Chapter 17
A General SEM Framework for Integrating Moderation and Mediation: The Constrained Approach

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Abstract Modeling the combination of latent moderating and mediating effects is a significant issue in the social and behavioral sciences. Chen and Cheng (Structural Equation Modeling: A Multidisciplinary Journal 21: 94–101, 2014) generalized Jöreskog and Yang's (Advanced structural equation modeling: Issues and techniques (pp. 57–88). Mahwah, NJ: Lawrence Erlbaum, 1996) constrained approach to allow for the concurrent modeling of moderation and mediation within the context of SEM. Unfortunately, due to restrictions related to Chen and Cheng's partitioning scheme, their framework cannot completely conceptualize and interpret moderation of indirect effects in a mediated model. In the current study, the Chen and Cheng (abbreviated below as C & C) framework is extended to accommodate situations in which any two pathways that constitute a particular indirect effect in a mediated model can be differentially or collectively moderated by the moderator variable(s). By preserving the inherent advantage of the C & C framework, i.e., the matrix partitioning technique, while at the same time further generalizing its applicability, it is expected that the current framework enhances the potential usefulness of the constrained approach as well as the entire class of the product indicator approaches.

Keywords Moderation • Mediation • The constrained approach

In recent years, the social and behavioral sciences have witnessed a trend toward an increasing number of empirical studies related to mediated moderation (a moderating effect is transmitted through a mediator variable, Baron and Kenny 1986) or moderated mediation (mediation relations are contingent on the level of a moderator, James and Brett 1984). As one example of a study incorporating mediated moderation, Pollack et al. (2012) examined the psychological experience

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of entrepreneurs in response to economic stress, finding that the indirect effect of economic stress on entrepreneurial withdrawal intentions through depressed affect is moderated by the level of business-related social ties. Exemplifying the use of moderated mediation, Cole et al. (2008) investigated affective mechanisms linking dysfunctional behavior to performance in work teams. The researchers specified negative team affective tone as a mediator between dysfunctional team behavior and team performance, whereas nonverbal negative expressivity was found to be contingent on the relation of team affective tone on team performance. In general, examples of empirical studies integrating mediation and moderation are quite abundant in the literature of various disciplines, including business and management (e.g., Cole et al. 2008; Pollack et al. 2012), psychology (e.g., Luszczynska et al. 2010), and marketing communications (e.g., Slater et al. 2007), among others. From the above-cited empirical examples, one can surmise that the substantive variables of interest utilized to establish causal connections in the corresponding theoretical models are generally treated as latent variables, each of which is a theoretical definition of a concept and measured by observed indicators.

Complementing the above empirical research examples are studies which propose various analytical procedures aimed at integrating moderation and mediation in the context of moderated regression or path analysis (e.g., Edwards and Lambert 2007; Fairchild and MacKinnon 2009; Preacher et al. 2007). In particular, Hayes (2013) systemically introduced the concepts of mediation analysis and moderation analysis, as well as their combination (i.e., conditional process analysis) and further demonstrated a computational tool (PROCESS macro) for estimation and inference. However, under regression or path analytical frameworks, all variables are assumed to be treated as manifest (non-latent) variables and measured without error. In the presence of measurement error (i.e., unreliability of measures), the regression coefficient of an interactive or moderating effect may produce a biased estimate and reduce the power of statistical tests of significance. One possible reason for this is that the reliability of the nonlinear term is heavily dependent on the reliability of its individual measures (Busemeyer and Jones 1983; Jaccard and Wan 1995). If the problem of measurement error is not corrected, these frameworks may be of limited use in social and behavioral science research such as psychology. Given that structural equation modeling (SEM) is equipped to deal with multivariate models and multiple measures of latent variables while controlling for measurement errors in observed variables (Bollen and Noble 2011), it should be appropriate to introduce SEM as a preferred alternative to regression or path analysis.

Out of multiple recent lines of SEM-based research, a variety of approaches have been developed for the estimation of latent nonlinear effects. Most approaches can be divided into several major categories: product indicator approaches (e.g., Algina and Moulder 2001; Coenders et al. 2008; Jöreskog and Yang 1996; Kelava and Brandt 2009; Kenny and Judd 1984; Marsh et al. 2004, 2006; Wall and Amemiya 2001), maximum likelihood (ML) estimation methods (e.g., Klein and Moosbrugger 2000; Klein and Muthén 2007; Lee and Zhu 2002), and Bayesian estimation methods (e.g., Arminger and Muthén 1998; Lee et al. 2007), among others. A cursory inspection seems to indicate that most of these approaches have been developed primarily to estimate interaction and/or quadratic effects of exogenous latent variables, leaving nonlinear effects of endogenous latent variables unaccounted for. Unfortunately, this means that there is a growing divide between the capability of available nonlinear SEM approaches, on the one hand, and the interests of social science empirical researchers on the other. In particular, current scholars, while continuing to do interaction research involving exogenous latent variables, have increasingly attempted to model endogenous latent variables as moderators in their theoretical models.

While each of the previously mentioned nonlinear SEM approaches could conceivably be developed to incorporate latent nonlinear relations involving endogenous latent variables, the underlying mathematical theories derived by each of these approaches may become overly complex which could lead to some potential problems. For example, within the classes of ML and Bayesian estimation methods, Wall (2009) mentioned that when latent nonlinear models increase in complexity (e.g., larger number of latent variables), the computational algorithms are likely to fail to reach convergence, and even if they do, may become less numerically precise. As another example, within the class of product indicator approaches, the elaborate and tedious nature of the model specification procedure may limit its potential usefulness. Even so, considering the ever-increasing number of complicated latent nonlinear relations (e.g., a combination of mediation and moderation) appearing in empirical applications, it is still imperative that research efforts be made to develop and generalize current nonlinear SEM approaches.

Pursuing this research aim, Chen and Cheng (2014) generalized Jöreskog and Yang's (1996) constrained approach, one of the product indicator methods, to process interaction and/or quadratic effects involving endogenous latent variables. The Chen and Cheng (abbreviated below as C & C) framework thus allows for the concurrent modeling of moderation and mediation within the context of SEM. Unfortunately, however, due to restrictions related to their partitioning scheme, the C & C framework cannot completely conceptualize and interpret moderation of indirect effects in a mediated model. For example, the two moderated mediation models (i.e., the first and second stage moderation model and the total effect moderation model) from Edwards and Lambert (2007) cannot be embedded into the C & C framework.

In the current study, further progress is made on the C & C framework to accommodate situations in which any two pathways that constitute a particular indirect effect in a mediated model (e.g., simple mediator models, parallel multiple mediator models, serial multiple mediator models) can be differentially or collectively moderated by the moderator variable(s). To simplify the model specification procedure without sacrificing generality, the latent variable versions of all the models from Edwards and Lambert (2007) are utilized to demonstrate the proposed partitioning scheme. The present research leverages key attributes of the two above-mentioned studies to create a highly general latent nonlinear framework. First of all, the models considered by Edwards and Lambert are relatively exhaustive and include many forms that integrate moderation and mediation that may be of interest to empirical researchers. Secondly, the proposed approach retains one of the major advantages of the C & C framework: in contrast to specifying constraints in equation form as in Jöreskog and Yang's (1996) constrained approach, we specify constraints in matrix form to simplify the constraint specification procedure and the process of model specification on the part of the researcher.

17.1 Model Partitioning Scheme

In this section, the partitioning scheme of the current nonlinear framework is demonstrated through latent variable versions of the great majority of the models in Edwards and Lambert (2007), with conceptual and statistical diagrams shown in Fig. 17.1. The present partitioning scheme allows latent variables that are themselves nonlinear functions of other latent variables to also influence other endogenous variables in the model in a nonlinear fashion (e.g., in Models D and H from Edwards and Lambert, the term M is influenced by the interaction term XZ, and in turn, interacts with Z to affect Y).

More specifically, with regard to the structural part of the current partitioning scheme (the conceptual diagram shown in Fig. 17.2), latent variables are partitioned into three subvectors (denoted as $\boldsymbol{\eta}_{\mathbf{F}}$, $\boldsymbol{\eta}_{\mathbf{S}}$ and $\boldsymbol{\eta}_{\mathbf{T}}$, where the subscripts, respectively, stand for "First layer," "Second layer," and "Third layer") to support the integration of the two vectors of latent nonlinear variables (denoted as $\boldsymbol{\eta}_{\mathbf{F}*}$ and $\boldsymbol{\eta}_{\mathbf{S}*}$). Here, $\boldsymbol{\eta}_{\mathbf{F}*}$ and $\boldsymbol{\eta}_{\mathbf{S}*}$ are, respectively, defined as $\mathbf{W_1}\text{vech}(\boldsymbol{\eta}_{\mathbf{F}}\boldsymbol{\eta}_{\mathbf{F}}^{\mathrm{T}})$ and $\mathbf{W_2}\text{vec}(\boldsymbol{\eta}_{\mathbf{S}}\boldsymbol{\eta}_{\mathbf{F}}^{\mathrm{T}})$, where $\mathbf{W_1}$ and $\mathbf{W_2}$ serve as filter matrices (Chen and Cheng 2014) to select a set of latent nonlinear terms that the researcher is interested in. Also note that the vech operator vectorizes a square matrix by stacking the columns from its lower triangle part while the vec operator vectorizes a matrix by stacking its columns (Seber 2007). Examining the interrelations among these partitions, the effects among $\boldsymbol{\eta}_{\mathbf{F}}$, $\boldsymbol{\eta}_{\mathbf{S}}$ and $\boldsymbol{\eta}_{\mathbf{T}}$ are presumed to be unidirectional in the sense that $\boldsymbol{\eta}_{\mathbf{F}}$ can influence $\boldsymbol{\eta}_{\mathbf{S}}$ and/or $\boldsymbol{\eta}_{\mathbf{T}}$, but $\boldsymbol{\eta}_{\mathbf{S}}$ and $\boldsymbol{\eta}_{\mathbf{T}}$ cannot affect $\boldsymbol{\eta}_{\mathbf{F}}$; likewise, $\boldsymbol{\eta}_{\mathbf{S}}$ can influence $\boldsymbol{\eta}_{\mathbf{T}}$, but $\boldsymbol{\eta}_{\mathbf{T}}$ cannot affect $\boldsymbol{\eta}_{\mathbf{S}}$. Meanwhile, with regard to the two vectors of latent nonlinear variables, it is assumed that $\boldsymbol{\eta}_{\mathbf{F}*}$ can influence $\boldsymbol{\eta}_{\mathbf{S}}$ and/or $\boldsymbol{\eta}_{\mathbf{T}}$ while $\boldsymbol{\eta}_{\mathbf{S}*}$ can only influence $\boldsymbol{\eta}_{\mathbf{T}}$. On the whole, as revealed in Fig. 17.2, the current partitioning scheme has the capability to incorporate moderation Models B to H from Edwards and Lambert (2007).

With regard to the measurement part of the present partitioning scheme, observed indicators are partitioned into three subvectors utilized as observed indicators of $\boldsymbol{\eta}_{\mathbf{F}}$, $\boldsymbol{\eta}_{\mathbf{S}}$ and $\boldsymbol{\eta}_{\mathbf{T}}$ (denoted as $\mathbf{y}_{\mathbf{F}}$, $\mathbf{y}_{\mathbf{S}}$ and $\mathbf{y}_{\mathbf{T}}$, respectively) which in turn support the integration of product indicators of $\boldsymbol{\eta}_{\mathbf{F}*}$ and $\boldsymbol{\eta}_{\mathbf{S}*}$ (denoted as $\mathbf{y}_{\mathbf{F}*}$ and $\mathbf{y}_{\mathbf{S}*}$, respectively). Here, $\mathbf{y}_{\mathbf{F}*}$ and $\mathbf{y}_{\mathbf{S}*}$ are, respectively, defined as $\mathbf{W_3}\text{vech}(\mathbf{y}_{\mathbf{F}}\mathbf{y}_{\mathbf{F}}^{\mathrm{T}})$ and $\mathbf{W_4}\text{vec}(\mathbf{y}_{\mathbf{S}}\mathbf{y}_{\mathbf{F}}^{\mathrm{T}})$, where $\mathbf{W_3}$ and $\mathbf{W_4}$ serve as filter matrices (Chen and Cheng 2014) to provide the researcher a convenient way of selecting the product indicators associated with $\boldsymbol{\eta}_{\mathbf{F}*}$ and $\boldsymbol{\eta}_{\mathbf{S}*}$. In the current partitioning scheme, the effect relating $\mathbf{y}_i$ to $\boldsymbol{\eta}_j$ (for $i = \mathbf{F}, \mathbf{S}, \mathbf{T}$ and $j = \mathbf{F}, \mathbf{S}, \mathbf{T}, \mathbf{F}^*, \mathbf{S}^*$) is assumed to be null for $i \neq j$.

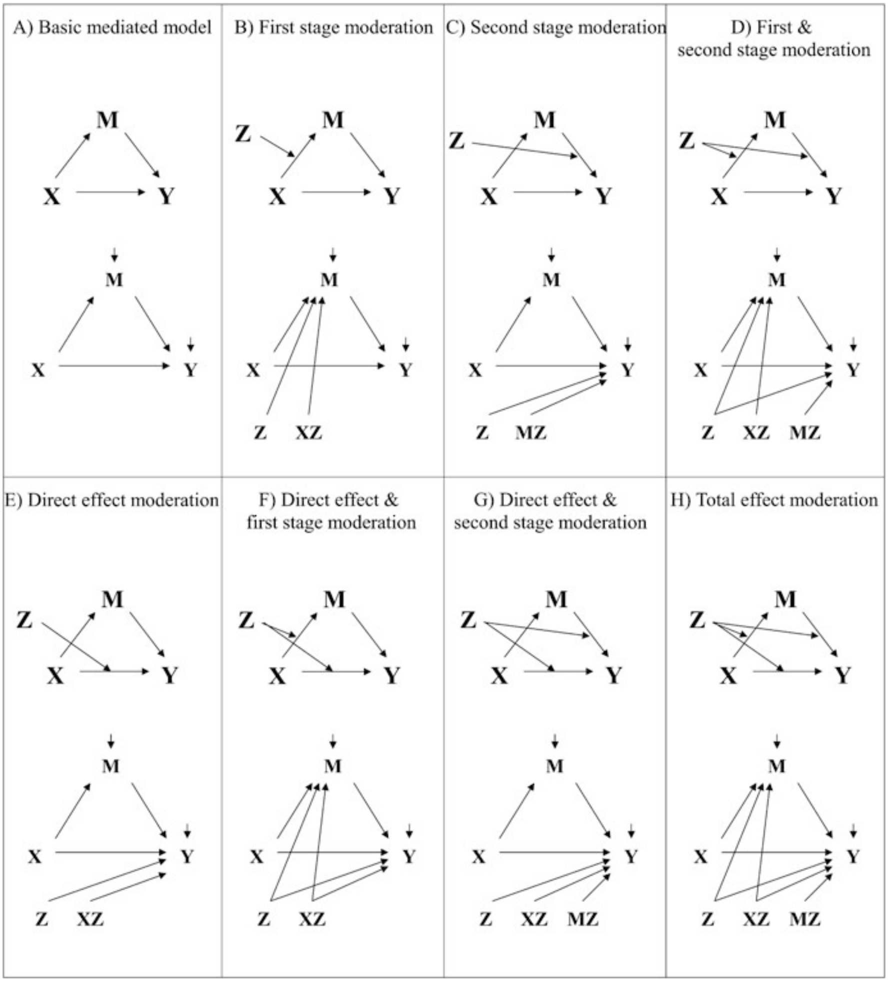

Fig. 17.1 Conceptual and statistical diagrams from Edwards and Lambert

Putting it all together, the current nonlinear framework can be established by integrating the partitioned vectors of latent variables $\boldsymbol{\eta} = \left[\boldsymbol{\eta}_{\mathbf{F}}^{\mathrm{T}} \mid \boldsymbol{\eta}_{\mathbf{S}}^{\mathrm{T}} \mid \boldsymbol{\eta}_{\mathbf{T}}^{\mathrm{T}} \mid \boldsymbol{\eta}_{\mathbf{F}*}^{\mathrm{T}} \mid \boldsymbol{\eta}_{\mathbf{S}*}^{\mathrm{T}}\right]^{\mathrm{T}}$ and observed indicators $\mathbf{y} = \left[\mathbf{y}_{\mathbf{F}}^{\mathrm{T}} \mid \mathbf{y}_{\mathbf{S}}^{\mathrm{T}} \mid \mathbf{y}_{\mathbf{T}}^{\mathrm{T}} \mid \mathbf{y}_{\mathbf{F}*}^{\mathrm{T}} \mid \mathbf{y}_{\mathbf{S}*}^{\mathrm{T}}\right]^{\mathrm{T}}$. The actual construction of this framework and the specification of constraints in matrix form will be illustrated in the next section.

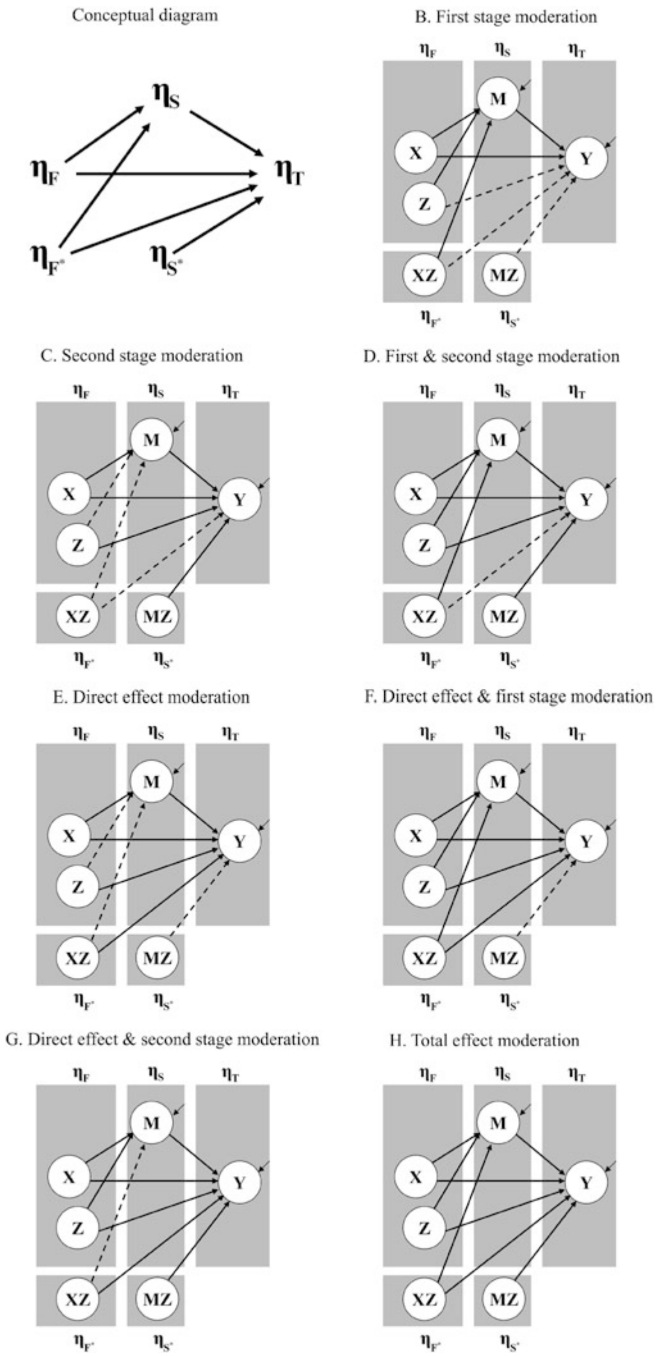

Fig. 17.2 Conceptual diagram and superimposition of the current framework on Models B and H from Edwards and Lambert

17.2 Model Specification

The current partitioned nonlinear framework adopts the notation of Muthén (1984) Case A. The structural and measurement parts, respectively, composed of $(f + s + t + f^* + s^*) \times 1$ and $(p + q + r + p^* + q^*) \times 1$ partitioned vectors of latent variables $\boldsymbol{\eta} = \left[\boldsymbol{\eta}_\mathbf{F}^\mathrm{T} \mid \boldsymbol{\eta}_\mathbf{S}^\mathrm{T} \mid \boldsymbol{\eta}_\mathbf{T}^\mathrm{T} \mid \boldsymbol{\eta}_{\mathbf{F}*}^\mathrm{T} \mid \boldsymbol{\eta}_{\mathbf{S}*}^\mathrm{T}\right]^\mathrm{T}$ and observed indicators $\mathbf{y} = \left[\mathbf{y}_\mathbf{F}^\mathrm{T} \mid \mathbf{y}_\mathbf{S}^\mathrm{T} \mid \mathbf{y}_\mathbf{T}^\mathrm{T} \mid \mathbf{y}_{\mathbf{F}*}^\mathrm{T} \mid \mathbf{y}_{\mathbf{S}*}^\mathrm{T}\right]^\mathrm{T}$, are shown in Eqs. (17.1) and (17.2).

$$\underset{\boldsymbol{\eta}}{\begin{bmatrix} \boldsymbol{\eta}_\mathbf{F} \\ \boldsymbol{\eta}_\mathbf{S} \\ \boldsymbol{\eta}_\mathbf{T} \\ \boldsymbol{\eta}_{\mathbf{F}*} \\ \boldsymbol{\eta}_{\mathbf{S}*} \end{bmatrix}} = \underset{\boldsymbol{\alpha}}{\begin{bmatrix} \boldsymbol{\alpha}_\mathbf{F} \\ \boldsymbol{\alpha}_\mathbf{S} \\ \boldsymbol{\alpha}_\mathbf{T} \\ \mathbf{W}_1\mathbf{L}_f\mathbf{F}_1\mathrm{vec}\left(\boldsymbol{\alpha}_\mathbf{F}\boldsymbol{\alpha}_\mathbf{F}^\mathrm{T}\right) \\ \mathbf{W}_2\mathbf{S}_1\mathrm{vec}\left(\boldsymbol{\alpha}_\mathbf{S}\boldsymbol{\alpha}_\mathbf{F}^\mathrm{T}\right) \end{bmatrix}} + \underset{\mathbf{B}}{\begin{bmatrix} \mathbf{B}_\mathbf{FF} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_\mathbf{SF} & \mathbf{B}_\mathbf{SS} & \mathbf{0} & \mathbf{B}_{\mathbf{SF}*} & \mathbf{0} \\ \mathbf{B}_\mathbf{TF} & \mathbf{B}_\mathbf{TS} & \mathbf{B}_\mathbf{TT} & \mathbf{B}_{\mathbf{TF}*} & \mathbf{B}_{\mathbf{TS}*} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_2\mathbf{S}_1\mathbf{S}_2 & \mathbf{0} & \mathbf{0} & \mathbf{W}_2\mathbf{S}_1\mathbf{S}_3 & \mathbf{0} \end{bmatrix}} \underset{\boldsymbol{\eta}}{\begin{bmatrix} \boldsymbol{\eta}_\mathbf{F} \\ \boldsymbol{\eta}_\mathbf{S} \\ \boldsymbol{\eta}_\mathbf{T} \\ \boldsymbol{\eta}_{\mathbf{F}*} \\ \boldsymbol{\eta}_{\mathbf{S}*} \end{bmatrix}} + \underset{\boldsymbol{\zeta}}{\begin{bmatrix} \boldsymbol{\zeta}_\mathbf{F} \\ \boldsymbol{\zeta}_\mathbf{S} \\ \boldsymbol{\zeta}_\mathbf{T} \\ \boldsymbol{\zeta}_{\mathbf{F}*} \\ \boldsymbol{\zeta}_{\mathbf{S}*} \end{bmatrix}}. \tag{17.1}$$

Here, $\boldsymbol{\alpha}_i$ and $\boldsymbol{\zeta}_i$ are vectors of intercepts and disturbance terms associated with $\boldsymbol{\eta}_i$ (for $i = \mathbf{F}, \mathbf{S}, \mathbf{T}$). $\mathbf{B}_{ij}$ represents the coefficient matrix relating $\boldsymbol{\eta}_i$ to $\boldsymbol{\eta}_j$ (for $i = \mathbf{F}, \mathbf{S}, \mathbf{T}$ and $j = \mathbf{F}, \mathbf{S}, \mathbf{T}, \mathbf{F}^*, \mathbf{S}^*$). More specifically, $\mathbf{B}_{ij}$ (for $i = j$) is specified as a non-null matrix with all diagonal elements restricted to zero; $\mathbf{B}_{ij}$ (for $i \neq j$) is specified as a null or non-null matrix in accordance with the inter-partition relations established for the current partitioning scheme (see Fig. 17.2). The expanded forms of the nonlinear vectors $\boldsymbol{\eta}_{\mathbf{F}*}$ and $\boldsymbol{\eta}_{\mathbf{S}*}$ shown in Eq. (17.1) were obtained by plugging the equations $\boldsymbol{\eta}_\mathbf{F} = \boldsymbol{\alpha}_\mathbf{F} + \mathbf{B}_\mathbf{FF}\boldsymbol{\eta}_\mathbf{F} + \boldsymbol{\zeta}_\mathbf{F}$ and $\boldsymbol{\eta}_\mathbf{S} = \boldsymbol{\alpha}_\mathbf{S} + \mathbf{B}_\mathbf{SF}\boldsymbol{\eta}_\mathbf{F} + \mathbf{B}_\mathbf{SS}\boldsymbol{\eta}_\mathbf{S} + \mathbf{B}_{\mathbf{SF}*}\boldsymbol{\eta}_{\mathbf{F}*} + \boldsymbol{\zeta}_\mathbf{S}$ from Eq. (17.1) into the equations $\boldsymbol{\eta}_{\mathbf{F}*} = \mathbf{W}_1\mathrm{vech}\left(\boldsymbol{\eta}_\mathbf{F}\boldsymbol{\eta}_\mathbf{F}^\mathrm{T}\right)$ and $\boldsymbol{\eta}_{\mathbf{S}*} = \mathbf{W}_2\mathrm{vec}\left(\boldsymbol{\eta}_\mathbf{S}\boldsymbol{\eta}_\mathbf{F}^\mathrm{T}\right)$ established in the previous section. Note that the details of these expansions can be found in Appendix A.

$$\underset{\mathbf{y}}{\begin{bmatrix} \mathbf{y}_\mathbf{F} \\ \mathbf{y}_\mathbf{S} \\ \mathbf{y}_\mathbf{T} \\ \mathbf{y}_{\mathbf{F}*} \\ \mathbf{y}_{\mathbf{S}*} \end{bmatrix}} = \underset{\boldsymbol{\nu}}{\begin{bmatrix} \boldsymbol{\nu}_\mathbf{F} \\ \boldsymbol{\nu}_\mathbf{S} \\ \boldsymbol{\nu}_\mathbf{T} \\ \mathbf{W}_3\mathbf{L}_p\mathrm{vec}\left(\boldsymbol{\nu}_\mathbf{F}\boldsymbol{\nu}_\mathbf{F}^\mathrm{T}\right) \\ \mathbf{W}_4\mathrm{vec}\left(\boldsymbol{\nu}_\mathbf{S}\boldsymbol{\nu}_\mathbf{F}^\mathrm{T}\right) \end{bmatrix}} + \underset{\boldsymbol{\Lambda}}{\begin{bmatrix} \boldsymbol{\Lambda}_\mathbf{FF} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_\mathbf{SS} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_\mathbf{TT} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_3\mathbf{L}_p\mathbf{E}_1 & \mathbf{0} & \mathbf{0} & \mathbf{W}_3\mathbf{L}_p\mathbf{E}_2\mathbf{D}_f\mathbf{W}_1^\mathrm{T} & \mathbf{0} \\ \mathbf{W}_4\mathbf{A}_1 & \mathbf{W}_4\mathbf{A}_2 & \mathbf{0} & \mathbf{0} & \mathbf{W}_4\mathbf{A}_3\mathbf{W}_2^\mathrm{T} \end{bmatrix}} \underset{\boldsymbol{\eta}}{\begin{bmatrix} \boldsymbol{\eta}_\mathbf{F} \\ \boldsymbol{\eta}_\mathbf{S} \\ \boldsymbol{\eta}_\mathbf{T} \\ \boldsymbol{\eta}_{\mathbf{F}*} \\ \boldsymbol{\eta}_{\mathbf{S}*} \end{bmatrix}} + \underset{\boldsymbol{\varepsilon}}{\begin{bmatrix} \boldsymbol{\varepsilon}_\mathbf{F} \\ \boldsymbol{\varepsilon}_\mathbf{S} \\ \boldsymbol{\varepsilon}_\mathbf{T} \\ \boldsymbol{\varepsilon}_{\mathbf{F}*} \\ \boldsymbol{\varepsilon}_{\mathbf{S}*} \end{bmatrix}}. \tag{17.2}$$

Here, $\boldsymbol{\nu}_i$ and $\boldsymbol{\varepsilon}_i$ are vectors of intercepts and measurement errors associated with $\mathbf{y}_i$ (for $i = \mathbf{F}, \mathbf{S}, \mathbf{T}$). Meanwhile, $\boldsymbol{\Lambda}_{ij}$, the factor loading matrix relating $\mathbf{y}_i$ to $\boldsymbol{\eta}_j$ (for $i = \mathbf{F}, \mathbf{S}, \mathbf{T}$ and $j = \mathbf{F}, \mathbf{S}, \mathbf{T}, \mathbf{F}^*, \mathbf{S}^*$), is presumed to be non-null for $i = j$ and null for $i \neq j$. The expanded forms of the nonlinear vectors $\mathbf{y}_{\mathbf{F}*}$ and $\mathbf{y}_{\mathbf{S}*}$ shown in Eq. (17.2) were obtained by plugging the equations $\mathbf{y}_\mathbf{F} = \boldsymbol{\nu}_\mathbf{F} + \boldsymbol{\Lambda}_\mathbf{FF}\boldsymbol{\eta}_\mathbf{F} + \boldsymbol{\varepsilon}_\mathbf{F}$ and $\mathbf{y}_\mathbf{S} = \boldsymbol{\nu}_\mathbf{S} + \boldsymbol{\Lambda}_\mathbf{SS}\boldsymbol{\eta}_\mathbf{S} + \boldsymbol{\varepsilon}_\mathbf{S}$ from Eq. (17.2) into the equations $\mathbf{y}_{\mathbf{F}*} = \mathbf{W}_3\mathrm{vech}\left(\mathbf{y}_\mathbf{F}\mathbf{y}_\mathbf{F}^\mathrm{T}\right)$ and $\mathbf{y}_{\mathbf{S}*} = \mathbf{W}_4\mathrm{vec}\left(\mathbf{y}_\mathbf{S}\mathbf{y}_\mathbf{F}^\mathrm{T}\right)$ established in the previous section. Also note that the details of these expansions can be found in Appendix A.

In the current partitioning nonlinear framework, the vectors of disturbance terms and measurement errors from Eqs. (17.1) and (17.2) are assumed to have a multivariate normal distribution as follows:

$$\begin{bmatrix} \boldsymbol{\zeta}_{\mathbf{F}} \\ \boldsymbol{\zeta}_{\mathbf{S}} \\ \boldsymbol{\zeta}_{\mathbf{T}} \\ \boldsymbol{\varepsilon}_{\mathbf{F}} \\ \boldsymbol{\varepsilon}_{\mathbf{S}} \\ \boldsymbol{\varepsilon}_{\mathbf{T}} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Psi}_{\mathbf{FF}} & & & & & \\ \boldsymbol{\Psi}_{\mathbf{SF}} & \boldsymbol{\Psi}_{\mathbf{SS}} & & & & \\ \boldsymbol{\Psi}_{\mathbf{TF}} & \boldsymbol{\Psi}_{\mathbf{TS}} & \boldsymbol{\Psi}_{\mathbf{TT}} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Theta}_{\mathbf{FF}} & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Theta}_{\mathbf{SF}} & \boldsymbol{\Theta}_{\mathbf{SS}} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Theta}_{\mathbf{TF}} & \boldsymbol{\Theta}_{\mathbf{TS}} & \boldsymbol{\Theta}_{\mathbf{TT}} \end{bmatrix} \right). \tag{17.3}$$

Having developed the generalized nonlinear framework as shown in Eqs. (17.1) and (17.2), it is now possible to proceed to examining the specification of constraints in the context of these two equations. The six partitioned matrices ($\boldsymbol{\alpha}$, $\mathbf{B}$, $\boldsymbol{\Psi}$, $\boldsymbol{\nu}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Theta}$) along with their respective constraints are described in Eqs. (17.4) through (17.7) and Appendix B. Constraints embedded into $\boldsymbol{\alpha}$, $\boldsymbol{\nu}$, $\boldsymbol{\Psi}$ and $\boldsymbol{\Theta}$ are derived based on the assumption of Expression (17.3) as well as extensions of several properties of the multivariate normal distribution (Magnus and Neudecker 1979; Tracy and Sultan 1993; Ghazal and Neudecker 2000). The details of the derivations of these partitioned matrices are available from the author upon request.

The partitioned vector $\boldsymbol{\alpha}$ is a 5×1 array of the form $\left[\boldsymbol{\alpha}_{\mathbf{F}}^{\mathrm{T}} \mid \boldsymbol{\alpha}_{\mathbf{S}}^{\mathrm{T}} \mid \boldsymbol{\alpha}_{\mathbf{T}}^{\mathrm{T}} \mid \boldsymbol{\alpha}_{\mathbf{F}*}^{\mathrm{T}} \mid \boldsymbol{\alpha}_{\mathbf{S}*}^{\mathrm{T}}\right]^{\mathrm{T}}$. $\boldsymbol{\alpha}_{\mathbf{F}*}$ and $\boldsymbol{\alpha}_{\mathbf{S}*}$, the vectors of intercepts of $\boldsymbol{\eta}_{\mathbf{F}*}$ and $\boldsymbol{\eta}_{\mathbf{S}*}$, encompass the non-null expected values of $\boldsymbol{\zeta}_{\mathbf{F}*}$ and $\boldsymbol{\zeta}_{\mathbf{S}*}$ to meet the assumption inherent to SEM that the expected values of disturbance terms are set to null. The resultant form of $\boldsymbol{\alpha}$ is expressed as

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_{\mathbf{F}} \\ \boldsymbol{\alpha}_{\mathbf{S}} \\ \boldsymbol{\alpha}_{\mathbf{T}} \\ \boldsymbol{\alpha}_{\mathbf{F}*} \triangleq \mathbf{W}_1 \mathbf{L}_f \mathbf{F}_1 \mathrm{vec}\left(\boldsymbol{\alpha}_{\mathbf{F}} \boldsymbol{\alpha}_{\mathbf{F}}^{\mathrm{T}} + \boldsymbol{\Psi}_{\mathbf{FF}}\right) \\ \boldsymbol{\alpha}_{\mathbf{S}*} \triangleq \mathbf{W}_2 \mathbf{S}_1 \left[\mathrm{vec}\left(\boldsymbol{\alpha}_{\mathbf{S}} \boldsymbol{\alpha}_{\mathbf{F}}^{\mathrm{T}}\right) + \mathbf{S}_6 \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right) + \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{SF}}\right)\right] \end{bmatrix} \tag{17.4}$$

(here "$\triangleq$" is the symbol for "defined as").

The general form of the partitioned vector $\boldsymbol{\nu}$ is $\left[\boldsymbol{\nu}_{\mathbf{F}}^{\mathrm{T}} \mid \boldsymbol{\nu}_{\mathbf{S}}^{\mathrm{T}} \mid \boldsymbol{\nu}_{\mathbf{T}}^{\mathrm{T}} \mid \boldsymbol{\nu}_{\mathbf{F}*}^{\mathrm{T}} \mid \boldsymbol{\nu}_{\mathbf{S}*}^{\mathrm{T}}\right]^{\mathrm{T}}$, where $\boldsymbol{\nu}_{\mathbf{F}*}$ and $\boldsymbol{\nu}_{\mathbf{S}*}$ are the vectors of intercepts of $\mathbf{y}_{\mathbf{F}*}$ and $\mathbf{y}_{\mathbf{S}*}$. $\boldsymbol{\nu}_{\mathbf{F}*}$ and $\boldsymbol{\nu}_{\mathbf{S}*}$ subsume the non-null expected values of $\boldsymbol{\varepsilon}_{\mathbf{F}*}$ and $\boldsymbol{\varepsilon}_{\mathbf{S}*}$ to avoid a violation of the assumption inherent to SEM that expected values of measurement errors are set to null. The resultant form of $\boldsymbol{\nu}$ is expressed as

$$\boldsymbol{\nu} = \begin{bmatrix} \boldsymbol{\nu}_{\mathbf{F}} \\ \boldsymbol{\nu}_{\mathbf{S}} \\ \boldsymbol{\nu}_{\mathbf{T}} \\ \boldsymbol{\nu}_{\mathbf{F}^*} \triangleq \mathbf{W}_3\mathbf{L}_p \mathrm{vec}\left(\boldsymbol{\nu}_{\mathbf{F}}\boldsymbol{\nu}_{\mathbf{F}}^{\mathrm{T}} + \boldsymbol{\Theta}_{\mathbf{FF}}\right) \\ \boldsymbol{\nu}_{\mathbf{S}^*} \triangleq \mathbf{W}_4 \mathrm{vec}\left(\boldsymbol{\nu}_{\mathbf{S}}\boldsymbol{\nu}_{\mathbf{F}}^{\mathrm{T}} + \boldsymbol{\Theta}_{\mathbf{SF}}\right) \end{bmatrix}. \tag{17.5}$$

The partitioned coefficient matrices **B** and **Λ** are taken directly from Eqs. (17.1) and (17.2) to be expressed by Eqs. (17.6) and (17.7), respectively.

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{\mathbf{FF}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{\mathbf{SF}} & \mathbf{B}_{\mathbf{SS}} & \mathbf{0} & \mathbf{B}_{\mathbf{SF}^*} & \mathbf{0} \\ \mathbf{B}_{\mathbf{TF}} & \mathbf{B}_{\mathbf{TS}} & \mathbf{B}_{\mathbf{TT}} & \mathbf{B}_{\mathbf{TF}^*} & \mathbf{B}_{\mathbf{TS}^*} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{\mathbf{S}^*\mathbf{F}} \triangleq \mathbf{W}_2\mathbf{S}_1\mathbf{S}_2 & \mathbf{0} & \mathbf{0} & \mathbf{B}_{\mathbf{S}^*\mathbf{F}^*} \triangleq \mathbf{W}_2\mathbf{S}_1\mathbf{S}_3 & \mathbf{0} \end{bmatrix}. \tag{17.6}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_{\mathbf{FF}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_{\mathbf{SS}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_{\mathbf{TT}} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Lambda}_{\mathbf{F}^*\mathbf{F}} \triangleq \mathbf{W}_3\mathbf{L}_p\mathbf{E}_1 & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_{\mathbf{F}^*\mathbf{F}^*} \triangleq \mathbf{W}_3\mathbf{L}_p\mathbf{E}_2\mathbf{D}_f\mathbf{W}_1^{\mathrm{T}} & \mathbf{0} \\ \boldsymbol{\Lambda}_{\mathbf{S}^*\mathbf{F}} \triangleq \mathbf{W}_4\mathbf{A}_1 & \boldsymbol{\Lambda}_{\mathbf{S}^*\mathbf{S}} \triangleq \mathbf{W}_4\mathbf{A}_2 & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_{\mathbf{S}^*\mathbf{S}^*} \triangleq \mathbf{W}_4\mathbf{A}_3\mathbf{W}_2^{\mathrm{T}} \end{bmatrix}. \tag{17.7}$$

Due to space considerations, details of the partitioning and constraint specification of the disturbance covariance matrix **Ψ** and the measurement error covariance matrix **Θ** are presented in Appendix B.

On the whole, constraints embedded into the six resultant matrices are represented as either the null matrix or a constraint matrix which is a function of one or more of the submatrices associated with $\boldsymbol{\eta}_{\mathbf{F}}$ and $\boldsymbol{\eta}_{\mathbf{S}}$ (i.e., $\boldsymbol{\alpha}_{\mathbf{F}}$, $\boldsymbol{\alpha}_{\mathbf{S}}$, $\mathbf{B}_{\mathbf{FF}}$, $\mathbf{B}_{\mathbf{SF}}$, $\mathbf{B}_{\mathbf{SS}}$, $\boldsymbol{\Psi}_{\mathbf{FF}}$, $\boldsymbol{\Psi}_{\mathbf{SS}}$, $\boldsymbol{\Psi}_{\mathbf{TF}}$ and $\boldsymbol{\Psi}_{\mathbf{TS}}$) and/or submatrices associated with $\mathbf{y}_{\mathbf{F}}$ and $\mathbf{y}_{\mathbf{S}}$ (i.e., $\boldsymbol{\nu}_{\mathbf{F}}$, $\boldsymbol{\nu}_{\mathbf{S}}$, $\boldsymbol{\Lambda}_{\mathbf{FF}}$, $\boldsymbol{\Lambda}_{\mathbf{SS}}$, $\boldsymbol{\Theta}_{\mathbf{FF}}$, $\boldsymbol{\Theta}_{\mathbf{SF}}$, $\boldsymbol{\Theta}_{\mathbf{SS}}$, $\boldsymbol{\Theta}_{\mathbf{TF}}$ and $\boldsymbol{\Theta}_{\mathbf{TS}}$).

In light of the above discussion, it can be seen that the process of constraint specification is neatly incorporated into the nonlinear framework of the present study. It should be noted that the forms of the derived constraint matrices are kept the same regardless of the number and type of latent nonlinear effects and product indicators selected. In the following section, an artificial model is illustrated to demonstrate the usage and validity of the current approach. This model will be implemented in OpenMx, taking advantage of the capability of this SEM package to readily support model construction in matrix form.

17.3 Artificial Interaction Model

The total effect moderation model (Model H) from Edwards and Lambert (2007) was used to validate the extended partitioned scheme detailed earlier. It was assumed that each latent variable was associated with two observed indicators. For a complete graphical depiction of the illustrated model, refer to the path diagram in Fig. 17.3.

In this model, the disturbance terms ($\zeta_1, \ldots, \zeta_4$) and measurement errors ($\varepsilon_1, \ldots, \varepsilon_8$) were assumed to have a multivariate normal distribution with mean zero and have a covariance matrix composed of the elements in $diag(\psi_{11}, \ldots, \psi_{44}, \theta_{11}, \ldots, \theta_{88})$ and the covariance term ψ_{21}. Simulated data were generated by PRELIS 2 with sample size set at 500 observations, while the population parameter values were shown in Table 17.1. The sample mean and covariance matrix for each replication were calculated from noncentered observed variables. Estimates and standard errors of parameters from maximum likelihood (ML) estimation were taken for the first 500 replications in which the estimation procedure converged.

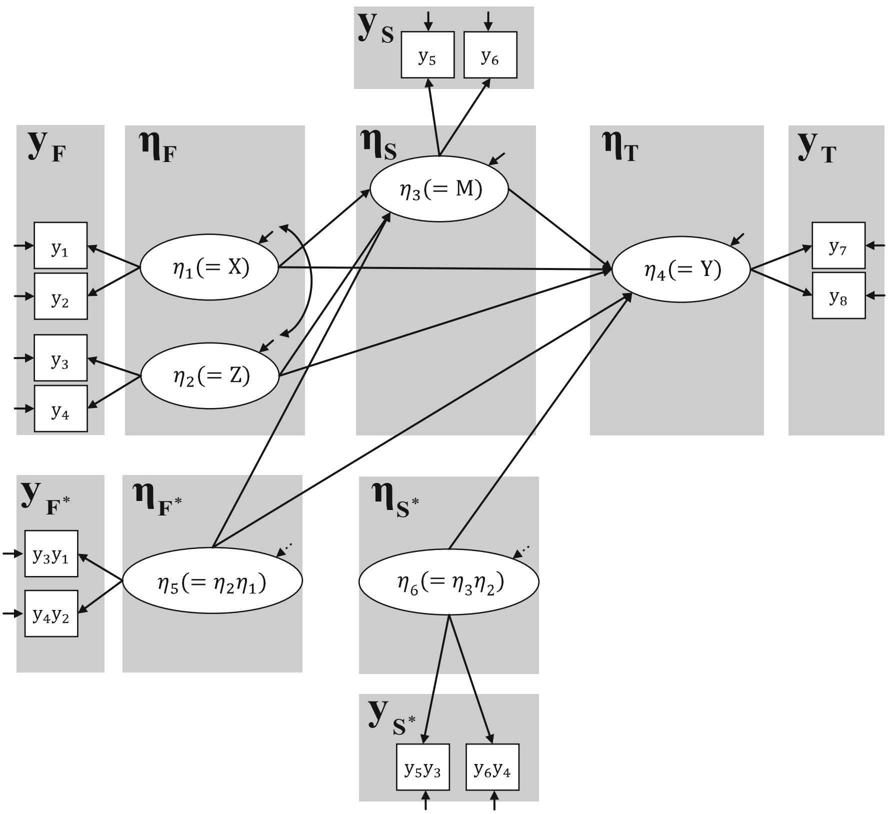

Fig. 17.3 Artificial interactive model

Table 17.1 Total effect moderation model

Parameter (true value)	Bias	*SE*	*SD*	Parameter (true value)	Bias	*SE*	*SD*
$\alpha_1(0.00)$	–	–	–	$\beta_{31}(0.40)$	0.001	0.048	0.070
$\alpha_2(0.00)$	–	–	–	$\beta_{32}(0.40)$	−0.002	0.049	0.064
$\alpha_3(0.00)$	–	–	–	$\beta_{41}(0.40)$	0.002	0.086	0.098
$\alpha_4(0.00)$	–	–	–	$\beta_{42}(0.40)$	−0.007	0.084	0.094
				$\beta_{43}(0.40)$	0.015	0.128	0.143
$\upsilon_1(1.00)$	0.000	0.037	0.064	$\beta_{35}(0.20)$	−0.004	0.041	0.056
$\upsilon_2(1.00)$	0.000	0.030	0.054	$\beta_{45}(0.20)$	0.002	0.096	0.109
$\upsilon_3(1.00)$	0.002	0.031	0.063	$\beta_{46}(0.20)$	0.003	0.084	0.093
$\upsilon_4(1.00)$	0.001	0.025	0.053				
$\upsilon_5(1.00)$	0.001	0.040	0.062	$\psi_{11}(1.00)$	0.002	0.097	0.142
$\upsilon_6(1.00)$	0.001	0.032	0.055	$\psi_{21}(0.30)$	0.000	0.039	0.069
$\upsilon_7(1.00)$	0.004	0.059	0.073	$\psi_{22}(1.00)$	−0.002	0.091	0.132
$\upsilon_8(1.00)$	0.000	0.047	0.056	$\psi_{33}(0.36)$	−0.007	0.060	0.077
				$\psi_{44}(0.36)$	−0.015	0.073	0.076
$\lambda_{11}(1.00)$	–	–	–				
$\lambda_{21}(0.70)$	0.006	0.053	0.072	$\theta_{11}(0.51)$	−0.003	0.074	0.093
$\lambda_{32}(1.00)$	–	–	–	$\theta_{22}(0.51)$	0.000	0.044	0.059
$\lambda_{42}(0.70)$	0.003	0.051	0.069	$\theta_{33}(0.51)$	−0.010	0.076	0.091
$\lambda_{53}(1.00)$	–	–	–	$\theta_{44}(0.51)$	−0.004	0.043	0.052
$\lambda_{63}(0.70)$	0.010	0.047	0.070	$\theta_{55}(0.51)$	−0.001	0.059	0.075
$\lambda_{74}(1.00)$	–	–	–	$\theta_{66}(0.51)$	−0.006	0.037	0.046
$\lambda_{84}(0.70)$	0.002	0.040	0.042	$\theta_{77}(0.51)$	−0.003	0.070	0.071
				$\theta_{88}(0.51)$	0.001	0.044	0.041

Notes. SD = empirical standard error; *SE* = average estimated standard error. *Rate of fully proper solutions* = 95.8 %. To fix scales, α_1 to α_4 are set to zero, and λ_{11}, λ_{32}, λ_{53} and λ_{74} are set to one

Due to space considerations, the OpenMx syntax used in implementing the total effect moderation model example is not provided here, but will be provided by the author upon request.

In order to confirm the validity of the current approach, the bias (calculated as the difference between the mean of the 500 parameter estimates and the population parameter value), the empirical standard deviation (*SD*), and average estimated standard error (*SE*) for each parameter from the simulation study are presented in Table 17.1.

The simulation results indicated that the mean estimates of all parameters were close to the population parameter values and the absolute biases were less than 0.02, confirming the validity of the current approach. The average estimated standard errors (*SE*) tended to be smaller than empirical standard deviations (*SD*), meaning that the estimated standard errors underestimated empirical standard deviation and thus wrongly inflated the level of significance of testing parameters (increasing the Type I error rate). It is important to mention that various simulation studies (cf., Yang-Wallentin and Jöreskog 2001; Moosbrugger et al. 2009; Chen and Cheng 2014) similarly showed that average estimated standard errors of parameters were underestimated when estimating the interaction and/or quadratic effect(s) of latent variables under the constrained approach. Thus, more research is needed to determine whether or not this phenomenon is an inherent feature of the constrained approach.

17.4 Conclusion

The current study established a more general latent nonlinear framework for integrating moderation and mediation. The proposed matrix specification scheme encapsulates many possible forms of moderation models that can accommodate situations in which any two pathways that constitute a particular indirect effect in a mediated model can be differentially or collectively moderated by the moderator variable(s), thereby further broadening the potential usefulness of the class of product indicator approaches, most notably the constrained approach.

Although the proposed framework provides a major step forward in the development of the constrained approach, there are a few caveats to take into consideration. First and foremost, the constraint specification procedure of the proposed framework is based on the assumption that $\boldsymbol{\zeta}_{\mathbf{F}}$, $\boldsymbol{\zeta}_{\mathbf{S}}$, $\boldsymbol{\zeta}_{\mathbf{T}}$, $\boldsymbol{\varepsilon}_{\mathbf{F}}$, $\boldsymbol{\varepsilon}_{\mathbf{S}}$ and $\boldsymbol{\varepsilon}_{\mathbf{T}}$ are multivariate normally distributed. If this assumption is violated, applying the proposed approach might result in unknown bias in the parameter estimates. Secondly, the current framework, focusing on the specification of latent interaction and/or quadratic effects, has no capability to deal with higher-order latent nonlinear effects. Finally, it is important to keep in mind that the current framework estimates latent nonlinear effects through a more generalized version of Jöreskog and Yang's (1996) constrained approach, which is but one of a multitude of approaches (e.g., the unconstrained approach, Marsh et al. 2004, 2006; the latent moderated structural equations approach, Klein and Moosbrugger 2000) that can potentially be used. Further research should be conducted with the aim of possibly developing other approaches that can likewise be used to estimate complex latent nonlinear effects.

Appendix A: Expansions of $\boldsymbol{\eta}_{\mathbf{F}*}$, $\boldsymbol{\eta}_{\mathbf{S}*}$, $\mathbf{y}_{\mathbf{F}*}$, and $\mathbf{y}_{\mathbf{S}*}$

Before $\boldsymbol{\eta}_{\mathbf{F}^*}$, $\boldsymbol{\eta}_{\mathbf{S}^*}$, $\mathbf{y}_{\mathbf{F}^*}$, and $\mathbf{y}_{\mathbf{S}^*}$ are discussed in the subsequent paragraph, it is necessary to gain familiarity with the notation of four basic types of matrices. The $n \times n$ identity matrix will be denoted as $\mathbf{I}_n$ and the $mn \times mn$ commutation matrix will be indicated as $\mathbf{K}_{mn}$ for $m \neq n$ and $\mathbf{K}_n$ for $m = n$ (see definition 3.1 of Magnus and Neudecker 1979). The $n\,(n+1)\,/2 \times n^2$ elimination matrix and $n^2 \times n\,(n+1)\,/2$ duplication matrix will be denoted as $\mathbf{L}_n$ and $\mathbf{D}_n$, respectively (see definitions 3.1a and 3.2a of Magnus and Neudecker 1980).

The expansions of $\boldsymbol{\eta}_{\mathbf{F}^*}$, $\boldsymbol{\eta}_{\mathbf{S}^*}$, $\mathbf{y}_{\mathbf{F}^*}$, and $\mathbf{y}_{\mathbf{S}^*}$ can be obtained with the aid of several theorems and properties of the Kronecker product, vech and vec operators shown in Magnus and Neudecker (1980, 1988). The resulting forms of these expansions are expressed as below.

$$\boldsymbol{\eta}_{\mathbf{F}^*} = \mathbf{W}_1\mathbf{L}_f\mathbf{F}_1\text{vec}\left(\boldsymbol{\alpha}_{\mathbf{F}}\boldsymbol{\alpha}_{\mathbf{F}}^{\mathrm{T}}\right) + \boldsymbol{\zeta}_{\mathbf{F}^*},$$

$$\boldsymbol{\eta}_{\mathbf{S}^*} = \mathbf{W}_2\mathbf{S}_1\text{vec}\left(\boldsymbol{\alpha}_{\mathbf{S}}\boldsymbol{\alpha}_{\mathbf{F}}^{\mathrm{T}}\right) + \mathbf{W}_2\mathbf{S}_1\mathbf{S}_2\boldsymbol{\eta}_{\mathbf{F}} + \mathbf{W}_2\mathbf{S}_1\mathbf{S}_3\boldsymbol{\eta}_{\mathbf{F}}^* + \boldsymbol{\zeta}_{\mathbf{S}^*},$$

$$\mathbf{y}_{\mathbf{F}^*} = \mathbf{W}_3\mathbf{L}_p\text{vec}\left(\boldsymbol{\nu}_{\mathbf{F}}\boldsymbol{\nu}_{\mathbf{F}}^{\mathrm{T}}\right) + \mathbf{W}_3\mathbf{L}_p\mathbf{E}_1\boldsymbol{\eta}_{\mathbf{F}} + \mathbf{W}_3\mathbf{L}_p\mathbf{E}_2\mathbf{D}_f\mathbf{W}_1^{\mathrm{T}}\boldsymbol{\eta}_{\mathbf{F}^*} + \boldsymbol{\varepsilon}_{\mathbf{F}^*},$$

$$\mathbf{y}_{\mathbf{S}^*} = \mathbf{W}_4\text{vec}\left(\boldsymbol{\nu}_{\mathbf{S}}\boldsymbol{\nu}_{\mathbf{F}}^{\mathrm{T}}\right) + \mathbf{W}_4\mathbf{A}_1\boldsymbol{\eta}_{\mathbf{F}} + \mathbf{W}_4\mathbf{A}_2\boldsymbol{\eta}_{\mathbf{S}} + \mathbf{W}_4\mathbf{A}_3\mathbf{W}_2^{\mathrm{T}}\boldsymbol{\eta}_{\mathbf{S}^*} + \boldsymbol{\varepsilon}_{\mathbf{S}^*},$$

where $\boldsymbol{\zeta}_{\mathbf{F}^*} \triangleq \mathbf{W}_1\mathbf{L}_f\mathbf{F}_1\left(\mathbf{F}_2\boldsymbol{\zeta}_{\mathbf{F}} + \boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}}\right)$,

$$\boldsymbol{\zeta}_{\mathbf{S}^*} \triangleq \mathbf{W}_2\mathbf{S}_1\left[\mathbf{S}_4\boldsymbol{\zeta}_{\mathbf{F}} + \mathbf{S}_5\boldsymbol{\zeta}_{\mathbf{S}} + \mathbf{S}_6\left(\boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}}\right) + \mathbf{S}_7\left(\boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}}\right) + \boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{S}}\right],$$

$$\boldsymbol{\varepsilon}_{\mathbf{F}^*} \triangleq \mathbf{W}_3\mathbf{L}_p\left[\mathbf{E}_3\boldsymbol{\varepsilon}_{\mathbf{F}} + \mathbf{E}_4\left(\boldsymbol{\varepsilon}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}}\right) + \mathbf{E}_5\left(\boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\varepsilon}_{\mathbf{F}}\right) + \boldsymbol{\varepsilon}_{\mathbf{F}} \otimes \boldsymbol{\varepsilon}_{\mathbf{F}}\right],$$

$$\boldsymbol{\varepsilon}_{\mathbf{S}^*} \triangleq \mathbf{W}_4\Big[\mathbf{A}_4\boldsymbol{\varepsilon}_{\mathbf{F}} + \mathbf{A}_5\boldsymbol{\varepsilon}_{\mathbf{S}} + \mathbf{A}_6\left(\boldsymbol{\varepsilon}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}}\right) + \mathbf{A}_7\left(\boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\varepsilon}_{\mathbf{S}}\right) + \mathbf{A}_8\left(\boldsymbol{\varepsilon}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{S}}\right)$$
$$+ \mathbf{A}_9\left(\boldsymbol{\varepsilon}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}} \otimes \boldsymbol{\zeta}_{\mathbf{F}}\right) + \boldsymbol{\varepsilon}_{\mathbf{F}} \otimes \boldsymbol{\varepsilon}_{\mathbf{S}}\Big]$$

(here "$\triangleq$" is the symbol for "defined as") in which $\mathbf{F}_1 = [(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}}) \otimes (\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})]^{-1}$, $\mathbf{F}_2 = \mathbf{I}_f \otimes \boldsymbol{\alpha}_{\mathbf{F}} + \boldsymbol{\alpha}_{\mathbf{F}} \otimes \mathbf{I}_f$, $\mathbf{S}_1 = [(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}}) \otimes (\mathbf{I}_s - \mathbf{B}_{\mathbf{SS}})]^{-1}$, $\mathbf{S}_2 = \boldsymbol{\alpha}_{\mathbf{F}} \otimes \mathbf{B}_{\mathbf{SF}}$, $\mathbf{S}_3 = \boldsymbol{\alpha}_{\mathbf{F}} \otimes \mathbf{B}_{\mathbf{SF}^*}$, $\mathbf{S}_4 = \mathbf{I}_f \otimes [\boldsymbol{\alpha}_{\mathbf{S}} + \mathbf{B}_{\mathbf{SF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1}\boldsymbol{\alpha}_{\mathbf{F}} + \mathbf{B}_{\mathbf{SF}^*}\mathbf{W}_1\mathbf{L}_f\mathbf{F}_1(\boldsymbol{\alpha}_{\mathbf{F}} \otimes \boldsymbol{\alpha}_{\mathbf{F}})]$, $\mathbf{S}_5 = \boldsymbol{\alpha}_{\mathbf{F}} \otimes \mathbf{I}_s$, $\mathbf{S}_6 = \mathbf{I}_f \otimes [\mathbf{B}_{\mathbf{SF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1} + \mathbf{B}_{\mathbf{SF}^*}\mathbf{W}_1\mathbf{L}_f\mathbf{F}_1\mathbf{F}_2]$, $\mathbf{S}_7 = \mathbf{I}_f \otimes (\mathbf{B}_{\mathbf{SF}^*}\mathbf{W}_1\mathbf{L}_f\mathbf{F}_1)$, $\mathbf{E}_1 = \boldsymbol{\Lambda}_{\mathbf{FF}} \otimes \boldsymbol{\nu}_{\mathbf{F}} + \boldsymbol{\nu}_{\mathbf{F}} \otimes \boldsymbol{\Lambda}_{\mathbf{FF}}$, $\mathbf{E}_2 = \boldsymbol{\Lambda}_{\mathbf{FF}} \otimes \boldsymbol{\Lambda}_{\mathbf{FF}}$, $\mathbf{E}_3 = \mathbf{I}_p \otimes [\boldsymbol{\nu}_{\mathbf{F}} + \boldsymbol{\Lambda}_{\mathbf{FF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1}\boldsymbol{\alpha}_{\mathbf{F}}] + [\boldsymbol{\nu}_{\mathbf{F}} + \boldsymbol{\Lambda}_{\mathbf{FF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1}\boldsymbol{\alpha}_{\mathbf{F}}] \otimes \mathbf{I}_p$, $\mathbf{E}_4 = \mathbf{I}_p \otimes (\boldsymbol{\Lambda}_{\mathbf{FF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1})$, $\mathbf{E}_5 = (\boldsymbol{\Lambda}_{\mathbf{FF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1}) \otimes \mathbf{I}_p$, $\mathbf{A}_1 = \boldsymbol{\Lambda}_{\mathbf{FF}} \otimes \boldsymbol{\nu}_{\mathbf{S}}$, $\mathbf{A}_2 = \boldsymbol{\nu}_{\mathbf{F}} \otimes \boldsymbol{\Lambda}_{\mathbf{SS}}$, $\mathbf{A}_3 = \boldsymbol{\Lambda}_{\mathbf{FF}} \otimes \boldsymbol{\Lambda}_{\mathbf{SS}}$, $\mathbf{A}_4 = \mathbf{I}_p \otimes [\boldsymbol{\nu}_{\mathbf{S}} + \boldsymbol{\Lambda}_{\mathbf{SS}}(\mathbf{I}_s - \mathbf{B}_{\mathbf{SS}})^{-1}(\boldsymbol{\alpha}_{\mathbf{S}} + \mathbf{B}_{\mathbf{SF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1}\boldsymbol{\alpha}_{\mathbf{F}} + \mathbf{B}_{\mathbf{SF}^*}\mathbf{W}_1\mathbf{L}_f\mathbf{F}_1(\boldsymbol{\alpha}_{\mathbf{F}} \otimes \boldsymbol{\alpha}_{\mathbf{F}}))]$, $\mathbf{A}_5 = [\boldsymbol{\nu}_{\mathbf{F}} + \boldsymbol{\Lambda}_{\mathbf{FF}}(\mathbf{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1}\boldsymbol{\alpha}_{\mathbf{F}}] \otimes \mathbf{I}_q$, $\mathbf{A}_6 = \mathbf{I}_p \otimes [\boldsymbol{\Lambda}_{\mathbf{SS}}(\mathbf{I}_s - \mathbf{B}_{\mathbf{SS}})^{-1}(\mathbf{B}_{\mathbf{SF}}(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}})^{-1} + \mathbf{B}_{\mathbf{SF}^*}\mathbf{W}_1\mathbf{L}_f\mathbf{F}_1\mathbf{F}_2)]$, $\mathbf{A}_7 = (\boldsymbol{\Lambda}_{\mathbf{FF}}\left(\mathrm{I}_f - \mathbf{B}_{\mathbf{FF}}\right)^{-1}) \otimes \mathbf{I}_q$, $\mathbf{A}_8 = \mathbf{I}_p \otimes (\boldsymbol{\Lambda}_{\mathbf{SS}}(\mathbf{I}_s - \mathbf{B}_{\mathbf{SS}})^{-1})$ and $\mathbf{A}_9 = \mathbf{I}_p \otimes (\boldsymbol{\Lambda}_{\mathbf{SS}}(\mathbf{I}_s - \mathbf{B}_{\mathbf{SS}})^{-1}\mathbf{B}_{\mathbf{SF}^*}\mathbf{W}_1\mathbf{L}_f\mathbf{F}_1)$.

Here, $\boldsymbol{\zeta}_{\mathbf{F}^*}$ and $\boldsymbol{\zeta}_{\mathbf{S}^*}$ are vectors of disturbance terms of $\boldsymbol{\eta}_{\mathbf{F}^*}$ and $\boldsymbol{\eta}_{\mathbf{S}^*}$, while $\boldsymbol{\varepsilon}_{\mathbf{F}^*}$ and $\boldsymbol{\varepsilon}_{\mathbf{S}^*}$ are vectors of measurement errors of $\mathbf{y}_{\mathbf{F}^*}$ and $\mathbf{y}_{\mathbf{S}^*}$. Meanwhile, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{S}_1$ to $\mathbf{S}_7$, $\mathbf{E}_1$ to $\mathbf{E}_5$, and $\mathbf{A}_1$ to $\mathbf{A}_9$ are all constant matrices.

Appendix B: Partitioned Matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Theta}$

The disturbance covariance matrix $\boldsymbol{\Psi}$ is partitioned into a 5×5 array of submatrices as expressed below:

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_{\mathbf{FF}} & & & & \\ \boldsymbol{\Psi}_{\mathbf{SF}} & \boldsymbol{\Psi}_{\mathbf{SS}} & & & \\ \boldsymbol{\Psi}_{\mathbf{TF}} & \boldsymbol{\Psi}_{\mathbf{TS}} & \boldsymbol{\Psi}_{\mathbf{TT}} & & \\ \boldsymbol{\Psi}_{\mathbf{F^*F}} & \boldsymbol{\Psi}_{\mathbf{F^*S}} & \boldsymbol{\Psi}_{\mathbf{F^*T}} & \boldsymbol{\Psi}_{\mathbf{F^*F^*}} & \\ \boldsymbol{\Psi}_{\mathbf{S^*F}} & \boldsymbol{\Psi}_{\mathbf{S^*S}} & \boldsymbol{\Psi}_{\mathbf{S^*T}} & \boldsymbol{\Psi}_{\mathbf{S^*F^*}} & \boldsymbol{\Psi}_{\mathbf{S^*S^*}} \end{bmatrix},$$

where $\boldsymbol{\Psi}_{\mathbf{F^*F}} \triangleq \mathbf{W_1 L}_f \mathbf{F_1 F_2} \boldsymbol{\Psi}_{\mathbf{FF}}$, $\boldsymbol{\Psi}_{\mathbf{F^*S}} \triangleq \mathbf{W_1 L}_f \mathbf{F_1 F_2} (\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}}$, $\boldsymbol{\Psi}_{\mathbf{F^*T}} \triangleq \mathbf{W_1 L}_f \mathbf{F_1 F_2} (\boldsymbol{\Psi}_{\mathbf{TF}})^{\mathrm{T}}$,

$$\boldsymbol{\Psi}_{\mathbf{F^*F^*}} \triangleq \mathbf{W_1 L}_f \mathbf{F_1} \left[\mathbf{F_2} \boldsymbol{\Psi}_{\mathbf{FF}} \mathbf{F_2^{\mathrm{T}}} + \left(\mathbf{I}_{f^2} + \mathbf{K}_{ff}\right) (\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}})\right] \mathbf{F_1^{\mathrm{T}} L}_f^{\mathrm{T}} \mathbf{W_1^{\mathrm{T}}},$$

$$\begin{aligned} \boldsymbol{\Psi}_{\mathbf{S^*F}} \triangleq \mathbf{W_2 S_1}\big[&\mathbf{S_4} \boldsymbol{\Psi}_{\mathbf{FF}} + \mathbf{S_5} \boldsymbol{\Psi}_{\mathbf{SF}} \\ &+ \mathbf{S_7} \Delta_{f^3 \times f} \left[\mathbf{K}_{ff^3} \left(\mathrm{vec}\left(\left(\mathbf{I}_{f^2} + \mathbf{K}_{ff}\right) (\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}})\right)\right.\right. \\ &+ \left.\left. \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{FF}}) \otimes \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{FF}})\right)\right]\big], \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Psi}_{\mathbf{S^*S}} \triangleq \mathbf{W_2 S_1}\big[&\mathbf{S_4} (\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}} + \mathbf{S_5} \boldsymbol{\Psi}_{\mathbf{SS}} \\ &+ \mathbf{S_7} \Delta_{f^3 \times s} \left[\mathbf{K}_{sf^3} \left(\mathrm{vec}\left((\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{SF}}) + \mathbf{K}_{fs} (\boldsymbol{\Psi}_{\mathbf{SF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}})\right)\right.\right. \\ &+ \left.\left. \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{FF}}) \otimes \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{SF}})\right)\right]\big], \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Psi}_{\mathbf{S^*T}} \triangleq \mathbf{W_2 S_1}\big[&\mathbf{S_4} (\boldsymbol{\Psi}_{\mathbf{TF}})^{\mathrm{T}} + \mathbf{S_5} (\boldsymbol{\Psi}_{\mathbf{TS}})^{\mathrm{T}} \\ &+ \mathbf{S_7} \Delta_{f^3 \times t} \left[\mathbf{K}_{tf^3} \left(\mathrm{vec}\left((\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{TF}}) + \mathbf{K}_{ft} (\boldsymbol{\Psi}_{\mathbf{TF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}})\right)\right.\right. \\ &+ \left.\left. \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{FF}}) \otimes \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{TF}})\right)\right]\big], \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Psi}_{\mathbf{S^*F^*}} \triangleq \mathbf{W_2 S_1}\big[&\mathbf{S_4} \boldsymbol{\Psi}_{\mathbf{FF}} \mathbf{F_2^{\mathrm{T}}} + \mathbf{S_5} \boldsymbol{\Psi}_{\mathbf{SF}} \mathbf{F_2^{\mathrm{T}}} \\ &+ \mathbf{S_7} \Delta_{f^3 \times f} \left[\mathbf{K}_{ff^3} \left(\mathrm{vec}\left(\left(\mathbf{I}_{f^2} + \mathbf{K}_{ff}\right) (\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}})\right) + \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{FF}}) \otimes \mathrm{vec}\,(\boldsymbol{\Psi}_{\mathbf{FF}})\right)\right] \mathbf{F_2^{\mathrm{T}}} \\ &+ \mathbf{S_6} \left(\mathbf{I}_{f^2} + \mathbf{K}_{ff}\right) (\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}) + \boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{SF}} + \mathbf{K}_{fs} (\boldsymbol{\Psi}_{\mathbf{SF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}})\big] \mathbf{F_1^{\mathrm{T}} L}_f^{\mathrm{T}} \mathbf{W_1^{\mathrm{T}}}, \end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Psi}_{\mathbf{S^*S^*}} \underline{\underline{\Delta}}\ & \mathbf{W_2S_1}\big[\mathbf{S_4}\boldsymbol{\Psi}_{\mathbf{FF}}\mathbf{S_4^T} + \mathbf{S_5}\boldsymbol{\Psi}_{\mathbf{SS}}\mathbf{S_5^T} + \mathbf{S_6}\left(\mathbf{I}_{f^2} + \mathbf{K}_{ff}\right)\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\mathbf{S_6^T} \\
& + \mathbf{S_7}\big[\left(\mathbf{I}_{f^3} + \mathbf{K}_{ff^2} + \mathbf{K}_{f^2f} + \mathbf{I}_f \otimes \mathbf{K}_{ff} + \mathbf{K}_{ff} \otimes \mathbf{I}_f + \left(\mathbf{I}_f \otimes \mathbf{K}_{ff}\right)\mathbf{K}_{ff^2}\right) \\
& \cdot \left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right) + \left(\mathbf{I}_{f^3} + \mathbf{K}_{ff^2} + \mathbf{K}_{f^2f}\right)\left(\left(\mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right)\mathrm{vec}(\boldsymbol{\Psi}_{\mathbf{FF}})^{\mathrm{T}}\right) \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right) \\
& \cdot \left(\mathbf{I}_{f^3} + \mathbf{K}_{ff^2} + \mathbf{K}_{f^2f}\right)\big]\mathbf{S_7^T} + \left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{SS}}\right) + \mathbf{K}_{fs}\left(\boldsymbol{\Psi}_{\mathbf{SF}} \otimes (\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}}\right) \\
& + \mathbf{S_5}\boldsymbol{\Psi}_{\mathbf{SF}}\mathbf{S_4^T} + \mathbf{S_4}(\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}}\mathbf{S_5^T} \\
& + \mathbf{S_7}\Delta_{f^3\times f}\left[\mathbf{K}_{ff^3}\left(\mathrm{vec}\left(\left(\mathbf{I}_{f^2} + \mathbf{K}_{ff}\right)\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\right) + \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right) \otimes \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right)\right)\right]\mathbf{S_4^T} \\
& + \mathbf{S_4}\Delta_{f\times f^3}\left[\mathbf{K}_{f^3f}\left(\mathrm{vec}\left(\left(\mathbf{I}_{f^2} + \mathbf{K}_{ff}\right)\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\right) + \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right) \otimes \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right)\right)\right]\mathbf{S_7^T} \\
& + \mathbf{S_7}\Delta_{f^3\times s}\left[\mathbf{K}_{sf^3}\left(\mathrm{vec}\left(\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{SF}}\right) + \mathbf{K}_{fs}\left(\boldsymbol{\Psi}_{\mathbf{SF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\right) + \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right) \otimes \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{SF}}\right)\right)\right]\mathbf{S_5^T} \\
& + \mathbf{S_5}\Delta_{s\times f^3}\left[\mathbf{K}_{f^3s}\left(\mathrm{vec}\left[\left((\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right) + \mathbf{K}_{ff}\left((\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\right]\right.\right. \\
& \left.\left. + \mathrm{vec}\left((\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}}\right) \otimes \mathrm{vec}\left(\boldsymbol{\Psi}_{\mathbf{FF}}\right)\right)\right]\mathbf{S_7^T} \\
& + \left[\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{SF}}\right) + \mathbf{K}_{fs}\left(\boldsymbol{\Psi}_{\mathbf{SF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\right]\mathbf{S_6^T} \\
& + \mathbf{S_6}\left[\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes (\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}}\right) + \mathbf{K}_{ff}\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes (\boldsymbol{\Psi}_{\mathbf{SF}})^{\mathrm{T}}\right)\right]\big]\mathbf{S_1^T}\mathbf{W_2^T}
\end{aligned}$$

(here the symbol "$\Delta_{n\times m}$" is used to transform a $nm \times 1$ vector into an $n \times m$ matrix).

The measurement error covariance matrix $\boldsymbol{\Theta}$ is partitioned into a 5×5 array of submatrices as expressed below:

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\Theta}_{\mathbf{FF}} & & & & \\ \boldsymbol{\Theta}_{\mathbf{SF}} & \boldsymbol{\Theta}_{\mathbf{SS}} & & & \\ \boldsymbol{\Theta}_{\mathbf{TF}} & \boldsymbol{\Theta}_{\mathbf{TS}} & \boldsymbol{\Theta}_{\mathbf{TT}} & & \\ \boldsymbol{\Theta}_{\mathbf{F^*F}} & \boldsymbol{\Theta}_{\mathbf{F^*S}} & \boldsymbol{\Theta}_{\mathbf{F^*T}} & \boldsymbol{\Theta}_{\mathbf{F^*F^*}} & \\ \boldsymbol{\Theta}_{\mathbf{S^*F}} & \boldsymbol{\Theta}_{\mathbf{S^*S}} & \boldsymbol{\Theta}_{\mathbf{S^*T}} & \boldsymbol{\Theta}_{\mathbf{S^*F^*}} & \boldsymbol{\Theta}_{\mathbf{S^*S^*}} \end{bmatrix},$$

where $\boldsymbol{\Theta}_{\mathbf{F^*F}} \underline{\underline{\Delta}}\ \mathbf{W_3L}_p\mathbf{E_3}\boldsymbol{\Theta}_{\mathbf{FF}}$, $\boldsymbol{\Theta}_{\mathbf{F^*S}} \underline{\underline{\Delta}}\ \mathbf{W_3L}_p\mathbf{E_3}(\boldsymbol{\Theta}_{\mathbf{SF}})^{\mathrm{T}}$, $\boldsymbol{\Theta}_{\mathbf{F^*T}} \underline{\underline{\Delta}}\ \mathbf{W_3L}_p\mathbf{E_3}(\boldsymbol{\Theta}_{\mathbf{TF}})^{\mathrm{T}}$,

$$\begin{aligned}
\boldsymbol{\Theta}_{\mathbf{F^*F^*}} \underline{\underline{\Delta}}\ & \mathbf{W_3L}_p\big[\mathbf{E_3}\boldsymbol{\Theta}_{\mathbf{FF}}\mathbf{E_3^T} + \mathbf{E_4}\left(\boldsymbol{\Theta}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\mathbf{E_4^T} + \mathbf{E_5}\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Theta}_{\mathbf{FF}}\right)\mathbf{E_5^T} \\
& + \left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)\left(\boldsymbol{\Theta}_{\mathbf{FF}} \otimes \boldsymbol{\Theta}_{\mathbf{FF}}\right) + \mathbf{E_5K}_{fp}\left(\boldsymbol{\Theta}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\mathbf{E_4^T} \\
& + \mathbf{E_4K}_{pf}\left(\boldsymbol{\Psi}_{\mathbf{FF}} \otimes \boldsymbol{\Theta}_{\mathbf{FF}}\right)\mathbf{E_5^T}\big]\mathbf{L}_p^{\mathrm{T}}\mathbf{W_3^T},
\end{aligned}$$

$$\boldsymbol{\Theta}_{\mathbf{S^*F}} \underline{\underline{\Delta}}\ \mathbf{W_4}\left[\mathbf{A_4}\boldsymbol{\Theta}_{\mathbf{FF}} + \mathbf{A_5}\boldsymbol{\Theta}_{\mathbf{SF}} + \mathbf{A_9}\Delta_{(pf^2)\times p}\left[\mathbf{K}_{p(pf^2)}\mathrm{vec}\left(\mathbf{K}_{fp}\left(\boldsymbol{\Theta}_{\mathbf{FF}} \otimes \boldsymbol{\Psi}_{\mathbf{FF}}\right)\right)\right]\right],$$

$$\mathbf{\Theta_{S^*S}} \triangleq \mathbf{W_4}\left[\mathbf{A_4}(\mathbf{\Theta_{SF}})^{\mathrm{T}} + \mathbf{A_5}\mathbf{\Theta_{SS}} + \mathbf{A_9}\Delta_{(pf^2)\times q}\left[\mathbf{K}_{q(pf^2)}\mathrm{vec}\left(\mathbf{K}_{fq}\left(\mathbf{\Theta_{SF}}\otimes\mathbf{\Psi_{FF}}\right)\right)\right]\right],$$

$$\mathbf{\Theta_{S^*T}} \triangleq \mathbf{W_4}\left[\mathbf{A_4}(\mathbf{\Theta_{TF}})^{\mathrm{T}} + \mathbf{A_5}(\mathbf{\Theta_{TS}})^{\mathrm{T}} + \mathbf{A_9}\Delta_{(pf^2)\times r}\left[\mathbf{K}_{r(pf^2)}\mathrm{vec}\left(\mathbf{K}_{fr}\left(\mathbf{\Theta_{TF}}\otimes\mathbf{\Psi_{FF}}\right)\right)\right]\right],$$

$$\begin{aligned}
\mathbf{\Theta_{S^*F^*}} \triangleq \mathbf{W_4}\big[&\mathbf{A_4}\mathbf{\Theta_{FF}}\mathbf{E_3^{\mathrm{T}}} + \mathbf{A_5}\mathbf{\Theta_{SF}}\mathbf{E_3^{\mathrm{T}}}\\
&+ \mathbf{A_9}\Delta_{(pf^2)\times p}\left[\mathbf{K}_{p(pf^2)}\mathrm{vec}\left(\mathbf{K}_{fp}\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{FF}}\right)\right)\right]\mathbf{E_3^{\mathrm{T}}}\\
&+ \mathbf{A_6}\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{FF}}\right)\mathbf{E_4^{\mathrm{T}}} + \mathbf{A_7}\mathbf{K}_{fq}\left(\mathbf{\Theta_{SF}}\otimes\mathbf{\Psi_{FF}}\right)\mathbf{E_4^{\mathrm{T}}} + \mathbf{A_8}\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{SF}}\right)\mathbf{E_4^{\mathrm{T}}}\\
&+ \mathbf{A_6}\mathbf{K}_{pf}\left(\mathbf{\Psi_{FF}}\otimes\mathbf{\Theta_{FF}}\right)\mathbf{E_5^{\mathrm{T}}} + \mathbf{A_7}\left(\mathbf{\Psi_{FF}}\otimes\mathbf{\Theta_{SF}}\right)\mathbf{E_5^{\mathrm{T}}} + \mathbf{A_8}\mathbf{K}_{ps}\left(\mathbf{\Psi_{SF}}\otimes\mathbf{\Theta_{FF}}\right)\mathbf{E_5^{\mathrm{T}}}\\
&+ \left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Theta_{SF}}\right) + \mathbf{K}_{pq}\left(\mathbf{\Theta_{SF}}\otimes\mathbf{\Theta_{FF}}\right)\big]\mathbf{L}_p^{\mathrm{T}}\mathbf{W_3^{\mathrm{T}}},
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Theta_{S^*S^*}} = \mathbf{W_4}\big[&\mathbf{A_4}\mathbf{\Theta_{FF}}\mathbf{A_4^{\mathrm{T}}} + \mathbf{A_5}\mathbf{\Theta_{SS}}\mathbf{A_5^{\mathrm{T}}} + \mathbf{A_6}\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{FF}}\right)\mathbf{A_6^{\mathrm{T}}}\\
&+ \mathbf{A_7}\left(\mathbf{\Psi_{FF}}\otimes\mathbf{\Theta_{SS}}\right)\mathbf{A_7^{\mathrm{T}}} + \mathbf{A_8}\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{SS}}\right)\mathbf{A_8^{\mathrm{T}}}\\
&+ \mathbf{A_9}\big[\left(\mathbf{I}_{pf^2} + \mathbf{I}_p\otimes\mathbf{K}_{ff}\right)\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{FF}}\otimes\mathbf{\Psi_{FF}}\right)\\
&+ \mathbf{K}_{p(f^2)}\left(\left(\mathrm{vec}\left(\mathbf{\Psi_{FF}}\right)\mathrm{vec}(\mathbf{\Psi_{FF}})^{\mathrm{T}}\right)\otimes\mathbf{\Theta_{FF}}\right)\mathbf{K}_{(f^2)p}\big]\mathbf{A_9^{\mathrm{T}}}\\
&+ \left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Theta_{SS}}\right) + \mathbf{K}_{pq}\left(\mathbf{\Theta_{SF}}\otimes(\mathbf{\Theta_{SF}})^{\mathrm{T}}\right)\\
&+ \mathbf{A_5}\mathbf{\Theta_{SF}}\mathbf{A_4^{\mathrm{T}}} + \mathbf{A_9}\Delta_{(pf^2)\times p}\left[\mathbf{K}_{p(pf^2)}\mathrm{vec}\left(\mathbf{K}_{fp}\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{FF}}\right)\right)\right]\mathbf{A_4^{\mathrm{T}}}\\
&+ \mathbf{A_4}(\mathbf{\Theta_{SF}})^{\mathrm{T}}\mathbf{A_5^{\mathrm{T}}} + \mathbf{A_9}\Delta_{(pf^2)\times q}\left[\mathbf{K}_{q(pf^2)}\mathrm{vec}\left(\mathbf{K}_{fq}\left(\mathbf{\Theta_{SF}}\otimes\mathbf{\Psi_{FF}}\right)\right)\right]\mathbf{A_5^{\mathrm{T}}}\\
&+ \mathbf{A_7}\mathbf{K}_{fq}\left(\mathbf{\Theta_{SF}}\otimes\mathbf{\Psi_{FF}}\right)\mathbf{A_6^{\mathrm{T}}} + \mathbf{A_8}\left(\mathbf{\Theta_{FF}}\otimes\mathbf{\Psi_{SF}}\right)\mathbf{A_6^{\mathrm{T}}}\\
&+ \mathbf{A_6}\mathbf{K}_{pf}\left(\mathbf{\Psi_{FF}}\otimes(\mathbf{\Theta_{SF}})^{\mathrm{T}}\right)\mathbf{A_7^{\mathrm{T}}} + \mathbf{A_8}\mathbf{K}_{ps}\left(\mathbf{\Psi_{SF}}\otimes(\mathbf{\Theta_{SF}})^{\mathrm{T}}\right)\mathbf{A_7^{\mathrm{T}}}\\
&+ \mathbf{A_6}\left(\mathbf{\Theta_{FF}}\otimes(\mathbf{\Psi_{SF}})^{\mathrm{T}}\right)\mathbf{A_8^{\mathrm{T}}} + \mathbf{A_7}\mathbf{K}_{fq}\left(\mathbf{\Theta_{SF}}\otimes(\mathbf{\Psi_{SF}})^{\mathrm{T}}\right)\mathbf{A_8^{\mathrm{T}}}\\
&+ \mathbf{A_4}\Delta_{p\times(pf^2)}\left[\mathbf{K}_{(pf^2)p}\left(\mathrm{vec}\left(\mathbf{\Theta_{FF}}\right)\otimes\mathrm{vec}\left(\mathbf{\Psi_{FF}}\right)\right)\right]\mathbf{A_9^{\mathrm{T}}}\\
&+ \mathbf{A_5}\Delta_{q\times(pf^2)}\left[\mathbf{K}_{(pf^2)q}\left(\mathrm{vec}\left((\mathbf{\Theta_{SF}})^{\mathrm{T}}\right)\otimes\mathrm{vec}\left(\mathbf{\Psi_{FF}}\right)\right)\right]\mathbf{A_9^{\mathrm{T}}}\big]\mathbf{W_4^{\mathrm{T}}}.
\end{aligned}$$

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