Prime Model with No Degree of Autostability Relative to Strong Constructivizations

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Abstract. We build a decidable structure $\mathcal{M}$ such that $\mathcal{M}$ is a prime model of the theory $Th(\mathcal{M})$ and $\mathcal{M}$ has no degree of autostability relative to strong constructivizations.

Keywords: Autostability · Decidable structure · Prime model · Autostability spectrum · Autostability relative to strong constructivizations · Degree of categoricity · Categoricity spectrum · Decidable categoricity

1 Introduction

The study of autostable structures goes back to the works of Fröhlich and Shepherdson [1], and Mal'tsev [2,3]. Since then, the notion of *autostability* has been relativized to the levels of the hyperarithmetical hierarchy, and to arbitrary Turing degrees $\mathbf{d}$, and has been the subject of much study.

Definition 1. Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathcal{A}$ is $\mathbf{d}$-*autostable* if, for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. $\mathbf{0}$-autostable structures are also called *autostable.*

The *autostability spectrum* of the structure $\mathcal{A}$ is the set

$$\mathrm{AutSpec}(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-autostable}\}.$$

A Turing degree $\mathbf{d}_0$ is the *degree of autostability* of $\mathcal{A}$ if $\mathbf{d}_0$ is the least degree in $\mathrm{AutSpec}(\mathcal{A})$.

Autostability spectra and degrees of autostability were introduced by Fokina, Kalimullin, and Miller [4]. Note that much of the literature (see, e.g., [4–8]) uses the terms *categoricity spectrum* and *degree of categoricity* in place of autostability spectrum and degree of autostability, respectively. In this paper, we follow the terminology of [9].

Suppose that n is a natural number and α is a computable ordinal. Fokina, Kalimullin, and Miller [4] proved that every Turing degree $\mathbf{d}$ that is d.c.e. in and above $\mathbf{0}^{(n)}$ is the degree of autostability of a computable structure. This result

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was extended by Csima, Franklin, and Shore [5] to hyperarithmetical degrees. They proved that every degree that is d.c.e. in and above $\mathbf{0}^{(\alpha+1)}$ is a degree of autostability. They also showed that $\mathbf{0}^{(\alpha)}$ is a degree of autostability.

Miller [10] constructed the first example of a computable structure with no degree of autostability. He proved that there exists a computable field F which is not autostable and such that for some $\mathbf{c}_0, \mathbf{c}_1 \in \mathrm{AutSpec}(F)$, $\mathbf{c}_0 \wedge \mathbf{c}_1 = \mathbf{0}$. For more results on autostability spectra, see the survey [11].

Recall that a computable structure $\mathcal{A}$ is *decidable* if its complete diagram $D^c(\mathcal{A})$ is a computable set. The following definition describes the notion of autostability restricted to decidable copies of a structure. This notion is a natural one, as it simply changes the word *computable* to *decidable*.

Definition 2. Suppose that $\mathbf{d}$ is a Turing degree. A decidable structure $\mathcal{A}$ is $\mathbf{d}$-*autostable relative to strong constructivizations* ($\mathbf{d}$-*SC-autostable*) if, for every decidable copy $\mathcal{B}$ of $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism $f\colon \mathcal{A} \to \mathcal{B}$. In case $\mathbf{d} = \mathbf{0}$, we say that $\mathcal{A}$ is *SC-autostable*.

The *autostability spectrum relative to strong constructivizations* (*SC-autostability spectrum*) is the set

$$\mathrm{AutSpec}_{SC}(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-}SC\text{-autostable}\}.$$

A Turing degree $\mathbf{d}_0$ is the *degree of autostability relative to strong constructivizations* (*degree of SC-autostability*) of $\mathcal{A}$ if $\mathbf{d}_0$ is the least degree in the spectrum $\mathrm{AutSpec}_{SC}(\mathcal{A})$.

The study of SC-autostability spectra was initiated by Goncharov [9]. In particular, he proved that every c.e. Turing degree is the degree of SC-autostability of some decidable prime model. In [12] the author announced the following result: for a computable successor ordinal α, every degree $\mathbf{d}$ that is c.e. in and above $\mathbf{0}^{(\alpha)}$ is a degree of SC-autostability.

Suppose that L is a language. If $\mathcal{M}$ is an L-structure, then $Th(\mathcal{M})$ is the first-order theory of $\mathcal{M}$. A structure $\mathcal{M}$ is a *prime model* (of the theory $Th(\mathcal{M})$) if, for every model $\mathcal{N}$ of $Th(\mathcal{M})$, there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. A structure $\mathcal{M}$ is an *almost prime model* if there exists a finite tuple $\bar{c}$ from $\mathcal{M}$ such that $(\mathcal{M}, \bar{c})$ is a prime model.

Our work is concerned with the following problem.

Problem 1. (Goncharov [9]). Suppose that $\mathcal{M}$ is a decidable almost prime model and $\bar{c}$ is a tuple from $\mathcal{M}$ such that $(\mathcal{M}, \bar{c})$ is a prime model of the theory $Th(\mathcal{M}, \bar{c})$. Let $\mathbf{d}$ be the Turing degree of the collection of complete formulas of $Th(\mathcal{M}, \bar{c})$. It is not difficult to see that $\mathbf{d}$ is a c.e. degree and $\mathcal{M}$ is $\mathbf{d}$-SC-autostable. Is it always true that $\mathbf{d}$ is the degree of SC-autostability of $\mathcal{M}$?

We give the negative answer to this question by proving the following result.

Theorem 1. *There exists a decidable structure $\mathcal{M}$ such that $\mathcal{M}$ is a prime model of the theory $Th(\mathcal{M})$ and $\mathcal{M}$ has no degree of SC-autostability.*

2 Preliminaries

Suppose that S is a countable set. A *numbering* of S is a map ν from the set ω of natural numbers *onto* the set S. A numbering ν is a *Friedberg numbering* if ν is 1-1.

γ denotes the standard numbering of the family of all finite subsets of ω. In particular, if $n_0 < n_1 < \ldots < n_k < \omega$, then

$$\gamma\left(2^{n_0} + 2^{n_1} + \ldots + 2^{n_k}\right) = \{n_0, n_1, \ldots, n_k\}.$$

For a set $A \subseteq \omega$, we use $|A|$ to denote the cardinality of A. We assume $\{\varphi_e\}_{e\in\omega}$ to be a standard effective enumeration of all unary partial computable functions. We also assume $\langle\cdot,\cdot\rangle$ to be a standard computable pairing function over ω. For a function f, δf denotes the domain of f and ρf denotes the range of f.

An L-structure $\mathcal{M}$ is an *atomic model* if, for any tuple $\bar{a} = a_0, \ldots, a_n$ from $\mathcal{M}$, there exists an L-formula $\phi(x_0, \ldots, x_n)$ such that $\mathcal{M} \models \phi(\bar{a})$, and every L-formula $\psi(x_0, \ldots, x_n)$ satisfies the following condition: if $\mathcal{M} \models \psi(\bar{a})$, then

$$\mathcal{M} \models \forall x_0 \ldots \forall x_n\, (\phi(x_0, \ldots, x_n) \to \psi(x_0, \ldots, x_n)).$$

Such a formula ϕ is called a *complete formula* of the theory $Th(\mathcal{M})$. Recall Vaught's theorem on the relationship of prime and atomic models (see [13]).

Theorem 2. (Vaught). *Suppose that $\mathcal{M}$ is an L-structure. $\mathcal{M}$ is a prime model if and only if $\mathcal{M}$ is a countable atomic model.*

We identify the set $\omega^{<\omega}$ with a tree with the following ordering: $\sigma \preccurlyeq \tau$ iff σ is an initial segment of τ. For any $\sigma, \tau \in \omega^{<\omega}$, we use $\sigma\widehat{\ }\tau$ to denote the concatenation of σ and τ. Suppose that T is a subtree of $\omega^{<\omega}$. We use $b(\sigma; T)$ to denote the *branching function* of T which is defined as follows. If $\sigma \in T$, then:

$$b(\sigma; T) = |\{n \in \omega : \sigma\widehat{\ }\langle n\rangle \in T\}|.$$

The following is a relativization of the Low Basis Theorem due to Jockusch and Soare (see [14,15]).

Theorem 3. (Jockusch and Soare). *Suppose that $V \subseteq \omega$, and $\mathcal{T}$ is a family of all V-computable finite-branching subtrees T of $\omega^{<\omega}$ with a V-computable branching function $b(\sigma; T)$. Then there exists a Turing degree $\mathbf{d}$ with $\mathbf{d}' \leq \deg_T(V')$ such that every infinite tree $T \in \mathcal{T}$ has a $\mathbf{d}$-computable path. (Such a degree is known as a PA-degree relative to V). Furthermore, there exist two PA-degrees $\mathbf{d}_0$ and $\mathbf{d}_1$ relative to V such that*

$$\forall \mathbf{c}\, ((\mathbf{c} \leq \mathbf{d}_0 \,\&\, \mathbf{c} \leq \mathbf{d}_1) \;\to\; \mathbf{c} \leq \deg_T(V)). \tag{1}$$

We refer the reader to [16,17] for further background on computable and decidable structures.

2.1 Colored Algebras

Let L_{BA} be the language $\{\vee^2, \wedge^2, \mathrm{C}^1; 0, 1\}$. We treat Boolean algebras as L_{BA}-structures. If $\mathcal{L}$ is a linear ordering, then $\mathrm{Int}(\mathcal{L})$ denotes the corresponding *interval algebra*. For a Boolean algebra $\mathcal{B}$, $\mathrm{Atom}(\mathcal{B})$ denotes the set of atoms of $\mathcal{B}$. If a is an element of $\mathcal{B}$, then $\widehat{a}_{\mathcal{B}}$ denotes the *relative algebra* with the universe $\{b \in \mathcal{B} : b \leq_{\mathcal{B}} a\}$. For further information on computable Boolean algebras, see [18].

Let k be a non-zero natural number. A *k-partition* of an element a in a Boolean algebra $\mathcal{B}$ is a sequence $b_1, \dots, b_k$ of pairwise disjoint non-zero elements (i.e., $b_i \wedge b_j = 0$ when $i \neq j$, and $b_i \neq 0$) such that $a = b_1 \vee \dots \vee b_k$. The formula $(b_1, \dots, b_k \mid a)$ denotes that $b_1, \dots, b_k$ is a k-partition of a.

Consider the new computable language $L_0 = L_{BA} \cup \{P_k^1 : k \in \omega\}$, where P_k^1 is a computable predicate.

Definition 3. Let $\mathcal{B}$ be a Boolean algebra. An L_0-structure $\mathcal{B}^c = (\mathcal{B}, P_k)_{k\in\omega}$ is a *colored algebra* if there exists a computable sequence of L_{BA}-formulas $\{\Phi_k(x, \bar{y}_k)\}_{k\in\omega}$ such that for any k, there is a tuple $\bar{b}_k$ from $\mathcal{B}$ with the property

$$\mathcal{B}^c \models \forall x \left(P_k(x) \leftrightarrow \Phi_k(x, \bar{b}_k) \right). \tag{2}$$

Such a sequence $\{\Phi_k\}_{k\in\omega}$ is called a *coloring sequence* of $\mathcal{B}^c$. The Boolean algebra $\mathcal{B}$ is called the *underlying algebra* of $\mathcal{B}^c$.

Colored algebras were introduced in [12]. The informal explanation of the term "colored algebra" is as follows. We treat the predicates P_k as colors and assign these colors to elements of a Boolean algebra $\mathcal{B}$. Note the important difference between our coloring and the graph coloring: we do not require that an element of $\mathcal{B}$ have only one color.

A colored algebra $\mathcal{B}^c$ is *atomic* if its underlying algebra $\mathcal{B}$ is an atomic Boolean algebra. We use $\mathrm{Col}(\mathcal{B}^c)$ to denote the set $\bigcup_{k\in\omega} P_k$ of all colored elements of $\mathcal{B}^c$.

Ershov [19] obtained the following result: a computable atomic Boolean algebra $\mathcal{B}$ is decidable iff the set of atoms $\mathrm{Atom}(\mathcal{B})$ is computable. It is not difficult to show that Ershov's result yields the following corollary.

Proposition 1. *Suppose that $\mathcal{B}^c = (\mathcal{B}, P_k)_{k\in\omega}$ is a computable atomic colored algebra, and $\{\Phi_k(x, \bar{y}_k)\}_{k\in\omega}$ is a coloring sequence of $\mathcal{B}^c$. The structure $\mathcal{B}^c$ is decidable if and only if it satisfies the following conditions:*

(i) the set of atoms $\mathrm{Atom}(\mathcal{B})$ *is computable; and*
(ii) there exists a computable function $g(x)$ such that for any k, the value $g(k)$ is equal to the Gödel number of some tuple $\bar{b}_k$ with the property (2).

3 The Proof of Theorem 1

We will build two decidable atomic colored algebras $\mathcal{A}^c$ and $\mathcal{B}^c$ such that $\mathcal{A}^c$ and $\mathcal{B}^c$ are isomorphic but not computably isomorphic. Lemmas 2, 5, and 8 guarantee that $\mathcal{A}^c$ satisfies Theorem 1. The construction uses the ideas of Miller [10, Theorem 3.4] and Steiner [20, Theorem 2.8].

We fix a computable atomless Boolean algebra $\mathcal{C} = (\omega; \vee, \wedge, \mathrm{C}; 0, 1)$. For clarity, we use $\leq_{\mathcal{C}}$ and $\leq_{\omega}$ when we need to differentiate between the ordering of the Boolean algebra $\mathcal{C}$ and the standard ordering of ω. We also fix a computable subalgebra $\mathcal{C}^0 \leq \mathcal{C}$ such that $\mathcal{C}^0$ is isomorphic to $\mathrm{Int}(\omega)$ and $\mathcal{C}_0$ has a computable set of atoms $\mathrm{Atom}(\mathcal{C}_0) = \{a_0 <_{\omega} a_1 <_{\omega} a_2 <_{\omega} \ldots\}$. For a set $X \subseteq \omega$, we use $\mathrm{gr}(X)$ to denote the subalgebra of $\mathcal{C}$ generated by X.

Consider a computable language $L_c = \left\{P_k^1 : k \in \omega\right\} \cup \left\{Q_{k,j}^1 : k, j \in \omega\right\}$. We will construct two L_c-structures $\mathcal{A}^c = \left(\mathcal{A}, P_k^A, Q_{k,j}^A\right)_{k,j\in\omega}$ and $\mathcal{B}^c = \left(\mathcal{B}, P_k^B, Q_{k,j}^B\right)_{k,j\in\omega}$ such that $\mathcal{A}^c \cong \mathcal{B}^c$, $\mathcal{C}^0 \leq \mathcal{A} \leq \mathcal{C}$, and $\mathcal{C}^0 \leq \mathcal{B} \leq \mathcal{C}$.

At stage s we define computable Boolean algebras $\mathcal{A}_s$ and $\mathcal{B}_s$. The universe of $\mathcal{A}_s$ is denoted by A_s, and the universe of $\mathcal{B}_s$ is denoted by B_s. For every $k \in \omega$, we also define the number $f_{k,s}$, elements $c_{k,s}^A, d_{k,s}^A$ from $\mathcal{A}_s$, and elements $c_{k,s}^B, d_{k,s}^B$ from $\mathcal{B}_s$. In addition, we build the predicates P_k^A, P_k^B, $Q_{k,j}^A$, and $Q_{k,j}^B$ in such a way that, for any predicate R, there is a unique stage t which deals with R.

Notation. We say that we *don't change k-parameters at stage $s+1$* if we define $f_{k,s+1} = f_{k,s}$, $c_{k,s+1}^A = c_{k,s}^A$, $d_{k,s+1}^A = d_{k,s}^A$, $c_{k,s+1}^B = c_{k,s}^B$, and $d_{k,s+1}^B = d_{k,s}^B$.

Construction *Stage* 0. Define $\mathcal{A}_0 = \mathcal{B}_0 = \mathcal{C}^0$. For every $k \in \omega$, set $P_k^A = P_k^B = \{a_{2k}, a_{2k+1}\}$, $f_{k,0} = 0$, $c_{k,0}^A = c_{k,0}^B = a_{2k}$, and $d_{k,0}^A = d_{k,0}^B = a_{2k+1}$.

Stage $s+1$. Suppose that $s = \langle k, t\rangle$. Consider the following four cases.

Case 1. Suppose that $f_{k,s} = 0$, and t is the least natural number such that $\varphi_{k,t}(a_{2k}) \downarrow = a_{2k}$ and $\varphi_{k,t}(a_{2k+1}) \downarrow = a_{2k+1}$. Find the following partitions in the Boolean algebra $\mathcal{C}$: $(c_1, c_2 \mid c_{k,s}^A)$, $(d_1, d_2, d_3 \mid d_{k,s}^A)$, $(c_1', c_2', c_3' \mid c_{k,s}^B)$, and $(d_1', d_2' \mid d_{k,s}^B)$. Define

$$\begin{gathered} \mathcal{A}_{s+1} = \mathrm{gr}(A_s \cup \{c_1, c_2, d_1, d_2, d_3\}), \quad \mathcal{B}_{s+1} = \mathrm{gr}(B_s \cup \{c_1', c_2', c_3', d_1', d_2'\}), \\ Q_{k,t}^A = \{c_1, d_1\}, \quad Q_{k,t+1}^A = \{c_2, d_2\}, \quad Q_{k,t+2}^A = \{d_3\}, \\ Q_{k,t}^B = \{c_1', d_1'\}, \quad Q_{k,t+1}^B = \{c_2', d_2'\}, \quad Q_{k,t+2}^B = \{c_3'\}, \\ Q_{k,t+l+3}^A = Q_{k,t+l+3}^B = \emptyset, \quad l \in \omega. \end{gathered}$$

Set $f_{k,s+1} = 1$. For any l, do not change l-parameters (except the parameter $f_{k,s+1}$).

Case 2. Suppose that $f_{k,s} = 0$, and t is the least number such that $\varphi_{k,t}(a_{2k}) \downarrow = a_{2k+1}$ and $\varphi_{k,t}(a_{2k+1}) \downarrow = a_{2k}$. Find the following partitions in $\mathcal{C}$: $(c_1, c_2 \mid c_{k,s}^A)$, $(d_1, d_2, d_3 \mid d_{k,s}^A)$, $(c_1', c_2' \mid c_{k,s}^B)$, and $(d_1', d_2', d_3' \mid d_{k,s}^B)$. The definitions of $\mathcal{A}_{s+1}$, $f_{k,s+1}$, $Q_{k,t+l}^A$ (where $l \in \omega$), and $Q_{k,t+l}^B$ (where $l \neq 2$) are the same as in the Case 1. Define

$$\mathcal{B}_{s+1} = \mathrm{gr}(B_s \cup \{c_1', c_2', d_1', d_2', d_3'\}), \quad Q_{k,t+2}^B = \{d_3'\}.$$

For any l, don't change l-parameters (except $f_{k,s+1}$).

Case 3. Suppose that $f_{k,s} = 0$ and neither of Cases 1 and 2 hold. Find the following partitions in $\mathcal{C}$: $(c_1, c_2 \mid c^A_{k,s})$, $(d_1, d_2 \mid d^A_{k,s})$, $(c'_1, c'_2 \mid c^B_{k,s})$, and $(d'_1, d'_2 \mid d^B_{k,s})$. Set

$$\begin{aligned}\mathcal{A}_{s+1} = \mathrm{gr}(A_s \cup \{c_1, c_2, d_1, d_2\}), \quad \mathcal{B}_{s+1} = \mathrm{gr}(B_s \cup \{c'_1, c'_2, d'_1, d'_2\}),\\ Q^A_{k,t} = \{c_1, d_1\}, \quad c^A_{k,s+1} = c_2, \quad d^A_{k,s+1} = d_2,\\ Q^B_{k,t} = \{c'_1, d'_1\}, \quad c^B_{k,s+1} = c'_2, \quad d^B_{k,s+1} = d'_2.\end{aligned}$$

For any l, don't change any l-parameters.

Case 4. If $f_{k,s} \neq 0$, then set $\mathcal{A}_{s+1} = \mathcal{A}_s$, $\mathcal{B}_{s+1} = \mathcal{B}_s$, and don't change l-parameters for any l.

We have described the construction. Define Boolean algebras $\mathcal{A} = \mathrm{gr}\left(\bigcup_{s\in\omega} A_s\right)$ and $\mathcal{B} = \mathrm{gr}\left(\bigcup_{s\in\omega} B_s\right)$. It is easy to see that the sets A_s and B_s, $s \in \omega$, are uniformly computable; therefore, we may assume that the structures $\mathcal{A}$ and $\mathcal{B}$ are computable. Consider the structures $\mathcal{A}^c = \left(\mathcal{A}, P^A_k, Q^A_{k,j}\right)_{k,j\in\omega}$ and $\mathcal{B}^c = \left(\mathcal{B}, P^B_k, Q^B_{k,j}\right)_{k,j\in\omega}$. It is not difficult to show that $\mathcal{A}^c$ and $\mathcal{B}^c$ are computable structures.

Verification. It is easy to verify the following properties of the construction.

Lemma 1. (a) *For any $k, j \in \omega$, we have $\left|P^A_k\right| = \left|P^B_k\right| \leq 2$ and $\left|Q^A_{k,j}\right| = \left|Q^B_{k,j}\right| \leq 2$. Moreover, there exist computable functions $f_A(x)$ and $f_B(x)$ such that for any k and j, $P^A_k = \gamma(f_A(\langle k, 0\rangle))$, $Q^A_{k,j} = \gamma(f_A(\langle k, j+1\rangle))$, $P^B_k = \gamma(f_B(\langle k, 0\rangle))$, and $Q^B_{k,j} = \gamma(f_B(\langle k, j+1\rangle))$.*

(b) $\mathrm{Atom}(\mathcal{A}) = \bigcup_{k,j\in\omega} Q^A_{k,j}$ *and* $\mathrm{Atom}(\mathcal{B}) = \bigcup_{k,j\in\omega} Q^B_{k,j}$.

(c) *Suppose that R and S are distinct predicates from the language L_c. Then $R^A \cap S^A = \emptyset$ and $R^B \cap S^B = \emptyset$.*

(d) *Every $k \in \omega$ satisfies one of the following two conditions.*

(d.1) *There exists a number $t \geq 3$ such that each of the algebras $\widehat{(a_{2k})}_{\mathcal{A}}$, $\widehat{(a_{2k+1})}_{\mathcal{A}}$, $\widehat{(a_{2k})}_{\mathcal{B}}$, and $\widehat{(a_{2k+1})}_{\mathcal{B}}$ is isomorphic either to* $\mathrm{Int}(t)$ *or to* $\mathrm{Int}(t+1)$. *Moreover, $\widehat{(a_{2k})}_{\mathcal{A}} \not\cong \widehat{(a_{2k+1})}_{\mathcal{A}}$ and $\widehat{(a_{2k})}_{\mathcal{B}} \not\cong \widehat{(a_{2k+1})}_{\mathcal{B}}$.*

(d.2) *Each of the algebras $\widehat{(a_{2k})}_{\mathcal{A}}$, $\widehat{(a_{2k+1})}_{\mathcal{A}}$, $\widehat{(a_{2k})}_{\mathcal{B}}$, and $\widehat{(a_{2k+1})}_{\mathcal{B}}$ is isomorphic to* $\mathrm{Int}(\omega)$.

(e) $\mathcal{A} = \mathrm{gr}\left(\mathrm{Atom}(\mathcal{A}) \cup \bigcup_{k\in\omega} P^A_k\right)$ *and* $\mathcal{B} = \mathrm{gr}\left(\mathrm{Atom}(\mathcal{B}) \cup \bigcup_{k\in\omega} P^B_k\right)$.

(f) *$\mathcal{A}$ is isomorphic to* $\mathrm{Int}(\omega^2)$.

Lemma 2. *Structures $\mathcal{A}^c$ and $\mathcal{B}^c$ are decidable colored algebras.*

Proof. Consider the function $f_A(x)$ from Lemma 1(a). Define the following sequence of L_{BA}-formulas.

$$\Phi_{k,j}(x, \bar{y}) = \begin{cases} (x = y_1) \vee (x = y_2), & \text{if } |\gamma(f_A(\langle k, j\rangle))| = 2,\\ x = y_1, & \text{if } |\gamma(f_A(\langle k, j\rangle))| = 1,\\ x \neq x, & \text{otherwise.}\end{cases}$$

Lemma 1(a) implies that the sequence $\{\Phi_{k,j}\}_{k,j\in\omega}$ is the coloring sequence for each of the structures $\mathcal{A}^c$ and $\mathcal{B}^c$. Hence, $\mathcal{A}^c$ and $\mathcal{B}^c$ are colored algebras. Lemma 1(a,b) also implies that $\mathcal{A}^c$ and $\mathcal{B}^c$ satisfy the conditions of Proposition 1; therefore, our structures are decidable.

The proof of Lemma 2 actually shows that every computable copy of $\mathcal{A}^c$ satisfies Proposition 1.

Corollary 1. *Every computable copy of $\mathcal{A}^c$ is decidable. In particular, the spectrum* $\mathrm{AutSpec}_{SC}(\mathcal{A}^c)$ *is equal to* $\mathrm{AutSpec}(\mathcal{A}^c)$.

Definition 4. Given an element a from the set $\mathrm{Col}(\mathcal{A}^c)$, we define the L_c-formula $\phi^a(x)$ as follows. First, we find a predicate R from L_c such that $a \in R^A$.

(i) If a is an atom of $\mathcal{A}$, then we define $\phi^a(x) = R(x)$.
(ii) If $a \notin \mathrm{Atom}(\mathcal{A})$, then $R = P_k$ for some $k \in \omega$. Consider the following two cases.
  (ii.a) Suppose that the Boolean algebra $\widehat{a}_{\mathcal{A}}$ is finite and it has exactly t atoms. We define

$$\phi^a(x) = P_k(x) \,\&\, \exists y\,((y \wedge x = y) \,\&\, Q_{k,t-1}(y)) \,\&\, \neg\exists z((z \wedge x = z) \,\&\, Q_{k,t}(z)).$$

  (ii.b) If $\widehat{a}_{\mathcal{A}}$ is an infinite algebra, then we set $\phi^a(x) = P_k(x)$.

Note that Lemma 1(b,c,d) implies that the formulas ϕ^a are well-defined. It is not difficult to prove the following lemma.

Lemma 3. *Suppose that $\mathcal{A}_1^c = \left(\mathcal{A}_1, P_k^{A_1}, Q_{k,j}^{A_1}\right)_{k,j\in\omega}$ is a colored algebra with the universe contained in ω. Suppose also that F is a bijection from* $\mathrm{Col}(\mathcal{A}^c)$ *onto* $\mathrm{Col}(\mathcal{A}_1^c)$ *with the following properties:*

(a) for any $a, b \in \mathrm{Col}(\mathcal{A}^c)$, $a \leq_{\mathcal{A}} b$ iff $F(a) \leq_{\mathcal{A}_1} F(b)$;
(b) for any $a, b \in \mathrm{Col}(\mathcal{A}^c)$, $\mathcal{A}^c \models \phi^a(b)$ iff $\mathcal{A}_1^c \models \phi^a(F(b))$.

Then there exists a unique isomorphism $F^c\colon \mathcal{A}^c \to \mathcal{A}_1^c$ such that $F^c \supseteq F$. Moreover, F^c can be constructed effectively from F and the atomic diagram of $\mathcal{A}_1^c$.

Lemma 4. *Colored algebras $\mathcal{A}^c$ and $\mathcal{B}^c$ are isomorphic but not computably isomorphic. In particular, $\mathcal{A}^c$ is not SC-autostable.*

Proof. It is easy to construct a $\mathbf{0}'$-computable bijection F from $\mathrm{Col}(\mathcal{A}^c)$ onto $\mathrm{Col}(\mathcal{B}^c)$ satisfying the conditions of Lemma 3. Therefore, $\mathcal{A}^c$ and $\mathcal{B}^c$ are $\mathbf{0}'$-computably isomorphic.

Note that for any $k \in \omega$ and any isomorphism $G\colon \mathcal{A}^c \to \mathcal{B}^c$, G maps a_{2k} to a_{2k} and a_{2k+1} to a_{2k+1}, or vice versa. Therefore, Cases 1 and 2 of the construction guarantee that φ_k is not an isomorphism. For example, Case 1 ensures that if $\varphi_k(a_{2k}) = a_{2k}$ and $\varphi_k(a_{2k+1}) = a_{2k+1}$, then the relative algebras $\widehat{(a_{2k})}_{\mathcal{A}}$ and $\widehat{(a_{2k})}_{\mathcal{B}}$ are not isomorphic.

Lemma 5. *$\mathcal{A}^c$ is a prime model.*

Proof. By Theorem 2, it is sufficient to prove that $\mathcal{A}^c$ is an atomic model. Given a tuple $\bar{a} = a_0, \dots, a_n$ from $\mathcal{A}^c$, we will construct a complete L_c-formula $\Phi(\bar{x})$ such that $\mathcal{A}^c \models \Phi(\bar{a})$. Lemma 1(e) implies that we can choose a tuple $\bar{b} = b_0, \dots, b_m$ from $\mathrm{Col}(\mathcal{A}^c)$ such that $\bar{a} \in \mathrm{gr}(\{b_0, \dots, b_m\})$. For $i \leqslant n$, fix an L_{BA}-term $t_i(\bar{y})$ such that $a_i = t_i(\bar{b})$. We define the formula

$$\Phi(\bar{x}) = \exists y_0 \dots \exists y_m \Psi(\bar{x}, y_0, \dots, y_m),$$

where Ψ is the conjunction of the following formulas:

1. $x_i = t_i(\bar{y})$ for $i \leqslant n$,
2. $\phi^{b_j}(y_j)$ for $j \leqslant m$,
3. $y_j \wedge y_k = y_j$ for all j and k with the property $b_j \leq_{\mathcal{C}} b_k$,
4. $y_j \wedge y_k \neq y_j$ for all j and k with the property $b_j \not\leq_{\mathcal{C}} b_k$.

It is easy to see that $\mathcal{A}^c \models \Phi(\bar{a})$. Suppose that $\mathcal{A}^c \models \Phi(\bar{c})$ for some $\bar{c}$. Using Lemma 3, it is not difficult to show that the structures $(\mathcal{A}^c, \bar{a})$ and $(\mathcal{A}^c, \bar{c})$ are isomorphic. Hence, Φ is a complete formula.

Definition 5. Let $\mathcal{A}_1^c = \left(\mathcal{A}_1, P_k^{A_1}, Q_{k,j}^{A_1}\right)_{k,j\in\omega}$ be a copy of $\mathcal{A}^c$ with universe $A_1 \subseteq \omega$. A *special numbering* of $\mathrm{Col}(\mathcal{A}_1^c)$ is a Friedberg numbering of $\mathrm{Col}(\mathcal{A}_1^c)$ with the following properties: ν is a computable function, and for all $x, k, j \in \omega$, if $\nu(x) \in Q_{k,j+1}^{A_1}$, then there exist y_0, y_1, z_0, z_1 such that $y_0 < y_1 < z_0 < z_1 < x$, $P_k^{A_1} = \{\nu(y_0), \nu(y_1)\}$, and $Q_{k,j}^{A_1} = \{\nu(z_0), \nu(z_1)\}$.

Note. If $\mathcal{A}_1^c$ is a computable copy of $\mathcal{A}^c$, then there exists a special numbering ν_1 of $\mathrm{Col}(\mathcal{A}_1^c)$. Moreover, ν_1 can be constructed effectively from the atomic diagram of $\mathcal{A}_1^c$.

We fix a special numbering ν of $\mathrm{Col}(\mathcal{A}^c)$. For a number s, $\nu[s]$ denotes the set $\{\nu(0), \dots, \nu(s)\}$. The following definition is based on [10, Definition 5.1] and [20, Definition 2.16].

Definition 6. Let $\mathcal{A}_1^c$ be a computable copy of $\mathcal{A}^c$ with universe A_1. The universe of the *isomorphism tree* T_{A,A_1} is the set of all functions f with the following properties.

(a) $\delta f = \nu[s]$ for some s, and $\rho f \subseteq A_1$;
(b) Suppose that L_f is a language $\left\{R \in L_c : \exists a \in \delta f \left(a \in R^A\right)\right\}$. Then f is an isomorphic embedding from the L_f-structure $\left(\delta f, L_f^A\right)$ into the L_f-structure $\left(A_1, L_f^{A_1}\right)$.
(c) For every $a, b \in \delta f$, $a \leq_{\mathcal{A}} b$ iff $f(a) \leq_{\mathcal{A}_1} f(b)$.

The ordering of the tree T_{A,A_1} is standard, i.e., $f \preccurlyeq g$ iff $f \subseteq g$. We identify the tree T_{A,A_1} with a computable subtree of $\omega^{<\omega}$. We may assume that T_{A,A_1} is built effectively from the atomic diagram of $\mathcal{A}_1^c$.

The following lemma justifies the choice of the term "isomorphism tree."

Lemma 6. *Suppose that* $\mathcal{A}_1^c$ *is a computable copy of* $\mathcal{A}^c$*. Let* I *be the set of all isomorphisms from* $\mathcal{A}^c$ *onto* $\mathcal{A}_1^c$*, and* P *be the set of all paths through the tree* T_{A,A_1}*. Then there exists a bijection* Ψ *from* P *onto* I *such that for any* $\pi \in P$*,* π *is Turing equivalent to* $\Psi(\pi)$*.*

Proof. Here we omit the details and give only general idea of the proof. Given a path π through T_{A,A_1}, build a bijection $F_\pi\colon \mathrm{Col}(\mathcal{A}^c) \to \mathrm{Col}(\mathcal{A}_1^c)$ such that F_π satisfies the conditions of Lemma 3 and for any $a \in \mathrm{Col}(\mathcal{A}^c)$, there is a finite function $f \prec \pi$ with the property $F_\pi(a) = f(a)$. The function F_π yields an isomorphism $F_\pi^c\colon \mathcal{A}^c \to \mathcal{A}_1^c$. Set $\Psi(\pi) = F_\pi^c$.

It is not difficult to verify the following claim.

Lemma 7. *The tree* T_{A,A_1} *is a finite-branching tree with a computable branching function* $b(\sigma; T_{A,A_1})$*. Moreover, for any* $\sigma \in T_{A,A_1}$*, we have* $b(\sigma; T_{A,A_1}) \leq 2$*.*

Lemma 8. *(1) Suppose that* $\mathbf{d}$ *is a PA-degree relative to* $\emptyset$*. Then* $\mathcal{A}^c$ *is* $\mathbf{d}$*-autostable.*
(2) $\mathcal{A}^c$ *has no degree of SC-autostability.*

Proof. Let $\mathcal{A}_1^c$ be a computable copy of $\mathcal{A}^c$. By Theorem 3 and Lemma 7, the isomorphism tree T_{A,A_1} has a $\mathbf{d}$-computable path π. By Lemma 6, there is a $\mathbf{d}$-computable isomorphism $\Psi(\pi)$ from $\mathcal{A}^c$ onto $\mathcal{A}_1^c$. Therefore, $\mathcal{A}^c$ is $\mathbf{d}$-autostable.

We fix two PA-degrees $\mathbf{d}_0$ and $\mathbf{d}_1$ relative to $\emptyset$ with the property (1) (where $V = \emptyset$). We already proved that $\mathcal{A}^c$ is $\mathbf{d}_0$-SC-autostable and $\mathbf{d}_1$-SC-autostable. Note that (1) implies that if $\mathcal{A}^c$ has a degree of SC-autostability, then $\mathcal{A}^c$ is SC-autostable. Therefore, by Lemma 4, $\mathcal{A}^c$ has no degree of SC-autostability.

This completes the proof of Theorem 1. In conclusion, we formulate some open questions related to Problem 1.

Question 1. Suppose that $\mathcal{M}$ is a decidable almost prime model and $\bar{c}$ is a tuple from $\mathcal{M}$ such that $(\mathcal{M}, \bar{c})$ is a prime model of the theory $Th(\mathcal{M}, \bar{c})$. Let $\mathbf{d}$ be the Turing degree of the collection of complete formulas of $Th(\mathcal{M}, \bar{c})$. Suppose also that $\mathcal{M}$ has the degree of SC-autostability $\mathbf{c}$. Is it possible that $\mathbf{c} < \mathbf{d}$?

Question 2. Is every d.c.e. degree a degree of SC-autostability for some almost prime model?

Note that the positive answer to Question 2 yields the positive answer to Question 1.

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References

1. Fröhlich, A., Shepherdson, J.C.: Effective procedures in field theory. Philos. Trans. Roy. Soc. London. Ser. A. **248**, 407–432 (1956)
2. Mal'tsev, A.I.: Constructive algebras. I. Russ. Math. Surv. **16**, 77–129 (1961)
3. Mal'tsev, A.I.: On recursive abelian groups. Sov. Math. Dokl. **32**, 1431–1434 (1962)
4. Fokina, E.B., Kalimullin, I., Miller, R.: Degrees of categoricity of computable structures. Arch. Math. Logic. **49**, 51–67 (2010)
5. Csima, B.F., Franklin, J.N.Y., Shore, R.A.: Degrees of categoricity and the hyperarithmetic hierarchy. Notre Dame J. Formal Logic. **54**, 215–231 (2013)
6. Bazhenov, N.A.: Degrees of categoricity for superatomic Boolean algebras. Algebra Logic. **52**, 179–187 (2013)
7. Anderson, B.A., Csima, B.F.: Degrees that are not degrees of categoricity. Notre Dame J. Formal Logic. (to appear)
8. Fokina, E., Frolov, A., Kalimullin, I.: Categoricity spectra for rigid structures. Notre Dame J. Formal Logic. (to appear)
9. Goncharov, S.S.: Degrees of autostability relative to strong constructivizations. Proc. Steklov Inst. Math. **274**, 105–115 (2011)
10. Miller, R.: **d**-computable categoricity for algebraic fields. J. Symb. Log. **74**, 1325–1351 (2009)
11. Fokina, E.B., Harizanov, V., Melnikov, A.: Computable model theory. In: Downey, R. (ed.) Turing's Legacy: Developments from Turing Ideas in Logic. Lecture Notes Logic, vol. 42, pp. 124–194. Cambridge University Press, Cambridge (2014)
12. Bazhenov, N.A.: Autostability spectra for Boolean algebras. Algebra Logic. **53**, 502–505 (2014)
13. Chang, C.C., Keisler, H.J.: Model Theory. North-Holland, Amsterdam (1973)
14. Jockusch, C.G., Soare, R.I.: Π^0_1 classes and degrees of theories. Trans. Amer. Math. Soc. **173**, 33–56 (1972)
15. Cenzer, D.: Π^0_1 classes in computability theory. In: Griffor, E.R. (ed.) Handbook of Computability Theory. Studies Logic Foundations Mathematics, vol. 140, pp. 37–85. Elsevier Science B.V., Amsterdam (1999)
16. Ash, C.J., Knight, J.F.: Computable Structures and the Hyperarithmetical Hierarchy. Elsevier Science B.V, Amsterdam (2000)
17. Ershov, Y.L., Goncharov, S.S.: Constructive Models. Kluwer Academic/Plenum Publishers, New York (2000)
18. Goncharov, S.S.: Countable Boolean Algebras and Decidability. Consultants Bureau, New York (1997)
19. Ershov, Y.L.: Decidability of the elementary theory of distributive lattices with relative complements and the theory of filters. Algebra Logic. **3**, 17–38 (1964)
20. Steiner, R.M.: Effective algebraicity. Arch. Math. Logic. **52**, 91–112 (2013)