Base-Complexity Classifications of QCB_0-Spaces (Extended Abstract)

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Abstract. We define and study new classifications of qcb_0-spaces based on the idea to measure the complexity of their bases. The new classifications complement those given by the hierarchies of qcb_0-spaces introduced in [7,8] and provide new tools to investigate non-countably based qcb_0-spaces. As a by-product, we show that there is no universal qcb_0-space and establish several apparently new properties of the Kleene-Kreisel continuous functionals of countable types.

Keywords: QCB_0-spaces · Y-based spaces · Hyperspaces · Scott topology · Hyperprojective hierarchy · Kleene-Kreisel continuous functionals

1 Introduction

A basic notion of Computable Analysis [10] is the notion of an *admissible representation* of a topological space X. This is a partial continuous surjection δ from the Baire space $\mathcal{N}$ onto X satisfying a certain universality property. Such a representation of X often induces a reasonable computability theory on X, and the class of admissibly represented spaces is wide enough to include most spaces of interest for Analysis or Numerical Mathematics. This class coincides with the class of the so-called qcb_0-spaces, i.e. T_0-spaces which are quotients of countably based spaces, and it forms a cartesian closed category with the continuous functions as morphisms [5]. Thus, among qcb_0-spaces one meets many important function spaces including the continuous functionals of finite types [3,4] interesting for several branches of logic and computability theory. In addition

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to being cartesian closed, the category QCB_0 of qcb_0-spaces is also closed under countable limits, countable colimits, and many other important constructions, making it a very convenient category of topological spaces. However, along with the benefits of this generality comes the challenge of developing comprehensive theories that provide a deeper understanding of arbitrary qcb_0-spaces.

Classical descriptive set theory [2] has proven to be extremely useful for classifying and studying separable metrizable spaces. Every separable metrizable space can be topologically embedded into a Polish space (a complete separable metrizable space), for example by taking the completion of a compatible metric. We can therefore classify a separable metrizable space according to the complexity of defining it as a subspace of some Polish space, where topological complexity can be quantified using natural hierarchies such as the Borel or Luzin (projective) hierarchies. This method of classification is topologically invariant (it does not depend on which Polish space we embed into) because of the remarkable fact that a subspace of a Polish space is Polish if and only if it is of level $\mathbf{\Pi}^0_2$ in the Borel hierarchy. We can even generalize this approach to the entire class of countably based T_0-spaces (abbreviated cb_0-spaces) by using quasi-Polish spaces [1], which have the same $\mathbf{\Pi}^0_2$ absoluteness property as Polish spaces. In fact, for classifying cb_0-spaces we can restrict ourselves to the algebraic domain $P\omega$ of all subsets of natural numbers (denoted ω), which is quasi-Polish and universal for cb_0-spaces.

Unfortunately, this approach to classifying topological spaces does not immediately generalize to the entire category of qcb_0-spaces. First of all, as we will see in this paper, there is no universal qcb_0-space to serve as a basis for comparing topological complexity. A second critical problem is that the $\mathbf{\Pi}^0_2$ absoluteness property of Polish and quasi-Polish spaces does not apply to subspaces of non-countably based spaces. For example, in [9] it is shown that the space $\mathcal{O}(\mathcal{N})$, the lattice of open subsets of $\mathcal{N}$ with the Scott-topology, contains *singleton* subsets which are $\mathbf{\Pi}^1_1$-complete even though they are trivially Polish with respect to the subspace topology. It is possible to use similar methods to construct qcb_0-spaces that have singleton subsets of arbitrarily high complexity in the hyperprojective hierarchy.

Important progress towards classifying qcb_0-spaces was made in [7,8], where the Borel, projective, and hyperprojective hierarchies of qcb_0-spaces were introduced. The major insight was to classify qcb_0-spaces according to the complexity of the equivalence relation on the elements of $\mathcal{N}$ induced by an admissible representation of the space, which elegantly sidesteps the problem of finding a universal space. This approach works well because the universal property of admissible representations causes them to reflect many important topological properties of the underlying space. In fact, it was shown in [7,8] that for cb_0-spaces, the newly introduced classification approach using admissible representations is equivalent to the approach described above that uses topological embeddings into $P\omega$.

However, the hierarchies defined in [7,8] do not differentiate between countably based qcb_0-spaces and non-countably based spaces. In particular, the problem of placing an upper bound on the relative complexity of even very simple subsets (such as singletons) of non-countably based spaces can not be settled using this approach. Thus, although the Borel, projective, and hyperprojective hierarchies quantify one important aspect of the complexity of qcb_0-spaces, there appears to be an additional dimension of complexity that is mostly apparent in the large difference between countably based and non-countably based spaces.

In this paper we attempt to capture this additional dimension of complexity by introducing methods to classify a topological space according to the complexity of defining a basis for its topology. Our hope is that by combining the basis-complexity measures introduced in this paper with the hierarchies defined in [7,8], we can obtain a more complete measure of the topological complexity of qcb_0-spaces.

The basic idea of our approach is a natural generalization of the definition of a countable basis. Given a topological space X, a countable basis for X can be viewed as a mapping ϕ from ω to the set $\mathcal{O}(X)$ of open subsets of X such that the range of ϕ is a basis for the topology of X. As a first approach to generalizing this definition to non-countably based spaces, we can replace the index set ω with an arbitrary topological space Y and consider whether or not a basis for X can be indexed by some mapping $\phi\colon Y \to \mathcal{O}(X)$ which is continuous with respect to the Scott-topology on $\mathcal{O}(X)$. The class of spaces that have such an indexing for a basis will be called *Y-based spaces*, and the complexity of Y according to the hierarchies in [7,8] provides an indication of the complexity of the spaces in this class. This definition is very natural and we will show that it has several useful properties, but unfortunately it can be difficult to use in practice. We therefore also introduce a second related concept that we call *sequentially Y-based* spaces, which requires a more complicated definition but behaves much better when working with sequential spaces. In particular, we will show that universal spaces exist for the class of sequentially Y-based spaces for each qcb_0-space Y. We expect this observation will be useful for future development of a descriptive theory of qcb_0-spaces that avoids the problems mentioned earlier in this introduction.

We will provide a detailed analysis of the relationship between the proposed hierarchies and the previous ones, and provide some applications. The newly introduced basis-complexity classifications can be particularly useful when determining whether one space can be embedded into another space. We will demonstrate this claim by investigating the existence of certain classes of universal qcb_0-spaces, by showing that every qcb_0-space can be embedded into a space with a total admissible representation, and by establishing several apparently new properties of the Kleene-Kreisel continuous functionals of countable types.

In Sect. 2 we discuss the notions of topological and sequential embeddings. In Sects. 3 and 4 we first introduce and study some versions of the notion of a Y-based space, and then define and investigate the two relevant classifications of qcb_0-spaces. In Sect. 5 we study which levels of the the new and old hierarchies have a universal (or sequentially universal) space. Because of the strict space bounds, we omit all proofs and use some notation and notions from [8] without definition.

2 Topological Embeddings Versus Sequential Embeddings

In this section we briefly discuss two notions of embedding for sequential spaces relevant to this paper. The first one is the usual topological embedding which is used in Sect. 3. The second one is a lesser known sequential embedding which is more natural for sequential spaces and results in a more satisfactory theory in Sect. 4 than the theory based on topological embeddings.

We say that a space X *embeds topologically* into Y, if X is homeomorphic to a topological subspace M of Y; the corresponding homeomorphism seen as a function e from X to Y is called a *topological embedding* of X into Y. When dealing with sequential spaces (in particular, qcb$_0$-spaces), it is natural to consider the following modification of the topological embeddings:

Definition 1. Let X, Y be sequential spaces.

(1) The space X is *a sequential subspace* of Y, if $X \subseteq Y$ and, whenever $(x_n)_n$ is a sequence in X and $x_\infty \in X$, convergence of $(x_n)_n$ to x_∞ in X is equivalent to convergence of $(x_n)_n$ to x_∞ in Y.
(2) We say that X *embeds sequentially into* Y, if there is an injection $e\colon X \to Y$ such that convergence of $(x_n)_n$ to x_∞ in X is equivalent to convergence of $(e(x_n))_n$ to $e(x_\infty)$ in Y. In this case we call e a *sequential embedding of* X *into* Y.

The distinction between topological subspace and sequential subspace is subtle, but very important. It can be shown that if X and Y are sequential spaces, then X embeds sequentially into Y if and only if there is a topological subspace $S \subseteq Y$ such that X is homeomorphic to the sequentialisation of S.

It is easy to check that, for all sequential spaces X, Y, if $e\colon X \to Y$ is a topological embedding then it is also a sequential embedding, but the converse does not hold in general. If $e\colon X \to Y$ is a surjective sequential embedding, then e is a homeomorphism.

3 Y-based Spaces

In this section we introduce and study the notion of a Y-based space, where Y is a topological space. This provides a natural generalization of the notion of a countably based space which can be applied to classifying non-countably based qcb$_0$-spaces.

Let $\mathbb{S}$ be the Sierpinski space and $\mathcal{O}(X)$ be the hyperspace of open subsets of a space X topologised with the ω-Scott topology. If X is a sequential space (in particular a qcb$_0$-space), then $\mathcal{O}(X)$ is homeomorphic to $\mathbb{S}^X$.

Definition 2. Let X, Y be topological spaces. A continuous function $\phi\colon Y \to \mathcal{O}(X)$ is a Y*-indexing* of a basis for X, if the range of ϕ is a basis for the topology on X. The space X is Y*-based* if there is a Y*-indexing* of a basis for X.

These notions are purely topological and apply to arbitrary topological spaces. It is easily shown that if X is Y-based and Y is a continuous image of a space Z then X is Z-based, and that any topological subspace of a Y-based space is Y-based.

The next proposition generalizes the fact that any countably-based T_0-space embeds topologically into $P\omega$, which is homeomorphic to $\mathcal{O}(\omega)$.

Theorem 1. *Let X, Y be sequential T_0-spaces such that X is Y-based. Then X topologically embeds into $\mathcal{O}(Y)$.*

The next basic fact characterizes qcb$_0$-spaces in terms of these notions.

Theorem 2. *The following are equivalent for any sequential T_0-space X:*

(1) *X is Y-based for some zero-dimensional cb$_0$-space Y (i.e., some $Y \subseteq \mathcal{N}$).*
(2) *X is Y-based for some qcb$_0$-space Y.*
(3) *X topologically embeds into $\mathcal{O}(Y)$ for some cb$_0$-space Y.*
(4) *X is a qcb$_0$-space.*

As $\mathcal{O}(Y)$ has a total admissible representation for any cb$_0$-space Y, we obtain:

Corollary 1. *Every qcb$_0$-space topologically embeds into a space with a total admissible representation.*

For any qcb$_0$-space Y, let $Based(Y)$ denote the class of Y-based qcb$_0$-spaces. For a class $\mathcal{S}$ of qcb$_0$-spaces, let $Based(\mathcal{S}) = \bigcup_{Y\in\mathcal{S}} Based(Y)$. Theorem 2 induces some natural classifications of qcb$_0$-spaces. For example, one can relate to any family of pointclasses $\mathbf{\Gamma}$ the classes $Based(\mathbf{\Gamma}(\mathcal{N}))$ and $Based(\mathsf{QCB}_0(\mathbf{\Gamma}))$ and easily check that the classes coincide. Here, the notation $\mathsf{QCB}_0(\mathbf{\Gamma})$ refers to the hierarchies of qcb$_0$-spaces defined in [7,8]. Thus, the classical hierarchies of subsets of the Baire space induce corresponding hierarchies of qcb$_0$-spaces, in particular the "hyperprojective base-hierarchy" $Based(\mathbf{\Sigma}^1_\alpha(\mathcal{N}))$, for which we use the simpler notation $Based(\mathbf{\Sigma}^1_\alpha)$.

We now establish a relationship between $Based(\mathbf{\Sigma}^1_\alpha)$ and the qcb$_0$-space $\mathbb{N}\langle\alpha\rangle$, which is *the space of continuous functionals of type* α over ω. The spaces $\mathbb{N}\langle\alpha\rangle$ are defined by induction on countable ordinals α as follows [8]:

$$\mathbb{N}\langle 0\rangle := \omega,\ \mathbb{N}\langle \beta+1\rangle := \omega^{\mathbb{N}\langle\beta\rangle} \text{ and } \mathbb{N}\langle\lambda\rangle := \textstyle\prod_{\alpha<\lambda}\mathbb{N}\langle\alpha\rangle,$$

where ω denotes the discrete space of natural numbers, $\beta, \lambda < \omega_1$ and λ is a limit ordinal. Obviously, for $k < \omega$ the space $\mathbb{N}\langle k\rangle$ coincides with the space of Kleene-Kreisel continuous functionals of type k extensively studied in the literature, and $\mathbb{N}\langle 1\rangle$ coincides with the Baire space $\mathcal{N}$. Moreover, we consider a standard admissible representation $\delta_\alpha \colon D_\alpha \to \mathbb{N}\langle\alpha\rangle$ for $\mathbb{N}\langle\alpha\rangle$ derived by a natural construction as presented in [8]. We will also deal with the coproduct spaces $\mathbb{N}\langle{<}\lambda\rangle := \bigoplus_{\alpha<\lambda}\mathbb{N}\langle\alpha\rangle$, where λ is a countable ordinal limit.

Proposition 1. *For any $\alpha < \omega_1$, $Based(D_{\alpha+1}) = Based(\mathbf{\Pi}^1_\alpha) = Based(\mathbf{\Sigma}^1_{\alpha+1}) = Based(\mathbb{N}\langle\alpha+1\rangle)$. For any limit ordinal $\lambda < \omega_1$, $Based(D_\lambda) = Based((\mathbf{\Pi}^1_{<\lambda})_\delta) = Based(\mathbf{\Sigma}^1_\lambda) = Based(\mathbb{N}\langle\lambda\rangle)$.*

By Theorem 1, any space from $\mathit{Based}(Y)$ topologically embeds into $\mathcal{O}(Y)$. A principal question is: for which qcb$_0$-spaces Y do we have that the space $\mathcal{O}(Y)$ is Y-based? Clearly, this is equivalent to saying that $\mathit{Based}(Y)$ is the class of spaces topologically embeddable into $\mathcal{O}(Y)$. Unfortunately, the assertion does not hold for all Y: one easily checks that the space $\mathcal{O}(\mathbb{Q})$ is not $\mathbb{Q}$-based. Nevertheless, the assertion $\mathcal{O}(Y) \in \mathit{Based}(Y)$ might hold for some natural spaces Y, in particular a positive answer to the following problem would clarify the nature of the hierarchy $\{\mathit{Based}(D_\alpha)\}_{\alpha<\omega_1}$ considerably:

Problem 1. Does the assertion $\mathcal{O}(D_\alpha) \in \mathit{Based}(D_\alpha)$ hold for all $\alpha < \omega_1$?

If the answer is positive, $\mathit{Based}(D_\alpha)$ would coincide with the class of spaces topologically embeddable into $\mathcal{O}(D_\alpha)$. For $\alpha = 0$ the assertion holds because $\mathcal{O}(\omega)$ is homeomorphic to $P\omega$, and we will show below that the assertion is also true for $\alpha = 1$. For $\alpha \geq 2$ we still do not know the answer. This is an obstacle to answering the principal question on the non-collapse of the introduced hierarchy $\{\mathit{Based}(D_\alpha)\}_{\alpha<\omega_1}$. By the non-collapse property we mean that the inclusion $\mathit{Based}(D_\alpha) \subseteq \mathit{Based}(D_\beta)$ is proper for each $\alpha < \beta < \omega_1$. The next result (along with the assertion $\mathcal{O}(D_1) \in \mathit{Based}(D_1)$) implies, in particular, that $\mathit{Based}(D_0) \subsetneqq \mathit{Based}(D_1)$.

Proposition 2. *For any $\alpha < \omega_1$, $\mathcal{O}(D_{\alpha+1}) \notin \mathit{Based}(D_\alpha)$. For any limit ordinal $\lambda < \omega_1$, $\mathcal{O}(D_\lambda) \notin \mathit{Based}(\bigoplus_{\alpha<\lambda} D_\alpha)$.*

The following relation between the hyperprojective hierarchy of qcb$_0$-spaces and the hierarchy $\{\mathit{Based}(D_\alpha)\}_{\alpha<\omega_1}$ is interesting in its own right and also implies a weak non-collapse property:

Proposition 3. *For any $\alpha < \omega_1$, $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \subseteq \mathit{Based}(\mathbf{\Pi}^1_{\alpha+1}) = \mathit{Based}(D_{\alpha+2})$.*

Conversely, for each ordinal $\alpha < \omega_1$, $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$ does not even contain all of $\mathit{Based}(\omega)$, as $D_{\alpha+2} \in \mathit{Based}(\omega) \setminus \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$ by Theorem 2 in [8]. The second item of the next corollary is the weak version of the non-collapse property.

Corollary 2. *Let $\alpha < \omega_1$. Then we have $\mathcal{O}(D_\alpha) \in \mathit{Based}(D_{\alpha+2})$ and the inclusion $\mathit{Based}(D_\alpha) \subset \mathit{Based}(D_{\alpha+3})$ is proper.*

Problem 2. For which $\alpha < \omega_1$ can the inclusion from Proposition 3 be improved to $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \subseteq \mathit{Based}(D_{\alpha+1})$ or even to $\mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \subseteq \mathit{Based}(D_\alpha)$?

Next we further investigate the important class $\mathit{Based}(\mathcal{N}) = \mathit{Based}(D_1)$ of $\mathcal{N}$-based qcb$_0$-spaces, which includes many natural non-countably based spaces. As an example, we state an interesting property of the class of quasi-Polish spaces [1], which includes both Polish spaces and ω-continuous domains.

Proposition 4. *If X is quasi-Polish then $\mathcal{O}(X)$ is $\mathcal{N}$-based.*

For metrizable spaces $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ we have the following complete characterization of when $\mathcal{O}(X)$ is $\mathcal{N}$-based.

Proposition 5. *Let $X \in \mathsf{CB}_0(\mathbf{\Pi}^1_1)$ be metrizable. Then $\mathcal{O}(X)$ is $\mathcal{N}$-based if and only if X is Polish.*

Corollary 3. *A qcb$_0$-space is $\mathcal{N}$-based if and only if it embeds topologically in $\mathcal{O}(\mathcal{N})$. In particular, Based$(D_0) \subsetneqq$ Based(D_1).*

4 Sequentially Y-based Spaces

In this section we consider some modifications of the notion of Y-based spaces from the previous section which are more suitable to the nature of sequential spaces (in particular, qcb$_0$-spaces). This will be sufficient to settle the analogues of the open questions in Sect. 3 for the sequential embeddings in place of topological embeddings.

One could define several modifications of the notion of Y-based space. For instance, for qcb$_0$-spaces X, P we could say that a function $\phi\colon P \to \mathcal{O}(X)$ is a *P-indexed sequential basis* for X, if ϕ is continuous and range(ϕ) is a subbasis for a topology τ on X such that the sequentialisation of τ is the Scott topology in $\mathcal{O}(X)$. Under this definition, some interesting facts may be established, e.g., one can show that for any $\alpha < \omega_1$ the space $\mathbb{N}\langle\alpha + 1\rangle$ has an $\mathbb{N}\langle\alpha\rangle$-indexed sequential basis (see Corollary 5). We also consider the following deeper modification:

Definition 3. Let X, P be sequential spaces.

(1) We call a collection $\mathcal{B}$ of open subsets of X *a sequential basis* for X if $\mathcal{B}$ is a subbase of a topology τ on the set X such that the sequentialisation of τ is equal to $\mathcal{O}(X)$.
(2) A function $\phi\colon P \to \mathcal{O}(X)$ is called a P-indexed sequential basis for X if ϕ is continuous and its range $rng(\phi)$ is a sequential basis for X.
(3) For a function $\phi\colon P \to \mathcal{O}(X)$, we define $\mathcal{B}_\phi$ to consist of all intersections of the form $\bigcap_{n \le \infty} \phi(p_n)$, where $(p_n)_n$ converges to p_∞ in P.
(4) A function $\phi\colon P \to \mathcal{O}(X)$ is called a *P-indexed generating system for X* if ϕ is continuous and $\mathcal{B}_\phi$ is a sequential basis for X.
(5) X is called *sequentially P-based* if there is a P-indexed generating system for X.

By Proposition 2.2 in [6] the elements of $\mathcal{B}_\phi$ are open in X, if ϕ is continuous, because $(\phi(p_n))_n$ converges to $\phi(p_\infty)$ in $\mathcal{O}(X)$.

Now we study for which spaces P the existence of a P-indexed generating system implies the existence of a P-indexed sequential basis.

Lemma 1. *Let P be a sequential space such that there is a continuous surjection from P onto $P^{\mathbb{N}_\infty}$. Then any sequential space X is sequentially P-based if, and only if, X has a P-indexed sequential basis.*

The spaces $\mathbb{N}\langle\alpha\rangle$ and $\mathbb{N}\langle{<}\lambda\rangle$ can be shown to fulfill the requirement of Lemma 1. We obtain:

Corollary 4. (1) *For any $\alpha < \omega_1$, a sequential space X is sequentially $\mathbb{N}\langle\alpha\rangle$-based if, and only if, there is an $\mathbb{N}\langle\alpha\rangle$-indexed sequential basis for X.*
(2) *For any limit ordinal $\lambda < \omega_1$, a sequential space X is sequentially $\mathbb{N}\langle<\lambda\rangle$-based if, and only if, there is an $\mathbb{N}\langle<\lambda\rangle$-indexed sequential basis for X.*

Although Definition 3 (4) is very technical, it is justified by several nice properties the main of which is the following theorem:

Theorem 3. *Let X and P be sequential T_0-spaces. Then X is sequentially P-based if, and only if, X embeds sequentially into $\mathcal{O}(P)$.*

Theorem 3 solves in the positive the "sequential analogue" of the question "is $\mathcal{O}(P) \in \mathit{Based}(P)$ for each P?" discussed in the previous section.

For any sequential space P we of course have $\mathit{Based}(P) \subseteq \mathit{SBased}(P)$. An interesting question is "for which P is this inclusion proper?" One example is $\mathbb{Q}$, because $\mathcal{O}(\mathbb{Q})$ is sequentially $\mathbb{Q}$-based by Theorem 3, but not countably based, as $\mathbb{Q}$ is a non-locally-compact metrizable space. This observation can be improved to the following:

Proposition 6. *The space $\mathcal{O}(\mathbb{Q})$ is sequentially $\mathcal{N}$-based, but not $\mathcal{N}$-based.*

We now show how to construct generating systems for countable products and function spaces (formed in the category Seq of sequential spaces).

Proposition 7. *Let X_i, P_i be sequential T_0-spaces such that X_i is sequentially P_i-based. Then the sequential product $\prod_{i\in\omega} X_i$ is sequentially $(\bigoplus_{i\in\omega} P_i)$-based.*

Proposition 8. *Let X, Y, P be sequential T_0-spaces such that Y is sequentially P-based. Then Y^X is sequentially $(P \times X)$-based.*

We obtain the following nice property of the spaces of functionals:

Corollary 5. *For any $\alpha < \omega_1$, the space $\mathbb{N}\langle\alpha+1\rangle$ is sequentially $\mathbb{N}\langle\alpha\rangle$-based. For any limit ordinal $\lambda < \omega_1$, the space $\mathbb{N}\langle\lambda\rangle$ is sequentially $\mathbb{N}\langle<\lambda\rangle$-based.*

For any qcb_0-space Y, let $\mathit{SBased}(Y)$ denote the class of sequentially Y-based qcb_0-spaces. For a class $\mathcal{S}$ of qcb_0-spaces Y, let $\mathit{SBased}(\mathcal{S}) = \bigcup_{Y\in\mathcal{S}} \mathit{SBased}(Y)$. Obviously, $\mathit{Based}(Y) \subseteq \mathit{SBased}(Y)$ for each qcb_0-space Y. Theorem 3 induces some natural classifications of qcb_0-spaces. For example, one can relate to any family of pointclasses $\mathbf{\Gamma}$ the classes $\mathit{SBased}(\mathbf{\Gamma}(\mathcal{N}))$, $\mathit{SBased}(\mathsf{QCB}_0(\mathbf{\Gamma}))$ and show that they coincide.

Thus, the classical hierarchies of subsets of the Baire space induce the corresponding hierarchies of qcb_0-spaces, in particular the "hyperprojective sequential-based-hierarchy" $\mathit{SBased}(\mathbf{\Sigma}^1_\alpha(\mathcal{N}))$; we simplify the notation to $\mathit{SBased}(\mathbf{\Sigma}^1_\alpha)$ and relate this hierarchy to the admissible representations $\delta_\alpha : D_\alpha \to \mathbb{N}\langle\alpha\rangle$. The next assertion is an analogue of Proposition 1.

Proposition 9. *For any $\alpha < \omega_1$, $\mathit{SBased}(D_{\alpha+1}) = \mathit{SBased}(\mathbf{\Pi}^1_\alpha) = \mathit{SBased}(\mathbf{\Sigma}^1_{\alpha+1}) = \mathit{SBased}(\mathbb{N}\langle\alpha+1\rangle)$. For any limit ordinal $\lambda < \omega_1$, $\mathit{SBased}(D_\lambda) = \mathit{SBased}(\mathbf{\Sigma}^1_\lambda) = \mathit{SBased}((\mathbf{\Pi}^1_{<\lambda})_\delta) = \mathit{SBased}(\mathbb{N}\langle\lambda\rangle)$.*

Next we solve the principal question on the non-collapse property of the hierarchy $\{SBased(D_\alpha)\}_{\alpha<\omega_1}$. Remember that the corresponding result for the hierarchy $\{Based(D_\alpha)\}_{\alpha<\omega_1}$ remained open.

Proposition 10. *The hierarchy $\{SBased(D_\alpha)\}_{\alpha<\omega_1}$ does not collapse. More precisely, $SBased(D_\alpha) \subsetneqq SBased(D_{\alpha+1})$ for all $\alpha < \omega_1$ and $SBased(\bigoplus_{\alpha<\lambda} D_\alpha) \subsetneqq SBased(D_\lambda)$ for each limit ordinal $\lambda < \omega_1$.*

The next fact shows that the class $SBased(\mathcal{N})$ is rather rich.

Proposition 11. *Let X be a qcb_0-space having a total admissible representation $\xi\colon \mathcal{N} \to X$. Then $\mathcal{O}(X)$ embeds sequentially into $\mathcal{O}(\mathcal{N})$.*

Problem 3. We know from Theorem 3 and Proposition 6 that $Based(\mathbb{Q}) \subsetneqq SBased(\mathbb{Q})$ and $Based(\mathcal{N}) \subsetneqq SBased(\mathcal{N})$. We would like to know which sequential spaces X satisfy $Based(X) \subsetneqq SBased(X)$. In particular, we conjecture that $Based(D_\alpha) \subsetneqq SBased(D_\alpha)$ for all non-zero ordinals $\alpha < \omega_1$, and $SBased(\bigoplus_{\alpha<\lambda} D_\alpha) \subsetneqq SBased(\bigoplus_{\alpha<\lambda} D_\alpha)$ for all limit ordinals $\lambda < \omega_1$. Good possible witnesses seem to be $\mathbb{N}\langle\alpha+1\rangle$ and $\mathbb{N}\langle\lambda\rangle$ respectively (see Corollary 5).

By Proposition 10, the spaces $\mathcal{O}(D_\alpha)$ are natural witnesses for the non-collapse property of the hierarchy $\{SBased(D_\alpha)\}_{\alpha<\omega_1}$. Next we observe that the spaces $\mathbb{N}\langle\alpha\rangle$ provide other natural witnesses for this property showing that Corollary 5 is in a sense optimal.

Theorem 4. (1) *For any $\alpha < \omega_1$, $\mathbb{N}\langle\alpha+2\rangle \in SBased(\mathbb{N}\langle\alpha+1\rangle) \setminus SBased(\mathbb{N}\langle\alpha\rangle)$.*

(2) *For any limit ordinal $\lambda < \omega_1$, $\mathbb{N}\langle\lambda+1\rangle \in SBased(\mathbb{N}\langle\lambda\rangle) \setminus SBased(\mathbb{N}\langle<\lambda\rangle)$.*

(3) *For any limit ordinal $\lambda < \omega_1$, $\mathbb{N}\langle\lambda\rangle \in SBased(\bigoplus_{\alpha<\lambda} \mathbb{N}\langle\alpha\rangle) \setminus \bigcup_{\alpha<\lambda} SBased(\mathbb{N}\langle\alpha\rangle)$.*

We also can deduce the following corollary about the continuous functionals.

Corollary 6. *For all $\alpha < \beta < \omega_1$, $\mathbb{N}\langle\beta\rangle$ does not sequentially embed into $\mathbb{N}\langle\alpha\rangle$.*

5 On Universal Spaces

In this section we discuss which classes of qcb_0-spaces have and which do not have a universal space. This is of interest because universal spaces are noticeable in several branches of set-theoretic topology.

Definition 4. (1) Let $\mathcal{S}$ be a class of topological spaces. A space X is *universal in $\mathcal{S}$*, if $X \in \mathcal{S}$ and any space from $\mathcal{S}$ embeds topologically in X.

(2) Let $\mathcal{S}$ be a class of sequential spaces. A space X is *sequentially universal in $\mathcal{S}$*, if $X \in \mathcal{S}$ and any space from $\mathcal{S}$ embeds sequentially in X.

The first notion above is well-known in topology. E.g., $P\omega$ is universal in the class of cb_0-spaces, while the class of all topological spaces has no universal space. The second notion is a "sequential version" of the first one which is natural when dealing with sequential spaces or qcb_0-spaces. As $P\omega$ is universal in the class of cb_0-spaces and Y-based spaces are designed as a natural generalization of countably based spaces, it is natural to ask for which $Y \subseteq \mathcal{N}$ the class of Y-based spaces has a universal space. At least, we can prove:

Corollary 7. *Let $Y \subseteq \mathcal{N}$ be such that the space $\mathcal{O}(Y)$ is Y-based. Then $\mathcal{O}(Y)$ is universal in the class of Y-based topological spaces. In particular, the space $\mathcal{O}(\mathcal{N})$ is universal in the class of $\mathcal{N}$-based spaces.*

For the sequential version, we derive from Theorem 3:

Corollary 8. *For any $Y \in \mathsf{QCB}_0$, $\mathcal{O}(Y)$ is sequentially universal in $SBased(Y)$.*

It is still open whether or not $Based(D_\alpha)$ contains a universal space when $\alpha > 1$. However, we see that each level of the hierarchy $\{SBased(D_\alpha)\}_{\alpha<\omega_1}$ contains a sequentially universal space $\mathcal{O}(D_\alpha)$ with a total admissible representation. The same applies to the hierarchies of cb_0-spaces in [7] (obviously, $P\omega$ is a universal space in $\mathsf{CB}_0(\mathbf{\Gamma})$ for each family of pointclasses $\mathbf{\Gamma}$ that contains $\mathbf{\Pi}^0_2$).

For the hierarchies of qcb_0-spaces in [7,8] the situation is more complicated. Currently we do not know which of the classes $\mathsf{QCB}_0(\mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is a level of the Borel or hyperprojective hierarchy, have a universal (or a sequentially universal) space. Nevertheless, we can show that the class of all qcb_0-spaces, as well as some natural pointclasses related to the hyperprojective hierarchy of qcb_0-spaces, do not have universal spaces. Recall from [7,8] that $\mathsf{QCB}_0(\mathbf{P}) := \bigcup_{n<\omega} \mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$ and $\mathsf{QCB}_0(\mathbf{HP}) := \bigcup_{\alpha<\omega_1} \mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)$ denote the classes of projective and of hyperprojective qcb_0-spaces, respectively.

Theorem 5. (1) *There is no universal (nor a sequentially universal) qcb_0-space.*
(2) *For any limit ordinal $\lambda < \omega_1$, there is no universal (nor a sequentially universal) space in $\mathsf{QCB}_0(\mathbf{\Sigma}^1_{<\lambda})$.*
(3) *There is no universal (nor a sequentially universal) space in $\mathsf{QCB}_0(\mathbf{P})$ (nor in $\mathsf{QCB}_0(\mathbf{HP})$).*

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