see that the 1-generic sets have recursive measure zero because for every such set A there are infinitely many n such that $A \cap [n, 2n] = \emptyset$. On the other hand, a variation of the construction in the proof of Proposition 17 shows that the 1-generic sets are not i.o. X-subuniform for any X:

Proposition 18. *The 1-generic sets are not i.o. X-subuniform for any set X.*

Proof. Let $P \subseteq \{0,1,2\}^\omega$ be an X-recursive parametrization and let A be 1-generic relative to X (so that A is in particular 1-generic). Then for every n, if $P_n(x) \neq 2$ for infinitely many x then

$$\left\{\sigma \in 2^{<\omega} : \exists x\,[P_n(x) \neq 2 \wedge P_n(x) \neq \sigma(x)]\right\}$$

is X-recursive and dense, hence A meets this set of conditions and consequently P does not i.o. parameterize A. □

Now both the example from Proposition 17 and the 1-generic sets are counterexamples to the implication "measure 0 $\Rightarrow$ i.o. subuniform" because of the *set structure* of the elements in the class. One might think that for classes closed downwards under Turing reducibility (i.e. classes defined by information content rather than set structure) the situation could be different, i.e. that for $\mathcal{A}$ closed downwards under $\leqslant_T$ the implication "X covers $\mathcal{A} \Rightarrow \mathcal{A}$ i.o. X-subuniform" would hold. Note that for X recursive this is not interesting, since any nonempty class closed downwards under Turing reducibility contains REC and REC does not have recursive measure zero. However, this is also not true: Let $\mathcal{A}$ be the class

$$\{A : A \leqslant_T G \text{ for some 1-generic} G\}.$$

Clearly $\mathcal{A}$ is closed downwards under Turing reducibility and it follows from proofs by Kurtz [11] and by Demuth and Kučera [2] (a proof is also given by Terwijn [19]), that $\mathcal{A}$ is a Martin-Löf nullset, and that in particular $\emptyset'$ covers $\mathcal{A}$. However, by Proposition 18 the 1-generic sets are not i.o. $\emptyset'$-subuniform so that in particular $\mathcal{A}$ is not i.o. $\emptyset'$-subuniform.

3.2 A Nonrecursive Set that Does Not Cover REC

It follows from Proposition 15 and Proposition 16 that if A is of hyperimmune degree then A covers REC. In particular every nonrecursive set comparable with $\emptyset'$ covers REC. We see that if A cannot cover REC then A must have hyperimmune-free degree. We now show that there are indeed nonrecursive sets that do not cover REC. Indeed, the following result establishes that there are natural examples of such sets.

Theorem 19. *If A is Martin-Löf random then there is no martingale $M \leqslant_{tt} A$ which covers* REC. *In particular if A is Martin-Löf random and of hyperimmune-free Turing degree then it does not cover* REC.

Proof. Let A be Martin-Löf random and M^A be truth-table reducible to A by a truth-table reduction which produces on every oracle a savings martingale, that is, a martingale which never goes down by more than 1. Without loss of generality, the martingale starts on the empty string with 1 and is never less than or equal to 0. Note that because of the truth-table property, one can easily define the martingale N given by

$$N(\sigma) = \int_{E \subseteq \omega} M^E(\sigma)\, dE.$$

As one can replace the E by the strings up to $use(|\sigma|)$ using the recursive use-function use of the truth-table reduction, one has that

$$N(\sigma) = \sum_{\tau \in \{0,1\}^{use(|\sigma|)}} 2^{-|\tau|} M^{\tau}(\sigma)$$

and N is clearly a recursive martingale. Let B be a recursive set which is adversary to N, that is, B is defined inductively such that

$$\forall n\, [N(B{\upharpoonright}(n+1)) \leqslant N(B{\upharpoonright}n)].$$

Define the uniformly r.e. classes S_n by

$$S_n = \{E : M^E \text{ reaches on } B \text{ a value beyond } 2^n + 1\}.$$

By the savings property, once M^E has gone beyond $2^n + 1$ on B, M^E will stay above 2^n afterwards. It follows that the measure of these E can be at most 2^{-n}. So $\mu(S_n) \leqslant 2^{-n}$ for all n and therefore the S_n form a Martin-Löf test. Since A is Martin-Löf random, there exists n such that $A \notin S_n$, and hence M^A does not succeed on B. □

We note that the set $\{A \in 2^\omega : A \text{ covers REC}\}$ has measure 1. This follows from Proposition 15 and the fact that the hyperimmune sets have measure 1 (a well-known result of Martin, cf. [4, Theorem 8.21.1]). We note that apart from the hyperimmune degrees, there are other degrees that cover REC:

Proposition 20. *There are sets of hyperimmune-free degree that cover the class* REC*.*

Proof. As in Proposition 15, take a PA-complete set A of hyperimmune-free degree. Then the recursive sets are A-subuniform, so by Proposition 16 A covers REC. □

3.3 Computing Covers Versus Uniform Computation

We have seen above that in general the implication "X covers $\mathcal{A} \Rightarrow \mathcal{A}$ i.o. X-subuniform" does not hold, even if $\mathcal{A}$ is closed downwards under Turing reducibility. A particular case of interest is whether there are sets that can cover REC but relative to which REC is not i.o. subuniform.

Theorem 21. *There exists a set A that covers* REC *but relative to which* REC *is not i.o. A-subuniform.*

Theorem 22. *We have the following picture of implications:*

$$\begin{array}{ccc} & & A \textit{ is PA-complete} \\ & & \Downarrow \\ A \textit{ is high} & \Rightarrow & \text{REC} \textit{ is } A-\textit{subuniform} \\ \Downarrow & & \Downarrow \\ A \textit{ has hyperimmune degree} & \Rightarrow & \text{REC} \textit{ is i.o. } A-\textit{subuniform} \\ & & \Downarrow \\ A \textit{ is nonrecursive} & \Leftarrow & A \textit{ covers } \text{REC} \\ & & \Downarrow \\ \multicolumn{3}{c}{A \textit{ has hyperimmune degree or } A \textit{ is not Martin-Löf random}} \end{array}$$

No other implications hold than the ones indicated.

The following interesting question is still open.

Question 23. *Are there sets A such that A covers* REC*, but not the class of recursively enumerable sets* RE*?*

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