A Note on the Computable Categoricity of ℓ^p Spaces

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Abstract. Suppose that p is a computable real and that $p \geq 1$. We show that in both the real and complex case, ℓ^p is computably categorical if and only if $p = 2$. The proof uses Lamperti's characterization of the isometries of Lebesgue spaces of σ-finite measure spaces.

1 Introduction

When p is a positive real number, let ℓ^p denote the space of all sequences of complex numbers $\{a_n\}_{n=0}^{\infty}$ so that

$$\sum_{n=0}^{\infty} |a_n|^p < \infty.$$

ℓ^p is a vector space over $\mathbb{C}$ with the usual scalar multiplication and vector addition. When $p \geq 1$ it is a Banach space under the norm defined by

$$\|\{a_n\}_n\| = \left(\sum_{n=0}^{\infty} |a_n|^p \right)^{1/p}.$$

Loosely speaking, a computable structure is *computably categorical* if all of its computable copies are computably isomorphic. In 1989, Pour-El and Richards showed that ℓ^1 is not computably categorical [10]. It follows from a recent result of A.G. Melnikov that ℓ^2 is computably categorical [8]. At the 2014 Conference on Computability and Complexity in Analysis, A.G. Melnikov asked "For which computable reals $p \geq 1$ is ℓ^p computably categorical?" The following theorem answers this question.

Theorem 1. *Suppose p is a computable real so that $p \geq 1$. Then, ℓ^p is computably categorical if and only if $p = 2$.*

We prove Theorem 1 by proving the following stronger result.

Theorem 2. *Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. Suppose C is a c.e. set. Then, there is a computable copy of ℓ^p, $\mathcal{B}$, so that C computes a linear isometry of ℓ^p onto $\mathcal{B}$. Furthermore, if an oracle X computes a linear isometry of ℓ^p onto $\mathcal{B}$, then X must also compute C.*

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These results also hold for ℓ^p-spaces over the reals. In a forthcoming paper it will be shown that ℓ^p is Δ_2^0-categorical.

The paper is organized as follows. Section 2 covers background and motivation. Section 3 presents the proof of Theorem 2. Concluding remarks are presented in Sect. 4.

2 Background

2.1 Background from Functional Analysis

Fix p so that $1 \leq p < \infty$. A *generating set* for ℓ^p is a subset of ℓ^p with the property that ℓ^p is the closure of its linear span.

Let e_n be the vector in ℓ^p whose $(n+1)$st component is 1 and whose other components are 0. Let $E = \{e_n : n \in \mathbb{N}\}$. We call E the *standard generating set* for ℓ^p.

Recall that an *isometry* of ℓ^p is a norm-preserving map of ℓ^p into ℓ^p. We will use the following classification of the surjective linear isometries of ℓ^p.

Theorem 3. (Banach/Lamperti). *Suppose p is a real number so that $p \geq 1$ and $p \neq 2$. Let T be a linear map of ℓ^p into ℓ^p. Then, the following are equivalent.*

1. *T is a surjective isometry.*
2. *There is a permutation of $\mathbb{N}$, ϕ, and a sequence of unimodular points, $\{\lambda_n\}_n$, so that $T(e_n) = \lambda_n e_{\phi(n)}$ for all n.*
3. *Each $T(e_n)$ is a unit vector and the supports of $T(e_n)$ and $T(e_m)$ are disjoint whenever $m \neq n$.*

In his seminal text on linear operators, S. Banach stated Theorem 3 for the case of ℓ^p spaces over the reals [2]. He also stated a classification of the linear isometries of $L^p[0,1]$ in the real case. Banach's proofs of these results were sketchy and did not easily generalize to the complex case. In 1958, J. Lamperti rigorously proved a generalization of Banach's claims to real and complex L^p-spaces of σ-finite measure spaces [7]. Theorem 3 follows from J. Lamperti's work as it appears in Theorem 3.2.5 of [4]. Note that Theorem 3 fails when $p = 2$. For, ℓ^2 is a Hilbert space. So, if $\{f_0, f_1, \ldots\}$ is any orthonormal basis for ℓ^2, then there is a unique surjective linear isometry of ℓ^2, T, so that $T(e_n) = f_n$ for all n.

2.2 Background from Computable Analysis

We assume the reader is familiar with the fundamental notions of computability theory as covered in [3].

Suppose $z_0 \in \mathbb{C}$. We say that z_0 is *computable* if there is an algorithm that given any $k \in \mathbb{N}$ as input computes a rational point q so that $|q - z_0| < 2^{-k}$. This is equivalent to saying that the real and imaginary parts of z_0 have computable decimal expansions.

Our approach to computability on ℓ^p is equivalent to the format in [10] wherein a more expansive treatment may be found.

Fix a computable real p so that $1 \leq p < \infty$. Let $F = \{f_0, f_1, \ldots\}$ be a generating set for ℓ^p. We say that F is an *effective generating set* if there is an algorithm that given any rational points $\alpha_0, \ldots, \alpha_M$ and a nonnegative integer k as input computes a rational number q so that

$$q - 2^{-k} < \left\| \sum_{j=0}^{M} \alpha_j f_j \right\| < q + 2^{-k}.$$

That is, the map

$$\alpha_0, \ldots, \alpha_M \mapsto \left\| \sum_{j=0}^{M} \alpha_j f_j \right\|$$

is computable. Clearly the standard generating set is an effective generating set.

Suppose $F = \{f_0, f_1, \ldots\}$ is an effective generating set for ℓ^p. We say that a vector $g \in \ell^p$ is *computable with respect to* F if there is an algorithm that given any nonnegative integer k as input computes rational points $\alpha_0, \ldots, \alpha_M$ so that

$$\left\| g - \sum_{j=0}^{M} \alpha_j f_j \right\| < 2^{-k}.$$

Suppose $g_n \in \ell^p$ for all n. We say that $\{g_n\}_n$, is *computable with respect to* F if there is an algorithm that given any $k, n \in \mathbb{N}$ as input computes rational points $\alpha_0, \ldots, \alpha_M$ so that

$$\left\| g_n - \sum_{j=0}^{M} \alpha_j f_j \right\| < 2^{-k}.$$

When $f \in \ell^p$ and $r > 0$, let $B(f; r)$ denote the open ball with center f and radius r. When $\alpha_0, \ldots, \alpha_M$ are rational points and r is a positive rational number, we call $B\left(\sum_{j=0}^{M} \alpha_j f_j; r\right)$ a *rational ball*.

Suppose $F = \{f_0, f_1, \ldots\}$ and $G = \{g_0, g_1, \ldots\}$ are effective generating sets for ℓ^p. We say that a map $T : \ell^p \to \ell^p$ is *computable with respect to* (F, G) if there is an algorithm P that meets the following three criteria.

- **Approximation:** Given a rational ball $B(\sum_{j=0}^{M} \alpha_j f_j; r)$ as input, P either does not halt or produces a rational ball $B(\sum_{j=0}^{N} \beta_j g_j; r')$.
- **Correctness:** If $B(\sum_{j=0}^{N} \beta_j g_j; r')$ is the output of P on input $B(\sum_{j=0}^{M} \alpha_j f_j; r)$, then $T(f) \in B(\sum_{j=0}^{N} \beta_j g_j; r')$ whenever $f \in B(\sum_{j=0}^{M} \alpha_j f_j; r)$.
- **Convergence:** If U is a neighborhood of $T(f)$, then f belongs to a rational ball $B_1 = B(\sum_{j=0}^{M} \alpha_j f_j; r)$ so that P halts on B_1 and produces a rational ball that is included in U.

When we speak of an algorithm accepting a rational ball $B(\sum_{j=0}^{M} \alpha_j f_j; r)$ as input, we of course mean that it accepts some representation of the ball such as a code of the sequence $(r, M, \alpha_0, \ldots, \alpha_M)$.

All of these definitions have natural relativizations. For example, if $F = \{f_0, f_1, \ldots\}$ is an effective generating set, then we say that X computes a vector $g \in \ell^p$ with respect to F if there is a Turing reduction that given the oracle X and an input k computes rational points $\alpha_0, \ldots, \alpha_M$ so that $\left\| g - \sum_{j=0}^{M} \alpha_j f_j \right\| < 2^{-k}$.

2.3 Background from Computable Categoricity

For the sake of motivation, we begin by considering the following simple example. Let ζ be an incomputable unimodular point in the plane. For each n, let $f_n = \zeta e_n$. Let $F = \{f_0, f_1, \ldots\}$. Thus, F is an effective generating set. However, the vector ζe_0 is computable with respect to F even though it is not computable with respect to the standard generating set E. In fact, the only vector that is computable with respect to E and F is the zero vector. The moral of the story is that different effective generating sets may yield very different classes of computable vectors and sequences. However, there is a surjective linear isometry of ℓ^p that is computable with respect to (E, F); namely multiplication by ζ. Thus, E and F give the same computability theory on ℓ^p even though they yield very different classes of computable vectors. This leads to the following definition.

Definition 4. *Suppose p is a computable real so that $p \geq 1$. We say that ℓ^p is* computably categorical *if for every effective generating set F there is a surjective linear isometry of ℓ^p that is computable with respect to (E, F).*

The definitions just given for ℓ^p can easily be adapted to any separable Banach space. Suppose $G = \{g_0, g_1 \ldots, \}$ is an effective generating set for a Banach space $\mathcal{B}$. The pair $(\mathcal{B}, G)$ is called a *computable Banach space*. Suppose that $\mathcal{B}$ is linearly isometric to ℓ^p, and let T denote a linear isometric mapping of $\mathcal{B}$ onto ℓ^p. Let $f_n = T(g_n)$, and let $F = \{f_0, f_1, \ldots\}$. Then, F is an effective generating set for ℓ^p, and T is computable with respect to (G, F). Thus, Theorem 2 can be rephrased as follows.

Theorem 5. *Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. Suppose C is a* c.e. *set. Then, there is an effective generating set for ℓ^p, F, so that with respect to (E, F), C computes a surjective linear isometry of ℓ^p. Furthermore, any oracle that computes a surjective linear isometry of ℓ^p with respect to (E, F) must also compute C.*

A.G. Melnikov and K.M. Ng have investigated computable categoricity questions with regards to the space $C[0, 1]$ of continuous functions on the unit interval with the supremum norm [8,9]. The study of computable categoricity for countable structures goes back at least as far as the work of Goncharov [5]. The text of Ash and Knight has a thorough discussion of the main results of this line of inquiry [1]. The survey by Harizanov covers other directions in the countable computable structures program [6].

3 Proof of Theorems 1 and 2

We begin by noting the following easy consequence of the definitions and Theorem 3.

Proposition 6. *Suppose p is a computable real so that $p \geq 1$ and so that $p \neq 2$. Let F be an effective generating set for ℓ^p. Then, the following are equivalent.*

1. *There is a surjective linear isometry of ℓ^p that is computable with respect to (E, F).*
2. *There is a permutation of $\mathbb{N}$, ϕ, and a sequence of unimodular points $\{\lambda_n\}_n$, so that $\{\lambda_n e_{\phi(n)}\}_n$ is computable with respect to F.*
3. *There is a sequence of unit vectors $\{g_n\}_n$ so that $\{g_n\}_n$ is computable with respect to F, $G = \{g_0, g_1, \dots\}$ is a generating set for ℓ^p, and so that the supports of g_n and g_m are disjoint whenever $n \neq m$.*

Proof. Parts (2) and (3) just restate each other. It follows from Theorem 3 that (1) implies (2).

Suppose (3) holds. Let T be the unique linear map of the span of E onto the span of G so that $T(e_n) = g_n$ for all n. Since the supports of $g_0, g_1, \dots$ are pairwise disjoint, and since each g_n is a unit vector, T is isometric. It follows that there is a unique extension of T to a unique linear isometry of ℓ^p; denote this extension by T as well. We claim that T is computable with respect to (E, F). For, suppose a rational ball $B(\sum_{j=0}^{M} \alpha_j e_j; r)$ is given as input. Since $\{g_n\}_n$ is computable with respect to F, it follows that we can compute a non-negative integer N and rational points $\beta_0, \dots, \beta_N$ so that $\left\| \sum_{j=0}^{M} \alpha_j g_j - \sum_{j=0}^{N} \beta_j f_j \right\| < r$. We then output $B(\sum_{j=0}^{N} \beta_j g_j; 2r)$. It follows that the Approximation, Correctness, and Convergence criteria are satisfied and so T is computable with respect to (E, F). □

We now turn to the proof of Theorem 5 which, as we have noted, implies Theorem 2. Our construction of F is a modification of the construction used by Pour-El and Richards to show that ℓ^1 is not computably categorical [10]. Let C be an incomputable c.e. set. Without loss of generality, we assume $0 \notin C$. Let $\{c_n\}_{n \in \mathbb{N}}$ be an effective one-to-one enumeration of C. Set

$$\gamma = \sum_{k \in C} 2^{-k}.$$

Thus, $0 < \gamma < 1$, and γ is an incomputable real. Set:

$$f_0 = (1 - \gamma)^{1/p} e_0 + \sum_{n=0}^{\infty} 2^{-c_n/p} e_{n+1}$$
$$f_{n+1} = e_{n+1}$$
$$F = \{f_0, f_1, f_2, \dots\}$$

Since $1 - \gamma > 0$, we can use the standard branch of $\sqrt[p]{\ }$.

We divide the rest of the proof into the following lemmas.

Lemma 7. *F is an effective generating set.*

Proof. Since

$$(1-\gamma)^{1/p}e_0 = f_0 - \sum_{n=1}^{\infty} 2^{-c_{n-1}/p} f_n$$

the closed linear span of F includes E. Thus, F is a generating set for ℓ^p. Note that $\|f_0\| = 1$.

Suppose $\alpha_0, \ldots, \alpha_M$ are rational points. When $1 \leq j \leq M$, set

$$\mathcal{E}_j = |\alpha_0 2^{-c_{j-1}/p} + \alpha_j|^p - |\alpha_0|^p 2^{-c_{j-1}}.$$

It follows that

$$\begin{aligned}\|\alpha_0 f_0 + \ldots + \alpha_M f_M\|^p &= |\alpha_0|^p \|f_0\|^p + \mathcal{E}_1 + \ldots + \mathcal{E}_M \\ &= |\alpha_0|^p + \mathcal{E}_1 + \ldots + \mathcal{E}_M.\end{aligned}$$

Since $\mathcal{E}_1$, ..., $\mathcal{E}_M$ can be computed from $\alpha_0, \ldots, \alpha_M$, $\|\alpha_0 f_0 + \ldots + \alpha_M f_M\|$ can be computed from $\alpha_0, \ldots, \alpha_M$. Thus, F is an effective generating set. □

Lemma 8. *Every oracle that with respect to F computes a scalar multiple of e_0 whose norm is 1 must also compute C.*

Proof. Suppose that with respect to F, X computes a vector of the form λe_0 where $|\lambda| = 1$. It suffices to show that X computes $(1-\gamma)^{-1/p}$.

Fix a rational number q_0 so that $(1-\gamma)^{-1/p} \leq q_0$. Let $k \in \mathbb{N}$ be given as input. Compute k' so that $2^{-k'} \leq q_0 2^{-k}$. Since X computes λe_0 with respect to F, we can use oracle X to compute rational points $\alpha_0, \ldots, \alpha_M$ so that

$$\left\| \lambda e_0 - \sum_{j=0}^{M} \alpha_j f_j \right\| < 2^{-k'}. \tag{1}$$

We claim that $|(1-\gamma)^{-1/p} - |\alpha_0|| < 2^{-k}$. For, it follows from (1) that $|\lambda - \alpha_0(1-\gamma)^{1/p}| < 2^{-k'}$. Thus, $|1 - |\alpha_0|(1-\gamma)^{1/p}| < 2^{-k'}$. Hence,

$$|(1-\gamma)^{-1/p} - |\alpha_0|| < 2^{-k'}(1-\gamma)^{-1/p} \leq 2^{-k'} q_0 \leq 2^{-k}.$$

Since X computes α_0 from k, X computes $(1-\gamma)^{-1/p}$. □

Lemma 9. *If X computes a surjective linear isometry of ℓ^p with respect to (E, F), then X must also compute C.*

Proof. By Lemma 8 and the relativization of Proposition 6. □

Lemma 10. *With respect to F, C computes e_0.*

Proof. Fix an integer M so that $(1-\gamma)^{-1/p} < M$.

Let $k \in \mathbb{N}$. Using oracle C, we can compute an integer N_1 so that $N_1 \geq 3$ and

$$\left\| \sum_{n=N_1}^{\infty} 2^{-c_{n-1}/p} e_n \right\| \leq \frac{2^{-(kp+1)/p}}{2^{-(kp+1)/p} + M}.$$

We can use oracle C to compute a rational number q_1 so that $|q_1 - (1-\gamma)^{-1/p}| \leq 2^{-(kp+1)/p}$. Set

$$g = q_1 \left[f_0 - \sum_{n=1}^{N_1 - 1} 2^{-c_{n-1}/p} f_n \right].$$

It suffices to show that $\|e_0 - g\| < 2^{-k}$. Note that since $1-\gamma < 1$, $|q_1(1-\gamma)^{1/p} - 1| \leq 2^{-(kp+1)/p}$. Note also that $|q_1| < M + 2^{-(kp+1)/p}$. Thus,

$$\begin{aligned}\|e_0 - g\|^p &= \left\| e_0 - q_1(1-\gamma)^{1/p} e_0 - q_1 \sum_{n=N_1}^{\infty} 2^{-c_{n-1/p}} e_n \right\|^p \\ &\leq |q_1(1-\gamma)^{1/p} - 1|^p + |q_1|^p \left\| \sum_{n=N_1}^{\infty} 2^{-c_{n-1}/p} e_n \right\|^p \\ &< 2^{-(kp+1)} + 2^{-(kp+1)} = 2^{-kp}\end{aligned}$$

Thus, $\|e_0 - g\| < 2^{-k}$. This completes the proof of the lemma. □

Lemma 11. *With respect to (E, F), C computes a surjective linear isometry of ℓ^p.*

Proof. By Lemma 10 and the relativization of Proposition 6. □

4 Concluding Remarks

We note that all of the steps in the above proofs work just as well over the real field.

Lamperti's result on the isometries of L^p spaces hold when $0 < p < 1$. For these values of p, ℓ^p is a metric space under the metric

$$d(\{a_n\}_n, \{b_n\}_n) = \sum_{n=0}^{\infty} |a_n - b_n|^p.$$

The steps in the above proofs can be adapted to these values of p as well.

In a forthcoming paper it will be shown that ℓ^p is Δ_2^0-categorical.

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