Studies in Fuzziness and Soft Computing

Dan E. Tamir
Naphtali D. Rishe
Abraham Kandel *Editors*

Fifty Years of Fuzzy Logic and its Applications

Springer

Studies in Fuzziness and Soft Computing

Volume 326

Series editor

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Fifty Years of Fuzzy Logic and its Applications

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ISSN 1434-9922 ISSN 1860-0808 (electronic)
Studies in Fuzziness and Soft Computing
ISBN 978-3-319-19682-4 ISBN 978-3-319-19683-1 (eBook)
DOI 10.1007/978-3-319-19683-1

Library of Congress Control Number: 2015940726

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

To my parents Rivka and Ezra Tamir

Dan E. Tamir

To the joyous Bar Mitzvah of Joshua Meir Yosef ben Tajana and Naphtali

Naphtali D. Rishe

To my grandchildren Kfeer, Maya, Riley, Liam, Leo, and Marley
With Love

Abraham Kandel

Recognizing the 50th anniversary of Florida International University — a peer of Fuzzy Logic

Preface

As early as in the era of Lady Ada Loveable and Babbage, scientists seriously considered the possibility of assigning certain complex activities performed by human beings to machines. This direction of research has significantly intensified, with the development of digital computers, through immense contributions by Turing and von Neumann and the progress in the discipline of artificial intelligence (AI). Following a period of enthusiasm about the possibility on one hand, and computer phobia on the other hand, many of the active AI researchers have faced, and attempted to resolve, an apparent obstacle. The formal philosophical and mathematical paradigms applied in AI and related research areas seemed to fall short of the capability to emulate human reasoning. In some sense, the issue was that the formalisms were very rigid and did not match the fuzzy nature of human perception of sets and inference.

A pioneer of artificial intelligence, L.A. Zadeh was concerned with the dichotomy between human reasoning and classical-logic/mathematical/machine precision. As early as 1961 (and most likely before) Zadeh attempted to resolve this dichotomy with a formal, mathematical theory of imprecision, aka Fuzzy Set Theory and Fuzzy Logic. The first documented reference to the need for this theory appears in his 1962 paper "From Circuit Theory to System Theory." The first formulation of a solution to the dichotomy is proposed in his seminal paper "Fuzzy Sets" published in *Information and Control* in 1965. These concepts, as well as several derivatives of the ideas, such as linguistic variables, Type-2 Fuzzy Logic, and Z-numbers, introduced by Zadeh, have opened the door to highly fruitful directions of research, development, and deployment in several areas.

Zadeh's 1965 paper and subsequent papers have sparked the interest of numerous researchers and practitioners and resulted in rapid developments in the fields of Fuzzy Set Theory, Fuzzy Logic, Fuzzy Systems, and related disciplines. The five decades that followed the 1965 paper and his pioneering work have produced a multitude of research work and applications related to artificial intelligence, control theory, inference, and reasoning. In recent years, Fuzzy Logic has been applied in many areas, including neural networks, clustering, data mining, and software testing.

The present volume, entitled "Fifty Years of Fuzzy Logic and its Applications," was conceived as a way of academic celebration of the fifty years' anniversary of the 1965 paper. It includes papers from pioneers and prominent scholars engaged in research on the theory and applications of fuzzy logic and uncertainty management. The papers cover a wide range of the spectrum and gamut of "Fuzziness."

The volume editors extend sincere gratitude to the distinguished chapter authors for their invaluable contributions and their kind patience in complying with the bureaucratic procedures involved in publishing this volume.

Acknowledgments

This work is based in part upon work supported by the National Science Foundation under grants I/UCRC IIP-1338922, AIR IIP-1237818, SBIR IIP-1330943, III-Large IIS-1213026, MRI (CNS-1429345, CNS-0821345, CNS-1126619), and CREST HRD-0833093 and by DHS S&T at TerraFly (http://terrafly.com) and the NSF CAKE Center (http://cake.fiu.edu).

Contents

Toward a Restriction-Centered Theory of Truth and Meaning (RCT)

Lotfi A. Zadeh

Abstract What is truth? The question does not admit a simple, precise answer. A dictionary-style definition is: The truth value of a proposition, p, is the degree to which the meaning of p is in agreement with factual information, F. A precise definition of truth will be formulated at a later point in this paper. The theory outlined in the following, call it RCT for short, is a departure from traditional theories of truth and meaning. In RCT, truth values are allowed to be described in natural language. Examples. Quite true, more or less true, almost true, largely true, possibly true, probably true, usually true, etc. Such truth values are referred to as linguistic truth values. Linguistic truth values are not allowed in traditional logical systems, but are routinely used by humans in everyday reasoning and everyday discourse. The centerpiece of RCT is a deceptively simple concept—the concept of a restriction. Informally, a restriction, R(X), on a variable, X, is an answer to a question of the form: What is the value of X? Possible answers: X = 10, X is between 3 and 20, X is much larger than 2, X is large, probably X is large, usually X is large, etc. In RCT, restrictions are preponderantly described in natural language. An example of a fairly complex description is: It is very unlikely that there will be a significant increase in the price of oil in the near future. The canonical form of a restriction, R(X), is X isr R, where X is the restricted variable, R is the restricting relation, and r is an indexical variable which defines the way in which R restricts X. X may be an n-ary variable and R may be an n-ary relation. The canonical form may be interpreted as a generalized assignment statement in which what is assigned to X is not a value of X, but a restriction on the values which X can take. A restriction, R(X), is a carrier of information about X. A restriction is precisiated if X, R and r are mathematically well defined. A key idea which underlies RCT is referred to as the meaning postulate, MP. MP postulates that the meaning of a proposition drawn from a natural language, p—or simply p—may be represented as a restriction, p → X isr R. This expression is referred to as the

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_1

canonical form of p, CF(p). Generally, the variables X, R and r are implicit in p. Simply stated, MP postulates that a proposition drawn from a natural language may be interpreted as an implicit assignment statement. MP plays an essential role in defining the meaning of, and computing with, propositions drawn from natural language. What should be underscored is that in RCT understanding of meaning is taken for granted. What really matters is not understanding of meaning but precisiation of meaning. Precisiation of meaning is a prerequisite to reasoning and computation with information described in natural language. Precisiation of meaning is a desideratum in robotics, mechanization of decision-making, legal reasoning, precisiated linguistic summarization with application to data mining, and other fields. It should be noted that most—but not all—propositions drawn from natural language are precisiable. In RCT, truth values form a hierarchy. First order (ground level) truth values are numerical, lying in the unit interval. Linguistic truth values are second order truth values and are restrictions on first order truth values. n^{th} order truth values are restrictions on (n-1) order truth values, etc. Another key idea is embodied in what is referred to as the truth postulate, TP. The truth postulate, TP, equates the truth value of p to the degree to which X satisfies R. This definition of truth value plays an essential role in RCT. A distinguishing feature of RCT is that in RCT a proposition, p, is associated with two distinct truth values—internal truth value and external truth value. The internal truth value relates to the meaning of p. The external truth value relates to the degree of agreement of p with factual information. To compute the degree to which X satisfies R, it is necessary to precisiate X, R and r. In RCT, what is used for this purpose is the concept of an explanatory database, ED. Informally, ED is a collection of relations which represent the information which is needed to precisiate X and R or, equivalently, to compute the truth value of p. Precisiated X, R and p are denoted as X^*, R^* and p^*, respectively. X and R are precisiated by expressing them as functions of ED. The precisiated canonical form, $CF^*(p)$, is expressed as $X^* isr^* R^*$. At this point, the numerical truth value of p, nt_p, may be computed as the degree to which X^* satisfies R^*. In RCT, the factual information, F, is assumed to be represented as a restriction on ED. The restriction on ED induces a restriction, t, on nt_p which can be computed through the use of the extension principle. The computed restriction on nt_p is approximated to by a linguistic truth value, lt_p. Precisiation of propositions drawn from natural language opens the door to construction of mathematical solutions of computational problems which are stated in natural language.

Keywords Precisiation of meaning • Computation with restrictions • Assessment of truth values • Formalization of everyday reasoning

1 Introduction

The concepts of truth and meaning are of fundamental importance in logic, information analysis and related fields. The theory outlined in this paper, call it RCT for short, is a departure from traditional theories of truth and meaning, principally correspondence theory, coherence theory, Tarski semantics, truth-conditional semantics and possible-world semantics [1–3, 5–9].

In large measure, traditional theories of truth and meaning are based on bivalent logic. RCT is based on fuzzy logic. Standing on the foundation of fuzzy logic, RCT acquires a capability to enter the realm of everyday reasoning and everyday discourse—a realm which is avoided by traditional theories of truth and meaning largely because it is a realm that does not lend itself to formalization in the classical tradition.

In RCT, truth values are allowed to be described in natural language. Examples. Quite true, very true, almost true, probably true, possibly true, usually true, etc. Such truth values are referred to as linguistic truth values. Linguistic truth values are not allowed in traditional logical systems.

The centerpiece of RCT is the deceptively simple concept—the concept of a restriction. The concept of a restriction has greater generality than the concept of interval, set, fuzzy set and probability distribution. An early discussion of the concept of a restriction appears in [12]. Informally, a restriction, R(X), on a variable, X, is an answer to a question of the form: What is the value of X? Example. Robert is staying at a hotel in Berkeley. He asks the concierge, "How long will it take me to drive to SF Airport?" Possible answers: 1 h, one hour plus/minus 15 min, about 1 h, usually about 1 h, etc. Each of these answers is a restriction on the variable, Driving. time. Another example. Consider the proposition, p: Most Swedes are tall. What is the truth value of p? Possible answers: true, 0.8, about 0.8, high, likely high, possibly true, etc. In RCT, restrictions are preponderantly described as propositions drawn from a natural language. Typically, a proposition drawn from a natural language is a fuzzy proposition, that is, a proposition which contains fuzzy predicates, e.g., tall, fast, heavy, etc., and/or fuzzy quantifiers, e.g., most, many, many more, etc., and/or fuzzy probabilities, e.g., likely, unlikely, etc. A zero-order fuzzy proposition does not contain fuzzy quantifiers and/or fuzzy probabilities. A first-order fuzzy proposition contains fuzzy predicates and/or fuzzy quantifiers and/or fuzzy probabilities. It is important to note that in the realm of natural languages fuzzy propositions are the norm rather than exception. Traditional theories of truth and meaning provide no means for reasoning and computation with fuzzy propositions.

Basically, R(X) may be viewed as a limitation on the values which X can take. Examples.

X = 5
X is between 3 and 7
X is small
X is normally distributed with mean m and variance σ^2
It is likely that X is small

Summers are usually cold in San Francisco (X is implicit)
Robert is much taller than most of his friends (X is implicit)

As a preview of what lies ahead, it is helpful to draw attention to two key ideas which underlie RCT. The first idea, referred to as the meaning postulate, MP, is that of representing a proposition drawn from a natural language, p, as a restriction expressed as

$$p \to X \text{ isr } R,$$

where X is the restricted variable, R is the restricting relation, and r is an indexical variable which defines the way in which R restricts X. X may be an n-ary variable, and R may be an n-ary relation. Generally, X and R are implicit in p. Basically, X is the variable whose value is restricted by p. X is referred to as the focal variable. In large measure, the choice of X is subjective, reflecting one's perception of the variable or variables which are restricted by p. However, usually there is a consensus. It should be noted that a semantic network representation of p may be viewed as a graphical representation of an n-ary focal variable and an n-ary restricting relation. The expression on the right-hand side of the arrow is referred to as the canonical form of p, CF(p). CF(p) may be interpreted as a generalized assignment statement [17]. The assignment statement is generalized in the sense that what is assigned to X is not a value of X, but a restriction on the values which X can take. Representation of p as a restriction is motivated by the need to represent p in a mathematically well-defined form which lends itself to computation.

The second key idea is embodied in what is referred to as the truth postulate, TP. The truth postulate equates the truth value of p to the degree to which X satisfies R. The degree may be numerical or linguistic. As will be seen in the sequel, in RCT the truth value of p is a byproduct of precisiation of the meaning of p.

Note. To simplify notation in what follows, in some instances no differentiation is made between the name of a variable and its instantiation. Additionally, in some instances no differentiation is made between a proposition, p, and the meaning of p.

2 The Concept of a Restriction—A Brief Exposition

The concept of a restriction is the centerpiece of RCT. As was stated earlier, a restriction, R(X), on a variable, X, may be viewed as an answer to a question of the form: What is the value of X? The concept of a restriction is closely related to the concept of a generalized constraint [18].

R(X) may be viewed as information about X. More concretely, R(X) may be expressed in a canonical form, CF(R(X)),

$$CF(R(X)):\ X \text{ isr } R,$$

where X is the restricted variable, R is the restricting relation, and r is an indexical variable which defines the modality of R, that is, the way in which R restricts X. X may be an n-ary variable and R may be an n-ary relation. A restriction is precisiated if X, R and r are mathematically well defined. Precisiation of restrictions plays a pivotal role in RCT. Precisiation of restrictions is a prerequisite to computation with restrictions. Here is an example of a simple problem which involves computation with restrictions.

Usually Robert leaves his office at about 5 pm.
Usually it takes Robert about an hour to get home from work.
At what time does Robert get home?

Humans have a remarkable capability to deal with problems of this kind using approximate, everyday reasoning. One of the important contributions of RCT is that RCT opens the door to construction of mathematical solutions of computational problems which are stated in a natural language.

2.1 Types of Restrictions

There are many types of restrictions. A restriction is singular if R is a singleton. Example. X = 5. A restriction is nonsingular if R is not a singleton. Nonsingularity implies uncertainty. A restriction is direct if the restricted variable is X. A restriction is indirect if the restricted variable is of the form f(X). Example.

$$R(p):\ \int_a^b \mu(u)p(u)du \ \text{ is likely,}$$

is an indirect restriction on p.

Note. In the sequel, the term restriction is sometimes applied to R.

The principal types of restrictions are: possibilistic restrictions, probabilistic restrictions and Z-restrictions.

Possibilistic restriction (r = blank)

$$R(X):\ X \text{ is } A,$$

where A is a fuzzy set in a space, U, with the membership function, μ_A. A plays the role of the possibility distribution of X,

$$\text{Poss}(X = u) = \mu_A(u).$$

Example.

X is small
↑ ↑
restricted variable restricting relation (fuzzy set)

The fuzzy set small plays the role of the possibility distribution of X. (Fig. 1)

Example.

Leslie is taller than Ixel →

(Height(Leslie), Height(Ixel)) is taller
↑ ↑
restricted variable restricting relation (fuzzy relation)

The fuzzy relation taller is the possibility distribution of ((Height(Leslie), Height (Ixel)).

Probabilistic restriction (r = p)

$$R(X): X \text{ is } p\, p,$$

where p is the probability density function of X,

$$\text{Prob}(u \le X \le u + du) = p(u)du.$$

Example.

$$X \text{ isp } \frac{1}{\sqrt{2\pi}}\exp(-(X-m)^2/2\sigma^2).$$
↑ ↑
restricted variable restricting relation (probability density function)

Z-restriction (r = z, s is suppressed)
X is a real-valued random variable.
A Z-restriction is expressed as

$$R(X): X \text{ iz } Z,$$

where Z is a combination of possibilistic and probabilistic restrictions defined as

$$Z: \text{Prob}(X \text{ is } A) \text{ is } B,$$

in which A and B are fuzzy numbers. Usually, A and B are labels drawn from a natural language. The ordered pair, (A,B), is referred to as a Z-number [19]. The first component, A, is a possibilistic restriction on X. The second component, B, is a

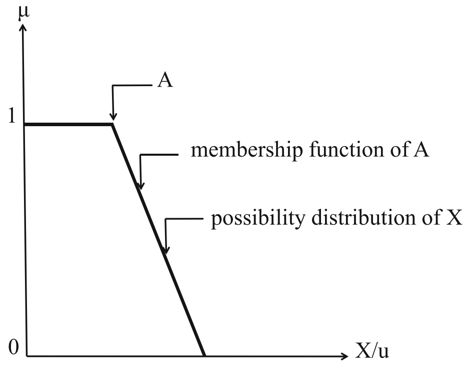


Fig. 1 Possibilistic restriction on X

possibilistic restriction on the certainty (probability) that X is A. A Z-interval is a fuzzy number in which the first component is a fuzzy interval.

Examples.

$$\text{Probably Robert is tall} \rightarrow \text{Height(Robert) iz (tall, probable)}$$
$$\text{Usually temperature is low} \rightarrow \text{Temperature iz (low, usually)}$$

Note.
Usually X is A,
is a Z-restriction when A is a fuzzy number.

A Z-valuation is an ordered triple of the form (X,A,B), and (A,B) is a Z-number. Equivalently, a Z-valuation, (X,A,B), is a Z-restriction on X,

$$(X, A, B) \rightarrow X \text{ iz } (A, B).$$

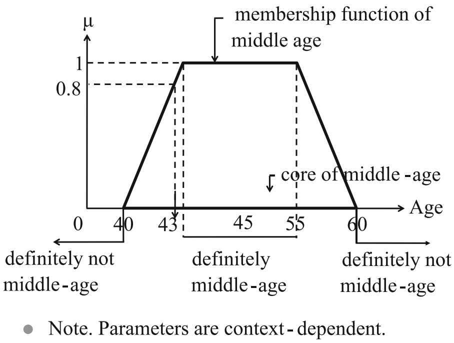
Examples.

(Age(Robert), young, very likely)
(Traffic, heavy, usually).

Note. A natural language may be viewed as a system of restrictions. In the realm of natural languages, restrictions are predominantly possibilistic. For this reason, in this paper we focus our attention on possibilistic restrictions. For simplicity, possibilistic restrictions are assumed to be trapezoidal.

Example. Figure 2 shows a possibilistic trapezoidal restriction which is associated with the fuzzy set middle-age.

Fig. 2 Trapezoidal possibilistic restriction on Age



2.2 Computation with Restrictions

Computation with restrictions plays an essential role in RCT. In large measure, computation with restrictions involves the use of the extension principle [10, 13]. A brief exposition of the extension principle is presented in the following. The extension principle is not a single principle. The extension principle is a collection of computational rules in which the objects of computation are various types of restrictions. More concretely, assume that Y is a function of X, Y = f(X), where X may be an n-ary variable. Assume that what we have is imperfect information about X, implying that what we know is a restriction on X, R(X). The restriction on X, R(X), induces a restriction on Y, R(Y). The extension principle is a computational rule which relates to computation of R(Y) given R(X). In what follows, we consider only two basic versions of the extension principle. The simplest version [10] is one in which the restriction is possibilistic and direct. This version of the extension principle reduces computation R(Y) to the solution of a variational problem,

$$Y = f(X)$$
$$R(X):\ X \text{ is } A$$
$$R(Y):\ \mu_Y(v) = \sup_u(\mu_A(u))$$

subject to

$$v = f(u),$$

where μ_A and μ_Y are the membership functions of A and Y, respectively. Simply stated,

$$\text{If } X \text{ is } A \text{ then } Y \text{ is } f(A),$$

where f(A) is the image of A under f. A simple example is shown in Fig. 3.

An inverse version of this version of the extension principle is the following.

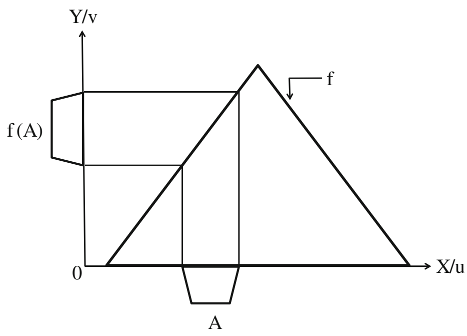


Fig. 3 Possibilistic version of the basic extension principle. f(A) is the image of A under f. What is shown is a trapezoidal approximation to f(A)

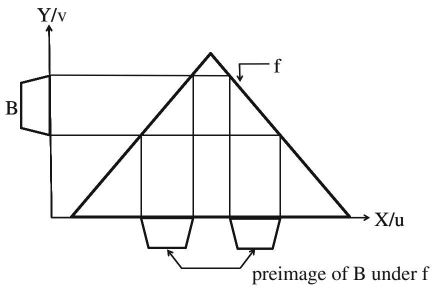


Fig. 4 Inverse version of the basic possibilistic extension principle. The induced restriction on X is the preimage of B, the restriction on Y

$$Y = f(X)$$
$$R(Y): Y \text{ is } B$$
$$R(X): \mu_A(u) = (\mu_B(f(u)))$$

Simply stated, A is the preimage of B under f. (Fig. 4)

A slightly more general version [13] is one in which R(X) is possibilistic and indirect.

$$Y = f(X)$$
$$R(X): g(X) \text{ is } A$$
$$R(Y): \mu_Y(v) = \sup_u(\mu_A(g(u)))$$

subject to

$$v = f(g(u)).$$

Example.

Given, p: Most Swedes are tall.
Question, q: What is the average height of Swedes?

The first step involves precisiation of p and q. For this purpose, it is expedient to employ the concept of a height density function, h.

$$h(u)du = \text{fraction of Swedes whose height lies in the interval } [u,\ u + du].$$

If h_{min} and h_{max} are, respectively, the minimum and maximum heights in the population, we have

$$\int_{h_{min}}^{h_{max}} h(u)du = 1.$$

In terms of the height density function, precisiations of q and p, q* and p*, may be expressed as

$$q^{*}: ?\, h_{ave} = \int_{h_{min}}^{h_{max}} uh(u)du,$$

$$p^{*} \int_{h_{min}}^{h_{min}} \mu_{tall}(u)h(u)du \text{ is most},$$

where μ_{tall} is the membership function of tall. Applying the basic, indirect, possibilistic version of the extension principle, computation of h_{ave} is reduced to the solution of the variational problem

$$\mu_{h_{ave}}(v) = \sup_{h} \mu_{most}\left(\int_{h_{min}}^{h_{max}} \mu_{tall}(u)h(u)du\right),$$

subject to

$$V = \int_{h_{min}}^{h_{max}} uh(u)du,$$

and

$$\int_{h_{min}}^{h_{max}} h(u)du = 1.$$

In RCT, for purposes of reasoning and computation what are needed—in addition to possibilistic versions of the extension principle—are versions in which restrictions are probabilistic restrictions and Z-restrictions. These versions of the extension principle are described in [21].

3 Truth and Meaning

It is helpful to begin with a recapitulation of some of the basic concepts which were introduced in the Introduction.

There is a close relationship between the concept of truth and the concept of meaning. To assess the truth value of a proposition, p, it is necessary to understand the meaning of p. However, understanding the meaning of p is not sufficient. What is needed, in addition, is precisiation of the meaning of p. Precisiation of the meaning of p involves representation of p in a form that is mathematically well defined and lends itself to computation. In RCT, formalization of the concept of truth is a byproduct of formalization of the concept of meaning. In the following, unless stated to the contrary, p is assumed to be a proposition drawn from a natural language. Typically, propositions drawn from a natural language are fuzzy propositions, that is, propositions which contain fuzzy predicates and/or fuzzy quantifiers and/or fuzzy probabilities.

The point of departure in RCT consists of two key ideas: The meaning postulate, MP, and the truth postulate, TP. MP relates to precisiation of the meaning of p. More concretely, a proposition is a carrier of information. Information is a restriction. Reflecting these observations, MP postulates that the precisiated meaning of p—or simply precisiated p—may be represented as a restriction. In symbols, p may be expressed as

$$p \rightarrow X \text{ isr } R,$$

where X, R and r are implicit in p. The expression X isr R is referred to as the canonical form of p, CF(p). In general, X is an n-ary variable and R is a function of X. Basically, X is a variable such that p is a carrier of information about X. X is referred to as a focal variable of p. In large measure, the choice of X is subjective. It should be noted that when X is an n-ary variable, a semantic network representation of p may be viewed as a graphical representation of the canonical form of p.

Examples.

p: Robert is young ⟶ Age(Robert) is young
↑ X ↑ R

p: Most Swedes are tall ⟶

Proportion(tall Swedes/Swedes) is most
↑ X ↑ R

p: Robert is much taller than most of his friends ⟶ Height(Robert) is much taller than most of his friends

p: Usually it takes Robert about an hour to get home from work ⟶ Travel time from office to home iz (approximately 1 hr., usually).

The truth postulate, TP, relates the truth value of p to its meaning. More concretely, consider the canonical form

$$\text{CF(p): X isr R.}$$

TP postulates that the truth value of p is the degree to which X satisfies R.

In RCT, truth values form a hierarchy: First-order (ground level), second order, etc. First order truth values are numerical. For simplicity, numerical truth values are assumed to be points in the interval. (Fig. 5)

A generic numerical truth value is denoted as nt. Second order truth values are linguistic. Examples. Quite true, possibly true. A generic linguistic truth value is denoted as lt. In RCT, linguistic truth values are viewed as restrictions on numerical truth values. In symbols, lt = R(nt). A generic truth value is denoted as t. t can be nt or lt.

3.1 Precisiation of X, R and P

Typically, X and R are described in a natural language. To compute the degree to which X satisfies R it is necessary to precisiate X and R. In RCT, what is used for this purpose is the concept of an explanatory database, ED [16, 20]. Informally, ED is a collection of relations which represent the information which is needed to precisiate X and R or, alternatively, to compute the truth value of p. Example. Consider the proposition, p: Most Swedes are tall. In this case, the information consists of three relations, TALL[Height;μ], MOST[Proportion;μ] and POPULATION[Name; Height]. In TALL, μ is the grade of membership of Height in tall. In MOST, μ is the grade of membership of Proportion—a point in the unit interval—in most. In POPULATION, Height is the height of Name, where Name is a variable which ranges

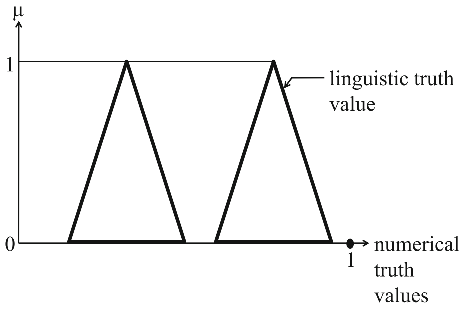


Fig. 5 Hierarchy of truth values. A numerical truth value is a first-order (ground level) truth value. A linguistic truth value is a second-order truth value. A linguistic truth value is a restriction on numerical truth values. Typically, a linguistic truth value is a fuzzy set or, equivalently, a possibility distribution

over the names of Swedes in a sample population. Equivalently, and more simply, ED may be taken to consist of the membership function of tall, μ_{tall}, the membership function of most, μ_{most}, and the height density function, h. h is defined as the fraction, h(u)du, of Swedes whose height is in the interval [u,u + du].

X and R are precisiated by expressing them as functions of ED. Precisiated X, R and p are denoted as X^*, R^* and p^*, respectively. Thus,

$$X^* = f(ED), \; R^* = g(ED).$$

The precisiated canonical form, $CF^*(p)$, is expressed as $X^* isr^* R^*$. At this point, the numerical truth value of p, nt_p, may be computed as the degree to which X^* satisfies R^*. In symbols,

$$nt_p = tr(ED)$$

in which tr is referred to as the truth function (Fig. 6).

What this equation means is that an instantiation of ED induces a value of nt_p. Varying instantiations of ED induces what is referred to as the truth distribution of p, denoted as Tr(p|ED). The truth distribution of p may be interpreted as the possibility distribution of ED given p, expressed as Poss(ED|p). Thus, we arrive at an important equality

$$Tr(p|ED) = Poss(ED|p).$$

In RCT, the precisiated meaning of p is expressed in three equivalent forms. First, as the precisiated canonical form, $CF^*(p)$. Second, as the truth distribution of p, Tr(p| ED). Third, as the possibility distribution, Poss(ED|p). These representations of the precisiated meaning of p play an essential role in RCT. The precisiated meaning of p may be viewed as the computational meaning of p. Of the three equivalent

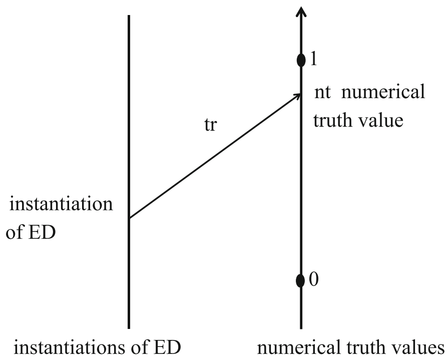


Fig. 6 A numerical truth value, nt, is induced by an instantiation of ED. tr is the truth function

definitions stated above, the definition that is best suited for computational purposes is that which involves the possibility distribution of ED. Adopting this definition, what can be stated is the following.

• Definition. The precisiated (computational) meaning of p is the possibility distribution of ED, Poss(ED|p), which is induced by p.

A simple example. Consider the proposition, p: Robert is tall. In this case, ED consists of Height(Robert) and the relation TALL[Height; μ] or, equivalently, the membership function μ_{tall}. We have,

$$X = \text{Height(Robert)}, \quad R = \text{tall}.$$

The canonical form reads

$$\text{Height (Robert) is tall.}$$

The precisiated X and R are expressed as

$$X^* = \text{Height(Robert)}, \quad R^* = \text{tall},$$

where tall is a fuzzy set with the membership function, μ_{tall}.

The precisiated canonical form reads

$$\text{Height (Robert) is tall.}$$

Note that in this case the unprecisiated and precisiated canonical forms are identical. The truth distribution is defined by

$$nt_p = \mu_{tall}(h),$$

where h is a generic value of Height(Robert).

The basic equality reads

$$\mathrm{Tr}(p|h) = \mathrm{Poss}(h|p).$$

More specifically, if h = 175 cm and $\mu_{tall(175cm)} = 0.9$, then 0.9 is the truth value of p given h = 175 cm, and the possibility that h = 175 cm given p (Fig. 7).

Example. Robert is handsome. In this case, assume that we have a sample population of men, $\mathrm{Name}_1, \ldots, \mathrm{Name}_n$ with μ_i being the grade of membership of Name_i in the fuzzy set handsome. The meaning of p is the possibility distribution associated with the fuzzy set handsome—the possibility distribution which is induced by p. The possibility that Name_i is handsome is equal to the grade of membership of Name_i in handsome.

A less simple example. Consider the proposition, p: Most Swedes are tall. In this case, X = Proportion(tall Swedes/Swedes) and R = most. The canonical form of p is

$$\text{Proportion (tall Swedes/Swedes) is most.}$$

The precisiated X and R may be expressed as

$$X^* = \int_{h_{min}}^{h_{max}} h(u)\mu_{tall}(u)du,$$
$$R^* = \text{most},$$

where most is a fuzzy set with a specified membership function, μ_{most}.
The precisiated canonical form reads

$$CF^*: \int_{h_{min}}^{h_{max}} h(u)\mu_{tall}(u)du \text{ is most.}$$

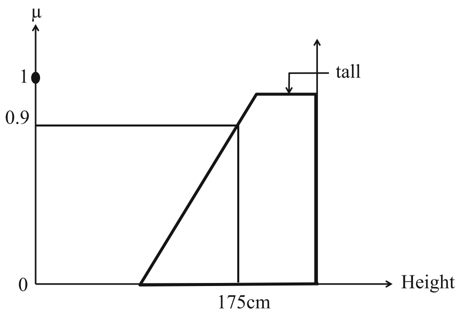


Fig. 7 0.9 = truth value of the proposition Robert is tall, given that Robert's height is 175 cm. 0.9 = possibility that Robert's height is 175 cm, given the proposition Robert is tall

The truth distribution, Tr(p|ED), is defined by computing the degree, nt_p, to which X^* satisfies R^*,

$$nt_p = \mu_{most}\left(\int_{h_{min}}^{h_{max}} h(u)\mu_{tall}du\right)$$

Note that an instantiation of ED induces a numerical truth value, nt_p.

Another example. Consider the proposition, p: Robert is much taller than most of his friends. In this case, assume that X = Proportion of friends of Robert in relation to whom Robert is much taller, and R = most. The explanatory database, ED, consists of the relations FRIENDS[Name;μ], HEIGHT[Name;Height], MUCH. TALLER[$Height_1$;$Height_2$;μ], and Height(Robert). Equivalently, ED may be expressed as $\mu_F(Name_i)$, h_i, and $\mu_{MT}(h,h_i)$, i = 1, …, n. In this ED, h = Height (Robert), h_i = Height($Name_i$), $\mu_F(Name_i)$ = grade of membership of $Name_i$ in the fuzzy set of friends of Robert, and $\mu_{MT}(h,h_i)$ = grade of membership of (h, hi) and the fuzzy set much taller. Precisiated X and R are expressed as,

$$X^* = \left(\frac{1}{n}\sum_i \mu_{MT}(h, h_i) \wedge \mu_F(Name_i)\right), \quad R^* = most,$$

The precisiated meaning of p is expressed as,

$$Poss(ED|p) = \mu_{most}\left(\frac{1}{n}\sum_i \mu_{MT}(h, h_i) \wedge \mu_F(i)\right),$$

where $\wedge$ denotes conjunction.

Note. The concept of an instantiated ED in RCT is related to the concept of a possible world in traditional theories. Similarly, the concept of a possibility distribution of the explanatory database is related to the concept of intension.

Precisiation of meaning is the core of RCT and one of its principal contributions. A summary may be helpful.

3.2 Summary of Precisiation

The point of departure is a proposition, p, drawn from a natural language. The objective is precisiation of p.

1. Choose a focal variable, X, by interpreting p as an answer to the question: What is the value of X? Identify the restricting relation, R. R is a function of X. At this point, X and R are described in a natural language.
2. Construct the canonical form, $CF(p) = X \text{ isr } R$.

3. Construct an explanatory database, ED. To construct ED, ask the question: What information is needed to express X and R as functions of ED? Alternatively, ask the question: What information is needed to compute the truth value of p?
4. Precisiate X and R by expressing X and R as functions of ED. Precisiated X and R are denoted as X^* and R^*, respectively.
5. Construct the precisiated canonical form, $CF^*(p)$: X^* isr^* R^*.
6. Equate precisiated p to $CF^*(p)$.
7. $CF^*(p)$ defines the possibility distribution of ED given p, $Poss(ED|p)$.
8. $CF^*(p)$ defines the truth distribution of the truth value or p given ED, $Tr(p|ED)$.
9. $Poss(ED|p) = Tr(p|ED)$.
10. Define the precisiated (computational) meaning of p as the possibility distribution of ED given p, $Poss(ED|p)$. More informatively, the precisiated (computational) meaning of p is the possibility distribution, $Poss(ED|p)$, together with the procedure which computes $Poss(ED|p)$.

3.3 Truth Qualification. Internal and External Truth Values

A truth-qualified proposition is a proposition of the form t p, where t is the truth value of p. t may be a numerical truth value, nt, or a linguistic truth value, lt. Example. It is quite true that Robert is tall. In this case, t = quite true and p = Robert is tall. A significant fraction of propositions drawn from a natural language are truth-qualified. An early discussion of truth-qualification is contained in [14]. Application of truth-qualification to a resolution of Liar's paradox is contained in [15].

In a departure from tradition, in RCT a proposition, p, is associated with two truth values—internal truth value and external truth value. When necessary, internal and external truth values are expressed as Int(truth value) and Ext(truth value), or Int(p) and Ext(p).

Informally, the internal numerical truth value is defined as the degree of agreement of p with an instantiation of ED. Informally, an external numerical truth value of p is defined as the degree of agreement of p with factual information, F. More concretely, an internal numerical truth value is defined as follows.

Definition.

$$Int(nt_p) = tr(ED).$$

In this equation, ED is an instantiation of the explanatory database, $Int(nt_p)$ is the internal numerical truth value of p, and tr is the truth function which was defined earlier.

More generally, assume that we have a possibilistic restriction on instantiations of ED, Poss(ED). This restriction induces a possibilistic restriction on nt_p which can be computed through the use of the extension principle. The restriction on nt_p may be expressed as tr(Poss(ED)). The fuzzy set, tr(Poss(ED)), may be approximated by the membership function of a linguistic truth value. This leads to the following definition of an internal linguistic truth value of p.

Definition.

$$\mathrm{Int}\left(lt_p\right) \approx \mathrm{tr}(\mathrm{Poss}(\mathrm{ED})).$$

In this equation, ≈ should be interpreted as a linguistic approximation. In words, the internal linguistic truth value, $\mathrm{Int}(lt_p)$, is the image—modulo linguistic approximation—of the possibility distribution of ED under the truth function, tr. It is important to note that the definition of linguistic truth value which was stated in the previous subsection is, in fact, the definition of internal linguistic truth value of p (Fig. 8).

Note. Poss(ED), tr(Poss(ED)) and lt_p are fuzzy sets. For simplicity, denote these fuzzy sets as A, B and C, respectively. Using the extension principle, computation of lt_p reduces to the solution of the variational problem,

$$\mu_B(v) = \sup_u \mu_A(u)$$

subject to

$$v = \mathrm{tr}(u)$$

$$\mu_C \approx \mu_B.$$

The external truth value of p, Ext(p), relates to the degree of agreement of p with factual information, F. In RCT, factual information may be assumed to induce a

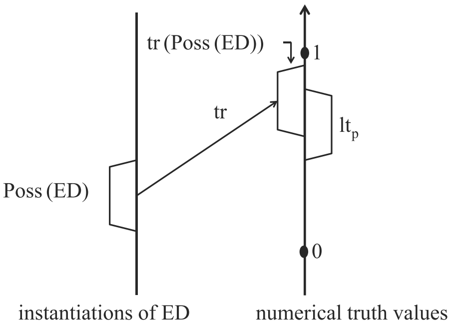


Fig. 8 A linguistic truth value, ltp, is induced by a possibilistic restriction on instantiations of ED, Poss(ED). ltp is a linguistic approximation to the image of Poss(ED) under tr

possibilistic restriction on ED, Poss(ED|F). In particular, if F instantiates ED, then the external truth value is numerical. This is the basis for the following definition.

Definition. The external numerical truth value of p is defined as

$$\mathrm{Ext}\left(\mathrm{nt_p}\right) = \mathrm{tr}(\mathrm{ED}|\mathrm{F}),$$

where ED is an instantiation of the explanatory database induced by F.

Simple example. In Fig. 7, if the factual information is that Robert's height is 175 cm, then the external numerical truth value of p is 0.9.

More generally, if F induces a possibilistic restriction on instantiations of ED, Poss(ED|F), then the external linguistic truth value of p may be defined as follows.

Definition.

$$\mathrm{Ext}\left(\mathrm{lt_p}\right) \approx \mathrm{tr}(\mathrm{Poss}(\mathrm{ED}|\mathrm{F})).$$

In this equation, ≈ should be interpreted as a linguistic approximation. In words, the external linguistic truth value of p is—modulo linguistic approximation—the image of Poss(ED|F) under tr.

Example. Consider the proposition, p: Most Swedes are tall. Assume that the factual information is that the average height of Swedes is around 170 cm. Around 170 cm is a fuzzy set defined by its membership function, $\mu_{ar.170cm}$. In terms of the height density function, h, the average height of Swedes may be expressed as

$$h_{ave} = \int_{h_{min}}^{h_{max}} uh(u)du.$$

The explanatory database consists of μ_{tall}, μ_{most} and h. Assuming that μ_{tall} and μ_{most} are fixed, the possibilistic restriction on ED is induced by the indirect possibilistic restriction

$$\int_{h_{min}}^{h_{max}} uh(u)du \text{ is around } 170\text{cm}$$

which is equivalent to the possibility distribution of h expressed as

$$\mathrm{Poss}\left(h|h_{ave}\right) = \mu_{ar.\,170\,cm}\left(\int_{h_{min}}^{h_{max}} uh(u)du\right).$$

An important observation is in order. An internal truth value modifies the meaning of p. An external truth value does not modify the meaning of p; it places in evidence the factual information, with the understanding that factual information is a possibilistic restriction on the explanatory database.

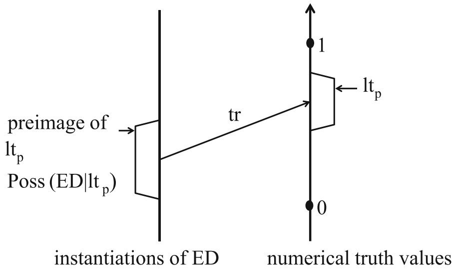


Fig. 9 Modification of meaning of p. Modified meaning of p is the preimage of ltp under tr

How does an internal truth value, t, modify the meaning of p? Assume that the internal truth value is numerical. The meaning of p is the possibility distribution, Poss(ED|p). The meaning of nt p is the preimage of nt under the truth function, tr. In other words, the meaning of p, expressed as the possibility distribution, Poss(ED|p), is modified to the possibility distribution Poss(ED|nt_p). If the internal truth value is linguistic, lt_p, the modified meaning is the preimage of lt_p, Poss(ED|lt_p), under tr (Fig. 9). More concretely, using the inverse version of the basic extension principle, we can write

$$\mu_{Poss\left(ED|lt_p\right)}(u) = \mu_{tr(Poss\left(ED|lt_p\right))}(tr(u)),$$

where u is an instantiation of ED, $\mu_{Poss(ED|ltp)}$ and $\mu_{tr(Poss(ED|ltp))}$ are the membership functions of Poss(ED|lt_p) and tr(Poss(ED|lt_p)), respectively.

Simple example. In Fig. 7, the preimage of 0.9 is 175 cm. The meaning of p is the possibility distribution of tall. The truth value 0.9 modifies the possibility distribution of tall to Height(Robert) = 175 cm. More generally, when the truth value is linguistic, lt_p, the modified meaning of p is the preimage of lt_p under tr (Fig. 10).

There is a special case which lends itself to a simple analysis. Assume that lt is of the form h true, where h is a hedge exemplified by quite, very, almost, etc. Assume that p is of the form X is A, where A is a fuzzy set. In this case, what can be postulated is that truth-qualification modifies the meaning of p as follows.

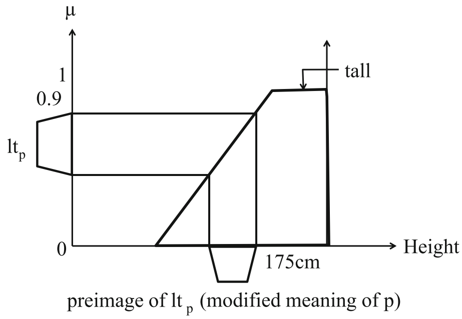
$$\text{h true}(\text{X is A}) = \text{X is h A}.$$

h A may be computed through the use of techniques described in early papers on hedges [4, 11].

Example.

$$(\text{usually true})\ \text{Snow is white} = \text{snow is usually white}.$$

Fig. 10 An internal linguistic truth value modifies the meaning of p



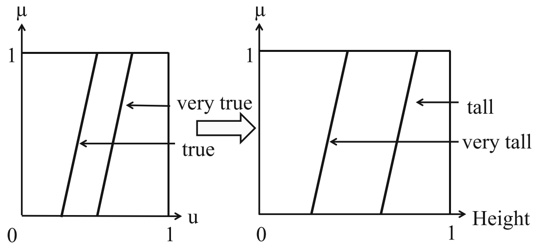


Fig. 11 Meaning-modification induced by hedged truth-qualification

Example. (Fig. 11).

$$\text{It is very true that Robert is tall} = \text{Robert is very tall.}$$

A word of caution is in order. Assume that there is no hedge. In this case, the equality becomes

$$\text{true}(\text{X is A}) = \text{X is A}.$$

If truth is bivalent, and true is one of its values, this equality is an agreement with the school of thought which maintains that propositions p and p is true have the same meaning. In RCT, p and p is true do not have the same meaning. There is a subtle difference. More concretely, the meaning of p relates to the agreement of p with a possibilistic restriction on ED. The meaning of p is true relates to a possibilistic restriction which is induced by factual information.

When lt_p is an external truth value, the meaning of p is not modified by lt_p. In RCT, a simplifying assumption which is made regarding the factual information, F, is that F may be described as a possibility distribution of instantiations of ED, Poss(ED|F). The external truth value, lt_p, identifies the factual information as the preimage of lt_p under tr,

$$\mathrm{Ext}\left(\mathrm{lt_p}\right) = \mathrm{tr}(\mathrm{Poss}(\mathrm{ED}|\mathrm{F}))$$
$$\mathrm{F} = \mathrm{Poss}\left(\mathrm{ED}|\mathrm{Ext}\left(\mathrm{lt_p}\right)\right).$$

In conclusion, truth-qualification in RCT is paralleled by probability-qualification in probability theory and by possibility-qualification in possibility theory. Truth-qualification, probability-qualification and possibility-qualification are intrinsically important issues in logic, information analysis and related fields.

4 Concluding Remark

The theory outlined in this paper, RCT, may be viewed as a step toward formalization of everyday reasoning and everyday discourse. Unlike traditional theories—theories which are based on bivalent logic—RCT is based on fuzzy logic. Fuzzy logic is the logic of classes with unsharp(fuzzy) boundaries. In the realm of everyday reasoning and everyday discourse, fuzziness of class boundaries is the rule rather than exception. The conceptual structure of RCT reflects this reality.

The theory which underlies RCT is not easy to understand, largely because it contains many unfamiliar concepts. However, once it is understood, what is revealed is that the conceptual structure of RCT is simple and natural.

Acknowledgments To Luis Magdalena and Enric Trillas.

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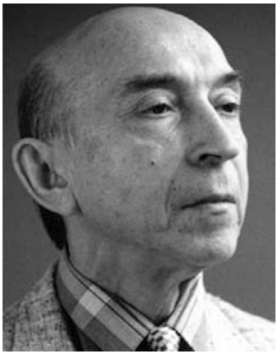
Author Biography



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Lotfi Zadeh is known as the inventor of fuzzy logic. His first paper, Fuzzy Sets, 1965, is the highest cited paper in Computer Science (Web of Science) and the seventh highest cited paper in Science (Web of Science). He has published extensively (over 240 single-authored papers) on a wide variety of subjects relating to the conception, design and analysis of information/intelligent systems, and is serving on the editorial boards of over seventy journals.

Prior to the publication of his first paper on fuzzy sets in 1965, Lotfi Zadeh'swork was concerned in the main with systems analysis, decision analysis and information systems. His current research is focused on fuzzy logic, semantics of natural languages, computational theory of perceptions, computing with words, extended fuzzy logic and Z-numbers.

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Functional Solution of the Knowledge Level Control Problem: The Principles of Fuzzy Logic Rules and Linguistic Variables

Ali M. Abbasov and Shahnaz N. Shahbazova

Abstract This paper addresses the problem of imitating a teacher evaluating the students' levels of knowledge. It proposes application of fuzzy logic to construct and manage a knowledge control system used for generating evaluating questions. The system contains a knowledge base with relevant information and a set of rules. Students build the rules based on the analysis of answers and relevant reactions to questions. The algorithms governing the system allow for an automatic selection of sequences of appropriate and customized questions. The presented system for knowledge control and generating questions is comparable in quality and efficiency with the real teacher's questioning process.

Keywords Decision making • Uncertainty • Fuzzy logic • Neuro-fuzzy expert systems • Complex systems • Expert knowledge

1 Introduction

A system for evaluating learnt knowledge and managing it is one of the main elements of successful educational or research activities. An efficient and effective process of appraising gained knowledge – called hereafter a knowledge control process – influences all aspects of education and research as they rely on the outcome of learning activities.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_2

From a practical point of view, procedures of knowledge control should allow for questioning students with the aim of verifying their knowledge and skills in the field of study. The effective control of knowledge should be able to mimic a teacher of the relevant subject. Therefore, a system that is able to simulate the teacher's behavior is the most reasonable to develop. The analysis of the teacher's behavior in performing an evaluation process leads to construction of a system for an automated knowledge control. This system is able to conduct evaluation of students' knowledge, and determine correctness and incorrectness of provided answers.

2 Implementation of Intelligent System in Educational Process

The quality of a teaching system depends on the precise definition of the characteristics of several key factors defining the student's knowledge level and abilities: results of absorbing material recently presented to her, mastering the material presented in the past, and current moral and psychological state of the student.

The problem of selecting further actions is solved by the system based on these key factors. The possible actions may be: continuation of the teaching process, asking questions related to the previous material, repetition of already asked and answered questions, or completion of the training process.

The system's electronic catalog stores all data related to the student's abilities, her test schedule, etc. In addition, it contains personal data of the student.

In the process of working with the program, the student is able not only to test her knowledge, but also to learn. This is achieved by the way questions are asked, and by the presence of all of the questions, comments, and explanations given by the teacher. Access to the Internet provides the student with the ability to explore information anywhere in the world, including the best libraries, archives, etc.

Once the sustainable results are achieved for a certain part of the material, the student can proceed to the next level of difficulty of questions. This transition will allow the student to continue her learning process further.

This step-wise training process gives the student the necessary time to fully master and strengthen the knowledge of a given material, and then to move to a new, more difficult material. Each transition is accompanied by a small test on the previous material, and an analysis of its mastery.

To this date, the work has been done on learning and testing of a group of students at the same time. A teacher creates shared folders on a particular subject or subjects. Using these shared folders, the teacher can give tasks and exercises to a group of students, as well as check their solutions and results. The shared folders are structured in a way that simplifies the work with groups of students. The teacher has access to the working directories of students, and is allowed to deal individually with each student. The same thing happens when a re-take of a course is recommended, or a more detailed analysis of errors in the shared directories is required. The sophistication of the Intelligent Information System of Learning and Control of

Knowledge (IISLCK) allows the teacher to add and edit her material, and to make corrections according to the latest achievements of science and culture [1].

One of possible ways to improve the functionality of the systems of technical control of knowledge is the application of intelligent technologies in particular methods based on the diverse hybrid Expert Systems (ESs). Hybrid ESs represent different kinds of knowledge and are equipped with conceptual, expert, and factual methods of its processing.

The main task of the development of hybrid systems is to combine different forms of representing knowledge and methods of its processing, and merge them with decision-making approaches of ES. This means, that the actual problem is to investigate the possibilities of optimal connection of different mechanisms of knowledge processing to improve the quality, mobility and efficiency of ES in solving problems of a knowledge control process in conditions of uncertainty.

The mobility of ES is due to the mobility of the knowledge base (KB) and its ability to replenish material/facts/data from different information components (database, bases of expert knowledge (BEK), the base of conceptual knowledge (BCK), dynamic files, etc.), as well as various procedures of drawing conclusions. The concretization of knowledge processing in solving problems decomposes them into accurate and inaccurate, complete and incomplete, static and dynamic, single-valued and multi-valued, etc. In addition, the expert knowledge is inaccurate due to their subjective character. The approximation and multiple meanings of knowledge processing means that the ES has to deal with several alternative areas. Therefore, the processing of incomplete knowledge can use several sources of knowledge.

The application of a fuzzy logic hybrid ES for knowledge control may have at least three implementations:

(1) Processing of fuzzy uncertainty of expert expressions, i.e. when the pre-condition is fuzzy variables, but an inference machine is a data extraction mechanism from these preconditions.
(2) Using a matrix of fuzzy connections, determining a number of factors and preconditions. The matrix contains the fuzzy relations between variables, represented as real numbers [0, 1], and determines the cause of a condition. The matrix and factors form equations of fuzzy relations. The resulting system is solved using minimum-maximum fuzzy inference mechanism.
(3) Using fuzzy conclusions. This approach is most often used in the construction of fuzzy knowledge bases [2].

The application of fuzzy hybrid ES to solve problems and control parameters of knowledge processing extends the capabilities of this class of intelligent systems, as well as increases their flexibility and mobility. This allows conducting expert evaluation of a large number of variants, increasing the credibility and accuracy of the evaluation of the results.

In this paper, the main principles of construction of a neuro-fuzzy hybrid ES with diverse knowledge and its analysis in conditions of uncertainty of its parameters are considered. Additionally, the application of a dynamic knowledge base combined with neural networks (NN) is being investigated [3].

In the neuro-fuzzy hybrid ES, standard model (SM) is stored in the knowledge base containing processed knowledge, and is refined in the process of acquiring new knowledge. The real model is formed in a database environment, and communication with the EM is achieved via the user's requests. Solving the problem of designing an intelligent system for quality knowledge control built based on a hybrid ES is done with taking into account the characteristics of the environment of ES.

The hybrid ES consists of the following parts: a database that stores standard and factual evidence about the process; the results of their comparison, conceptual, physical and info logical models; knowledge base (KB) – its static part (knowledge is stored in the form of expert knowledge (of products) as well as formulas, facts, dependencies, tables, concepts specific subject area), and its dynamic part (the knowledge is stored in combined models of NN in the form of standard of dynamic processes taking into account the partial or complete uncertainly parameter of control); a mechanism of logic inference that is based on an algorithm for generating cause and effect network of events functional-structural model; adaptation mechanism to coordinate the work of the database (DB) and KB in the process of logical inference depending on the situation, explaining the mechanism, which is an interpretation of the process of logical inference; planner coordinating the process of solving the problem; solver for finding effective solutions to positive, negative and mixed statements of problems.

The content, form and algorithms for representing information inside the hybrid ES are flexible and depend on the complexity of a situation being modeled, and the specific and individual characteristics of the user.

The expert presents her knowledge in the form of sets of examples. A derivation tree is used as the internal form of presentation of the knowledge. A set of examples is described by attributes. All examples of the same structure, as defined by its attributes, are linked by logical transitions. In this case, the relevant trees of inference are combined in such a way that at the terminal vertex of one tree another tree is added.

The Computational Model of the ES and the DB in solving problems under uncertainty is given in the form:

$$W = <A, D, B, F, H>, \tag{2.1}$$

where A – is a set of attributes of DB and KB; D – denotes domains (attribute values of DB and KB); B – is a set of functional dependencies defined over the attributes; F – denotes descriptions of all types used in the functional dependencies B; and H – is a set of fuzzy relations over a set of attributes A [4].

One of the most difficult aspects to achieve in the hybrid ES is the requirement of dealing with different forms of knowledge representation, such as frames, semantic networks, databases, the concepts presented in KB, neural networks, fuzzy logic, genetic algorithms. All of these components have to share a single information space in the hybrid ES. For example, in the hybrid ES, diverse knowledge is stored in static components of ES, while dynamic knowledge about the current state of information is stored in neural networks. The modern information and database technology (for example, Object Linking and Embedding paradigm) can easily share diverse knowledge within a single information space [5].

It should be noted that the approach considered here, i.e., the application of hybrid ES as the basis for the intelligent system for knowledge control in the presence of uncertainty allows for:

(1) Actively applying the diverse knowledge (conceptual, structural, procedural, factual, base rule with membership function, rules and fuzzy rules of DB, KB, BEK procedures) together with inference mechanisms for finding effective solutions to the problem of determining the level of student's knowledge;
(2) Summarizing and improving the conceptual model of representation of diverse knowledge among relational DB and the managed DBMS; and interacting with the core of hybrid ES;
(3) Effectively solving the problem of optimizing and distributing information streams among individual subsystems of the hybrid ES under the conditions of uncertainty.

The methodology of constructing diverse knowledge storage for hybrid neuro-fuzzy ES includes the following stages [6]:

(1) The formalization of the domain (the development of a conceptual model);
(2) The description of knowledge model as individual concepts (knowledge) in the KB;
(3) The formation of KB with the base rule as a managing components of intelligent core;
(4) The description of diverse information to control the student's knowledge in the individual sub-systems of the hybrid ES (DB, KB, EKB, a graphical DB, the computed files);
(5) Selecting a neural-network model and learning rules;
(6) Development of fuzzy logic procedures;
(7) Distribution of information streams between the ES and its individual subsystems;
(8) Testing individual subsystems of the ES;
(9) Testing the neuro-fuzzy hybrid ES.

3 The Methodology for Knowledge Control

An important element of the learning system is its ability to make decisions regarding the level of difficulty of questions which should be posed to students. This should be preformed based on the results of answering previous questions. The solution to this problem depends on numerous parameters, most of which are unknown to the system. A fairly accurate answer can be found with the help of the mathematical apparatus of fuzzy logic [7].

The analysis of the current situation depends on following:

(1) Questions answered correctly by a student;
(2) Questions answered incorrectly by a student;
(3) Question answered incorrectly to previous questions by a student;
(4) Preliminary analysis of a student's ability;
(5) The number of correct answers coupled with their difficulty and in respect to erroneous answers.

This list reflects the real computational tasks. A decision-making process is carried out in order to select questions, which according to the program, corresponds to student's ability. An incorrect answer triggers a re-valuation process of the data about the student and leads to less difficult questions to be asked in the next time. In the case of a correct answer, the program asks questions with progressive difficulty. This decision-making method allows an individual to make a progress during the learning process [8]. Furthermore, it gives the most accurate evaluation of the student abilities.

At the end of the evaluation process, when both student and teacher want to sum up the result of the educational session, the program analyses the number of correct answers and their complexity. It starts with updating the relevant database record of the student, and then begins the process of analysis that aims at providing updated and correct information about the student.

This information can include: the current level of mastery of the subject of the student; comparison with previous results of analysis of the student's incorrect responses, the visualization of the correct answers with commentary, as well as comments provided by the teacher while entering questions into the database [9].

The importance of evaluation of the executed test could be adjusted by the program and/or by the teacher. This approach allows for performing individual pretests and tests at different levels of difficulty.

As stated, due to the large number of external parameters a decision-making process is done with the help of the mathematical apparatus of fuzzy logic. The responsible subsystem also includes conducting tests that satisfy the following requirements [10, 11]:

(1) Protecting answers from unauthorized access;
(2) Preventing a student from modification of the number of correct answers to questions;
(3) Providing equal conditions for the tests.

During the process of testing, the next question is read from the database based on the inference result obtained from a knowledge base located in the network. The question is displayed in a form convenient for the student (Fig. 1).

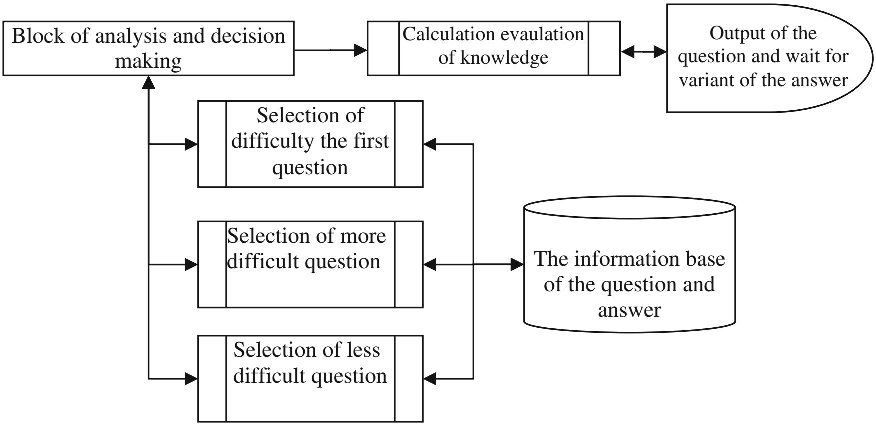

Fig. 1 Simplified block-schema of the control system of knowledge

These expressions can be represented in the form of conditional statements of complicated structure. As a very simple example is the expression of the form:

$$\textbf{\textit{If the Previous Answer}} = \textbf{\textit{Right}},$$
$$\textbf{\textit{THEN Correct Answers}} = \textbf{\textit{Correct Answers}} + \mathbf{1}$$

The next level of complication of the statements is to generate weighted questions (complexity):

$$\textbf{\textit{If the Previous Answer}} = \textbf{\textit{right}},$$
$$\textbf{\textit{Then Weight Correct Answer}} = \textbf{\textit{Weight Correct Answer}} +$$
$$\textbf{\textit{Table Weight}}\ (\textbf{\textit{Index Current Answer}})$$

The level of intelligence of the subsystems can be increased by adding records of the elapsed time and other parameters, and providing complicated logic expressions. In such a case the level of testing provided by subsystems can be compared with surveys conducted by the real teacher [12]. Additional parameters in mathematical expressions provide the descriptions of the following characteristics: the ability to remember, attentiveness, reaction speed, decision-making speed, reading speed, etc.

In the process of working with the program, the student cannot only test her knowledge, but also learn. This is accomplished when the question is asked, and also with the access to all of the comments and explanations given by the teacher.

Once stable results are obtained for a certain group of questions, the student can move on to the questions on the next level of complexity [13]. This transition enables the student's further development without being stack at the achieved results.

A gradual learning process gives the student the necessary time to complete mastering and strengthening the material, and then transit to a new, more complex material. Each transition is accompanied by a small test containing questions related to the previous material, and an analysis of its mastering.

4 Decision Making and the Knowledge Control in the Managing System

The algorithm of choosing the first and subsequent questions uses the results of carrying out the following tasks.

- preliminary analysis – used to evaluate the level of student's knowledge for making a decision regarding the first question (students lagging in knowledge assimilation are asked questions from a group of simple questions, while prepared students are given more difficult questions) [1] (Fig. 2).

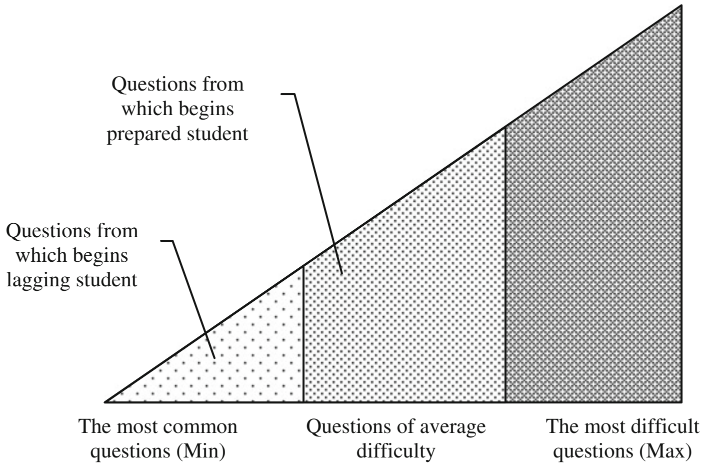

Fig. 2 Strategy for selection of the first question

- The formula below represents one of functions of the decision making block. Its essence comes down to choosing the next question, which corresponds to the student's level of knowledge. If an incorrect answer is chosen during the evaluation, the student will be given a less difficult question (2). If the correct answer is selected, the program will choose the more difficult question (1). The decision process can be described in the following way.

- A student answers the previous question correctly: in such a case the student is asked a question of increased difficulty (Fig. 3). The formula for selecting the next question is [6]:

$$Q = \frac{(Max(A^{+}) + Max(A^{-}))}{2} \pm 2\,\% \tag{4.1}$$

where, Q is the next question, (A+) is the level of difficulty of a correctly answered question, Max(A+) is the maximum level of difficulty of the questions to which the student gave the correct answers, (A-) is the level of difficulty of an incorrectly answered question, and Max(A-) is the maximum level of difficulty of the question to which the student gave the incorrect answer. In case the student has not given an incorrect answer yet, the assigned value is the maximum level of difficulty of the questions for this course. ± 2 % is the maximum deviation in the level of the next asked question, and it represents randomness in a selection process.

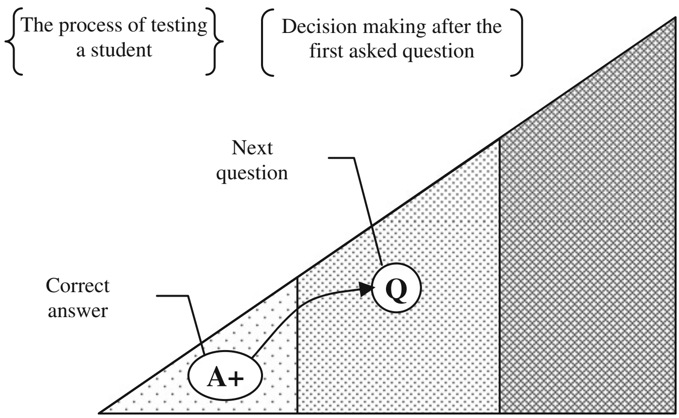

Fig. 3 The strategy of selection question after the correct answer

- A student answers the previous question incorrectly: in this case the student is asked a less difficult question (Fig. 4). The selection formula of the next question is [7]:

$$Q = \frac{(Max(A^{-}) + Min(A^{+}))}{2} \pm 2\,\% \tag{4.2}$$

where, Min(A+) is the minimum level of difficulty of the correctly answered question, while Max(A-) is the maximum level of difficulty of the question to which the student gave the incorrect answer. If the correct answer was not given by the student, the assigned value is the minimum level of difficulty of the questions for this course.

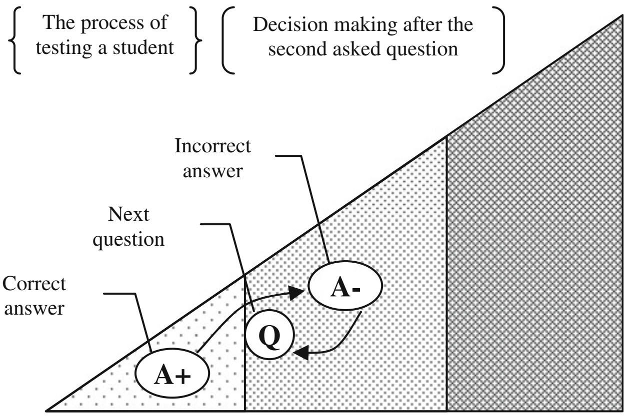

Fig. 4 A strategy for selecting the next question after an incorrect answer

The deviation included in the formulas ensures that for every student group there are no be two students that are given the same questions, even if the order of correct and incorrect answers are the same [2].

- processing the results and making a decision related to the final evaluation or continuation of testing – the number of correctly answered questions multiplied by their difficulty in relation to the number of mistakes and sets of correctly and incorrectly answered questions are the input to the decision-making subprogram; this results in a final evaluation or, if there remains a high probability of uncertainty, in continuation of testing (according to formula 4).

$$Z + jP = f\left(\frac{\sum_{i=1}^{N}\left(A_i^+\right)}{N}, \frac{\sum_{j=1}^{M}\left(A_j^-\right)}{M}, \left[A_1^+, A_2^+, \ldots, A_N^+\right], \left[A_1^-, A_2^-, \ldots, A_M^-\right]\right) \tag{4.3}$$

where, Z – evaluation of knowledge, P – uncertainty of evaluation, f – the decision making subprogram which works based on the following characteristics of conducted testing, (A_i+) – the set of difficulty levels of correctly answered questions, (A_j–) – the set of difficulty levels of incorrectly answered questions, N – number of questions with the correct answers, M – number of questions with the incorrect answers [5].

The result of the formula 4 is a complex number. This number represents the response of the decision making subprogram and indicates a degree of uncertainty in the students' knowledge [3]. This uncertainty provides a level of confidence in the evaluation process. Its value relates to the coverage of questions in the learning process. Higher the coverage less the uncertainty, which depends on the number of asked questions (Fig. 5). For example, the student can answer only a few questions and obtain an excellent score, but the uncertainty in this case would be very high, due to the fact that only several questions covering a significant amount of extremely difficult course material have been asked.

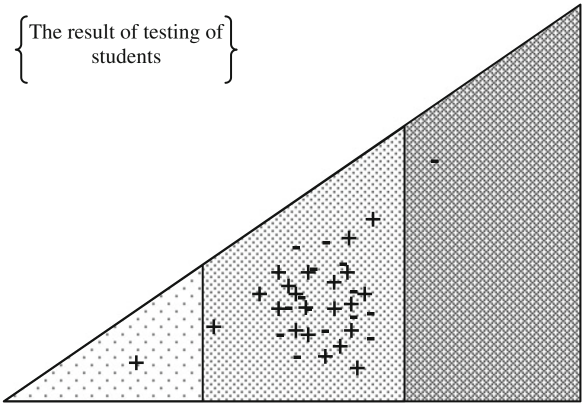

Fig. 5 Example distribution of answers after testing

Thus, the system containing the flexible algorithm of questioning, allows the teacher to decide about the volume of material covered in the course and the number of questions that should be asked to students in order to make an accurate determination of students' knowledge [4]. The preformed test may be an intermediate exam related to a small amount of learning material (10–20 questions in 20–30 min), or a full-scale exam based on the entire volume of the studied material (100–150 questions in 3–4 h).

5 Conclusion

The developed decision-making algorithm can determine the level of knowledge of a tested person on the basis of questioning with the minimum possible number of questions. This allows providing an evaluation of the students' knowledge level, over a short period of time, with a high degree of reliability when compared to the traditional method of questioning conducted by a teacher. Hence, an ingenious system of knowledge control has been developed via the application of fuzzy logic. It is very close to imitating the teacher's behavior in the process of student's questioning. It includes ability and precision that have not been seen before in any automated system. The proposed system integrates elements of expert systems, processes of development and populating a database, as well as construction of powerful and flexible rules. All system's aspects described above indicate effectiveness and flexibility of algorithms and functions used to build the knowledge control system. and confirm usefulness of applied newest technologies.

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Authors Biography

Professor Dr. Ali M. Abbasov - Minister of Communications and High Technologies of the Republic of Azerbaijan. He has a long and remarkable career in academia, education and the public sector. His recent experience includes the following positions: Director of Institute of Information Technologies of the National Academy of Sciences of the Republic of Azerbaijan (1991–2000), Rector of Azerbaijan State Economic University (2000–2004), Minister of Communications and High Technologies of the Republic of Azerbaijan (2004– present).

As a minister, Professor Dr. Ali Abbasov pays special attention to the acceleration of transition to the information society and formation of the digital economy, application of e-government solutions and new technologies, development of broadband services and human recourses. He is an initiator of a few national and regional projects and conducts their implementation. He also played special role in establishment and further development of the internet in Azerbaijan.

Professor Dr. Ali M. Abbasov has lectures and researches in the fields of digital economy, network design and application, artificial intelligence, Big data and some others. He did his PhD in microelectronics at the Academy of Sciences of Ukraine. In 1994, he defended his thesis, entitled "Information processing and systems management", and was awarded with degree of doctor of technical sciences. Dr. A.M. Abbasov is a full-time professor since 1996, and is a full member of the National Academy of Sciences of Azerbaijan since 2001. Furthermore, he is full member The World Academy of Sciences, International Informatization Academy, International Telecommunication Academy, International Academy of Engineering and as a fellow of the IEEE.

Professor A.M. Abbasov was a member of the National Parliament from 2000 through 2004, and was a member of Parliamentary Assembly of the Council of Europe for the period of 2001–2004. He is a member of the Strategy Council of the Global Alliance for ICT and Development since 2008, and is a Commissioner of the Broadband Commission for Digital Development since 2010.

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Her research interests include Artificial Intelligence, Soft Computing, Intelligent system, Machine Learning Techniques to Decision Making and Fuzzy Neural Network. She was awarded research grants in the following countries: India (1998), Germany-DAAD (1999), Germany-DAAD (2003), USA, UC Berkeley, Fulbright Visiting Scholar (2007/2008), Germany-DAAD (2010), Scientific Development Foundation Under the President of Azerbaijan (2014/2015). She is a member of Berkeley Initiative in Soft Computing group (BISC), New York Academy of Sciences, member of "Defined Candidate Dissertation" Society and member of

International Women Club, Azerbaijan. Dr. Shahbazova was invited to serve as a Program Committee member for over 30 international conferences and as reviewer for nearly 40 international journals. She was an elected board director of the North American Fuzzy Information Processing Society (NAFIPS) and served as Program Chair of Organizing committee for the NAFIPS Conferences. Dr. Shahbazova is a general chair of World Conference on Soft Computing. Also, she is an international expert UNESCO of implementation ICT in educational environment in Azerbaijan. She is co-editor of the books: “Soft Computing: State of the Art Theory and Novel Applications”, (Springer, Series Title: Studies in Fuzziness and Soft Computing, 2011), “Recent Developments and New Directions in Soft Computing” (Springer, Series Title: Studies in Fuzziness and Soft Computing, 2014) and Soft Computing: New Directions in Foundations and Applications (Springer, Series Title: Studies in Fuzziness and Soft Computing, 2015).

Dr. Shahnaz N. Shahbazova is a member of following projects: “Application ICT in education environment” (USA, Indiana, 2001), Information-communication technologies and higher education-priorities of modern society development”(Saint Petersburg, 2009), “Establishment of Educational Network System of Azerbaijan Technical University” (Korea, KOICA, 2011), “The problem of intelligent fusion of multi-source information in the social network model based on fuzzy sets” (Baku, 2014/2015).

Learning Systems with FUZZY

Sang Wan Lee and Z. Zenn Bien

Abstract Fuzzy techniques have been proven to effectively tackle the problems of uncertainty in relationships among variables in systems that learn to adapt to a changing environment. This paper outlines our challenges for the last 25 years to design learning systems with fuzzy techniques and their applications to many real world problems. We then focus on the development of human-in-the-loop systems, such as a smart home or an assistive robotic environment, that involve different types of learning strategies. This warrants a full consideration of learning mechanisms in humans that mediate action-selection. We envisage that the principles of fuzzy theory, when combined with what we know about computational learning mechanisms in the human brain, will offer a practical guidance on how we design learning systems to advance user's experience in real-world scenarios.

Keywords Learning system • Fuzzy • Human-in-the-loop system • System design • Human brain • Smart home • Multiple learning systems

1 Introduction

Many real-world problems pose daunting challenges for the design of engineering systems. The crux of them is arguably uncertainty inasmuch as one variable may not be exclusively dedicated to a single part of a system. There is a grey area where one variable has multiple roles in a system while there is a black and white area where contribution of a variable to the system is distinctive. Fuzzy theory directly

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_3

addresses the problem of uncertain nature of the relationship among variables, affording the fuzzy systems the leverage to model uncertain environments.

One other problem that raises challenges is the fact that a situation changes over time. This is why a system lacking an ability to cope with it sometimes fails to deliver credible performance in real-world situations even though the system has been previously proven to work well in a controlled environment. The system therefore needs to learn to adapt to a changing environment.

This paper describes our challenges for the last 25 years to design learning systems with fuzzy techniques as well as their applications to many real world problems. First, we introduce a few examples of designing fuzzified controllers that replace conventional control systems. These techniques are also applied to build human-in-the-loop systems in two different levels – one focusing on action recognition and the other focusing on intention reading underlying those actions. We then show how these ideas lead to an invention of integrated systems, such as a smart home or an assistive robotic environment. Since we have learned that design of such integrated systems essentially involves a combination of multiple learning systems, we finally suggest a new direction of system design based on what have been known about learning mechanisms in the human brain.

2 Reinventing Control Systems

Performance of an inference system depends on how accurately it describes relationships among variables. Considering that the degree of complexity exponentially increases with the number of internal variables, designing such systems inevitably entails the risk of overfitting. This means that the system is vulnerable to noise or change in an environment. Fuzzy logic remedies this problem by quantifying the amount of uncertainty in the relationships among internal variables of an inference system [1, 2]. The fuzzy technique has also been proven to be effective in designing an adaptive controller for nonlinear systems [3].

The idea of encoding uncertainty by means of fuzzy rule bases has been successfully applied to designing a fuzzy controller for many real-world applications. The technique allows a system to effectively resolve inconsistency in fuzzy rule bases [4]. This encourages us to deal with more realistic issues, such as multi-objective or time-delayed system design [5, 6].

Another line of research is to maintain reliable system performance in a dynamic environment. To meet this need, we attempted to design a nonlinear multi-input-multi-output system (MIMO) in such a way that incorporates learning capability [7]. The first challenge was that it is difficult to determine whether we create a new rule base or modify existing ones when we have a new set of observations. We solved this problem by borrowing an idea from rough set theory, by which we can find a minimal set of rules given new examples [8]. The next challenge was that the learning requires supervision for fine-tuning. The reinforcement learning, a semi-supervised learning technique based on Markovian decision process [9], have been shown to be a necessary component for dealing with system uncertainty [10, 11]. The combination of fuzzy systems and reinforcement learning techniques evolved into more sophisticated system designs incorporating a various types of neural networks, demonstrating that the combined system yields dramatic performance improvement in many real-world applications, such as gesture recognition, facial expression recognition, and even general-purpose on-line adaptive system, [12–15].

3 Learning in Human-in-the-Loop Systems

There has been a steadily-growing interest in developing service robotic systems that are capable of serving for human directly. For effective control and management, the robots and human are often equally considered as subsystems of the system: this type of system is called a “human-in-the-loop system”, where the occurrence of two-way interactions between human and robots is inevitable (for example, see Fig. 1a). In our studies, we restricted our attention to service robotic systems that are intended to assist the elderly or the persons with physical disability, and advocated that one of the major considerations for designing such robotic systems is “human-friendliness”. In doing so, the robot agents need to exert all the possible functional capabilities when interaction takes place, including sensing, recognition, and decision making, to the extent that the system places a minimum-possible burden to the on-users.

Technical challenge arises when designing each individual robot agents. In order for the robot agents to function as a viable observer or controller, it is necessary to learn to recognize various forms of human physical motion. However, difficulty arises when there is uncertainty either in human motions or in environment. Substantial progress has been made for the last twenty years toward developing reliable systems that directly tackle this challenge.

The study focuses on two different levels of recognition. The lower level of recognition deals with physical motions, such as gesture or footprint, and the higher level of recognition considers subtle features that underlies such motions, such as facial expressions, emotional states, or action planning. It is noted that the former approach enables us to design more reliable and robust system, while the latter creates an opportunity for more efficient two-way interactions between a human and robots by making predictions about future actions.

3.1 Learning to Recognize Observable States – Physical Motions

In many real world applications, an image processing lacking robustness against variability of color or edge is often doomed to failure of recognition. Our previous studies have demonstrated that fuzzy technique is an efficient tool for robust image

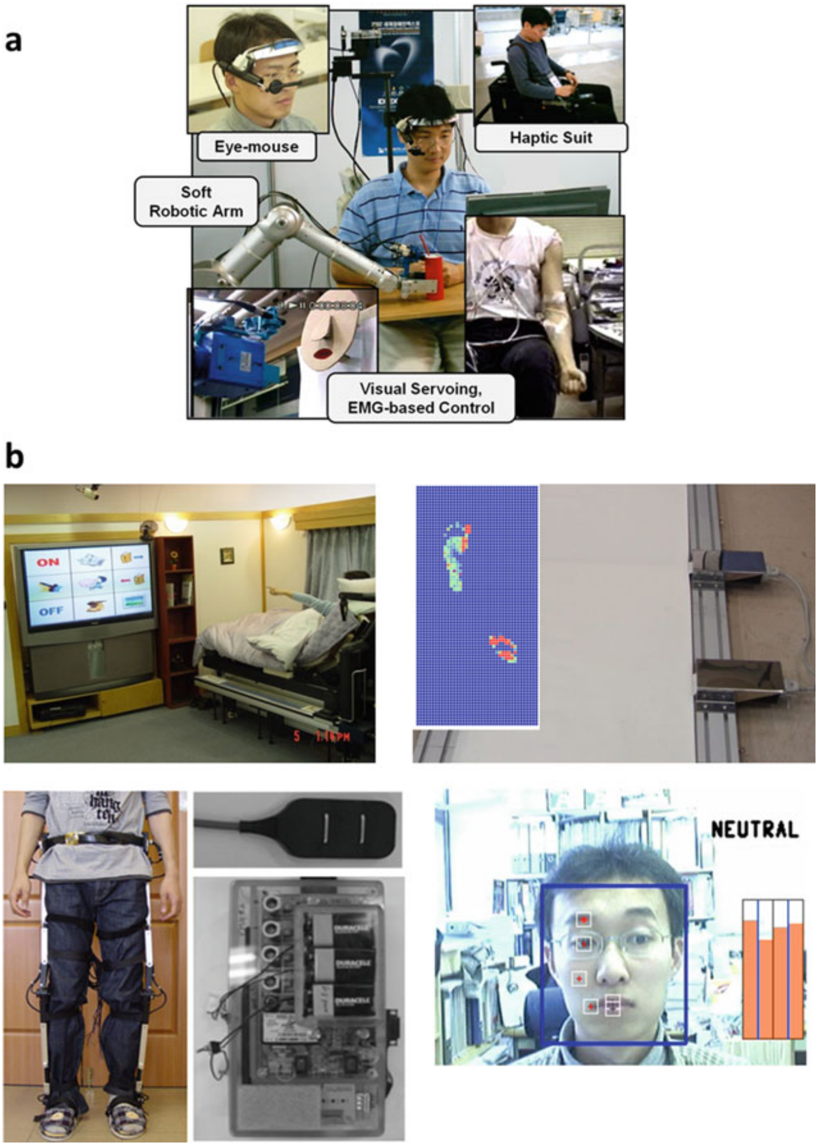

Fig. 1 Different types of human-in-the-loop systems. **(a)** KARES robotic agent [35]. The system is equipped with a robotic arm, an eye-mouse, and EMG-based control module. **(b)** Multiple types of recognition systems. The fuzzy technique has been applied to various recognition problems, demonstrating its effectiveness in resolving uncertainty in patterns of physical motions, such as hand gestures (upper-left), walking (upper-right), electromyography (EMG) signals (lower-left). The fuzzy learning technique has been also shown to be useful for reading-out of human intention, such as facial expressions (lower-right)

recognition [16–18]. This is perhaps not a surprising success considering how efficiently the fuzzy rule deals with uncertainty in these variables.

We then focused on developing a robust gesture recognition system: how does the system successfully learn to recognize hand commands or a sign language when faced with subjective and noisy representation of gestures? We tackled this problem by combining a multi-layered neural network and a fuzzy rule base that translates dynamic trajectories of gestures into discrete entities [13, 19]. This implementation has led to more interesting idea that the gesture can be used as a soft remote controller in service robotic environment [20] (Fig. 1b; upper-left), and later evolved to design of a beat gesture recognition system that interacts with an automatic music agent, such as a piano playing robot [21]. This technique has also been applied to systems with adaptation capability, in which the system can explore a new type of gestures [22] and automatically learn to recognize a new user's gestures [23]. It is noted that our studies on gesture recognition over the 10 years ensued an integrated sign language recognition system that is capable of recognizing and rendering more than 400 sign word gestures in real-time [13, 19, 24], as well as a patent on baby sign-language recognition [25] and a spin-off product which is now manufactured by a company.

The developments for dynamic hand gestures recognition and learning systems afford insight into how the fuzzy technique and learning mechanisms serve to resolve uncertainty in human motions in general. It stimulates another type of studies that focus on person identification based on dynamic patters of walking (Fig. 1b; upper-right). Specifically, the system recognizes a sequence of footprints by means of an estimation of foot shapes and a trajectory of center of gravity [26, 27]. By virtue of the fact that the recognition process requires a minimal effort but natural walking, it suggests an alternative to conventional identification based on finger prints or eyes which require extra processes for authentication. It also opens up a possibility of providing personalized services in a service robotic environment [28].

Whilst a camera and a pressure sensor have been demonstrated to be an effective means to learn from gesture and walking patterns, respectively, Electromyography (EMG) offers us more detailed guidance on what motions they actually plan to perform (Fig. 1b; lower-left). We have demonstrated that application of fuzzy techniques surmount a difficulty in counteracting adverse effects, such as fatigue or class inseparability [18, 29]. It is noted that this type of systems is particularly useful for the disabled or amputees given that the brain sends a distinctive signal pertaining to an intended motion to peripheral muscles.

3.2 Learning to Recognize Latent States – Intention Reading

The above mentioned systems are expected to function to learn from observations. However, perhaps more fundamental challenge to the study of human motions is how these actions take place in the first place, in other words, what are the hidden states underlies these actions. We hypothesized that motivational/emotional states play a pivotal role in galvanizing us into those actions.

A large amount of literatures assume that facial expression is an embodiment of emotional states, and this premise indeed helped us work out a practical solution to many human-robot interaction problems [30, 31]. In particular, wrinkles or stretched shapes of a face are known to be very effective features for recognizing facial emotions [32]. We have made a series of attempts to take these features into account (Fig. 1b; lower-right). First off, a system was proposed to establish a solid knowledge base of human experts, namely "fuzzy observer", which indirectly quantifies the amount of uncertainty in linguistic variables of the knowledge base [33]. This system is built upon a multiplayer neural network to perform parameter adjustment of the fuzzy observer. The idea has then developed into an adaptive learning scheme, dubbed as "personalized" facial expression recognition, where an addition of a new classifier, a modification of an existing classifier, and a feature selection process are streamlined and guided in an integrated fashion [15, 34].

4 Design of Integrated Learning Systems

4.1 Application to Smart Homes for Aiding the Disables

The "human-in-the-loop" system, in which both service robots and a human are considered as a part of its control loop, essentially addresses a need for seamless operation in our living environment, such as a smart home. It is particularly useful for people with movement disabilities because the system is required to engage in daily activities of the users with minimal interruption.

Our first effort has been made to design an intelligent robotic agent as a means to offer various kinds of proactive assistance [35]. The crux of the design was to balance usability with complexity of functions of a system; a user would be overwhelmed by the system if it had a complex user interface, regardless of how versatile the system is. We argued that the remedy to this problem lies in the psychological implications of how much the users feel comfortable to access a various functions of the system, called "human-friendly service" [36, 37]. A variety of human-robot interfaces have been implemented accordingly, including eye-mouse, head and shoulder user interfaces, and EMG signal interfaces, meeting the needs of different levels of disability.

From solicited feedbacks on these system from potential end-users with spinal cord injury, we learned that, in the presence of multiple robotic agents, an intermediate decision maker is required to facilitate effective communication between a user and individual modules and also to increase accessibility for novice users [37]. This stimulated designing a new type of a service robot, called "Steward robot"

[38]. The steward robot system has two novel features. First, it learns from user's behavioral patterns on a daily basis, providing personalized services. Second, the user interface is equipped with an emotional interaction module, which is intended to offer a human-friendly environment, as well as to enable the system to collect natural behavioral data. Taken together, we have demonstrated that a proper combination of fuzzy learning techniques is essential for efficient human-machine interaction in a smart home environment.

4.2 Integrating Multiple Learning Systems

We have undergone a transition from the design of each individual learning agent to the integration of these learning systems. Our earlier studies have focused on a low-level communication architecture of the smart home for the disabled, where a single control unit controls communication among multiple devices and robotic agents [39]. In a subsequent study, we have proposed higher level functional architecture to exert control over multiple robotic modules [40]. The proposed integrated system spanned all levels of control, from user interfaces to action units. The first layer, functioning as input devices, provides a user with human-machine interactions in manifold forms, such as a soft remote controller operated by hand gestures, a voice recognition system, and other types of sensory devices (a joystick or a touch screen). The second layer, functioning as monitoring devices, consists of a pressure sensor-based bed that detects body movement and postures and a health monitoring system that collect bio-signals. This layer also deals with environment parameters (illumination, humidity, and temperature). The next layer, a central control unit, receives inputs from the first two layers to execute a command for a variety of action units. The action units is the last layer of this architecture, which include a bed-mounted rehabilitation robot, mobile robots, a wheelchair-mounted robot, an intelligent bed, a robotic hoist, home appliances.

A considerable challenge arises when designing the last action unit layer is how to organize low-level commands (e.g., "move the robotic hoist", "turn off the light", or "position the wheelchair in front of the hoist") that is necessary for achieving an abstract-level goal set by a user (e.g., "I want to go out" or "I want to go to bed"). Motivated by the way humans draft a plan, the task knowledge organization system has been proposed by combining a top-down scenario analysis and a bottom-up commands development [41, 42]. The top-down process develops specific task structure by configuring task knowledge from the user's point of view, and then in the subsequent bottom-up process, the system simulates the user's scenario to assess validity of the developed tasks before actually executing a complete task sequence. This idea has been demonstrated in the KAIST's intelligent sweet home (ISH; Fig. 2) scenarios [42].

Fig. 2 KAIST's intelligent sweet home scenario. Multiple agents interact to provide a user with disabled lower-limbs a variety of proactive services, which consists of multiple low-level commands carried out by a transferring robot (shown in the upper left), a steward robot (shown in the upper right), a hand gesture-based soft remote control system (shown in the lower left), and an intelligent bed with a robotic arm (shown in the lower right)

4.3 Multiple Learning Systems: Algorithms Versus Human Brains

The integrated learning system in the last resort needs to be built upon the understanding of how humans learn and choose different strategies to reinforce behavior in a coherent manner. This leads to an emergence of an applicability of neural theory to the design of learning system. A series of seminal studies in neuroscience, in which dopamine neurons in behaving non-human primates and those target areas in humans implement a prediction error from a temporal difference reinforcement learning algorithm [43, 44], encourage us to utilize those types of learning model for system design. Subsequent studies combining the learning algorithms and neural data also found that multiple learning systems were implemented in human brains as opposed to just a single system [45]. This remarkable resemblance between the learning algorithm and the human brains, combined with our computational proposal of control of multiple learning algorithms to guide integrated behaviors [46], merits a test to understand how human brains exert control over these multiple learning systems. Our study recently demonstrated that such control mechanism is indeed implanted in the brain, specifically the allocation of control is based on the relative degree of "uncertainty" in the estimates from the two learning systems [47]. We thus envision that fuzzy theory might be useful again as we begin to understand that human brains use uncertainty information to implement a control for multiple learning strategies.

5 Outlook

The last three decades is the crucial period in the evolution of fuzzy theory. The effectiveness in dealing with uncertainty in system variables indeed enables itself to make a significant contribution to designing learning systems. Our developments using fuzzy techniques span a wide range of learning systems, demonstrating effectiveness in handling human users as a component in the control loop. The techniques also lend themselves well to formulating a design principle of an integrated system for the smart home for the disabled, in which multiple learning agents effectively interact to provide aids for the user.

On the other hand, we begin to understand how our brain learns from experiences, and more importantly, when and how it exerts control over multiple types of learning strategies based on uncertainty information. A few recent studies, directly pitting different types of computational learning models against each other to understand how the human brain arbitrate multiple learning systems, have provided us with an insight into how we design an integrated learning systems for real-world applications.

Taken all together, we envisage that the principles of fuzzy theory, when combined with what we know about computational learning mechanisms in the human brain, will not only advance user's experience in real-world scenarios but also offer a practical guidance on how we design learning systems.

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Fuzzy Modifiers at the Core of Interpretable Fuzzy Systems

Bernadette Bouchon-Meunier and Christophe Marsala

Abstract Fuzzy modifiers associated with linguistic hedges have been introduced by L.A. Zadeh at the early stage of approximate reasoning and they are fundamental elements in the management of interpretable systems. They can be regarded as a solution to the construction of fuzzy sets slightly different from original ones. We first present the main definitions of modifiers based on mathematical transformations of membership functions, mainly focusing on so-called post-modifiers and pre-modifiers, as well as definitions based on fuzzy relations. We show that measures of similarity are useful to evaluate the proximity between the original fuzzy sets and their modified form and we point out links between modifiers and similarities. We then propose an overview of application domains which can take advantage of fuzzy modifiers, for instance analogy-based reasoning, rule-based systems, gradual systems, databases, machine learning, image processing, and description logic. It can be observed that fuzzy modifiers are either constructed in a prior way by means of formal definitions or automatically learnt or tuned, for instance in hybrid systems involving genetic algorithm-based methods.

Keywords Fuzzy modifiers · Linguistics hedges · Similarity

1 Introduction

Human beings are very efficient in coping with real world complexity and human-like automated systems have been constructed for decades now, with the purpose of managing large size data, subjective and imperfect information and ill-known

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_4

environments. Concepts managed by human beings are often imprecise, with a core of easy to classify instances and a shadow of other instances [1].

Fuzzy modeling is well-suited for their representation and their use in automated systems. It is the reason why fuzzy systems have been a key solution to the management of complex systems since the introduction of linguistic variables and fuzzy if-then rules by Lotfi A Zadeh [2–5]. They are based on approximate descriptions of fuzzy variables by means of fuzzy modalities and relations between such descriptions. Qualities of fuzzy systems such as their expressiveness and their capacity to manage gradual knowledge have taken a large part in their success in real-world applications. Similarity, or its brother concepts resemblance, closeness, proximity, analogy, has been pointed out as fundamental in a number of domains, such as linguistics, semiology, psychology. It is particularly useful in computational intelligence, and especially in fuzzy modeling in which it takes part in the modeling of imprecision and classes with unsharp boundaries.

We focus in this paper on a particular representation of similarity between concepts by means of the utilization of fuzzy modifiers. In his seminal paper [6], L.A. Zadeh introduced the concept of modifier to represent linguistic hedges such as *very*, *more or less*, *slightly*, by means of a mathematical transformation of membership functions based on power functions. Psychometrical analyses and empirical studies were then proposed by [7–9]. The concept of modifier was extensively studied from a psychometrical or an empirical point of view [10–13] and gave rise to various works on mathematical, algebraic, or logical approaches [14, 15] as well as proposals to use fuzzy modifiers in soft computing.

In the first section of this paper, we first summarize the main formal definitions of modifiers, based on mathematical transformations or fuzzy relations. Then we consider modifiers from the point of view of measures of similarity. The main purpose of fuzzy modifiers being to take a part in the interpretability of fuzzy systems, we devote the second part to application domains which have made good use of modifiers. We conclude on the importance to preserve the link with fuzzy modifiers and linguistic hedges.

2 Fuzzy Modifiers

Let U be an ordered universe of discourse and $F(U)$ the set of fuzzy sets of U. For instance, U could be the set of real numbers $\mathbb{R}$, or a subset of that set. By convention, we use the same symbol A for a fuzzy set and its membership function. Given the various forms of linguistic hedges (*very*, *more or less*, *strongly*, *at least*, *extremely*, etc.), we summarize several approaches to their formal definition.

For membership degrees $A(x)$ associated with elements x of U, the general idea is to construct a new membership function, denoted by $m \circ A$ deduced from "close" elements of x in U or to consider a proximity between $A(x)$ and $m \circ A(x)$ for every x. In a more complex approach, it is also possible to consider a proximity between $A(x)$ and $m \circ A(y)$, for elements y "close" to x.

2.1 Fuzzy Modifiers Defined by Mathematical Transformations

To represent these two proximities, one on U and the other one on [0, 1], we consider the following generic definition of fuzzy modifiers [16].

A fuzzy modifier can be regarded as a pair $m = (g, h)$, where $g : [0, 1] \longrightarrow [0, 1]$ and $h : U \longrightarrow U$ are functions.

If A is a fuzzy set of U, then $m \circ A$ is also a fuzzy set of U defined for every x in U by: $m \circ A(x) = g \circ A \circ h(x)$. This definition is very general and corresponds to various transformations of a given fuzzy set, not necessarily related to proximities.

Complementation is one of them, even though the transformation is extreme, associated with the identity function h and the function defined by $g(x) = 1 - x$, representing the linguistic hedge *not*.

Normalization is another one, not associated with a linguistic hedge, but to a technical transformation defined by the function $h(x) = kx$, k being the largest value of x in U where A attains its maximum and $g(x) = \frac{x}{A(k)}$ in [17]. We can also imagine the simple normalization based on the identity function h and the function defined by $g(x) = \frac{x}{a}$, with $a = \max_x A(x)$.

Sharpeners or contrast intensification operators [18] are also fuzzy modifiers, such that h is again the identity function and g is a function such that $g(x) \geq x$ if $x \geq \frac{1}{2}$ and $g(x) < x$ if $x < \frac{1}{2}$. The most drastic sharpener corresponds to $g(x) = 1$ if $x \geq \frac{1}{2}$ and $g(x) = 0$ if $x < \frac{1}{2}$.

We present in the sequel the most important classes of fuzzy modifiers used in formal or applied research, with a focus on their interpretability, associated with linguistic hedges.

The following contrast enhancement operator is indicated by [19] for information fusion in signal and image processing:

$$g(x) = 2x^2 \text{ if } x < \frac{1}{2}$$

and

$$g(x) = 1 - 2(1 - x)^2 \text{ otherwise.}$$

2.2 Post-modifiers

If h is the identity function on U, then m is called a post-modifier [20]. Typical post-modifiers are reinforcing modifiers and weakening modifiers [21, 22] which extend the seminal forms of modifiers introduced by Zadeh [6], defined by $g(x) = x^{\alpha}$, with respectively $\alpha \geq 1$ in the case of reinforcing modifiers, and $\alpha \leq 1$ in the case of weakening modifiers.

More generally, reinforcing modifiers are such that $m \circ A(x) \leq A(x)$ for every x in U and correspond to linguistic modifiers such as *very*, *really*, or *strongly* with the idea of a more restrictive view of the underlying concept. The fuzzy set $m \circ A$ is more specific and/or more precise than A. The category described by $m \circ A$ is included in the category described by A.

Weakening modifiers are such that $m \circ A(x) \geq A(x)$ for every x in U and correspond to linguistic modifiers such as *approximately*, *rather* or *about* [21] which yield $m \circ A$ less specific and/or less precise than A.

Examples of functions g are homotheties applied to membership functions, either preserving the support of A and decreasing its specificity by extending its kernel (*approximately*), or preserving the kernel of A and extending its support to decrease its precision (*rather*). A softer form of modifier extends both kernel and support to decrease the specificity and the precision of A (*about*). In all these cases, the category represented by $m \circ A$ is wider than the category represented by A, with less sharp boundaries.

2.3 Pre-modifiers

It should be remarked that such fuzzy modifiers do not cover all forms of linguistic hedges. In some cases, a reinforcement or a weakening of the description represented by A corresponds to a decrease or an increase of the values of U in its kernel or its support. For instance, if U is a universe of length, it is very common to consider that the fuzzy set representing *very small* has a kernel or a support "before" the kernel or the support of small, with respect to the order on U. Symmetrically, *very long* will be represented by a fuzzy set with a kernel or a support "after" the one of long. This example shows the complexity of linguistic modifiers and the necessity to introduce another form of fuzzy modifiers, as follows.

If g is the identity function on [0, 1], then m is called a pre-modifier [20]. Typical forms of pre-modifiers are translatory modifiers [23] associated with functions h defining translations on U to the right or to the left, $h(x) = x + t$ for every x in U, for a positive or negative parameter t, to answer the above remark on the direction of the necessary modification according to the meaning of the description represented by A. Such modifiers are neither reinforcing nor weakening. The fuzzy set $m \circ A$ represents a category which is shifted to the upper zones of U or to the lower ones with respect to the original category represented by A. Such an approach can be useful in case of evolving categories, progressively moving on U over time.

2.4 Fuzzy Relation-Based Modifiers

The original concept of modifier was introduced to handle similar categories we can distinguish by means of subtle differences, for instance expressed by linguistic hedges. It is therefore natural to use measures of similarity [24] or fuzzy relations [25] to introduce and study fuzzy modifiers.

Let us consider a fuzzy relation R on U. [25] introduces families of relation-based modifiers. Let us focus on the family based on a conjonction operator $\top$, such that:

$$m \circ A(x) = \sup_{y \in U} \top(A(y), R(y,x)), \text{ for every } x \in U. \quad (1)$$

The particular case where $\top$ is the minimum yields a membership degree of every x to $m \circ A$ defined as the maximum value of membership degrees assigned to elements of U in relation R with x.

Some of these relation-based modifiers are expansive or restrictive. In this case, we can imagine R as a similarity relation, $m \circ A$ being then the maximum membership degree of all elements of U similar to x.

In the case where $\top$ is a t-norm and E a $\top -$ *equivalence* (reflexive, transitive and $\top$-transitive), we can consider a fuzzy ordering R such that $R(x,y) \geq E(x,y)$ and $\top(R(x,y), R(y,x)) \leq E(x,y)$ for every x and y in U [26]. Then Equation (1) yields a fuzzy modifier expressed as *at least* A. If we take the inverse ordering defined by $R^{(-1)}(x,y) = R(y,x)$, then we obtain a representation of the linguistic hedge *at most*.

3 Similarites and Modifiers

Another manner to take resemblances into account [24] consists in evaluating the "closeness" of A and $m \circ A$ in order to measure their similarity. We consider a fuzzy set measure $M : F(U) \longrightarrow \mathbb{R}^+$ such that $M(\emptyset) = 0$ and M is monotonous with respect to the classic inclusion of fuzzy sets $\subseteq$. We also consider a difference $\ominus$ between fuzzy sets, such that $A \ominus B$ is monotonous with respect to A and $A \subseteq B$ implies $A \ominus B = \emptyset$.

Such a measure can be used to evaluate the similarity between A and $m(A)$ for a modifier m. We restrict ourselves to so-called $(M, \varepsilon, \lambda)$-modifiers [27] such that $M(m \circ A \ominus A) = 1 - \varepsilon$ and $M(A \ominus m \circ A) = 1 - \lambda$, for two parameters ε and λ in $[0, 1]$.

Particular cases of such $(M, \varepsilon, \lambda)$-modifiers are $(M, \varepsilon, 1)$-modifiers which are expansive, $(M, 1, \lambda)$-modifiers which are restrictive, and translatory modifiers being particular cases of $(M, \varepsilon, \varepsilon)$-modifiers. $(M, 1, 1)$-modifiers can be regarded as providing the closest modified forms of the primitive fuzzy set.

An example is the linguistic hedge *approximately* represented by

$$m_1 \circ A(x) = \min(1, \varepsilon \cdot A(x)),$$

for every x in U, for $\varepsilon \in [0, 1]$, corresponds to a modifier preserving the support of A and extending its kernel to decrease its specificity.

It is easy to see [27] that it can be considered as a $(M_1, 1, 1)-$modifier if we define $M_1(A) = \int_{x \in U} A(x)dx$ and the difference between fuzzy sets as $A \ominus_2 B = A(x)$ if $B(x) = 0$ and $A \ominus_2 B = 0$ if $B(x) > 0$, It can also be regarded as a $(M_2, \frac{1}{\varepsilon}, 1)$ when we define $M_2(A) = \sup_{x \in U} A(x)$ with the difference $A \ominus_1 B = \max(0, A(x) - B(x))$. This points out the role of the perception of "similarity", strongly dependent on parameters ε and λ, as well as chosen operators M and $\ominus$. Another simple form of modifier is the representation of uncertainty as follows:

$$m_2 \circ A(x) = \max(A(x), \varepsilon),$$

for every x in U, which can be expressed as *A with an uncertainty* ε. Such modifiers are $(M_2, 1 - \varepsilon, 1)$-modifiers with the difference $\ominus_1$, which shows that they are not far from the original description A.

Going further in the use of similarities between fuzzy descriptions and their modified forms, we consider a measure of similarity S on U defined as a function $S : F(U) \times F(U) \longrightarrow [0, 1]$, such that $S(A, B) = F(M(A \cap B), M(A \ominus B), M(B \ominus A))$ is non-decreasing with respect to $M(A \cap B)$ and non-increasing with respect to $M(A \ominus B)$ and to $M(B \ominus A)$.

Particular measures are defined, according to their specific properties [28]. A measure of satisfiability is exclusive, which means that $S(A, B) = 0$ when $A \cap B = \emptyset$, and independent of $M(A \ominus B)$. A measure of inclusion is also exclusive. In addition, it is independent of $M(B \ominus A)$. They correspond to the idea that A is a reference to which B is compared, the measure of inclusion being only interested in the extent to which B can be considered as a particular case of A. A measure of resemblance is symmetric in $M(A \ominus B)$ and $M(B \ominus A)$, which means that there is no reference and both A and B have the same status in the research of similarity. It is easy to see that, if the modifier m is expansive, then A and $m \circ A$ will be compared through a measure of satisfiability. If m is restrictive, A and $m \circ A$ will be compared through a measure of inclusion. If m is translatory, A and $m \circ A$ will be compared by means of a measure of resemblance [24].

4 Application Domains Using Modifiers

4.1 Analogy-Based Reasoning

Evaluations of the proximity or similarity between A and $m(A)$ mentioned previously can for instance be used in an analogy-based reasoning, or case-based reasoning, to construct interpretable conclusions from observations [24].

Starting from a rule such as *If X is A*, *then Y is B*, for $i = 1, \ldots, n$, or cases such that X is A at the same time as Y is B, an observation A' will be compared to A to evaluated their similarity $S(A, A')$. One way to determine the description B' of Y is to assume that $S(A, A') = S(B, B')$ and to make a choice among all fuzzy sets B' satisfying this constraint.

Using modifiers limits the number of solutions and provides an easily interpretable solution. According to the relations between measures of similarity and modifiers, we use an expansive modifier m to describe $B' = m \circ B$ if we choose a measure of satisfiability S.

As an example, let us consider the following measure of satisfiability: $S(A, A') = 1 - M_2(A' \ominus A)$. If the value of $S(A, A')$ is σ, then we can use a modifier such as m_1 for the parameter $\frac{1}{\sigma}$ interpreted as *approximately,* or m_2 for the parameter $1 - \sigma$, interpreted as A *with an uncertainty* $1 - \sigma$.

If we use a measure of inclusion S, a restrictive modifier is convenient to express the difference between the two fuzzy sets. For instance, let us consider the following measure of inclusion: $S(A, A') = 1 - M_2(A \ominus A')$. If σ is the obtained value, we can think of a modifier m such as $m \circ B(x) = min(\sigma, B(x))$.

If we use a measure of resemblance, we associate it with a translatory modifier that we will not describe in detail.

4.2 Rule-Base Systems

Rule-based systems and the particular case of fuzzy control are the first domains where modifiers help to obtain interpretable results. The interpretability of rule-based systems is complex and has given rise to various analyses. We can mainly point out three important factors of this interpretability: the easily understandable linguistic description of variables, the number of rules and the number of premises in each rule.

One of the first attempts to manage linguistic labels in a fuzzy knowledge-based system was the linguistic approximation used in the MILORD system to deal with both uncertainty and imprecision [29].

With regard to easily understandable descriptions of variables, reasoning with modifiers provides interpretable conclusions when using generalized modus ponens [16, 22] with rules of the form : *If X is* A_i *then Y is* B_i, for $i = 1, \dots, n$.

In particular, if we use restrictive modifiers, we obviously obtain a conclusion identical with the conclusion of the rule with all classic fuzzy implications [21]. When we use expansive modifiers such as *approximately* defined by $m_1 \circ A_i$ to describe observations, for instance, such rules provide conclusions of the form $m_1 \circ B_i$ itself, or $m_2 \circ B_i$ (representing an uncertainty on B_i), or $m_3 \circ B_i$ for some other expansive modifier m_3. These forms of conclusions are easily interpretable, which is not the case when using general modus ponens with any observation. We can conclude that, if an observation is similar to a premise A_i through a modifier, the obtained conclusion is also similar to the conclusion of the rule through a modifier when using the most classic fuzzy implications.

Fuzzy modifiers are also useful to adjust the shape of membership functions dynamically in the design of a fuzzy controller. In [30], the number of premises is limited to three to simplify the design of fuzzy controllers. A so-called linguistic hedge module is added to the fuzzy controller to play the role of powered modifiers and dynamically modify the shape of membership functions to the feedback signal, to provide additional information without increasing the number of rules. Genetic algorithms are used to tune membership functions by means of modifiers in the design of the rule base.

In a different approach, [31] proves that modifiers are useful to find a trade-off between interpretability and accuracy in the construction of rule-based systems. The authors propose to extend the initial list of linguistic descriptions and to create new rules of the form: *If* X_1 *is* $m_{i1} \circ A_{i1}$, *and* X_2 *is* $m_{i2} \circ A_{i2}$ *and* ... *then* Y *is* $m_i \circ B_i$, for $i = 1, \ldots, n$. They consider powered, expansive or restrictive modifiers, as well as translatory ones.

In [32], the authors propose fuzzy modifiers to be learnt to obtain the best fuzzy rule-based classification system. After a prior rule base is constructed on the basis of pre-defined linguistic descriptions represented by fuzzy sets, a genetic algorithm-based method is used to select the best subset of rules and to learn the set of linguistic modifiers to apply to the linguistic variables for the considered fuzzy logic system.

Gonzalez et al. [33] consider also linguistic hedges included in a genetic algorithm in the framework of the inductive learning algorithm called Structural Learning Algorithm on Vague Environment. They help the user to learn and tune fuzzy rules.

In [26], it is proposed to introduce modifiers to reduce the size of a fuzzy rule base and thus to enhance its interpretability and expressiveness. The proposal is, first of all, to group rules leading to the same output decision, and to rank them according to the input parameters they involve. Then, such "neighboring" rules can be merged and replaced by a rule which summarizes their premises into a single one constructed by means of a modifier. For instance, in a PD-style fuzzy controller set of rules, "Negative Big", "Negative Small" and "Zero" are grouped and these three fuzzy values are replaced by "*at most* Zero". Moreover, the authors highlight the fact that modifiers could be very useful in interpolative reasoning when the fuzzy rule base has been constructed incomplete. The proposed work in this paper is based on the use of a fuzzy ordering (see Sect. 2.4).

4.3 Gradual Systems

Their ability to manage graduality is an important property of fuzzy set-based representations. The most basic view of graduality corresponds to the unsharp boundaries of a category represented by A, in which we enter progressively with slightly increasing membership degrees and from which we get out with slightly decreasing membership degrees when we progress along U. This graduality is handled by the concept of fuzzy set itself.

A second type of graduality corresponds to a change in the boundaries of a category, reducing or extending it according to the context or the observations, and this graduality is clearly handled by means of weakening or reinforcing post-modifiers.

A third type of graduality corresponds to a progression along U, in the case where we have a family of fuzzy sets $A_1, \ldots, A_n$, describing a variable defined on U, for instance classes of a fuzzy partition of U. This graduality corresponds to values of the variable smoothly evolving from a category A_i to the next one A_{i+1}. This graduality is clearly related to the use of adaptive pre-modifiers.

A deductive management of graduality [22] enables to take into account various forms of fuzzy rules. One of the most natural forms of gradual knowledge for human experts is the following: *The more (less) X is A*, *the more (less) Y is B*, for which solutions are not obvious in automated systems.

Generalized modus ponens and the use of modifiers to represent *more or less* leads to an automated deductive system translating such rules directly or with consideration of uncertainties, according to the chosen fuzzy implication, in rules of the form: *It is rather certain that the less X is A*, *the less X is B*, or *the more certain X is A*, *the more certain Y is B*. This graduality can be considered as restricted to one rule and then local.

A global graduality can be managed through a collection of rules such as: *The more (less) X is* A_i, *the more (less) Y is* B_i, for i between 1 and n, the conclusion moving progressively from B_i to B_{i+1} or B_{i-1} when the observation varies from A_i to A_{i+1} or A_{i-1}.

4.4 Other Uses of Modifiers

More generally, modifiers have been introduced in several other kinds of applications.

Databases is a domain where linguistic hedges are often seen useful (for instance, in [34, 35]). In [34], modifiers are integrated in SQLf, a fuzzy extension of SQL, the well-known query language for databases. Here, the “where” part of a select query is associated with a fuzzy condition that could be defined by means of a modifier. A modifier is used to define a partition of the tuples from the database that can be searched for, for instance, “select * from ... where salary is *more or less equal to* ...”. As a consequence, querying becomes more naturally expressed thanks to interpretable conditions.

An example of use of modifiers in a machine learning task could be found in [36]. In this work, the linguistic hedges (Zadeh’s form) have been introduced in a mining fuzzy association rules process. Various hedges are generated in order to increase a fuzzy taxonomy associated with a transaction. The mining of fuzzy association rules is done with this augmented fuzzy taxonomy in order to find a maximum of rules, taking into account linguistic descriptions.

In [37], a fuzzy description logic is introduced that handles hedges. Concept modifiers are introduced as a chain of hedges. The sign of a hedge is used to position the hedge with regards to the associated primitive concept. For instance, the sign of *very large* is positive with regards to *large*, and the sign of *more or less large* is negative. Modifier Memberships are those of Zadeh (exponentiation). The authors introduce a semantic based on hedge algebras and propose a decision procedure and an approach to determine the satisfiability of fuzzy constraints in their fuzzy description logic.

Image processing or image-based retrieval, in which a linguistic description of images could be very useful to summarize their content, or to enhance their interpretability, are often based on the use of linguistic hedges. Among existing works, we can cite [38] which introduces linguistic hedges to model the feedback of the user when the retrieved image is not fully satisfactory. The user proposes a modifier on the current query to create a new query that could be processed to enhance the search.

In the acoustics domain, [39] has proposed the used of modifiers in a vocabulary used to describe noise annoyance expresses by users with different languages. Similarity measures are then used to match foreign terms and find correspondence among them.

5 Conclusion

We have presented the main solutions to construct fuzzy modifiers associated with linguistic hedges, in an attempt to expand the vocabulary available to describe objects, either in a knowledge-base or in the outcomes of a fuzzy system. The purpose is to provide a flexible way to represent knowledge without increasing the complexity of the system. The main qualities of fuzzy systems are kept in mind: first their interpretability and their ability to handle easy to understand descriptions of objects, and second the graduality inherent in their definitions, providing soft transitions between descriptions and mimicing a very natural facet of human reasoning.

Fuzzy modifiers are defined in a prior way on the basis of expert knowledge or psychometric analyses. Among the various approaches enabling to provide their definitions, we have given priority to those which are involved in the interpretability of fuzzy systems. We have shown that fuzzy modifiers can also be learnt or tuned automatically, for instance with the help of genetic algorithms in hybrid systems. It is clear that these ways to obtain linguistic modifiers are similar to those providing membership functions of fuzzy sets representing linguistic descriptions of variables. They can be regarded as a kind of standardization of linguistic terms easy to handle and to share with experts or end users of fuzzy systems.

Future works can focus on the specific methods to use fuzzy modifiers in new areas, such as fuzzy case-based reasoning, fuzzy inductive learning or fuzzy summarization, to name but a few.

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Human and Machine Intelligence — Between Fuzzy Logic and Daoist Thought

Liya Ding and Xiaogan Liu

Abstract The theory of fuzziness, offering an important scientific approach in building intelligent machines, has been researched and developed in the past fifty years from various perspectives and applied for real world problem solving in many areas. Daoist thought, being one of the most influential schools of Chinese philosophy, has been studied for more than two thousand years and its wisdom exploited from generation to generation. Would a natural echo exist between the modern fuzzy thinking and the ancient oriental Daoist thought?

Keywords Fuzzy logic · Precisiation · Computing with words · Daoism · Nonaction

1 Fuzziness in Modeling Reality

Mathematical models are built for human dealing with the real world. However, the subtle behavior of the natural world (as we perceive it) cannot be modelled by rigid axioms. Most human concepts lack a rigorous definition for its vagueness and imprecision as well as its changing meaning reflecting evolving nature and human societies, the more rigorous the model, the less similar to reality. The great achievement of the theory of fuzziness is to have succeeded to build models for entities that lack a rigorous definition.

The famous Turing Test remains a dream of artificial intelligence until today. Human intelligence has been taken as the gold standard of machine intelligence, but such gold standard also lacks a rigorous definition. We apprehend the inner and the outer world by vague feelings, which become progressively more precise. "… the

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_5

machinery of fuzzy logic is needed for mechanization of human reasoning. In this perspective, fuzzy logic is of direct relevance to achievement of *human level machine intelligence*." (Zadeh) [1]

To further emphasize why and how we can expect that fuzzy logic plays the role of a bridge from natural to machine intelligence, Professor Zadeh has put forward the following words:

> *"Science deals not with reality but with models of reality. In large measure, scientific progress is driven by a quest for better models of reality. In the real world, imprecision, uncertainty and complexity have a pervasive presence. In this setting, construction of better models of reality requires a better understanding of how to deal effectively with imprecision, uncertainty and complexity. To a significant degree, development of fuzzy logic has been, and continues to be, motivated by this need.*" (Zadeh) [2]

What can be read out from the words above are guiding ideas and thought that are significant both technically and philosophically. From technical perspective, the development of a good model of reality needs to take into account imprecision, uncertainty and complexity. From philosophical perspective, humans deal with reality not directly but through models that are only approximation of the real world, and the better we handle imprecision, uncertainty and complexity, the better the model we may be able to build.

It is the philosophized thought of fuzzy logic that inspires us to further discuss some fundamental questions about human and machine intelligence, and explore a natural echo between fuzzy logic and Daoist thought, being one of the most influential schools of Chinese philosophy.

2 Admission of Imperfection

> *"A concept which has a position of centrality in fuzzy logic is that of a fuzzy set. Informally, a fuzzy set is a class with a fuzzy boundary, implying a gradual transition from membership to nonmembership. A fuzzy set is precisiated through graduation, that is, through association with a scale of grades of membership. Thus, membership in a fuzzy set is a matter of degree. Importantly, in fuzzy logic everything is or is allowed to be graduated, that is, be a matter of degree. Furthermore, in fuzzy logic everything is or is allowed to be granulated, with a granule being a clump of attribute-values drawn together by indistinguishability, equivalence, similarity, proximity or functionality. Graduation and granulation form the core of fuzzy logic."* (Zadeh) [2]

The concept of "graded membership" applies to a class that lacks a rigorous definition. Such concept is often confused with the concept of probability, which is caused by an underlying random process or by lack of information. A situation of lack of information may be improved when more information and data become available, especially with the growth of big data and information technologies. The lack of definition, however, relates to a more fundamental limitation in human's cognitive ability.

Although imprecision is a phenomena rooted at the limitation in human's cognitive ability, the idea of fuzziness is not universally accepted even today. To this situation, Professor Zadeh says, "there are many misconceptions and misunderstandings regarding fuzzy set theory and fuzzy logic. To some, I was advocating an abandonment of the deep-seated tradition of striving for rigor and precision."[1] While most scientists are used to consider that accuracy and certainty are essential principle for exploring true nature of the world, Daoist[2] thinkers believe that the ultimate source and true nature of the universe is uncertain and obscure.

As many philosophical and religious traditions, Daoism has its reflection and doctrine about the source and ground of the universe, which is not so certain and acknowledgeable. One significant piece on this theme, which is found in the *Laozi* [3], is not providing merely theory about the beginning of our world and myriad things in the universe, but also the foundation of all other Daoist teachings and characters. The unique speculation and reflection about the possible entity and state, from which the universe comes, is a key for understanding the general attributes and uniqueness of Daoist philosophy. There are noticeable points of Daoist thought, and one comes of the most significant points is concerning the true nature of the universe.

There was something undifferentiated and yet complete,
Which existed before heaven and earth. (Chap. 25)

The word "undifferentiated" in the above quotation is a render of the Chinese word "hun" which may suggest mixture, unclear, or shapeless etc. The state of the origin and the formation of the beginning of the world, and the ground that sustains myriad things are merely ambiguous. In the *Laozi*, all sentences related to the ultimate reason and truth share the same style. Thus Daoist statements about the true nature of the universe are always hesitating; at least they seem to reflect non-perfect confidence. Thus, the *Laozi* further states:

It may be considered the mother of the universe.
I do not know its name; I style it "Dao" (Tao, Way).
If forced to give it a name, I shall call it Great. (ibid.)

The author of *Laozi* frankly admits that he does not know what is the origin or the mother of the universe. *Dao* is just a styled symbol or nickname of the source

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[1]Lotfi A. Zadeh has distributed the same message repeatedly, the quotation here is from his message distributed to BISC online discussion group on Oct 2013.

[2]The word *Daoism* suggests complicated thought system (or *Daoist* philosophy) and Daoist religious movements. In this essay, the Daoist theories are mainly based on the first Daoist text *Daodejing* (Tao-te-ching) or the *Laozi* (Lao-tzu). This has nothing to do with Daoist religious teachings. The quotations from the *Laozi* are based on Liu's complication and adaptation from various versions and translations, unless specific citations provided otherwise.

and ground of the universe. It has no proper name. If being forced to give it a name, he will call it *Great*. Obviously, great, similar to good, bad, big, small, etc., cannot be used as a proper name. What the author repeatedly emphasizes is that no one knows exactly the foundation and true nature of the world, though the author does believe there is something functioning as the ultimate source and ground of all beings in the world.

This makes Daoism different from many religious and philosophical doctrines. The Bible describes clearly and precisely the processing of God's creation of the universe; Plato's theory sees it specifically and systematically as the relation between the transcendent kingdom of perfect ideas and the empirical world of imperfect myriad things. Even in other Chinese or oriental religious and philosophical schools, we find such differences from Daoism. Buddhism confidently disserts that the truth of world is essentially empty, all existence we can see and feel are just delusion and untrue; Confucian thinkers strongly believe that the moral doctrine *tian-li* (heavenly principle) is the ultimate truth of the world.

Further examining, we find some key concepts in which both the theory of fuzziness and the thought of Daoism surprisingly coincide. Here we list a few.

A. True and False

In fuzzy logic everything is, or is allowed to be graduated, that is, be a matter of degree. Fuzzy logic allows no clear distinguishing between true and false, and introduces a numerical grade or word to indicate the degree of truth of a piece of knowledge or information. In fuzzy inference systems, a perfect truth is not required for data, information and knowledge, and the executions of inference are done on an approximate basis. This greatly extends the ability of precise reasoning in handling real world problems with imprecision, incompleteness, or partial truth. In such, fuzzy logic establishes fuzzy reasoning containing precise reasoning as its special case.

From the Daoist viewpoint, there is nothing absolutely right and true, yet true and false dose not distinct clearly. The *Laozi* claims in Chap. 41:

> *The Dao which is bright appears to be dark.*
> *The Dao which goes forward appears to fall backward.*
> *The Dao which is level like a valley (hollow).*
> *Great purity appears like disgrace.*
> *Far-reaching virtue appears as if insufficient.*
> *True substance appears to be changeable.*
> *Solid virtue appears as if unsteady.*
> *True substance appears to be changeable.*
> ([4]: 160)

B. Good and Bad

The idea that "membership in a fuzzy set is a matter of degree" made a fundamental revolution to classical set theory, that is membership and non-membership are not in an absolute distinguishing. This is depicted in Fig. 1.

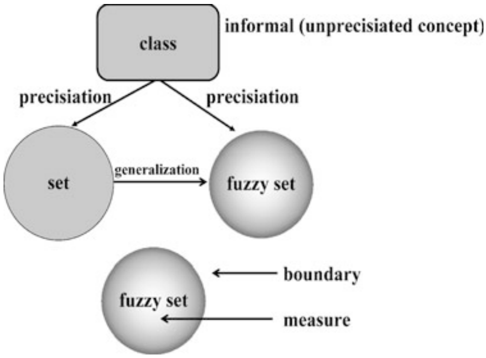

Fig. 1 The concepts of a set and a fuzzy set are derived from the concept of a class through precisiation. A fuzzy set has a fuzzy boundary. A fuzzy set is precisiated through graduation (Fig. 1 from Zadeh in [2])

In our everyday life, *good* and *bad* are often treated as two opposite concepts. However, when the two are applied to make evaluation with the same criteria, they both appear to be fuzzy sets with unsharpened boundary and possible overlap in between. The same spirit applies to many pairs of concepts that are usually considered opposite, such as *long* and *short*, *big* and *small*, *high* and *low*, *fast* and *slow*, etc. The distinguishing factors between these paired concepts are only relative. Some well accepted and widely adopted schemes in our life, such as “passing” or “failure” of school examination, are a binary precisiation of “good-bad” defined as crisp sets on numerical scores. The theory of fuzzy set provides a mathematical tool to handle concepts that are not fully distinguishable and differentiable.

The sentences previously quoted from *Laozi* Chap. 41 reveal Daoist position that everything appears as containing the opposite elements, which are in contradiction and mutual transformation. In addition, the uncertainty comes from the transformation of the oppositional elements in all beings or things. For example, the *Laozi* argues in Chap. 58:

Calamity is that upon which happiness depends;
Happiness is that in which calamity is latent. …
Then the correct again becomes the perverse
And the goodness will again become evil.
([4]: 167)

Similar argument stated in Chap. 2 of *Laozi*:

When the people of the world all know [certain] beauty as beauty,
There arises the recognition of ugliness.
When they all know the good as good,
There arises the recognition of evil.
Therefore:
Being and non-being produce each other;
Difficult and easy complete each other;
Long and short contrast each other;

High and low distinguish each other;
Front and back follow each other.
Therefore the sage manages affairs without action.
And spreads doctrines without words. ([4]: 140)

According to the *Laozi*, all seemingly quite different oppositions are actually in transition and mutually effected. All these oppositional transformations suggest the difficulty of our knowledge and recognition in reality. It should be noticed that when mentioning "manages affairs without action" and "spreads doctrines without words" the *Laozi* does not mean to do nothing, but special action different from regular actions by common rulers and people. We shall relate to this point in Sect. 4 with fuzzy decision.

The important finding here is that both Fuzzy Logic and Daoist thought are aware of the fundamental limitation in human's cognitive ability in handling the complexity of real world. While Daoism establishes the philosophical foundation for the discussion of this limitation, fuzzy logic provides a practical mathematical tool to describe concepts that are not fully distinguishable and differentiable. In other words, Daoism explores the limitations and the reasons that they exist, and fuzzy logic discusses how we can act under such limitations.

3 Precisiation and Description

In order to have a machinery to deal with imprecision in reality, we need *precisiation* for our understanding and description of reality. Informally, precisiation is an operation which transforms an object, p, into another object, p^*, which is more precisely defined, in some specified sense, than p. [2]. This is depicted in Fig. 2.

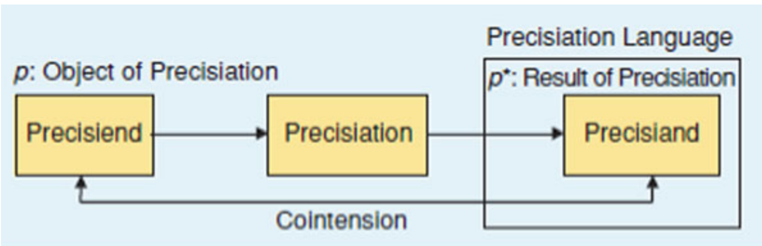

Fig. 2 Basic concepts relating to precisiation and cointension. (Fig. 14 from Zadeh in [2])

The significance of the theory of precisiation may be understood from two aspects: a technical aspect, and a philosophical aspect. From the technical aspect, with the utilization of fuzzy sets, fuzzy truths, fuzzy numbers, type-2 or higher order fuzziness, it allows but not ignores or rejects undifferentiation; it provides a spectrum between true and false, know and don't-know, to more naturally reflect the human knowledge imperfection; it offers tools for describing highly complex situation with mixed types of imperfection involved.

From the philosophical aspect, it accepts the limitations in human cognitive ability, therefore sets up a rational foundation for discussion of machine intelligence having human intelligence as the gold standard. There are many things in the universe that we don't know, and more importantly there are also many things we don't know how much we know, or don't know how to describe or evaluate what we know. Precisiation is a process through which we describe, and evaluate the world we perceive with our limited capability and resources. The result of precisiation is a simplified model of reality but not the true reality, referring to Fig. 2, we have $p^* \neq p$.

The philosophical thinking behind the fuzziness and precisiation finds an echo from the Daoist admission of the limitation of human's cognition.

The general feature of Dao combines both being and nonbeing, though not in a strait forward way. All descriptions of Dao seem to suggest non-being (*wu*), in the sense that human beings cannot grasp it because it is not any concrete thing humans can perfectly capture. The *Laozi*, in Chap. 14, describes it this way:

We look at it and do not see it,
its name is the invisible.
We listen to it and do not hear it,
its name is the inaudible.
We touch it and do not find it,
its name is the subtle (formless)….
Infinite and boundless,
it cannot be given any name ['unnameability']
It reverts to nothingness,
this is called shape without shape,
image without entity…
([4]: 146)

"It cannot be given any name" or it cannot be nameable implies the impossibility for human beings to cognize Dao, because it is transcendent and out of our approach. Similar ideas are presented in the *Laozi* Chap. 25, which reveals the Daoist conception of the source and ground of the universe. Here it reads:

There was something undifferentiated and yet complete,
Which existed before heaven and earth.
Soundless and formless, it depends on nothing, and does not change.
It may be considered the mother of the universe.
I do not know its name; I style it "Dao" (Tao, Way).
If forced to give it a name, I shall call it Great.
Now being great means functioning everywhere.
Functioning everywhere means far-reaching.
Being far-reaching means returning to the original state.

This is a reflection of Daoist attitude towards human being's capacity of cognition and understanding. Daoism has little dogmatic assertion; instead, Daoism intends to faithfully represent the difficulty and limitation of human beings observation and understanding.

The words “soundless and formless” in the above-quoted passage is also a kind of *description* of Dao in coping with the ambiguity. Without a definition a “description” leaves a room of tolerance for imprecision and vagueness. The limitation of human beings’ intelligence and cognition is an important concern of Daoism. We can further examine *knowing* and *unknowing* in dealing the reality to better understand Dao. In Chap. 1, the *Laozi* argues:

The way can be spoken of,
but it will not be the constant way;
The name can be named,
but is will not be the constant name.
The nameless is the beginning of myriad things,
The named is the mother of myriad things. ([5]: 267)

Here the *nameless* and the *named* are two characteristic aspects of Dao, which refer to the human faculty of recognition of Dao and also its limitation.

Furthermore from the admission of the fundamental limitation in human’s cognitive ability, another important finding here is that both Fuzzy Logic and Daoist thought are aware of the gap existing between any description (model) and the reality being described. While Daoism has this idea supported by its ancient wisdom, fuzzy logic names precisiation as one of the unique characteristics in human thinking towards modern machine intelligence.

4 Decision with Imperfection and Action with Nonaction

Fuzzy decision [6], or decision in fuzzy environment, is the confluence of fuzzy goals and constraints that reflect naturally real world situations. One of the key features found in fuzzy decision is that the decision maker is not forced to give a precise formulation, merely for the sake of mathematical reasons. A fuzzy decision is made with a compromise of the satisfaction of multiple constraints and objectives that are described with fuzziness. The importance or proportion of contribution of each constraint or objective may be arranged in appropriate ways.

There are two key ideas here that attract our attention. The first is that one is not forced to provide a precise formulation, if the original problem comes from a fuzzy environment; the second is that a decision is made as the confluence of fuzzy goals and constraints for a compromise.

Let’s examine Daoist perspectives in decision and action. Viewing the inevitable transformation of everything in the world, the *Laozi* argues for a distinctive way of actions, of which the complicated meaning cannot be translated, therefore we take a compromised way using nonaction as a token for the Chinese term *wu-wei*. Literally, it sounds like no action at all, but actually, it implies a special way of action, a negation of regular ways, for transcendent and better results. That is summed as a famous saying: “To do nothing yet leave nothing undone.” This way is different or opposite from common ways to deal with governance, as well as general affairs.

The spirit of nonaction is to do business for better or distinctive outcomes and minimum side or bad effects.

Aiming at a confluence of fuzzy goals and constraints allows one to make an optimal decision with multiple criteria and objectives taken into account. Without requiring forced precision in modeling fuzzy decision making, one will be able to keep more information (with imprecision and vagueness) from reality and bring that to final decision; and will also be able to minimize the extra inaccuracy introduced through forced precision in early stages. In other words, no forced action for precision leads to more accuracy to reality. At this point, we find another consistency between the spirits of fuzzy logic and Daoism in their deep roots.

For the Daoist aspects, a typical manner of nonaction is *assisting* (*fu* in the original *Laozi* literature, which can be translated as to assist, help, or support, etc.) From the *Laozi* Chap. 64, we read:

> *Therefore the sage desires not to desire,*
> *And does not value goods that are hard to come by;*
> *He studies what is not studied,*
> *And makes good of the mistakes of the multitude.*
> *And so the sage is*
> ***able** to assist the myriad things'* ***naturalness**,*
> *but is **unable** to act* [in the common manner]

As for the meaning of *fu* or assisting, it is better understood as a spectrum between two extremes. One extreme is restraint, manipulation, interruption, interference, exploitation, control, and oppression; the other is pampering, spoiling, indulgence, permissiveness, and over-protection. Thus, *fu* or assistance is the careful and prudent art of sagely leadership; its purpose and objective are completely aimed at benefiting the myriad things, no aspect of which shows off the sage's own importance and intelligence or accrues personal benefits. This *fu* or assistance is a typical example of nonaction for better results from unusual actions. The key point lies in the last sentence: the sage *assists* the myriad creatures to realize their natural prosperousness, but *dares not to act* generally in the manner of the common people. In this way, the sage seems to do nothing yet reaches the best result: all things get the right chance to develop themselves in a harmony condition.

In the past fifty years Lotfi A. Zadeh, the founding father of fuzzy logic, has selflessly provided his support, and assistance to thousands of scientists and researchers by guiding new directions, encouraging discussions, and listening to comments and even disagreements. Without his guidance and continuous efforts in establishing a friendly, an encouraging, and a harmony environment in the fuzzy logic community, the achievement of fuzzy logic research would never be the same. The success of fuzzy logic witnesses how the spirit of *fu* helps our development in science and technology.

The study of fuzzy decision has made an important foundation of utilizing human intelligence for decision making in a fuzzy environment. "*Humans have many remarkable capabilities. Among them there are two that stand out in importance. First, the capability to converse, communicate, reason and make rational decisions in an environment of imprecision, uncertainty, incompleteness of* *information, partiality of truth and partiality of possibility. …"* (Zadeh) [2]. In spite of the limited cognitive ability in describing the real world with imprecision, uncertainty, and partial truth, humans are able to approximate the reality for their problem solving with amazingly simplified models, through observation and intuition, as well as rational analyses. One important evidence is that a fuzzy rule-based system in general uses a far less number of rules compared to a typical rule-based expert system for the same application, through appropriate fuzzy granulation. *Computing with words* [1, 2] opens the door for further exploring potential utilization of human wisdom toward the development of human level machine intelligence.

5 Between Fuzzy Logic and Daoist Thought

Fuzzy logic as a principal member of soft computing has been considered to be positioned in a "soft" branch of science among the others. On the other hand, Chinese philosophy has been considered less strict compared to Western philosophy, and Daoist thought is one of the most influential schools of Chinese philosophy. Inspired by the philosophized thought of fuzzy logic, we have made an attempt to explore an echo between fuzzy logic and Daoist thought, and compare similarities of some key concepts from both theories. Table 1 briefly summarizes our key findings.

Table 1 Similarities of concepts of fuzzy logic and daoism

	Similarities	Differences
1	**Understanding the world:** Admission of ambiguity, complex, and transience of the world surrounding us.	Daoist philosophy is supported by the metaphysical concept of Dao, though Dao's certain features are the representatives of our empirical world.
		Fuzzy logic makes generalization of classical mathematical tools to deal with reality, with tolerance of imprecision.
2	**Describing the world:** Human's common talent or capability is not enough for recognizing the objective world. In other words, human being's cognitive ability is limited.	Daoism emphasizes the infinite and transience of myriad things in the world.
		Fuzzy Logic aims at practical approach for human intellectual activities through precisiation.
3	**Acting:** Human beings should and can try to approximate the truth through irregular way of knowing. This approximation is more accurate than certain precise descriptions or claims because many boundaries in the real world are not clear.	Daoist thought develops its theory through rational observation and analyses, as well as intuition to approximate the true nature of the world.
		Fuzzy Logic has graduation and granulation as keys to approach the infinite and complex reality.

In summary, our motivation and the significance of such comparison are supported by several points.

(1) Finding the mutual support between scientific approaches and philosophical wisdom, between the modern and antique, and between the West and East cultures.
(2) Having found more broad and solid theories to correct dogmatic and romantic belief about infinite human knowledge and capability.
(3) Further exploring human wisdom in rational action with knowledge imperfection toward future human level machine intelligence.
(4) The research in Daoism and fuzzy logic can be inspired from each other to further develop their theories and argumentations.

6 Conclusion

We have argued that important concepts in fuzzy logic and Daoist thought echo each other from afar. The values of Daoism to modern society have recently been significantly recognized in critical issues, such as environment protection, social harmony, and management science [7–9]. Would the application of fuzzy thinking make it more executable in modern society for some of the key ideas of Daoist thought? Would it be possible to further exploit human wisdom in ancient thoughts through the channel of fuzzy thinking, to support future development of machine intelligence? We do not yet have concrete answers and our findings are still very preliminary, but we believe such discussion will be beneficial for building a more bright future of the world, either from system engineering or human society point of view.

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Developing Fuzzy State Models as Markov Chain Models with Fuzzy Encoding

Dimitar Filev, Ilya Kolmanovsky and Ronald Yager

Abstract This paper examines the relationship and establishes the equivalence between a class of dynamic fuzzy models, called Fuzzy State Models (FSM), and recently introduced Markov Chain models with fuzzy encoding. The equivalence between the two models leads to a methodology for learning FSMs from data and a systematic way for model based design of rule-based fuzzy controllers. The proposed approach is demonstrated on a case study of vehicle adaptive cruise control system in which an FSM is identified from simulation data and a fuzzy feedback controller is generated by exploiting the Stochastic Dynamic Programming (SDP).

Keywords Automotive applications • Fuzzy systems • Granular computing • Markov models • Possibility theory • Belief functions • Stochastic dynamic programming

The research of Ilya Kolmanovsky was supported in part by the National Science Foundation Award Number ECCS 1404814 to the University of Michigan.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_6

1 Introduction

Fuzzy systems are widely used in process modeling and control as tools for handling system complexity and accounting for information uncertainty. Most of the dynamic system applications of the fuzzy systems exploit the following two interpretations of system dynamics within the *If … Then* rule framework.

The first approach follows Mamdani's concept of fuzzy control [1, 2] in which a family of *If … Then* rules with fuzzy predicates are used to define a control algorithm realizing nonlinear PI, PD or PID-like control strategies. These are nonlinear mappings from the state space to the space of control variables that implement intuitive control strategies with no requirements for an explicit plant model.

The second approach is based on the Takagi-Sugeno (TS) models [3] which exploit families of rules with fuzzy predicates and functional consequences. The antecedents of the rules decompose the state space into a set of regions with corresponding linear deterministic models. The state of the TS model is a nonlinear combination of the states of the subsystem models. This approach may be viewed as a generalization of the gain-scheduling technique in which piecewise linear models that are associated with multiple fuzzily defined regions of the state space are combined.

Both types of fuzzy models are focused on the deterministic, i.e. the defuzzified value of the system output. In recent years much progress has been made on techniques for improving the performance of fuzzy control algorithms, stability analysis, and systematic design of fuzzy controllers based on TS state and input-output models of the plant. With the fuzzy decomposition, the non-linear system is represented by a polytopic nonlinear system of coupled linear models [4, 5]. This polytopic representation leads to sufficient stability conditions for the TS systems and a systematic design methodology that is based on solving Linear Matrix Inequalities (LMIs), e.g. [6]. The TS approach, with its strong theoretical underpinnings, addresses some of the major criticisms regarding the lack of rigorous analytical framework of the fuzzy control and places fuzzy control as one of the tools of modern control theory. However, despite the progress made towards the development of formal analytical model-based approaches for designing TS fuzzy control systems, most of the practical fuzzy system applications remain centered around heuristic rule-based control utilizing *If … Then* rules. One of the reasons for this is that the TS approach, although based on the methodology of approximate reasoning, is mostly focused on linear models with little if any ability to incorporate subjective information and heuristics. It seems that this observation only confirms the original assertion of Mamdani who introduced fuzzy logic control as a powerful tool to "convert heuristic control rules stated by a human operator into an automatic control strategy" [2].

The progress in Markov Chain models with fuzzy encoding [7–9] suggests that the *If … Then* rules that have been mostly used as static mappings or as fuzzy controllers have the potential for addressing some of the deficiencies of the existing fuzzy models, especially when they are applied to dynamic systems. This includes the ability to represent the system states as possibility distributions, to analytically describe the evolution of the states, and to formulate and solve general multistage decision problems in uncertain environment, including optimization problems that were originally formulated by Bellman and Zadeh in [10].

Specifically, we are interested in determining an optimal control strategy for a general time invariant finite state dynamic system in which the state variable x takes values from a finite set of states $A = \{A_1, A_2, \ldots, A_n\}$ and the control variable u ranges over a finite set $B = \{B_1, B_2, \ldots, B_r\}$. We assume that the states $A_i, i \in \{1, \ldots, n\}$, and the controls $B_j, j \in \{1, \ldots, r\}$, are fuzzy subsets of the state and control universes X and U. We also assume that system dynamics are described by a set of rules expressing the following equation,

$$x^+ = f(x, u),$$

where $f: X \times U \to X$ is a specified *random* function defining the transition from the current state x to the next state x^+ under the control u. We refer to this special type of dynamic system models as the Fuzzy State Models (FSMs).

The considered encoding of the state and control variables into fuzzy subsets is inspired by the ability of the fuzzy partitioning to address the uncertainty in the coding of the continuous signals [7, 9]. Our interest in dynamic optimization problems for FSMs is motivated by the growing interest in the applications of the stochastic dynamic programming and stochastic model predictive control [11–14] based on Markov Chain Models for on-board applications, especially for the automotive and aerospace systems exposed to rapid transients and disturbances.

In this paper we address the basics of the FSMs by expanding the recent results on Markov Chain models with fuzzy encoding [7, 9]. We propose a calculus for formalizing the FSMs by using concepts and results from the Dempster-Shafer theory of evidence [15, 16]. By examining the relationship between the possibility and probability theory [17, 18] we demonstrate the equivalence between the FSMs and the Markov Chain models with fuzzy encoding [9]. We further investigate the mechanism of propagating possibility distributions by FSMs and derive a recursive analytical expression for the possibility distribution that is inferred by a model of this type. The developments in the paper pertain to the critical question [19] challenging the ability of the theory of approximate reasoning to deal with propagating the possibility distributions by a fuzzy system. This important theoretical problem is discussed throughout the paper by utilizing the established relationships and similarities between fuzzy systems, belief structures, and Markov Chain models. From that perspective our approach differs from the works on the abstract dynamic fuzzy system theory, e.g. [20]. Based on the proven equivalence between the FSMs and the Markov Chain models with fuzzy encoding we propose a systematic approach to learning FSM from data. We also reveal how the learned FSMs can be used in conjunction with SDP for model based design of fuzzy controllers. Results are demonstrated on a case study exploiting a FSM of car following dynamics and followed by SDP based synthesis of an adaptive cruise controller implementing rule base type fuzzy control algorithm.

2 Fuzzy State Models

The conventional fuzzy rule models of the type,

$$\textit{If } u \textit{ is } A_i \textit{ then } y \textit{ is } B_i,\ i=\{1,2,\ \ldots,m\}, \tag{1}$$

define mappings from the fuzzy partitioning of the input space (rule antecedents) to a corresponding partitioning of the output space (rule consequents), and can be viewed as approximations of static input-output functions of the form, $y=T(u)$. Many of the fuzzy control applications exploit the Mamdani controller [1, 2] approach. This approach uses static model (1) representing controller dynamics by rules mapping the state space (the vector of the error and its derivative) to the space of control variables:

$$\textit{If } x \textit{ is } A_i \textit{ then } u \textit{ is } B_i,\ i=\{1,2,\ \ldots,m\},$$

i.e., models of the form $u=G(x)$. These models generalize the structure of the widely used industrial look-up table (LUT) controllers [21]. Although highly efficient for designing practical heuristic control strategies, these models cannot be used for plant modeling, model-based design, and for analysis of the stability and performance of feedback control systems. Their main drawback is the lack of efficient mechanism to describe the state dynamics and the interaction between the system inputs, states, and outputs.

Yet a large number of fuzzy models that are used to approximate system dynamics belong to the TS type [3]:

$$\textit{If } x \textit{ is } A_i \textit{ then } x^+=F_i x+G_i u,\ \ i=\{1,2,\ \ldots,m\} \tag{2}$$

Such models use nonlinear weighting functions to combine multiple linear state models, i.e.,

$$x^+ = \sum_{i=1}^{n} \nu_i(x)(F_i x+G_i u).$$

These models result in polytopic representations of the nonlinear dynamics and have become one of the common tools for modeling and control of piecewise linear systems. Their main drawback is in their limited interpretability since the entire model dynamics are captured by the linear subsystems and the role of the fuzziness is to determine the regions where the linear subsystems are defined. Thus one of the prospects of introducing the fuzzy system concept – the opportunity of infusing heuristic information and human knowledge in the control system design – is not completely utilized in the TS models (2) and the associated control design methods.

Multiple attempts to develop fuzzy system models of Mamdani type (1) that can be applied to model based design of fuzzy controllers, see e.g. [18, 22–25] have not resulted in a practical systematic methodology for identification of fuzzy models and subsequent synthesis of fuzzy controllers For example, in [18] we examined a family of fuzzy state models defined by static rules for representing the state space dynamics,

$$\begin{aligned} &If\ u\ is\ B_s\ and\ x\ is\ A_i\ Then\ x^+\ is\ \overline{A}_j \\ &If\ u\ is\ B_s\ and\ x\ is\ A_i\ Then\ y\ is\ D_t, \end{aligned}$$

where $B_s, A_i, \bar{A}_j$ and D_t are fuzzy subsets defined in the input, current /next state, and output domain. Some of the main difficulties with the practical use of this type of models remained the lack of a well-defined mechanism for determining the two families of fuzzy subsets A_i and $\bar{A}_j$ that describe the logic of state transitions, the mappings between the input, state, and output space, and the complexity of dealing with MIMO fuzzy systems.

In this paper we consider the FSMs of the Mamdani type (1) as a different class of dynamic fuzzy models which can represent the state dynamics in cases where the system states are vaguely defined and are formalized as fuzzy subsets. We limit the discussion to rule-based models that involve families of rules of the form:

$$\begin{aligned} If\ x\ is\ A_i\ Then\ x^+\ is\ &A_1\ with\ probability\ p_{i1} \\ &A_2\ with\ probability\ p_{i2} \\ &\ldots \\ &A_n\ with\ probability\ p_{in}, i=\{1,2,\ \ldots,n\}. \end{aligned} \tag{3}$$

The main reason for considering this kind of models is that they can be viewed as representing an uncertain dynamic system of the type,

$$x^+ = f(x),$$

where $x \in X$ is a state variable taking values from the universe of all states X, $x^+ \in X$ is a variable representing the next state, and $A_1, A_2, \cdots A_n$ are fuzzy subsets of X. The fuzzy subsets are defined by their membership functions, $a_i(x)$, on the universe of x. The probabilities p_{ij} are the conditional probabilities satisfying

$$p_{ij} = P\left(x^+ \in A_j | x \in A_i\right), \quad \sum_{i=1}^{n} p_{ij} = 1 (for\ all\ 1 \le j \le n), \tag{4}$$

and describing the probability of transitions between the current and the next states. The definition of the probabilities of fuzzy states $A_1, A_2, \ldots, A_n$ follows Zadeh's definition [26] of the probability $P(F)$ of a fuzzy event F as the Lebesgue-Stieltjes integral of the membership function $\mu_F(x)$ of F,

$$P(F) = \int_X \mu_F(x) dP.$$

We can expect that the possibility distribution inferred by the FSM (3) depends on the transition probabilities and the fuzzy subsets $A_1, A_2, \ldots, A_n$. One special case of the FSM (3) is the model with no randomness:

$$If\ x\ is\ A_i,\ then\ x^+\ is\ A_j, \quad i,j = \{1, 2, \ldots, n\}.$$

This model is obtained from the FSM (3) for the case when only one of the possible transitions from the state i is 100 % certain, i.e. $p_{ij} = 1$ and $p_{is} = 0$ for $s \in \{1, \ldots, n\}$, $s \neq j$. Thus the assumption of no randomness essentially transforms the FSM (3) into the conventional fuzzy model (2).

Further generalizations of the FSM concept (3) can include the dependence of the transition probabilities on time and external control and disturbance inputs; these generalizations are not addressed in the present paper, but will be pursued in the future publications.

As an example, consider a FSM which represents the average traffic conditions on a certain road section during the day [27]. The typical traffic states can be described as fuzzy variables (Freely Flowing, Slightly Congested, Moderately Congested, Jammed) defined on the Navteq Jam Factor Scale (this is a 0–10 scale, similar to the Richter Scale, characterizing the overall traffic conditions with 0 and 10 being, respectively, the best and the worst traffic conditions) See Fig. 1.

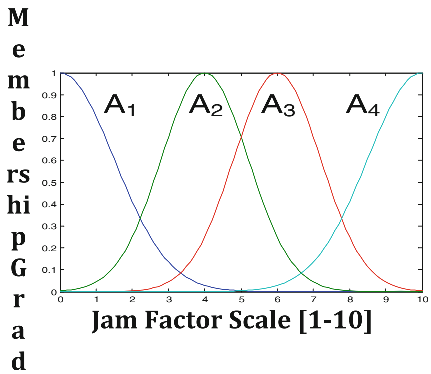


Fig. 1 Fuzzy variables: Freely Flowing (A_1), Slightly Congested (A_2), Moderately Congested (A_3), Jammed (A_4) Traffic defined on the Navteq Jam Factor scale

The average traffic dynamics can be summarized by the set of rules that capture all possible combinations between the states while taking into account the information about the probabilities of transitioning between states:

$$If\ x\ is\ A_i,\ 1 \le i \le 4,\ then\ x^+\ is \quad \begin{array}{l} A_1\ with\ probability\ p_{i1}; \\ A_2\ with\ probability\ p_{i2}; \\ A_3\ with\ probability\ p_{i3}; \\ A_4\ with\ probability\ p_{i4}. \end{array}$$

The conditional probabilities p_{ij}. can be learned from the normalized sigma-counts c_{ij} [7, 9] of observed transitions between the states A_i and A_j,

$$p_{ij} \approx \frac{c_{ij}}{c_{oi}},$$

over a given time interval, where

$$c_{0i} = \sum_{j=1}^{n} c_{ij},$$

is the total number of transitions that are initiated from the state A_i.

For example, a change in the traffic Jam Factor from $x = 7$ to $x^+ = 2$ affects to different degree the current $(A_1(7) = 0, A_2(7) = 0.04, A_3(7) = 0.71, A_4(7) = 0.14)$ and next $(A_1(2) = 0.41, A_2(2) = 0.25, A_3(2) = 0, A_4(2) = 0)$ states and changes the sigma counts, respectively the probabilities, associated with the transitions between the states in Fig. 1 as follows:

$c_{11} = c_{11} + 0;$ $c_{12} = c_{12} + 0;$
$c_{13} = c_{13} + 0;$ $c_{14} = c_{14} + 0;$
$c_{21} = c_{21} + 0.04*0.41;$ $c_{22} = c_{22} + 0.04*0.25;$
$c_{23} = c_{23} + 0;$ $c_{24} = c_{24} + 0;$
$c_{31} = c_{31} + 0.71*0.41;$ $c_{32} = c_{32} + 0.71*0.25;$
$c_{33} = c_{33} + 0;$ $c_{34} = c_{34} + 0;$
$c_{41} = c_{41} + 0.14*0.41;$ $c_{42} = c_{42} + 0.14*0.25;$
$c_{43} = c_{43} + 0;$ $c_{44} = c_{44} + 0;$
$c_{01} = c_{01} + 0;$ $c_{02} = c_{02} + 0;$

Models of the type (3) are dynamic models that have not been commonly used in the literature. In what follows, we apply the Dempster aggregation rule in order to derive a method for formalizing the FSM dynamic fuzzy model (3) that is consistent with the theory of approximate reasoning.

In the basic fuzzy model (1), the consequent of each rule consists of a fuzzy subset B_i. The use of a fuzzy subset implies a special kind of uncertainty associated with the output of a rule. This kind of uncertainty is called a possibilistic uncertainty, and it is a reflection of a lack of precision in describing the output. The use of this imprecision allows one to represent a complex nonlinear function in terms of a collection of simpler fuzzy rules. The consequent of the FSM (3) includes possibilistic uncertainty - a collection of fuzzy subsets $A_1, \ldots, A_n$. In addition to the possibilistic uncertainty, the consequent of the FSM features an additional probabilistic uncertainty that is represented by the probability of selecting between multiple consequent fuzzy subsets. A natural way to deal with both types of uncertainty is to consider the consequents to be fuzzy Dempster–Shafer granules (see, for instance, [18, 27, 28]). Therefore, the output of each rule can be viewed as a belief structure M_i with focal elements $A_1, \ldots, A_n$ that are fuzzy subsets of the universe X and have weights $\boldsymbol{M}_i(A_j)$. The FSM (3) is then replaced by the following set of rules and belief structures

$$If\, x\, is\, A_i\, then\, x^+\, is\, \boldsymbol{M_i}, \quad \boldsymbol{i} = \{\boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{n}\}, \tag{5}$$

where each of the belief structures $\boldsymbol{M}_i$ includes n focal variables that coincide with the states A_i and are assigned weights as follows:

$$\begin{array}{lll}
\boldsymbol{M}_1 & & \\
 & A_1 & M_1(A_1) = p_{11} \\
 & A_2 & M_1(A_2) = p_{12} \\
 & \ldots & \\
 & A_n & M_1(A_n) = p_{1n} \\
\boldsymbol{M}_2 & & \\
 & A_1 & M_2(A_1) = p_{21} \\
 & A_2 & M_2(A_2) = p_{22} \\
 & \ldots & \\
 & A_n & M_2(A_n) = p_{2n} \\
 & \ldots & \\
\boldsymbol{M}_n & & \\
 & A_1 & M_n(A_1) = p_{n1} \\
 & A_2 & M_n(A_2) = p_{n2} \\
 & \ldots & \\
 & A_n & M_n(A_n) = p_{nn}.
\end{array}$$

We note the antecedent portion of the rules in (3) and (5) is unchanged. The inclusion of a belief structure to model the output of a rule essentially means that $M_i(A_j)$ is the probability that the output of the ith rule belongs to the set A_j. So rather than being certain as to in which set the output of the ith rule lies we introduce some degree of randomness in the determination of the outcome set. As we mentioned above, the use of a Dempster-Shafer belief structure to model the consequent of a rule brings with it the option of fusing multiple types of uncertainty. The first type of uncertainty is the randomness associated with determining which of the focal elements of the belief structure M_i is in effect if the rule fires. This selection is essentially determined by a random experiment that uses the corresponding weights, the $M_i(A_j)$, as the associated probabilities.

The weights $M_i(A_j)$ correspond to the conditional probabilities defined in (4). In order to simplify the exposition we consider the case of two states, $n = 2$, and a model defined by the following rules and belief structures:

$$\begin{aligned} &\mathit{If}\ x\ \mathit{is}\ A_1\ \mathit{then}\ x^+\ \mathit{is}\ \boldsymbol{M}_1 \\ &\mathit{If}\ x\ \mathit{is}\ A_2\ \mathit{then}\ x^+\ \mathit{is}\ \boldsymbol{M}_2 \end{aligned} \tag{6}$$

where

$$\begin{array}{lll} \boldsymbol{M}_1 & & \\ & A_1 & M_1(A_1) = p_{11} \\ & A_2 & M_1(A_1) = p_{12} \\ \boldsymbol{M}_2 & & \\ & A_1 & M_1(A_1) = p_{11} \\ & A_2 & M_1(A_2) = p_{12} \end{array}$$

and where

$$p_{11} + p_{12} = 1,\ p_{21} + p_{22} = 1.$$

Suppose that for a given crisp value x_0 or a possibility distribution $\chi_0(x)$ of the state x, the firing levels of the two rules are τ_1 and τ_2 where

$$\tau_i = a_i(x_0),\ \text{resp.}\ \tau_i = \bigvee_X (a_i(x)\chi_0(x)). \tag{8}$$

The output of each rule can now be viewed as a new belief structure $\widehat{M}_i = \tau_i M_i$ defined on X. The focal elements of $\widehat{M}_i$ are

$$\begin{aligned} F_{i1} &= \tau_i a_1, \\ F_{i2} &= \tau_i a_2, \end{aligned}$$

where F is a fuzzy subset of X. The weights associated with these new focal elements remain the same as the ones in $\widehat{M}_i$ [28], i.e. $\widehat{M}_i(F_{ij}) = M_i(A_j)$. Then following [27] we obtain the possibility distribution χ^+ inferred by the rules (assuming a summation type of aggregation and a normalizing coefficient q that scales χ^+ to the unit interval):

$$\chi^+ = \frac{1}{q}\left(\widehat{M}_1 + \widehat{M}_2\right) = \frac{1}{q}(\tau_1 M_1 + \tau_2 M_2)$$

that can be expressed for each of the focal elements as follows:

$$\widehat{M}_1$$

$$F_{11} = \tau_1 a_1$$
$$M_1(A_1) = p_{11}$$
$$F_{12} = \tau_1 a_2$$
$${}_1(A_2) = p_{12}$$

$$\widehat{M}_2$$

$$F_{21} = \tau_2 a_1$$
$$M_2(A_1) = p_{21}$$
$$F_{22} = \tau_2 a_2$$
$$M_2(A_2) = p_{22}$$

We further obtain summation of these two belief structures. The focal elements of M are obtained according to the Dempster rule as follows:

$$\begin{array}{ll} E_1 = F_{11} + F_{21} = \tau_1 a_1 + \tau_2 a_1 & M(E_1) = p_{11} * p_{21} \\ E_2 = F_{11} + F_{22} = \tau_1 a_1 + \tau_2 a_2 & M(E_2) = p_{11} * p_{22} \\ E_3 = F_{12} + F_{21} = \tau_1 a_2 + \tau_2 a_1 & M(E_3) = p_{12} * p_{21} \\ E_4 = F_{12} + F_{22} = \tau_1 a_2 + \tau_2 a_2 & M(E_4) = p_{12} * p_{22} \end{array}$$

By aggregating the focal elements and taking into account (7) we obtain:

$$\begin{aligned} \chi^+ &= \frac{1}{q}((\tau_1 a_1 + \tau_2 a_1) p_{11} p_{21} + (\tau_1 a_1 + \tau_2 a_2) p_{11} p_{22} + (\tau_1 a_2 + \tau_2 a_1) p_{12} p_{21} \\ &\qquad + (\tau_1 a_2 + \tau_2 a_2) p_{12} p_{22}) \\ &= \frac{1}{q}(\tau_1 a_1 p_{11} + \tau_2 a_1 p_{21} + \tau_1 a_2 p_{12} + \tau_2 a_2 p_{22}) \end{aligned}$$

Apparently,

$$q = \sum_{i=1}^{2} \sum_{j=1}^{2} \tau_j p_{ji} = \tau_1 + \tau_2$$

is one possible normalizing coefficient that scales χ^+ to the unit interval since the terms

$$\frac{\tau_j p_{ji}}{q} = \frac{\tau_j p_{ji}}{\tau_1 + \tau_2}, i, j = \{1, 2\}$$

sum to one, i.e. this type of normalization is equivalent to a weighted aggregation of the focal elements.

The extension of this analytical expression for the possibility distribution inferred by a FSM for $n > 2$ is straightforward:

$$\chi^+ = \frac{1}{q}\left(\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \tau_j p_{ji}\right), q = \sum_{i=1}^{n} \tau_j.$$

Alternatively, the above inferred possibility distribution can be formalized using an equivalent vector/matrix expression:

$$\chi^+ = \frac{\tau\Pi a}{\sum_{i=1}^{n} \tau_i} = \frac{\tau\Pi a}{\tau\Pi e} = \overline{\tau}\Pi \text{a}, \quad (9)$$

where $a = [a_1(x) a_{12}(x) \ldots a_n(x)]^T$ is a matrix of the uniformly sampled membership functions of the subsets $A_1, A_2, \cdots,$ and A_n; τ is a row vector of the firing levels,

$$\tau = [\tau_1 \tau_2 \ldots \tau_n],$$

$$\overline{\tau} = \frac{\tau}{\sum_{i=1}^{n} \tau_i},$$

is the vector of the normalized firing levels, Π is the matrix formed by the conditional probabilities p_{ij}, and e is the column vector of ones of size n.

3 Markov Chain Models with Fuzzy Encoding and Fuzzy State Models Are Equivalent Concepts

Markov Chain models with fuzzy encoding were introduced in [7], motivated by the approximation properties of fuzzy granules. Following the theory of approximate reasoning [18], we considered the partitioning the universes of x and x^+ into n fuzzy subsets, A_j. Subsets A_j are defined by their membership functions,

$$a_j(x) : X \to [0, 1]; \forall x \in X, \exists j, 1 \le j \le n, a_j(x) \ne 0.$$

As we now discuss, the FSMs that use a set of appropriately defined rules and belief structures are similar to Markov Chain models with states being defined as fuzzy subsets See Fig. 2.

More specifically, the Markov Chain models with fuzzy granulation [29] can be expressed as a collection of n^2 rules with fuzzy predicates of the form,

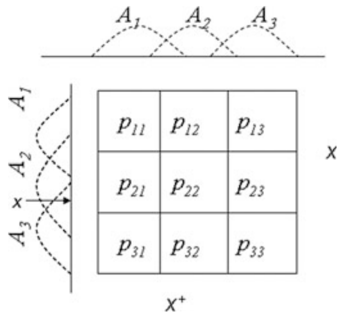


Fig. 2 Markov Chain Model with fuzzy granulation; the states are fuzzy subsets [9]

$$IF\ x\ is\ A_i\ Then\ x^+\ is\ A_j\ with\ probability\ p_{ij}$$

where probabilities $p_{ij}, i,j \in \{1, \cdots n\}$, , are the elements of the transition probability matrix, Π. The antecedent and consequent subsets are defined by their membership functions $a(x)$ on the universe of x. Note that each of the cells of the Markov Chain Model with fuzzy encoding corresponds to a rule expressing the relationship between the possible antecedents and consequents. In [9] we showed that when the current state of the Markov Chain with fuzzy encoding is deterministic, the expressions for the next state (fuzzy or deterministic) are given by:

$$\chi^+(x) = \frac{\sum_{i=1}^{n} a_i(x_0) \sum_{i=1}^{n} p_{ij} \sum_{i=1}^{n} a_j(x)}{\sum_{i=1}^{n} a_i(x_0)}, \tag{10}$$

$$x_0^+ = \frac{\sum_{i=1}^{n} a_i(x_0) \sum_{i=1}^{n} p_{ij}\bar{x}_j}{\sum_{i=1}^{n} a_i(x_0)}. \tag{11}$$

Taking into account that $a_i(x_0)$ essentially corresponds to the degree of firing τ_i of the rule with predicate $a_i(x)$ by (8) we can rewrite (10) into a vector/matrix form:

$$\chi^+ = \frac{\tau \Pi a}{\sum_{i=1}^{n} \tau_j} = \frac{\tau \Pi a}{\tau \Pi e} = \bar{\tau} \Pi a,$$

that is identical to (9). Furthermore, the deterministic output of the Markov Chain with fuzzy encoding (11) corresponds to the defuzzified valued of the fuzzy state model (9).

We summarize this important result in the following proposition.

Proposition: The Fuzzy State Model and the Markov Chain model with fuzzy encoding are equivalent models.

In [9, 30] we showed that for Markov Chain models with fuzzy encoding there exists an analytical expression of the mapping between a given possibility distribution $\chi_0(x)$ and the inferred possibility distribution, χ^+:

$$\overline{\chi}^{+} = \alpha^{T} \Pi \overline{a} = \frac{\overline{\chi}\overline{a}^{T} \Pi \overline{a}}{\overline{\chi}\overline{a}_{0}^{T}}, \tag{12}$$

where the membership function of $\chi_0(x)$ is represented by a row vector $\overline{\chi}$ and $\overline{a}_0 = \sum_{i=1}^{n} \overline{a}_i$. This result was obtained under the following assumptions:

A1. Uniformly sampled (discrete) membership functions a_j and possibility distributions χ and χ^{+} yielding s –dimensional vectors $\overline{a}_j = [\overline{a}_{1j}\overline{a}_{2j} \ldots \overline{a}_{sj}]$, $\overline{\chi}$ and $\overline{\chi}^{+}$.

A2. Use of a correlation type measure

$$\overline{\chi}\overline{a}^{T} = \overline{\chi}[\overline{a}_1^{T}\overline{a}_2^{T} \ldots \overline{a}_n^{T}],$$

of compatibility between the vectors $\overline{\chi}$ and $\overline{a}_j^{T}$ instead of the more conventional *max-product* or *max-min* type similarity measures.

The assumption (A1) simplifies the exposition and can be easily relaxed, for instance, by replacing the inner product of possibilistic vectors by an integral of a product of two possibilistic distribution functions over a domain. The assumption (A2), on the correlation measure, is more critical, but it is reasonable in treating many application problems. In addition to replacing the nonlinear *maximum* operation by a linear inner product operation, this correlation measure may in many problems provide a more complete characterization of the overall similarity between the vectors $\overline{\chi}$ and $\overline{a}_j^{T}$ (see, for instance, [18, 25]). Under the above assumptions, the degrees of firing τ and their normalized counterparts $\bar{\tau}$ in (8) can be expressed as follows:

$$\tau = [\tau_1\, \tau_2\, \ldots\, \tau_{\mathrm{n}}] = \bar{\chi}[\bar{a}_1^{T}\, \bar{a}_2^{T}\, \ldots\, \bar{a}_n^{T}] = \bar{\chi}\, \bar{a}^{T}$$

$$\bar{\tau} = \frac{\tau}{\sum_{i=1}^{n} \tau_i} = \frac{\bar{\chi}\, \bar{a}^{T}}{\sum_{i=1}^{n} \bar{\chi}\, \bar{a}^{T}} = \frac{\bar{\chi}\, \bar{a}^{T}}{\bar{\chi}\, \bar{a}_o^{T}}$$

where $\bar{a}_o = \sum_{i=1}^{n} \bar{a}_i$. Henceforth, under those assumptions expression (9) can be rewritten in a vector form:

$$\bar{\chi}^{+} = \bar{\tau}\, \Pi\, \bar{a} = \frac{\bar{\chi}\bar{a}^{T} \Pi\, \bar{a}}{\bar{\chi}\bar{a}_o^{T}} \tag{13}$$

that is identical to (13).

Therefore, the FSM and the MC with fuzzy encoding are also equivalent under the assumptions A1 and A2. Consequently, the calculus and conclusions related to the latter can be applied to the former. The following figure summarizes the result of this section (Fig. 3).

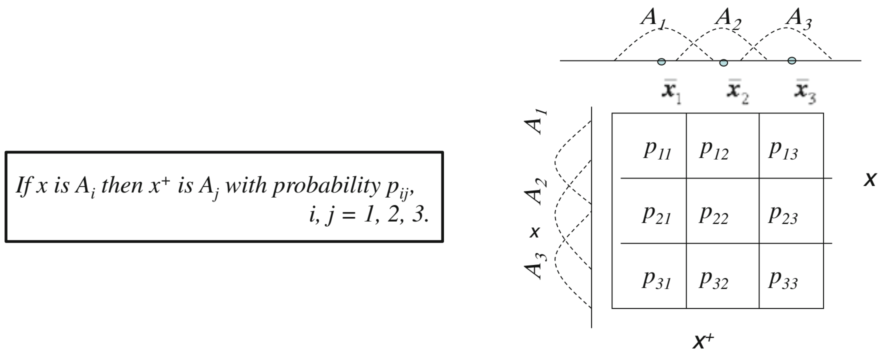


Fig. 3 Equivalence between Fuzzy State Models and Markov Chain models with fuzzy encoding

4 Generalized Dynamic Fuzzy Models

The demonstrated equivalency between Markov models with fuzzy encoding and the FSMs has thus far covered only one special class of first order dynamic systems,

$$x(k+1) = f(x(k)),$$

where the scalar variable x denotes the state of the system and f is a nonlinear mapping. We now discuss an extension to systems with multiple state variables (x is an n-dimensional vector rather than a scalar).

First, we show that higher order dynamic fuzzy models, e.g.:

$$If\ x_1\ is\ A_i^1\ and\ x_2\ is\ A_j^2\ Then\ x_1^+\ is\ A_s^1\ and\ x_2^+\ is\ A_t^2, \tag{14}$$

where n_1 subsets A_i^1 and n_2 subsets A_j^2 partition the universes of the state variables x_1 and x_2, and are defined by the respective membership functions,

$$a_i^1(x_1): X_1 \to [0, 1]; \forall x_1 \in X_1, \exists i, 1 \le i, s \le n_1, a_i^1(x_1) \ne 0,$$
$$a_j^2(x_2): X_2 \to [0, 1]; \forall x_2 \in X_2, \exists j, 1 \le j, t \le n_2, a_j^j(x_2) \ne 0,$$

can be transformed to the first order model (9). By aggregating the state variables x_1 and x_2, and corresponding subsets A_i^1, A_i^2, and substituting in (12) we obtain the following rule,

$$If\ z\ is\ C_i\ Then\ z^+\ is\ C_j, \tag{15}$$

where the new variable z is defined in the two-dimensional Cartesian space $X_1 \times X_2$. C_i and C_j are fuzzy subsets (granules) of $X_1 \times X_2$. Numerically, they are represented by the vector products of the discretized membership functions $\bar{a}^1_{i_1}$ and $\bar{a}^2_{j_1}$ of the subsets, $A^1_{i_1}$ and $A^2_{j_1}$,

$$\overline{C}_i = \bar{a}^{1T}_{i_1} a^2_{j_1}. \tag{16}$$

By rearranging the elements of $\overline{C}_i$ in a vector $\overline{c_i}$ we obtain a vector representation for the membership function of the subset C_i that is formally identical to the single dimensional case and satisfying assumption A1 and A2. We will further denote by the symbol $\overline{\times}$ the combined operation of vector product followed by a transformation to a single dimensional vector, i.e.:

$$\bar{c}_i = \bar{a}^1_{i_1} \times a^2_{j_1}, \tag{17}$$

where $\bar{c}_i$ stands for the vector expression of the membership function of the subset C_i. Therefore, if the granules are considered Markov states and they satisfy the Markov assumption the system can be modeled as a Markov chain with fuzzy encoding:

$$If\ z\ is\ C_i\ Then\ z^+\ is\ C_j\ with\ probability\ p_{ij}. \tag{18}$$

The probabilities, p_{ij}, $i,j = 1, \cdots, n$, where $n = n_1 n_2$, are the elements of the transition probability matrix Π covering the transitions between the subsets, C_i. $p_{ij} = P\left(z^+ \in C_j | z \in C_i\right)$, $\sum_{j=1}^n p_{ij} = 1$. Consequently, expression (11) applies and determines the possibility distribution inferred by the dynamic fuzzy model (14).

Similarly, the Markov chain theory can also be extended to the case of fuzzy modeling of stochastic dynamic multiple-state, multiple-input systems:

$$x(k+1) = f(x(k), u(k)), \tag{19}$$

where x and u are of dimensions n and r, respectively. In order to simplify the notations we consider dynamic fuzzy models of the type:

$$\begin{aligned} &If\ u_1\ is\ B^1_l\ and\ u_2\ is\ B^2_m\ and\ x_1\ is\ A^l_i\ and\ x_2\ is\ A^2_j \\ &\qquad Then\ x^+_1\ is\ A^1_s\ and\ x^+_2\ is\ A^2_t, \end{aligned} \tag{20}$$

where $u_1 \in U_1$ and $u_2 \in U_2$ and B^1_l and B^2_m are fuzzy subsets of U_1 and U_2 with cardinalities r_1 and r_2. Based on the discussion about higher order state model we can assume that the state variables are granulated as in (12). Similar aggregation of the input variables u_1 and u_2 into a new variable, w, and corresponding subsets B^1_l and B^2_m into a new subset D_s yields:

$$If\ w\ is\ D_s\ and\ z\ is\ C_i\ Then\ z^+\ is\ C_j \tag{21}$$

Alternatively, this rule can be rewritten in the form:

$$\begin{aligned} &If\ w\ is\ D_s\ Then \\ &\qquad If\ z\ is\ C_i\ Then\ z^+\ is\ C_j. \end{aligned} \tag{22}$$

Such a granulation of the inputs and states defines a partitioning of the compound Cartesian input/state space into rn fuzzy granules, where $r = r_1 r_2$. Assuming further that the granules C_i satisfy the Markov assumption, the system can be modeled as a Markov chain with fuzzy encoding

$$\begin{aligned} &If\ w\ is\ D_s\ Then \\ &\qquad If\ z\ is\ C_i\ Then\ z^+\ is\ C_j\ with\ probability\ p_{ij}^{(s)} \end{aligned} \tag{23}$$

where the transition probabilities $p_{ij}^{(s)}$ and corresponding transition probability matrices $\Pi^{(s)}$ the are defined as the following condition probabilities:

$$p_{ij}^{(s)} = P\left(z^+ \in C_j | z \in C_i\right), w \in D_s, s = 1, 2, \ldots, r; i, j = 1, 2, \ldots n. \tag{24}$$

In order to include the impact of the input w on the inferred possibility distribution by (14) we aggregate the corresponding possibility distributions that are inferred under different degrees of membership of the input w to the subsets $D_s, s = 1, 2, \ldots, r$, i.e.,

$$\bar{\chi}^+ = \sum_{s=1}^{r} \frac{\nu_s}{\sum_{t=1}^{r} \nu_t} \bar{\tau}\, \Pi^{(s)}\, \bar{a} = \sum_{s=1}^{r} \frac{\nu_s}{\sum_{t=1}^{r} \nu_t} \frac{\bar{\chi}\bar{a}^T \Pi^{(s)}\, \bar{a}}{\bar{\chi}\bar{a}_o^T} \tag{25}$$

where ν_s is the degree of membership of the input w in the subsets $D_s, s = 1, 2, \ldots, r$.

We can visualize the transition probabilities $p_{ij}^{(s)}$ as r transition probability matrices $\Pi^{(s)}$ of size $(n \times n)$ that are associated with the corresponding subsets D_s proportionally to the degrees of membership of the input w in D_s, $s = 1, 2, \ldots, r$.

Figure 2 illustrates the transition probabilities for a system with 2 input and 2 state variables that are defined on continuous universes (Fig. 4).

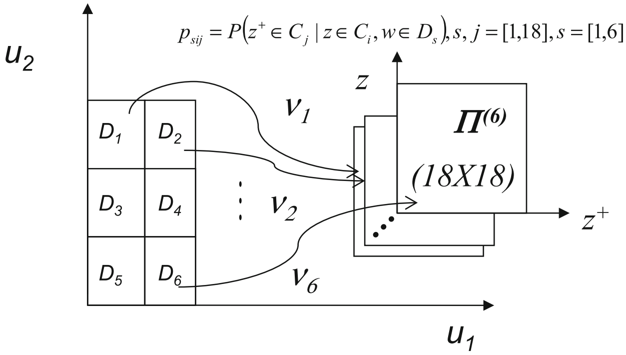


Fig. 4 Example of a vehicle model with 2 state variables (x_1 and x_2) and 2 input variables (u_1 and u_2) that are defined on continuous universes, e.g. speed and acceleration can be thought as the state variables and the accelerator and brake pedal positions can be thought of as the inputs. The inputs are partitioned into 2 and 3 intervals respectively defining 6 input granules, D_s. The ranges of possible values of the state variables are partitioned into 6 and 3 fuzzy subsets, respectively, defining 18 fuzzy subsets, C_i. For different input conditions and states, the transition probabilities are described by 6 transition probability matrices $\Pi^{(s)}$ with the elements, $p_{sij} = P(z^+ \in C_j | z \in C_i, w \in D_s), i, j \in \{1, 2, \cdots, 18\}, s \in \{1, 2, \cdots, 6\}$

5 Learning Generalized Dynamic Fuzzy Models from Data

In the fuzzy models that were discussed above we considered the case of a second order system with 2 inputs. The formal extension to the multiple input systems of a higher order is straightforward and is omitted.

The method and algorithm for learning the transition probabilities for Markov Chains with fuzzy encoding were discussed in detail in [9]. Leveraging the equivalence between both models and by applying the learning algorithm from [9] we get for the transition probability matrix in (18),

$$\Pi(k) = diag(F_0(k))^{-1} F(k), \tag{26}$$

where

$$F(k) = F(k-1) + \beta(\tau(k)\gamma(k)^T - F(k-1)), \tag{27}$$

$$F_o(k) = F_o(k-1) + \beta\ (\tau(k)\gamma(k)^T 1_M - F_o(k-1)), \tag{28}$$

β is the learning rate, and $\tau(k)$ and $\gamma(k)$ are the vectors of membership of the aggregated vectors $z = z(k)$ and $z^+ = z(k+1)$ in the subsets C_i and C_j. The algorithm is initialized as follows:

$$F(0) = \epsilon E; F_o(0) = F(0)\, 1_M \tag{29}$$

with E being a matrix of compatible size and unit elements, and ϵ being a small nonnegative constant introduced to avoid singularity.

As follows from (24), for the case of multiple inputs, the number of transition probability matrices corresponds to the number of fuzzy subsets D_s, $s = 1, 2, \ldots, r$ of the aggregated input variables. Apparently, all transition probability matrices are created in the same way since they summarize the transitions between the states. However, since the input vector can belong to each of the subsets D_s with a different degree of membership ν_s we use this membership to weigh the contributions of corresponding transitions.

Assuming a set of observations *(w(k), z(k)), k = 1, 2, …, K* and fuzzy subsets $D_s, s = 1, 2, \ldots, r$ for w, then for each of the r transition probability matrices we get

$$\Pi_s(k) = diag(F_{s0}(k))^{-1} F_s(k), \tag{30}$$

where

$$F_s(k) = F_s(k-1) + \beta(\nu_s(k)\tau(k)\gamma(k)^T - F_s(k-1)), \tag{31}$$

$$F_{so}(k) = F_{so}(k-1) + \beta(\nu_s(k)\tau(k)\gamma(k)^T 1_M - F_{so}(k-1)), \tag{32}$$

β is the learning rate,$\tau(k)$ and $\gamma(k)$ are the vectors of membership of the aggregated vectors $z = z(k)$ and $z^+ = z(k+1)$ in the subsets C_i and C_j, and ν_s is the degree of membership of the input w in the subsets D_s, $s = 1, 2, \ldots, r$. The algorithm is initialized as follows:

$$F_s(0) = \epsilon E; F_{so}(0) = F_s(0)\, 1_M \tag{33}$$

6 Case Study

We consider an example of adaptive cruise control of a vehicle following another vehicle in traffic. The model is given by

$$\begin{aligned} d(t+1) &= d(t) + \Delta T v(t), \\ v(t+1) &= v(t) + \Delta T(u(t) - w(t)), \end{aligned} \tag{34}$$

where $\Delta T = 0.25$ s is the sampling period, $d(t) = \rho(t) - \rho_{nom}$ is the deviation of the relative distance, $\rho(t)$, from the nominal safe following distance ρ_{nom} (for simplicity, we refer to $d(t)$ as the relative distance), $v(t)$ is the relative speed, $w(t)$ is the acceleration of the lead vehicle in traffic, and $u(t)$ is the acceleration of the follower vehicle that hosts the control algorithm.

Our objective is to develop a model-based control algorithm for the host vehicle acceleration, $u(t)$, that maintains the relative distance and vehicle following. As a first step, we define a fuzzy state model of the two vehicles. We model the two states (the relative distance to a vehicle in traffic, $d(t)$, and the relative speed, $v(t)$) and the control input (the host vehicle acceleration, $u(t)$) by partitioning their ranges into 3 fuzzy subsets. The rules derived from this partitioning are of the form,

$$\begin{aligned} &\textit{If } u \textit{ is } D_s \textit{ and } v \textit{ is } A_i^1 \textit{ and } s \textit{ is } A_j^2 \\ &\qquad \textit{Then } v^+ \textit{ is } A_k^1 \textit{ and } s^+ \textit{ is } A_l^2,, i,j,k,l \in \{1,2,3\} \end{aligned}$$

where D_s stands for the fuzzy subsets ***Negative***, ***Zero***, *and* ***Positive*** of the *Host Vehicle Acceleration u,* A_i^1 *and* A_k^1 stand for the fuzzy subsets ***Negative***, ***Zero***, *and* ***Positive*** of the *Relative velocity v, and* A_j^2 and A_t^2 stand for the fuzzy subsets ***Negative***, ***Zero***, *and* ***Positive*** of the *Relative Distance d.* The fuzzy subsets are modeled by the membership functions depicted on Fig. 5. A discrete set of control actions, representing the follower vehicle accelerations,

$$u \in U = \{-0.5, 0, 0.5\}$$

was considered.

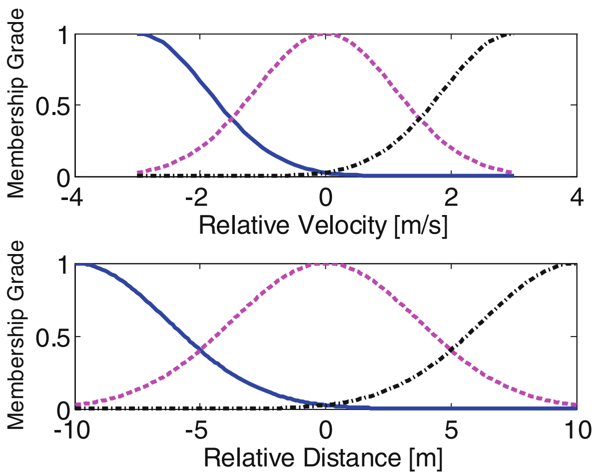


Fig. 5 Membership functions of Relative Velocity and Relative Distance

Following the discussion in the previous section we introduce the Cartesian product subsets of the state variables,

$$C_t = A_i^1 \textit{ and } A_j^2, i,j \in \{1,2,3\}, 1 \le t \le 9,$$

representing the aggregated states v and s. For example, the fuzzy subset $C_2 = A_1^1$ *and* A_2^2 represents the following conjunction, *Relative Speed is* ***Negative*** *and Relative Distance is* ***Zero***.

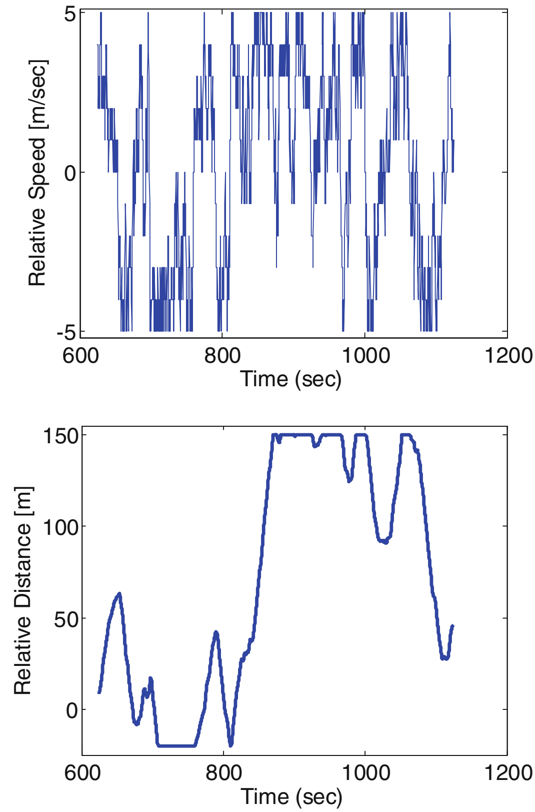
Therefore, the dynamic fuzzy model describing the relative distance and velocity of the two vehicles can be written as a set of the following 243 rules:

$$\begin{aligned} &\textit{If } u \textit{ is } D_s \textit{ Then} \\ &\qquad \textit{If } z \textit{ is } C_i \text{ Then } z^+ \textit{ is } C_j \text{ with probaility } p_{ij}^{(s)}, \end{aligned} \tag{35}$$

where the transition probabilities $p_{ij}^{(s)}$ and corresponding transition probability matrices $\Pi^{(s)}$ the are defined as the following conditional probabilities:

$$p_{ij}^{(s)} = P(z^+ \in C_j | z \in C_i, u \in D_s), s \in \{1, 2, 3\}, i, j \in \{1, 2, \cdots, 9\}$$

Fig. 6 Sample trajectories of the Relative Speed and Relative Distance



To determine the transition probabilities, first a speed trajectory of the lead vehicle was defined varying between 20 and 30 m/s based on a Markov Chain with −1, 0 and 1 m/s vehicle speed change per time step with respective probabilities of 0.3, 0.4 0.3 except for the lower (upper) boundaries, where staying at the same value or transitioning to upper (lower) value was permitted with respective probabilities of 0.5. Then model (34) was simulated for three different values of $u \in U = \{-0.5, 0, 0.5\}$. To avoid values outside of the range of interest, the relative distance state was saturated between −20 and 150 m while the relative velocity state was saturated between −5 and 5 m/s. The transition probability matrices were learned from the simulated trajectories of lead and host vehicle acceleration, relative distance, and relative speed (selected sections of state trajectories for $u = 0.5$ are shown in Fig. 6) by applying the learning algorithm (30)–(33). These transition probability matrices are visualized in Fig. 7.

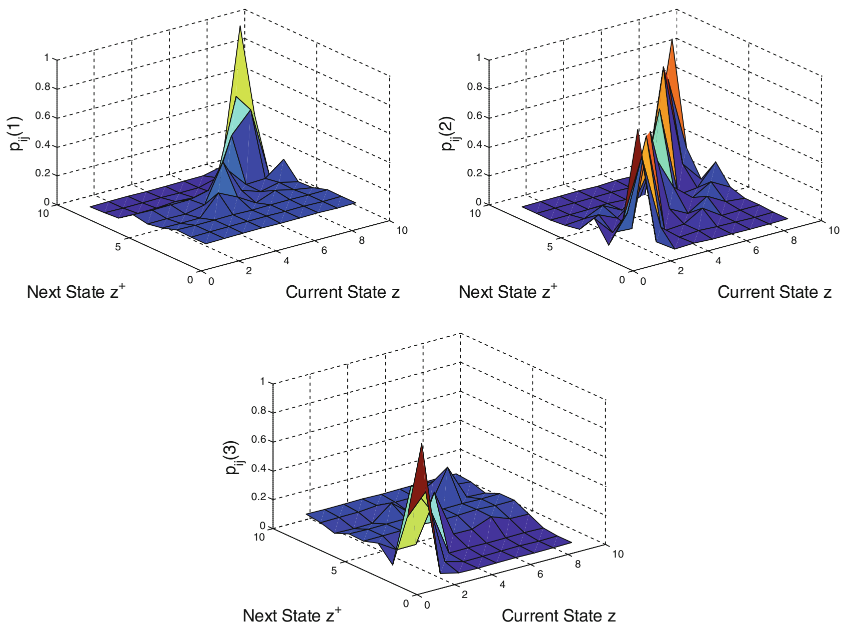


Fig. 7 Transition Probability Matrices, $\Pi^{(s)}$, $s = 1, 2, 3$

The resulting rules reflect the car following dynamics. As an example, noting that

$$C_7 = A_3^1 \; and \; A_1^2 = \textit{Relative Speed is } \textbf{\textit{Positive}} \textit{ and Relative Distance is } \textbf{\textit{Negative}},$$

$$C_8 = A_3^2 \; and \; A_2^2 = \textit{Relative Speed is } \textbf{\textit{Positive}} \textit{ and Relative Distance is } \textbf{\textit{Zero}}$$

the rule

$$\begin{gathered} \textit{If } u \textit{ is } D_1 \textit{ Then} \\ \textit{If } z \textit{ is } C_7 \textit{ Then } z^+ \textit{ is } C_8 \textit{ with probability } p_{78}^{(1)}, \end{gathered}$$

is equivalent to the following rule

*If Host Vehicle Acceleration is **Negative** Then*
*If Relative Speed is **Positive** and Relative Distance is* **Negative** *Then*
*Next Relative Speed is **Positive** and*
*Relative Distance is **Zero** with probability* 0.41

or

*If Host Vehicle Acceleration is **Negative** and Relative Speed is **Positive***
*Relative Distance is **Negative** Then*
*Next Relative Speed is **Positive** and Next Relative Distance is **Zero***
with probability 0.41.

Low rule probability indicates low weight (impact) of a specific rule and allows it to be eliminated from the model. In this example 71 rules with probabilities less than *0.01* were eliminated without affecting the model performance.

In the next step we use the model (35) to design a fuzzy controller that maintains the relative distance and speed. We apply Stochastic Dynamic Programming (SDP) to determine the control actions corresponding to the aggregated fuzzy subsets, C_t, $t \in \{1, 2, \cdots, 9\}$. The penalty function, $L(C_t)$, corresponding to the fuzzy subsets, C_t, $t \in \{1, 2, \cdots, 9\}$, was chosen to discourage subsets that are further away from the origin, i.e. away from the state of $v = 0, d = 0$. We assign lower penalty to the "self-correcting states" corresponding to relative distance and velocity with opposite signs. This leads to the choice of penalty for individual states presented in Table 1.

Table 1 Penalty selection for different states of relative distance and velocity

d/v	*Negative*	*Zero*	*Positive*
Negative	1000	500	0
Zero	200	0	200
Positive	0	500	1000

To generate the control policy, the Value Iteration Algorithm was applied to the Markov Chain model with fuzzy encoding with transition probability matrices visualized in Fig. 7 and the penalty function defined in Table 1. The value iterations take the following form,

$$V_n(C) = \min_{u \in U}\{L(C) + qE[V_{n-1}(C^+)]\},$$

with discount factor $q = 0.90$. Figure 8 shows the value iteration convergence.

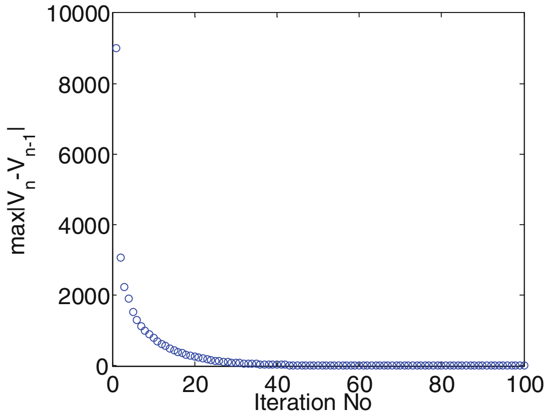


Fig. 8 Maximum difference between value functions in two subsequent iterations over the domain

The optimal control values for each of the fuzzy subsets C_t, $t \in \{1, 2, \cdots, 9\}$, we determined from SDP are summarized in Table 2.

Table 2 Consequent centroids corresponding to the different states of relative distance and velocity as calculated from the optimal policy derived by the application of the Value Iteration Algorithm

d/v	***Negative***	***Zero***	***Positive***
Negative	–0.5	–0.5	0
Zero	–0.5	0	0.5
Positive	0	0.5	0.5

Thus the optimal controller comprises the following rule base:

If Relative Speed is ***Negative*** *and Relative Distance is* ***Negative*** *then* $u = -0.50$
If Relative Speed is ***Negative*** *and Relative Distance is* ***Zero*** *then* $u = -0.50$
If Relative Speed is ***Negative*** *and Relative Distance is* ***Positive*** *then* $u = 0$
If Relative Speed is ***Zero*** *and Relative Distance is* ***Negative*** *then* $u = -0.50$
If Relative Speed is ***Zero*** *and Relative Distance is* ***Zero*** *then* $u = 0$
If Relative Speed is ***Zero*** *and Relative Distance is* ***Positive*** *then* $u = 0.50$
If Relative Speed is ***Positive*** *and Relative Distance is* ***Negative*** *then* $u = 0$
If Relative Speed is ***Positive*** *and Relative Distance is* ***Zero*** *then* $u = 0.50$
If Relative Speed is ***Positive*** *and Relative Distance is* ***Positive*** *then* $u = 0.50$

By using the Simplified Reasoning Method [18] (weighed average aggregation of the centers of gravity of rule consequents), the controller rule base produces a control policy, $u^{SDP}(v,d)$, as a function of the relative speed, v, and relative distance, d shown in Fig. 9.

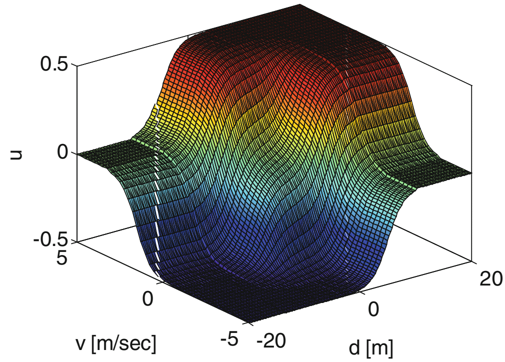


Fig. 9 Control surface derived from the optimal controller rule-base under the Simplified Reasoning Method

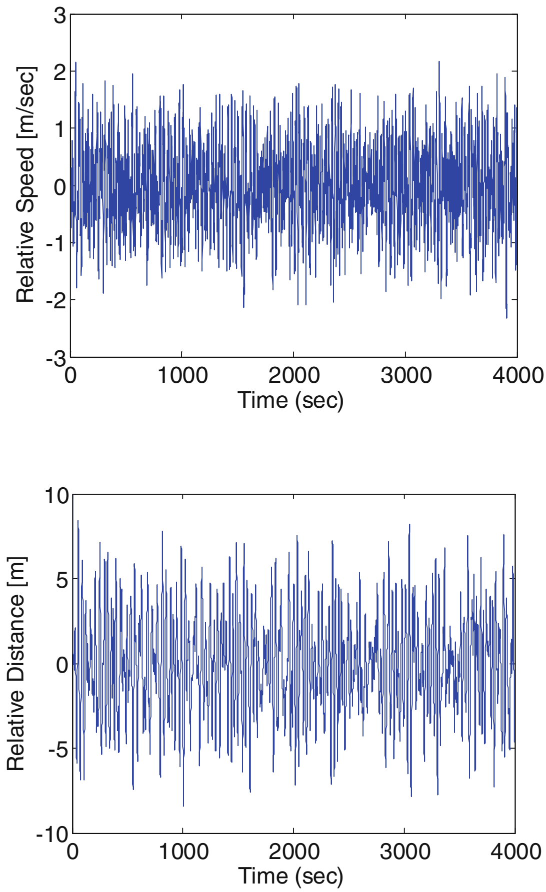


Fig. 10 Relative Speed (top) and Relative Distance (bottom) under the application of the fuzzy controller with consequents calculated by the SDP algorithm when following a randomly accelerating vehicle with acceleration varying between −1 and 1 m/s^2

Fig. 11 Relative Speed (top) and Relative Distance (bottom) under the application of the fuzzy controller with consequents calculated by the SDP algorithm when following a lead vehicle moving at constant speed

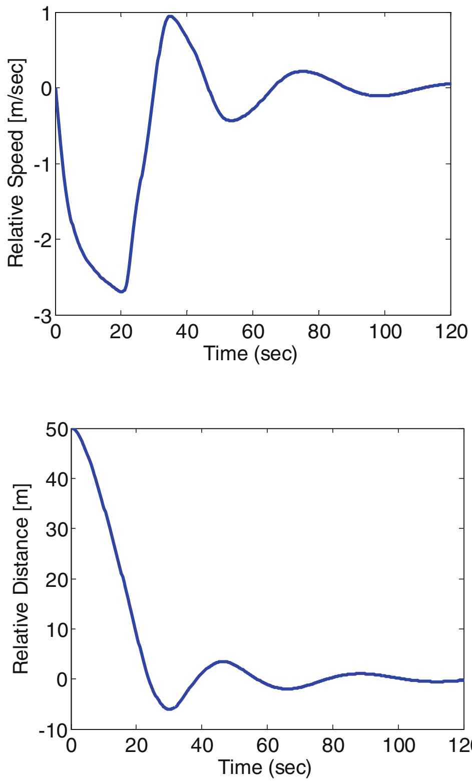


Figure 10 illustrates that the host vehicle is able to maintain relative speed and relative distance in a small range while following a randomly accelerating lead vehicle with acceleration magnitude between -1 and 1 m/s^2 exceeding the follower vehicle acceleration authority.

Figure 11 illustrates the closed-loop response when the lead vehicle is moving at a constant speed, i.e., $w = 0$. The closed-loop trajectories converge so that $v(t) \to 0$ and $d(t) \to 0$ m as $t \to \infty$, where the equilibrium values of the speed and distance are the roots of $u^{SDP}(v, d) = 0$. The response is lightly damped but asymptotically stable. By linearizing the model (34) with $u = u^{SDP}(v, d)$ and $w = 0$ numerically, the closed-loop eigenvalues are $0.9928 \pm 0.0740j$ and are inside the unit disk, confirming that the closed-loop system is asymptotically stable.

7 Summary and Conclusions

In this paper we analyzed the equivalency between the Fuzzy State Models and the Markov Chains with fuzzy encoding and demonstrated that these approaches are identical under certain assumptions. This allowed us to analytically describe the propagation of possibility distribution by a dynamic feedback fuzzy system and to derive an analytical model of the possibility distribution inferred by a Fuzzy State Model. We also showed that the methodology for developing and learning Markov Chain models with fuzzy encoding can be extended to FSMs of higher order and multiple inputs. Results were illustrated on a case study where a FSM for vehicle following dynamics has been learned from sample trajectory data and Stochastic Dynamic Programming (SDP) was applied to generate a fuzzy controller stabilizing the relative speed and distance between two vehicles. We believe that these developments will lead to a framework for systematically addressing the problems of model-based design, stability, and optimal control of fuzzy systems.

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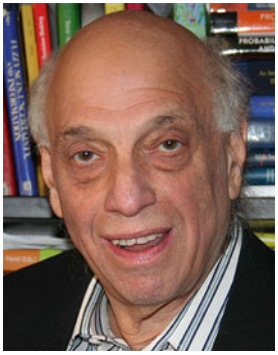

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Incremental Granular Fuzzy Modeling Using Imprecise Data Streams

Daniel Leite and Fernando Gomide

Abstract System modeling in dynamic environments needs processing of streams of sensor data and incremental learning algorithms. This paper suggests an incremental granular fuzzy rule-based modeling approach using streams of fuzzy interval data. Incremental granular modeling is an adaptive modeling framework that uses fuzzy granular data that originate from unreliable sensors, imprecise perceptions, or description of imprecise values of a variable in the form fuzzy intervals. The incremental learning algorithm builds the antecedent of functional fuzzy rules and the rule base of the fuzzy model. A recursive least squares algorithm revises the parameters of a state-space representation of the fuzzy rule consequents. Imprecision in data is accounted for using specificity measures. An illustrative example concerning the Rossler attractor is given.

1 Introduction

Data produced by real world systems result from nonlinear, uncertain, and time-varying dynamic processes. The description of the underlying dynamical behavior using data models derived from first-principles remains unrealistic. Data-driven orientation is becoming increasingly important as a key to complement first-principles orientation. Modeling from data streams requires adaptive adjustment of models to the dynamic variation of the data. Stream-based modeling algorithms need to be developed with emphasis on the evolution of the data. The modeling process should account for data distribution drifts and shifts triggered by the dynamics and the context of the data. Because the volume of data increases continuously, it is not feasible

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_7

to process the data efficiently using multiple passes. Typically learning procedures must be designed to operate with one pass of the data.

Granular data emerge as a consequence of the concepts of indistinguishability, similarity, proximity and functionality [1]. Data granulation can be viewed as a form of lossy data compression in an environment of imprecision. In many cases, data streams contain more information than is needed for a particular purpose [2, 3]. For example, in practice, measurements do not contain more details than the sensors can distinguish.

A granular mapping is defined on information granules and a quotient structure. Mapping of granular data consists in associating a set of granules expressed in some input space to another set of granules draw in an output space. Granular mappings are frequently encountered in rule-based systems, where the mapping is given by If-Then type of statements. Computing with granules emphasizes multiple levels of understanding, analyzing and representing information. Fuzzy granular computing [4–6] hypothesizes that accepting some level of imprecision may be beneficial and therefore suggests a balance between precision and uncertainty.

Linguistic and functional rule-based systems are widely known types of fuzzy systems, which emerged years ago from studies in linguistic modeling and control systems. Both systems share the same rule antecedent structure, but differ in the way the consequents are formed. Linguistic fuzzy rules use fuzzy set-based consequents whereas functional fuzzy rules use functions of the antecedent variables as consequent [7]. Linguistic and functional rule-based systems have been used in granular data modeling [8, 9].

This chapter addresses system modeling using streams of fuzzy interval data. The idea is to start with imprecise description of the values of data attributes and represent them in terms of formal fuzzy objects and functional fuzzy rules whose consequents are discrete-time state space models. The purpose is to represent nonlinear dynamic time-varying processes using conceptual entities, such as data granules and association rules, with no prior assumption about statistical properties of data. Granular fuzzy models rely on the concepts of information granule and granular mapping to encapsulate the imprecision in data streams, and to turn information granules into knowledge in the form of fuzzy rules.

The chapter is structured as follows. Section 2 addresses an incremental, evolving modeling approach able to process imprecise data streams. The approach is a continuous learning algorithm that process pointwise or fuzzy data; does not store previous samples; does not depend upon prior structural knowledge; self-adapts the model structure whenever needed; is independent of statistical properties of data; and does not require 'prototype' initialization. A specificity-weighted recursive least squares algorithm is used to handle imprecise data when updating the parameters of the rule consequents. Section 3 presents an illustrative application on one-step estimation of the Rossler system. Section 4 concludes the chapter summarizing the ideas and suggesting issues for further development.

2 Incremental Granular Modeling

2.1 Fuzzy Modeling

Incremental, evolving modeling concerns the gradual development of the model structure (the fuzzy rule-base) and its parameters. Because data streams often are non-stationary, the structure of the underlying data model should also be dynamic. Model adaptation should continuously learn from the information contained in new data and integrate new information in the current model.

A functional fuzzy rule-based model built from a stream of data is attractive whenever the underlying process is unknown or changes over time. Usually, a finite number of past states $\mathbf{x}(k), \mathbf{x}(k-1), ..., \mathbf{x}(k-m)$, outputs and other exogenous variables can be part of the fuzzy rules antecedents. This chapter assumes functional fuzzy rules of the form

R^i: IF $x_1(k)$ is $\mathscr{M}_1^i$ AND ... AND $x_\Psi(k)$ is $\mathscr{M}_\Psi^i$

THEN $\mathbf{x}^i(k+1) = A^i\mathbf{x}(k)$

where $\mathbf{x}(k) = [x_1(k) \; ... \; x_\psi(k) \; ... \; x_\Psi(k)]^T$ is the state at k; $i = 1, ..., c$ is the number of rules. In incremental modeling, A^i is a matrix of appropriate dimension with variable entries; $\mathscr{M}_\psi^i$, $\psi = 1, ..., \Psi$, are membership functions built using the data available. The number of rules R^i, $i = 1, ..., c$, is also variable. Superscript i on the left-hand side of the consequent equation means a local estimation. We assume that all state variables $\mathbf{x}(k)$ are measurable. State observers are not addressed in this chapter.

Consequent matrices and the state vector can be extended to include affine terms as follows:

$$\widetilde{A}^i = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{a}_0^i & A^i \end{bmatrix}, \quad \widetilde{\mathbf{x}} = \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}, \tag{1}$$

where $\mathbf{a}_0^i = [a_{10}^i \; ... \; a_{\psi 0}^i \; ... \; a_{\Psi 0}^i]^T$. Rules R^i can be rewritten as:

R^i: IF $x_1(k)$ is $\mathscr{M}_1^i$ AND ... AND $x_\Psi(k)$ is $\mathscr{M}_\Psi^i$

THEN $\widetilde{\mathbf{x}}^i(k+1) = \widetilde{A}^i\widetilde{\mathbf{x}}(k)$

In the rest of the paper we omit the tilde from the notation for short, and consider affine models. For the same reason, the time index k is omitted from the time-varying membership functions $\mathscr{M}_\psi^i$ and matrices A^i.

The state estimate from the functional fuzzy model is found as the weighted average:

$$\mathbf{x}(k+1) = \sum_{i=1}^{c} \mu^{ir}\mathbf{x}^i(k+1), \tag{2}$$

where μ^{ir} is the rescaled activation degree of the i-th rule,

$$\mu^{ir} = \frac{\mu^i}{\sum_{i=1}^{c} \mu^i}, \text{ so that } \mu^{ir} \geq 0 \text{ and } \sum_{i=1}^{c} \mu^{ir} = 1. \quad (3)$$

Activation degrees μ^i are computed using any conjunctive aggregation operator, typically a t-norm [7, 10]. t-norms are commutative, associative and monotone operators on the unit hypercube $[0, 1]^n$ whose boundary conditions are $T(\omega, \omega, ..., 0) = 0$ and $T(\omega, 1, ..., 1) = \omega$, $\omega \in [0, 1]$. The neutral element of t-norms is $e = 1$. In this work we use the product t-norm. Thus

$$\mu^i = \prod_{\psi=1}^{\Psi} \mu^i_{\psi}. \quad (4)$$

where μ^i_{ψ} is the degree of membership of $x_{\psi}(k)$ in $\mathscr{M}^i_{\psi}$. While it is common to assume that the activation degree μ^i of at least one rule R^i is nonzero, this is not the case in evolving modeling because no fuzzy set exists *a priori*. Fuzzy sets and rules are created and developed to gradually cover the input data domain. The number of rules c increases by a unit if $\mu^i = 0\ \forall i$. In this case, $\mu^{c+1} = 1$, that is, the fuzzy sets of the new rule match the input data. Incremental development of fuzzy sets and rules is taken up in the next sections.

2.2 Fuzzy Data

Fuzzy data may originate from measurements of unreliable sensors, expert judgment, imprecision introduced in pre-processing steps, and summarization of numeric data over time periods (time granulation). Fuzzy data modeling generalizes pointwise data modeling by allowing fuzzy interval granulation [4, 5].

This chapter concerns fuzzy functional rule-based models and trapezoidal fuzzy data. A trapezoidal fuzzy set $\mathscr{N} = (l,\lambda,\Lambda,L)$ allows the modeling of a wide class of granular objects [11]. A triangular fuzzy set is a trapezoid where $\lambda{=}\Lambda$; an interval is a trapezoid where $l{=}\lambda$ and $\Lambda{=}L$; a singleton is a trapezoid where $l{=}\lambda{=}\Lambda{=}L$. Additional features that make the trapezoidal representation attractive include: (*i*) ease of acquiring the necessary parameters: only four parameters need to be captured. A trapezoidal fuzzy set can be formed straightforwardly from a trapezoidal datum; and (*ii*) many operations on trapezoids can be performed using the endpoints of intervals, which are level sets of trapezoids. The piecewise linearity of the trapezoidal representation allows calculation of only two level sets (core and support) to obtain a complete instance.

A fuzzy set $\mathscr{N} : X \rightarrow [0, 1]$ is upper semi-continuous if the set $\{x \in X | \mu(x) > \alpha\}$ is closed, that is, if the α-cuts of $\mathscr{N}$ are closed intervals. If the universe X is the set of real numbers and $\mathscr{N}$ is normal, $\mu(x) = 1 \; \forall x \in [\lambda, \Lambda]$, then $\mathscr{N}$ is a model of a fuzzy interval, with monotone increasing function $\zeta_{\mathscr{N}}$: $[l, \lambda[\rightarrow [0, 1]$, monotone decreasing function $\iota_{\mathscr{N}}$: $]\Lambda, L] \rightarrow [0, 1]$, and zero otherwise [7]. A fuzzy interval $\mathscr{N}$ has the following canonical form:

$$\mathscr{N} \; : \; x \rightarrow \mu(x) = \begin{cases} \zeta_{\mathscr{N}}, & x \in [l, \lambda[\\ 1, & x \in [\lambda, \Lambda] \\ \iota_{\mathscr{N}}, & x \in \;]\Lambda, L] \\ 0, & \text{otherwise} \end{cases} , \tag{5}$$

where x is a real number in X. The fuzzy interval $\mathscr{N}$ satisfies the conditions of normality ($\mu(x) = 1$ for at least one $x \in X$) and convexity ($\mu(\kappa x^1 + (1 - \kappa)x^2) \geq min\{\mu(x^1), \mu(x^2)\}, x^1, x^2 \in X, \kappa \in [0, 1]$). If

$$\zeta_{\mathscr{N}} = \frac{x - l}{\lambda - l} \text{ and} \tag{6}$$

$$\iota_{\mathscr{N}} = \frac{L - x}{L - \Lambda}, \tag{7}$$

then the fuzzy interval (5) reduces to the model of a trapezoidal membership function. Moreover, when $\lambda = \Lambda$, $\mu(x) = 1$ for a single element x in X. In this case, the corresponding fuzzy entity is a fuzzy number.

Let $x = (\underline{\underline{x}}, \underline{x}, \overline{x}, \overline{\overline{x}})$ be a trapezoidal datum. The membership degree of x in the fuzzy set $\mathscr{N}$ can be obtained from (5) if x is degenerated into a singleton. Otherwise, if x is a symmetric object, i.e. if $\underline{x} - \underline{\underline{x}} = \overline{\overline{x}} - \overline{x} \neq 0$, its membership degree in $\mathscr{N}$ can be computed using the midpoint of x:

$$mp(x) = \frac{\overline{x} + \underline{x}}{2}. \tag{8}$$

The center of gravity

$$CoG(x) = \frac{\underline{\underline{x}} + 5\underline{x} + 5\overline{x} + \overline{\overline{x}}}{12} \tag{9}$$

is useful when x is asymmetric. Even though it is apparent that these approximations of the true value are useful to facilitate computations, they contradict the purpose of taking into account the data uncertainty into fuzzy models. Additionally, in some situations, as that shown in Fig. 1, the midpoint (or center of area) approximation can give zero (or low) membership degree to significantly overlapped fuzzy objects. A measure of similarity between fuzzy granular data is needed to properly consider all relevant situations.

2.3 Similarity Between Fuzzy Sets

Similarity is a fundamental notion to construct rule-based systems from streams of data. In this work, data are trapezoidal fuzzy sets. A possible similarity measure for trapezoids, say x and $\mathscr{M}^i$, is:

$$S(x, \mathscr{M}^i) = 1 - D(x, \mathscr{M}^i), \tag{10}$$

where $D(x, \mathscr{M}^i)$ is a distance measure computed as follows:

$$D(x, \mathscr{M}^i) = \frac{|\underline{\underline{x}} - l^i| + 2|\underline{x} - \lambda^i| + 2|\overline{x} - \Lambda^i| + |\overline{\overline{x}} - L^i|}{6}. \tag{11}$$

The value of S equals 1 for identical trapezoids and indicates the maximum degree of matching between them. S decreases linearly as x and $\mathscr{M}^i$ depart from each other. In particular, (11) is a Hamming-like distance where the parameters of the trapezoids are directly compared. Core parameters have double weight in relation to support parameters. Although (10) - (11) are simple to compute, involving only basic arithmetic operations, there are no strong principled reasons to choose this measure. In fact, there is no generally accepted consensus on a best similarity measure [12].

Let the expansion region of a set $\mathscr{M}^i$ be denoted by

$$E^i = [L^i - \rho,\ l^i + \rho], \tag{12}$$

where ρ is the maximum width that the set $\mathscr{M}^i$ is allowed to expand to fit a datum x; $L^i - l^i \leq \rho$ at any k. Define the membership degree of the datum x in the fuzzy set $\mathscr{M}^i$ as $\mu^i = S(x, \mathscr{M}^i)$ if $x \in E^i$, and $\mu^i = 0$ otherwise.

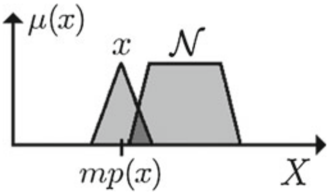


Fig. 1 Case where the membership degree of the fuzzy datum x in the fuzzy set $\mathscr{N}$ obtained by (8)is zero despite their significant similarity

The similarity measure (10) can be generalized for vectors of trapezoids, say $\mathbf{x} = [x_1 \ldots x_\psi \ldots x_\Psi]^T$ and $\mathscr{M}^i = [\mathscr{M}^i_1 \ldots \mathscr{M}^i_\psi \ldots \mathscr{M}^i_\Psi]^T$, as follows

$$S(\mathbf{x}, \mathscr{M}^i) = 1 - \frac{1}{6\Psi} \sum_{\psi=1}^{\Psi} (|\underline{\underline{x}}_\psi - l^i_\psi| + 2|\underline{x}_\psi - \lambda^i_\psi| + \\ + 2|\overline{x}_\psi - \Lambda^i_\psi| + |\overline{\overline{x}}_\psi - L^i_\psi|), \tag{13}$$

then $\mu^i = S(\mathbf{x}, \mathscr{M}^i)$ if $\mathbf{x} \in \mathbf{E}^i$. Refer to [12] for a thorough discussion about similarity measures.

2.4 Incremental Adaptation

The purpose of simultaneously adapting the structure and parameters of dynamic fuzzy models is to use current information about the process to keep its representation updated. This section develops model structure identification and antecedent parameter estimation. An incremental learning method is introduced to avoid time consuming training common in multiple passes learning methods.

Expansion regions $\mathbf{E}^i$, see (12), help to verify if new input data belong to a granule in the input space. Different values of ρ produce different representations of the same data set in different levels of granularities. For normalized data, ρ assumes values in [0, 1]. If ρ is equal to 0, then granules are not expanded. Learning creates a new rule for each sample, which causes overfitting and excessive complexity. If ρ is equal to 1, then a single granule covers the entire data domain. Evolvability is reached choosing intermediate values for ρ.

A rule is created whenever one or more entries of $\mathbf{x}$ are not within the expansion regions $\mathbf{E}^i$ of $\mathscr{M}^i$, $i = 1, ..., c$. A new associated granule $\mathscr{M}^{c+1}$ is constructed from fuzzy sets $\mathscr{M}_\psi^{c+1}$, $\psi = 1, ..., \Psi$, whose parameters match $\mathbf{x}$, that is,

$$\mathscr{M}_\psi^{c+1} = (l_\psi^{c+1}, \lambda_\psi^{c+1}, \Lambda_\psi^{c+1}, L_\psi^{c+1}) = (\underline{\underline{x}}_\psi, \underline{x}_\psi, \overline{x}_\psi, \overline{\overline{x}}_\psi). \tag{14}$$

Adaptation of an existing granule $\mathscr{M}^i$ consists in expanding the support $[l_\psi^i, L_\psi^i]$ and updating the core $[\lambda_\psi^i, \Lambda_\psi^i]$ of its fuzzy sets. Among all granules $\mathscr{M}^i$ that can be expanded to include a sample $\mathbf{x}$, the one with highest similarity according to (13) is chosen. Adaptation proceeds depending on where the datum x_ψ is placed. The conditions to expand the support are:

$$\text{If } \underline{\underline{x}}_\psi \in [L_\psi^i - \rho, l_\psi^i] \text{ then } l_\psi^i(\text{new}) = \underline{\underline{x}}_\psi, \text{ and}$$

$$\text{If } \overline{\overline{x}}_\psi \in [L_\psi^i, l_\psi^i + \rho] \text{ then } L_\psi^i(\text{new}) = \overline{\overline{x}}_\psi.$$

The parameters of the core are recursively updated using:

$$\lambda_\psi^i(\text{new}) = \frac{(w^i - 1)\lambda_\psi^i + \underline{x}_\psi}{w^i} \text{ and} \tag{15}$$

$$\Lambda_\psi^i(\text{new}) = \frac{(w^i - 1)\Lambda_\psi^i + \overline{x}_\psi}{w^i}, \tag{16}$$

where w^i is the number of times that the granule $\mathscr{M}^i$ was chosen to be adapted. Figure 2 shows seven possible adaptation situations. In the figure, the datum $x = (\underline{\underline{x}}, \underline{x}, \overline{x}, \overline{\overline{x}})$ places either outside, partially inside or inside the fuzzy set $\mathscr{M}^i$. The learning procedure creates a new granule $\mathscr{M}^{c+1}$ or adapts the parameters of $\mathscr{M}^i$ accordingly.

2.5 Specificity-Weighted Recursive Least Squares Algorithm

A recursive least squares-like (RLS) algorithm is used to adapt the parameters of the rule consequents as follows.

Consider the consequent of rule R^i:

$$\mathbf{x}^i(k+1) = A^i \mathbf{x}(k) \tag{17}$$

where $\mathbf{x} = [1\ x_1\ ...\ x_\psi\ ...\ x_\Psi]^T$. The elements of A^i are denoted $a^i_{\psi_1\psi_2}$, $\psi_1, \psi_2 = 0, ..., \Psi$. Rule R^i is chosen to be adapted whenever its antecedent part $\mathscr{M}^i$ is more similar to $\mathbf{x}(k)$ than the antecedent part of the remaining rules. When instance $\mathbf{x}(k+1)$ becomes known, equation (17) can be solved for A^i.

Expanding the ψ-th row of (17) we have

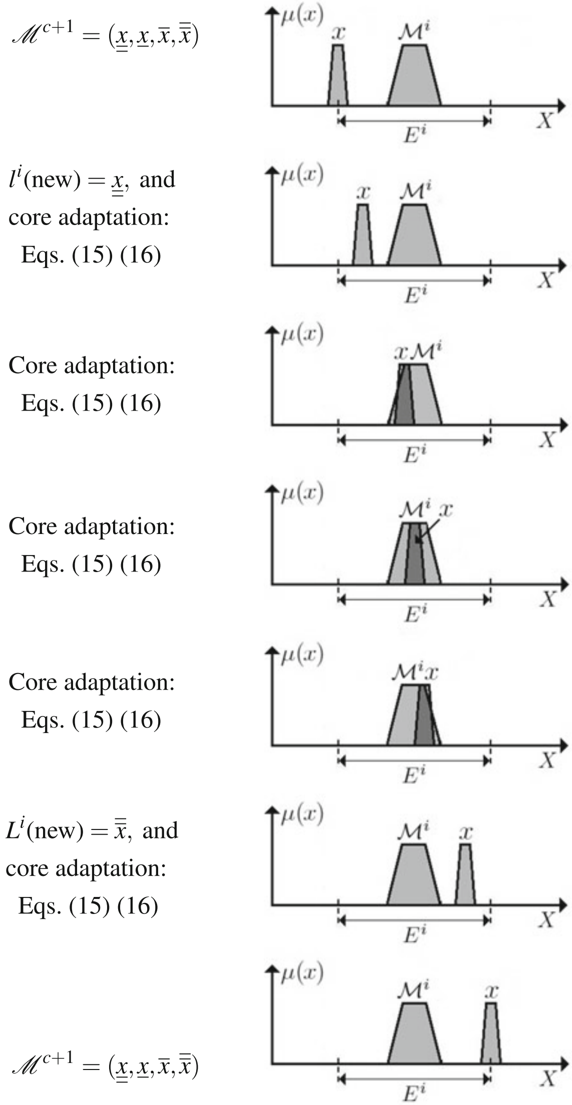
$$x_\psi(k+1) = a^i_{\psi 0} + a^i_{\psi 1} x_1(k) + ... + a^i_{\psi\Psi} x_\Psi(k). \tag{18}$$

The standard RLS algorithm can be used for each row of (17) if we replace the trapezoids x_ψ by their midpoint (8) or center of gravity (9), depending on their symmetry. Imprecision in the data can be accounted for by weighing the adjustment of $a^i_{\psi_1\psi_2}$ using specificity measures. Specificity measures refer to the amount of information conveyed by a fuzzy datum [13]. A highly imprecise fuzzy datum (lower specificity) may not be as important as a more precise (higher specificity) datum.

Let $\mathbf{a}^i_\psi = [a^i_{\psi 0}\ a^i_{\psi 1}\ ...\ a^i_{\psi\Psi}]^T$ be the vector of unknown coefficients; $\mathfrak{X} = [1\ CoG(x_1)(k)\ ...\ CoG(x_\Psi)(k)]$ be the regression vector; and $\mathfrak{Y} = [CoG(x_\psi)(k+1)]$. Then, in matrix form, equation (18) becomes

$$\mathfrak{Y} = \mathfrak{X}\mathbf{a}^i_\psi. \tag{19}$$

Fig. 2 Creation and recursive adaptation of fuzzy sets



To estimate the coefficients $\mathbf{a}_{\psi}^{i}$ we let

$$\mathfrak{Y} = \mathfrak{X}\mathbf{a}_{\psi}^{i} + \Xi, \tag{20}$$

where $\Xi := [\varepsilon_{\psi}(k+1)]$ and

$$\varepsilon_{\psi}(k+1) = CoG(x_{\psi})(k+1) - CoG(\hat{x}_{\psi})(k+1) \tag{21}$$

is the approximation error. While in batch estimation the rows in $\mathfrak{Y}$, $\mathfrak{X}$ and Ξ increase with the number of available samples, in recursive mode only two rows are kept and we reformulate equations (19)-(21) as follows:

$$\mathfrak{Y} = \begin{bmatrix} CoG(x_{\psi})(k) \\ CoG(x_{\psi})(k+1) \end{bmatrix}, \ \Xi = \begin{bmatrix} \varepsilon_{\psi}(k) \\ \varepsilon_{\psi}(k+1) \end{bmatrix} \text{ and}$$

$$\mathfrak{X} = \begin{bmatrix} 1 & CoG(x_1)(k-1) & \ldots & CoG(x_{\Psi})(k-1) \\ 1 & CoG(x_1)(k) & \ldots & CoG(x_{\Psi})(k) \end{bmatrix}. \tag{22}$$

The rows of the matrices in (22) refer to values before and just after adaptation. The RLS algorithm chooses $\mathbf{a}_{\psi}^{i}$ to minimize the functional

$$J(\mathbf{a}_{\psi}^{i}) = \Xi^{T}\Xi. \tag{23}$$

$\mathbf{a}_{\psi}^{i}$ is given by

$$\mathbf{a}_{\psi}^{i} = (\mathfrak{X}^{T}\mathfrak{X})^{-1}\mathfrak{X}^{T}\mathfrak{Y}. \tag{24}$$

Let $Q = (\mathfrak{X}^{T}\mathfrak{X})^{-1}$. From the matrix inversion lemma [14] we avoid inverting $\mathfrak{X}^{T}\mathfrak{X}$ using:

$$Q(\text{new}) = Q(\text{old}) \left[I - \frac{\mathfrak{X}^{T}\mathfrak{X}Q(\text{old})}{1 + \mathfrak{X}_{(2)}Q(\text{old})\mathfrak{X}_{(2)}^{T}} \right], \tag{25}$$

where I is the identity matrix, and $\mathfrak{X}_{(2)}$ is the second row of $\mathfrak{X}$. In practice it is usual to choose large initial values for the entries of the main diagonal of Q. We use $Q(0) = 10^{3}I$ as the default value.

Performing simple mathematical transformations, the vector of coefficients can be rearranged recursively as

$$\mathbf{a}_{\psi}^{i}(\text{new}) = \mathbf{a}_{\psi}^{i}(\text{old}) + Q(\text{new})\mathfrak{X}^{T}(\mathfrak{Y} - \mathfrak{X}\mathbf{a}_{\psi}^{i}(\text{old})) \tag{26}$$

or, similarly,

$$\mathbf{a}^i_\psi(\text{new}) = \mathbf{a}^i_\psi(\text{old}) + Q(\text{new})\mathfrak{X}^T \Xi. \tag{27}$$

Yager [11] defines the specificity of a trapezoid x_ψ as

$$sp(x_\psi) = 1 - wdt(x_{\psi(0.5)}). \tag{28}$$

This form of specificity measure means one minus the width of the 0.5 level set of x_ψ. In terms of the parameters of x_ψ we get

$$sp(x_\psi) = 1 - \frac{(\overline{x}_\psi + \overline{\overline{x}}_\psi) - (\underline{x}_\psi + \underline{\underline{x}}_\psi)}{2}. \tag{29}$$

Let the specificity of $\mathbf{x}$ be the diagonal matrix:

$$sp(\mathbf{x}) = diag([1 \;\; sp(x_1) \;\; \ldots \;\; sp(x_\Psi)]). \tag{30}$$

Thus, we may add specificity into equation (27) to account for data uncertainty as follows:

$$\mathbf{a}^i_\psi(\text{new}) = \mathbf{a}^i_\psi(\text{old}) + sp(\mathbf{x})Q(\text{new})\mathfrak{X}^T \Xi \tag{31}$$

Figure 3 gives the idea of the specificity-weighted RLS algorithm. In the figure in the left, the coefficients $\mathbf{a}^i(\text{old})$ of the approximation function result from recursive adaptation based on $x(1)$, $x(2)$ and $x(3)$. Note that the data granules $x(1)$, $x(2)$ and $x(3)$ are of the same size and thus have the same specificity. When the new datum $x(4)$ arrives (with the same specificity as that of previous data), the algorithm weights its contribution equivalently to the contribution of previous data to adapt $\mathbf{a}^i(\text{old})$ and yield $\mathbf{a}^i(\text{new})$. Conversely, on the right side, the specificity of the new datum $x(4)$ is lower than that of $x(1)$, $x(2)$ and $x(3)$. The higher uncertainty on the value of $x(4)$ causes a smaller adjustment of the approximation function toward $x(4)$.

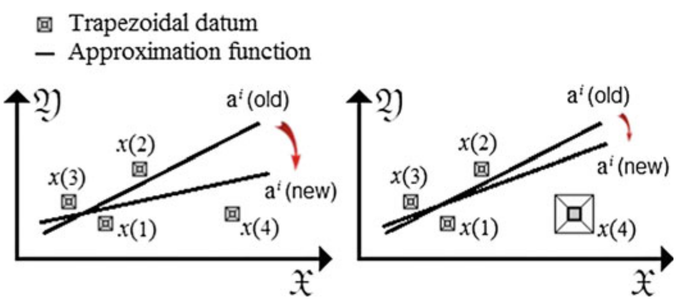


Fig. 3 Specificity-weighted RLS algorithm

The specificity-weighted RLS algorithm described in this section is repeated for $\psi = 1, ..., \Psi$ at each step. Detailed derivations of the RLS algorithm can be found in [15]. A convergence analysis is given in [16].

3 The Rossler Attractor

This section addresses an application example to show the potential of the evolving fuzzy granular modeling approach. The Rossler attractor [17] is a system of three nonlinear ordinary differential equations that exhibits chaotic dynamics. The equations have been commonly used as a model of equilibrium in chemical reactions. An orbit within the attractor follows an outward spiral around an unstable fixed point, close to the $x_1 - x_2$ plane. Once the orbit spirals out enough, it is influenced by a second fixed point that causes a rise and twist in the x_3 dimension. In the time domain, irregular oscillations bounded in a range of values are perceptible.

Here, we use the Rossler equations only to generate a data stream. The objective is to obtain a fuzzy model of an "unknown" nonlinear dynamical system based on the data stream. The discrete-time Rossler equations are:

$$\begin{aligned} x_1(k+1) &= x_1(k) + (-x_2(k) - x_3(k))dt + \eta_1 \\ x_2(k+1) &= x_2(k) + (x_1(k) + ax_2(k))dt + \eta_2 \\ x_3(k+1) &= x_3(k) + (b + x_1(k)x_3(k) - cx_3(k))dt + \eta_3 \end{aligned} \quad (32)$$

The nonlinearity is x_1x_3. Similar to many articles, we considered $a = b = 0.1$, and $c = 14$. dt is the sampling period; η_i is a random value in $[-0.5, 0.5]$. The initial state $\mathbf{x}(0)$ is $(1; 0; 0)$. The error introduced by the discretization of the original equations is negligible for sampling periods dt sufficiently small compared with the significant time constant of the system. As shown in Fig. 4, the trajectory of the system states in the phase space settles into an aperiodic oscillation. Trajectories are confined to a fractal set.

In a first experiment a fuzzy model is evolved to approximate (32). The equations are perceived through pointwise input $([x_1(k)\ x_2(k)\ x_3(k)])$ and output $([x_1(k+1)\ x_2(k+1)\ x_3(k+1)])$ data. Data become available gradually to simulate a data stream. No data is available before learning starts. In addition, no data is stored during the entire learning process. The one-step forecasting given by the evolving fuzzy model using the maximum width allowed for granules, ρ, equal to 2 is shown in Fig. 5. The sampling period was chosen to be $dt = 0.005$ in this experiment. The figure shows the results for $k = 10500, ..., 16000$. The root mean square error, calculated as

$$RMSE = \frac{1}{k_c} \sum_{k=1}^{k_c} \sqrt{\sum_{j=1}^{3} (x_j(k+1) - \hat{x}_j(k+1))^2}, \quad (33)$$

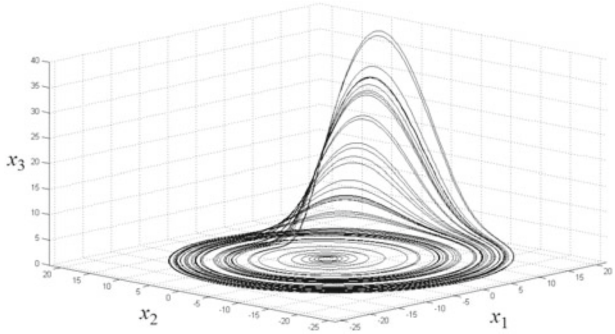


Fig. 4 Rossler chaotic system: phase space trajectory

is $RMSE = 0.0372$ for $k_c = 60000$. Five rules were developed during the simulation period. Their parameters are:

Rule 1:

$\mathscr{M}_1^1 = (-0.9981, 0.0010, 0.0010, 1.0000)$
$\mathscr{M}_2^1 = (0, 0.7781, 0.7781, 1.5562)$
$\mathscr{M}_3^1 = (-0.0178, 0.0066, 0.0066, 0.0310)$

$$A^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.0494 & 0.0049 & -0.9385 & -2.7209 \\ -0.0151 & 1.0415 & 0.1366 & 0.1307 \\ 0.1351 & 0.0073 & -0.0709 & -14.0967 \end{bmatrix}$$

Rule 2:

$\mathscr{M}_1^2 = (-1.1365, -0.1378, -0.1378, 0.8608)$
$\mathscr{M}_2^2 = (-1.2667, -0.3420, -0.3420, 0.5826)$
$\mathscr{M}_3^2 = (-0.0086, 0.0077, 0.0077, 0.0241)$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.0243 & -0.0065 & -0.9531 & 3.9008 \\ -0.0994 & 1.0198 & 0.0227 & 2.8007 \\ 0.2135 & 0.0194 & 0.0153 & -22.7749 \end{bmatrix}$$

Rule 3:

$\mathscr{M}_1^3 = (0.8690, 1.1200, 1.1200, 1.3710)$
$\mathscr{M}_2^3 = (-0.9937, 0.0061, 0.0061, 1.0058)$
$\mathscr{M}_3^3 = (-0.0141, 0.0073, 0.0073, 0.0287)$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.1717 & 0.1292 & -1.0691 & 1.1173 \\ -0.1491 & 1.1051 & 0.1056 & 4.7427 \\ 0.7875 & -0.5095 & 0.1014 & -17.0389 \end{bmatrix}$$

Rule 4:

$\mathscr{M}_1^4 = (-1.6920, -1.3501, -1.3501, -1.0083)$
$\mathscr{M}_2^4 = (-0.7079, 0.2867, 0.2867, 1.2813)$
$\mathscr{M}_3^4 = (-0.0043, 0.0098, 0.0098, 0.0238)$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1070 & 0.0906 & -1.0105 & 5.5327 \\ 0.6310 & 1.4638 & 0.1224 & 1.2655 \\ 1.7437 & 0.8991 & -0.3307 & -25.7244 \end{bmatrix}$$

Rule 5:

$\mathscr{M}_1^5 = (-1.5243, -0.5816, -0.5816, 0.3612)$
$\mathscr{M}_2^5 = (-1.7995, -1.2594, -1.2594, -0.7193)$
$\mathscr{M}_3^5 = (-0.0082, 0.0088, 0.0088, 0.0257)$

$$A^5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.6108 & 0.0624 & -0.6953 & -8.7299 \\ -0.7245 & 0.8414 & -0.2828 & 5.8742 \\ -0.4074 & -0.2262 & -0.2866 & -23.5195 \end{bmatrix}$$

From Fig. 5, the effectiveness of the evolving approach in predicting nonlinear systems without prior knowledge about the data and system equations can be verified. The error signals have relatively small amplitudes compared to the amplitudes of the system states. An important point in this experiment is that due to exponential divergence of the trajectories for small differences in the measurements, parameters or initial states, a non-evolving (offline-trained) modeling method is unlike to track the trajectory of the states. Another point is that the higher the number of granules and rules, the more accurate the state estimation tends to be. However, the state estimation depends on the availability of sufficient data for setting local parameters.

A second experiment consisted in evaluating the ability of the modeling approach in handling fuzzy data, and detecting and reacting to concept drifts and shifts. We considered the data as perceptions of the values of a variable. Imprecision of the values of x_j is represented by a fuzzy object of the form $(x_j - 0.5, x_j, x_j, x_j + 0.5)$. At $k = 11500$, the parameters of the Rossler equations are shifted to $a = b = 0.2$ and $c = 3$ to simulate a concept shift. At each step after $k = 14000$, an offset of 0.02 is

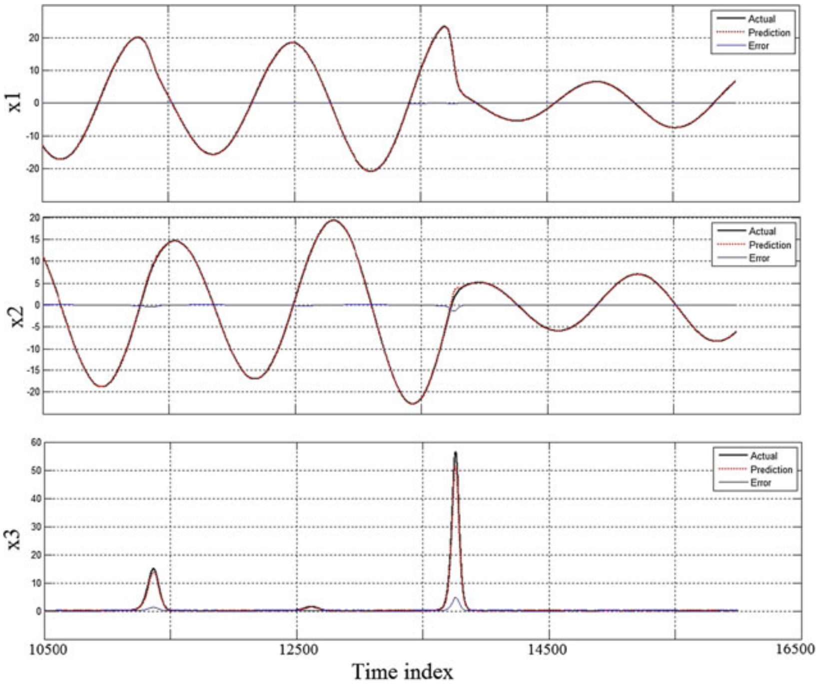


Fig. 5 One-step estimation of the Rossler map

added to c whereas an offset of -0.02 is added to b and c to produce gradual change of parameters. Figure 6 shows the results for the state variable x_1. The results for the remaining state variables are essentially the same.

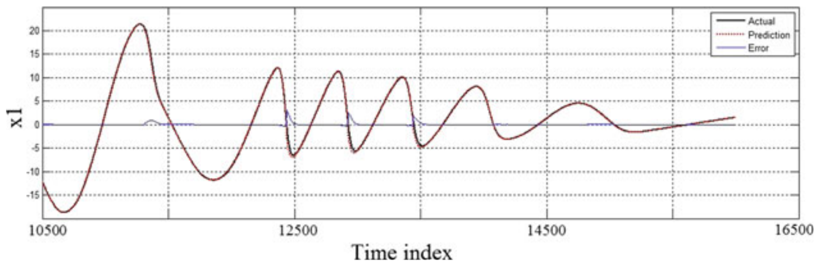


Fig. 6 One-step estimation of the variable x_1 of the Rossler system subject to abrupt ($k = 11500$) and gradual ($k = 14000, ...$) changes of parameters

Note from Fig. 6 that the oscillations of the error rate are stronger after the concept shift, but the quality of state estimates improves after few time steps. To maintain an acceptable level of prediction performance when the large and unknown change occurred, the learning algorithm created an additional fuzzy rule - a 6^{th} rule. Conversely, when gradual change of the values of the parameters occurred, the learning algorithm basically adapted the parameters of existing rules to track the trajectory of the states. The evolving granular modeling method has shown to be robust to time-varying parameters and able to handle fuzzy data streams.

The granular incremental modeling was compared with alternative state-of-the-art evolving modeling approaches. The following models were considered: evolving Takagi-Sugeno (eTS) [18], Dynamic evolving Neuro-Fuzzy Inference System (DeNFIS) [19], extended Takagi-Sugeno (xTS) [20], and the evolving Granular Fuzzy Model (abbreviated in Table 1 as eGFM) described in this paper. We prioritized model compactness and estimation performance. The models were developed from scratch, with no rules nor pre-training. Table 1 summarizes the results of one-step state forecasting of the Rossler chaos. The RMSE is calculated over non-normalized data and averaged over 10 runs. The number of samples, k_c, see (33), is equal to 60000 in each of the simulations.

Table 1 Rossler Chaos - Prediction Performance

Model	Avg. Rules	RMSE Best	RMSE Avg.
xTS	23.7	0.0727	0.0744 ± 0.0015
eTS	**5.5**	0.0511	0.0619 ± 0.0096
DENFIS	34.7	0.0485	0.0528 ± 0.0032
eGFM	**5.5**	**0.0303**	**0.0407 ± 0.0100**

The results of Table 1 show that, strictly speaking, eGFM is the most accurate model from the best and average RMSE point of view. The eGFM produces an average of 5.5 rules, a rule base as compact as that of eTS. In other words, the granular modeling approach does not take advantage from a large amount of local processing units (granules/rules) to achieve the average performance of 0.0407. eGFM benefits from a combination of ingredients concerning with structural assumptions, peculiarities of the learning algorithm, and fuzzy granular framework to attain that performance. The effectiveness of the granular evolving approach in one-step estimation without prior knowledge about the data is verified in this experiment.

4 Conclusion

This chapter has introduced an incremental fuzzy granular approach for evolving modeling of nonlinear time-varying systems. The approach is capable to process and learn from numeric and/or fuzzy data incrementally. Imprecise data is handled using specificity measures of the input data during learning. Experiments with the time-varying Rossler attractor show the usefulness of the method developed; meanwhile, comparisons with state-of-the-art evolving approaches show its effectiveness.

Further research is needed to manage unmeasurable state variables. A systematic design method for evolving fuzzy observers using input-output data shall be considered. We will also look into issues related to different kinds of nonstationarities and uncertainties in data streams. Stability analysis and stabilization of time-varying nonlinear systems is also an important issue to be investigated.

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Fuzzy Measures and Integrals: Recent Developments

Michel Grabisch

Abstract This paper surveys the basic notions and most important results around fuzzy measures and integrals, as proposed independently by Choquet and Sugeno, as well as recent developments. The latter includes bases and transforms on set functions, fuzzy measures on set systems, the notion of horizontal additivity, basic Choquet calculus on the nonnegative real line introduced by Sugeno, the extension of the Choquet integral for nonmeasurable functions, and the notion of universal integral.

1 Introduction

This paper gives a survey of the research done on fuzzy measures and integrals since Sugeno proposed in 1974 the concept of fuzzy measure, with an emphasis on recent results. This field of research lies at the intersection of several independent domains, which makes it very active and attractive, namely, measure theory, theory of aggregation functions, cooperative game theory, combinatorial optimization, pseudo-Boolean functions and more generally theoretical computer sciences. As an illustration of this fact, the word "fuzzy measure" which was coined by Sugeno, has many different names according to the field where it is used: nonadditive measure, capacity, monotone game, pseudo-Boolean function, rank function of a polymatroid, etc. Evidently, this short paper cannot make a complete account of all the research undertaken in this area, a whole book will hardly suffices. Indeed, the author is preparing a monograph on this topic, with the title: "Set functions, games and capacities in decision making", to be published by Springer around the end of 2015. This paper gives a kind of quick and necessarily simplified summary of selected topics. We recommend the interested reader to consult the main (available) monographs dealing with fuzzy measures and integrals: Pap [1], Denneberg [2], Wang and Klir [3], the Handbook of measure theory edited by Pap [4], as well as the edited book [5],

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_8

and the survey paper [6]. The latter focusses on application in multicriteria decision making, an aspect which is not covered by this paper, restricting to theory.

To avoid intricacies, in the whole paper the universal set X is finite, with $|X| = n$. We often use $\vee, \wedge$, which collapse to maximum and minimum on finite sets.

2 Fuzzy Measures

Fuzzy measures introduced by Sugeno [7] are generalization of classical measures, i.e., additive and nonnegative set functions, whose domain is an algebra $\mathcal{F}$ on X. As we will see in Sect. 2.4, the structure of algebra is not needed here, and various structures can be thought of. For simplicity, we assume $\mathcal{F} = 2^X$ in the first subsections, the general case will be addressed in the last one.

2.1 Definition, Main Families and Properties

A *fuzzy measure* on X is a set function $\mu : 2^X \to \mathbb{R}$ such that $\mu(\varnothing) = 0$ and μ obeys monotonicity: $A \subseteq B \subseteq X$ implies $\mu(A) \leqslant \mu(B)$. Fuzzy measures are also called capacities (after Choquet [8]), nonadditive measures (Denneberg [2]), monotone measures (Wang and Klir [3]), etc. If in addition $\mu(X) = 1$, then the fuzzy measure is said to be *normalized*.

If monotonicity is dropped from the definition, we obtain nonmonotonic fuzzy measures, more commonly called *games*, denoted usually by v.

One of the most important property of fuzzy measures (or games as well) is *convexity*, a.k.a. *supermodularity*. A fuzzy measure μ is convex if for all $A, B \in 2^X$, $\mu(A \cup B) + \mu(A \cap B) \geqslant \mu(A) + \mu(B)$. If the reverse inequality holds, μ is said to be *concave* or *submodular*. Convexity is generalized by the so-called *k-monotonicity* property: μ is k-monotone for some fixed $2 \leqslant k \leqslant n$ if for any family of k sets $A_1, \ldots, A_k \in 2^X$,

$$\mu\Big(\bigcup_{i=1}^{k} A_i\Big) \geqslant \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} \mu\Big(\bigcap_{i \in I} A_i\Big). \tag{1}$$

Moreover, μ is *totally monotone* if it is k-monotone for every $k \geqslant 2$ (in fact, $2 \leqslant k \leqslant 2^n - 2$ suffices). The *k-alternating property* is defined similarly, interchanging $\bigcap$ and $\bigcup$ and reversing inequality. Lastly, μ is said to be *maxitive* if $\mu(A \cup B) = \mu(A) \vee \mu(B)$, and *minitive* if $\mu(A \cap B) = \mu(A) \wedge \mu(B)$.

The simplest fuzzy measures which can be thought of are 0-1-fuzzy measures: their range is simply $\{0, 1\}$. In game theory, they are called *simple games* and are useful in voting theory. Among them, particularly useful are *unanimity games* (a.k.a. *simple support functions*): for any $\varnothing \neq A \subseteq X$, the unanimity game u_A is defined by

$$u_A(B) = \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{otherwise.} \end{cases}$$

The next remarkable families are *possibility* and *necessity measures*: a possibility (resp., necessity) measure is a normalized maxitive (resp., minitive) fuzzy measure (Zadeh [9], Dubois and Prade [10]). Necessity measures are particular cases of *belief functions*, as proposed by Shafer [11] (similarly, plausibility functions generalize possibility measures). Mathematically speaking, a belief (resp., plausibility) function is a normalized totally monotone (resp., alternating) fuzzy measure.

2.2 Transforms and Bases

The set of games, as well as the set of set functions, form a vector space of dimension $2^n - 1$ (resp., 2^n). This is not the case for the set of fuzzy measures, which is only a cone, while the set of normalized capacities is a polytope, whose vertices are the 0-1 fuzzy measures (Stanley [12], Radojevic [13]). In the rest if this section, we deal with the vector space of set functions (the results can be however easily adapted to the set of games).

A *transform* is a mapping $\Psi : \mathbb{R}^{(2^N)} \to \mathbb{R}^{(2^N)}$, assigning to any set function ξ the set function Ψ^ξ. If the transform is linear and invertible, then it induces a basis of the vector space of set functions (and similarly for games). Conversely, any basis induces a linear invertible transformation. This is explicited in the next lemma.

Lemma 1 *(Faigle and Grabisch [14]) For every basis $\{b_S\}_{S\in 2^X}$ of $\mathbb{R}^{2^X}$, there exists a unique linear invertible transform Ψ such that for any $\xi \in \mathbb{R}^{2^X}$,*

$$\xi = \sum_{S\in 2^X} \Psi^\xi(S) b_S, \tag{2}$$

whose inverse Ψ^{-1} is given by $\xi \mapsto (\Psi^{-1})^\xi = \sum_{T\in 2^X} \xi(T) b_T$.

Conversely, to any transform Ψ corresponds a unique basis $\{b_S\}_{S\in 2^X}$ such that (2) holds, given by $b_S = (\Psi^{-1})^{\delta_S}$, where δ_S is a 0-1-valued set function defined by $\delta_S(T) = 1$ if and only $T = S$.

It is well known that the set of unanimity games forms a basis of the set of games. Adding the 0-1-valued set function $u_\varnothing$ defined by $u_\varnothing(S) = 1$ if and only if $S = \varnothing$, we get a basis for the vector space of set functions. By Lemma 1, the corresponding transform, denoted by m, satisfies

$$\xi(A) = \sum_{B\subseteq A} m^{\xi}(B) \qquad (A \in 2^X),$$

which yields

$$m^{\xi}(A) = \sum_{B\subseteq A} (-1)^{|A\setminus B|} \qquad (A \in 2^X).$$

This transform is known as the *Möbius transform*, famous in combinatorics. Among the many existing transforms, at least two of them have a special interest. The *interaction transform* [15], generalizing the Shapley value [16] and the interaction index of Murofushi and Soneda [17], has the following expression:

$$I^{\xi}(A) := \sum_{B\subseteq X\setminus A} \frac{(n-b-a)!b!}{(n-a+1)!} \Delta_A \xi(B) = \sum_{K\subseteq X} \frac{|X\setminus (A\cup K)|!|K\setminus A|!}{(n-a+1)!}(-1)^{|A\setminus K|}\xi(K)$$

for all $A \subseteq X$, where a, b, k are cardinalities of subsets A, B, K, respectively, and $\Delta_A \xi(B) = \sum_{K\subseteq A}(-1)^{|A\setminus K|}\xi(B\cup K)$. This transform enables the interpretation of fuzzy measures in a multicriteria decision making context [6, 18]. The inverse transform is given by

$$(I^{-1})^{\xi}(S) = \sum_{K\subseteq X} \beta^{|K|}_{|S\cap K|}\xi(K),$$

with coefficients β^l_k given by

$$\beta^l_k = \sum_{j=0}^{k} \binom{k}{j} B_{l-j} \qquad (k \leqslant l),$$

where the B_j's are the Bernoulli numbers. It follows from Lemma 1 that the corresponding basis is

$$b^I_T(S) = \beta^{|T|}_{|T\cap S|} \qquad (S, T \in 2^X).$$

The interaction transform of ξ can be expressed in a simple way through its Möbius transform:

$$I^{\xi}(A) = \sum_{B\supseteq A} \frac{1}{b-a+1} m^{\xi}(B). \tag{3}$$

The second transform of interest is the so-called *Fourier transform*, well known in computer sciences (see, e.g., de Wolf [19] and O'Donnell [20]). The Fourier transform of a set function ξ is defined by

$$F^{\xi}(S) = \frac{1}{2^n} \sum_{K \subseteq X} (-1)^{|S \cap K|} \xi(K).$$

Interestingly enough, it is auto-inverse up to the factor $1/2^n$:

$$(F^{-1})^{\xi}(S) = \sum_{K \subseteq X} (-1)^{|S \cap K|} \xi(K).$$

The corresponding basis is therefore

$$b_T^F(S) = \sum_{K \subseteq X} (-1)^{|S \cap K|} \delta_T(K) = (-1)^{|S \cap T|} \qquad (S, T \in 2^X).$$

The vectors of this basis (not that these are not games) are called *parity functions* in the literature of computer sciences. They are up to a recoding equal to the *Walsh functions* $w_S(T) = (-1)^{|S \setminus T|}$ (indeed, $b_T^F(S) = w_S(X \setminus T)$). These are a finite version of the original functions proposed by Walsh (see Hurst et al. [21]), who form a orhonormal basis of the set of square integrable functions on [0, 1]. The major advantage of the Fourier (or Walsh) basis is that it is orthonormal, in the sense that $\langle b_T^F, b_S^F \rangle = 1$ if $S = T$, and 0 otherwise, where the inner product is defined by

$$\langle \xi, \xi' \rangle = \frac{1}{2^n} \sum_{S \in 2^X} \xi(S)\xi'(S).$$

Another remarkable property is that the Fourier transform turns the convolution product into an ordinary product (like with the original definition of the Fourier transform):

$$F^{\xi * \xi'} = F^{\xi} F^{\xi'}$$

where the convolution product of two set functions is defined by

$$(\xi * \xi')(S) = \frac{1}{2^n} \sum_{T \in 2^X} \xi(S \Delta T)\xi'(T)$$

($S \Delta T$ is the symmetric difference, i.e., $(S \cup T) \setminus (S \cap T)$).

We finish this section by giving the bounds of the Möbius transform for a normalized fuzzy measure. Surprisingly, the interval in which the Möbius transform of a normalized fuzzy measure can vary is not $[-1, 1]$, but its bounds grow rapidly with n, approximately in $\frac{4^{\frac{n}{2}}}{\sqrt{\frac{\pi n}{2}}}$, as shown in [22] (corrected version of an earlier publication [23]). The precise result is as follows.

Theorem 1 *For any normalized fuzzy measure μ, its Möbius transform satisfies for any $A \subseteq N$, $|A| > 1$:*

$$-\binom{|A|-1}{l'_{|A|}} \leqslant m^{\mu}(A) \leqslant \binom{|A|-1}{l_{|A|}},$$

with

$$l_{|A|} = 2\left\lfloor \frac{|A|}{4} \right\rfloor, \quad l'_{|A|} = 2\left\lfloor \frac{|A|-1}{4} \right\rfloor + 1 \tag{4}$$

and for $|A| = 1 < n$:

$$0 \leqslant m^{\mu}(A) \leqslant 1,$$

and $m^{\mu}(A) = 1$ if $|A| = n = 1$. These upper and lower bounds are attained by the normalized fuzzy measures μ_A^*, μ_{A*}, respectively:

$$\mu_A^*(B) = \begin{cases} 1, & \text{if } |A| - l_{|A|} \leqslant |B \cap A| \leqslant |A| \\ 0, & \text{otherwise} \end{cases},$$

$$\mu_{A*}(B) = \begin{cases} 1, & \text{if } |A| - l'_{|A|} \leqslant |B \cap A| \leqslant |A| \\ 0, & \text{otherwise} \end{cases}$$

for any $B \subseteq N$.

We give in Table 1 the first values of the bounds.

Table 1 Lower and upper bounds for the Möbius transform of a normalized fuzzy measure

$\|A\|$	1	2	3	4	5	6	7	8	9	10	11	12
u.b. of $m^{\mu}(A)$	1	1	1	3	6	10	15	35	70	126	210	462
l.b. of $m^{\mu}(A)$	1(0)	−1	−2	−3	−4	−10	−20	−35	−56	−126	−252	−462

2.3 k-additive and p-symmetric Fuzzy Measures

A fuzzy measure μ is *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for every disjoint $A, B \in 2^X$. Normalized additive fuzzy measures therefore coincide with probability measures. Observing that the Möbius transform of an additive fuzzy measure μ satisfies $m^{\mu}(A) = 0$ for all $A \in 2^X$ such that $|A| > 1$, a natural generalization of additivity is k-additivity: a fuzzy measure μ is *k-additive* ($1 \leq k \leq n$) if $m^{\mu}(A) = 0$ for all $A \in 2^X$ such that $|A| > k$, and there exists at least one $A \in 2^X$ such that $m^{\mu}(A) \neq 0$ (Grabisch [15]). It follows that a k-additive fuzzy measure needs only $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$ coefficients to be defined, instead of $2^n - 1$.

Due to (3), an equivalent definition is: μ is k-additive if its interaction transform I^μ vanishes for subsets of more than k elements, and there exists a subset A of k elements such that $I^\mu(A) \neq 0$. Since the interaction transform has a clear interpretation in the context of multicriteria decision making, k-additive fuzzy measures are of particular interest. Especially, 2-additive fuzzy measure have the advantage of being the simplest fuzzy measures (in terms of number of free coefficients) able to represent interaction between two elements.

k-additive fuzzy measures are families of fuzzy measures which are of polynomial complexity instead of the exponential complexity of general fuzzy measures. Another set of such families is provided by the concept of p-symmetric fuzzy measure (Miranda and Grabisch [24, 25]). A fuzzy measure μ is *symmetric* if $\mu(A) = \mu(B)$ whenever $|A| = |B|$. Furthermore, two distinct elements $i, j \in X$ are *symmetric* w.r.t. a fuzzy measure μ (denoted by $i \sim_\mu j$) if $\mu(A \cup i) = \mu(A \cup j)$ for every $A \subseteq X \setminus \{i, j\}$. Note that $\sim_\mu$ is an equivalence relation, and let us consider its equivalence classes, which forms a partition of X. Clearly, a symmetric fuzzy measure has only one such equivalence class, which is X. A natural generalization is: a fuzzy measure is *p-symmetric* if $\sim_\mu$ has p equivalence classes. It follows that any fuzzy measure is p-symmetric for some $1 \leq p \leq n$ (by the way, also k-additive for some $1 \leq k \leq n$).

Consider a p-symmetric fuzzy measure μ, with set of equivalence classes $\{A_1, \ldots, A_p\}$, and a subset $B \subseteq X$. Clearly, the value $\mu(B)$ depends uniquely on the numbers $b_1, \ldots, b_p$, with $b_i := |A_i \cap B|$. Since $0 \leqslant b_i \leqslant |A_i|$, it follows that μ needs $\prod_{i=1}^{p}(|A_i| + 1)$ coefficients to be defined.

2.4 Fuzzy Measures on Set Systems

A *set system* $\mathcal{F}$ on X is a subcollection of 2^X containing $\varnothing$ and covering X, that is, $\bigcup_{A \in \mathcal{F}} A = X$. We consider in this section fuzzy measures whose domain is a set system.

We begin by introducing the main families of set systems of interest. The most classical example borrowed from measure theory is algebra. An *algebra* is a set system closed under finite union and complementation. Although complementation is fundamental in classical measure theory, this is no more the case for fuzzy measures and games, so that other algebraic structures arise:

(i) **Set systems closed under union and intersection:** (Faigle and Kern [26]) It follows that such set systems contain X and are distributive lattices. Under the additional condition that there is no macro-element (i.e., a subset $M \subset X$ with $|M| > 1$ such that for any $A \in \mathcal{F}$, either $M \subseteq A$ or $A \cap M = \varnothing$), from Birkhoff's representation theorem, the set of all such set systems is in bijection with the set of partial orders on X. In other words, any such $\mathcal{F}$ is generated by a partial order on X, which can be interpreted as a kind of hierarchy of the elements in X. This is particularly meaningful when X is a set of players, agents, etc., or criteria.

(ii) **Weakly union-closed set systems:** (Algaba [27], Faigle and Grabisch [28, 29]) $\mathcal{F}$ is weakly union-closed if $A, B \in \mathcal{F}, A \cap B \neq \varnothing$ imply $A \cup B \in \mathcal{F}$. This larger family is motivated by communication graphs. Suppose that a graph (X, E) is defined on X, with X being the set of nodes, and E being the set of edges, i.e., pairs $\{i, j\}$ with $i, j \in X$ and $i \neq j$. Say that a subset $A \subseteq X$ is *connected* if for any distinct $i, j \in A$, there exists a sequence $i = i_1, i_2, \ldots, i_q = j$ of elements of X such that $\{i_k, i_{k+1}\} \in E$ for $k = 1, \ldots, q-1$. Defining $\mathcal{F}$ as the set of connected subsets of X, it follows that $\mathcal{F}$ is weakly union-closed (this is however not a characterizing property).

(iii) **Regular set systems:** [30, 31] a set system $\mathcal{F}$ is regular if it contains X and any maximal chain[1] from $\varnothing$ to X has length n. Every distributive lattice is a regular set system. The motivation for such sets systems is more mathematical: it happens that many concepts around games and fuzzy measures are based on maximal chains of length n (Shapley value, marginal vectors, Choquet integrals, etc.).

If $\mathcal{F}$ is a lattice (in particular, if $\mathcal{F}$ is closed under union and intersection), the definition of k-monotonicity is easily adapted by substituting $\cup, \cap$ in (1) by $\vee, \wedge$ of the lattice. It is well-known that when $\mathcal{F} = 2^X$, there is an equivalence between total monotonicity and the nonnegativity of the Möbius transform. It has been for a long time an unsolved issue whether this equivalence still holds if $\mathcal{F}$ is a lattice, only recently solved:

Theorem 2 *Let μ be fuzzy measure on a lattice $\mathcal{F}$. Then μ is totally monotone if and only if it has a nonnegative Möbius transform.*

The "only if" part was shown by Barthélemy [32], and the "if part" recently by Zhou [33].

3 The Choquet and Sugeno Integrals

The term "fuzzy integral" has been introduced by Sugeno [7] in 1974, and is now most commonly called the Sugeno integral. However, Choquet already in 1954 proposed a functional w.r.t. a fuzzy measure (or capacity), referred now as the Choquet integral. As we will see in Sect. 3.8, other integrals w.r.t. fuzzy measures have been proposed. We study in detail the Choquet and Sugeno integrals, which can be considered as the most representative (and still very different) fuzzy integrals. Except for Sect. 3.7, we assume that fuzzy measures are defined on $\mathcal{F} = 2^X$.

[1] A *chain* from $\varnothing$ to X is a sequence $\varnothing = A_0, A_1, \ldots, A_q = X$ of sets in $\mathcal{F}$ such that $A_0 \subset A_1 \subset \cdots \subset A_q$. Its *length* is q, and the chain is *maximal* if no other chain from $\varnothing$ to X contains it.

3.1 Definitions and Basic Properties

We begin by introducing the general definition, which is valid for arbitrary spaces. For this, we need decumulative distribution functions. Let μ be a fuzzy measure and $f : X \to \mathbb{R}$. The *decumulative distribution* of f w.r.t. μ is

$$G_{\mu,f}(t) = \mu(\{x \in X \mid f(x) \geqslant t\} \qquad (t \in \mathbb{R}).$$

We consider first nonnegative functions. Let $f : X \to \mathbb{R}_+$ and μ be a fuzzy measure. The *Choquet integral* of f w.r.t. μ is defined by

$$\int f \, \mathrm{d}\mu = \int_0^\infty G_{\mu,f}(t) \, \mathrm{d}t, \tag{5}$$

where the right hand-side integral is the Riemann integral. The *Sugeno integral* of f w.r.t. μ is defined by

$$⨍ f \, \mathrm{d}\mu = \bigvee_{t \geqslant 0} (G_{\mu,f}(t) \wedge t) = \bigwedge_{t \geqslant 0} (G_{\mu,f}(t) \vee t).$$

In words, the Sugeno integral is the abscissa of the intersection point between the diagonal and the decumulative function, while the Choquet integral is the area below the decumulative function. It can be proven that it is equivalent to consider a strict inequality in the definition of $G_{\mu,f}$. Another equivalent formula for the Sugeno integral is

$$⨍ f \, \mathrm{d}\mu = \bigvee_{A \in \mathcal{F}} \Big(\bigwedge_{x \in A} f(x) \wedge \mu(A) \Big).$$

Note that the Choquet integral can be defined w.r.t. games as well. However, since the decumulative function is no more monotone with games, the definition of the Sugeno integral is restricted to fuzzy measures. An elementary property is that for every $A \subseteq X$, $\int 1_A \, \mathrm{d}\mu = \mu(A)$, where 1_A is the characteristic function of A. The latter property holds also for the Sugeno integral, provided μ is normalized. In view of this property, the Choquet and Sugeno integrals can be considered as extensions of fuzzy measures.

When $X = \{x_1, \ldots, x_n\}$, the formulas can be made more explicit. For a function $f : X \to \mathbb{R}_+$, let f_i denotes $f(x_i)$ for simplicity, and take a permutation σ on $\{1, \ldots, n\}$ such that $f_{\sigma(1)} \leqslant \cdots \leqslant f_{\sigma(n)}$. Define $A_\sigma^\uparrow(i) = \{x_{\sigma(i)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\}$, $i = 1, \ldots, n$. The Choquet integral is given by

$$\int f \, \mathrm{d}\mu = \sum_{i=1}^{n} (f_{\sigma(i)} - f_{\sigma(i-1)}) \mu(A_\sigma^\uparrow(i)) \tag{6}$$

$$= \sum_{i=1}^{n} f_{\sigma(i)} \big(\mu(A_\sigma^\uparrow(i)) - \mu(A_\sigma^\uparrow(i+1)) \big), \tag{7}$$

with the conventions $f_{\sigma(0)} = 0$ and $A_\sigma^\uparrow(n+1) = \varnothing$.

For the Sugeno integral, we obtain:

$$⨍ f \, \mathrm{d}\mu = \bigvee_{i=1}^{n} \big(f_{\sigma(i)} \wedge \mu(A_\sigma^\uparrow(i)) \big) \tag{8}$$

$$= \bigwedge_{i=0}^{n} \big(f_{\sigma(i)} \vee \mu(A_\sigma^\uparrow(i+1)) \big) \tag{9}$$

with the same conventions.

We consider now the case of real-valued integrands. For any $f : X \to \mathbb{R}$, we write

$$f = f^+ - f^-, \text{ with } f^+ = 0 \vee f, \quad f^- = (-f)^+.$$

Then the *symmetric Choquet integral* (a.k.a. *Šipoš integral* [34]) is defined by

$$\check{\int} f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu. \tag{10}$$

The *asymmetric Choquet integral*, which is the usual definition, is defined by

$$\int f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\overline{\mu}, \tag{11}$$

where $\overline{\mu}$ is the *conjugate* fuzzy measure, defined by $\overline{\mu}(A) = \mu(X) - \mu(X \setminus A)$ for any $A \in 2^X$. The asymmetric Choquet integral is translation invariant (it is the only extension having this property), while the symmetric integral satisfies

$$\check{\int} (-f) \, \mathrm{d}\mu = -\check{\int} f \, \mathrm{d}\mu.$$

The case of the Sugeno integral is more cumbersome, essentially due to the following problem. The Sugeno integral is defined through the $\vee, \wedge$ operators, playing the rôle of addition and product respectively (compare (6) with (8)). Remembering that on the ring of real numbers, $a - b$ is shorthand for $a + (-b)$, a transposition of formula (10) for the Sugeno integral would read

$$\check{\int} f \, d\mu = \int f^+ \, d\mu \circledvee \Big(-\int f^- \, d\mu \Big) \tag{12}$$

where $\circledvee$ is an extension of $\vee$ for real numbers (i.e., $a \circledvee b = a \vee b$ whenever $a, b \geqslant 0$) such that $a \circledvee (-a) = 0$. Surprisingly, such an operator $\circledvee$ would be necessarily nonassociative. Indeed,

$$((-3) \circledvee 3) \circledvee 2) = 0 \circledvee 2 = 0 \vee 2 = 2$$
$$(-3) \circledvee (3 \circledvee 2) = (-3) \circledvee (3 \vee 2) = (-3) \circledvee 3 = 0.$$

The lack of associativity forbids to infer the so-called *rule of sign*, i.e., $(-a) \circledvee (-b) = -(a \circledvee b)$, which is necessary for the symmetry of the integral:

$$\begin{aligned}\check{\int}(-f) \, d\mu &= \int f^- \, d\mu \circledvee \Big(-\int f^+ \, d\mu \Big) = -\Big(\Big(-\int f^- \, d\mu \Big) \circledvee \int f^+ \, d\mu \Big) \\ &= -\check{\int} f \, d\mu.\end{aligned} \tag{13}$$

It can be shown [35] that the best operator (in the sense that it is associative on the largest domain) satisfying the above requirements (including the rule of sign) is the *symmetric maximum*, defined by

$$a \circledvee b = \begin{cases} -(|a| \vee |b|), & \text{if } b \neq -a \text{ and either } |a| \vee |b| = -a \text{ or } = -b \\ 0, & \text{if } b = -a \\ |a| \vee |b|, & \text{otherwise.} \end{cases} \tag{14}$$

The *symmetric Sugeno integral* [36] is therefore defined by (12) and $\circledvee$. Up to now, there is no adequate definition of an asymmetric Sugeno integral.

3.2 The Choquet Integral as a Linear Interpolator

Consider the following problem: a function $I : [0,1]^n \to [0,1]$ is known only on the vertices of the hypercube $[0,1]^n$ (in particular $I(\mathbf{0}) = 0$, where $\mathbf{0}$ is the 0 vector), and has to be determined everywhere in the hypercube. This is an interpolation problem, and there exists many ways to make the interpolation. Noting that the vertices of the hypercube correspond bijectively to the subsets of X (with $|X| = n$), it follows that I is necessarily an extension of a game v: $I(1_A) = v(A)$ for every $A \in 2^X$. Hence the Choquet and Sugeno integrals could be candidate.

Even if we restrict to a linear interpolation, there are still many ways of doing the interpolation, depending on which vertices are chosen, but there exist two extreme ways. If all vertices are used for each point $f \in [0,1]^n$, we get the *multilinear model* (owen, citeowe88), given by:

$$I(f) = \sum_{A \subseteq X, A \neq varnothing} m^v(A) \prod_{i \in A} f_i$$

where m^v is the Möbius transform of v, defined by $v(A) = I(1_A)$ for every $A \in 2^X$. The other extreme case would be to take the minimum number of vertices so that the considered vector x is contained in the convex hull of the selected vertices (parsimonious interpolation). Then this number is $n+1$, the number of vertices of a n-dimensional simplex, and the problem of choosing the right simplices for each f amounts to the triangulation problem of the hypercube. There is one triangulation of particular interest since it leads to an interpolation where all constant terms are 0, the triangulation in the $n!$ canonical simplices, where each simplex is induced by a permutation σ on $\{1, \ldots, n\}$:

$$S_\sigma = \{f \in [0,1]^n \mid f_{\sigma(1)} \leqslant f_{\sigma(2)} \leqslant \cdots \leqslant f_{\sigma(n)}\}.$$

Then it can be shown that the parsimonious linear interpolation based on the canonical simplices is the Choquet integral. This fact was remarked by Singer [37], and also Marichal [38].

3.3 Expression W.r.t Transforms

The Choquet integral being linear w.r.t. the game, it is easy to get its expression when the game is expressed by some linear invertible transform (equivalently, in some other basis). Let Ψ be a linear invertible transform, and $\{b_A^\Psi\}_{A \in 2^X}$ the corresponding basis of set functions given by Lemma 1. Since these set functions are not necessarily games, and the Choquet integral needs games to be well defined, we build a basis of games $\{b_A'^\Psi\}_{A \in 2^X \setminus \{\varnothing\}}$ as follows:

$$b'_S(T) = \begin{cases} b_S(T), & \text{if } T \neq \varnothing \\ 0, \text{otherwise} \end{cases} \qquad (S \in 2^X \setminus \{\varnothing\}). \tag{15}$$

Then for every $f \in \mathbb{R}^X$ and every game v,

$$\int f \,\mathrm{d}v = \int f \,\mathrm{d}\Big(\sum_{\varnothing \neq A \subseteq X} \Psi^v(A) b_A'^\Psi\Big) = \sum_{\varnothing \neq A \subseteq X} \Psi^v(A) \int f \,\mathrm{d}b_A'^\Psi. \tag{16}$$

It is therefore sufficient to compute $\int f \,\mathrm{d}b_A'^\Psi$ for every $A \subseteq X, A \neq \varnothing$.

Applying this to the Möbius transform immedaitely yields the following well-known formula:

$$\int f \,\mathrm{d}v = \sum_{A \subseteq X} m^v(A) \bigwedge_{i \in A} f_i. \tag{17}$$

The same methodology is not applicable to the Sugeno integral since it is not linear w.r.t. the fuzzy measure. It is possible however to obtain a formula similar to (17), by means of the *ordinal Möbius transform*. The ordinal Möbius transform of a fuzzy measure μ is the interval $[m] := [m_*, m^*]$, with $m^* = \mu$, and

$$m_*(A) = \begin{cases} \mu(A), & \text{if } \mu(A) > \mu(A \setminus i), \forall i \in A \\ 0, & \text{otherwise} \end{cases} \qquad (A \subseteq X). \tag{18}$$

The above formula has been first proposed in [39, 40], then developed in [35]. Then, it can be proved that the Sugeno integral takes the form:

$$⨍ f \,\mathrm{d}\mu = \bigvee_{A \subseteq X} \Big(\bigwedge_{i \in A} f_i \wedge m(A) \Big) \tag{19}$$

where m is any function in $[m_*, m^*]$.

3.4 Properties

The next propositions summarize the main elementary properties of Choquet and Sugeno integrals. In the whole section, X is supposed to be finite, and $\mathcal{F} = 2^X$.

Theorem 3 *Let $f : X \to \mathbb{R}$ be a function and a game v. The following properties hold for the Choquet integral.*

(i) Positive homogeneity:

$$\int \alpha f \,\mathrm{d}v = \alpha \int f \,\mathrm{d}v \qquad (\alpha \geqslant 0)$$

(ii) Homogeneity of the symmetric Choquet integral:

$$\check{\int} \alpha f \,\mathrm{d}v = \alpha \check{\int} f \,\mathrm{d}v \qquad (\alpha \in \mathbb{R})$$

(iii) Translation invariance:

$$\int (f + \alpha 1_X) \,\mathrm{d}v = \int f \,\mathrm{d}v + \alpha v(X) \qquad (\alpha \in \mathbb{R})$$

(iv) *Asymmetry:*

$$\int (-f)\,\mathrm{d}v = -\int f\,\mathrm{d}\overline{v}$$

where $\overline{v}$ is the conjugate game;

(v) *Scale inversion:*

$$\int (\alpha 1_X - f)\,\mathrm{d}v = \alpha v(X) - \int f\,\mathrm{d}\overline{v} \qquad (\alpha \in \mathbb{R})$$

(vi) *Monotonicity w.r.t. the integrand: for any fuzzy measure μ,*

$$f \leqslant f' \Rightarrow \int f\,\mathrm{d}\mu \leqslant \int f'\,\mathrm{d}\mu$$

(vii) *Monotonicity w.r.t. the game for nonnegative integrands: if $f \geqslant 0$,*

$$v \leqslant v' \Rightarrow \int f\,\mathrm{d}v \leq \int f\,\mathrm{d}v'$$

(viii) *Linearity w.r.t. the game:*

$$\int f\,\mathrm{d}(v + \alpha v') = \int f\,\mathrm{d}v + \alpha \int f\,\mathrm{d}v', \qquad (\alpha \in \mathbb{R})$$

(ix) *Boundaries:* $\inf f$ *and* $\sup f$ *are attained:*

$$\inf f = \int f\,\mathrm{d}\mu_{\min}, \quad \sup f = \int f\,\mathrm{d}\mu_{\max},$$

with $\mu_{\min}(A) = 0$ for all $A \subset X$, and $\mu_{\max}(A) = 1$ for all nonempty $A \subseteq X$;

(x) *Continuity.*

Theorem 4 *Let $f : X \to \mathbb{R}_+$, and μ a fuzzy measure on X. The following properties hold for the Sugeno integral.*

(i) *Positive $\wedge$-homogeneity:*

$$⨍ (\alpha 1_X \wedge f)\,\mathrm{d}\mu = \alpha \wedge ⨍ f\,\mathrm{d}\mu \qquad (\alpha \geqslant 0)$$

(ii) *Positive $\vee$-homogeneity if* $\sup f \leqslant \mu(X)$*:*

$$⨍ (\alpha 1_X \vee f)\,\mathrm{d}\mu = \alpha \vee ⨍ f\,\mathrm{d}\mu \qquad (\alpha \in [0, \sup f]).$$

(iii) *Hat function: for every* $\alpha \geqslant 0$ *and for every* $A \in \mathcal{F}$,

$$\underline{\int} \alpha 1_A \, \mathrm{d}\mu = \alpha \wedge \mu(A)$$

(iv) *Scale inversion: if* $\sup f \leqslant \mu(X)$,

$$\underline{\int} (\mu(X)1_X - f) \, \mathrm{d}\mu = \mu(X) - \underline{\int} f \, \mathrm{d}\overline{\mu},$$

where $\overline{\mu}$ *is the conjugate fuzzy measure;*

(v) *Scale translation:*

$$\underline{\int} (f + \alpha 1_X) \, \mathrm{d}\mu \leqslant \underline{\int} f \, \mathrm{d}\mu + \underline{\int} \alpha \, \mathrm{d}\mu = \underline{\int} f \, \mathrm{d}\mu + \alpha \wedge \mu(X) \qquad (\alpha \geqslant 0)$$

(vi) *Monotonicity w.r.t. the integrand:*

$$f \leqslant f' \Rightarrow \underline{\int} f \, \mathrm{d}\mu \leqslant \underline{\int} f' \, \mathrm{d}\mu \qquad (f, f' \in B^+(\mathcal{F}))$$

(vii) *Monotonicity w.r.t. the fuzzy measure:*

$$\mu \leqslant \mu' \Rightarrow \underline{\int} f \, \mathrm{d}\mu \leq \underline{\int} f \, \mathrm{d}\mu'$$

(viii) *Max-min linearity w.r.t. the fuzzy measure:*

$$\underline{\int} f \, \mathrm{d}(\mu \vee (\alpha \wedge \mu')) = \underline{\int} f \, \mathrm{d}\mu \ \vee \ \left(\alpha \wedge \int f \, \mathrm{d}\mu'\right) \qquad (\alpha \geqslant 0)$$

(ix) *Boundaries:* $\inf f$ *and* $\sup f$ *are attained:*

$$\inf f = \underline{\int} f \, \mathrm{d}\mu_{\min}, \quad \sup f = \underline{\int} f \, \mathrm{d}\mu_{\max},$$

with $\mu_{\min}, \mu_{\max}$ *defined as in Theorem 3;*

(x) *Lipschitz continuity:*

$$\left| \underline{\int} f \, \mathrm{d}\mu - \underline{\int} g \, \mathrm{d}\mu \right| \leqslant \mu(X) \wedge \|f - g\| \qquad (f, g \in B^+(\mathcal{F}))$$

with $\|f\| = \sup_{x \in X} |f(x)|$ *(Chebyshev norm). Hence, if* μ *is normalized and* f, g *are valued on* $[0, 1]$, *we obtain that the Sugeno integral is 1-Lipschitzian for the Chebyshev norm.*

A fundamental feature of both Choquet and Sugeno integrals is their relation with comonotonic functions. Two functions $f, g : X \to \mathbb{R}$ are *comonotonic* if there is no $x, x' \in X$ such that $f(x) < f(x')$ and $g(x) > g(x')$ (equivalently, in the case of a finite universe, if there exists a permutation σ on X such that $f_{\sigma(1)} \leqslant \cdots \leqslant f_{\sigma(n)}$ and $g_{\sigma(1)} \leqslant \cdots \leqslant g_{\sigma(n)}$).

Theorem 5 *Let f, g be comonotonic functions on X (finite). Then for any game v, the Choquet integral is comonotonically additive, and the Sugeno integral is comonotonically maxitive and minitive for any fuzzy measure μ:*

$$\int (f+g)\,\mathrm{d}v = \int f\,\mathrm{d}v + \int g\,\mathrm{d}v$$
$$⨍ (f \vee g)\,\mathrm{d}\mu = ⨍ f\,\mathrm{d}\mu \ \vee ⨍ g\,\mathrm{d}\mu$$
$$⨍ (f \wedge g)\,\mathrm{d}\mu = ⨍ f\,\mathrm{d}\mu \ \wedge ⨍ g\,\mathrm{d}\mu.$$

A more recently introduced type of additivity is called horizontal additivity (see Šipoš [34], and Benvenuti et al. [41]). Given a function $f : X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$, the *horizontal min-additive decomposition* of f is:

$$f = (f \wedge c1_X) + (f - (f \wedge c1_X)).$$

This amounts to "cut" horizontally the function at level c. Similarly, the *horizontal max-additive decomposition* of f is:

$$f = (f \vee c1_X) + (f - (f \vee c1_X)).$$

A functional $I : \mathbb{R}^X \to \mathbb{R}$ is *horizontally min-additive* if for every $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$,

$$I(f) = I(f \wedge c1_X) + I(f - (f \wedge c1_X)).$$

Horizontal max-additivity is defined similarly. It turns out that these notions are equivalent to comonotonic additivity, as shown by Couceiro and Marichal [42]. A related notion is horizontal median-additivity, introduced by Couceiro and Marichal [42]. Lastly, we introduce comonotonic modularity. A functional $I : \mathbb{R}^X \to \mathbb{R}$ is *modular* if for every $f, g : X \to \mathbb{R}$,

$$I(f \vee g) + I(f \wedge g) = I(f) + I(g).$$

It can be easily shown that the Choquet integral is comonotonically modular, i.e., for any comonotonic functions f, g it holds

$$\int (f \vee g)\, \mathrm{d}v + \int (f \wedge g)\, \mathrm{d}v = \int f\, \mathrm{d}v + \int g\, \mathrm{d}v.$$

This also holds for the Sugeno integral.

The next theorem clarifies the important case of supermodular fuzzy measures for the Choquet integral.

Theorem 6 *For any game v, the following conditions are equivalent:*

(i) *v is supermodular;*

(ii) *The Choquet integral is superadditive, that is,*

$$\int (f + g)\, \mathrm{d}v \geqslant \int f\, \mathrm{d}v + \int g\, \mathrm{d}v$$

for all $f, g : X \to \mathbb{R}$

(iii) *The Choquet integral is supermodular, that is,*

$$\int (f \vee g)\, \mathrm{d}v + \int (f \wedge g)\, \mathrm{d}v \geqslant \int f\, \mathrm{d}v + \int g\, \mathrm{d}v$$

for all $f, g : X \to \mathbb{R}$;

(iv) *The Choquet integral is concave, that is,*

$$\int (\lambda f + (1-\lambda) g)\, \mathrm{d}v \geqslant \int \lambda f\, \mathrm{d}v + (1-\lambda) \int g\, \mathrm{d}v$$

for all $\lambda \in [0, 1], f, g : X \to \mathbb{R}$.

(v) *The Choquet integral yields the lower expected value onthe core of v:*

$$\int f\, \mathrm{d}v = \min_{\phi \in \mathsf{core}(v)} \int f\, \mathrm{d}\phi, \tag{20}$$

where $\mathsf{core}(v)$ is the set of additive games ϕ on X such that $\phi(X) = v(X)$ and $\phi(S) \geqslant v(S)$ for all $S \in 2^X$.

Lastly, we give the properties of the Sugeno integral concerning maxitivity and minitivity.

Theorem 7 *The following holds:*

(i) $⨍(f \vee g)\, \mathrm{d}\mu = ⨍ f\, \mathrm{d}\mu \vee ⨍ g\, \mathrm{d}\mu$ *for all $f, g \in B^+(\mathcal{F})$ if and only if μ is maxitive;*

(ii) $⨍(f \wedge g)\, \mathrm{d}\mu = ⨍ f\, \mathrm{d}\mu \wedge ⨍ g\, \mathrm{d}\mu$ *for all $f, g \in B^+(\mathcal{F})$ if and only if μ is minitive.*

3.5 Characterizations

The most famous characterization of the Choquet integral is due to Schmeider [43], whose adaptation to the finite case ($|X| = n$) and $\mathcal{F} = 2^X$ is as follows.

Theorem 8 *Let $I : \mathbb{R}^X \to \mathbb{R}$ be a functional. Define the set function $v(A) = I(1_A)$ on 2^X. The following propositions are equivalent:*

(i) I is monotone and comonotonically additive;
(ii) v is a fuzzy measure, and for all $f \in \mathbb{R}^N$, $I(f) = \int f \, \mathrm{d}v$.

The discrete version (with a redundant axiom) was shown by de Campos and Bolaños [44]. A similar characterization for the Choquet integral w.r.t. games was obtained by Murofushi et al. [45].

In the discrete case, a characterization using comonotonic modularity was obtained by Couceiro and Marichal [46, 47].

Theorem 9 *Let $|X| = n$ and $\mathcal{F} = 2^X$, and let $I : \mathbb{R}^X \to \mathbb{R}$ be a functional. Define the set function $v(A) = I(1_A)$, $A \subseteq X$. The following propositions are equivalent:*

(i) I is comonotonically modular and satisfies $I(\alpha 1_S) = |\alpha| I(\mathrm{sign}\,(\alpha) 1_S)$ for all $\alpha \in \mathbb{R}$ and $S \subseteq X$, and $I(1_{X\setminus S}) = I(1_X) + I(-1_S)$;
(ii) v is a game and $I(f) = \int f \, \mathrm{d}v$.

The Sugeno integral was characterized in the discrete case by de Campos and Bolaños [44]. Here follows a simplified and more general version.

Theorem 10 *Let $|X| = n$, $\mathcal{F} = 2^X$, and let $I : (\mathbb{R}_+)^X \to \mathbb{R}_+$ be a functional. Define the set function $\mu(A) = I(1_A)$, $A \subseteq X$. The following propositions are equivalent:*

(i) I is comonotonically maxitive, satisfies $I(\alpha 1_A) = \alpha \wedge I(1_A)$ for every $\alpha \geqslant 0$ and $A \subseteq X$, and $I(1_X) = 1$;
(ii) μ is a normalized fuzzy measure on X and $I(f) = ⨍ f \, \mathrm{d}\mu$.

The next characterization is due to Marichal [48]. Still others can be found in this reference.

Theorem 11 *Let $|X| = n$, $\mathcal{F} = 2^X$, and let $I : [0, 1]^X \to [0, 1]$ be a functional. Define the set function $\mu(A) = I(1_A)$, $A \subseteq X$. The following propositions are equivalent:*

(i) I is nondecreasing, $\vee$-homogeneous and $\wedge$-homogeneous;
(ii) μ is a normalized fuzzy measure on X and $I(f) = ⨍ f \, \mathrm{d}\mu$.

3.6 The Choquet Integral on the Nonnegative Real Line

As remarked by Sugeno in two recent papers [49, 50], so far there is no "Choquet integral calculus", similar to classical integral calculus, even if one restricts to functions and measures on the real line. By means of the Laplace transform, Sugeno established in these two papers the basis of Choquet integral calculus. For this, the Choquet integral on a restricted domain is used:

$$\int_A f \, \mathrm{d}\mu = \int_0^\infty \mu(\{x \geqslant t\} \cap A) \, \mathrm{d}t$$

for some $A \subseteq X$. We give now the fundamental theorem.

Theorem 12 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing and continuously differentiable, and let μ be a continuous fuzzy measure on $\mathbb{R}_+$, such that $\mu([\tau, t])$ is differentiable w.r.t. τ on $[0, t]$ for every $t > 0$, and $\mu(\{t\}) = 0$ for every $t \geqslant 0$. Then*

$$\int_{[0,t]} f \, \mathrm{d}\mu = -\int_0^t \frac{\partial \mu}{\partial \tau}([\tau, t]) f(\tau) \, \mathrm{d}\tau \qquad (t > 0),$$

where the righthand side integral is the Riemann integral. In particular, for a distorted Lebesgue measure μ_h with h being continuously differentiable, we obtain

$$\int_{[0,t]} f \, \mathrm{d}\mu_h = \int_0^t \frac{\partial h}{\partial \tau}(t - \tau) f(\tau) \, \mathrm{d}\tau. \tag{21}$$

Equation (21) can be computed very easily through the Laplace transform. Denoting by $\mathcal{L}^{-1}$ the inverse Laplace transform, and by $H(s)$ and $F(s)$ the Laplace transforms of h and f, we have:

$$\int_{[0,t]} f \, \mathrm{d}\mu_h = \mathcal{L}^{-1}(sH(s)F(s)).$$

3.7 The Choquet Integral of Nonmeasurable Functions

So far we have considered that $\mathcal{F} = 2^X$, so that every subset is measurable and consequently any function is measurable too (i.e., its level sets belong to $\mathcal{F}$). In the case where $\mathcal{F} \subset 2^X$, what about the integral of a nonmeasurable function? The question may appear quite odd, but makes sense in practical situations, for example in multicriteria decision making. In this field, X is the set of criteria and $\mu(A)$ for some $A \subseteq X$ is interpreted as the overall evaluation of an alternative being satisfactory on criteria in A, and unsatisfactory or neutral on the others. It may be the case that such an alternative is not conceivable, and so no value can be assigned to $\mu(A)$. However, when computing the overall score of an alternative, knowing the vector f of its scores on every criterion, the set A may be a level set of f (i.e., $A = \{x \in X \mid f(x) \geqslant t\}$ for some t), so that f is not measurable and its Choquet integral cannot be computed. In this section we indicate how to extend the Choquet integral to nonmeasurable functions. This work is based on [28].

Let $\mathcal{F}$ be a fixed set system. We decompose any game v on $\mathcal{F}$ as $v = v^+ + v^-$, where v^+, v^- are two totally monotone fuzzy measures:

$$v^+ = \sum_{A\in\mathcal{F}|m^v(A)>0} m^v(A)u_A, \quad v^- = \sum_{A\in\mathcal{F}|m^v(A)<0} (-m^v(A))u_A. \tag{22}$$

We first define the Choquet integral w.r.t. a totally monotone fuzzy measure b on $\mathcal{F}$ as follows (f is assumed to be nonnegative):

$$\int_{\mathcal{F}} f\,\mathrm{d}b = \max\left\{\sum_{A\in\mathcal{F}} \alpha_A b(A) \mid \sum_{A\in\mathcal{F}} \alpha_A 1_A \leqslant f, \alpha_A \geqslant 0, \forall A \in \mathcal{F}\right\} \tag{23}$$

$$= \min\left\{\sum_{i\in X} P_i f_i \mid \sum_{i\in A} P_i \geqslant b(A), \forall A \in \mathcal{F}, P_i \geqslant 0, \forall i \in X\right\}. \tag{24}$$

It can be proved that this is the smallest functional I satisfying positive homogeneity, superadditivity and $I(1_A) \geqslant b(A)$ for all $A \in \mathcal{F}$. Now, the Choquet integral for any function $f : X \to \mathbb{R}$ w.r.t. a game v is defined by

$$\int_{\mathcal{F}} f\,\mathrm{d}v = \int_{\mathcal{F}} f\,\mathrm{d}v^+ - \int_{\mathcal{F}} f\,\mathrm{d}v^-. \tag{25}$$

We summarize the main properties of this integral.

Theorem 13 *Let $f : X \to \mathbb{R}_+$ be a function and v be a game on $(X, \mathcal{F})$, where $\mathcal{F}$ is any set system. The following properties hold.*

(i) Positive homogeneity:

$$\int_{\mathcal{F}} \alpha f\,\mathrm{d}v = \alpha \int_{\mathcal{F}} f\,\mathrm{d}v \qquad (\alpha \geqslant 0)$$

(ii) For any $S \in \mathcal{F}$,

$$\int_{\mathcal{F}} f\,\mathrm{d}u_S = \min_{i\in S} f_i$$

where u_S is the unanimity game w.r.t. S;

(iii) If $\mathcal{F}$ is weakly union-closed,

$$\int_{\mathcal{F}} f\,\mathrm{d}v = \sum_{S\in\mathcal{F}} m^v(S) \min_{i\in S} f_i$$

where m^v is the Möbius transform of v;

(iv) If $\mathcal{F}$ is weakly union-closed,

$$\int_{\mathcal{F}} f \, \mathrm{d}v = \int f \, \mathrm{d}\hat{v}$$

where the right-hand side integral is the ordinary Choquet integral, and $\hat{v}$ is a game on $(X, 2^X)$ defined by

$$\hat{v}(S) = \int_{\mathcal{F}} 1_S \, \mathrm{d}v = \sum_{F \text{ maximum in } \mathcal{F}(S)} v(F) \qquad (S \in 2^X),$$

with $\mathcal{F}(S) = \{F \in \mathcal{F} \mid F \subseteq S\}$.

(v) If $\mathcal{F}$ is weakly union-closed, $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is superadditive if and only if it is concave if and only if $\hat{v}$ is supermodular.

From (iv) we see that this integral is essentially the Choquet integral w.r.t. a modified game $\hat{v}$, and therefore inherits all of its properties. Moreover, $\hat{v}$ is an extension of v in the sense that it coincides with v on $\mathcal{F}$. It turns out that this integral yields the Choquet integral for measurable functions, and is indeed an extension of the Choquet integral. Note however that if v is monotone, $\hat{v}$ is not necessarily so.

More results can be obtained if $\mathcal{F}$ is closed under union. In this case, it can be shown that a fuzzy measure μ on $\mathcal{F}$ is supermodular if and only if $\hat{\mu}$ is, where supermodularity for μ is defined as follows: for any $S, T \in \mathcal{F}$,

$$\mu(S \cup T) + \mu((S \cap T)') \geqslant \mu(S) + \mu(T),$$

where $(S \cap T)'$ is the largest subset of $S \cap T$ in $\mathcal{F}$. Moreover, the following holds.

Theorem 14 *Let $\mathcal{F}$ be a set system closed under union, and μ be a fuzzy measure on $(X, \mathcal{F})$. The following are equivalent:*

(i) For every function $f : X \to \mathbb{R}_+$,

$$\int_{\mathcal{F}} f \, \mathrm{d}\mu = \max\Big\{ \sum_{S \in \mathcal{F}} \lambda_S \mu(S) \mid \sum_{S \in \mathcal{F}} \lambda_S 1_S \leqslant f, \lambda \geqslant \mathbf{0} \Big\}$$
$$= \min\Big\{ \sum_{i \in X} P_i f_i \mid P(S) \geqslant \mu(S), \forall S \in \mathcal{F}, P \geqslant \mathbf{0} \Big\},$$

where $\mathbf{0}$ indicates the 0 vector.

(ii) $\int_{\mathcal{F}} \cdot \, \mathrm{d}\mu$ is superadditive;

(iii) μ is supermodular.

3.8 Other Integrals

We describe briefly other kinds of integrals defined with respect to fuzzy measures.

Pseudo-additive integrals and fuzzy t-conorm integrals It is possible to define other integrals by simply replacing the operations used in the definitions of Choquet and Sugeno integrals (sum, product, max, min) by other ones, generally speaking, by *pseudo-additions* and *pseudo-multiplications*. There has been many studies in this direction, starting from Weber [51] and Kruse [52], then later Sugeno and Murofushi [53], Murofushi and Sugeno (fuzzy t-conorm integral) [54], Klement, Mesiar and Pap ((S,U)-integral) [55], Benvenuti et al. [41], and more recently the impressive study by Sander and Siedekum [56–58].

Basically, the (S, U)-integral uses as basis operators a continuous t-conorm S and a uninorm U which is distributive w.r.t. S in the following sense:

$$U(x, S(y, z)) = S(U(x, y), U(x, z))$$

for all $x, y, z \in [0, 1]$ such that $S(y, z) < 1$.

The fuzzy t-conorm integral proposed by Murofushi and Sugeno uses three continuous t-conorms S_1, S_2, S_3 which are either the maximum or Archimedean, plus a pseudo-multiplication $\odot$, being nondecreasing in each place, continuous on $]0, 1]^2$, and satisfying $a \odot x = 0$ implies either $a = 0$ or $x = 0$, and two distributivity properties:

(D1) $S_1(a, b) < 1$ implies $(S_1(a, b)) \odot x = S_3((a \odot x), (b \odot x))$
(D2) $S_2(x, y) < 1$ implies $a \odot (S_2(x, y)) = S_3((a \odot x), (a \odot y))$.

The definition of the fuzzy t-conorm integral is then:

$$(S_1, S_2, S_3, \odot) \int f \, d\mu := \mathop{\mathrm{S}_3}_{i=1}^{n} (f_{\sigma(i)} \stackrel{S_1}{-} f_{\sigma(i-1)}) \odot \mu(A_{\sigma(i)})$$

with same notation as above, and $\stackrel{S_1}{-}$ is the residuated difference w.r.t. S_1, defined by

$$a \stackrel{S_1}{-} b := \inf\{c \mid S_1(b, c) \geqslant a\}$$

for any (a, b) in $[0, 1]^2$. The Choquet integral is recovered with $S_1, S-2, S_3$ being the Łukasiewicz t-conorm, and $\odot$ the usual product. The Sugeno integral is recovered with $S_1 = S_2 = S_3 = \max$ and $\odot = \min$, and the Shilkret [59] integral is obtained when $\odot$ is the ordinary product.

The integral proposed by Benvenuti et al. is similar.

Universal integrals Universal integrals, proposed by Klement et al. [60] (see also a more recent work [61]), try to answer the following question: *What is an integral w.r.t. a fuzzy measure?*. The answer given by Klement et al. is axiomatic: they propose a list of axioms a functional should satisfy to be considered as a integral.

The name "universal" comes from the fact that the integral should be defined for any measurable space $(X, \mathcal{A})$ where $\mathcal{A}$ is a σ-algebra.

They first define a pseudo-multiplication as an operator $\otimes : [0, \infty]^2 \to [0, \infty]$ satisfying the following properties: it is nondecreasing in each place, 0 is an annihilator of $\otimes$, i.e., $a \otimes 0 = 0 \otimes a = 0$, and $\otimes$ has a neutral element $e \neq 0$, i.e., $a \otimes e = e \otimes a = a$.

Let us denote by $\mathcal{D}$ the set of all Cartesian products $\mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A})$ for every measurable space $(X, \mathcal{A})$, where $\mathcal{M}(X, \mathcal{A})$ is the set of fuzzy measures on $(X, \mathcal{A})$, and $\mathcal{F}(X, \mathcal{A})$ is the set of $\mathcal{A}$-measurable functions. A functional $I : \mathcal{D} \to [0, \infty]$ is called a *universal integral* of it satisfies the three following axioms:

(i) For any measurable space $(X, \mathcal{A})$, its restriction to $\mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A})$ is nondecreasing in each place
(ii) There exists a pseudo-multiplication $\otimes$ such that for all $(\mu, c \cdot 1_A) \in \mathcal{D}$, $I(\mu, c \cdot 1_A) = c \otimes \mu(A)$
(iii) $I(\mu, f) = I(\mu', f')$ if $G_{\mu,f} = G_{\mu',f'}$.

Obviously, the Choquet integral and the Sugeno integrals are universal integrals. It is not difficult to see that a universal integral is a distortion of the decumulative function by a function J begin nondecreasing and satisfying $J(d \cdot 1_{]0,c]} = c \otimes d$. The Sugeno and Shilkret integrals belong to the set of smallest universal integrals (in the sense of the usual partial order on functions), given by

$$I_{\otimes}(\mu, f) = \sup_{t \in]0, infty]} (t \otimes G_{\mu,f}(t)).$$

It can be shown that all integrals of the form (5), with product and addition being replaced by a pseudo-multiplication $\otimes$ and a pseudo-addition $\oplus$ being continuous, associative, nondecreasing, having 0 as neutral element and being left-distributive w.r.t. $\otimes$, are universal integrals.

The concave integral and decomposition integral Recenty, in a series of papers Lehrer presented the concave integral [62–64], and a more general concept called the decomposition integral [65], encompassing both the concave integral and the Choquet integral, as well as the Shilkret integral.

We first introduce the concave integral. Let $f : X \to \mathbb{R}_+$ and μ be a fuzzy measure. The *concave integral* of f w.r.t. μ is given by:

$$\int^{\mathrm{cav}} f \,\mathrm{d}\mu = \sup \left\{ \sum_{S \subseteq X} \alpha_S \mu(S) \mid \sum_{S \subseteq X} \alpha_S 1_S = f, \quad \alpha_S \geqslant 0, \forall S \subseteq X \right\}. \qquad (26)$$

In words, the concave integral is the value achieved by the best decomposition of the integrand into hat functions. Note that for totally monotone fuzzy measures, the concave integral and the integral proposed by Faigle and Grabisch coincide (see Sect. 3.7).

Its main properties are given below.

Theorem 15 *The following properties hold for the concave integral:*

(i) *For every fuzzy measure* μ, *the concave integral* $\int^{\text{cav}} \cdot \, d\mu$ *is a concave and positively homogeneous functional, and satisfies* $\int^{\text{cav}} 1_S \, d\mu \geqslant \mu(S)$ *for all* $S \in 2^X$*;*

(ii) *For every* $f \in \mathbb{R}_+^X$ *and fuzzy measure* μ*,*

$$\int^{\text{cav}} f \, d\mu = \min \big\{ I(f) \mid I : \mathbb{R}_+^X \to \mathbb{R} \text{ concave, positively homogeneous, and such that } I(1_S) \geqslant \mu(S), \forall S \subseteq X \big\}$$

(iii) *For every* $f \in \mathbb{R}_+^X$ *and fuzzy measure* μ*,*

$$\int^{\text{cav}} f \, d\mu = \min_{P \text{ additive }, P \geqslant \mu} \int f \, dP$$

(iv) *For every* $f \in \mathbb{R}_+^X$ *and fuzzy measure* μ*,*

$$\int f \, d\mu \leqslant \int^{\text{cav}} f \, d\mu,$$

and equality holds for every $f \in \mathbb{R}_+^X$ *if and only if* μ *is supermodular.*

Property (iv) clearly shows that unless the fuzzy measure is supermodular, the Choquet integral and the concave integral differ.

As for the decomposition integral, the idea is simply to fix a "vocabulary" for the decompositions. If only chains are allowed for the decomposition of a function, then the Choquet integral obtains as the best achievable value for such decompositions. If no restriction applies, then the concave integral is obtained. Also, the Shilkret integral can also be recovered. We refer the reader to [65] for full details on this complex notion.

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Important New Terms and Classifications in Uncertainty and Fuzzy Logic

Madan M. Gupta and Ashu M.G. Solo

Abstract Human cognitive and perception processes have a great tolerance for imprecision or uncertainty. For this reason, the notions of perception and cognition have great importance in solving many decision making problems in engineering, medicine, science, and social science as there are innumerable uncertainties in real-world phenomena. These uncertainties can be broadly classified as either *type one uncertainty* arising from the random behavior of physical processes or *type two uncertainty* arising from human perception and cognition processes. Statistical theory can be used to model the former, but lacks the sophistication to process the latter. The theory of fuzzy logic has proven to be very effective in processing type two uncertainty. New computing methods based on fuzzy logic can lead to greater adaptability, tractability, robustness, a lower cost solution, and better rapport with reality in the development of intelligent systems. Fuzzy logic is needed to properly pose and answer queries about quantitatively defining imprecise linguistic terms like *middle class, poor, low inflation, medium inflation,* and *high inflation.* Imprecise terms like these in natural languages should be considered to have *qualitative definitions, quantitative definitions, crisp quantitative definitions, fuzzy quantitative definitions, type-one fuzzy quantitative definitions,* and *interval type-two fuzzy quantitative definitions.* There can be *crisp queries, crisp answers, type-one fuzzy queries, type-one fuzzy answers, interval type-two fuzzy queries,* and *interval type-two fuzzy answers.*

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_9

1 Introduction

For a long time, engineers and scientists have learned from nature and tried to mimic some of the capabilities observed in humans and animals in electrical and mechanical machines. The Wright brothers started their work on the first airplane by studying the flight of birds. Most scientists of the time thought that it was the flapping of wings that was the principle component of flying. However, the Wright brothers realized that wings were required to increase the buoyancy in air.

In biomedical engineering, the principles of natural science and engineering are applied to the benefit of the health sciences. The opposite approach, *reverse biological engineering,* is used to apply biological principles to the solution of engineering and scientific problems. In particular, engineers and scientists use this reverse engineering approach on humans and animals in developing intelligent systems.

The principle attributes of a human being can be classified in three categories (3 Hs): hands, head, and heart. The hands category refers to the physical attributes of humans. These physical attributes have been somewhat mimicked and somewhat improved on to surpass the restrictive physical limitations of humans through such mighty machines as the tractor, assembly line, and aircraft. The head category refers to the perception and cognition abilities of the brain. The restrictive reasoning limitations of humans have been supplemented through the ongoing development of microprocessors. However, the challenge of creating an intelligent system is still in its incipient stages. Finally, the heart category refers to emotions. Machines can display simple emotional behavior, but they can't really feel.

One of the most exciting engineering endeavors involves the effort to mimic human intelligence. Intelligence implies the ability to comprehend, reason, memorize, learn, adapt, and create. It is often said that everybody makes mistakes, but an intelligent person learns from his mistakes and avoids repeating them. This fact of life emphasizes the importance of comprehension, reasoning, learning, and the ability to improve one's performance autonomously in the definition of intelligence.

There are essentially two computational systems: carbon-based organic brains, which have existed in humans and animals since their inception, and electronics-based computers, which have rapidly evolved over the latter half of the twentieth century and beyond. Technological advances in recent decades have made it possible to develop computers that are extremely fast and efficient for numerical computations. However, these computers lack the abilities of humans and animals in processing cognitive information acquired by natural sensors. For example, the human brain routinely performs tasks like recognizing a face in an unfamiliar crowd in 100–200 ms whereas a computer can take days to perform a task of lesser complexity. While the information perceived through natural sensors in humans is not numerical, the brain can process such cognitive information efficiently and cause the human to act on it accordingly. Modern day computers fail miserably in processing such cognitive information.

This leads engineers to wonder if some of the functions and attributes of the human sensory system, cognitive processor, and motor neurons can be emulated in an intelligent system. For such an emulation process, it is necessary to understand the biological and physiological functions of the brain. Hardware can be developed to model aspects of neurons, the principle element of the brain. Similarly, new theories and methodologies can be developed to model the human thinking process.

Many advances have been made in mimicking human cognitive abilities. These advances were mostly inspired by certain biological aspects of the human brain. One of the intriguing aspects of human perception and cognition is its tolerance for imprecision and uncertainty [1–10], which characterize most real-world phenomena.

2 Certainty and Precision

The excess of precision and certainty in engineering and scientific research and development is often providing unrealizable solutions. Certainty and precision have much too often become an absolute standard in design, decision making, and control problems. One of the fundamental aims in science and engineering has been to move from perceptions to measurements in observations, analysis, and decision making.

Through the methodology of precise measurements, engineers and scientists have had many remarkable accomplishments. These include putting people on the moon and returning them safely to Earth, sending spacecraft to the far reaches of the solar system, sending rovers to explore the surface of Mars, exploring the oceans depths, designing computers that can perform billions of computations per second, developing the nuclear bomb, mapping the human genome, and constructing a scanning tunneling microscope that can move individual atoms. However, the path of precision, as manifested in the theories of determinism and stochasticism, has often caused engineers to be ineffectual and powerless as well as lose scientific creativity.

3 Uncertainty and Imprecision in Perception and Cognition

The attribute of certainty or precision does not exist in human perception and cognition. Alongside many startling achievements using the methodology of precise measurements, there have been many abysmal failures that include modeling the behavior of economic, political, social, physical, and biological systems. Engineers have been unable to develop technology that can decipher sloppy handwriting, recognize oral speech as well as a human can, translate between languages as well as a human interpreter can, drive a motorcycle in heavy traffic, walk with the agility of a human or animal, replace the combat infantry soldier, determine the veracity of a statement by a human subject with an acceptable degree of accuracy, replace judges and juries, summarize a complicated document, and explain poetry or song lyrics. Underlying these failures is the inability to manipulate imprecise perceptions instead of precise measurements in computing methodologies.

Albert Einstein wrote, "So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality [11]."

There are various types of uncertainty. However, they can be classified under two broad categories: *type one uncertainty* and *type two uncertainty* [8–10].

3.1 Type One Uncertainty

Type one uncertainty deals with information that arises from the random behavior of physical systems. The pervasiveness of this type of uncertainty can be witnessed in random vibrations of a machine, random fluctuations of electrons in a magnetic field, diffusion of gases in a thermal field, random electrical activities of cardiac muscles, uncertain fluctuations in the weather pattern, and turbulent blood flow through a damaged cardiac valve. Type one uncertainty has been studied for centuries. Complex statistical mathematics has evolved for the characterization and analysis of such random phenomena.

3.2 Type Two Uncertainty

Type two uncertainty deals with information or phenomena that arise from human perception and cognitive processes or from cognitive information in general. This subject has received relatively little attention. Perception and cognition through biological sensors (eyes, ears, nose, etc.), perception of pain, and other similar biological events throughout our nervous system and neural networks deserve special attention. The perception and cognition phenomena associated with these processes are characterized by many great uncertainties and cannot be described by conventional statistical theory. A person can linguistically express perceptions experienced through the senses, but these perceptions cannot be described using conventional statistical theory.

Type two uncertainty and the associated cognitive information involve the activities of neural networks. It may seem strange that such familiar notions have recently become the focus of intense research. However, it is the relative unfamiliarity of these notions and their technological applications in intelligent systems that have led engineers and scientists to conduct research in the field of type two uncertainty and its associated cognitive information.

4 Fuzzy Logic

Fuzzy logic [12–37] has proven to be a very promising tool for dealing with type two uncertainty. Stochastic theory is only effective in dealing with type one uncertainty. The theory of fuzzy logic is based on the notion of relative graded membership, as inspired by the processes of human perception and cognition. Lotfi A. Zadeh published his first famous paper on fuzzy sets [12] in 1965.

Fuzzy logic can deal with information arising from computational perception and cognition that is uncertain, imprecise, vague, partially true, or without sharp boundaries. Fuzzy logic allows for the inclusion of vague human assessments in computing problems. Also, it provides an effective means for conflict resolution of multiple criteria and better assessment of options. New computing methods based on fuzzy logic can be used in the development of intelligent systems for decision making, identification, recognition, optimization, and control.

Measurements are crisp numbers, but perceptions are fuzzy numbers or fuzzy granules, which are groups of objects in which there can be partial membership and the transition of a membership function is gradual, not abrupt. A granule is a group of objects put together by similarity, proximity, functionality, or indistinguishability.

Fuzzy logic is extremely useful for many people involved in research and development including engineers (electrical, mechanical, civil, chemical, aerospace, agricultural, biomedical, computer, environmental, geological, industrial, mechatronics), mathematicians, computer software developers and researchers, natural scientists (biology, chemistry, earth science, physics), medical researchers, social scientists (economics, management, political science, psychology), public policy analysts, business analysts, jurists, etc. Indeed, the applications of fuzzy logic, once thought to be an obscure mathematical curiosity, can be found in many engineering and scientific works. Fuzzy logic has been used in numerous applications such as facial pattern recognition, washing machines, vacuum cleaners, antiskid braking systems, transmission systems, control of subway systems and unmanned helicopters, knowledge-based systems for multiobjective optimization of power systems, weather forecasting systems, models for new product pricing or project risk assessment, medical diagnosis and treatment plans, and stock trading. This branch of mathematics has instilled new life into scientific disciplines that have been dormant for a long time.

5 Qualitative Definitions, Crisp Quantitative Definitions, Type-One Fuzzy Quantitative Definitions, and Interval Type-Two Fuzzy Quantitative Definitions of Imprecise Words

Type-one fuzzy logic or *interval type-two fuzzy logic* [38–40] can be used to properly quantitatively define many imprecise linguistic terms including temperature, speed, unemployment levels, and inflation. Fuzzy logic is needed to quantitatively define imprecise linguistic terms like *high unemployment, moderate unemployment, low unemployment, very high unemployment, high inflation, medium inflation, low inflation, extremely low inflation, fast speed, low speed,* etc. *Type-one fuzzy sets* and *interval type-two fuzzy sets* have been used for imprecise linguistic terms in many intelligent systems applications, but this research chapter proposes the use of type-one fuzzy sets and interval type-two fuzzy sets for the application of posing and answering queries about quantitatively defining imprecise linguistic terms in natural languages.

An imprecise word should be considered to have *qualitative definitions* and *quantitative definitions* [41–44].

There are multiple qualitative definitions because a word can have multiple meanings and because different ways of defining a word can be employed. That is, different dictionaries use different descriptions to convey the meaning of the same word.

An imprecise word should be considered to have two types of quantitative definitions: *crisp quantitative definitions* and *fuzzy quantitative definitions* [41–44].

Crisp quantitative definitions are those made with crisp sets. There are multiple crisp quantitative definitions because different individuals have different perceptions of the crisp set for imprecise words. A crisp quantitative definition of annual inflation levels is in Fig. 1.

Fuzzy quantitative definitions are those made with fuzzy sets. There are multiple fuzzy quantitative definitions because different individuals have different perceptions of the fuzzy set for imprecise words. Fuzzy quantitative definitions of annual inflation levels are in Fig. 2 and Fig. 3.

It should be realized that while quantitative definitions of imprecise words can be made with crisp sets or fuzzy sets, only fuzzy sets can model the imprecision of words, so crisp sets have extremely limited value in modeling imprecise words.

An imprecise word should be considered to have two types of fuzzy quantitative definitions: *type-one fuzzy quantitative definitions* and *interval type-two fuzzy quantitative definitions* [44].

Type-one fuzzy quantitative definitions are those made with type-one fuzzy sets. Figure 2 shows type-one fuzzy quantitative definitions.

Interval type-two fuzzy quantitative definitions are those made with interval type-two fuzzy sets. Figure 3 shows interval type-two fuzzy quantitative definitions.

It is important to distinguish between qualitative definitions and quantitative definitions, crisp quantitative definitions and fuzzy quantitative definitions, and type-one fuzzy quantitative definitions and interval type-two fuzzy quantitative definitions.

6 Crisp Quantitative Definitions of Inflation Levels

6.1 Inflation Levels as Crisp Sets

Crisp sets can be arbitrarily defined for *low inflation, medium inflation,* and *high inflation.* These crisp sets are as illustrated in Fig. 1 and are the crisp quantitative definitions for *low inflation, medium inflation,* and *high inflation.*

For annual inflation rates less than 2.5 %, there is a membership of 1 in the *low inflation* crisp set and a membership of 0 in the other crisp sets. For annual inflation rates between 2.5 % and 5.5 %, there is a membership of 1 in the *medium inflation* crisp set and a membership of 0 in the other crisp sets. For annual inflation rates greater than 5.5 %, there is a membership of 1 in the *high inflation* crisp set and a membership of 0 in the other crisp sets. These crisp sets could be defined with different parameters.

With these crisp sets, an annual inflation rate of 2.4999 % is considered *low inflation* whereas an annual inflation rate of 2.5001 is considered *medium inflation.* This extremely sudden transition from low inflation to medium inflation for extremely small differences in annual inflation doesn't make sense and can be rectified using type-one fuzzy sets or interval type-two fuzzy sets, as can be seen in the next sections.

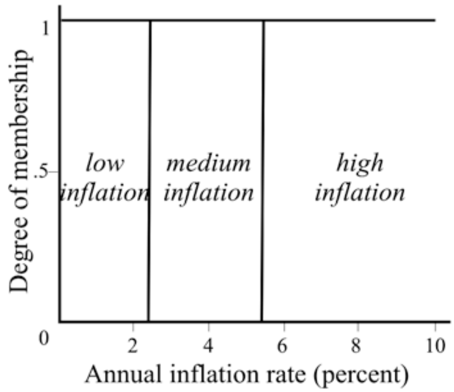


Fig. 1 Crisp sets for annual inflation levels.

6.2 Crisp Query About Quantitatively Defining Inflation Levels with Crisp Sets

A single *crisp query* for quantitatively defining annual inflation rates with fuzzy sets could be articulated as follows: "Using historical data on annual inflation rates, classify different annual inflation rates into low inflation, medium inflation, or high inflation."

6.3 Crisp Answer in Quantitatively Defining Inflation Levels with Crisp Sets

A *crisp answer* could be articulated as follows: “An annual inflation rate less than 2.5 % is low inflation. An annual inflation rate between 2.5 % and 5.5 % is medium inflation. An annual inflation rate greater than 5.5 % is high inflation.”

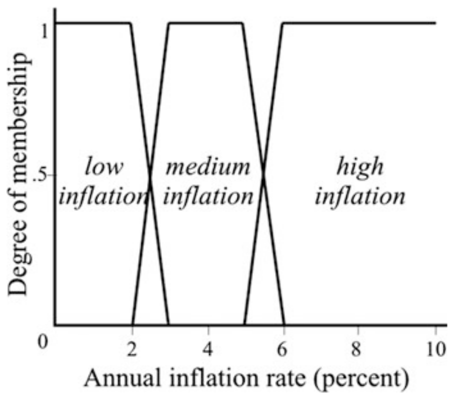
7 Type-One Fuzzy Quantitative Definitions of Inflation Levels

7.1 Inflation Levels as Type-One Fuzzy Sets

A type-one fuzzy set uses a membership function to assign a degree of membership from 0 to 1 to each domain value. Type-one fuzzy sets can be arbitrarily defined for *low inflation, medium inflation,* and *high inflation.* These type-one fuzzy sets are as illustrated in Fig. 2 and are the type-one fuzzy quantitative definitions for *low inflation, medium inflation,* and *high inflation.*

For annual inflation rates less than 2 %, there is a membership of 1 in the *low inflation* fuzzy set. As annual inflation increases from 2 % to 3 %, its membership in the *low inflation* fuzzy set steadily decreases from 1 to 0 with constant slope and its membership in the *medium inflation* fuzzy set steadily increases from 0 to 1 with constant slope. For annual inflation rates between 3 % and 5 %, there is a membership of 1 in the *medium inflation* fuzzy set. As annual inflation increases from 5 % to 6 %, its membership in the *medium inflation* fuzzy set steadily decreases from 1 to 0 with constant slope and its membership in the *high inflation* fuzzy set steadily increases from 0 to 1 with constant slope. For annual inflation rates greater than 6 %, there is a membership of 1 in the *high inflation* fuzzy set. These fuzzy sets could be defined with different parameters.

Fig. 2 Type-one fuzzy sets for annual inflation levels.



7.2 Type-One Fuzzy Query About Quantitatively Defining Inflation Levels with Type-One Fuzzy Sets

A *type-one fuzzy query* for quantitatively defining annual inflation rates with fuzzy sets could be articulated as follows: "Give me a range of inflation rates that are definitely low inflation. Give me a range of inflation rates that are partially low inflation and partially medium inflation. Give me a range of inflation rates that are definitely medium inflation. Give me a range of inflation rates that are partially medium inflation and partially high inflation. Give me a range of inflation rates that are definitely high inflation."

7.3 Type-One Fuzzy Answer About Quantitatively Defining Inflation Levels with Type-One Fuzzy Sets

A *type-one fuzzy answer* could be articulated as follows: "An annual inflation rate less than 2 % is low inflation. An annual inflation rate between 2 % and 3 % is partially low inflation and partially medium inflation. As annual inflation increases from 2 % to 3 %, its degree of being low inflation steadily decreases and its degree of being medium inflation steadily increases. An annual inflation rate between 3 % and 5 % is medium inflation. An annual inflation rate between 5 % and 6 % is partially medium inflation and partially high inflation. As annual inflation increases from 5 % to 6 %, its degree of being medium inflation steadily decreases and its degree of being high inflation steadily increases. An annual inflation rate greater than 6 % is high inflation."

8 Interval Type-Two Fuzzy Quantitative Definitions of Inflation Levels

8.1 Type-Two Fuzzy Sets and Interval Type-Two Fuzzy Sets

A type-two fuzzy set allows the inclusion of uncertainty into the parameters of a membership function. The membership function of a type-two fuzzy set is in itself a fuzzy set. A type-two fuzzy set is three-dimensional where the third dimension indicates the degree of membership of the two-dimensional membership function at each point in its two-dimensional domain.

In a type-two fuzzy set, a footprint of uncertainty indicates the upper and lower bounds in the two-dimensional domain of a type-two fuzzy set. A footprint of uncertainty in a type-two fuzzy set is a region bounded by an upper membership function and lower membership function.

An interval type-two fuzzy set is a type-two fuzzy set in which the third dimension is constant in value meaning the degree of membership is constant for the two-dimensional membership function at each point in its two-dimensional domain. Therefore, the third dimension is ignored.

It would be extremely difficult to linguistically describe an imprecise linguistic term with a type-two fuzzy set because there is a third dimension that indicates the degree of membership of the two-dimensional membership function at each point in its two-dimensional domain. It is much less difficult to linguistically describe an imprecise linguistic term with an interval type-two fuzzy set because the third dimension is constant in value and can be ignored. Because it is impractical to attempt to linguistically describe a type-two fuzzy set for an imprecise linguistic term, this research chapter only covers the usage of interval type-two fuzzy sets for describing imprecise linguistic terms.

8.2 Inflation Levels as Interval Type-Two Fuzzy Sets

Interval type-two fuzzy sets can be arbitrarily defined for *low inflation, medium inflation,* and *high inflation.* These interval type-two fuzzy sets are as illustrated in Fig. 3 and are the interval type-two fuzzy quantitative definitions for *low inflation, medium inflation,* and *high inflation.*

For annual inflation rates less than an inflation rate between 1.75 % and 2.25 %, there is a membership of 1 in the *low inflation* fuzzy set. As annual inflation increases from an inflation rate between 1.75 % and 2.25 % to an inflation rate between 2.75 % and 3.25 %, its membership in the *low inflation* fuzzy set steadily decreases from 1 to 0 with a constant slope and its membership in the *medium inflation* fuzzy set steadily increases from 0 to 1 with a constant slope. For annual inflation rates from an inflation rate between 2.75 % and 3.25 % to an inflation rate between 4.75 % and 5.25 %., there is a membership of 1 in the *medium inflation* fuzzy set. As annual inflation increases from an inflation rate between 4.75 % and 5.25 % to an inflation rate between 5.75 % and 6.25 %, its membership in the *medium inflation* fuzzy set steadily decreases from 1 to 0 with a constant slope and its membership in the *high inflation* fuzzy set steadily increases from 0 to 1 with a constant slope. For annual inflation rates greater than an inflation rate between 5.75 % and 6.25 %, there is a membership of 1 in the *high inflation* fuzzy set. These interval type-two fuzzy sets are as illustrated in Fig. 3 and are the interval type-two fuzzy quantitative definitions for *low inflation, medium inflation,* and *high inflation.* These interval type-two fuzzy sets could be defined with different parameters.

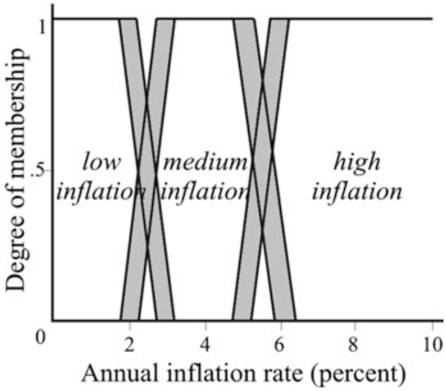


Fig. 3 Interval type-two fuzzy sets for annual inflation levels.

8.3 *Interval Type-Two Fuzzy Query About Quantitatively Defining Inflation Levels with Interval Type-Two Fuzzy Sets*

An *interval type-two fuzzy query* for quantitatively defining annual inflation rates with fuzzy sets could be articulated as follows: "Give me a range of annual inflation rates below which there is definitely low inflation. Give me a range of annual inflation rates between which there is partially low inflation and partially medium inflation. Give me a starting range and ending range of annual inflation rates between which there is definitely medium inflation. Give me a range of annual inflation rates between which there is partially medium inflation and partially high inflation. Give me a range of annual inflation rates above which there is definitely high inflation."

8.4 *Interval Type-Two Fuzzy Answers About Quantitatively Defining Inflation Levels with Interval Type-Two Fuzzy Sets*

An *interval type-two fuzzy answer* could be articulated as follows: "An annual inflation less than an inflation rate between 1.75 % and 2.25 % is low inflation. As annual inflation increases from an inflation rate between 1.75 % and 2.25 % to an inflation rate between 2.75 % and 3.25 %, there is partially low inflation and partially medium inflation. As annual inflation increases from an inflation rate between 1.75 % and 2.25 % to an inflation rate between 2.75 % and 3.25 %, its degree of being low inflation steadily decreases and its degree of being medium inflation steadily increases. An annual inflation rate between an inflation rate between 2.75 % and 3.25 % to an inflation rate between 4.75 % and 5.25 % is medium inflation. An annual inflation from an inflation rate between 4.75 % and 5.25 % to an inflation rate between 5.75 % and 6.25 % is partially medium inflation and partially high inflation. As annual inflation increases from an inflation rate between 4.75 % and 5.25 % to an inflation rate between 5.75 % and 6.25 %, its degree of being medium inflation steadily decreases and its degree of being high inflation steadily increases. An annual inflation rate greater than an inflation rate between 5.75 % and 6.25 % is high inflation."

9 Conclusion

Uncertainty is an inherent phenomenon in the universe and in peoples' lives. To some, it may become a cause of anxiety, but to engineers and scientists it becomes a frontier full of challenges. Engineers and scientists attempt to comprehend the language of this uncertainty through mathematical tools, but these mathematical tools are still incomplete. In the past, studies of cognitive uncertainty and cognitive information were hindered by the lack of suitable tools for modeling such information. However, fuzzy logic, neural networks, and other methods have made it possible to expand studies in this field. Whereas stochastic theory is effective in dealing with type one uncertainty, fuzzy logic is needed for type two uncertainty.

Humans think in imprecise and vague terms. Consequently, human language is inherently imprecise and vague. A major problem arises when people try to bring precision into situations where it doesn't apply, such as defining human linguistic terms like *high inflation* as being greater than a single precise annual income. An understanding of the basic principles of type-one fuzzy logic and interval type-two fuzzy logic can be extremely useful in posing proper questions and giving proper answers about quantitatively defining imprecise linguistic terms. Imprecise linguistic terms in natural languages should be considered to have qualitative definitions, crisp quantitative definitions, fuzzy quantitative definitions, type-one fuzzy quantitative definitions, and interval type-two fuzzy quantitative definitions.

Crisp queries, crisp answers, and crisp quantitative definitions are simpler than type-one fuzzy queries, type-one fuzzy answers, and type-one fuzzy quantitative definitions. It's easier to define an imprecise linguistic term with a crisp set than with a type-one fuzzy set, but a type-one fuzzy set allows for the inclusion of uncertainty in a membership function. If one wants to include uncertainty in a membership function, then a type-one fuzzy set should be used.

Type-one fuzzy queries, type-one fuzzy answers, and type-one fuzzy quantitative definitions are simpler than interval type-two fuzzy queries, interval type-two fuzzy answers, and interval type-two fuzzy quantitative definitions. It's easier to define an imprecise linguistic term with a type-one fuzzy set than with an interval type-two fuzzy set, but an interval type-two fuzzy set allows for the inclusion of uncertainty about the bounds of the membership function. If one wants to include uncertainty about the bounds of the membership function in a quantitative definition of an imprecise linguistic term, then an interval type-two fuzzy set should be used.

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Dr. Gupta was elected fellow of the Institute of Electrical and Electronics Engineers (IEEE) for his contributions to the theory of fuzzy sets and adaptive control systems and for the advancement on the diagnosis of cardiovascular disease. He was elected fellow of the International Society for Optical Engineering (SPIE) for his contributions to the field of neuro-control and neuro-fuzzy systems. Also, he was elected fellow of the International Fuzzy Systems Association (IFSA) for his contributions to fuzzy-neural computing systems. In 1998, Dr. Gupta was honored by the III-Kaufmann Prize and Gold Medal for his research in the field of fuzzy logic. This Gold Medal was presented by the Foundation FEGI (Fundacio per a l'Estudi de la Gestio en la Incertesa: Fuzzy Management Research Foundation) and SIGEF (Sociedad Internacional de Gestion Economia: Fuzzy, International Association for Fuzzy Set Management and Economy) in Reus, Spain. In 1991, Dr. Gupta was the co-recipient of the Institute of Electrical Engineering Kelvin Premium.

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Formalization and Visualization of Kansei Information Based on Fuzzy Set Approach

Fangyan Dong and Kaoru Hirota

Abstract Kansei or affective-computing related information is easy to express in terms of fuzzy sets. Three examples of Kansei information, e.g., emotion, atmosphere, and Kansei texture, are formalized by using fuzzy set concept on $[-1,1]^3$ space. They are also visualized by using shape-brightness-size, shape-color-size, and contour-shape-gradation models, respectively. Their applications to agent to agent communication, multiagent communication, and online shopping are also introduced.

1 Introduction

Kansei engineering has been studied originally in Japan and nowadays in worldwide, and is sometimes referred to affective computing. Its main purpose is to introduce information processing capability related to human ambiguity or subjectivity in the computer science/engineering field, accordingly it has a good matching with fuzzy set approach. The authors' group has been studying Kansei information processing in various IT fields. In this article, three topics are introduced, i.e., emotion understanding in man-machine interaction, atmosphere analysis/understanding in humans-robots interaction, and Kansei texture in online shopping.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_10

2 Emotion Understanding in Man-Machine Interaction

Emotion understanding has been studied in man-machine, human-robot, or generally agent-agent interaction using different type of devices. The ones that are more close to human way of understand the emotions are those which are based on voice, face, and gesture information [1, 2]. But the part missing with these approaches is a lack of experience [3]; learning from the interaction and creating knowledge is what gives humans the power to understand deeply the emotions of other person. Humans emotions are complex, and in many situations the emotion displayed in the face, voice or body gesture sometimes may not indicate the real or absolute emotion of the individuals [2], making the necessity to create an algorithm to model this human ability to improve human-robot interaction [4]. To address and to make a model of this problem, understanding by using information from face, voice, gesture, and others is called surface level emotion understanding, whereas a deep level emotion understanding is also proposed [5], where customized learning knowledge from communication history and a basic knowledge base about the observed agent are utilized with the observed visual/acoustic/gesture information input (Fig. 1).

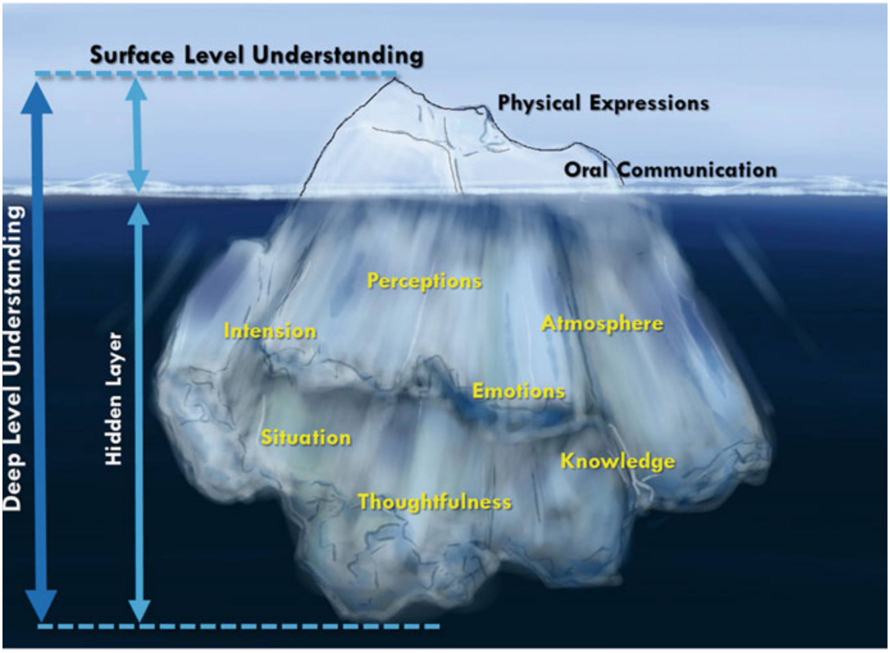


Fig. 1 Concept of deep level emotion understanding

There are many proposals to represent the emotion in so called emotion space. The resulting emotions are displayed in the arousal pleasure-affinity space [6] as shown in Fig. 2 to understand where the emotion and the intension is placed, which is defined as

$$E = \left(e_{\_affinity}, e_{\_pleasure}, e_{\_arousal}\right)\ e_{\_affinity}, e_{\_pleasure}, e_{\_arousal} \in [-1, 1], \quad (1)$$

where E is the emotion state vector, $e_{\_affinity}$, $e_{\_pleasure}$, and $e_{\_arousal}$ are the values for "Affinity- No-affinity", "Pleasure-Displeasure", and "Arousal-Sleep" axes, respectively. The E is a 3D vector in $[-1,1]^3$ as shown in the emotion centroid in Fig. 2. But the human emotion is complex and sometimes varies according to the situation. So it maybe natural to represent the emotion by a fuzzy set as shown by the cone (generally a distorted cone) in Fig. 2. The emotion represented as a fuzzy set has generally a complex shaped membership function, but it may be possible to approximate the complex shaped membership function by an emotion centroid, i.e., an average vector, and emotion standard deviation, i.e., a standard deviation vector in $[0,1]^3$ whose component indicates the standard deviation of the distorted cone along each axis.

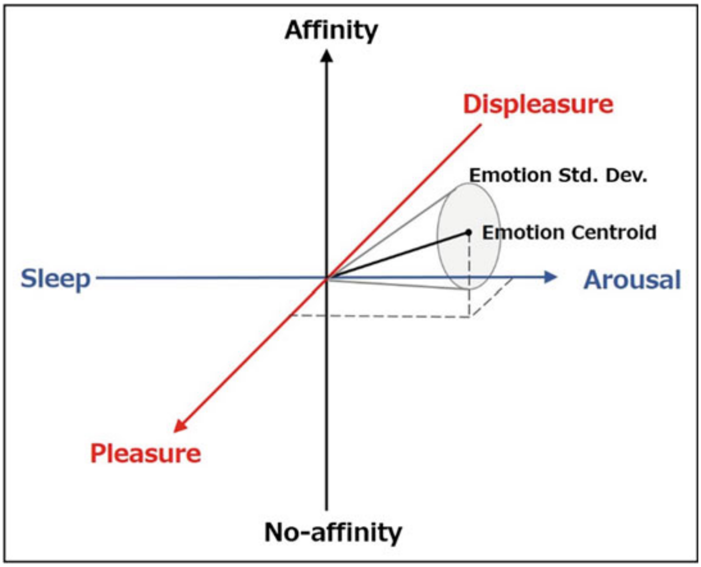


Fig. 2 Affinity Pleasure - Arousal space [6]

The most important information is indicated by the average vector in $[-1,1]^3$ and is illustrated by a visualization method using shape-brightness-size model as shown in Fig. 3. For the pleasure-displeasure axis [-1, 1], some meaningful shapes are accepted. Based on the culture in Japan where the authors' group are studying, circle represents a positive or good answer, while the X-shape represents a negative or bad answer. That makes the shape a suitable way to represent the pleasure (= 1) or displeasure (= -1). In between displeasure and pleasure, continuous deformed shape from X to circle is used as shown in the upside of Fig. 3.

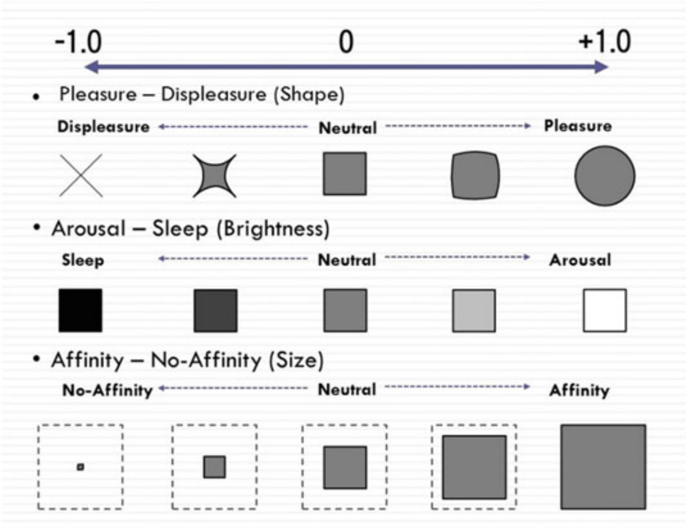


Fig. 3 Visualization of emotion

Brightness inside of the shape is accepted to represent the arousal–sleep axis [-1, 1]. White is the brightest color that denotes vivid, activeness, and arousal (= 1), while black is the darkest color that denotes gloom, passiveness, and sleep (= -1). The degree from sleep to arousal is expressed by gray level degree as shown in the middle of Fig. 3. For the affinity- no-affinity axis [-1, 1], the size of the shape is used from the smallest in the case of no-affinity (= -1) to the full size in affinity (= 1) as indicated in the bottom of Fig. 3.

A scenario is created to demonstrate the concept of the proposed method, where the communication is done between a human employee (observed agent) and a robot secretary (emotion observer) in a company as shown in Fig. 4. The topic is a meeting room reservation requested by the employee to the secretary followed by the reservation change because of the employee's mistake. The employee's face/voice/body-gesture are captured by Kinect attached to the robot secretary. They provide the surface level emotion of the employee to the secretary robot by using three neural networks, and the deep level emotion is inferred by the secretary robot using fuzzy inference with the customized knowledge about the employee. The result is shown in both a vector in $[-1,1]^3$ (bottom left in Fig. 4) and the visualization method (bottom right in Fig. 4).

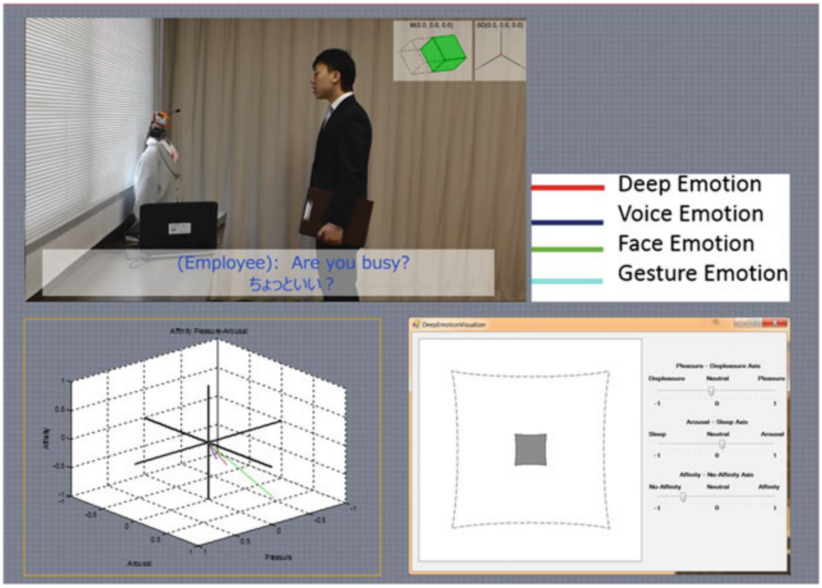


Fig. 4 Communication between employee and secretary robot

3 Atmosphere Analysis/Understanding in Humans-Robots Interaction

To make a smooth communication in human-robot/machine or generally agent to agent interaction, understanding the emotion of others is important. In the case of many to many agents communication, however, the atmosphere of the society may provide more important information than the emotion of each agent. Although many studies have been done on the emotion from viewpoints of cognitive science or human-machine interface, the atmosphere generated by the communication society/field by many agents has not been studied enough. The authors' group at Tokyo Institute of Technology has been studied on many robots and many humans communication through internet, where the atmosphere of the communication field/society by many (huge number of) individuals plays an important role for the smooth communication [7].

The concept of Fuzzy Atmosfield (FA), is proposed to express the atmosphere in such humans-robots communication field/society [8]. The "Atmosfield" is a new word from "atmosphere" and "field", and is created by the authors' group. It is characterized by a 3D fuzzy cubic space $[-1,1]^3$ as shown in Fig. 5 with "friendly-hostile", "lively-calm", and "casual-formal" axes by doing a cognitive science experiments and applying principle component analysis.

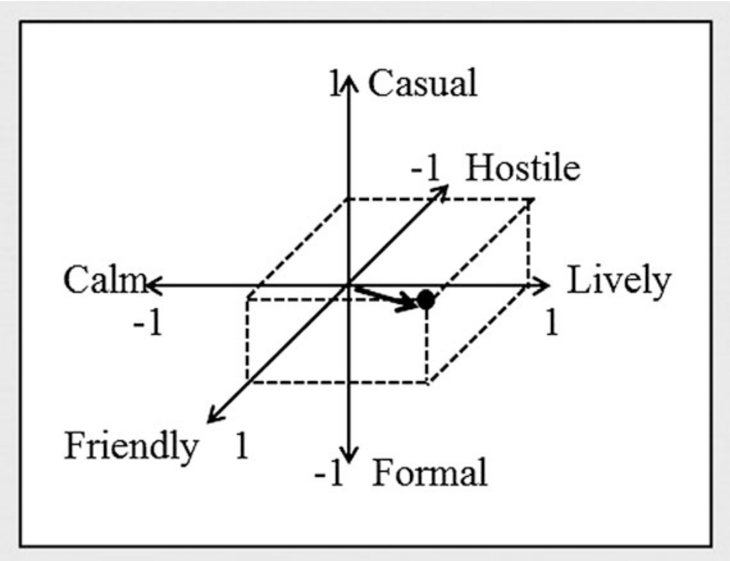


Fig. 5 Fuzzy atmosfield

The atmosphere in the communication field/society is expressed by a point in the 3D fuzzy cubic space $[-1,1]^3$ and maybe varying/moving in the space time by time. To understand easily such movement of the atmosphere, a graphical representation method is also proposed as shown in Fig. 6 by using a shape-color-size model, where "friendly-hostile" information is represented by "shape", "lively-calm" by "color", and "casual-formal" by "size".

To illustrate the FA and its visualization method, a demonstration scenario "enjoying home party by using a Mascot Robot System is introduced/performed. The Mascot Robot System consists of 5 robots, i.e., 4 fixed robots (placed on a TV, a darts game machine, an information terminal, and a mini-bar) and 1 mobile robot (Fig. 7). Each of them includes an eye robot, a speech recognition module, and a notebook PC that controls the robot and the speech recognition module. These robots are connected together with a server through the internet by RTM (Robot Technology Middleware developed by AIST, Japan), thus constituting the Robot System. The Mascot Robot System's functioning is demonstrated in an ordinary living room, where casual communication between 5 robots and 4 human beings (1 host, 2 guests, and 1 walk-in) is conducted based on speech recognition and mentality expression of eye robots.

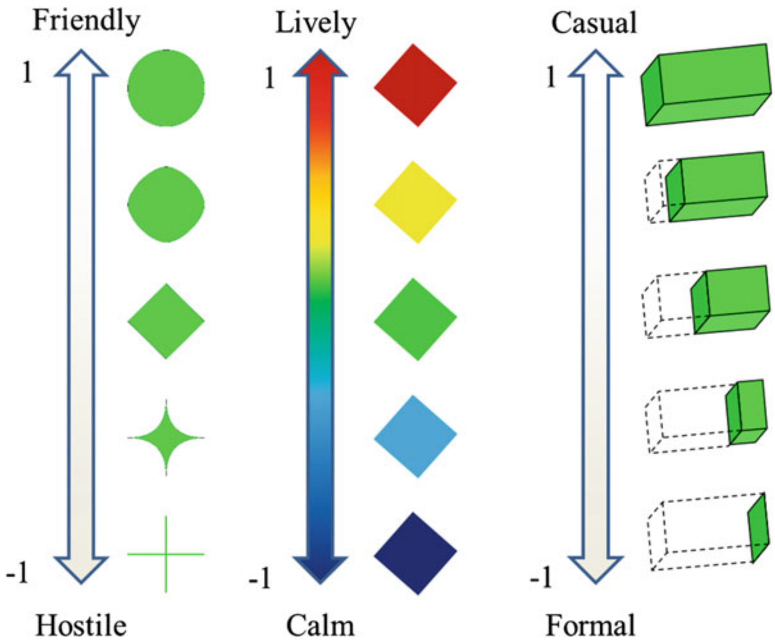


Fig. 6 Visualization of fuzzy atmosfield vector

An example scene of Demonstration Video “Enjoying Home Party” is shown in Fig. 8, where the atmosphere information is indicated in the top center (3D vector value) and top right (visualized illustration by a shape-color-size model).

4 Kansei Texture in Online Shopping

The online shopping market size becomes doubled in the last 10 years, because customers can easily purchase various kinds of products anytime and anywhere. In the online shopping, however, the customers have to imagine the sensation of the product from a few photos, price, specification, reviews, and so on. Therefore, the quality of the delivered product is sometimes different from imaged one. On the other hand, when customers purchase a product in the shop, they actually can observe and take it in their hand, and they can select suitable one based on their feeling about the value, the tactile sensation, the textures, and so on. It means that there exists information gap between real shop and online shop. In order to compensate the information gap in online shopping, Kansei Texture which adds new information on the present net shopping is proposed [9].

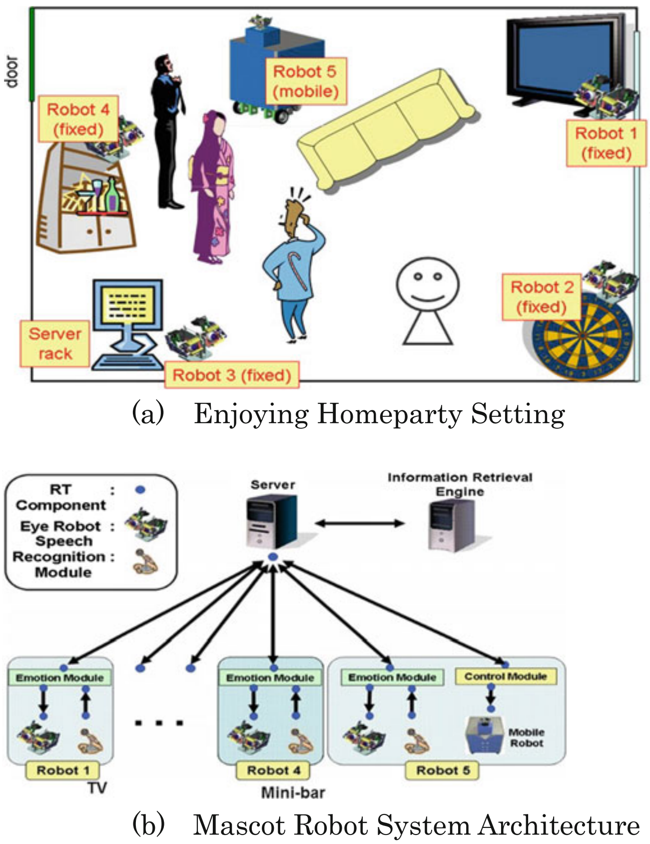


Fig. 7 Mascot robot system

The Kansei Texture ("Shokushitsu-kan" in Japanese language) is defined as the quality index of the feeling information on the tactile or vision sense when people see the photo/movie image of a product or a real object.

Firstly, many kinds of expression terms which contains the amount of feelings like onomatopoeia are gathering from the photos of the products. The expression terms are changed in the amount of feelings which are characterized by 5 tactile sensations in [-1, 1] scale, i.e., roughness, hardness, dryness, warmness, and glossiness, based on the result of the subjectivity evaluation questionnaire. The Kansei Texture is finally represented in 3-dimensional $[-1,1]^3$ space condensed from 5-dimensional $[-1,1]^5$ space, and Kansei Texture of a product is shown by the combination of each value of new defined 3 axes, i.e., "PuruPuru - GotsuGotsu", "KachiKachi - FuwaFuwa", and "ButsuButsu - PikaPika" as shown in Fig. 9.

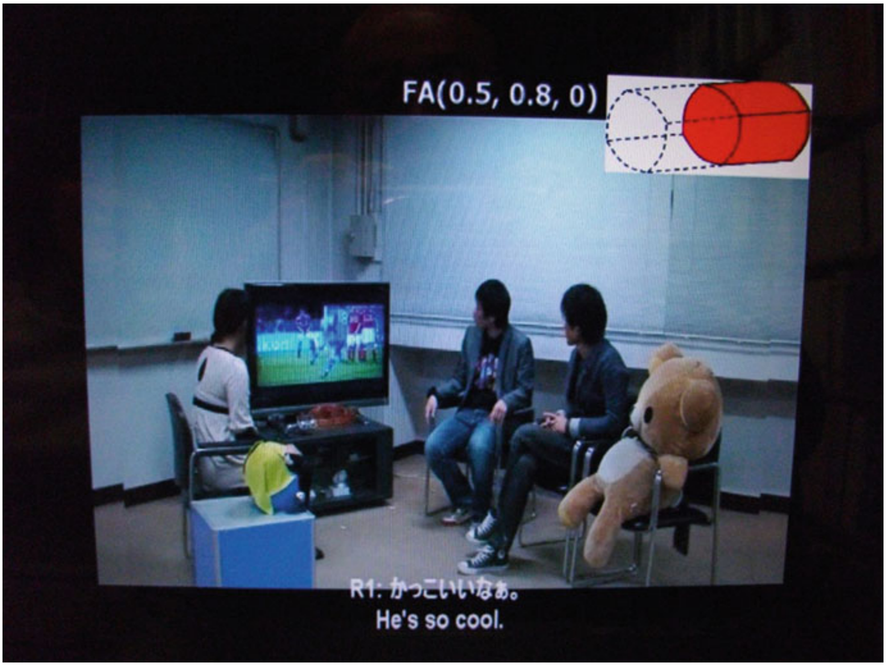
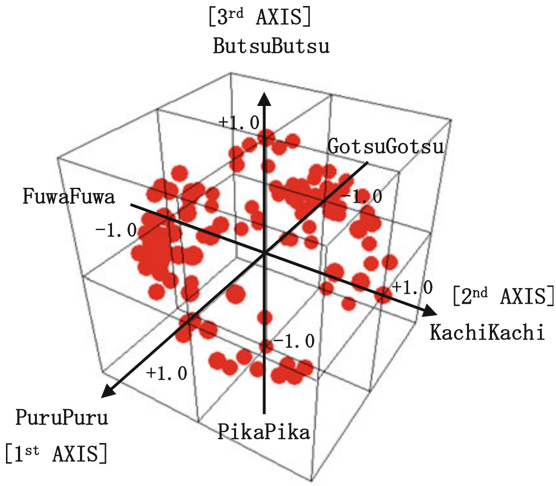


Fig. 8 An example of "enjoying home party"

Fig. 9 Kansei texture space [-1,1]^3



Again, it maybe not easy for general customers in online shopping to understand 3D vectors in $[-1,1]^3$, instead an easily understandable visualization method is developed [10]. For this purpose a contour-shape-gradation model is used as shown in Fig. 10. A program has been developed to generate the visualization illustration by inputting the 3D vector information in the Kansei texture space $[-1,1]^3$ as shown in Fig. 11. Several illustration examples generated by the program are shown in Fig. 12.

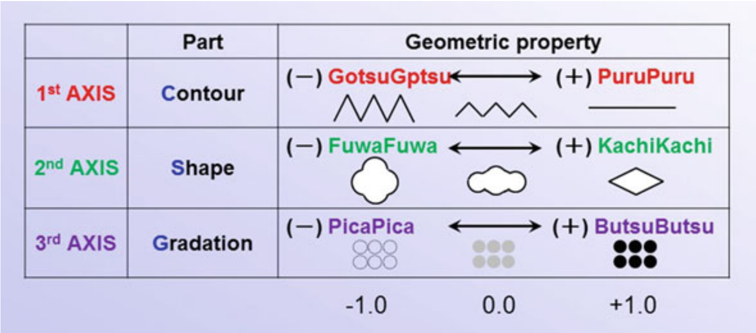


Fig. 10 Visualization of Kansei texture by CSG model

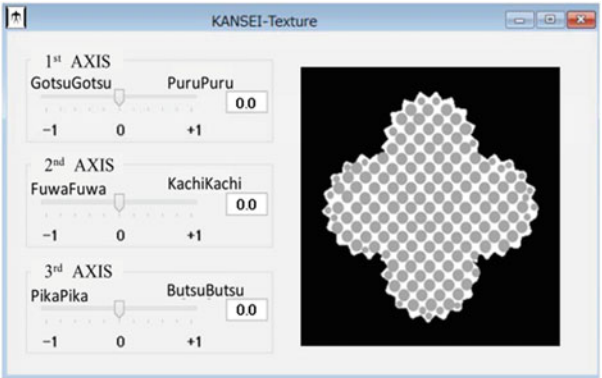


Fig. 11 Visualization program of Kansei texture

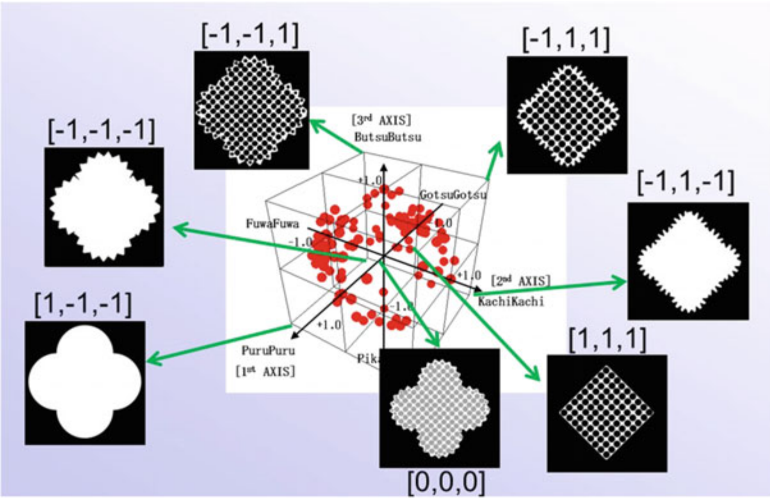


Fig. 12 Examples of visualization

The Kansei texture information is visualized in the online shopping screen as shown in Fig. 13 (bottom right). The customers are able to understand the Kanse texture about the good from the visualized Kansei texture information and to imagine the tactile quality of the good with the photo and the text data.

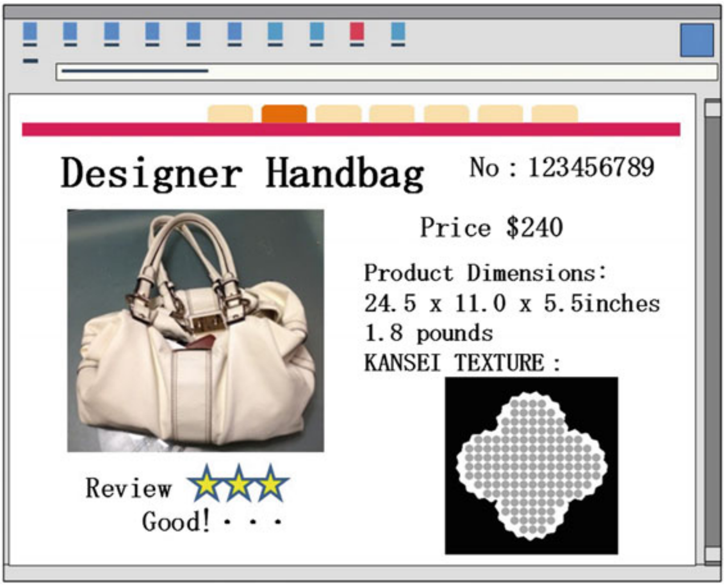


Fig. 13 Online shopping screen with Kansei texture information

5 Conclusions

Formalization procedures of Kansei information are introduced in terms of fuzzy set approach. Three examples of Kansei information are shown, i.e., emotion, atmosphere, and Kansei texture for the application purposes of man-machine interaction, humans-robots interaction, and online shopping, respectively. The visualization methods of the Kansei information are also presented.

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Cognitive Informatics: A Proper Framework for the Use of Fuzzy Dynamic Programming for the Modeling of Regional Development?

Janusz Kacprzyk

Abstract We advocate Wang's cognitive informatics as a potentially powerful general approach and paradigm to formulate, analyze and solve human centric systems modeling,decision and control problems. We show the use of fuzzy dynamic programming for solving a regional development problem in which many crucial aspects, in particular life quality indicators, are subject to objective and subjective, by the humans, judgments and evaluations which are closely related to human perceptions and cognitive abilities. We consider how a best (optimal) investment policy can be obtained under different development scenarios.

1 Introduction

The main purpose of this paper is to indicate a potential of Wang's [26, 27] (cf. also Wang et al. [29–33] *cognitive informatics* for providing a novel perspective through which some decision making and control applications can be viewed. To be more specific, we consider the use of multistage decision making (control) under fuzzy constraints and goals, notably by employing fuzzy dynamic programming (cf. Kacprzyk [10]) to regional development planning. That new perspective, should considerably enhanced other human centric and perception oriented perspectives proposed for solving the problem in question by Kacprzyk [8, 12, 14], Kacprzyk Francelin and Gomide [18], etc.

The main "pre-inspiration" of this paper, which is meant to show and emphasize some impprtant research directions that have occured over the last five decades of fuzzy sets/logic, may be what the founder of fuzzy sets theory and fuzzy logic, Professor Lotfi A. Zadeh, has been sating since the very baginning. Namely, namely

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*, Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_11

that – in his opinion – the very esence of fuzzy sets and fuzzy logic would make them particularly suited for all kinds of applications in broadly perceived human centric systems, that is, those in which the human being plays a crucial role, or – in a slightly broader mening - which are meant to analyze and solve problems that are important to the individuals or soail groups, organizations, etc. Unfortunately, that Zadeh's early belief has not been totally fulfilled as, for a strange reason, in the first breakthrough in the real world applications of fuzzy logic starting from the early 1980 s with the so called fuzzy boom in Japan, most applications have been in technology, notably in relatively simple control of various home appliances, cranes, etc.

However, even in that initial period some other application, more related to what might be called human centric, have also appeared and have been implemented to solve important real world problems. This paper is about such an application, to sustainable regional planning, which has been initiated by the author and his collaborators, cf. notably Kacprzyk and Straszak [20–23], in the end of the 1970s and beginning of the 1980 s at the International Institute for Applied Systems Analysis (IIASA) in Laxenburg, Asutria (www.iiasa.at). The models developed have been widely used by the author and his collaborators in many regional planning projects in various countries, exemplified by the Upper Noteć Region in Poland, Tisza Region in Hungary, Kinki region in Japan, to name a few. Those works have resulted in many research publications, among which the following ones can be quoted: Kacprzyk [8, 14], Kacprzyk, Francelin and Gomide [18], Kacprzyk and Straszak [20–23], etc. The models proposed have been not only widely used in practice, documented also in project reports of limited circulation, but they have also been mentioned as one of the most successful examples of fuzzy systems modeling in a Special Volume on the Fiftieth Anniversary opf the Britis Operational Research Society published in 1987 by Pergamon Press – cf Thomas [25]. This has been the second inspiration of this paper.

The third inspiration is a recent growth of interest in various types of more human centric and human consistent modeling, notably *cognitive informatics* and its related *cognitive modeling*, a new area which has been conceptualized and proposed by Wang [26, 27], and then considerably advanced over the next years by Wang and his numerous collaborators and followers. For our purposes, which are related to bradly perceived decision making and control, the work by Wang and Ruhe [32] is presumably the most relevant.

Briefly speaking, cognitive informatics is a multidisciplinary field within informatics, or computer science, that is based on results of cognitive and information sciences, and which deals with human information processing mechanisms and processes and their decision theoretic, engineering, etc. applications in broadly perceived computing, including multistage decision making processes which are of our interest. The agenda of ccgnitive informatics is to develop and implement models, tools and techniques, and technologies to facilitate and extend the information acquisition, comprehension and processing capacity of humans to overcome some cognitive difficulties related to the presence of the human being as a crucial part of the system. In our case, the system will be highly related to human judgments, and search for best (optimal) solutions. A limited comprehension, memorizing, learning, choice and decision making abilities, satisfaction with partial truth, allowing for not perfect solutions, etc. will be or relevance. Those issues are considered and solved using tools and techniques derived from many areas like psychology, behavioral science, neuroscience, artificial intelligence, linguistics, etc. In our case, we will concentrate on some cognitive informatics type elements that mostly have been inspired by psychology and behavioral sciences, as our problem is inherently related to human judgments and perceptions. Some relation to neuroeconomics can also be pursued, cf. Kacprzyk [15].

The first idea of such a cognitive informatics related perspective for mmore general dynamic modeling issues, notably related to dynamic programming under fuzzy constraints and goals, have been proposed in Kacprzyk's plenary talk at WCCI-2014 in Beijing, China (cf. http://www.ieee-wcci2014.org/files/Janusz.Kacprzyk.pdf). The very purpose of that talk was to propose of what might be called *cognitive fuzzy dynamic programming*. The purpose of this paper is more general, namely to propose some new perspective in fuzzy systems modeling which might be called *cognitive fuzzy modeling*. The new fuzzy dynamic programming models presented, in which the above human specific aspects will be shown and analyzed and a cognitive informatics perspective will be indicated, will be shown on a sustainable regional development considered in terms of expenditures, subsidies, life qualities, etc. For a slightly different approach within the cognitive modeling context, cf. Hotaling and Busemeyer [4].

2 Fuzzy Dynamic Programming as a Step Towards Perception Based and Cognitive Multistage Decision Making and Control

As a point of departure we take the famous Bellman and Zadeh's [1] model of decision making under fuzziness in which if $X = \{x\}$ is some set of possible *options* (alternatives, variants, choices, decisions, …), then the *fuzzy goal* is defined as a fuzzy set G in X, characterized by its membership function $\mu_G : X \longrightarrow [0, 1]$ such that $\mu_G(x) \in [0, 1]$ specifies the grade of membership of a particular option $x \in X$ in the fuzzy goal G, and the *fuzzy constraint* is similarly defined as a fuzzy set C in the set of options X, characterized by $\mu_C : X \longrightarrow [0, 1]$ such that $\mu_C(x) \in [0, 1]$ specifies the grade of membership of a particular option $x \in X$ in the fuzzy constraint C.

The general problem formulation is: "Attain G and satisfy C" which leads to the *fuzzy decision*

$$\mu_D(x) = \mu_G(x) \wedge \mu_C(x), \qquad \text{for each } x \in X \tag{1}$$

where "$\wedge$" stands for the minimum that may be replaced, for instance, a t-norm.

The *maximizing decision* is defined as an $x^* \in X$ such that

$$\mu_D(x^*) = \max_{x \in X} \mu_D(x) \tag{2}$$

The human cognition related aspect is that, first, the strict optimization in (2) may be viewed to strict and unnecessary and some sort of a satisfactory, good enough solution could be accepted. Second, in reality the satisfaction of constraints and attainment of goals have both an objective and subjective aspect. We will mainly deal with that second aspect.

The Bellman and Zadeh's [1] framework can therefore be extended by introducing: an *objective fuzzy goal* $\mu_{G_o}(x)$, a *subjective fuzzy goal* $\mu_{G_s}(x)$, an *objective fuzzy constraint* $\mu_{C_o}(x)$, and a *subjective fuzzy constraint* $\mu_{C_s}(x)$.

We wish therefore to "Attain $[G_o$ and $G_s]$ and satisfy $[C_o$ and $C_s]$" which leads to the fuzzy decision

$$\mu_D(x) = [\mu_{G_o}(x) \wedge \mu_{G_s}(x)] \wedge [\mu_{C_o}(x) \wedge \mu_{C_s}(x)], \qquad \text{for each } x \in X \tag{3}$$

and the *maximizing*, or *optimal* decision is defined as in (2); clearly, remarks on a relaxation of that condition of a strict optimality are valid here too, as well as throughout the paper.

This framework can be extended to handle multiple fuzzy constraints and fuzzy goals, and also fuzzy constraints and fuzzy goals defined in different spaces, cf. Kacprzyk's [10] book. Namely, if we have: $n_o > 1$ objective fuzzy goals – $G_o^1, \ldots, G_o^{n_o}$ defined in Y, $n_s > 1$ subjective fuzzy goals – $G_s^1, \ldots, G_s^{n_s}$ defined in Y, $m_o > 1$ objective fuzzy constraints – $C_o^1, \ldots, C_o^{m_o}$ defined in X, $m_s > 1$ subjective fuzzy constraints – $C_s^1, \ldots, C_s^{m_s}$ defined in X, and a function $f : X \longrightarrow Y$, $y = f(x)$, then

$$\begin{aligned}
&\mu_D(x) = \\
&= (\mu_{G_o^1}[f(x)] \wedge \cdots \wedge \mu_{G_o^{n_o}}[f(x)]) \wedge (\mu_{G_s^1}[f(x)] \wedge \cdots \wedge \mu_{G_s^{n_s}}[f(x)]) \wedge \\
&\quad \wedge[\mu_{C_o^1}(x) \wedge \cdots \wedge \mu_{C_o^{m_o}}(x)] \wedge [\mu_{C_s^1}(x) \wedge \cdots \wedge \mu_{C_s^{m_s}}(x)] \wedge \\
&\quad \wedge[\mu_{C_s^1}(x) \wedge \cdots \wedge \mu_{C_s^{m_s}}(x)], \qquad \text{for each } x \in X
\end{aligned} \tag{4}$$

and the *maximizing decision* is defined as (2), i.e. $\mu_D(x^*) = \max_{x \in X} \mu_D(x)$.

In the *control process* dealt with the decision (control) space is $U = \{u\} = \{c_1, \ldots, c_m\}$, the state (output) space is $X = \{x\} = \{s_1, \ldots, s_n\}$, and both are finite. We start from an initial state $x_0 \in X$, apply a decision (control) $u_0 \in U$, which is subjected to a fuzzy constraint $\mu_{C^0}(u_0)$, and attain a state $x_1 \in X$ via a known state transition equation of the system under control S; a fuzzy goal $\mu_{G^1}(x_1)$ is imposed on x_1. Next, we apply u_1, subjected to $\mu_{C^1}(u_1)$, and attain x_2, subjected to $\mu_{G^2}(x_2)$, etc.

The (deterministic) system under control is described by a *state transition equation*

$$x_{t+1} = f(x_t, u_t), \qquad t = 0, 1, \ldots \tag{5}$$

where $x_t, x_{t+1} \in X = \{s_1, \ldots, s_n\}$ are the states at t and $t + 1$, respectively, and $u_t \in U = \{c_1, \ldots, c_m\}$ is the decision (control) at t.

At t, $t = 0, 1, \ldots$, $u_t \in U$ is subjected to a *fuzzy constraint* $\mu_{C^t}(u_t)$, and on $x_{t+1} \in X$ a *fuzzy goal* is imposed, $\mu_{G^{t+1}}(x_{t+1})$. The fixed and specified in advance *initial state* is $x_0 \in X$, and the *termination time* (planning horizon), $N \in \{1, 2, \dots\}$, is finite, and fixed and specified in advance.

The *performance* of the particular decision making (control) stage t, $t = 0, 1, \dots, N-1$, is evaluated by

$$v_t = \mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1}) = \mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}[f(x_t, u_t)] \tag{6}$$

while the *performance* of the whole multistage decision making (control) process is given by the fuzzy decision

$$\begin{aligned} \mu_D(u_0, \dots, u_{N-1} \mid x_0) &= v_0 \wedge v_1 \wedge \dots \wedge v_{N-1} = \\ &= [\mu_{C^0}(u_0) \wedge \mu_{G^1}(x_1)] \wedge \dots \wedge [\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)] \end{aligned} \tag{7}$$

The problem is to find an optimal sequence of decisions (controls) $u_0^*, \dots, u_{N-1}^*$ such that

$$\mu_D(u_0^*, \dots, u_{N-1}^* \mid x_0) = \max_{u_0, \dots, u_{N-1} \in U} \mu_D(u_0, \dots, u_{N-1} \mid x_0) \tag{8}$$

Kacprzyk's [10] book provides and wide coverage of various aspects and extensions to this basic formulation.

In the case of an extention proposed in this paper and outlined in Sect. 2 in which the objective and subjective fuzzy constraints and fuzzy goals are assumed, which are inherently related to human judgment and cognition, we have, at each $t = 0, 1, \dots, N-1$: an objective fuzzy constraint $\mu_{C_t^1}(u_t)$ and a subjective fuzzy constraint $\mu_{C_s^t}(u_t)$, and an objective fuzzy goal $\mu_{G_o^{t+1}}(u_{t+1})$ and a subjective fuzzy constraint $\mu_{G_s^{t+1}}(u_{t+1})$.

The (extended) performance of the particular stage t, $t = 0, 1, \dots, N-1$, is then given by

$$\overline{v}_t = [\mu_{C_o^t}(u_t) \wedge \mu_{C_s^t}(u_t)] \wedge [\mu_{G_o^t}(x_t) \wedge \mu_{G_s^t}(x_t)] \tag{9}$$

which can be schematically shown as in Fig. 1.

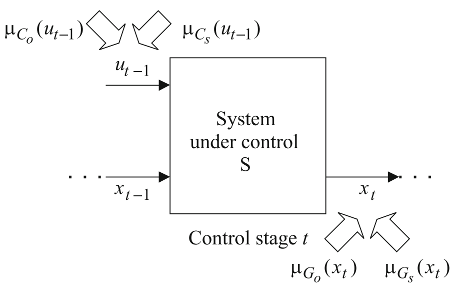


Fig. 1 Evaluation of (extended) performance of decision making (control) stage t

The (extended) performance of the whole multistage decision making (control) process is then given by the fuzzy decision

$$\begin{aligned}
&\mu_{\overline{D}}(u_0, \dots, u_{N-1} \mid tx_0) = \overline{v}_0 \wedge \overline{v}_1 \wedge \dots \wedge \overline{v}_{N-1} = \\
&= \{[\mu_{C_o^0}(u_0) \wedge \mu_{C_s^0}(u_0)] \wedge [\mu_{G_o^1}(x_1) \wedge \mu_{G_s^1}(x_1)]\} \wedge \dots \\
&= \wedge \{[\mu_{C_o^{N-1}}(u_{N-1}) \wedge \mu_{C_s^{N-1}}(u_{N-1})] \wedge [\mu_{G_o^N}(x_N) \wedge \mu_{G_s^N}(x_N)]\}
\end{aligned} \tag{10}$$

and we seek again an $u_0^*, \dots, u_{N-1}^*$ such that

$$\mu_{\overline{D}}(u_0^*, \dots, u_{N-1}^* \mid x_0) = \max_{u_0, \dots, u_{N-1} \in U} \mu_{\overline{D}}(u_0, \dots, u_{N-1} \mid x_0) \tag{11}$$

There is an extremely relevant aspect related to the subjective fuzzy constraints and fuzzy goals. We will consider the subjective fuzzy goals in which this is presumably much more pronounced than in the subjective fuzzy constraints. Namely, it often happens that the (subjective) human satisfaction resulting from the attainment of some level of x_{t+1}, a value of a life quality index, depends not only on the "objectively attained" value but on how this value is perceived, how it looks like in comparision with the past, what are future prospects, etc. For simplicity, let us concentrate in these perceptions and jidgments on the past only.

The trajectory of the multistage decision making (control) process from $t = 0$ to a current stage $t = k$ is

$$H_k = (x_0, u_0, C_o^0, C_s^0, x_1, G_o^1, G_s^1, \dots, u_{k-1}, C_o^{k-1}, C_s^{k-1}, x_k, G_o^k, G_s^k) \tag{12}$$

that is, it involves all aspects of what has happened in terms of decisions applied, states attained, and objective and subjective opinions of how well the fuzzy constraints have been satisfied and fuzzy goals attained. However, it is often sufficient to take into account the *reduced trajectory*

$$h_k = (x_{k-2}, u_{k-2}, C_o^{k-2}, C_s^{k-2}, x_{k-1}, G_o^{k-1}, G_s^{k-1}, u_{k-1}, C_o^{k-1}, C_s^{k-1}, x_k, G_o^k, G_s^k) \tag{13}$$

which only takes into account the current, $t = k$, and previous stage, $t = k - 1$. Let us assume this reduced trajectory. Such an approach has a long tradition, e.g. in all kinds of the Markov decision processes, and has proved to be effective and efficient.

A further simplification is that with a trajectory, or reduced trajectory, an evaluation function is associated, $E : S(H_k) \longrightarrow [0, 1]$ or $e : S(h_k) \longrightarrow [0, 1]$, where $S(H_k)$ and $S(h_k)$ are the sets of trajectories and reduced trajectories, respectively, such that $E(H_k) \in [0, 1]$ and $e(h_k) \in [0, 1]$ denote the satisfaction of the past development, from 1 for full satisfaction to 0 for full dissatisfaction, through all intermediate values. This is again consistent with the human perception.

The subjective fuzzy constraints and huzzy goals are now:

- when the (reduced) trajectory is accounted for

$$\begin{cases} \mu_{C_o^k}(u_k \mid h_k) & \text{and} \quad \mu_{C_s^k}(u_k \mid h_k) \\ \mu_{G_o^{k+1}}(x_{k+1} \mid h_k) & \text{and} \quad \mu_{G_s^{k+1}}(x_{k+1} \mid h_k) \end{cases} \tag{14}$$

- when the evaluation of the (reduced) trajectory is accounted for

$$\begin{cases} \mu_{C_o^k}[u_k \mid E(h_k)] & \text{and} \quad \mu_{C_s^k}[u_k \mid E(h_k)] \\ \mu_{G_o^{k+1}}[x_{k+1} \mid E(h_k)] & \text{and} \quad \mu_{G_s^{k+1}}[x_{k+1} \mid E(h_k)] \end{cases} \tag{15}$$

Problem (8) can be solved using the following two basic traditional techniques: dynamic programming (cf. Bellman and Zadeh [1], Kacprzyk [7, 10]), and branch-and-bound (Kacprzyk [5], and also using the two new ones: a neural network (cf. Francelin, Gomide and Kacprzyk [2, 3], and a genetic algorithm (cf. Kacprzyk [11, 12]. We will only briefly show the use of dynamic programming, and refer the reader for an extensive coverage on this and other solution techniques to Kacprzyk's [10] book.

First, we rewrite (8) as to find $u_0^*, \ldots, u_{N-1}^*$ such that

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{N-1} \mid x_0) = \\ &= \max_{u_0,\ldots,u_{N-1}} [\mu_{C^0}(u_0) \wedge \mu_{G^1}(x_1) \wedge \ldots \\ &\quad \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1}))] \end{aligned} \tag{16}$$

and then, since

$$\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1}))$$

depends only on u_{N-1}, then the maximization with respect to $u_0, \ldots, u_{N-1}$ in (16) can be split into:

- the maximization with respect to $u_0, \ldots, u_{N-2}$, and
- the maximization with respect to u_{N-1},

written as

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{N-1}^* \mid x_0) = \\ &= \max_{u_0,\ldots,u_{N-2}} \{\mu_{C^0}(u_0) \wedge \mu_{G^1}(x_1) \wedge \ldots \\ &\quad \ldots \wedge \mu_{C^{N-2}}(u_{N-2}) \wedge \mu_{G^{N-1}}(x_{N-1}) \wedge \\ &\quad \wedge \max_{u_{N-1}}[\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1}))]\} \end{aligned} \tag{17}$$

which may be continued for u_{N-2}, u_{N-3}, etc.

This backward iteration leads to the following set of fuzzy dynamic programming recurrence equations:

$$\begin{cases} \mu_{\overline{G}^{N-i}}(x_{N-i}) = \\ \qquad = \max_{u_{N-i}} [\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i}}(x_{N-i}) \wedge \mu_{\overline{G}^{N-i+1}}(x_{N-i+1})] \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); \qquad i = 0, 1, \ldots, N \end{cases} \tag{18}$$

where $\mu_{\overline{G}^{N-i}}(x_{N-i})$ is viewed as a fuzzy goal at control stage $t = N - i$ induced by the fuzzy goal at $t = N - i + 1$, $i = 0, 1, \ldots, N$; $\mu_{\overline{G}^N}(x_N) = \mu_{G^N}(x_N)$.

The $u_0, \ldots, u_{N-1}$ sought is given by the successive maximizing values of u_{N-i}, $i = 1, \ldots, N$ in (18) which are obtained as functions of x_{N-i}, i.e. as an *optimal policy*, $a_{N-i} : X \longrightarrow U$, such that $u_{N-i} = a_{N-i}(x_{N-i})$.

It easy to notice that if we use the subjective fuzzy constraints and fuzzy goals to extend the above fuzzy dynamic programming model, then the very idea of dynamic programming, i.e. the use of backward iteration represented by the recurrence equations (18), prohibits the use of subjective fuzzy constraints and subjective fuzzy goals defined as functions of the trajectory, or any evaluation of the trajectory, as both of them are somehow calculated on the basis of outcomes of control stages prior to those which have been accounted for so far since we proceed via backward iteration. Therefore, if we intend to employ fuzzy dynamic programming, as in this paper, we can only use the subjective fuzzy constraints and goals depending on the current value of decision (control) applied and state attained. The involvement of subjective fuzzy constraints and goals depending on the trajectory or its evaluation needs another approach as, e.g., the use of a genetic algorithm (cf. Kacprzyk [6, 9, 12, 17]) or a neural network based approach by Francelin, Gomide and Kacprzyk [2] or Francelin, Kacprzyk and Gomide [3].

Therefore, by involving the line of reasoning (16)–(18), using the objective and subjective fuzzy constraints and fuzzy goals: $\mu_{C_o^{N-i}}(u_{N-i})$ and $\mu_{C_s^{N-i}}(u_{N-i})$, and $\mu_{G_o^{N-i+1}}(x_{N-i+1})$ and $\mu_{G_s^{N-i+1}}(x_{N-i+1})$, for $i = 1, 2, \ldots, N$, we arrive at the following set of (extended) dynamic programming recurrent equations:

$$\begin{cases} \mu_{\overline{G}^{N-i}}(x_{N-i}) = \\ \qquad = \max_{u_{N-i}} \{[\mu_{C_o^{N-i}}(u_{N-i}) \wedge \mu_{C_s^{N-i}}(u_{N-i})] \wedge \\ \qquad\qquad [\mu_{G_o^{N-i}}(x_{N-i}) \wedge \mu_{G_s^{N-i}}(x_{N-i}) \wedge \mu_{\overline{G}^{N-i+1}}(x_{N-i+1})]\} \\ x_{N-i+1} = f(x_{N-i}, u_{N-i}); \qquad i = 0, 1, \ldots, N \end{cases} \tag{19}$$

3 Sustainable Socioeconomic Regional Development Planning Under Fuzziness

Regional development planning is a problem of crucial relevance in virtually all countries but is difficult to formalize and solve as it involves various aspects (political, economic, social, environmental, technological, etc.), different parties and agents (inhabitants, authorities of different levels, formal and informal groups, etc.), and many entities and aspects that are difficult to precisely single out, define and quantify. Needless to say that the sustainable regional development planning is even more complex but for a lack of space we will not discuss its specifics in more detail. To overcome these difficulties, the use of a fuzzy model was Kacprzyk and Straszak [21, 23], and then extended by Kacprzyk [10], and Kacprzyk, Francelin and Gomide [18].

Basically, they consider a (rural) region plagued by severe difficulties mainly related to a poor *life quality* perceived. Hence, life quality (or, in fact, a perception therof) should be improved, by some (mostly external) funds (investments) whose amount and their temporal distribution should be found. We will show now how the extended, cognitive type and perception based model developed above can be employed.

For our purposes the essence of socioeconomic regional development may be depicted as in Fig. 2.

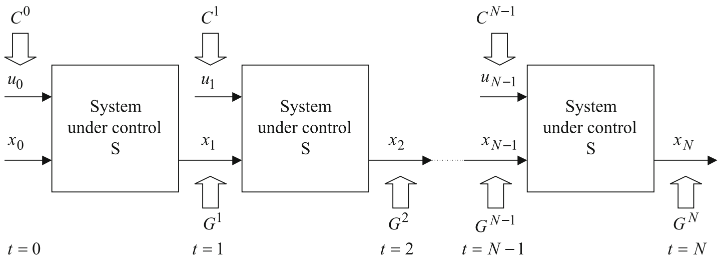


Fig. 2 Essential elements of socioeconomic regional development

The region is represented by a socioeconomic dynamic system under control the state of which at the development (planning) stage $t-1$, X_{t-1}, is characterized by a set of relevant socioeconomic life quality indicators. Then, the decision (investment), at $t-1$, u_{t-1}, changes X_{t-1} to X_t; $t = 1, \ldots, N$, and N is a finite, fixed and specified planning horizon.

The evaluation of a planning stage t, $t = 1, \ldots, N$, is performed by accounting for both the "goodness" of the u_{t-1} applied (i.e. costs), and the "goodness" of the X_t attained (i.e. benefits); the former has to do with how well some constraints are satisfied, and the latter with how well some goals are attained. We will involve, for simplicty, a subjective assessment for the attainment of fuzzy goals only.

First, the socioeconomic system is represented as in Fig. 3. Its state (output) X_t is equated with a *life quality index* that consists

of the following seven *life quality indicators* (i.e. $X_t = [x_t^1, \ldots, x_t^7]$):

- x_t^1 – economic quality (e.g., wages, salaries, income, …),
- x_t^2 – environmental quality,
- x_t^3 – housing quality,
- x_t^4 – health service quality,

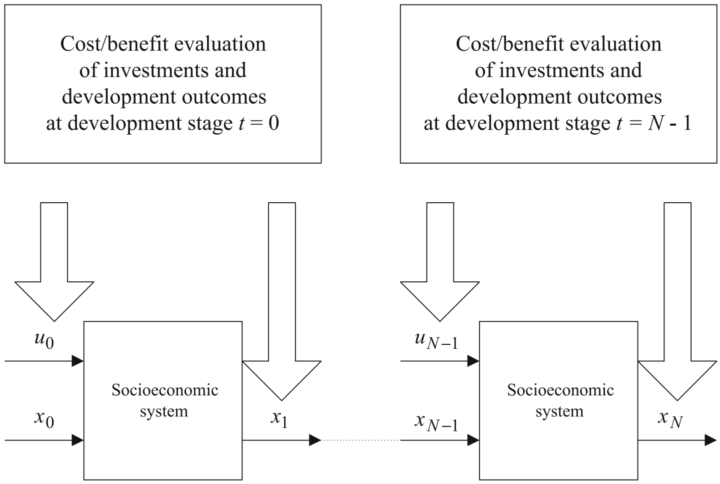
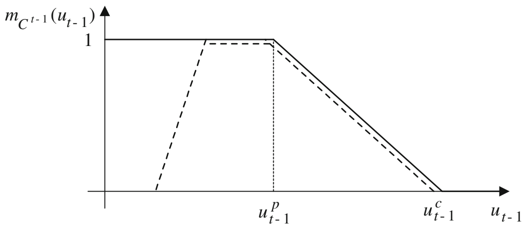


Fig. 3 Basic elements of the socioeconomic system under control

- x_t^5 – infrastructure quality,
- x_t^6 – work opportunity,
- x_t^7 – leisure time opportunity,

The decision at $t-1$, u_{t-1} is investment, and we impose on u_{t-1} a fuzzy constraint $\mu_{C^{t-1}}(u_{t-1})$ in a piecewise linear form as shown in Fig. 4 to be read as follows.

Fig. 4 Fuzzy constraints on investment u_{t-1}



The investment may be fully utilized up to u^p_{t-1}, hence $\mu_{C^{t-1}}(u_{t-1}) = 1$ for $0 < u_{t-1} < u^p_{t-1}$. However, this is usually insufficient and some additional contingency investment is needed, maximally up to u^c_{t-1} (the more the worse, of course). The fuzzy constraints are often as shown in the dotted line in Fig. 4 in that too low a use of available investments should also be avoided, for "political" reasons, as in all public funding related cases.

The $t-1$, u_{t-1} is partitioned into $u^1_{t-1}, \ldots, u^7_{t-1}$, devoted to improve the respective life quality indicators, but we will assume here that this rule is fixed.

The temporal evolution of the particular life quality indicators is governed by the state transition equation

$$x_t^i = f_{t-1}^i(x_{t-1}^i, u_{t-1}^i), \qquad i = 1, \ldots, 7; t = 1, \ldots, N \tag{20}$$

which may be derived by, e.g., using experts' opinions, past experience, mathematical models, etc.

The evaluation of development takes into account how well some predetermined goals are fulfilled, i.e. *effectiveness*, then be related to the investment spent, i.e. *efficiency* – cf. Kacprzyk's [10] book.

The effectiveness of regional development involves two aspects: the effectiveness of a particular development stage, and the effectiveness of the whole development.

The effectiveness of a particular development stage has both an objective and subjective aspect. The objective evaluation is basically the determination of how well the fuzzy constraints are fulfilled, and fuzzy goals are attained. The objective fuzzy goals concern desired values of the life quality indicators, i.e. concern objective entities; however, goal attainment is not clear-cut, and a fuzzy goal should rather be used.

For each life quality indicator at $t = 1, \ldots, N$, x_t^i, we define an *objective fuzzy subgoal* $G_o^{t,i}$ characterized by $\mu_{G_o^{t,i}}(x_t^i)$ as shown in Fig. 5

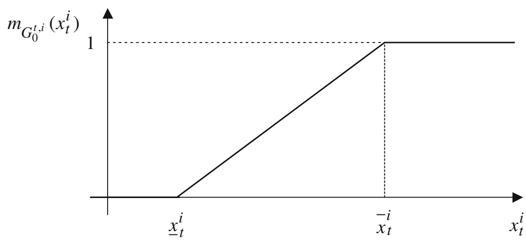


Fig. 5 Objective fuzzy subgoal

to be read as follows: $G_o^{t,i}$ is fully satisfied for $x_t^i \geq \overline{x}_t^u$, where $\overline{x}_o^i$ is some *aspiration level* for the indicator x_t^i; therefore, $\mu_{G_o^{t,i}}(x_t^i) = 1$, for $x_t^i \geq \overline{x}_t^i$. Less preferable are $\underline{x}_t^i < x_t^i < \overline{x}_t^i$ for which $0 < \mu_{G_o^{t,i}}(x_t^i) < 1$, and $x_t^i \leq \underline{x}_t^i$ are assumed to be impossible, hence $\mu_{G_o^{t,i}}(x_t^i) = 0$. Notice that an objective fuzzy (sub)goal may be relatively easily determined by experts by specifying two values only, $\underline{xt}_t^i$ and $\overline{x}_t^i$.

The objective evaluation of the life quality index at t, $X_t = [x_t^1, \ldots, x_t^7]$, is obtained by the aggregation of partial assessments of the particular life quality indicators, i.e.

$$\mu_{G_o^t}(X_t) = \mu_{G_o^{t,1}}(x_t^1) \wedge \ldots \wedge \mu_{G_o^{t,7}}(x_t^7) \tag{21}$$

and "$\wedge$" may be replaced here and later on by another suitable operation as, e.g., a t-norm [cf. Kacprzyk (1997a)] but this will not be considered here.

Basically, the use of "$\wedge$" (minimum) reflects a pessimistic, safety-first attitude, and a lack of substitutability (i.e. that a low value of one life quality indicator cannot be compensated by a higher value of another), which is often adequate.

Finally, note that the objective evaluation concerns more the authorities than the inhabitants by somehow “mechanically” checking the values of life quality indicators attained against some desired predetermined levels. The inhabitants’ assessment of the “goodness” of development conc‘erns in fact the (perception of) *social satisfaction* resulting from the life quality index attained. This is clearly subjective. The attained value of a particular life quality indicator at t, x_t^i, implies its corresponding partial social satisfaction s_t^i derived as in Fig. 6, and its interpretation is basically as for the objective evaluation shown in Fig. 5.

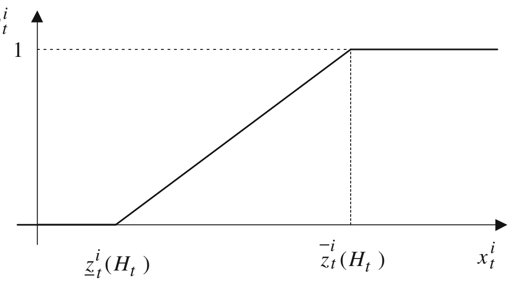


Fig. 6 Partial social satisfaction

In general, both $\underline{z}_t^i$ and $\overline{z}_t^i$ may be functions of the trajectory (history) of development [cf. (12)]

$$H_t = [(X_1, s_1, \mu_{G_o^1}(X_1), \mu_{G_s^1}(s_1)), \ldots, (X_t, s_t, \mu_{G_o^t}(s_t), \mu_{G_s^t}(s_t))]$$

where $s_k = [s_k^1, \ldots, s_k^7], k = 1, \ldots, t$, is the social satisfaction resulting from X_k. Basically, if H_t is encouraging, then the inhabitants may become more demanding, and $\underline{z}_t^i(H_t)$ and $\overline{z}_t^i(H_t)$ may move up. On the other hand, if H_t is discouraging, then $\underline{z}_t^i(H_t)$ and $\overline{z}_t^i(H_t)$ may move down (cf. Kacprzyk [7, 10]). Very often, however, one can limit the analysis to the reduced trajectory [cf. 13)]. This important aspect will not be considered here.

The social satisfaction at t is now

$$s_t = s_t^1 \wedge \ldots \wedge s_t^7 \tag{22}$$

where “$\wedge$” again reflects a pessimistic, safety-first attitude, and a lack of substitutability.

The social satisfaction s_t is subjected to a subjective fuzzy goal $\mu_{G_s^t}(s_t)$ which is meant similarly as its objective counterpart shown in Fig. 5.

The effectiveness of t is meant as a relation of what has been attained (the life quality indices and their respective social satisfactions) to what has been “paid for” (the respective investments), i.e. is a benefit–cost relationship. Formally, the (fuzzy) effectiveness of stage t is expressed as

$$\mu_{E^t}(u_{t-1}, X_t, s_t) = \mu_{C^{t-1}}(u_{t-1}) \wedge \mu_{G_o^t}(X_t) \wedge \mu_{G_s^t}(s_t) \tag{23}$$

and the aggregation reflects the nature of a compromise between the interests of the authorities (for whom the fuzzy constraints and the objective fuzzy goal matter), and those of the inhabitants (for whom the subjective fuzzy goal, and to some extent the objective fuzzy goal, matter); the minimum reflects a safety-first attitude, hence a "more just" compromise.

Then, the effectiveness measures of the particular $t = 1, \ldots, N$, $\mu_{E^t}(u_{t-1}, X_t, s_t)$ given by (23), are aggregated to yield the fuzzy effectiveness measure for the whole development

$$\mu_E(H_N) = \mu_{E^1}(u_0, X_1, s_1) \wedge \ldots \wedge \mu_{E^N}(u_{N-1}, X_N, s_n) \tag{24}$$

The fuzzy decision is

$$\begin{aligned} &\mu_D(u_0, \ldots, u_{N-1} \mid X_0, B_N) = \\ &= [\mu_{C^0}(u_0) \wedge \mu_{G_o^1}(X_1) \wedge \mu_{G_s^1}(s_1)] \wedge \ldots \\ &\quad \ldots \wedge [\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G_o^N}(X_N) \wedge \mu_{G_s^N}(s_N)] \end{aligned} \tag{25}$$

and it expresses some crucial compromises between, e.g.:

- the fuzzy constraints and (objective and subjective) fuzzy goals,
- the interests of the authorities and inhabitants, etc.

The problem is now to find an optimal sequence of controls (investments) $u_0^*, \ldots, u_{N-1}^*$ such that (under a given policy B_N; the optimization of policy is a separate problem which will not be cosnidered here):

$$\begin{aligned} &\mu_D(u_0^*, \ldots, u_{N-1}^* \mid X_0, B_N) = \\ &= \max_{u_0, \ldots, u_{u_{N-1}}} \{[\mu_{C^0}(u_0) \wedge \mu_{G_o^1}(X_1) \wedge \mu_{G_s^1}(s_1)] \wedge \ldots \\ &\quad \ldots \wedge [\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G_o^N}(X_N) \wedge \mu_{G_s^N}(s_N)]\} \end{aligned} \tag{26}$$

For illustration we will show a simple example that in its initial form was shown first in Kacprzyk's [10] book but will be changed with respect to numbers to account for different economic conditions in the present time. **Example:** The region, predominantly agricultural, has a population of ca. 120,000 inhabitants, and its arable land is ca. 450,000 acres. For simplicity, the region's development will be considered over the next 3 development stages (years, for simplicity). The life quality index consists of the four life quality indicators:

- x_t^{I} – average subsidies in US$ per acre (per year),
- x_t^{II} – sanitation expenditures (water and sewage) in US$ per capita (per year),
- x_t^{III} – health care expenditures in US$ per capita (per year), and
- x_t^{IV} – expenditures for paved roads (new roads and maintenance of the existing ones) in US$ (per year).

Suppose now that the investments are partitioned into parts devoted to the improvement of the above life quality indicators due to the fixed partitioning rule $A_{t-1}(u_{t-1}, i)$: 5 % for subsidies, 25 % for sanitation, 45 % for health care, and 25 % for infrastructure.

Let the initial, at $t = 0$, values of the life quality indicators be:

$$x_0^{\mathrm{I}} = 0.5 \quad x_0^{\mathrm{II}} = 15 \quad x_0^{\mathrm{III}} = 27 \quad x_0^{\mathrm{IV}} = 1,700,000$$

For clarity, we will only take into account the following two *scenarios* (policies):

- Policy 1: $u_0 = \$16,000,000$ $u_1 = \$16,000,000$ $u_2 = \$16,000,000$
- Policy 2: $u_0 = \$15,000,000$ $u_1 = \$16,000,000$ $u_2 = \$17,000,000$

Under Policy 1 and Policy 2, the values of the life quality indicators attained are:

Policy 1:

Year(t)	u_t	x_t^{I}	x_t^{II}	x_t^{III}	x_t^{IV}
0	\$16,000,000				
1	\$16,000,000	0.88	16.7	30	\$4,000,000
2	\$16,000,000	0.88	16.7	30	\$4,000,000
3		0.88	16.7	30	\$4,000,000

Policy 2:

Year(t)	u_t	x_t^{I}	x_t^{II}	x_t^{III}	x_t^{IV}
0	\$15,000,000				
1	\$16,000,000	0.83	15.6	28.1	\$3,500,000
2	\$17,000,000	0.88	16.7	30	\$8,000,000
3		0.94	17.7	31.9	\$2,250,000

For the evaluation of the above two development trajectories, for simplicity and readability we will only take into account the *effectiveness* of development, and the objective evaluation only. The consecutive fuzzy constraints and objective fuzzy subgoals are assumed piecewise linear, i.e. their definition requires two values only (cf. Figs. 4 and 5): the aspiration level (i.e. the fully acceptable value) and the lowest (or highest) possible (still acceptable) value) which are:

t				
0	$C^0 : u_0^p = \$15,000,000$			
	$u_0^c = \$17,000,000$			
1	$C^1 : u_1^p = \$16,500,000$			
	$u_1^c = \$18,000,000$	$G_o^{1,\mathrm{I}}$	$: \underline{x}_1^{\mathrm{I}} = 0.6$	$\overline{x}_1^{\mathrm{I}} = 0.85$
		$G_o^{1,\mathrm{II}}$	$: \underline{x}_1^{\mathrm{II}} = 14$	$\overline{x}_1^{\mathrm{II}} = 16$
		$G_o^{1,\mathrm{III}}$	$: \underline{x}_1^{\mathrm{III}} = 27$	$\overline{x}_1^{\mathrm{III}} = 29$
		$G_o^{1,\mathrm{IV}}$	$: \underline{x}_1^{\mathrm{IV}} = \$3,600,000$	$\overline{x}_1^{\mathrm{IV}} = \$3,800,000$

$$
\begin{array}{llll}
2\ C^2: u_2^p = \$16,000,000 & & & \\
u_1^c = \$20,000,000 & G_o^{2,\mathrm{I}}: \underline{x}_2^{\mathrm{I}} = 0.7 & & \bar{x}_1^{\mathrm{I}} = 0.9 \\
& G_o^{2,\mathrm{II}}: \underline{x}_2^{\mathrm{II}} = 15 & & \bar{x}_1^{\mathrm{II}} = 17 \\
& G_o^{2,\mathrm{III}}: \underline{x}_2^{\mathrm{III}} = 28 & & \bar{x}_1^{\mathrm{III}} = 30 \\
& G_o^{2,\mathrm{IV}}: \underline{x}_2^{\mathrm{IV}} = \$3,800,000 & & \bar{x}_2^{\mathrm{IV}} = \$4,000,000 \\
3 & G_o^{3,\mathrm{I}}: \underline{x}_3^{\mathrm{I}} = 0.75 & & \bar{x}_3^{\mathrm{I}} = 1 \\
& G_o^{3,\mathrm{II}}: \underline{x}_3^{\mathrm{II}} = 16 & & \bar{x}_1^{\mathrm{II}} = 18.5 \\
& G_o^{3,\mathrm{III}}: \underline{x}_3^{\mathrm{III}} = 29 & & \bar{x}_1^{\mathrm{III}} = 31 \\
& G_o^{3,\mathrm{IV}}: \underline{x}_3^{\mathrm{IV}} = \$3,800,000 & & \bar{x}_3^{\mathrm{IV}} = \$4,200,000
\end{array}
$$

Using the "$\wedge$" (minimum) to reflect a safety-first attitude, which is clearly preferable in the situation considered (a rural region plagued by aging of the society, out-migration to neighboring urban areas, economic decay, etc.), the evaluation of the two investment policies is:

- Policy 1

$$
\begin{aligned}
&\mu_D(\$16,000,000; \$16,000,000; \$16,000,000 \mid .) = \\
&= \mu_{C^0}(\$16,000,000) \wedge (\mu_{G_o^{1,\mathrm{I}}}(0.88) \wedge \\
&\quad \wedge \mu_{G_o^{1,\mathrm{II}}}(16.7) \wedge \mu_{G_o^{1,\mathrm{III}}}(30) \wedge \mu_{G_o^{1,\mathrm{IV}}}(\$4,000,000)) \wedge \\
&\quad \wedge \mu_{C^1}(\$16,000,000) \wedge (\mu_{G_o^{2,\mathrm{I}}}(0.88) \wedge \\
&\quad \wedge \mu_{G_o^{2,\mathrm{II}}}(16.7) \wedge \mu_{G_o^{2,\mathrm{III}}}(30) \wedge \mu_{G_o^{2,\mathrm{IV}}}(\$4,000,000)) \wedge \\
&\quad \wedge \mu_{C^2}(\$16,000,000) \wedge (\mu_{G_o^{3,\mathrm{I}}}(0.88) \wedge \\
&\quad \wedge \mu_{G_o^{3,\mathrm{II}}}(16.7) \wedge \mu_{G_o^{3,\mathrm{III}}}(30) \wedge \mu_{G_o^{3,\mathrm{IV}}}(\$4,000,000)) = \\
&= 0.5 \wedge (1 \wedge 1 \wedge 1 \wedge 1) \wedge 0.8 \wedge \\
&\quad \wedge (0.9 \wedge 0.85 \wedge 1 \wedge 1) \wedge 1 \wedge (0.52 \wedge 0.28 \wedge 0.5 \wedge 0.33) = \\
&= 0.5 \wedge 0.8 \wedge 0.28 = 0.28
\end{aligned}
$$

- Policy 2

$$
\begin{aligned}
&\mu_D(\$15,000,000; \$16,000,000; \$15,500,000 \mid .) = \\
&= \mu_{C^0}(\$15,000,000) \wedge (\mu_{G_o^{1,\mathrm{I}}}(0.83) \wedge \\
&\quad \wedge \mu_{G_o^{1,\mathrm{II}}}(15.6) \wedge \mu_{G_o^{1,\mathrm{III}}}(28.1) \wedge \mu_{G_o^{1,\mathrm{IV}}}(\$3,750,000)) \wedge \\
&\quad \wedge \mu_{C^1}(\$16,000,000) \wedge (\mu_{G_o^{2,\mathrm{I}}}(0.88) \wedge \\
&\quad \wedge \mu_{G_o^{2,\mathrm{II}}}(16.7) \wedge \mu_{G_o^{2,\mathrm{III}}}(30) \wedge \mu_{G_o^{2,\mathrm{IV}}}(\$4,000,000)) \wedge \\
&\quad \wedge \mu_{C^2}(\$17,000,000) \wedge (\mu_{G_o^{3,\mathrm{I}}}(0.94) \wedge \\
&\quad \wedge \mu_{G_o^{3,\mathrm{II}}}(17.7) \wedge \mu_{G_o^{3,\mathrm{III}}}(31.9) \wedge \mu_{G_o^{3,\mathrm{IV}}}(\$4,250,000)) =
\end{aligned}
$$

$$= 1 \wedge (0.92 \wedge 0.8 \wedge 0.55 \wedge 0.75) \wedge 0.8 \wedge$$
$$\wedge(0.9 \wedge 0.85 \wedge 1 \wedge 1) \wedge 0.75 \wedge (0.76 \wedge 0.68 \wedge 1 \wedge 1) =$$
$$= 0.55 \wedge 0.8 \wedge 0.68 = 0.55$$

The second policy is therefore better.

4 Concluding Remarks

We tried to show that Wang's cognitive informatics may be a potentially powerful general approach and paradigm to formulate, analyze and solve human centric systems modeling, decision and control problems. To be more specific, we showed the use of fuzzy dynamic programming for solving a regional development problem in which many crucial aspects, in particular life quality indicators, were subject to objective and subjective, by the humans, judgments and evaluations which are closely related to human perceptions and cognitive abilities. For illustration, we showed a simple example of regional development planning in which the problem was to determine a best (optimal) investment policy under different development scenarios, and subject to objective and subjective evaluations.

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On Discord Between Expected and Actual Developments in Applications of Fuzzy Logic During Its First Fifty Years

George J. Klir

Abstract Developments of applications of fuzzy logic during the first fifty years of its existence are examined in this paper with the aim of comparing the actual developments with the expected ones in various areas of human affairs. It is shown that in many of the examined areas the actual developments turned out to be very different from the expected ones. In each area, an attempt is made to explain reasons for this surprising discord between reasonable expectations and the actual developments.

Keywords Principle of bivalence • Fuzzy logic • Applications of fuzzy logic

1 Introduction

In this paper, the term *fuzzy logic* in used in its general, commonsense meaning, referring to all principles and methods for representing and manipulating knowledge that employ, in addition to the classical truth values—*true* and *false*—intermediary truth values that are interpreted as *degrees of truth*. The principal characteristic of fuzzy logic viewed in this way is the rejection of the *bivalence principle* of classical logic—the assumption, inherent in classical logic—that each declarative sentence has exactly two possible truth values, *true* and *false*.

Recognizing that any challenge of the bivalence principle in logic and mathematics is extremely radical explains why such challenges have been very rare in the long history of logic and mathematics. Prior to the 20th century, only a very few challenges of the bivalence principle have been discovered by historians of logic,

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_12

and these happened to be all inconsequential. In the 20th century, the bivalence principle was challenged more seriously by the emergence of the various many-valued logics and, eventually, by fuzzy logic.[1]

It is undeniable that the most significant challenge to the principle of bivalence is closely associated with the introduction of fuzzy set theory by Lotfi Zadeh [2]. The aim of this book is to examine how this theory has developed during the first fifty years of its existence and what is its impact on mathematics and other areas of human affairs. I chose to focus in this short note on a rather neglected aspect of the 50-year history of the theory—the discrepancy between the expected and actual applications of the theory in some basic areas of science, engineering, and other professions (medicine, business, etc.). I also try to explain the cause of this discrepancy in each of the examined areas.

It has often been emphasized by Zadeh that two distinct meaning of the term *fuzzy logic* should be recognized, and he introduced the terms *fuzzy logic in the narrow sense* and *fuzzy logic in the broad sense* for these two meanings. This distinction, which Zadeh described particularly well in [3], is useful and I consider it relevant for the discussion I intend to pursue in this article, so let me introduce it from the outset.

Fuzzy logic in the narrow sense is concerned with formal logical systems in which the truth of each proposition is a matter of degree. It studies the various propositional, predicate and other fuzzy logic systems that are sound and complete in a similar way as in classical, bivalent logic. These systems provide foundations for fuzzy logic in the broad sense, which has a considerably wider and highly pragmatic agenda.

Fuzzy logic in the broad sense can be loosely characterized as a research program that has been pursued under the leadership of Lotfi Zadeh since the publication of his seminal paper [2]. The primary aim of this program is to employ fuzzy set theory for emulating common-sense human reasoning in natural language and for utilizing it for various other purposes. In pursuing this aim, fuzzy logic in the broad sense often reaches beyond the established concepts and results in fuzzy logic in the narrow sense, which, in turn, motivates further research in fuzzy logic in the narrow sense.[2]

Fuzzy logic in the broad sense is a huge undertaking, which has been shaped over the years by many contributors. Among them, however, Lotfi Zadeh has played a leading role by continually introducing novel ideas, which gradually expanded the agenda of this research program. From 1965 until the mid 1990s, the genesis of these ideas is well documented in two large volumes of his collected papers, edited by Yager et al. [5] and Klir and Yuan [6]. After the mid 1990s, Zadeh

[1]For historical details regarding this very brief summary, see the recent book by Belohlavek, Dauben and Klir [1].

[2]However, this statement makes sense only since the 1990s, when the first systems of fuzzy logic in the narrow emerged through the work of Peter Hájek [4] and other logicians.

introduced a few additional prime ideas, including those of computing with words [7], computing with perceptions [8], and general theory of uncertainty [9, 10].

Clearly, the term *fuzzy logic* (in both its narrow and broad sense) represents a generic concept that characterizes a wide variety of special systems. In fuzzy logic in the narrow sense, these systems are distinguished from one another by various properties such as the set of truth degrees employed and its algebraic structure, truth functions employed for logic connectives, recognized inference rules, and the like. In fuzzy logic in the broad sense they are distinguished by the employed set of membership degrees and its algebraic structure, the employed aggregation operations on fuzzy sets, modifiers representing linguistic hedges, and the like.

In the next, rather short section, I examine theoretical developments in fuzzy logic over the last fifty years. This is followed by a considerably longer section concerned with applications of fuzzy logic, which is the core of this paper.

2 Theoretical Developments

The development of fuzzy logic in the broad sense began with the publication of the seminal paper on fuzzy sets by Zadeh [2]. The concept of a fuzzy set, as introduced in this paper, is an intuitive one, not an axiomatic one. Its meaning is described in the paper as follows (page 339):

> The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability…. Such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership.

The agenda of Zadeh's research program—fuzzy logic in the broad sense—derives rather naturally from the observations that fuzzy sets generalize ordinary (classical) sets and that this generalization expands potentially their applicability. In developing the agenda, Zadeh set on exploring these two observations, and it is significant that he has pursued these explorations in a systematic fashion in his many publications. It is typical for his publications that each contains not only some new ideas, but also an extensive overview of relevant previous ideas in the context of the new ideas. Through this consistent repetition of relevant previous ideas, his own or in some cases introduced by other contributors, the agenda of fuzzy logic in the broad sense has gradually evolved in a coherent way.

In his seminal paper [2], Zadeh introduced only a special class of fuzzy sets, the range of whose membership functions is always the unit interval [0,1]. These are usually referred to as *standard fuzzy sets*. However, he made a remark in a footnote of the paper that this range can be generalized to "a suitable partially ordered set." He also made another remark in the same footnote that if values of the membership function are interpreted as truth values, a multivalued logic is obtained with a continuum of truth values in [0,1]. These two remarks were taken seriously by

Joseph Goguen, a student of Zadeh, who fully elaborated on them in his two early papers. In the first paper [11], he generalized standard fuzzy sets to the so-called *L-fuzzy sets* by extending the unit interval of standard fuzzy sets to a more general and well-conceived algebraic structure of a complete residuated lattice of membership grades. In the second paper [12], he developed basic ideas of logic for reasoning with inexact concepts, in which fuzzy sets (standard or *L*-fuzzy) play the role of inexact predicates and quantifiers. These two papers are historically very important since they are closely associated not only with the genesis of fuzzy logic in the broad sense, but also with genesis of fuzzy logic in the narrow sense.

The developments of fuzzy logic from the two viewpoints—the broad one and the narrow one—have been pursued more or less independently from one another, primarily due to their very different agendas and because most researchers attracted to fuzzy logic were interested in either one or the other agenda. Although there have been some researchers who were interested in both agendas, such as Joseph Goguen, they were unfortunately very rare.

Contrary to fuzzy logic in the broad sense, the one in the narrow sense has a long prehistory, associated with the various many-valued logics that have been studied since the beginning of the 20th century, as is well documented in the book by Rescher [13]. The connection of many-valued logics with fuzzy logic in the narrow sense, which was for the first time recognized in the above-mentioned paper Goguen [11], is examined more completely in a large book by Gottwald [14] as well as in [1]. These books also describe in detail how the various formal systems of fuzzy logic were developed within the framework of many-valued logics. I do not cover these theoretical developments in this paper, which is primarily oriented to applications of fuzzy logic.

In the next section, which is the kernel of this paper, I examine applications of fuzzy logic in various areas of human affairs. In each area, I focus on the discrepancies between expectations and reality and I try to find plausible explanations for these discrepancies.

3 Applications of Fuzzy Logic

3.1 Motivations for Introducing Fuzzy Sets

The introduction of the concept of a fuzzy set by Zadeh in his seminal paper [2] was based on well-conceived and convincing motivations, which are expressed not only in the seminal paper, but also in several of his other early publications. Zadeh's earliest thought about the need for fuzzy sets is expressed in his 1962 paper [15], where he writes (page 857):

> For coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitudes more complex than man-made systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities.

He returned to this theme a few years later in [16], where he wrote (p. 199):

> One cannot help feeling that, on the whole, the degree of success achieved by the use of mathematical techniques in biosciences has been quite limited. What is more disturbing, however, is the possibility that classical mathematics—with its insistence on rigor and precision—may never be able to provide totally satisfying answers to the basic questions related to the behavior of animate systems.

The importance of fuzzy sets for biology, but also for psychology and other so-called soft sciences was also explicitly recognized by Goguen in his Introduction to [12]:

> The 'hard' sciences, such as physics and chemistry, construct exact mathematical models to make predictions. Certain aspects of reality always escape such models, and we look hopefully to future refinements. But sometimes there is an elusive fuzziness, a readjustment to context, or an effect of observer upon observed. These phenomena are particularly indigenous to natural language, and are common in the 'soft' sciences, such as biology and psychology.

In fact, this whole paper is devoted to the investigation of imprecise concepts, primarily from the psychological point of view.

In his seminal paper, Zadeh emphasized that fuzzy sets provides a natural tool for "dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables." The same year, he illustrated in [17] these problems by those of optimization under ill defined constrains. Two years later, in a joint paper with Bellman and Kalaba [18], they were illustrated by problems of abstraction and pattern classification and later, in another joint paper [19], by decision-making problems in which "the goals and/or constraints constitute classes of alternatives whose boundaries are not sharply defined." The suggested use of fuzzy set theory in dealing with these problem areas—decision making, pattern classification and/or recognition, and optimization—attracted quickly attention of a small group of enthusiastic researchers who made substantial advances in these areas already in the 1970s. This is well documented in the early monographs by Kickert [20] on fuzzy decision making and in the book by Bezdek [21] on fuzzy pattern recognition. The former also contains a survey of associated fuzzy optimization methods, such as fuzzy linear programming or fuzzy dynamic programming. Research on the use of fuzzy logic in these problem areas has even intensified since the 1970s and has produced a remarkable spectrum of important results. These applications are certainly among the most successful applications of fuzzy logic.

In the following, I examine the development of applications of fuzzy logic in various areas of science, engineering and other areas of human affairs. In each of considered areas, I focus on comparing the expected developments with the actual ones. I show that in many cases, the actual developments have turned out very differently from the expectations and I try in each such case to explain reasons for the discrepancy.

3.2 Engineering

Around the time fuzzy set theory emerged, engineering was not seen as an area in which the use of fuzzy sets was needed. Yet, one of the most successful and visible applications of fuzzy set theory was an engineering application—fuzzy control. Except for the applications in decision making, pattern recognition, and optimization, mentioned in Sect. 3.1, this was also one of the earliest applications of fuzzy set theory. Let me explain circumstances that led to this early and exceedingly successful, but highly unexpected application of fuzzy set theory.

Seven years after publishing his seminal paper, Lotfi Zadeh wrote a short, two-page note [22], in which he argued that the excessive concern with precision and mathematical rigor in conventional control theory has become counterproductive because it tends to focus the research in this area only on problems that allow of exact solution. Hence, problems that are too complex or ill defined to admit of precise mathematical analysis are avoided as mathematically intractable. He suggested dealing with such "intractable" control problems by fuzzy algorithms, which he already introduced in an earlier paper [23]. To make his suggestion more specific, he illustrates it by a simple fuzzy algorithm for guiding a blindfolded person from an initial position in a room with no obstacles to a desirable final position.

It seems from the way this short note was written that Zadeh did not expect that his suggestion would be actually pursued any time soon, but presented it rather as a long-term perspective. However, contrary to Zadeh's expectations, actual work on designing, implementing, and testing an experimental fuzzy controller for controlling a small steam engine began shortly after the publication of his note [22] at Queens Mary College in London by Ebrahim Mamdani with one of his students (S. Assilian). It was already described, together with some initial experiments in [24], three years after Zadeh's note. In his recollections [25], Mamdani describes circumstances that led to his pioneering work on fuzzy controllers (p. 340):

> It was Zadeh's paper [22] published at that time which persuaded us to use a fuzzy rule-based approach. Between reading and understanding Zadeh's paper and having a working controller took a mere week and it was "surprising" how easy it was to design a rule-based controller.

In 1880, the first commercial fuzzy controller, inspired by basic ideas of Mamdani's design, was permanently installed for controlling a cement kiln owned by F. L. Smidth & Company in Denmark. The controller successfully replaced control by human operators with computer-based control and even improved somewhat the performance and cut fuel consumption. This was a great success since the control by human operators was too expensive and inconvenient as it took about eight weeks to train a new operator, and the process to be controlled was in this case too complex and unwieldy for conventional controller. The fuzzy controller was designed by a Danish engineer at the University of Denmark who left the university to work at the company to develop a computer-based controller. He tried to do that by using conventional control theory, but he soon discovered that it was virtually impossible. Fortunately, he came across Mamdani's work on fuzzy controller described in [24], and the idea immediately appealed to him due to its focus on modeling knowledge of a well-trained operator rather than on the process to be controlled. He found that the rules described in the textbook commonly used for training human operators of cement kilns could be readily represented as if-then rules in a fuzzy controller of the Mamdani type. This first commercial fuzzy controller, which is described in [26], has been subsequently used for controlling many other kilns, mills, and other complex processes. This was undoubtedly at that time the most significant application of fuzzy logic.

However, much more significant applications were at the same time under development in Japan. One was a sophisticated fuzzy control, involving both feedback and feedforward features, of fully automatic operation of the subway system in the city of Sendai. This project was conceived in 1979 by two researchers at Hitachi Systems development Laboratory, Seiji Yasunobu and Shoji Miyamoto, who were inspired by the novelty of Mamdani's fuzzy controller. It is likely that the successful installation of fuzzy controller for controlling the cement kiln in Denmark helped them to convince the upper management to support this large and rather risky project. The city of Sendai switched from trains operated by human operators to fully automatic operation based on fuzzy control in 1987, and it was a huge success in all measures. Details of this sophisticated fuzzy controller are described in [27].

Success of this project motivated many Japanese industries to invest in various other applications of fuzzy logic. This resulted in a surprising variety of innovative and sometimes unexpected applications of fuzzy logic, especially fuzzy controllers, that turned out to be technically as well as commercially highly successful. One positive outcome of these developments, which are described in detail in a well-researched book by McNeill and Freiberger [28], was that the visibility of fuzzy logic tremendously increased and, as a consequence, industries and governments in some countries, not only in Japan, became more receptive to support research on fuzzy logic.

The enormous success of fuzzy controllers is my first example of discrepancy between expected and actual applications of fuzzy logic. Indeed, fuzzy control was in no way among the factors motivating the need for introducing the concept of a fuzzy set. Yet, it turned out to be an extraordinarily successful early application of fuzzy sets. Next, I am going to turn to three natural sciences, biology, chemistry, and physics.

3.3 Biology

As is explained in Sect. 3.1, the envisioned need for mathematics based on fuzzy logic in biology was one of the primary motivations for Zadeh to introduce fuzzy sets. Yet, the biological community has shown virtually no interest in exploring this emerging new mathematics. Biology is thus one area in which the reasonable expectations have not realized so far. This is surprising and not easy to explain.

However, as I see it, this lack of interest may in this case be at least partially explained by a huge gap between experimental and theoretical biology and by the strong dominance of the former one. Indeed, most biologists focus on experimental work, for which they are trained, and pay little or no attention to mathematics. Theoretical biologists are a small minority among biologists with little influence on biology at large. Most biologists are just not interested in the work of theoretical biologists. Moreover, none of the few theoretical biologists have shown any interest in exploring the utility of fuzzy set theory in their theories.

I should add that some interest in the use of fuzzy logic in biology has been shown for the last fifteen years or so, but only in the context of the rapidly growing new subarea of biology—bioinformatics—a rather narrow, but highly important subarea, which is closely connected with Human Genome Project. The objective of this large international collaborative program, which was implemented during the period from 1990 to 2003, was to determine structures—that is sequences of deoxyribonucleic acid (DNA) molecules—of all genes of human beings. The outcome of the project was a very large database containing structures of all human genes. This database and other biomolecular databases provide researchers in molecular genetics with huge amount of information. The challenge is to utilize this information for advancing biological knowledge by answering many profound biological questions, such as those regarding functions of the individual genes, processes leading to the three-dimensional structures of proteins, functions of these structures, and the like. It is this analytical part of bioinformatics, where the usefulness of fuzzy logic was suggested already in 2000 in two early papers [29, 30]. These papers were soon followed by a rapid growth of literature on various applications of fuzzy logic and soft computing in bioinformatics. Just during the first decade of the 21st century, three large edited books devoted to these applications were published. In 2008, the time was already ripe for publishing the first monograph on these applications [31].

3.4 Chemistry

Chemistry, similarly as physics, has always been considered as belonging to the so-called hard sciences. As such, the need for fuzzy logic in chemistry was definitely not among the motivations for introducing fuzzy sets (see, e.g., the excerpt from Goguen's paper [12] in Sect. 3.1). Hence, researchers in fuzzy logic paid virtually no attention to chemistry for long time after the emergence of fuzzy set theory in 1965. In the early 1990s, however, a few researchers in theoretical chemistry discovered fuzzy logic and began to recognize that it might be potentially useful in dealing with some unresolved problems in their area. These problems emanated from the conventional way of viewing some important concepts in chemistry, such as symmetry or chirality, as bivalent—either true or false in each of their applications in chemistry. For example, in paper [32], the authors discuss this issue with respect to the concept of symmetry as employed in chemistry (p. 7843):

> One of the most deeply-rooted paradigms of scientific thought is that Nature is governed in many of its manifestation by strict symmetry laws. The continuing justification of that paradigm lies with the very achievements of human knowledge it has created over the centuries. Yet we ague that the treatment on natural phenomena in terms of “either/or”, when it comes to a symmetry characteristic property, may become restrictive to the extent that some of the fine details of phenomenological interpretation may be lost. Atkins writes in his widely-used text on physical chemistry:[3] “Some objects are more symmetrical than others”, signaling that a scale, quantifying this most basic property, may be in order. The view we wish to defend in this report is that symmetry can be and, in many instances, should be treated as a “gray” property, and not necessarily as a black or white property which exists or does not exist. Why is such continuous symmetry measure important? In short, replacing a “yes or no” information processing filter, which acts as a threshold decision-making barrier which differentiate between two states, with a filter allowing a full range of “maybe’s”, enriches, in principle, the information content available for analysis.

In two follow-up papers published in the J. of the American Chemical Society—**115**(24), 1993, and **117**(1), 1995—the authors further elaborated on the continuous symmetry measure and introduced, in addition, a continuous chirality measure. Similar observations and arguments regarding the need to abandon the principle of bivalence for dealing with some chemical concepts were at the same time advanced by a fair number of other researchers in theoretical chemistry. It was also increasingly recognized that it should be beneficial to utilize fuzzy logic for dealing with these problems.

It was eventually decided to devote one of the annual Mathematical Chemistry Conferences fully to the role of fuzzy logic in chemistry. The title of the conference—Are the Concepts of Chemistry All Fuzzy?—captured quite well the primary issues discussed within the area of theoretical chemistry at that time. The conference was held in 1996 at conference facilities of a major distillery—very appropriate for a conference of this kind, in Pitlochry, Scotland, in 1996. I was invited to present a tutorial on fuzzy logic and to represent the fuzzy-logic community. A major outcome of this conference was a book entitled “Fuzzy Logic in Chemistry”, carefully edited by Dennis Rouvray [33]. It consists of nine rather extensive chapters that are loosely based on presentations at the conference. The book convincingly demonstrate that fuzzy logic is useful not only for representing realistically some fundamental chemical concepts, such as symmetry chirality, molecular structure, or molecular shape and size, but also for dealing properly and effectively with some methodological problems in chemistry, such as problems of molecular recognition, hierarchical classification, or computer-aided elucidation of molecular structures.

As far as I know, no additional books on fuzzy logic in chemistry have been published. However, after the publication of [33], fuzzy logic has been routinely utilized in chemistry not only for dealing with the above-mentioned conceptual problems, but with various other problems as well. In other words, fuzzy logic has been rather naturally recognized in chemistry as useful. This situation is clearly radically different from the corresponding situation in biology, which is described in Sect. 3.3. In biology, the utility of fuzzy logic was strongly anticipated, but it has

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[3] Atkins, P. W.: Physical Chemistry, 3rd Edition, p. 406. Oxford Univ. Press, Oxford (1986).

not yet been recognized by the biological community, with the exception of the very recent and still rather modest recognition of its role in bioinformatics. In chemistry, no use of fuzzy logic was anticipated and yet, its utility was discovered already in the early 1990s by researchers in theoretical chemistry and has gradually been accepted by the entire chemical community.

3.5 Physics

Physics, which is undoubtedly the most advanced and successful area of science, was certainly not among the areas in which the need for fuzzy logic was anticipated. It is perhaps due to the enormous success of physics why the developed measurement routines in physics have virtually never been questioned within the physics community. A rare dissenter was the outstanding American physicist Percy Williams Bridgman (1882–1961).[4] His rather unorthodox views about measurement in physics are captured reasonably well in the following excerpts from one of his papers ([34], 227–228, italics added):

> The physics of measurement and of the laboratory does not have the yes-no sharpness of mathematics, but nevertheless employs conventional mathematics as an indispensable tool. Every physicist combines in his own person, to greater or less degree, the experimental physicist who makes measurements in the laboratory, and the theoretical physicist who represents the results of the measurements by the numbers of mathematics. These numbers are things he says or writes on paper. The jump by which he passes from the operations of the laboratory to what he says about the operations is a jump which may not be bridged logically, and is furthermore a jump which ignores certain essential features of the physical situation. For the mathematics which the physicist uses does not exactly correspond to what happens to him. *In the laboratory every measurement is fuzzy because of error*. As far as reproducing what happens to him is concerned, the mathematics of the physicist might equally well be the mathematics of the rational numbers… Now one would certainly be going of one's way to attempt to force theoretical physics into a straightjacket of the mathematics of rational numbers as distinguished from the mathematics of all real numbers, but *by forcing it into the straightjacket of any kind of mathematics at all, with its yes-no sharpness, one is discarding an essential aspect of physical experience and to that extent renouncing the possibility of exactly reproducing that experience*. In this sense, the commitment of physics to the use of mathematics itself constitutes, paradoxically, a renunciation of the possibility of rigor….Now it appears to me, the linkage of error in every sort of physical measurement must be regarded as inevitable when it is considered that the knowledge of the measurement, which is all we can be concerned with, is a result of the coupling of the external situation with a human brain. Even if we had adequate knowledge of the details of this coupling we admittedly could not yet use this knowledge in formulating in detail how the *unavoidable fuzziness* should be incorporated in our description of the world nor how should we modify our present use of mathematics, but with the additional caveat to every equation, warning that things are not quite as they seem.

[4]Bridgman was an excellent experimental physicist who had been most of his academic career with Harvard University. In 1946, he won the Nobel Prize in Physics for his groundbreaking work on the physics of high pressures. He also wrote extensively on measurement in physics and on various other aspects of philosophy of science.

The use of the terms *fuzzy* and *fuzziness* by Bridgman in 1959, five years before Zadeh's seminal paper [2], is certainly interesting, especially because his use of these terms is quite similar to Zadeh's use. In fact, these terms appear even in some of Bridgman's earlier writings.

Bridgman's critical comments about measurement in physics did not have any visible impact on physics during his lifetime. However, since the early 1990s some physicists specializing on measurement have occasionally referred to Bridgman's criticism and to the potential role of fuzzy sets in physical measurement; see, for example a representative paper by Mari [35].

Finally, I should look at the potential role of fuzzy logic in quantum mechanics. There is an extensive literature on this topic, too large and complex to be even briefly surveyed in this paper. In any case, none of the many logics, fuzzy or non-fuzzy, which have been proposed for quantum mechanics thus far, has not been generally accepted as yet. The situation is well characterized in the monograph by Chiara et al. [36], which to my best knowledge is the only one that covers logics that recognize the principle of bivalence as well as those that do no recognize it. They are referred to in the book as *sharp quantum logics* and *unsharp quantum logics*, respectively. The authors seems to be well aware of the importance of the prospective unsharp quantum logics, and make this interesting observation (p. 5):

> Strangely enough, from the historical point of view, the abstract researches on fuzzy structures and on quantum structures have undergone quite independent developments for many decades during the 20th century.Only after the Eighties, there emerged an interesting convergence between the investigations about fuzzy and quantum structures, in the framework of the so-called *unsharp approach to quantum theory*. In this connection a significant conjecture has been proposed: perhaps some apparent *mysteries* of the quantum world should be described as special cases of some more general *fuzzy phenomena*, whose behavior has not yet been fully understood.

It is also significant that on page 37 of this book, the authors describe a specific example in quantum theory, in which the principle of bivalence fails.

Physics is thus an area of science in which the use of fuzzy logic was not expected at all. Nevertheless, its utility in physical measurements was recognized by some physicists, such as Bridgman a more recently Mari and others. Even more importantly, fuzzy logic is likely to play some role, potentially a very important role in some of its incarnations, in quantum mechanics, as suggested in the above quotation from [36].

3.6 Geology

Similarly, as it was not expected that fuzzy logic would play any useful role in chemistry, it was not expected that it would be of any use in geology. Now we know that both of these expectations were wrong. The similarities extend further. In both of these areas, the utility of fuzzy logic was not recognized by researchers outside these areas (e.g. those who worked on fuzzy logic and were searching for applications), but researchers within these very areas, and it was recognized in both areas within approximately the same period—roughly during the 1990s. It seems that this timing could be explained by the great success of fuzzy logic in the 1990s, as is explained in Sect. 3.2, which significantly increased its visibility.

With the emergence of computer technology around the middle of the 20th century, geology has undergone a major transformation regarding the nature of geological knowledge. Since the 19th century, geology has been preoccupied primarily with attempts to understand how the surface of the Earth had developed. A substantial amount of knowledge was produced by the work of many geologists via their extensive, systematic, and painstaking observations, combined with commonsense reasoning. Knowledge obtained in this way was of course expressed by the geologists in natural language, without any use of mathematics. After the emergence of computer technology, this knowledge was gradually dismissed as useless, as it could not be represented in a computer-acceptable language. This led to the development of mathematical geology. When some geologist discovered fuzzy logic around the mid 1990s and became familiar with its capabilities at that time, they tried to experiment with it by simulating some of the knowledge described verbally in the older geological books, often directly in the form of if-then statements. The first such simulation was described in 1996 in a paper by Nordland [37], where it was successfully illustrated by a particular example from the area of dynamic stratigraphic modeling.

The paper stimulated a fair number of other geologists to pursue similar studies not only in the same area but also in various other areas of geology. They were astonished by the excellent results they obtained and that motivated further research into the use of fuzzy logic in geology. As a result, the literature on applications of fuzzy logic in geology grew very rapidly at the end of 20th century, and it was generally felt that the time was ripe for a comprehensive book overview of this alternative approach to dealing geological problems, once abandoned and then rediscovered with the help of fuzzy logic. In fact, Lotfi Zadeh explicitly suggested that such a book be published.

The book on fuzzy logic in geology was eventually published in 2004 [38] and Zadeh wrote a wonderful Foreword to it. The book contains a tutorial on fuzzy logic for geologists, and a comprehensive overview with a literature review of all recognized applications of fuzzy logic in geology. In addition, it contains several chapters that describe in detail applications of fuzzy logic in the areas of stratigraphic modeling, hydrology and water resources, paleontology, and seismology, as well as the use of fuzzy logic for dealing with the problems of reef growth and ancient sea level estimation.

Since the publication of [38], the literature dealing with applications of fuzzy logic in geology as well as other areas of Earth sciences has substantially expanded including several monographs and edited volumes. This indicates that the utility of fuzzy logic is well recognized in this domain.

3.7 Psychology

Psychology and biology were two areas of science in which fuzzy logic was expected to play an important role. In fact, the need for mathematics based on fuzzy logic in these areas was among the main motivations for introducing the concept of a fuzzy set. As is explained in Sect. 3.3, the expectation has not realized in biology. It has not realized in psychology as well, but for very different and more complicated reasons. In the following, I am going to briefly survey the history of connections between psychology and fuzzy logic, which is quite extraordinary.

One year before the publication of Zadeh's seminal paper on fuzzy sets, Robert Duncan Luce (1925–2012), one of the preeminent figures in mathematical psychology, published a paper [39], in which he wrote (p. 376):

> The language of sets does not always seem adequate to formulate psychological problems. Put it so baldly, the statement is almost heretical since, in practice, set theory is the accepted way to formulate mathematical problems and, hence, applied mathematical problems. Still, we should not forget that set theory is really quite new—less than a century old. It could be an interim theory. Certainly, when I think about certain psychological problems, I wish it weren't the way it is. The boundaries of my "sets," and of ones that my subjects ordinarily deal with, are a good deal fuzzier than those in mathematics…. It is quite difficult to pin down just what elements are and are not members of that set, and I am not sure that it is possible in principle. Do we merely lack techniques adequate to answer that question today, or is it basically impossible to answer it?

Luce repeatedly returned in his many publications to the issues raised in this short excerpt, especially the question posed in the last sentence. Although he has frequently used in his writings the term "fuzzy" in its various forms, it is unfortunate that he was apparently not aware throughout his whole lifetime about the existence of fuzzy set theory and fuzzy logic.

Another connection between psychology, especially the psychology of concepts,[5] and fuzzy logic was introduced in the classic paper by Goguen [12], from which I use a few short excerpts (pp. 325–326):

> "Exact concepts" are the sort envisioned in pure mathematics, while "inexact concepts" are rampant in everyday life…. Ordinary logic is much used in mathematics, but applications to everyday life have been criticized because our normal language habits seem so different. Various modifications of orthodox logic have been suggested as remedies, particularly omission of the Law of Excluded Middle.Ordinary logic represents exact concepts *syntactically*: that is a concept is given a name (such as 'man') which becomes an object of manipulation in a formal language…. Another representation is the *semantic*, as in Cantorian set theory. Here we consider the collection or *set of elements* exemplifying the concept and study such manipulations as might be performed on actual physical collections: lumping together, removing a subcollection, and so on. The laws of set theory describe

[5]In general, a concept is viewed in psychology as a mental representation of a class of real or abstract entities, which is usually called a *concept category*. In the psychology of concepts, a concept is usually viewed more specifically as a body of knowledge regarding the entities in the associated concept category that is stored in the long-term memory (sometimes called a *semantic memory*) and employed by default in most of cognitive processes.

> general properties of these manipulations.Without a semantic representation for inexact concepts, it is hard to see that one modification of traditional logic really provides a more satisfactory syntactic theory of inexact concepts than another. However, such a representation is now available. Zadeh [2] has studied fuzzy sets, and suggested a number of concrete applications.

Another connection between psychology and fuzzy logic emerged from groundbreaking psychological experiments that were performed by Eleanor Rosch in the early 1970s and published in a series of articles, two of which are [40, 41]. These experiments consistently demonstrated that membership in concept categories is not a yes-or-no matter, but rather a matter of degree. This led to an almost universal rejection of the classical view of concepts, in which each concept category is defined by a collection of attributes that are both necessary and sufficient. Rosch's experiments also revealed that each concept category is associated with an ordering relation that reflects the typicality of individuals in the category as examples of the concept. The most typical individual(s) can be viewed as natural prototype(s) of the category. The above-mentioned typicality ordering can then be defined via a suitable similarity measure, as thoroughly investigated in the psychological context by Tversky [42]. This is briefly the essence of a prototype view of concepts that emerged form Rosch's experiments as a natural successor to the classical view of concepts.

Although the results obtained by Rosch and the emerging prototype view of concepts were suggestive of possible use of fuzzy sets in the psychology of concepts, Rosch herself did not seem to be interested in exploring it. However, her results have stimulated other psychologists to examine the potential role of fuzzy sets in psychology. This led to a lively discussion of this issue in psychological literature throughout the 1970s. However, this positive attitude toward fuzzy logic has visibly changed to negative attitude since the early 1980s. It is now well established that this change was triggered by a paper published in 1981 by two highly influential cognitive psychologists, Daniel Osherson and Edward Smith [43], whose aim was a critique of the prototype theory of concepts. This is how they describe the organization of their paper ([43], p. 36):

> We first present one version of prototype theory. We then show how it might be extended to account for conceptual combinations by means of principles derived from *fuzzy-set theory*. This extension is demonstrated to be fraught with difficulties. We then move on to the issue of truth conditions for thought, again using fuzzy-set theory as a means of implementing the prototype approach, and again demonstrating that this implementation won't work. In a final section, we establish that our analysis holds for virtually any version of prototype theory, and consider ways of reconciling previous evidence for this theory with the wisdom of the older kind of theory of concepts.

This paper had such a strong influence on attitudes toward fuzzy set theory in psychology that fuzzy set theory was virtually dismissed by the psychological community as useless. Only some twenty years after the paper by Osherson and Smith was published, some awkward mathematical errors were accidentally discovered in it. This led to a detailed analysis of all claims about fuzzy set theory in the paper, which revealed, surprisingly, that they were virtually all erroneous, as is shown in [44]. Further investigation [45] revealed how these erroneous claims were uncritically accepted within the psychological community and used often as arguments against fuzzy set theory. This extraordinary episode is fully documented in a book that I coedited with Radim Belohavek [46]. However, the primary aim of this book is to renew the dialog and hopefully a cooperation of researchers in psychology with those working in the area of fuzzy logic.

The up-and-downs regarding the use fuzzy logic in psychology certainly do not coincide with the original expectations. One reason might be that fuzzy logic is not yet properly developed for its specific use in psychology. This in fact was already extensively argued in the late 1980s by Fuhrmann in several of his papers (see, e.g., his paper [47]). If he is right, then the cooperation between the two areas will be essential.

3.8 Economics

Economics is generally viewed as the most advanced social science, primarily due to the extensive role that mathematics has played in it since the late 19th century. It is well known, however, that the mathematically ever more sophisticated economic theories have almost never produced accurate and practical economic predictions, while experience economists are often able to formulate fairly accurate and useful economic predictions in linguistic terms, such as “The rate of inflation is likely to increase substantially in the very near future.” Such predictions are based on common sense reasoning, employing the economist’s knowledge and relevant information, both expressed in natural language. Due to these observations, fuzzy logic was broadly expected, soon after it emerged in the mid 1960s, to play an important role in economics.

This expectation became in some sense a reality in the 1980s through the work of some French economists under the leadership of distinguished French economist Claude Ponsard (1927–1990). Influenced by the early publication of four-volume French book on fuzzy sets by Arnold Kaufmann,[6] Ponsard began to explore the use of fuzzy set theory in economics in the late 1970s. In one of his early papers [49], he shows, for example, how fuzzy sets can be used for reformulating the classical theory of consumer behavior in mathematical economics by discarding its two unrealistic assumptions, that the consumer can perfectly discriminate between different goods and that goods satisfying consumer’s needs are all supplied at a unique point in space. The result is a considerably more realistic theory of consumer behavior. During the first half of the 1980s, Ponsard published a series of papers, in which he fuzzified other areas of classical economics, and which culminated in the publication of a book he co-edited with his colleague Bernard Fustier [50]. The

[6]Introduction a la Theorie des Sous-Ensembles: vol. 1 (1973); vol. 2 (1975); vol. 3 (1975); vol. 4 (1977). Masson et Cie Editeurs. Only the first volume was published later in English [48].

books contains, ten important papers on fuzzy economics, all written by members of the Institute for Mathematical Economics at the University of Dijon, which was at that time directed by Ponsard. The book contains two papers by Ponsard, one on a theory of spatial general equilibrium in fuzzy economics and one on viewing spatial oligopoly[7] as a fuzzy game. One year after publishing the edited book, Ponsard generalized the famous Nash equilibrium concept[8] by showing that each *n*-person non-cooperative fuzzy game with mixed strategies has at least one equilibrium point.

In 1988, Ponsard begins his excellent survey paper [51] on established fuzzy models in economics with a question raised by Zadeh in his Foreword to the classical book by Zimmermann [52]: "Are there, in fact, any significant problem areas in which the use of the theory of fuzzy sets leads to results which could not be obtained by classical methods"? And he closes the paper by answering the question: "In economics, the answer is positive. The use of fuzzy subset theory leads to results which could not be obtained by classical methods."

In 1988, Ponsard also began to work on a major book with a tentative title "Fuzzy Economic Space: An Axiomatic Approach". When he unexpectedly passed away in 1990, the book manuscript was not yet fully completed. Fortunately, his main ideas are preserved and further developed in an important book by Billot [53], who was Ponsard's doctoral student at that time.

This section would be rather incomplete without mentioning the work by a British economist George Shackle on the theory of graded possibilities within the context of economics, long before the theory was interpreted in terms of fuzzy sets by Zadeh [54]. Due to the limited space of this paper, I take the liberty to refer to my paper [55], in which I outline Shacke's unorthodox approach to economics and describe in fair detail his work on the theory of graded possibilities.

The utility of fuzzy set theory in economics has not yet been fully recognized by mainstream economists. Nevertheless, the work by Ponsard and the other French economists, together with the work by Shackle, is sufficiently significant and convincing to conclude that the early expectations by the fuzzy community that fuzzy set theory would play an important role in economics have already been at least partially met.

3.9 Other Social Sciences

The usefulness of fuzzy set theory in all social sciences, not only in economics, was generally expected when the theory emerged in the mid 1960s. Zadeh, for example, devoted one of his early articles fully to this issue [56].

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[7]Market situation influenced by a few producers.

[8]Nash, J. F. Equilibrium points in *n*-person games. Proc. of the National Academy of Sciences **36**, 48-49 (1950).

The classical book by Smithson [57] attracted some attention to the useful role of fuzzy sets in social sciences, but for many years almost exclusively within the fuzzy-set community. Fortunately, it eventually attracted the attention of Charles Ragin, a social scientist, who became seriously interested in exploring the role of fuzzy sets for bridging the gap between the qualitative and quantitative methods in social-science research. This led him to write the whole book on this subject [58]. While some social scientists praised the book, most remained skeptical about his ideas, as is well captured by the following short excerpts from Ragin's Introduction to the book (p. 3):

> Social scientists generally stay away from anything labeled "fuzzy" because their work is so often described this way by others, especially by scholars in the "hard" sciences. My initial title for this book, *Fuzzy Social Science*, made so many of my colleagues cringe that I felt compelled to change it so that the adjective "fuzzy" applied to sets, not to social science.

Eight years later, another book by Ragin was published [59], in which he challenges the conventional approach to social science research and proposes an alternative approach based on fuzzy set theory that overcomes the various limitations of conventional quantitative as well as qualitative social-science research. He argues and demonstrates experimentally that the proposed approach has the capability to narrow the gap between knowledge obtained by qualitative social scientists and that obtained by quantitative social scientists.

When taken together, the two books by Ragin form an important statement about the utility of fuzzy set theory in social sciences. Unfortunately, the approach to social science research proposed by him has not yet been widely accepted by social scientists.

Perhaps the most important contribution to the use of fuzzy logic in social sciences at large is at this time the book by Badredine Arfi [60], a Finnish political scientist. He further develops in the book the idea of computing with words, first suggested by Zadeh [7, 8], and applies it to a wide range of problems in social sciences. In his methodology, he allows both membership grades and truth values to be linguistic variables. The book contains Forewords by Ragin and Zadeh who both highly praise it.

Interesting applications of mathematics based on fuzzy logic in political science emerged from collaboration of political scientists with mathematicians at Creighton University in Omaha, Nebraska. These applications, which are described in detail in [61], show that the concept of fuzzy geometry is superior for dealing with some problems in comparative politics in comparison with the traditional use of classical Euclidean geometry.

I consider it reasonable to conclude that the expected utility of fuzzy logic in social sciences has already been demonstrated, even though fuzzy logic has not yet been fully endorsed within these areas. It is interesting that the most negative attitude toward fuzzy logic is shown by the quantitative (or mathematical) social scientists.

3.10 Medicine

The need for fuzzy sets in medicine did not motivate, at least not explicitly, their introduction. However, the utility of fuzzy sets in medicine was recognized quite early, in about the mid 1970s, and primarily in the area of medical diagnosis.

The use of fuzzy set theory in medical diagnosis was first suggested and discussed in a doctoral dissertation by Albin [62], followed shortly by two early papers by Sanchez [63, 64]. In these papers, Sanchez formulated medical diagnosis in terms of fuzzy relational equations, which he introduced and investigated in his earlier paper [65]. A few additional papers regarding the use of fuzzy set theory in medical diagnosis were published in the 1970s. However, concentrated and systematic research on fuzzy-set-based system for computer-assisted medical diagnosis began only in the 1980 and mostly at the Department of medical Computer Science of the University of Vienna Medical School in Austria under the leadership of Klaus Peter Adlassnig. Accomplishments of this research over the last two decades of the 20th century are concisely described in [66]. Various other types of applications of fuzzy set theory in medicine were also developed during this period and are surveyed in [67].

The literature on applications of fuzzy set theory in medicine rapidly increased in the new millennium, including some specialized monographs, such as [68–70], and numerous edited volumes, exemplified by [71] A particularly significant is the scholarly work by Kazem Sadegh-Zadeh[9] in analytic philosophy of medicine. In many of his papers published since 2000, he has consistently argued on both medical and philosophical grounds that fuzzy logic is the only adequate logic for medical practice. This argumentation is completely and coherently covered in *Handbook of Analytic Philosophy of Medicine* [72], which is a sort of climax of his lifelong work. This large monograph, consisting of 1,133 pages, covers comprehensively and in considerable detail the principal philosophical issues associated with medicine.

About 40 % of the Handbook is devoted to logical issues involved in clinical reasoning. After showing that classical first-order predicate logic is hopelessly inadequate in medicine as it is capable of representing only a very small fraction of language employed in medicine, Sadegh-Zadeh then examines the various modal extensions of classical logic and shows that even with all these extensions classical logic is still not sufficiently expressive to represent medical language. Next, he examines non-classical logics, such as paraconsistent, intuitionistic, and many-valued logics, and shows that each individually would help to overcome some common difficulties in medicine, such as dealing with contradictory medical data or with situations in which the law of excluded middle does not hold. Finally, he examines fuzzy logic in detail from the medical point of view and shows that it has

[9]Kazem Sadegh-Zadeh was born in Tabriz, Iran in 1942. In the 1960s and 1970s, he studied medicine and philosophy at the German universities of Münster, Berlin, and Göttingen. He has been for many years with the University of Münster, where he worked in the area of analytic philosophy of medicine, and where he is now s professor-emeritus.

all the ingredients needed in medicine. This leads him to the conclusion that fuzzy logic (representing actually a class of logics) is the only one (one class) among all currently recognized logics that is fully adequate for clinical reasoning in medicine.

I should add that one year after the publication of Sadegh-Zadeh's Handbook, a companion volume to it was published [73], which contains 27 chapters on various applications of fuzzy logic in medicine.

Medicine is thus somewhat different from the other areas examined in previous sections of this paper as far as the difference between anticipated and actual utility of fuzzy logic. The need for fuzzy logic in medicine was not (at least explicitly) among the factors that contributed to the emergence of fuzzy logic in the mid 1960s. However, its potential utility in medicine was recognized quite early, in the mid 1970s, and it was expected that it would play an important role in medicine. It turned out, eventually, that fuzzy logic is the only adequate logic for medicine, which even exceeded the expectations.

3.11 Management and Business

The need for fuzzy logic in the areas of management and business was not among the motivations for introducing fuzzy sets. Although some scattered applications of fuzzy set theory to various problems related to these areas, such as optimization, scheduling and resource allocation, began to appear in the literature since the late 1970s, the role of fuzzy set theory in these areas was for the first time systematically discussed in a textbook by George and Maria Bojadziev [74] published 1997. Two years later, a very impressive survey of applications of fuzzy set theory in management, consisting of four large chapters, is included in [67]. Three of the chapters deal, respectively, with strategic planning, research and development planning, and production planning and scheduling. The forth one is devoted to the use of fuzzy sets in actuarial science, where the book by Ostaszewski [75] —the first and still the only book on using fuzzy set theory in actuarial science—should be highlighted.

In the new millennium, publications devoted to the use of fuzzy set theory in management and business virtually exploded. As far as management is concerned, the most important seems to the monograph by Carlsson et al. [76]. In business, two significant monographs deserve to be mentioned in this very short overview [77, 78]. The number of edited volumes in this area is too large to even mention a few representative samples.

In summary, the utility of fuzzy set theory was not initially recognized in either management or business. However, some applications of the theory, developed some ten years after fuzzy set theory emerged, were already indicative that fuzzy sets might be useful in these areas. This was more explicitly recognized and discussed in the late 1990s. Since that time, the development of applications of fuzzy set theory in management and business has been very rapid, which showed manifestly that the theory is of great utility in these areas. As in medicine, discussed in Sect. 3.10, the actual success of fuzzy set theory in these areas exceeded all expectations.

3.12 Music

Within the large variety of the arts, the use of fuzzy logic have occasionally been suggested in some branches of the arts, such as painting, sculpture, architecture, poetry, and others, but no significant interest resulted from these rather isolated and ad hoc suggestions. A rare exception is music, where the use of fuzzy logic turns out to be very natural and significant. In order to explain this rather bold claim, I begin with a short excerpt from Preface to one of the prime monographs on mathematical theory of tone systems [79]:

> There are four important and mutually interacting attributes that we can manipulate to create or describe any sound. And we can work with these attributes in two different ways: We can measure them and we can hear them. If we measure them, they are physical attributes; if we hear them, they are perceptual attributes. The four physical attributes are: frequency, amplitude, waveform, and duration. Their perceptual counterparts are: pitch, loudness, timbre, and (psychological) time. There is similarity between hearing and measuring these attributes; however, it is a complex correlation. The two are not exactly parallel.

As is well captured by this excerpt, the basic elements of music—musical tones—can be viewed and studied either as physical entities produced by various musical instruments or as perceptions of these physical entities by humans.

It is well established that human auditory perceptive capabilities are remarkably tolerant (or insensitive) to small deviations from the ideal (physical) frequencies representing individual tones. That is, tones whose actual frequencies are sufficiently close to the ideal frequency defining a particular tone in a given tone system are perceived as the same pitch. In a similar way, human perception is tolerant to small deviations from the ideal values of the other three physical attributes.

This fundamental dichotomy between physical and perceptual entities applies not only to individual tones, but to various systems of musical tones as well. The two most important characteristics of each tonal system are the pitch of each tone recognized in the system and the pitch differences of any two of the recognized tones. The former is a psychological concept that represents approximately the tone frequency. The latter, called musical intervals, are physically defined as ratios of their frequencies, which perceptually are approximate ratios.

A particular interval whose frequency ratio is 2 is called an octave. Two tones whose distance is equal to one or more octaves are viewed in the physical domain as equivalent, and are perceived as approximately equivalent. Most frequently, especially in Western classical music, 12 tones within each octave are chosen according to some rules that govern the intervals between consecutive tones in each octave. The notes together with their locations within each octave form a particular tonal system. When tones in these systems are viewed a physical entities, classical mathematics based on bivalent logic is perfectly adequate to deal with them. However, when the tones are viewed as perceptual entities, the best classical mathematics can do is to employ intervals of real numbers for representing the perceptual tolerances,[10] but this is a rather poor representation. Much better representation can be obtained by mathematics based on fuzzy logic. The concept of "being sufficiently closed to a number expressing the ideal frequency" can be approximated in a natural way by an appropriate fuzzy number (granule) constructed on the basis of available knowledge regarding characteristics of human auditory perception.

This fuzzy approximation plays an especially important role in the so-called well-tempered tuning within a given tonal system. The aim of this tuning is to make small deviations (tolerated by human perception) from perfect tuning in each key in order to achieve a perceptually acceptable tuning in all keys. This allows instruments such as pianos or harps, once tuned in a well-tempered way, to play compositions in any key and they are all perceived as well tuned.[11] Although in any well-tempered tuning, the sizes of comparable tone intervals in the physical domain cannot be the same in all keys, this is generally viewed as a musical advantage, as it gives a slightly distinctive character to compositions written in different keys.

Except for one early paper by Goguen [80], the literature pointing to the role fuzzy logic in music is almost exclusively associated with the 21st century. In addition to the book by Haluška [79], the three papers [81–83] seem to be the most visible representatives of the growing literature in this area.

3.13 Concluding Remarks

Due to the limited space of this paper, I had to make conscious decisions about which applications of fuzzy logic to include and which to omit. From the well-established applications, I chose to omit those in image analysis, spatial information processing, robotics, risk analysis, database systems, computer vision, and a few others. Moreover, I did not include any of the many applications that are promising, but have not yet been adequately developed. These include for example those in archaeology, paleontology, forensic science, humanities, and numerous other areas. I also did not include application areas such as the law profession, in which the utility of fuzzy logic is highly suggestive and potentially very significant, but where its actual use encounters various virtually insurmountable barriers, such as political, ethical, religious, and others.

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[10]This was actually suggested by the Russian musicologist N. A. Garbuzov in 1948 in his book entitled "Zonal Nature of the Human Aural Perception (in Russian), published by the Academy of Sciences of the USSR in Moscow and Leningrad.

[11]The famous systematic collection of 24 preludes and 24 fugues, each written in all 12 major and 12 minor keys, which are known under the German name "Das Wohltemperierte Klavier" (The Well-Tempered Clavier), were composed by Johann Sebastian Bach to provide an ultimate practical test that a piano is properly tuned in a well tempered way. After they are all played on the piano to be tested and each composition is perceived as well tuned, then the piano may be certified as perfectly well-tempered.

I hope that this review of how applications of fuzzy logic developed in some basic areas of human affairs during its first fifty years of existence, focusing especially on the often striking differences between the expected and actual developments in many of these areas, is an appropriate and potentially useful reflection on accomplishments of fuzzy logic on the 50th anniversary of its genesis.

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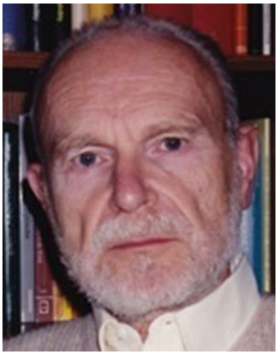
Author Biography



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Meta-Heuristic Optimization of a Fuzzy Character Recognizer

Alex Tormási and László T. Kóczy

Abstract Meta-heuristic algorithms are well researched and widely used in optimization problems. There are several meta-heuristic optimization algorithms with various concepts and each has its own advantages and disadvantages. Still it is difficult to decide which method would fit the best to a given problem. In this study the optimization of a fuzzy rule-base from a classifier, more specifically fuzzy character recognizer is used as the reference problem and the aim of the research was to investigate the behavior of selected meta-heuristic optimization techniques in order to develop a multi meta-heuristic algorithm.

Keywords Fuzzy systems · Fuzzy rule-base optimization · Bacterial evolutionary algorithm Big bang–big crunch algorithm · Imperialist competitive algorithm · Particle swarm optimization · Multi meta-heuristics

1 Introduction

Genetic [1], bacterial [2] and other evolutionary and population based meta-heuristic methods [3, 4] are widely used [5] in various computational intelligence related optimization problems including the tuning of fuzzy sets [6] and other parameters of fuzzy rule-based systems [7, 8]. It has both theoretical and technical significance to have a deep knowledge of the behavior of meta-heuristic optimization techniques in

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_13

fuzzy rule-base optimization in order to develop new, more efficient and accurate models. The aim of this study was not to find the optimal fuzzy rule-base for a discrete classifier, more precisely a character recognizer system [9], but to study how the meta-heuristic optimization algorithms handle a given problem under certain conditions, to find out which are their most sensitive parameters and how they could be improved – if possible at all.

The investigated meta-heuristics [2–4, 10] were selected according to actual trends in the field and to cover various approaches to optimization algorithms. The paper also includes results of a multi meta-heuristic [11–13] experiment, where there is a switch between various optimization algorithms, in order to achieve lower resource usage with faster convergence to the (quasy)optimum.

Various fuzzy systems are well researched and used in a wide scope of problems; and in many cases these solutions are more accurate and/or more efficient compared to other conventional methods. The simple way of knowledge representation by fuzzy sets makes these systems a great subject for experimenting with meta-heuristic optimization. A fuzzy rule-based classifier [14–16], more accurately, a fuzzy character recognizer [17] was selected as the sample problem used in the experiment. The reasons of this choice were the presence of fuzzy systems in the problem, the very wide applicability of the classifiers (including theoretical and technical aspects), and the previous in-detail knowledge of the system and of the dataset. The meta-heuristic algorithms had to be extended to work with multiple populations without the ability of migration in order to handle the special features of the problem.

The paper consists of five sections; after the introduction in Sect. 2, the studied meta-heuristic methods are summarized including the modifications made to them done in order to fit the sample problem. It is followed by the details of the optimization task (the recognizer engine), the used/investigated parameters and other aspects of the experiment. The results of the study are presented and interpreted from various aspects in Sect. 4. Section 5 summarizes the results of the presented work and discusses the possible directions for a future research.

2 Meta-Heuristic Methods Studied

2.1 Bacterial Evolutionary Algorithm

The Bacterial Evolutionary Algorithm (BEA) [2] is inspired by the evolutional processes of bacteria. Each bacterium in the population represents a solution in the problem space.

The algorithm uses two main evolutionary operators the bacterial mutation and the gene transfer; the first step of the algorithm is the bacterial mutation. Each bacterium is selected individually and cloned a maximal number of times. Each randomly selected allele of the clones is modified randomly, the modified allele of the alternative bacterium (or the original one's) together with the best result is copied to all other clones; this step is repeated until all the alleles have been selected.

The second step of the algorithm is the gene transfer (infection), in which the population is sorted by the goodness of the bacteria and divided into two subsets; the set of "good" and the set of "bad" bacteria. A randomly selected allele of a random bacterium from the group of good bacteria is copied to a randomly selected bad bacterium. This step is repeated until the algorithm reaches the maximum number of infections.

The above steps of the algorithm are repeated until the maximum number of generations specified previously has been reached or until other termination conditions (like 100 % result) have been satisfied (as seen on Fig. 1).

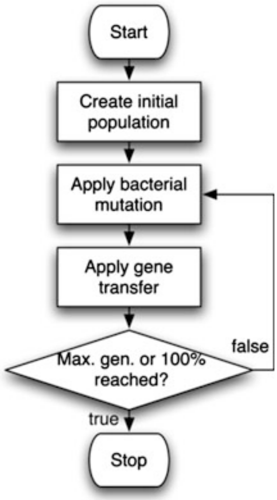

Fig. 1 Flowchart of the bacterial evolutionary algorithm

2.2 Big Bang–Big Crunch Algorithm

The Big Bang-Big Crunch optimization algorithm [3] uses the concept of a physical cosmology theory. In this theory the universe expands during the Big Bang event and then it collapses (Big Crunch) into a black hole (repeatedly). The candidate solutions are points in the search space; these entities are randomly generated in the (complete) problem space during the initialization step. In the beginning there could be a great number of points, which will decrease in the next generations of the universe.

The algorithm can be divided into two main parts, one is the Big Bang phase and the other is the Big Crunch phase. These steps are repeated until one of the termination conditions is satisfied.

The points weighted by their fitness are used to determine the center of gravity for the next generation of the universe. In the next phase the search space is narrowed down around this new center. One of the greatest advantages of this algorithm is the fast decrease of the search space, where the dimensions are handled individually; in the other hand this property of the method causes its main disadvantage: it often converges to a local optimum instead of the global optimum.

2.3 Imperialist Competitive Algorithm

The Imperialist Competitive Algorithm (ICA) [4] is an optimization algorithm inspired by imperialistic competition, where the points (candidate solutions) in the problem space are the countries. Initial countries are generated randomly over the problem space and their strength is calculated by the cost function. The countries with greater strength are the imperialists, while the weaker solutions are the colonies; empires are formed by imperialists taking control over colonies.

The algorithm uses two main operators in the first part: the assimilation and the revolution; during assimilation the colonies are approaching the imperialist country (in the problem space), while in the revolution phase the position of some colonies in the problem space are changed. Colonies may turn over the imperialists by reaching a better position during their movements in the search space caused by the previously described operators.

Imperialist competition is the second part of the algorithm. The colonies could be taken from the weakest empire by the stronger ones; the goal of each empire to eliminate others by taking over them. The power of an empire is calculated from the aggregated strength of the imperialist and the colonies. The method to calculate the power of an empire must ensure that its power will increase even if it takes over the weakest colony, in other words an empire cannot increase its power by losing its weakest colony. The above steps are repeated until the stop condition is not satisfied as in Fig. 2.

2.4 Particle Swarm Optimization

The Particle Swarm Optimization (PSO) [10] uses the simplified model of the dynamics of movements of various animal swarms (or particles). The solutions are represented by the particles in the search domain; each particle has a position and a speed vector. The evolution of the population does not use evolutionary operators unlike in genetic algorithms [1].

The orientation and the speed of the particles are influenced by all other particles. An individual particle moves towards the particle with the best local- or global solution and is influenced by its personal best position.

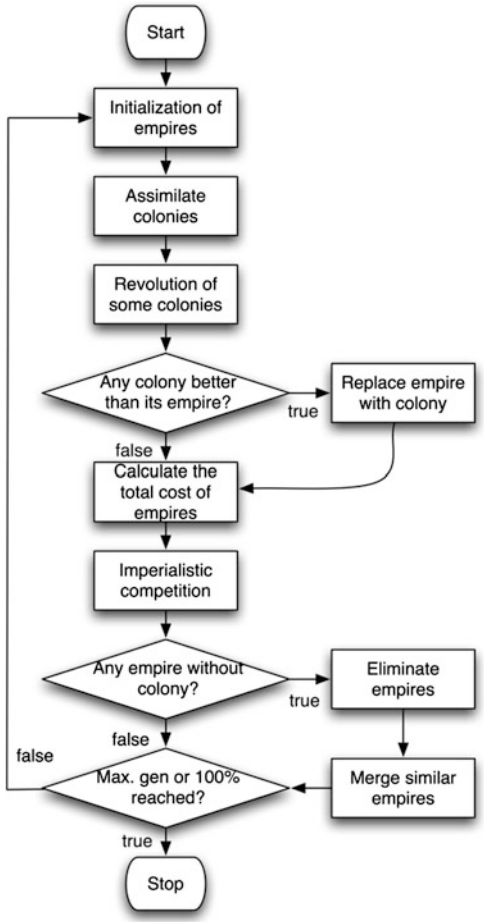

Fig. 2 Flowchart of the imperialist competitive algorithm

3 Optimization Task and Experiment Properties

3.1 Optimization Task

Basic Concept, Properties and Limitations.
Four key features were defined at the beginning of the development of the recognition engine:

1. Accuracy: it has to reach an acceptable recognition rate, i.e. at least as good as, or better than other accepted methods.
2. Efficiency: the methods must fit the user's requirements in response time and in resources of hardware (complex geometrical transformations and other mathematical functions should be avoided).

3. Flexibility of the alphabet: the alphabet must be easily modifiable to support various alphabets and context-sensitive recognition.
4. Learning: it should be able to learn user-specific writing style.

The general characteristics and properties of fuzzy systems enable them to satisfy all the features considered above. This fact led us to use fuzzy inference method for the recognition method. Fuzzy-Based Character Recognizer (FUBAR) is a family of algorithms of various single-stroke and multi-stroke hand printed (handwritten, non-cursive, capital letters) character recognition engines. The designed system is a personalized online recognizer, which means it processes digital ink and deploys user-specific information. The basic concept of the designed method is shown in Fig. 3.

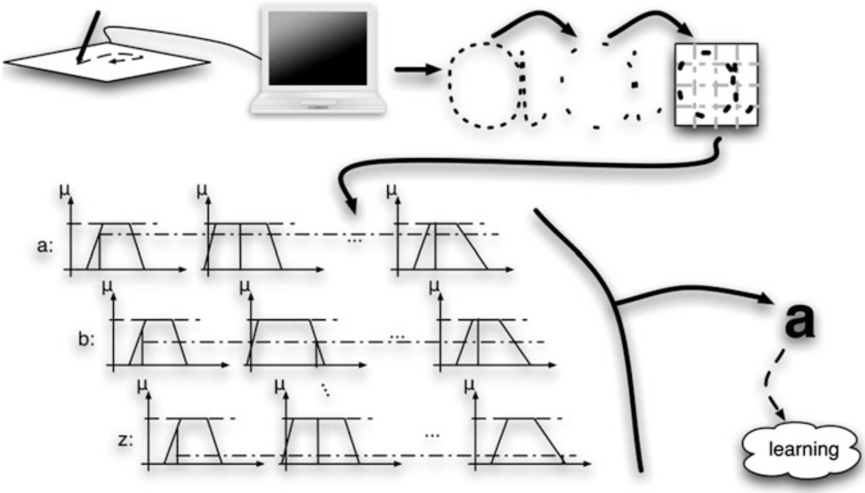

Fig. 3 Concept of the FUBAR engine

Input Conditioning and Handling.
The input signal of the algorithm consists of two-dimensional (x, y) coordinates in chronological order, representing the pen-movement (stroke). In unistroke (or single-stroke) recognizers, letters are represented by a single stroke; while in multi-stroke systems each symbol is represented by any numbers of strokes (sub-strokes). The FUBAR algorithm merges the multi-strokes into one unistroke and handles it accordingly.

Usually, the received signal is non-continuous as a result of the bottlenecks of the hardware, which causes information loss during recording the pen-movement; this information-loss causes difficulties in the processing, because the positions of the missing coordinates thus become non-deterministic. The received signal must be normalized for further processing and better recognition rate. In the FUBAR algorithm family the points of the received signal are re-sampled; the points between a given (Euclidean) distance from the reference point are filtered out. The re-sampling of the strokes also has an anti-aliasing property.

Feature Extraction.

FUBAR uses two kinds of stroke features for the recognition: (1) the width/height ratio of the stroke and (2) the average number of points in the rows and columns of the grid drawn around the stroke. The first member of the FUBAR family used a crisp grid (with sharp borders) for the feature extraction, but the system reached a low average recognition rate as some of the users started to write faster and use italic writing style The sampled points of the strokes of oblique and normal characters could be located in completely different rows and columns of the grid, which caused huge overlap between the features of various letters. Other methods are rotating the input characters to avoid the negative effects of the italic writing style, but those methods use complex mathematical transformations, which dramatically increase the computational complexity of the method. To resolve the problems caused by the italic writing style, fuzzy grids [18, 19] were proposed. In fuzzy grids the rows and columns of the grid are defined by fuzzy sets. It can be also considered as a transformation of the stroke into a fuzzy space. The points in a fuzzy grid may belong to two different columns or rows at the same time with various membership values as seen in Fig. 4.

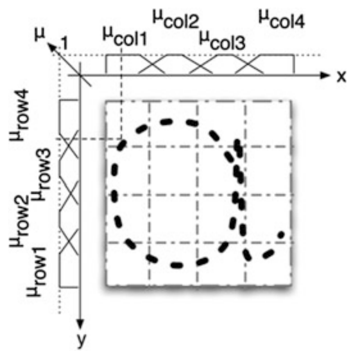

Fig. 4 Concept of fuzzy grids

Inference.

In the designed recognition engine a fuzzy rule-based inference method [7, 8] is used. Each symbol in the alphabet is represented by a single rule. The input parameters of the rules are the features described above; the output parameter of the rules is the *degree of matching* between the features of the input stroke and the stored rules as seen in Fig. 5. FUBAR returns the character associated to the best matching rule after the rule evaluation phase.

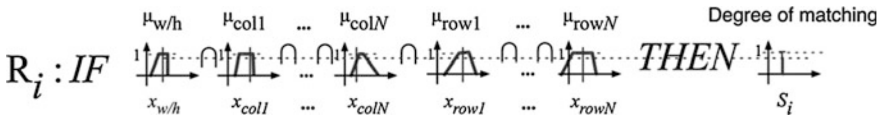

Fig. 5 Fuzzy rule describing a character

3.2 Extensions of Meta-Heuristic Methods

Each algorithm is extended to use multiple populations without the option of migration. This is an important modification in order to handle the various characters independently and to avoid the overlap between the fuzzy sets representing the features of various characters. During the evaluation of candidate solutions the best rules are used from each population.

The initial populations are randomly generated in the original algorithms; in order to switch between the meta-heuristics, the algorithms must support to use predetermined populations.

4 Results

4.1 Rule Optimization from Scratch

Bacterial Evolutionary Algorithm.

In this experiment various population sizes, number of clones and number of gene transfers (infections) were used. Tests were performed with the combination of each parameter with values between 10 and 30 (increased by 5 in each different test). The fourth parameter was affecting the bacterial mutation operator. If the mutation parameter was set to "tolerant", it accepted a new allele when the system had the same or better result as with the original allele value; otherwise ("strict") it accepted only the mutation from clones with better results. The maximum number of generations was set to 100.

The results reflected that the algorithm reached the same results (or with insignificant difference) for various parameter values, except for the mutation parameter. If it set to "strict", the algorithm was stuck at the same point during the process.

The best result (0.44 error rate) was achieved when the size of the population, the number of clones and the number of infections both were set to 10 and the mutation type was set to "tolerant". The average error rates/generations are shown in Fig. 6 for the training dataset and in Fig. 7 for the validation set.

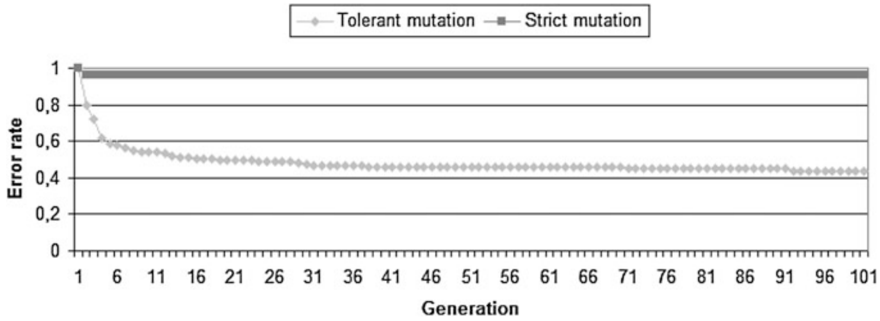

Fig. 6 Error rates of BEA from flat sets on the training data

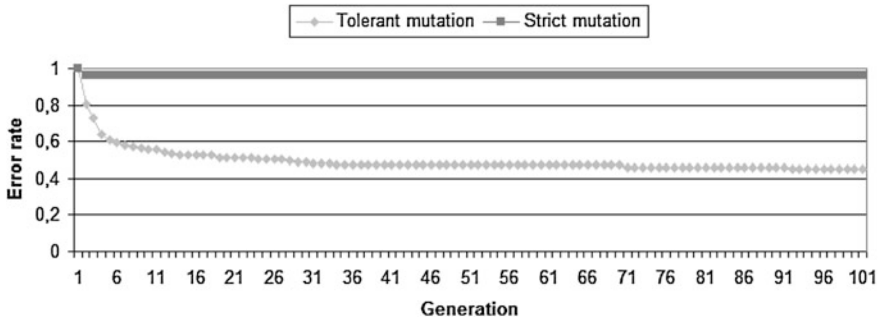

Fig. 7 Error rates of BEA from flat sets on the validation data

Big Bang–Big Crunch Algorithm.

The effect of the number of generations and the size of the population were investigated in this algorithm. The number of generations was changed between 50 and 100, while the size of the population was selected between 10 and 50 (step size was 10 in both parameters). The results showed that the change of these parameters does not significantly affect the results; the distribution of the error rate was the same for each scenario, however there was a slightly greater chance to find a better quasy-optimum after more generations. The best result (0.11 error rate) was achieved in 70 generations and population size 10. The best, the worst and the average convergence of the error rates/generations for the training dataset is in Fig. 8 and for the validation data is in Fig. 9.

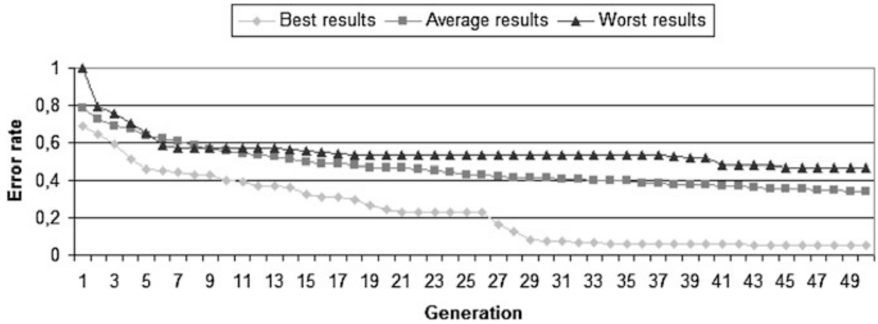

Fig. 8 Error rates of BBBC from flat sets on the training data

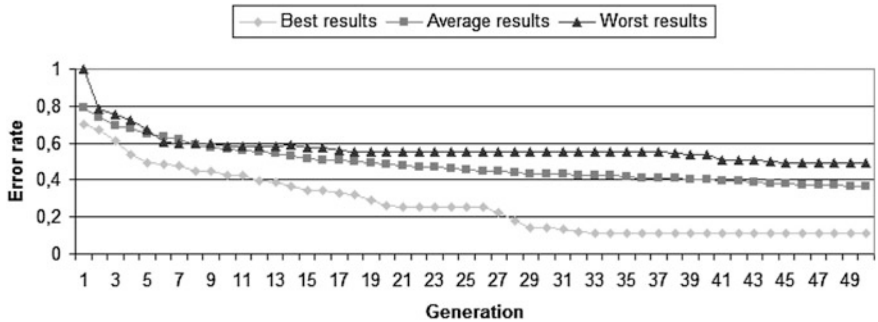

Fig. 9 Error rates of BBBC from flat sets on the validation data

The average error rates are more close to the worst case scenario compared to the best results as it can be seen in the figures above.

Imperialist Competitive Algorithm.

At the testing of the ICA algorithm the number of countries was changed between 10 and 30 and the number of generations was between 50 and 100 (the step was 10 in both cases). The results indicated that the number of generations over 50 did not have any effect on the results, while the greater number of the countries did result in lower error rates. The best result (0.04 error rate) was achieved in 50 generations and the number of countries was 30. Worst, average and best error rates/generations for the validation set are shown in Fig. 10 (with 10 countries), in Fig. 11 (with 20 countries) and in Fig. 12 (with 30 countries).

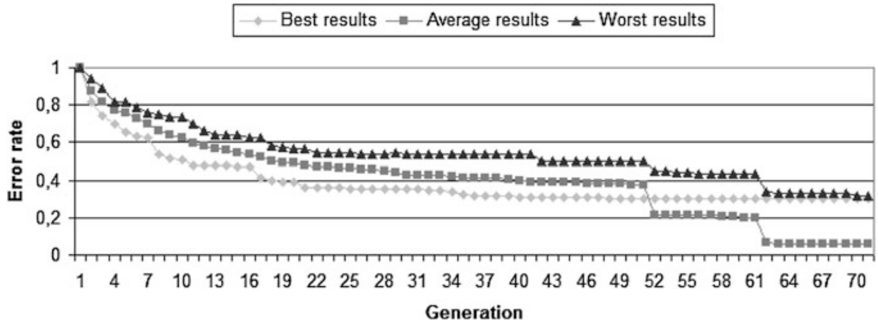

Fig. 10 Error rates of ICA from flat sets on the validation data with 10 countries

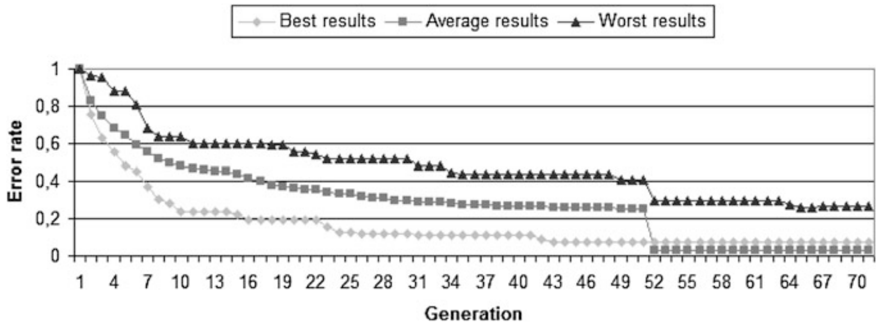

Fig. 11 Error rates of ICA from flat sets on the validation data with 20 countries

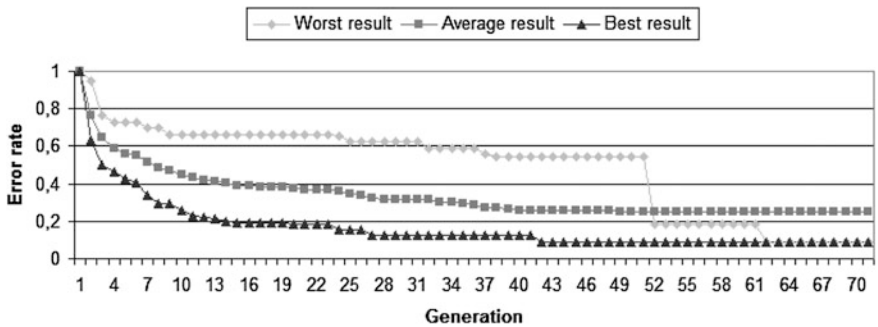

Fig. 12 Error rates of ICA from flat sets on the validation data with 30 countries

Particle Swarm Algorithm.

The number of generations was between 50 and 100, while the size of the swarm was between 10 and 30 during the test of the Particle Swarm method. The best result (0.32 error rate) was achieved with swarm size of 20 and in 60 generations. The results are indicating that, the larger swarm size slightly increases the probability of a better result. Worst, best and average error rates/generations for validation set are shown in the figures below (Figs. 13, 14 and 15).

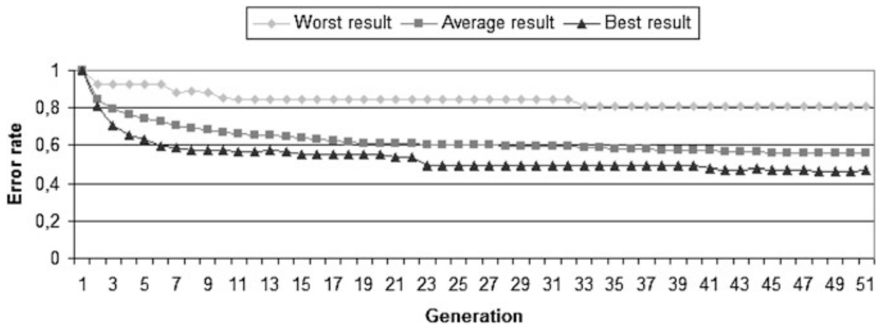

Fig. 13 Error rates of PSA from flat sets on the validation data with 10 particles

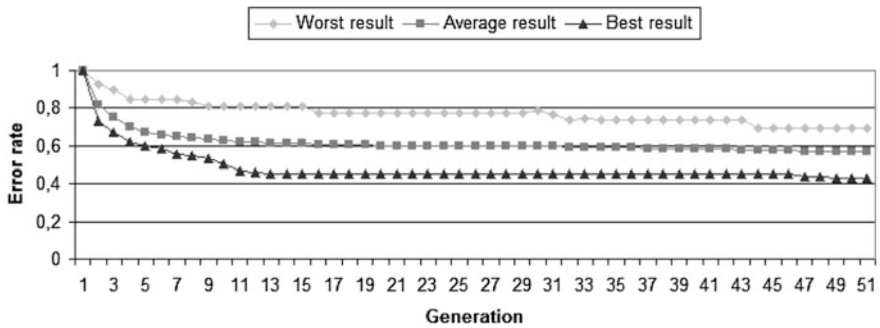

Fig. 14 Error rates of PSA from flat sets on the validation data with 20 particles

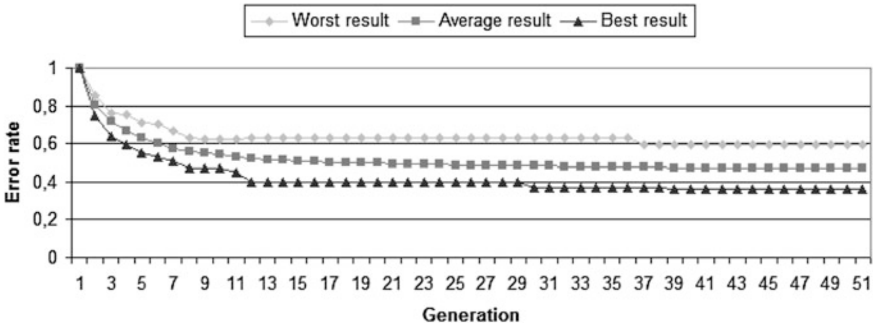

Fig. 15 Error rates of PSA from flat sets on the validation data with 30 particles

4.2 Rule Optimization with Multi Meta-Heuristics

In this experiment a previous test population was used as a starting point for each algorithm; this initial population was selected from a BBBC algorithm experiment. At generation 40 and at error rate of 0.12 the population was backed up and later loaded into each algorithm several times.

The lowest error rate in the worst cases was achieved by the ICA (0.09), while BEA, PSA and BBBC reached the error rate of 0.12, 0.13 and 0.13 respectively (Fig. 16).

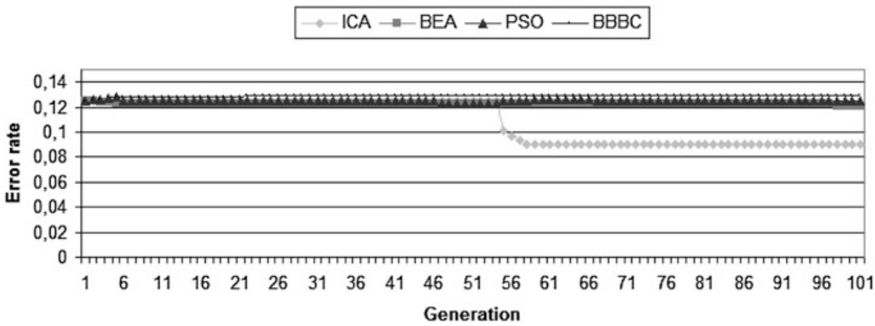

Fig. 16 Worst error rates of BEA, BBBC, ICA, PSA

The lowest error rate in the best scenarios was achieved by the PSA (0.048), but the BBBC and ICA algorithms were performing only slightly worse (0.05 error rate), while the BEA algorithm could not achieve better error rate than 0.078 (Fig. 17).

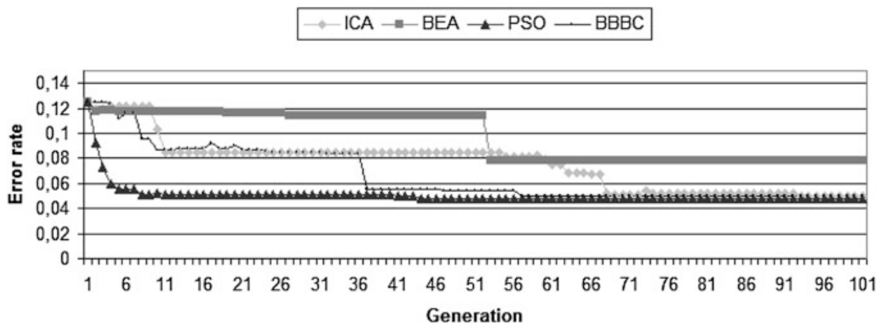

Fig. 17 Best error rates of BEA, BBBC, ICA, PSA

The best performing algorithm in average was the ICA with the error rate of 0.0778, the second one was the PSA with a slightly higher error rate 0.0779, while the BBBC reached 0.1 and the BEA produced the error rate of 0.114 (Fig. 18)

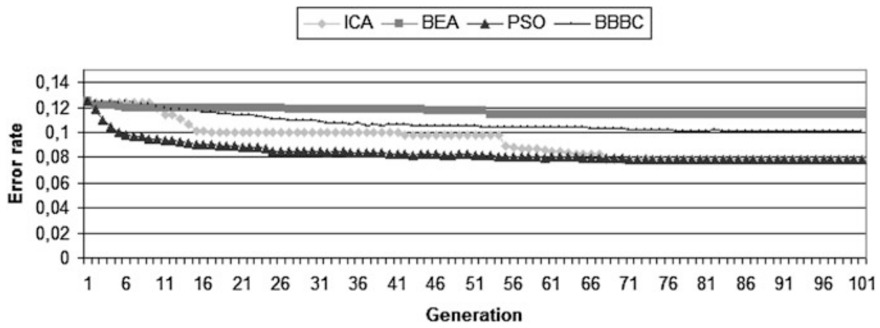

Fig. 18 Average error rates of BEA, BBBC, ICA, PSA

The Imperialist Competitive Algorithm and the Particle Swarm Algorithm performed best during the experiment in all scenarios, while the Big Bang–Big Crunch algorithm had the second worse results in average. The Bacterial Evolutionary Algorithm produced the worst results in all three scenarios, but all its results were close to each other.

5 Conclusions and Discussions

The aim of this research was to investigate the behavior of meta-heuristic algorithms applied on a fuzzy rule-based classifier (multi-stroke character recognizer) system.

The experiment suggests that the algorithms are not enough sensitive for their main parameters (population size, number of generations) to reach higher accuracy without a significant increase in the computational time. A more detailed test of other algorithm-specific parameters should be executed in order to find a low-cost way of enhancing their effectiveness.

The results are showing that the ICA algorithm could reach the lowest error rate (0.04), when the rule-base had to be created from scratch; the second best results (0.11) were achieved by the BBBC algorithm. The worst result was produced by the BEA with the error rate of 0.44, while the PSA could decrease the error rate to 0.32.

It is also important to consider that the average generations evaluated in one second for the BEA, BBBC, ICA and PSA were 0.0667, 4, 0.667 and 1 respectively (in the experiment environment). The ICA is 10 times faster than the BEA and it can reach much lower error rates. The PSA is 33 % faster and the BBBC is about 6 times faster than the ICA, while their optimization performances are very close. This means that it might be beneficial to combine these methods if we could switch between the algorithms according to the dynamics of the population, the properties of the problem and the algorithms.

The BBBC and PSA algorithms could be a good choice to start the optimization process, because it is able to reach average results with a very low cost, but due to its disadvantages (convergence to local optimum) it does not worth it to use it during the optimization. In some scenarios they could perform same or slightly better as the ICA, but it is more like a matter of random situations. BBBC and PSA should be modified to avoid local optimum solutions (i.e. “anomalies” in the BBBC algorithm which could move the center of the universe from these points or restore some parts of the universe) without significantly increasing their processing time.

The overall performance of ICA algorithm was the best, but the evaluation time of generations is significantly higher compared to BBBC and PSA. It might worth to use ICA instead of the other algorithms, but in some cases it does not perform that much better than BBBC and PSA, which would make it reasonable to use it (considering its processing time).

The general properties of the BEA algorithm made it “stable” and it has its own advantages despite its low results and high computational time. The algorithm should be improved by reducing its time consuming computations.

The next aim is to research a simple and automatic procedure to test and investigate the characteristics and other features of the population, where the presented algorithms are performing best (in terms of results and resource requirements). Using this knowledge a more extended alternative method for [BK] might be developed, which would be able to switch between more than two meta-heuristic optimization algorithms. The planned method would change to a new optimization algorithm if the properties of the “environment” and the population are indicating that another algorithm could be more successful (have better convergence or solution) in order to save resource.

Acknowledgments This paper is partially supported by the TÁMOP-4.2.2.A-11/1/KONV-2012-0012 and Hungarian Scientific Research Fund (OTKA) grants K105529, K108405.

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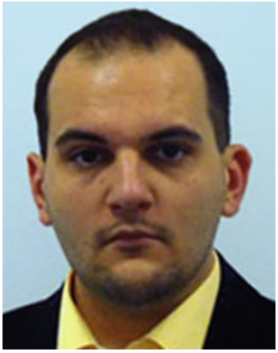
Authors Biography



Alex Tormási was born in Győr, Hungary in 1985; He received the B.S. and M.S. degrees in computer science engineering from the Széchenyi István University (SZE, Győr) in 2008 and 2010; later in 2010 started his Ph.D. studies at the Multidisciplinary Doctoral School of Engineering Sciences at SZE.

He was working on several software development projects as an entrepreneur between 2004 and 2013. However he has been a visiting lecturer from 2009 until 2011 at the Department of Mathematics and Computational Sciences and the Department of Automation at SZE. He started to work as a technical assistant at the Department of Information Technology at SZE from 2012, then as an assistant lecturer since 2013.

Mr. Tormási was a recipient of the Scholarship of the Hungarian Republic three times in academic years 2007/2008, 2008/2009 and 2009/2010 and was awarded with "Young Scientist Award" in 2013 by the International Fuzzy Systems Association (IFSA). He was an organizing member at the Fifth Győr Symposium and First Hungarian-Polish Joint Conference on Computational Intelligence in 2012 and at the Spring Wind 2014 (Engineering Sciences Section). He was member of the general chair at Széchenyi Doctoral Students' Conference in 2014 and member of the international program committee at the Sixth Győr Symposium and Third Hungarian-Polish and First Hungarian-Romanian Joint Conference on Computational Intelligence (ConfCI 2014). From 2013 until 2014 he was the head at the Section of the Engineering Sciences (formerly Engineering, Architectural and Earth Sciences) in the Association of Hungarian PhD and DLA Students (DOSz).

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He spent most of his career at BME until 2001 and from 2002 at Szechenyi Istvan University (Gyor, SZE). However, he has been a visiting professor at various universities abroad, namely in Australia (ANU and others), in Japan (TIT, where he was one of the LIFE Endowed Fuzzy Theory Chair Professors and at the same time an advisor to the Laboratory for International Fuzzy Engineering Research in Yokohama), in Korea (POSTECH), Austria, and Italy. His focus of research interest is fuzzy systems and Computational Intelligence topics (evolutionary algorithms, neural networks), as well as applications. He has published over 450 refereed papers and several text books on the subject. He did introduce the concept of rule interpolation in sparse fuzzy models, and hierarchical interpolative fuzzy systems, fuzzy Hough transform, and also fuzzy signatures and fuzzy situational maps among others. His research interests include applications of CI for telecommunication, transportation and logistics, vehicles and mobile robots, control, information retrieval, etc.

Prof. Koczy was an Associate Editor of IEEE TFS and is now of Fuzzy Sets and Systems, International Journal of Fuzzy Systems, Journal of Advanced Computational Intelligence, International Journal of of fuzzy Systems, etc. He was the General Chair of FUZZ-IEEE 2004 in Budapest, Special Session Chair of WCCI 2012 and Panel Chair of FUZZ-IEEE 2013, will be the

Poster Chair of WCCI 2016; was a chair, co-chair, PC member, etc. at many other scientific events. He served in the International Fuzzy Systems Association as President, and he was Administrative Committee member of IEEE Computational Intelligence Society for two cycles and he was another two cycles a member of the AdCom of the IEEE Systems Council. At SZE he is president of the University Doctoral Council and of the University Research Council, he has been appointed member of the Hungarian Accreditation Committee for Higher Education by the Prime Minister, where he chairs the Engineering Sub-Committee, he is a member of the National Doctoral Council (Hungary), of the National Fellowship Committee and of the National Advisory Board for Industrial Safety.

Additive Fuzzy Systems as Generalized Probability Mixture Models

Bart Kosko

Abstract Additive fuzzy systems generalize the popular mixture-density models of machine learning. Additive fuzzy systems map inputs to outputs by summing fired then-parts sets and then taking the centroid of the sum. This additive structure produces a simple convex structure: Outputs are convex combinations of the centroids of the fired then-part sets. Additive systems are uniform function approximators and admit simple learning laws that grow and tune rules from sample data. They also behave as conditional expectations with conditional variances and other higher moment that describe their uncertainty. But they suffer from exponential rule explosion in high dimensions. Extending finite-rule additive systems to fuzzy systems with continuum-many rules overcomes the problem of rule explosion if a higher-level mixture structure acts as a system of tunable meta-rules. Monte Carlo sampling can then compute fuzzy-system outputs.

Keywords Additive fuzzy system • Mixture density models • Compounding • Function approximation • Fuzzy approximation theorem • Learning laws • Conditional expectations • Convex sums • E-M algorithm • Monte carlo simulation • Importance sampling • Continuum-many fuzzy rules

1 Centroidal Fuzzy Systems as Statistical Estimators

This chapter reviews and extends the main mathematical properties of additive fuzzy systems [1–9]. Additive fuzzy systems exploit the convex-sum structure that results from additively combining fired if-then rules. They generalize mixture-density models from machine learning and pattern recognition because such mixtures are convex sums that do not depend on an input value. Additive fuzzy systems

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_14

admit simple gradient-descent learnings laws and are uniform function approximators on compact sets. But they suffer from exponential rule explosion in high dimensions. This curse of dimensionality limits modeling with additive fuzzy systems and limits tuning them with sample data. Extending finite-rule additive fuzzy systems to continuum-many fuzzy rules overcomes the problem of rule explosion. It allows the user to define higher-level fuzzy meta-rules with a mixture structure and then use modern statistical techniques to tune such continuum-rule additive systems. This analysis turns on a probabilistic interpretation of fuzzy systems. This first section shows that such a probabilistic interpretation applies to all centroidal fuzzy systems even if they do not additively combine fired rules.

A fuzzy system is a mapping $F: \mathbb{R}^n \to \mathbb{R}$. It uses a set of fuzzy if-then rules to convert a vector input x to an output $F(x)$. There is no loss of generality if the fuzzy system is scalar and thus if it maps to the real line $\mathbb{R}$. All results still hold with appropriate vector notation for vector-valued fuzzy systems $F: \mathbb{R}^n \to \mathbb{R}^p$.

We first show that *any* centroidal fuzzy system defines a conditional expectation and hence is a probabilistic or statistical system. The fuzzy system need not be additive. A non-additive system could combine rules through a maximum operation or through any other aggregation operation [10–13]. Early fuzzy systems often combined outputs with a maximum or supremum operation.

A centroidal output suffices to produce a conditional expectation. So the conditional-expectation result does not require an independent probabilistic assumption. It follows instead from just the nonnegativity and the integrability of the then-part fuzzy sets that all fuzzy if-then rules use. We first state some notation for fuzzy systems and then state and prove the conditional-expectation result as Theorem 1.

A centroidal fuzzy system $F: \mathbb{R}^n \to \mathbb{R}$ is a fuzzy system that computes the output $F(x)$ by taking the centroid of a finite number m of combined "fired" then-part sets: $F(x) = Centroid(B(x))$. Later we will drop the finite assumption. The term $B(x)$ stands for the combined fired then-parts. The argument x implies that the vector input x has fired the m rules. The fired combination $B(x)$ formally is any nonnegative function $b: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ that has a finite integral. The jth rule $R_{A_j \to B_j}$ has the linguistic form "If $X = A_j$ then $Y = B_j$" for if-part fuzzy set $A_j \subset \mathbb{R}^n$ and scalar then-part fuzzy set $B_j \subset \mathbb{R}$. The unfired then-part set B_j has set function $b_j: \mathbb{R} \to [0,\ 1]$. But its fired version $B_j(x)$ has a two-place argument and thus corresponds to the set function $b_j(x, y): \mathbb{R}^n \times \mathbb{R} \to [0,\ 1]$ for vector input $x \in \mathbb{R}^n$. But we still write the set function in single-argument notation $b_j(x)$ for simplicity. The rule $R_{A_j \to B_j}$ is a fuzzy subset of the input-output product space $\mathbb{R}^n \times \mathbb{R}$ because all input-output pairs (x, y) satisfy the rule to some degree. So the rule corresponds to a two-valued set function $r_{A_j \to B_j}:\ \mathbb{R}^n \times \mathbb{R} \to [0,\ 1]$.

The n-dimensional fuzzy set A_j corresponds to a joint set membership or multivalued indicator function $a_j: \mathbb{R}^n \to [0,\ 1]$. Users often assume in practice that the joint membership function factors into a product of scalar membership functions: $a_j(x) = \prod_{k=1}^{m} a_j^k(x^k)$ where each factor set $A_j^k \subset \mathbb{R}$ has set function $a_j^k: \mathbb{R} \to [0,\ 1]$ for row vector $x = (x^1, \ldots, x^n)$. Earlier fuzzy systems sometimes formed the joint set function a_j by taking pairwise minima $a_j(x) = \min(a_j^1(x^1), \ldots, a_j^k(x^n))$ or some other pairwise triangular-norm operation. The minimum function ignores the information in all scalar inputs except the smallest one when the inputs differ. The standard additive fuzzy systems below always work with the simpler product factorization. The product function preserves the relative values of the scalar inputs. The then-part set function can be generalized set function. The then-part fuzzy sets B_j need only have positive and integral set functions $b_j\colon \mathbb{R} \to \mathbb{R}^+$ because of the normalization involved in taking the centroid. They do not need to map to the unit interval.

Now suppose the vector input $x = (x^1, \ldots, x^n)$ activates the scalar fuzzy system $F\colon \mathbb{R}^n \to \mathbb{R}$ to produce the combined rule firings $B(x)$. Then Theorem 1 states that taking the centroid results in a conditional expectation for *any* fuzzy system that combines rules to produce $B(x)$.

Theorem 1 *Every centroidal fuzzy system is a conditional expectation*: $F(x) = E[Y \mid X = x]$.

Proof The theorem follows from the definition of the centroid and from the non-negativity and integrability of the then-part sets B_j. We also assume that the input x leads to nontrivial rule firings. So it leads to a nonzero combination of fired rules $B(x)$: $B(x) > 0$. Then □

$$F(x) = Centroid(B(x)) \tag{1}$$

$$= \frac{\int_{-\infty}^{\infty} y\, b(x)\, dy}{\int_{-\infty}^{\infty} b(x)\, dy} \tag{2}$$

$$= \frac{\int_{-\infty}^{\infty} y\, b(x,y)\, dy}{\int_{-\infty}^{\infty} b(x,y)\, dy} \tag{3}$$

$$= \int_{-\infty}^{\infty} y \left[\frac{b(x,y)}{\int_{-\infty}^{\infty} b(x,y)\, dy} \right] dy \tag{4}$$

$$= \int_{-\infty}^{\infty} y\, p(y \mid x)\, dy \tag{5}$$

$$= E[Y \mid X = x]. \tag{6}$$

The result follows because $p(y \mid x) = \dfrac{b(x,y)}{\int_{-\infty}^{\infty} b(x,y)\, dy}$ is nonnegative and because $\int_{-\infty}^{\infty} p(y \mid x)\, dy = 1$ holds from the nonnegativity and integrability of the b function if $b(x,y) > 0$. So $p(y \mid x)$ is a proper conditional probability density function. Then $E[Y \mid X = x]$ is a realization of the condition-expectation random variable $E[Y \mid X]$. Q.E.D.

Theorem 1 yields a system conditional variance $V[Y|X=x]$ and higher-order moments so long as the appropriate integrals exist:

$$V[Y|X=x] = E[Y^2|X=x] - E^2[Y|X=x]. \tag{7}$$

The conditional density $p(y\,|\,x)$ gives the second moment as

$$E[Y^2\,|\,X=x] = \int_{-\infty}^{\infty} y^2\, p(y\,|\,x)\, dy. \tag{8}$$

The next section gives closed forms for these moments and for other higher-order moments when the fuzzy systems are not just centroidal but additive.

A word is in order here about just how a given input x_0 fires a rule $R_{A_j \to B_j}$. Fuzzy models assume that the input x_0 belongs to the if-part set A_j to degree $a_j(x_0)$: $a_j(x_0) = Degree(x_0 \in A_j)$. Then this membership or "fit" (fuzzy unit) value $a_j(x_0)$ changes the corresponding then-part B_j to produce the fired then-part set $B_j(x_0)$ with set function value $b_j(x_0, y)$. Viewing the if-part set A_j as a probability density function would give the null result $a_j(x_0) = 0$ for all x_0 since a_j is continuous. We instead view the input x_0 as a *delta pulse* $\delta(x - x_0)$ centered at x_0. Then convolution gives the proper fit value $a_j(x_0)$ for the fired if-part set [1, 5]:

$$\int_{-\infty}^{\infty} \delta(x - x_0)\; a_j(x)\; dx = a_j(x_0). \tag{9}$$

This convolution result follows from the "sifting" property of the delta function. It also extends the *point* fuzzy system to a *set* fuzzy system that takes an arbitrary continuous fuzzy set A as input if we define the corresponding activation in terms of a more general inner product:

$$\int_{-\infty}^{\infty} a(x)\; a_j(x)\; dx = a_j(A). \tag{10}$$

Then the proof of Theorem 1 still gives the system output as $F(A) = E[Y\,|\,X=A]$. All the standard-additive results below admit such a set-input extension.

2 Additive Fuzzy Systems as Convex Combinations of Centroids

Additive fuzzy systems add fired then-part sets to compute the combined set $B(x)$. This leads to the central fact of additive systems: Their outputs equal the *convex combination* of the centroids of the fired then-part sets.

This convex structure carries over into many properties of centroidal additive systems. It starts with the basic "mixture" fact in Proposition 1 below that the global output conditional probability density $p(y\,|\,x)$ is itself a convex combination of the m then-part conditional probabilities. This convex structure leads in particular to simple and practical forms for the conditional expectation and conditional variance. Theorem 3 below shows that the global conditional mean (6) equals a convex sum of local conditional means. These conditional means are local or rule-specific in the sense that their conditional probability density arises from the shape of their corresponding fired then-part set.

Theorem 3 also shows that the conditional variance decomposes into two convex sums. The first sum averages the uncertainty that arises from the shapes of the then-part sets. So here set shape matters. This result differs from simple function approximation where only the shape of the if-part sets controls the approximation for default then-part sets. This then-part shape dependence contrasts with many fuzzy applications that simply replace the then-part sets with spikes centered at what would otherwise be a then-part set's centroid. Such a then-part spike simplifies some computations but it implicitly assumes total certainly about the then-part of the rule. The second sum averages the uncertainty that arises from interpolating between rule centroids to produce the system output. This term measures the inherent uncertainty in the fuzzy system that results from such interpolation. The other higher-order conditional moments in Theorem 3 involve similar convex sums and interpolations.

We first prove that all additive centroidal fuzzy systems are convex sums of fired then-part centroids. An additive fuzzy system combines the m fired then part sets by adding them:

$$B(x) = \textstyle\sum_{j=1}^{m} w_j\, B_j(x) \tag{11}$$

for positive rule weights $w_j>0$. The rule weights need not sum to unity. And they can depend on the input x. They drop out of the centroidal output $F(x)$ if they are all equal: $w_1 = \cdots = w_n$. Then the combined set $B(x)$ has a generalized set function $b(y\,|\,x)$ for each input x as y ranges over the range space $\mathbb{R}$:

$$b(y\,|\,x) = \textstyle\sum_{j=1}^{m} w_j\, b_j(y\,|\,x). \tag{12}$$

We here use the conditional notation $b_j(y\,|\,x)$: $\mathbb{R}^n \times \mathbb{R} \to [0,\ 1]$ for the set function of the fired then-part set $B_j(x)$. So the inputs x parametrize the fired then-part sets.

Each fired then-part set $B_j(x)$ has an area or volume $V_j(x)$:

$$V_j(x) = \int_{-\infty}^{\infty} b_j(y\,|\,x)\, dy. \tag{13}$$

We again assume that all such integrals are finite and positive. This gives in turn an input-dependent centroid $c_j(x)$ for fired then-part set $B_j(x)$:

$$c_j(x) = \frac{\int_{-\infty}^{\infty} y\, b_j(y\,|\,x)\; dy}{\int_{-\infty}^{\infty} b_j(y\,|\,x)\; dy} = \frac{1}{V_j(x)} \int_{-\infty}^{\infty} b_j(x, y)\; dy. \tag{14}$$

Then Theorem 2 states that all additive centroidal fuzzy systems equal a convex combination of fired then-part centroids.

Theorem 2 *Additive centroidal systems are convex combinations of fired then-part centroids*:

$$F(x) = \textstyle\sum_{j=1}^{m} p_j(x)\; c_j(x). \tag{15}$$

The convex coefficients $p_j(x)$ have the ratio form

$$p_j(x) = \frac{w_j\; V_j(x)}{\sum_{k=1}^{m} w_k\; V_k(x)}. \tag{16}$$

Proof.

$$F(x) = Centroid(B(x)) = \frac{\int_{-\infty}^{\infty} y\, b(y\,|\,x)\; dy}{\int_{-\infty}^{\infty} b(y\,|\,x)\; dy} \tag{17}$$

$$= \frac{\int_{-\infty}^{\infty} y\, \sum_{j=1}^{m} w_j\; b_j(y\,|\,x)\; dy}{\int_{-\infty}^{\infty} \sum_{k=1}^{m} b_j(y\,|\,x)\; dy} \tag{18}$$

$$= \frac{\sum_{j=1}^{m} w_j \int_{-\infty}^{\infty} y\, b_j(y\,|\,x)\; dy}{\sum_{k=1}^{m} w_k \int_{-\infty}^{\infty} b_j(y\,|\,x)\; dy} \tag{19}$$

$$= \frac{\sum_{j=1}^{m} w_j\; V_j(x) \left[\frac{\int_{-\infty}^{\infty} y\, b_j(y\,|\,x)\; dy}{V_j(x)}\right]}{\sum_{k=1}^{m} w_k\; V_k(x)} \tag{20}$$

$$= \frac{\sum_{j=1}^{m} w_j\; V_j(x)\; c_j(x)}{\sum_{k=1}^{m} w_k\; V_k(x)} \tag{21}$$

$$= \textstyle\sum_{j=1}^{m} \left[\frac{w_j\; V_j(x)}{\sum_{k=1}^{m} w_k\; V_k(x)}\right] c_j(x) \tag{22}$$

$$= \textstyle\sum_{j=1}^{m} p_j(x)\; c_j(x). \tag{23}$$

The coefficients $p_j(x)$ are convex because they are nonnegative and sum to unity by (16). **Q.E.D.** □

The convex-sum structure of Theorem 2 underlies much of the power of additive fuzzy systems. An important example is universal function approximation.

Centroidal additive fuzzy systems F can *uniformly* approximate any continuous function $f : K \subset \mathbb{R}^n \to \mathbb{R}$ on a compact set K [2, 5]: $|F(x) - f(x)| < \varepsilon$ for *all* x given any initial choice of the error level $\varepsilon > 0$. The heart of the proof in the scalar case is that convexity traps the output $F(x)$ between the smallest and largest centroids: $c_{\min}(x) \le F(x) \le c_{\max}(x)$. A similar result holds component-wise in higher dimensions because the vector output $F(x)$ lies in a centroidal hyper-rectangle. So in principle we can always find some set of fuzzy rules that makes an additive fuzzy system as close as we wish to any continuous function. The uniform approximation does not require either that the if-part sets be fuzzy or that the then-part sets be fuzzy. Binary rectangles still produce uniform function approximators even if the resulting fuzzy systems are not as smooth as fuzzy systems with sets based on Gaussian or Cauchy or other smooth functions. So a fuzzy system need not be fuzzy at all. The raw approximation power comes not from working with fuzzy matters of degree. It comes instead from the additive system's use of parallel if-then rules and its convexity.

A second example of this convex structure occurs in rule adaptation or learning. The ratio structure in (21) allows a direct application of the quotient rule of the differential calculus. Convexity leads to a simple form for the learning gradient term $\frac{\partial F}{\partial w_j}$ for the rule weight w_j [5, 8]: $\frac{\partial F}{\partial w_j} = \frac{p_j(x)}{w_j}\left[c_j(x) - F(x)\right]$. This leads to the squared-error-based learning law $w_j(t+1) = w_j(t) + \mu_t \varepsilon(t) \frac{p_j(x)}{w_j(t)}\left[c_j - F(x)\right]$ where the supervised error term $\varepsilon(t) = d(t) - F(x(t))$ requires knowledge of the system's desired outcome $d(t)$ at a given time instant. The learning coefficients μ_t usually decrease linearly in accord with convergence principles from stochastic approximation. The same form of learning law holds for the then-part volume $V_j(x)$ and the if-part set function a_j. But there are two more partial derivatives to unpack in the if-part set-function case because the set function factors and because each factor depends on shape parameters such as the location and scale of the scalar factor set. The centroid has an even simpler learning term: $\frac{\partial F}{\partial c_j} = p_j(x)$ because of the convex structure. Then gradient learning can use sample data to tune these system parameters. The simplest case minimizes the squared error $\frac{1}{2}(f(x) - F(x))^2$ for some known or unknown sample function f. Then the fuzzy system will quickly approximate the sampled function given enough representative samples and enough training iterations. The rules quickly move to cover the extrema or turning points of f. This reflects the theorem that optimal additive rules cover extrema [3]. Unsupervised clustering algorithms can help find these optimal rules in practice [1]. Such data-driven clusters area also a good way to initialize the fuzzy system. But the number of rules involved tends to grow exponentially with the input dimension because the fuzzy rules define a graph cover in the input-output product space. This curse of dimensionality tightly constrains the above learning laws when the fuzzy system has as few as three input dimensions [8].

The next theorem shows how the convex structure in Theorem 2 passes into the structure of the additive system's conditional expectation and conditional variance and indeed into all its higher conditional moments. The key insight is that additivity induces convexity in the global conditional probability density function $p(y\,|\,x)$. This density decomposes into a convex sum of the m local rule-specific conditional probability density functions $p_{B_j}(y\,|\,x)$:

$$p_{B_j}(y\,|\,x) = \frac{b_j(y\,|\,x)}{\int_{-\infty}^{\infty} b_j(y\,|\,x)\,dy}. \tag{24}$$

The normalizing denominator is just the input-dependent area or volume $V_j(x)$ from (13). This "mixture" result is important enough to state as a separate proposition.

Proposition 1 $p(y\,|\,x) = \sum_{j=1}^{m} p_j(x)\; p_{B_j}(y\,|\,x)$ for $p_j(x)$ in (16).

Proof.

$$p(y\,|\,x) = \frac{b(y\,|\,x)}{\int_{-\infty}^{\infty} b(y\,|\,x)\,dy} \tag{25}$$

$$= \frac{\sum_{j=1}^{m} w_j\; b_j(y\,|\,x)}{\int_{-\infty}^{\infty} \sum_{k=1}^{m} w_k\; b_j(y\,|\,x)\,dy} \tag{26}$$

$$= \frac{\sum_{j=1}^{m} w_j\; V_j(x)\; \left[\frac{b_j(y\,|\,x)}{V_j(x)}\right]}{\sum_{k=1}^{m} w_k\; V_k(x)} \tag{27}$$

$$= \sum_{j=1}^{m} \left[\frac{w_j\; V_j(x)}{\sum_{k=1}^{m} w_k\; V_k(x)}\right] p_{B_j}(y\,|\,x) \text{ from } (24) \tag{28}$$

$$= \sum_{j=1}^{m} p_j(x)\; p_{B_j}(y\,|\,x) \text{ from } (16).\;\; \mathbf{Q.E.D.} \tag{29}$$

We can now state and prove the key theorem on the conditional moments of all additive centroidal systems. □

Theorem 3 *All higher-order moments of additive centroidal fuzzy systems are convex sums*:

$$(a)\qquad E[Y\,|\,X=x] = \sum_{j=1}^{m} p_j(x)\; c_j(x) = F(x) \tag{30}$$

$$(b)\qquad V[Y\,|\,X=x] = \sum_{j=1}^{m} p_j(x)\; \sigma_{\mathbf{B_j}}^2(x) + \sum_{j=1}^{m} p_j(x)\; \left[c_j(x) - F(x)\right]^2 \tag{31}$$

$$(c)\qquad E[(Y - E[Y\,|\,X=x])^k | X=x] = \sum_{j=1}^{m} p_j(x)\; \sum_{l=1}^{k} \binom{k}{l}\; E_{B_j(x)}[(Y - c_j(x))^l](c_j(x) - F(x))^{k-l} \text{ for positive integer } k. \tag{32}$$

Proof. Result (a) restates Theorems 1 and 2. Result (b) uses Proposition 1:

$$V[Y\,|\,X=x] = \int_{-\infty}^{\infty} (y - E[Y\,|\,X=x])^2 p(y\,|\,x)\; dy \tag{33}$$

$$= \int_{-\infty}^{\infty} (y - E[Y\,|\,X=x])^2 \textstyle\sum_{j=1}^{m} p_j(x)\; p_{B_j}(y\,|\,x)\; dy \tag{34}$$

$$= \textstyle\sum_{j=1}^{m} p_j(x) \displaystyle\int_{-\infty}^{\infty} \left[(y - c_j(x)) + (c_j(x) - F(x))\right]^2 p_{B_j}(y\,|\,x)\; dy \tag{35}$$

from Theorem 1

$$\begin{aligned} &= \textstyle\sum_{j=1}^{m} p_j(x) \displaystyle\int_{-\infty}^{\infty} (y - c_j(x))^2 p_{B_j}(y\,|\,x)\; dy \\ &\quad + \textstyle\sum_{j=1}^{m} p_j(x)[c_j(x) - F(x)]^2 \displaystyle\int_{-\infty}^{\infty} p_{B_j}(y\,|\,x)\; dy \\ &\quad + 2\textstyle\sum_{j=1}^{m} p_j(x)(c_j(x) - F(x)) \displaystyle\int_{-\infty}^{\infty} (y - c_j(x)) p_{B_j}(y\,|\,x)\; dy \end{aligned} \tag{36}$$

$$= \textstyle\sum_{j=1}^{m} p_j(x)\; \sigma_{B_j}^2(x) + \textstyle\sum_{j=1}^{m} p_j(x)\; \left[c_j(x) - F(x)\right]^2 \tag{37}$$

because the cross term in (36) equals zero since $\int_{-\infty}^{\infty} y\, p_{B_j}(y\,|\,x)\; dy = c_j(x) = E_{B,j(x)}[Y]$ and because $\sigma_{B_j}^2(x) = \int_{-\infty}^{\infty} (y - E_{B_j(x)}[Y])^2 p_{B_j}(y\,|\,x)\; dy$.

The conditional higher-order moments (c) follow similarly from the binomial theorem $(p+q)^n = \sum_k^m \binom{n}{k} p^k q^{n-k}$ for real numbers p and q and the combinatorial coefficients $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. Then

$$E[(Y - E[Y\,|\,X=x])^k | X=x] = \int_{-\infty}^{\infty} (y - E[Y\,|\,X=x])^k \textstyle\sum_{j=1}^{m} p_j(x)\; p_{B_j}(y\,|\,x)\; dy \tag{38}$$

$$= \textstyle\sum_{j=1}^{m} p_j(x) \displaystyle\int_{-\infty}^{\infty} \left[(y - c_j(x)) + (c_j(x) - F(x))\right]^k p_{B_j}(y\,|\,x)\; dy \tag{39}$$

$$= \textstyle\sum_{j=1}^{m} p_j(x) \displaystyle\int_{-\infty}^{\infty} \textstyle\sum_l^k \binom{k}{l} (y - c_j(x))^k (c_j(x) - F(x))^{k-l} p_{B_j}(y\,|\,x)\; dy \tag{40}$$

$$= \textstyle\sum_{j=1}^{m} p_j(x) \sum_{l=0}^{k} \binom{k}{l} \left[\displaystyle\int_{-\infty}^{\infty} (y - c_j(x))^l p_{B_j}(y\,|\,x)\; dy\right] (c_j(x) - F(x))^{k-l} \tag{41}$$

$$= \sum_{j=1}^{m} p_j(x) \sum_{l=1}^{k} \binom{k}{l} E_{B_j(x)}[(Y - c_j(x))^l](c_j(x) - F(x))^{k-l}. \quad \textbf{Q.E.D.} \tag{42}$$

The penalty term $(c_j(x) - F(x))^k$ appears in the conditional variance and in all the conditional higher-order moments. It measures the uncertainty due to rule interpolation in the convex-sum output $F(x)$. The jth rule tries in effect to make the global output $F(x)$ look like its own centroidal output $c_j(x)$. The jth rule tends to have the most weight in the convex sum if the rule fires dead-on and thus if $p_j(x) \approx 1$ tends to hold. Then $c_j(x) \approx F(x)$ tends to hold and thus the interpolation penalty is small. This is easier to see with the simpler standard additive models in the next section. The quadratic penalty tends to be higher for rules whose if-parts fire only slightly. The penalty is most severe for inputs that occur where the if-part sets are sparse.

The conditional variance gives a natural measure of confidence in the fuzzy system output $F(x)$. There is no need to invoke Type-2 fuzzy sets or other ad hoc schemes to capture this second-order uncertainty of a given fuzzy output. Computing the conditional variance involves no more computation than what existing fuzzy models already use to compute the first-order output $F(x)$. So it has arguably been a needless oversight not to give users this confidence information. The first published plot of the conditional variance of a fuzzy system appeared as Fig. 4 in the 2005 paper [9]. The figure shows that rules involving one portion of the input space have substantially more confidence than the rest.

We conclude this section with two more extensions of the additive model. The first is the extension of the conditional covariance when the fuzzy system is a vector-valued fuzzy system $F: \mathbb{R}^n \to \mathbb{R}^p$. Then the conditional variance extends to a p-by-p conditional covariance matrix $K_{Y|X=x}$:

$$K_{Y|X=x} = \sum_{j=1}^{m} p_j(x)\, K_{Y|X=x, B_j(x)} + \sum_{j=1}^{m} p_j(x)\, (c_j(x) - F(x))(c_j(x) - F(x))^T. \tag{43}$$

Both the centroid $c_j(x)$ and output $F(x)$ are here p-dimensional column vectors with the same definitions as before. The local conditional covariance matrix $K_{Y|X=x, B_j(x)} = E_{B_j(x)}[(Y - c_j(x))(Y - c(x))^T]$ uses the conditional density $p_{B_j(x)}(y \mid x)$. Users can calculate these local matrices directly for simple shapes of the then-part sets. The weak law of large numbers also allows sample covariance matrices to approximate these population conditional covariance matrices if there are enough random samples and if all appropriate moments exist.

The second extension is to combining multiple fuzzy systems. The fuzzy systems themselves need not be additive.

Suppose there are q separate real-valued fuzzy systems $F_1, \ldots, F_q$. The kth fuzzy system F_k somehow produces its own combined rule firings $B^k(x)$. We also assign the nonnegative system weight w^k to F_k. Then we can additively combine the weighted rule firings $w^k B^k(x)$ to give the multi-system combined firings $B(x) = \sum_{k=1}^{q} w^k B^k(x)$. This allows the system to combine rules or *throughputs* rather than simply combining outputs. Taking the centroid gives the global output as a convex sum of the system centroids: $F(x) = \sum_{k=1}^{q} p_k(x)\ c_k(x)$ if c_k is the system centroid of $B^k(x)$. The result is $F(x) = \sum_{k=1}^{q} p_k(x)\ F_k(x)$ if the q systems are each centroidal. The output is richer still if each of the q systems is not just centroidal but additive [5]:

$$F(x) = \sum_{k=1}^{q} \sum_{j=1}^{m_k} p_j^k(x)\ c_j^k(x) \tag{44}$$

where the convex coefficients are now

$$p_j^k(x) = \frac{w^k w_j^k\ V_j^k(x)}{\sum_{u=1}^{q} \sum_{v=1}^{m_u} w^u w_v^u\ V_v^u(x)}. \tag{45}$$

The next section shows that all the above additive results become simpler and more practical still if the fuzzy systems are standard additive.

3 The Standard Additive Model (SAM): Rule Firing as Multiplicative Scaling

How exactly does the vector input $x \in \mathbb{R}^n$ fire the jth rule $R_{A_j \to B_j}$? Standard additive models give a simple and useful answer: multiplication. We say that an additive fuzzy system $F: \mathbb{R}^n \to \mathbb{R}$ is a *standard additive model* (SAM) if the fired if-part set value $a_j(x)$ multiplicatively scales the then-part B_j [1, 5]:

$$B_j(x) = a_j(x) B_j\ . \tag{46}$$

The multiplicative scaling shrinks the then-part set B_j over the same base. This scaling leaves the *relative* structure of the then-part set unchanged. The min-clip $\min(a_j(x), B_j)$ simply discards all then-part set information above the threshold $a_j(x)$. The same problem occurs for almost all other triangular-norm functions of $a_j(x)$ and B_j. So a user would need to produce some compelling reason for using some firing operator other than SAM multiplicative scaling.

The jth rule itself defines a Cartesian "patch" $A_j \times B_j$ or fuzzy subset of the product space $X \times Y\ :\ R_{A_j \to B_j} = A_j \times B_j = \{(x, y) \in X \times Y\!:\ x \in A_j\ \&\ y \in B_j\}$. Now suppose the vector input is x_0. Then convolution gives the fired rule as a product [5]:

$$\int_{-\infty}^{\infty} \delta(x - x_0)\ R_{A_j \to B_j}(x, y)\ dx = \int_{-\infty}^{\infty} \delta(x - x_0)\ a_j(x)\ b_j(y)\ dx \tag{47}$$

$$= b_j(y) \int_{-\infty}^{\infty} \delta(x - x_0)\ a_j(x)\ dx \tag{48}$$

$$= a_j(x_0)\ b_j(y). \tag{49}$$

So a traditional multiplicative Cartesian product leads to the SAM scaling $a_j(x)B_j$.

The SAM structure greatly simplifies the above results for additive fuzzy systems. The controlling fact is that $a_j(x)$ factors out of the key SAM calculations. This leads in turn to an important cancellation that converts the local conditional probability $p_{B_j}(y\,|\,x)$ to the *unconditional* probability $p_{B_j}(y)$:

$$p_{B_j}(y\,|\,x) = \frac{b_j(y\,|\,x)}{\int_{-\infty}^{\infty} b_j(y\,|\,x)\ dy} \tag{50}$$

$$= \frac{a_j(x)\ b_j(y)}{\int_{-\infty}^{\infty} a_j(x)\ b_j(y)\ dy} \tag{51}$$

$$= \frac{a_j(x)\ b_j(y)}{a_j(x)\int_{-\infty}^{\infty} b_j(y)\ dy} \tag{52}$$

$$= \frac{b_j(y)}{\int_{-\infty}^{\infty} b_j(y)\ dy} \tag{53}$$

$$= \frac{b_j(y)}{V_j} \tag{54}$$

$$= p_{B_j}(y) \tag{55}$$

if $a_j(x) > 0$. Zero conditional probability $p_{B_j}(y\,|\,x) = 0$ holds for zero if-part activations $a_j(x) = 0$ so long as the then-part set functions b_j are not trivial.

The SAM volume or area V_j is a constant that the user can pre-compute in advance of running the SAM fuzzy system. This also holds for the SAM then-part centroids c_j:

$$c_j(x) = \frac{\int_{-\infty}^{\infty} y\ b_j(y\,|\,x)\ dy}{\int_{-\infty}^{\infty} b_j(y\,|\,x)\ dy} \tag{56}$$

$$= \frac{a_j(x)\int_{-\infty}^{\infty} y\ b_j(y)\ dy}{a_j(x)\int_{-\infty}^{\infty} b_j(y)\ dy} \tag{57}$$

$$= c_j \tag{58}$$

since again $a_j(x)>0$ by assumption.

The SAM structure likewise simplifies Proposition 1 to a convex sum or mixture of unconditional probability density functions:

$$p(y\,|\,x) = \textstyle\sum_{j=1}^{m} p_j(x)\; p_{B_j}(y). \tag{59}$$

This is the result that directly generalizes the mixture probability model below.

These SAM simplifications now give the *SAM Theorem* [1, 5, 6] as a special case of Theorem 2:

$$F(x) = \frac{\sum_{j=1}^{m} w_j\; a_j(x)\; V_j\; c_j}{\sum_{k=1}^{m} w_k\; a_j(x)\; V_k} \tag{60}$$

$$= \textstyle\sum_{j=1}^{m} p_j(x)\; c_j. \tag{61}$$

But now the convex coefficients are simpler:

$$p_j(x) = \frac{a_j(x)\; w_j\; V_j}{\sum_{k=1}^{m} a_k(x)\; w_k\; V_k}. \tag{62}$$

This SAM system also enjoys simple forms for adaptation and statistics as well as for function approximation. One example is the combination of q-many SAM set systems using (10):

$$F(A) = \textstyle\sum_{k=1}^{q} \sum_{j=1}^{m_k} p_j^k(A)\; c_j^k \tag{63}$$

for input fuzzy set A. The SAM convex coefficients have the form

$$p_j^k(A) = \frac{w^k w_j^k\; a_j^k(A)\; V_j^k}{\sum_{u=1}^{q} \sum_{v=1}^{m_u} w^u w_v^u\; a_j(A)\; V_v^u} \tag{64}$$

if $a_j(A) = \int_{-\infty}^{\infty} a(x)\; a_j(x)\; dx$. Gradient tuning of set-function SAMs can lead to extremely complicated learning laws for updating the if-part sets [7]. This is not the case for tuning ordinary point SAMs.

The most important SAM special case historically has been the so-called "center of gravity" or COG fuzzy system because such systems were the basis of almost all early applications of fuzzy systems [5, 11, 14]:

$$F(x) = \frac{\sum_{j=1}^{m} a_j(x)\; c_j}{\sum_{k=1}^{m}\; a_j(x)}. \tag{65}$$

So a COG corresponds to SAM with equal volumes and equal weights since then all the volume and weight terms cancel out of the SAM ratio. Some fuzzy engineers have ignored the shape of the then-part sets and just worked with a "spike" centroid. This corresponds to interpreting the then-part set B_j as a delta function centered at the centroid: $B_j = \delta(y - c_j)$. Then the area or volume of B_j integrates to one and thus $Centroid(B_j) = c_j$ holds from the sifting property of the delta function.

Gaussian sets further simplify the COG model. Using factored if-part set functions $a_j(x) = \prod_{k=1}^{m} a_j^k(x^k)$ in the SAM/COG ratio then gives the popular *radial basis functions* found in the neural network literature [15–18] when both if-part sets and then-part sets are Gaussian or truncated-Gaussian sets. The idea for rule generation here comes out of the theory of probability density estimation: Center a Gaussian ball at the input vector x and center a Gaussian bell curve at the output value y for a given data pair (x, y). Then adding up such normalized terms gives a type of smoothed histogram of the sampled joint probability density function.

Rule weights can depend on then-part-set volumes. This often occurs in Gaussian SAMs so that a rule with a wide then-part set B_j will not have more influence than the same rule with a thinner then-part set. The wider then-part set has a larger volume V_j and thus has more influence on the output $F(x)$ because the SAM ratio (60) is increasing in V_j. The choice $w_j = \frac{1}{V_j}$ cancels the volumes in the SAM ratio. The more common choice $w_j = \frac{1}{V_j^2}$ makes the width or size of then-part sets vary inversely with their influence on the output. That roughly captures the intuition that more uncertain rules should have less overall influence. It also changes the SAM learning law for the volumes because then [4, 5]

$$\frac{\partial F}{\partial V_j} = p_j(x)\left[c_j(x) - F(x)\right]\left(\frac{1}{V_j} + \frac{1}{w_j}\frac{\partial w_j}{\partial V_j}\right) \tag{66}$$

$$= p_j(x)\left[c_j(x) - F(x)\right]\left(\frac{1}{V_j} - V_j^2\frac{2}{V_j^3}\right) \tag{67}$$

$$= -p_j(x)\left[c_j(x) - F(x)\right]\frac{1}{V_j}. \tag{68}$$

This leads to the volume learning law $V_j(t+1) = V_j(t) - \mu_t \varepsilon(t)\frac{p_j(x)}{V_j(t)}\left[c_j - F(x)\right]$. A cruder approach simply sets the rule weights equal to the inverse variance: $w_j = \frac{1}{\sigma_j^2}$.

The conditional variance of a SAM simplifies to

$$V[Y \mid X = x] = \sum_{j=1}^{m} p_j(x)\ \sigma_{\mathbf{B_j}}^2 + \sum_{j=1}^{m} p_j(x)\ \left[c_j(x) - F(x)\right]^2 \tag{69}$$

since the then-part set variances $\sigma_{B_j}^2$ no longer depend on the input x. The COG case simplifies further because all the then-part variances are the same and thus each then-part set B_j has the same variance σ^2. Then the convex sum structure gives the COG conditional variance as

$$V[Y \mid X = x] = \sigma^2 + \sum_{j=1}^{m} p_j(x)\ \left[c_j(x) - F(x)\right]^2. \tag{70}$$

This shows that the shape of the then-part sets matters for higher-order uncertainty even for the simplest COG models because different shapes give different inherent variances σ^2. The positive value σ^2 represents the minimal level of uncertainty that a COG can achieve.

SAM systems also allow *exact* representation of arbitrary bounded functions $f: \mathbb{R}^n \to \mathbb{R}$. The Watkins Representation Theorem [19–22] states that a SAM system F needs only *two* rules to exactly represent a bounded real function f in the sense that $F(x) = f(x)$ for all x. Such representation trivializes the usual problem of exponential rule explosion in fuzzy function approximation. The catch is that the two if-part set functions build the bounded function f directly into their definition: $a_1(x) = \frac{\sup f - f(x)}{\sup f - \inf f}$ and $a_2(x) = 1 - a_1(x)$. The two rules have the linguistic form "If $X = A$ then $Y = B_1$" and "If $X = not\ A$ then $Y = B_2$". The then-part sets B_1 and B_2 can have any shape so long as the infimum of f is the center of B_1 and the supremum is the center of B_2: $c_1 = \inf f$ and $c_2 = \sup f$. The volumes or areas can be any positive value so long as they are equal: $V_1 = V_2 > 0$. Then unity rule weights give

$$F(x) = \frac{\sum_{j=1}^{2} a_j(x)\ V_j\ c_j}{\sum_{k=1}^{2} a_j(x)\ V_k} \tag{71}$$

$$= \frac{a_1(x) \inf f\ -\ (1 - a_1(x)) \sup f}{a_1(x) + 1 - a_1(x)} \tag{72}$$

$$= \left(\frac{\sup f - f(x)}{\sup f - \inf f}\right)(\inf f - \sup f) + \sup f \tag{73}$$

$$= f(x) \text{ for all } x. \tag{74}$$

Such SAM representation is especially useful in modern Bayesian statistics because it allows just two rules to represent either a bounded prior or a bounded likelihood probability density function. A common conjugate prior is the beta probability density function $f(\theta) = Beta(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ for θ in the unit interval and for positive shape parameters α and β. Here Γ denotes the gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$. Then a two-rule SAM can exactly represent the beta density $f(\theta) = Beta(5, 8)$ with the set function [21] $a_1(\theta) = 1 - \frac{11^{11}}{7^7 4^4} \theta^7 (1-\theta)^4$ if $c_1 = \inf f = 0$ and $c_2 = \sup f = \frac{\Gamma(13)}{\Gamma(8)\Gamma(5)} \left(\frac{7}{11}\right)^7 \left(\frac{4}{11}\right)^4$. The approximation power of SAMs also allows users to go beyond closed-form densities and use rule-defined priors or likelihoods and still be assured that the resulting fuzzy system will uniformly approximate the underlying Bayesian posterior density $f(\theta | x)$ [21]. This even holds for hierarchical Bayes systems where the prior density itself depends on a hyperprior density [22] The above scheme for combining SAMs can combine rule-represented priors or likelihoods with rule-defined priors or likelihoods into a single SAM system.

4 Generalized Mixture Models and Continuum-Many Rules

We conclude by showing that additive fuzzy systems generalize mixture models and that they extend to fuzzy systems with continuum-many fuzzy rules.

Mixture models are finite convex combinations of probability density functions (pdfs) [23]. Such a convex combination mixture of m pdfs $f_1, \ldots, f_m$ gives a new pdf f with m modes if the pdfs are sufficiently spread out:

$$f(x) = \sum_{j=1}^{m} \pi_j f_j(x) \tag{75}$$

if the nonnegative *mixing weights* $\pi_1, \ldots, \pi_m$ sum to unity: $\sum_{j=1}^{m} \pi_j = 1$. This convex sum can model taking random samples from a population made up of m-many subpopulations such as m words or patterns. Then the estimation task is to find the mixture weights and the parameters of the mixed pdfs. The most popular mixture by far is the Gaussian mixture where f_j is a scalar or vector Gaussian $N(\mu_j, \theta_j^2)$.

The mixture sum is not arbitrary. It follows from the elementary theorem on total probability. Suppose m hypothesis sets $H_1, \ldots, H_m$ partition the sample space Ω. Suppose the set $E \subset \Omega$ represents some observed evidence. Then the theorem on total probability states that the unconditional probability of the evidence $P(E)$ equals the convex combination of the prior probabilities $P(H_j)$ and the likelihoods $P(E \mid H_j)$: $P(E) = \sum_{j=1}^{m} P(H_j)\, P(E \mid H_j)$. This corresponds to the mixture sum because the evidence is the input x and because π_j is the prior probability of the jth class or mixture element. So $f_j(x) = f(x \mid j)$ if the conditional density $f(x \mid j)$ is the likelihood that we would observe such an x if it came from the jth class or mixture density.

The ubiquitous Expectation-Maximization (EM) algorithm quite often estimates the mixing weights and means and variances by iteratively maximizing the likelihood function [23]. The class memberships of the m decision classes correspond to the hidden or latent variables in the EM algorithm. Then carefully injected noise can always speed up convergence of the EM algorithm [24–26] as it climbs the nearest hill of likelihood.

Mixture moments follow directly from the convexity of the mixed sum. This gives the *unconditional* mean and variance of the random variable X as [23]

$$E[X] = \sum_{j=1}^{m} \pi_j\, \mu_j \tag{76}$$

and

$$V[X] = \sum_{j=1}^{m} \pi_j\, \sigma_j^2 + \sum_{j=1}^{m} \pi_j\, \left[\mu_j - E[X]\right]^2 \tag{77}$$

if the mixture's underlying jth random variable X_j has pdf f_j and thus if it has mean μ_j and variance σ_j^2. We see at once that the mixture mean and variance are special cases of a SAM fuzzy system's *conditional* mean and variance.

The SAM and other additive systems generalize mixture models by making the mixture weights π_j depend on the input x: $\pi_j(x) = p_j(x)$. This in turn makes the mixture's means and variances (and other moments) depend on x and thus become conditional moments. So mixture models correspond to fixed-input centroidal additive systems. Then Proposition 1 gives back the defining mixture-density combination for unfired then-part sets:

$$p(y) = \sum_{j=1}^{m} p_j \, p_{B_j}(y). \tag{78}$$

Mixture models sample in effect from convex combinations of m suitably normalized then-part sets.

We can extend SAM models to systems with infinitely many fuzzy rules. The cardinality of the rules can be countably or uncountably infinite. We will work with the latter continuum case. This follows from the direct extension of mixture models to *compounding* models that weight one pdf with another and then integrate out the continuous mixture index [23]. Our approach will instead impose a higher-level mixture structure on the continuum of rules.

Suppose now that the real parameter θ indexes the continuum-many if-part set functions a_θ and the corresponding then-part sets B_θ in continuum-many rules of the form "If $X = A_\theta$ then $Y = B_\theta$". Then integration gives the combined rule firings:

$$b(y|x) = \int_{\theta=-\infty}^{\theta=\infty} w_\theta \, b_\theta(y \,|\, x) \, d\theta \tag{79}$$

if we assume the integral exists for appropriate nonnegative rule weights w_θ. The proof of Theorem 3 still goes through if (definite) integrals replace the finite sums:

$$F(x) = \int p_\theta(x) \, c_\theta \, d\theta \tag{80}$$

and

$$p_\theta(x) = \frac{a_\theta(x) \, w_\theta \, V_\theta}{\int a_\phi(x) \, w_\phi \, V_\phi \, d\phi}. \tag{81}$$

Consider a simple Gaussian set of rules for a scalar parameter θ. The rules have vector-Gaussian if-part set functions a_θ and scalar Gaussian then-part set functionsb_θ: $a_\theta(\cdot) = N(\theta \bullet 1, \; K_\theta)$ and $b_\theta(y) = N(\theta, \; \sigma^2)$ if $\theta \bullet 1$ denotes the n-vector with all elements equal to θ. K_θ is an n-by-n covariance matrix. It equals the identity matrix in the simplest or "white" case. Then $c_\theta = \theta$ and $V_\theta = 1$ since $b_\theta(y) = N(\theta, \; \sigma^2)$. This gives the output $F(x)$ as a simple unconditional expectation for each x: $F(x) = \int p_\theta(x) \; \theta \; d\theta = E_{p_\theta(x)}[\Theta]$. This approach extends at once to vector parameters.

Computing the convex integral for $F(x)$ is more complicated than in the simpler case of probabilistic compounding. Compounding allows the modeler to pick the weighting pdf p_θ as a normal or gamma or other well-behaved closed-form pdf. But the SAM convex-sum p_θ involves a highly nonlinear transformation of continuum-many if-part set functions a_θ. This transformation may not be tractable. Integrating it to produce $F(x)$ can only compound the computational intractability.

A practical solution is to rely on the weak law of large numbers (WLLN) through Monte Carlo simulation. The WLLN states that the sample mean $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ of independent and identically distributed finite-variance random variables $X_1, X_2, \ldots$ converges in probability to the population mean $E[X]$: $\lim_{n\to\infty} P(|\bar{X}_n - E[X]| > \varepsilon) = 0$ for all $\varepsilon > 0$. Monte Carlo simulation interprets an ordinary definite integral $\int_a^b g(x)\, dx$ as the expectation of a random variable that has a uniform distribution over (a, b) [23]:

$$\int_a^b g(x)\, dx = (b-a)\int_a^b g(x)\frac{dx}{b-a} = (b-a)E[X] \tag{82}$$

for $X \sim U(a, b)$. The user need not integrate the integrand $(b-a)g(x)$. The user need only compute values $(b-a)g(x_k)$ for random uniform draws x_k from (a, b). The random draws can come from any uniform random number generator. Then the WLLN ensures that $\frac{1}{n}\sum_{k=1}^{n}(b-a)g(x_k) \approx (b-a)E[X] = \int_a^b g(x)dx$ for enough random draws x_k. The variance in the WLLN estimate decreases linearly with the number n of draws.

Monte Carlo simulation can estimate the integrals in the continuum-rule SAM for a given input x. Assume there are n random draws of θ from some finite interval (a, b) for a fuzzy system defined on the compact interval $[a, b]$. Then

$$F(x) = \frac{\int a_\theta(x)\, w_\theta\, V_\theta\, c_\theta\, d\theta}{\int a_\phi(x)\, w_\phi\, V_\phi\, d\phi} \tag{83}$$

$$\approx \frac{\frac{1}{n}\sum_{k=1}^{n} w_k\, a_k(x)\, V_k\, c_k}{\frac{1}{n}\sum_{k=1}^{n} w_k\, a_k(x)\, V_k} \tag{84}$$

$$= \frac{\sum_{k=1}^{n} w_k\, a_k(x)\, V_k\, c_k}{\sum_{k=1}^{n} w_k\, a_k(x)\, V_k} \tag{85}$$

$$= \sum_{k=1}^{n} p_k(x)\, c_k. \tag{86}$$

The result has the same convex-sum form as the finite-rule SAM even though the sums use random choices of rules instead of firing all the rules.

The final task is to control and shape the overall distribution of the continuum of fuzzy rules. This allows the fuzzy engineer to define *meta-rules* at a much higher level of abstraction. An engineer can also give the wave-like groupings of rules a linguistic interpretation such as "small negative" or "medium positive" and the like. The engineer should be able to pick an initial set of such meta-rules just as in the case of setting up a finite SAM. Then there should be some practical way to tune these meta-rules with data to give different levels of control or function approximation. This requires a Bayesian-like approach that puts some probabilistic structure on the parameter θ: $\Theta \sim h(\theta)$. This corresponds to the old fuzzy-engineering task of picking the shapes of if-part and then-part sets.

Mixture densities offer a natural way to define fuzzy meta-rules over the continuum of fuzzy rules. The mixture variable is no longer x. It is now θ. Suppose the fuzzy engineer wants to impose k-many fuzzy meta-rules. This requires mixing k-many densities:

$$h(\theta) = \sum_{i=1}^{k} \pi_i f_i(\theta). \tag{87}$$

The engineer might center the mixture pdfs closer together in regions of the input space where he desires greater control. An early example of such proximity control was the fuzzy truck-backer-upper [27]. The truck-and-trailer rig backed up to a loading dock from a parking lot. Closer and narrower if-part sets near the loading dock gave finer control near that equilibrium point. A few wide if-part sets then covered much of the remaining parking lot. The engineer could distribute these meta-rule mixture pdfs in the same way.

Standard statistical techniques can then compute fuzzy outputs $F(x)$ and tune the fuzzy meta-rules. Monte Carlo simulation can estimate the output $F(x)$ for a given x. But the sampling now cannot be from a uniform density in general. That would always give the same output on average. The sampling must come instead from the meta-rule mixture density $h(\theta)$ itself to reflect the distribution of the meta-rules. This is just the well-known technique of *importance sampling* from mixtures [28]. Then the E-M algorithm or its variants can tune the mixture parameters based on sampled inputs x.

There is algorithmic irony in using mixture densities to control fuzzy meta-rules after seeing that the underlying fuzzy rule based systems generalize mixture densities. There is also a loss of the exponential rule explosion that plagues ordinary finite fuzzy systems. This loss holds because the growth in meta-rules is only *linear* in the number k of mixed densities. That shifts much of the computational burden to the sampling task involved in converting an input x to an output $F(x)$.

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Fuzzy Information Retrieval Systems: A Historical Perspective

Donald H. Kraft, Erin Colvin, Gloria Bordogna and Gabriella Pasi

Abstract The application of fuzzy set theory to information retrieval has been applied, specifically to Boolean models. This includes fuzzy indexing procedures defined to represent the varying significance of terms in synthesizing the documents' contents, the definition of query languages to allow the expression of soft selection conditions, and associative retrieval mechanisms to model fuzzy pseudo-thesauri, fuzzy ontologies, and fuzzy categorizations of documents.

Keywords Fuzzy · Information retrieval · Query · Imprecision · Vagueness · Indexing · Ememes · Geographic information retrieval

1 Introduction

The objective of this entry is to provide an overview of the application of fuzzy set theory to design an information retrieval system (IRS). We consider the representation of uncertainty, imprecision, vagueness and subjectivity, which are characteristics of the process of information searching and retrieval. Salton notes that IR deals with the representation, storage, and access to "documents" or representatives of documents (i.e., document surrogates) [49]. The elements of an IRS include a set of documents, document processing for content analysis, a query describing an

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_15

information need, a matching of the query and the documents, an output module with a ranked list of documents deemed relevant, and a user interface.

We will discuss some current trends and key issues in information retrieval, and give an overview of the basic notions of fuzzy set theory to model IRSs. We will also provide a description of the traditional fuzzy document representation and a fuzzy representation of documents structured into logical sections that can be adapted to the subjective needs of a user. In addition, we give a description of how the Boolean query language of IR can be extended so as to make it flexible and suitable to express soft constraints by capturing the vagueness of the user needs. Both numeric and linguistic selection conditions are introduced to qualify term's importance, and we will show how linguistic quantifiers are defined to specify soft aggregation operators of query terms. We will also discuss how fuzzy sets can serve to define associative mechanisms to expand the functionalities of IR systems, i.e., the capability to represent concepts and to model their semantic relationships. Fuzzy sets provide notions that can be applied to this purpose allowing to model fuzzy pseudothesauri and fuzzy ontologies and to build fuzzy categorizations of documents via fuzzy clustering techniques. Lastly, we present emerging applications of information retrieval modelled within the fuzzy framework such as geographic information retrieval, multi-dimensional relevance evaluation, and discovery of similar web pages contents in multiple searches and ememe identification and tracking and discuss fuzzy performance measures for IR systems.

2 Key Issues in Information Retrieval

Modeling the concept of relevance in IR is certainly a key issue, perhaps the most difficult one, and no doubt the most important one. What makes a document relevant to a given user is still not fully understood, specifically when one goes beyond topicality (i.e., the matching of the topics of the query with the topics of the document). This leads to the realization that relevance is imprecise as well as subjective.

A second key issue is the representation of the documents in a collection, as well as the representation of users' information needs, especially for the purpose of matching documents to the queries semantically. This implies introducing incompleteness, approximation, and managing vagueness and imprecision. Yet another key issue is how to evaluate an information retrieval system's performance properly, where imprecision also exists.

2.1 Imprecision, Vagueness, Uncertainty, and Inconsistency in Information Retrieval

Very often the terms imprecision, vagueness, uncertainty, and inconsistency are used as synonymous concepts. Nevertheless when they are used to qualify a characteristic of the information they have a distinct meaning [42].

There are several ways to represent imprecise and vague concepts. One can approach this indirectly by defining similarity or proximity relationships between each pair of imprecise and vague concepts. If we regard a document as an imprecise or vague concept, i.e., as bearing a vague content, a numeric value computed by a similarity measure can be used to express the closeness of any two pairs of documents. This is the way of dealing with the imprecise and vague document and query contents via the weights in IR's vector space model. In this context the documents and the query are represented as points in a vector space of terms and the distances between the query and the documents points are used to quantify their similarity [50].

Another way to represent vague and imprecise concepts is by means of the notion of fuzzy set. The notion of a fuzzy set is an extension to normal set theory [66]. The notion of fuzzy set has been used in the IR context to represent the vague concepts expressed in a flexible query for specifying soft selection conditions of the documents [59].

Uncertainty is related to the truth of a proposition, intended as the conformity of the information carried by the proposition with the considered reality. Possibility theory [27, 65] together with the concept of a linguistic variable defined within fuzzy set theory [67], provide a unifying formal framework to formalize the management of imprecise, vague and uncertain information [15].

The same information content can be expressed by choosing a trade-off between the vagueness and the uncertainty embedded in a proposition. A dual representation can eliminate imprecision and augment the uncertainty, like in the expression "it is *not completely probable* that document d fully satisfies the query q." One way to model IR is to regard it as an uncertain problem [33].

There are two alternative ways to model IR activity. One possibility is to model the query evaluation mechanism as an uncertain decision process. The concept of relevance is considered binary (crisp), as the query evaluation mechanism computes the probability of relevance of a document d to a query q. Such an approach, which does model the uncertainty of the retrieval process, has been introduced and developed using probabilistic IR models [26, 28, 60]. Another possibility is to interpret the query as the specification of soft "elastic" constraints that the representation of a document can satisfy to an extent, and to consider the term *relevant* as a gradual (vague) concept. This is the approach adopted in fuzzy IR models [9, 33]. In this latter case, the decision process performed by the query evaluation mechanism computes the degree of satisfaction of the query by the representation of each document.

This satisfaction degree, called the retrieval status value (RSV), is considered an estimate of the degree of relevance (or is at least proportional to the relevance) of a given document with respect to a given user query. An RSV of 1 implies maximum relevance; an RSV of 0 implies absolutely no relevance. And, an RSV in the interval (0, 1) implies an intermediate level or degree of relevance.

Inconsistency comes from the simultaneous presence of contradictory information about the same reality. An example can be observed when submitting the same query to several IRSs that adopt different representations of documents and produce different results. This is actually very common and often occurs when searching for information over the Internet using different search engines. To solve this kind of inconsistency, some fusion strategies can be applied to the ranked lists each search engine produces. In fact, this is what metasearch engines do [14, 64].

The document representation based on a selection of index terms is invariably incomplete. When synthesizing the content of a text manually by asking an expert to select a set of index terms, one introduces subjectivity in the representation. On the other hand, automatic full-text indexing introduces imprecision since the terms are not all fully significant in characterizing a document's content. However, these terms can have a partial significance that might also depend upon the context in which they appear, i.e., which document component. Modern retrieval systems may include natural language processing capabilities to try to deal with semantics. Thus, one can move from the notion of a document as a "bag of terms" to having a set of concepts. This leads to the idea of a taxonomy, i.e., a vocabulary and structure (e.g., a cat or a dog is a pet), and an ontology, i.e., a set of relationships and rules and constraints (i.e., dog chases cat). These ideas have their own sets of imprecision or vagueness.

3 Fuzzy Retrieval Models

Fuzzy retrieval models have been defined in order to reduce the imprecision that characterizes the Boolean indexing process, to represent the user's vagueness in queries, and to deal with discriminated answers estimating the partial relevance of the documents with respect to queries. Extended Boolean models based on fuzzy set theory have been defined to deal with one or more of these aspects [5, 10, 11, 16, 18, 21, 32, 47, 61]. Surveys of fuzzy extensions for IRSs and of fuzzy generalizations of the Boolean retrieval model can be found in [9, 33].

Fuzzy "knowledge based" models [35, 36], and fuzzy associative mechanisms [38–40, 43] have been defined to cope with the incompleteness that characterizes either the representation of documents or the users' queries. [37] illustrates a wide range of methods to generate fuzzy associative mechanisms.

It has been speculated that Boolean logic is passé', out of vogue. Yet, researchers have employed p-norms in the vector space model or Bayesian inference nets in the probabilistic model to incorporate Boolean logic. In addition, the use of Boolean logic to separate a collection of records into two disjoint classes has been considered, e.g., using the one-clause-at-a time (OCAT) methodology [56]. Even now retrieval systems such as Dialog and web search engines such as Google allow for Boolean connectives.

4 Fuzzy Techniques for Indexing

During the indexing process one wants to provide more specific and exhaustive representations of each document's information content. This means improving these representations beyond those generated by existing indexing mechanisms. We introduce the fuzzy interpretation of a weighted document representation and a fuzzy representation of documents structured in logical sections that can be adapted to a user that has a subjective criteria for interpreting the content of documents [11, 41]. In order to increase the effectiveness of IRSs, the indexing process plays a crucial role.

4.1 Vector Space, Probabilistic, and Generalized Boolean Indexing

The vector space model and the probabilistic models generally adopt a weighted document representation, which has improved the Boolean document representation by allowing the association of a numeric weight with each index term [54, 60]. The automatic computation of the index term [56, 57]. In this case, the indexing mechanism computes d for each document and t for each term by means of a function F. An example of F has the index term weight increasing with the frequency of term t in document d but decreasing with the frequency of the term in all the documents of the archive is given by

$F(d,t) = tf_{dt} \times g(IDF_t)$ where tf_{dt} is a normalized term frequency, which can be defined as:

$$tf_{dt} = \frac{OCC_{dt}}{MAXOCC_d}, \quad (1)$$

OCC_{dt} is the number of occurrences of t in d; $MAXOCC_d$ is the number of occurrences of the most frequent term in d, IDF_t is an inverse document frequency which can be defined as:

$$IDF_t = Log \frac{N}{NDOC_t}, \quad (2)$$

N is the total number of documents in the archive; $NDOC_t$ is the number of documents indexed by t; and g is a normalizing function.

To simplify the computation of this value, it is possible to heuristically approximate it: during the archive generation phase, with an expert indicating the estimated percentage of the average length of each section with respect to the average length of documents ($PERL_s$). Given the number of occurrences of the most frequent term in each document d, $MAXOCC_d$, an approximation of the number of occurrences of the most frequent term in section s of document d is $MAXOCC_{sd} = PERL_s * MAXOCC_d$.

The adoption of weighted indexes allows for an estimate of the relevance, or of the probability of relevance, of documents to a query [53, 60]. Based on such an indexing function, and by incorporating Boolean logic into the query, the fuzzy interpretation of an extended Boolean model has been to adopt a weighted document representation and to interpret it as a fuzzy set of terms [17]. From a mathematical point of view, this is a quite natural extension: the concept of the significance of index terms in describing the information content of a document can then be naturally described by adopting the function F, such as the one defined above, as the membership function of the fuzzy set representing a document's being in the subset of concepts represented by the term in question. Formally, a document is represented as a fuzzy set of terms:

$$R_d = \sum_{t \in T} \mu_{Rd}(t)/t, \tag{3}$$

in which the membership function is defined as $\mu_{Rd}: D \times T \to [0, 1]$. In this case, $\mu_{Rd}(t) = F(d, t)$, the membership value, can be obtained by the indexing function F which is expressed by a numeric score or RSV.

Fuzzy set theory has been applied to define new and more powerful indexing models than the one based on the function above. The definition of new indexing functions has been motivated by several considerations. First, the F functions do not take into account the idea that a term can play different roles within a text according to the distribution of its occurrences. The text can be considered as a black box, closed to a user's interpretation. This outlines the fact that relevance judgments are driven by a subjective interpretation of the document's structure, and supports the idea of *dynamic* and *adaptive* indexing [2, 11]. By adaptive indexing, we mean indexing procedures which take into account the users' desire to *interpret* the document contents and to "build" their synthesis on the basis of this interpretation.

An indexing model has been proposed where the occurrences of a term in the different documents' sections are taken into account according to specific criteria, and the user's interpretation of the text is modeled [11]. During the retrieval phase, the user can specify the distinct importance of the sections and decide that a term must be present in *all* the sections or in at least a certain number of them in order to consider the term fully significant. A section is a logical subpart identified by s_i, where $i \in 1,..,n$ and n is the total number of the sections in the documents. We assume here that an archive contains documents sharing a common structure.

4.2 Fuzzy Representation of Structured Documents

We also consider the synthesis of a fuzzy representation of structured documents that takes into account the user needs [11]. A document can be represented as an entity composed of sections (e.g., title, authors, introduction, and references). The information role of each term occurrence depends then on the semantics of the subpart where it is located. This means that to the aim of defining an indexing function for structured documents the single occurrences of a term may contribute differently to the significance of the term in the whole document. The document's subparts may have a different importance determined by the users' needs.

When generating an archive of a set of documents, it is necessary to define the sections one wants to employ to structure each document. The structure of the documents, i.e., the type and number of sections, depends on the semantics of the documents and on the accuracy of the indexing module. A formal representation of a document using a fuzzy binary relation: with each pair <section, term>, a significance degree in the interval [0,1] is computed to express the significance of that term in that document section. To obtain the overall significance degree of a term in a document, i.e., the index term weight, these values are dynamically aggregated by taking into account the indications that a user states in the query formulation. Other non-fuzzy approaches have also introduced the concept of a boosting factor to emphasize differently the contribution of the index terms occurrences depending on the document sections to the overall index term weights. However these approaches compute *static* index term weights during the indexing process, without taking into account the user interpretation.

On the contrary, in the fuzzy approach, the aggregation function, is defined on two levels. First, the user expresses preferences for the document sections (the equivalent of the boosting factors), second, the user should decide which aggregation function has to be applied to produce the overall significance degree. By adopting this document representation, the same query can select documents in different relevance order depending on the user's preferences.

Formally, a document is represented as a fuzzy binary relation, $R_d = \sum_{(t,s) \in T \times S} \mu_d(t,s)/(t,s)$, where the value $\mu_d(t,s) = F_s(d,t)$ expresses the significance of term t in section s of document d. An indexing function F_s: $DxT \rightarrow [0,1]$ is then defined for each section s. The overall significance degree F(d,t) is computed by combining the single significance degrees of the sections, the $F_s(d,t)$ s, through an aggregation function specified by the user.

5 Definition of Flexible Query Languages

The objective here is to define query languages that are more expressive and natural than classical Boolean logic. This is done to capture the vagueness of user needs as well as to simplify user system interaction. This has been pursued with two different approaches. First, there has been work on the definition of soft selection criteria (soft constraints), which allow the specification of the different importance of the search terms. Query languages based on numeric query term weights with different semantics have been first proposed as an aid to define more expressive selection criteria [5, 18, 21, 22, 61]. An evolution of these approaches has been defined that introduces linguistic query weights, specified by fuzzy sets such as *important* or *very important*, in order to express the different vague importance of the query terms [12]. Second, there is the approach of introducing soft aggregation operators for the selection criteria, characterized by a parametric behavior which can be set between the two extremes of intersection (AND) and union (OR) as adopted in Boolean logic. Boolean query languages have been extended and generalized by defining aggregation operators as linguistic quantifiers such as *at least k* or *about k* [10]. As a consequence of this extension, the exact matching that is employed by a classical Boolean IRS is softened using a partial matching mechanism that evaluates the degree of satisfaction of a user's query for each document. This degree of satisfaction is the RSV that is used for ranking.

5.1 Term Significance

To obtain the overall degree of significance of a term in a document, an aggregation scheme of the values has been suggested, based on a twofold specification of the user [11]. When starting a retrieval session, the user can specify her/his preferences on the sections s by numeric score $\alpha_s \in [0,1]$ where the most important sections have an importance weight close to 1.

Within fuzzy set theory linguistic quantifiers used to specify aggregations are defined as Ordered Weighted Averaging (OWA) operators [62]. When processing a query, the first step accomplished by the system for evaluating F(d,t) is the selection of the OWA operator associated with the linguistic quantifier l_q, OWA_{lq}. When the user does not specify any preferences on the documents' sections, the overall significance degree F(d,t) is obtained by applying directly the OWA_{lq} operator to the values $\mu_1(d,t), \ldots, \mu_n(d,t)$: $F(d,t) = OWA_{lq}(\mu_1(d,t), \ldots, \mu_n(d,t))$. When distinct preference scores $\alpha_1,\ldots,\alpha_n$ are associated with the sections, it is first necessary to modify the values $\mu_1(d,t), \ldots, \mu_n(d,t)$ in order to increase the "contrast" between the contributions due to important sections with respect to those of less important ones. The evaluation of the overall significance degree F(d,t) is obtained by applying the operator OWA_{lq} to the modified degrees $a_1,..,a_n$: $F(d,t) = OWA_{lq}(a_1,..,a_n)$.

5.2 Fuzzy Associative Mechanisms

These associative mechanisms allow automatically generating fuzzy pseudothesauri, fuzzy ontologies, and fuzzy clustering techniques to serve three distinct but compatible purposes. First, fuzzy pseudothesauri and fuzzy ontologies can be used to contextualize the search by expanding the set of index terms of documents to include additional terms by taking into account their varying significance in representing the topics dealt with in the documents. The degree of significance of these associated terms depends on the strength of the associations with a document's original descriptors. Second, an alternative use of fuzzy pseudothesauri and fuzzy ontologies is to expand the query with related terms by taking into account their varying importance in representing the concepts of interest. The importance of an additional term is dependent upon its strength of association with the search terms in the original query. Third, fuzzy clustering techniques, where each document can be placed within several clusters with a given strength of belonging to each cluster, can be used to expand the set of the documents retrieved in response to a query. Documents associated with retrieved documents, i.e., in the same cluster, can be retrieved. The degree of association of a document with the retrieved documents does influence its RSV. Another application of fuzzy clustering is that of providing an alternative way of presenting the results of a search. When the user does not specify any criterion to aggregate the single degrees of the sections, a default aggregation operator is used [10]. Since no importance is specified to differentiate the contributions of the sections, all of them are assumed to have the same importance weight of 1.

Associative retrieval mechanisms are defined to enhance the retrieval of IRSs. They work by retrieving additional documents that are not directly indexed by the terms in a given query but are indexed by other, related terms, sometimes called associated descriptors. The most common type of associative retrieval mechanism is based upon the use of a thesaurus to associate index or query terms with related terms. In traditional associative retrieval, these associations are crisp.

Fuzzy associative retrieval mechanisms obviously assume fuzzy associations. A fuzzy association between two sets $X = \{x_1,\ldots,x_m\}$ and $Y = \{y_1,\ldots,y_n\}$ is formally defined as a fuzzy relation.

$f : X \times Y \rightarrow [0,1]$, where the value f(x,y) represents the degree or strength of the association existing between the values $x \in X$ and $y \in Y$. In information retrieval, different kinds of fuzzy associations can be derived depending on the semantics of the sets X and Y.

5.3 Fuzzy Thesauri

A thesaurus is an associative mechanism that can be used to improve both indexing and querying. The development of thesauri is very costly, as it requires a large amount of human effort to construct and to maintain. In highly dynamic situations, i.e., volatile situations, terms are added and new meanings derived for old terms quite rapidly, so that the thesaurus needs frequent updates. For this reason, methods for automatic construction of thesauri have been proposed, named pseudothesauri, based on statistical criteria such as the terms' co-occurrences, i.e., the simultaneous appearance of pairs (or larger subsets) of terms in the same documents.

In a thesaurus, the relations defined between terms are of different types. If the associated descriptor has a more general meaning than the entry term, the relation is classified as broader term (BT), while a narrower term (NT) is the inverse relation. Synonyms and near-synonyms are parts of another type of relationship associated by a related term (RT) connection.

The concept of a fuzzy thesaurus has been suggested [37, 38, 44, 48], where the links between terms are weighted to indicate the relative strengths of these associations. Fuzzy pseudothesauri are generated when the weights of the links are automatically computed by considering document relationships rather than concept relationships [40, 44].

The first work on fuzzy thesauri introduced the notion of fuzzy relations to represent associations between terms [48, 49]. Let us look at a formal definition of a fuzzy thesaurus [38, 39]. Consider T to be the set of index terms and C to be a set of concepts. Each term t∈T corresponds to a fuzzy set of concepts h(t):

$$h(t) = \{<c,\ t(c)> \mid c \in C\}, \tag{4}$$

in which t(c) is the degree to which term t is related to concept c. A measure M is defined on all of the possible fuzzy sets of concepts, which satisfies: $M(\varnothing) = 0$, $M(C) < \infty$, $M(A) \le M(B)$, if $A \subseteq B$.

A typical example of M is the cardinality of a fuzzy set. The fuzzy RT relation is represented in a fuzzy thesaurus by the similarity relation between two index terms, t_1 and $t_2 \in T$ and is defined as:

$$s(t_1, t_2) = M[h(t_1) \cap h(t_2)] / M[h(t_1) \cup h(t_2)], \tag{5}$$

This definition satisfies the following: if terms t1 and t2 are synonymous, i.e., h(t1) = h(t2), then s(t1,t2) = 1; if t1 and t2 are not semantically related, i.e., $h(t1) \cap h(t2) = \varnothing$, then s(t1,t2) = 0; s(t2,t1) = s(t1,t2) for all t1,t2 ∈ T; and if t1 is more similar to term t3 than to t2, then s(t1,t3) > s(t1,t2). The fuzzy NT relation, indicated as nt, which represents grades of inclusion of a narrower term t_1 in another (broader) term t_2, is defined as:

$$nt(t_1, t_2) = M[h(t_1) \cap h(t_2)] / M[h(t_1)], \tag{6}$$

This definition satisfies the following: if term t1's concept(s) is completely included within term t2's concept(s), i.e. $h(t1) \subseteq h(t2)$, then nt(t1,t2) = 1; if t1 and t2 are not semantically related, i.e., $h(t1) \cap h(t2) = \varnothing$, then nt(t1,t2) = 0; and if the inclusion of t1's concept(s) in t2's concept(s) is greater than the inclusion of t1's concept(s) in t3's concept(s), then nt(t1,t2) > nt(t1,t3).

By assuming M as the cardinality of a set, s and nt are given as:

$$s(t_1, t_2) = \sum_{k=1}^{M} \min[t_1(c_k), t_2(c_k)] / \sum_{k=1}^{M} \max[t_1(c_k), t_2(c_k)], \tag{7}$$

$$nt(t_1, t_2) = \sum_{k=1}^{M} \min[t_1(c_k), t_2(c_k)] / \sum_{k=1}^{M} t_1(c_k), \tag{8}$$

A fuzzy pseudothesaurus can be defined by replacing the set C in the definition of h(t) above with the set of documents D, with the assumption that h(t) is the fuzzy set of documents indexed by term t. This yields

$$h(t) = \{(d,\ t(d)) \mid d \in D\}, \tag{9}$$

in which t(d) = F(d,t) is the index term weight defined above. F can be either a binary value defining a crisp representation, or it can be a value in [0,1] to define a fuzzy representation of documents. The fuzzy RT and the fuzzy NT relations now are defined as:

$$s(t_1, t_2) = \sum_{k=1}^{M} \min[F(t_1, d_k), F(t_2, d_k)] / \sum_{k=1}^{M} \max[F(t_1, d_k), F(t_2, d_k)], \quad (10)$$

$$nt(t_1, t_2) = \sum_{k=1}^{M} \min[F(t_1, d_k), F(t_2, d_k)] / \sum_{k=1}^{M} F(t_1, d_k), \quad (11)$$

Note that $s(t_1,t_2)$ and $nt(t_1,t_2)$ are dependent on the co-occurrences of terms t_1 and t_2 in the set of documents, D. The set of index terms of document d, i.e., $\{t \mid F(d,t) \neq 0$ and $t \in T\}$, can be augmented by those terms t_A which have $s(t,t_A) > \alpha$ and/or $nt(t,t_A) > \beta$ for parameters α and β∈[0,1].

A thesaurus can be generated based on the max-star transitive closure for linguistic completion of a thesaurus generated initially by an expert linking terms [4]. A probabilistic notion of term relationships can be employed by assuming that if one given term is a good discriminator between relevant and non relevant documents, then any term that is closely associated with that given term (i.e., statistically co-occurring) is likely to be a good discriminator, too [60]. Note that this implies that thesauri are collection-dependent.

One can also expand on Salton's [50] use of the F(d,t) values. Salton [51] infers term relationships from document section similarities. On the other hand, one can manipulate the F(d,t) values in order to generate co-occurrence statistics to represent term linkage weights [31]. Here, a synonym link is considered, defined as:

$$\mu_{synonym}(t_1, t_2) = \sum_{d \in D} [F(d, t_1) \leftrightarrow F(d, t_2)], \quad (12)$$

where $F(d, t_1) \leftrightarrow F(d, t_2) = \min[F(d, t_1) \rightarrow F(d, t_2),\ F(d, t_1) \leftarrow F(d, t_2)]$ and $F(d,t_1)$ $F(d,t_2)$ can be defined in variety of ways. For instance, $F(d,t_1)$ $F(d,t_2)$, the implication operator, can be defined as $[F(d, t_1)^c \vee F(d, t_2)]$, where $F(d, t_1)^c = 1 - F(d, t_1)$ is the complement of $F(d, t_1)$ and $\vee$ is the disjunctive (OR) operator defined as the max; or it can be defined as $\min(1, [1 - F(d, t_1) + F(d, t_2)])$. A narrower term link (where term t_1 is narrower than term t_2, so term t_2 is broader than term t_1), is defined as:

$$\mu_{narrower}(t_1, t_2) = \sum_{d \in D} [F(d, t_1) \rightarrow F(d, t_2)], \quad (13)$$

Note that fuzzy narrower relationships defined between fuzzy sets can help the purpose of identifying generalization and specialization of topics, while the fuzzy similarity relationship between fuzzy sets can be of aid to identify similar topics.

5.4 Fuzzy Clustering for Documents

Clustering in information retrieval is a method for partitioning D, a given set of documents, into groups using a measure of similarity (or distance) which is defined on every pairs of documents. Grouping like documents together is not a new phenomenon, especially for librarians. The similarity between documents in the same group should be large, while the similarity between documents in different groups should be small.

A common clustering method is based on the simultaneous occurrences of citations in pairs of documents. Documents are clustered using a measure defined on the space of the citations. Generated clusters can then be used as an index for information retrieval, i.e., documents which belong to the same clusters as the documents directly indexed by the terms in the query are retrieved.

Similarity measures have been suggested empirically or heuristically, sometimes analogously to the similarity measures for documents matched against queries [52, 54, 56]. When adopting a fuzzy set model, clustering can be formalized as a kind of fuzzy association. In this case, the fuzzy association is defined on the domain $D \times D$. By assuming R(d) to be the fuzzy set of terms representing a document d with membership function values $d(t) = F(d,t)$ being the index term weights of term t in document d, the symmetric fuzzy relation s, as originally defined above, is taken to be the similarity measure for clustering documents:

$$s(d_1, d_2) = \sum_{k=1}^{M} \min[d_1(t_k), d_2(t_k)] / \sum_{k=1}^{M} \max[d_1(t_k), d_2(t_k)], \quad (14)$$

$$nt(d_1, d_2) = \sum_{k=1}^{M} \min[F(t_k, d_1), F(t_k, d_2)] / \sum_{k=1}^{M} \max[F(t_k, d_1), F(t_k, d_2)], \quad (15)$$

in which T is the set of index terms in the vocabulary and M is the number of index terms in T.

In fuzzy clustering, documents can belong to more than one cluster with varying degree of membership [3]. Each document is assigned a membership value to each cluster. Modified fuzzy clustering, also called soft clustering, uses thresholding mechanisms to limit the number of documents belonging to each cluster. The main advantage of using modified fuzzy clustering is the fact that the degree of fuzziness is controlled.

6 A Query Evaluation Mechanism

Query processing within retrieval can be interpreted as a decision-making activity. Its aim is to evaluate a set of alternatives or possible solutions, in this case a set of documents, based upon some criteria or selection conditions in order to select the optimal list (perhaps ranked) of documents in response to a user's query.

In a Boolean query, the alternatives are the document representations as described based on the presence or absence of index terms or keywords. The selection conditions, as expressed by terms specified in a query, define a set of constraints requiring the presence or absence of these terms within a document's representation. These conditions are expressed connected by aggregation operators, i.e., the Boolean logic operators of AND, OR, and NOT. The decision process is performed through an exact matching function, which is strictly dependent on the system query language

Given a fuzzy approach to retrieval, query processing can be regarded as a decision activity affected by vagueness. The query can be seen as the specification of a set of soft constraints, i.e. vague selection conditions that the documents can satisfy to a partial extent. The documents described through the significance degrees of the index terms constitute the alternatives. The query evaluation mechanism is regarded as fuzzy decision process that evaluates the degree of satisfaction of the query constraints by each document representation by applying a partial matching function. This degree is the RSV and can be interpreted as the degree of relevance of the document to the query and is used to rank the documents. Then, as a result of a query evaluation, a fuzzy set of documents is retrieved in which the RSV is the membership value. In this case the definition of the partial matching function is strictly dependent on the query language, specifically on the semantics of the soft constraints.

A wish list of requirements that a matching function of an IRS must satisfy has been proposed [21, 61]. Included in this list is the separability property that the evaluation of an atomic selection condition for an individual term in a query should be independent of the evaluation of the other atomic components or their Boolean connectors. The matching function should be based solely upon a function evaluating atomic conditions. Following the calculation of these evaluations, one can then aggregate them based upon the Boolean operators in the query. It has been shown that this property guarantees a homomorphic mapping from the space of all single terms to the space of all possible Boolean queries using these terms [1]. This property has been considered widely within fuzzy retrieval models, especially in the definition of flexible query languages.

By designing the partial matching mechanism from the bottom-up the separability property is ensured. First, each atomic selection condition or soft constraint in the query is evaluated by a function E for a given document. Then the aggregation operators are applied to the results starting from the inmost operator in the query to the outermost operator by a function E*. This E function evaluates the soft constraints associated with the query atoms on the fuzzy set R_d representing each document, where these soft constraints are defined as fuzzy subsets. The membership value $\mu_{atom}(i)$ is the degree of satisfaction of the soft constraint associated with the atomic query *atom*, i.e., E(<*atom*>,d) = μ_{atom}(F(d,t)). In other words, E evaluates how well the term t, which has an indexing weight F(d,t) for document d, satisfies the soft constraint specified by *atom*. The result of the evaluation is a fuzzy set, $\sum d \in D$ μatom(F(d, t))/d in which μ_{atom}(F(d,t)) is interpreted as the RSV of document d with respect to the query atom.

The function E^*: $D \times Q \rightarrow [0, 1]$, where Q is the set of all the proper queries in the query language, evaluates the final RSV of a document, reflecting the satisfaction of the whole query. The definition of E* depends strictly upon the structure of the query language, specifically upon the aggregation operators used to combine the atomic components. The AND connective is classically defined as the minimum (min) operator, the OR connective as the maximum (max) operator, and the NOT connective as the one-minus (1-) or complement operator. These definitions preserve the idempotence property. A fuzzy generalization of the Boolean query structure has been defined in which the Boolean operators are replaced by linguistic quantifiers [10]. In this context, linguistic quantifiers are used as aggregation operators to determine the degree of satisfaction for the soft constraints. They allow to improve as well as to simplify the expressiveness of the Boolean query language.

7 Query Weights

To render a Boolean query language to be more user friendly and more expressive, one can extend the atomic selection conditions by introducing query term weights [5, 7, 20, 47]. An example of weighted query is the following: $<t_1, w_1>$ AND ($<t_2, w_2>$ OR $<t_3, w_3>$) in which t_1, t_2, t_3, are search terms with numeric weights w_1, w_2, and w_3 in the interval [0,1]. These weights are implicitly given as being equal to 1 in the classical Boolean query language.

The concept of query weights raises the problem of their interpretation. Several authors have realized that the semantics of query weights should be related to the concept of the "importance" of the terms. Being well aware that the semantics of the query term weights influences the definition of the partial matching function, specifically the E function, different semantics for the soft constraint imposed by a pair $<t,w>$ have been proposed in the literature trying to satisfy as much as possible properties of the wish list in particular the separability property.

Early on, query weights were interpreted as a relative importance weight where the separability property does not hold. Two distinct definitions of E have been proposed for conjunctive and disjunctive queries, respectively [5, 63]. Later, other models [20, 47, 61] used an interpretation of the query weights w as a threshold on the index term weight or as an ideal index term weight [7, 22].

7.1 Implicit Query Weights

The simplest extension of the Boolean model consists of the adoption of a weighted document representation with a classical Boolean query language [17]. This retrieval mechanism ranks the retrieved documents in decreasing order of their significance with respect to the user query. In this case, an atomic query consisting of a single term t is interpreted as the specification of a pair/$<t,1>$ in which $w = 1$

is implicitly specified. The soft constraint associated with <t,1> is then interpreted as the requirement that the index term weight be "close to 1" and its evaluation is defined as $\mu_w(F(d,t)) = F(d, t)$. This means that the desired documents are those with maximum index term weight for the specified term t, i.e., closest to 1. This interpretation implies that the evaluation mechanism tolerates the satisfaction of the soft constraint associated with <t,1> with a degree equal to F(d,t).

7.2 Relative Importance Query Weights

Here, query weights are interpreted as measures of the "relative importance" of each term with respect to the other terms in the query [5, 63]. This interpretation allows the IRS to rank documents so that documents are ranked higher if they have larger index term weights for those terms that have larger query weights. However, since it is not possible to have a single definition for the soft constraint μ_w that preserves the "relative importance" semantics independently of the Boolean connectors in the query, two distinct definitions of μ_w have been proposed, depending on the aggregation operators in the query. This approach, sadly, gives up the separability property. Two alternative definitions have been proposed for conjunctive and disjunctive queries [5, 63]. The first proposal [5] yields $\mu_w(F(d,t)) = [w * F(d, t)]$ for disjunctive queries and $\mu_w(F(d,t)) = \max(1, F(d, t)/w)$ for conjunctive queries; while the second proposal [56] yields $\mu_w(F(d,t)) = \min[w, F(d, t)]$ for disjunctive queries and $\mu_w(F(d,t)) = \max[(1- w), F(d, t)]$ for conjunctive queries. Notice that any weighted Boolean query can be expressed in disjunctive normal form (DNF) so that any query can be evaluated by using one of these two definitions.

7.3 Threshold Query Weights

To preserve the separability property, an approach treating the query weights as thresholds has been suggested [20, 47]. By specifying query weights as thresholds the user is asking to see all documents "sufficiently about" a topic. In this case, the soft constraint identified by the numeric query weight can be linguistically expressed as "more or less over w". Of course, the lower the threshold, the greater the number of documents retrieved. Thus, a threshold allows a user to define a point of discrimination between under- and over satisfaction.

The simplest formalization of threshold weights has been suggested as a crisp threshold [47].

$$\mu w(F(d,t)) = \begin{cases} 0 & \text{for} \quad F(d,t) < w \\ F(d,t) & \text{for} \quad F(d,t) \geq w \end{cases}, \tag{16}$$

In this case, the threshold defines the minimally acceptable document. Due to its inherent discontinuity, this formalization might lead to an abrupt variation in the number of documents retrieved for small changes in the query weights. To remedy this, continuous threshold formalization has been suggested [20]:

$$\mu_w(F(d,t)) = \begin{cases} P(w) * \frac{F(d,t)}{w} & \text{for} \quad F(d,t) < w \\ P(w) + Q(w) * \frac{(F(d,t) - w)}{(1 - w)} & \text{for} \quad F(d,t) \geq w \end{cases}., \tag{17}$$

where P(w) and Q(w) might be defined as $P(w) = \frac{1+w}{2}$ and $Q(w) = \frac{1-w^2}{4}$. For F(d,t) < w, the μ_w function measures the closeness of F(d,t) to w; for F(d,t) ≥ w, $\mu_w(F(d,t))$ expresses the degree of over satisfaction with respect to w, and under satisfaction with respect to 1.

7.4 Ideal Query Weights

Another interpretation for the query weights has been defined [7, 22]. Here, the pair <t,w> identifies a set of ideal or perfect documents so that the soft constraint μ_w measures how well F(d,t) comes close to w, yielding

$$\mu_w(F(d,t)) = e^{\ln(k) * (F(d,t) - w)^2}, \tag{18}$$

The parameter k in the interval [0,1] determines the steepness of the Gaussian function's slopes. As a consequence, k will affect the strength of the soft constraint "*close to w*". So, the larger the value of k is, the weaker the constraint becomes. This parametric definition makes it possible to adapt the constraint interpretation to the user concept of "*close to w*" [7]. The retrieval operation associated with a pair <t,w> corresponds in this model to the evaluation of a similarity measure between the importance value w and the significance value of t in R_d: w ≈ F(d,t).

7.5 Linguistic Query Weights

The main limitation of numeric query weights is their inadequacy in dealing with the imprecision which characterizes the concept of importance that they represent. In fact, the use of numeric query weights forces the user to quantify a qualitative and rather vague notion and to be aware of the weight semantics. Thus, a fuzzy retrieval model with linguistic query weights has been proposed [12] with a linguistic extension of the Boolean query language based upon the concept of a linguistic variable [67]. With this approach, the user can select the primary linguistic term "*important*" together with linguistic hedges (e.g., "*very*" or "*almost*") to qualify the desired importance of the search terms in the query. When defining such a query language the term set, i.e., the set of all the possible linguistic values of the linguistic variable importance, must be defined. Such a definition depends on the desired granularity that one wants to achieve. The greater the number of the linguistic terms, the finer the granularity of the concepts that are dealt with. Next, the semantics for the primary terms must be defined. A pair <t, important> , expresses a soft constraint $\mu_{\text{important}}$ on the term significance values (the F(d,t) values). The evaluation of the relevance of a given document d to a query consisting solely of the pair <t, important> is based upon the evaluation of the degree of satisfaction of the associated soft constraint $\mu_{\text{important.}}$

The problem of giving a meaning to numeric weights reappears here in associating a semantic with the linguistic term important. The $\mu_{\text{important}}$ function is defined based on the ideal semantics of the numeric weight to yield [12].

$$\mu_{important}(\mathrm{F(d,t)}) = \begin{cases} e^{\ln(k)*(F(d,t)-i)^2} & \text{for} \quad F(d,t) < i \\ 1 & \text{for} \quad i \le F(d,t) \le j \\ e^{\ln(k)*(F(d,t)-j)^2} & \text{for} \quad F(d,t) > j \end{cases}, \tag{19}$$

We see that if F(d,t) is less than the lower bound i or greater than the upper bound j, the constraint is under satisfied. The strength of the soft constraint $\mu_{\text{important}}$ depends upon both the width of the range [i, j] and the value of the k parameter. The values i and j delimit the level of importance for the user. We note that as the value |i - j| increases, the soft constraint becomes less precise. So, for the case of the ideal semantics of numeric query term weights, k determines the sharpness of the constraint in that ask increases, the constraint increases in fuzziness.

We can define the $\mu_{\text{important}}$ function based upon the threshold semantics to yield [34]

$$\mu_{important}(\mathrm{F(d,t)}) = \begin{cases} \frac{1+i}{2} * e^{\ln(k)*(F(d,t)-i)^2} & \text{for} \quad F(d,t) < i \\ \frac{1+F(d,t)}{2} & \text{for} \quad i \le F(d,t) \le j \\ \frac{1+j}{2} * \left(1 + \frac{F(d,t)-j}{2}\right) & \text{for} \quad F(d,t) > j \end{cases}. \tag{20}$$

We note that this compatibility function is continuous and non-decreasing in F(d,t) over the interval [0,1]. For F(d,t) < i, $\mu_{\text{important}}$ increases as a Gaussian function. For F (d,t) in the interval [i,j], μ_{import}ant increases at a linear rate. For F(d,t) > j, μimportant still increases, but at a lesser rate. The compatibility functions of non-primary terms, such as *very important* or *fairly important*, are derived by modifying the compatibility functions of primary terms. This is achieved by defining each linguistic hedge as a modifier operator. For example, the linguistic hedges are defined as translation operators in [34] to yield:

$\mu_{\text{very important}}(x) = \mu_{\text{important}}(x)$ with $i_{\text{very}} = i + 0.2$ and $j_{\text{very}} = j + 0.2$ and $\forall x \in [0,1]$.

$\mu_{averagely\ important}$ (x) = $\mu_{important}$ (x) with $i_{averagely}$ = i - 0.3 and $j_{averagely}$ = j - 0.3 and ∀x ∈ [0,1].

$\mu_{minimally\ important}$ (x) = $\mu_{important}$ (x) with $i_{minimally}$ = i - 0.5 and $j_{minimally}$ = j - 0.5 and ∀x ∈ [0,1],

in which i and j are values in [0,1] delimiting the range of complete satisfaction of the constraint $\mu_{important}$. With these definitions, any value F(d,t) of the basic domain of the importance variable fully satisfies at least one of the constraints defined by the linguistic query terms. In [30] a query language with linguistic query weights having heterogeneous semantics have been proposed so as to benefit the full potential offered of a fuzzy set to model subjective needs.

8 Linguistic Quantifiers to Aggregate the Selection Conditions

In a classical Boolean query language, the AND and OR connectives allow only for crisp (non-fuzzy) aggregations which do not capture any of the inherent vagueness of user information needs. For example, the AND used for aggregating M selection conditions does not tolerate the no satisfaction of but a single condition which could cause the no retrieval of relevant documents. To deal with this problem, additional extensions of Boolean queries have been provided which involves the replacement of the AND and OR connectives with soft operators for aggregating the selection criteria [46, 54, 55].

Within the framework of fuzzy set theory, a generalization of the Boolean query language has been defined based upon the concept of linguistic quantifiers that are employed to specify both crisp and vague aggregation criteria of the selection conditions [10]. New aggregation operators can be specified by linguistic expressions with self-expressive meaning, such as at least k and most of. They are defined to exist between the two extremes corresponding to the AND and OR connectives, which allow requests for *all* and *at least one of* the selection conditions, respectively. The linguistic quantifiers used as aggregation operators, are defined by ordered weighted averaging (OWA) operators.

Adopting linguistic quantifiers more easily and intuitively formulate the requirements of a complex Boolean query. A quantified aggregation function can be applied not only to single selection conditions, but also to other quantified expressions. Then, the E* function evaluating the entire query yields a value in [0,1] for each document d in the archive D. If S is the set of atomic selection conditions and Q is the set of legitimate Boolean queries over our vocabulary of terms, then the E* function can be formalized by recursively applying the following rules:

- if q∈ S then E * (d, s) = μ_w(F(d, t)) in which μ_w(F(d,t)) is the satisfaction degree of a pair <t,w> by document d with w being either a numeric weight or a linguistic weight.

- if q = quantifier (q_1,…,q_n) and q_1,…,q_n ∈ Q then E*(d,q) = OWA $_{quantifier}$(E*(d,q_1),…, E * (d, q_n))E * (d, NOTq) = 1 - E * (d, q) in which OWA $_{quantifier}$ is the OWA operator associated with quantifier.

The formal definition of the query language with linguistic quantifiers with the following quantifies has been generated [10]

- all replaces AND;
- at least k acts as the specification of a crisp threshold of value k on the number of selection conditions and is defined by a weighting vector $w_{at\ least\ k}$ in which $w_k = 1$, and $w_j = 0$, for $i \le k$ – noting that at least 1 selects the maximum of the satisfaction degrees so that it has the same semantics of OR;
- about k is a soft interpretation of the quantifier at least k in which the k value is not interpreted as a crisp threshold, but as a fuzzy one so that the user is fully satisfied if k or more conditions are satisfied but gets a certain degree of satisfaction even if k-1, k-2,…,1 conditions are satisfied - this quantifier is defined by a weighting vector $w_{about\ k}$ in which $w_i = \frac{i}{\sum_{j=1}^{k} j}$ for $i \le k$, and $w_i = 0$ for $i > k$;
- most is defined as a synonym of at least $\frac{2}{3}$ n in which n is the total number of selection conditions.

With respect to non-fuzzy approaches that tried to simplify the Boolean formulations, the fuzzy approach subsumes the Boolean language, allows reformulating Boolean queries in a more synthetic and comprehensible way, and improves the Boolean expressiveness by allowing flexible aggregations. Other authors have followed these ideas by proposing alternative formalization of linguistic query weights and flexible operators based on ordinal labels and ordinal aggregations [29], thus reducing the complexity of the evaluation mechanism.

9 Emerging Applications of Fuzzy Set Theory to Model Information Retrieval Tasks

9.1 Geographic Information Retrieval

An emerging task in information retrieval is retrieving documents relevant with respect to both a content based condition and a geographic condition.

Such applications involve the management of uncertainty and imprecision and the modeling of user preferences and context. Indexing the geographic content of documents implies dealing with the ambiguity, synonymy and homonymy of geographic names in texts. On the other side, the evaluation of queries specifying both content based conditions and spatial conditions on documents contents requires representing the vagueness and context dependency of spatial conditions and the personal user's preferences. The spatial condition can be specified linguistically in the query through vague terms such as "close to the North East of Milan", whose semantic depends on the user's context and perception of distance.

In [25] a geographic information retrieval model has been defined that represents both the uncertainty in indexing the geographic documents' content and the user's context and preferences in evaluating flexible spatial queries.

Finally, the system allows evaluating two types of queries flexibly combining the content based condition, such as "*vegetarian Restaurants*" with the spatial condition "*close to Milano Central Ralway station*". The spatial condition "*close*" is defined as a soft constraint on the user's perceived distance between the documents' footprint and query's footprint [8].

For each retrieved document, two relevance scores are computed with respect to the two query conditions that are flexibly combined to generate an overall ranked list of documents. The user can choose the semantic for the combination, that can be either an asymmetric "and possibly" aggregation between the mandatory content condition and the optional spatial condition, or a compensative "average" aggregation, defined as a linear combination of the two conditions; further, a relative preference between the conditions can be specified to achieve personalization and effectiveness.

9.2 Aggregation of Multi Dimensional Relevance Dimensions

Relevance assessment is usually based on the evaluation of multiple criteria, also called relevance dimensions, which are aimed to capture different aspects or properties of the considered document or document/user context. All the considered dimensions concur to estimate the utility of a document with respect to the considered user's query. The concept of page popularity in search engines is an example of a relevance dimension that is usefully exploited in the process of documents' relevance estimate. In the multidimensional relevance assessment each dimension is usually evaluated in an independent way, and a numeric score is associated with each dimension for each document. To obtain an overall relevance score the single scores will get aggregated into an overall score representing the document's RSV.

Among the aggregation operators, traditional non-compensatory operators, such as the min and the max operator allow to set up a pessimistic (e.g. min, as an example of T-norm operator) or optimistic (e.g. max, as an example of T-conorm operators) aggregation scheme, while traditional averaging aggregation operators are totally compensatory, i.e., a lack in the satisfaction of a criterion can be compensated by the surplus satisfaction of another one. This property is not very realistic in many real applications in general, and in particular in Information Retrieval (IR). In [23–25] two prioritized aggregation operators for multidimensional relevance assessment have been proposed (the *scoring* and the *prioritized and* operators), and they have been evaluated by a user-centered approach that has been conducted in a personalized IR setting, where four relevance dimensions have been considered (aboutness (or topicality), coverage, appropriateness and reliability). An interesting aspect in considering a personalized IR setting to evaluate the prioritized aggregation is that the priority over the considered relevance dimensions may be user dependent; by setting different priority orders over the four considered relevance dimensions, different types of users' can be identified, with distinct search intents. The main impact in making the prioritized aggregation scheme user dependent is that for a same query and a same user different document rankings can be obtained.

9.3 Discovery of Similar Contents in Web Pages Retrieved by Multiple Queries

Another recent application of fuzzy set in information retrieval is related with the task of organizing and discovering the contents of the Web pages retrieved by several query reformulations of the same information need.

It may often happen that by reformulating a query by slightly changing some words or adding new terms to a previous query new and already retrieved documents are represented in the list of results.

In order to discover the common contents retrieved by two or more queries and select only diversified contents by eliminating near duplicates an approach defined in [6] consists in applying soft operators to distinct lists of Web pages retrieved by either the same query submitted to distinct search engines or similar queries submitted to the same search engine.

The approach proposes [6] the soft ranked intersection to generate from two lists of web pages retrieved by two distinct searches a cluster of Web pages with similar contents, and the soft ranked union to generate a cluster of Web pages with diversified contents. The soft operators work on the representation of the Web pages provided by information granules consisting of the Web pages titles, url string, and snippets displayed in the result page, thus without the need to access the Web page content so as to achieve efficiency. The soft operators allow users to discover and reveal the hidden shared topics retrieved by multiple Web searches, possibly identifying near duplicates and selecting Web pages with diversified contents.

9.4 Ememe Identification and Tracking

A very up to date task in information retrieval is the identification and tracking of the evolution of Internet Meme, hereafter named ememe, intended as a unit of information (idea) replicated and propagated through the Web by one or more applications such as social networks and blogs. The Web constitutes a huge information repository, and several Internet based-services and applications represent a quite rich soil for memes' growth and evolution. Like blogs, e-mails, and social network applications.

The conceptual notion of ememe has been intended in a simple form as a replicated or paraphrased sentence by the scientific literature on meme tracking. The first attempt that considers an ememe not as merely a sequence of words or a quoted sentence but as a set of interrelated concepts has been proposed in [6] where a method based on the definition of an ememe by an OWL schema, and a process for retrieving ememe instances and fusing them by means of Ordered Weighted Averaging Operators has been proposed. Specifically, by taking inspiration from the anthropologic literature on memes a methodology to formally define, identify, revise and measure some characteristic properties of ememes on the Blogosphere was defined.

The proposed method for identifying ememes on the Blogosphere needs a preliminary user-system interaction [13]; the user is in fact asked to provide an "a priori" core definition of the ememe that he/she wants to identify; the core definition can be specified by a conceptual schema expressed in OWL, and it is provided in input to a pull mechanism that will search the candidate instances of the given ememe into one or several source repositories of the blogosphere; the blogs to be analysed can be also specified by the user, and if not provided the system will search on the whole blogosphere.

Once the OWL core definition has been completed, some textual queries are automatically generated and submitted to a search engine that will inquiry the blogosphere to retrieve blogs' posts that potentially contain ememe instances.

Finally, a filtering process is defined, which takes in input the retrieved blogs' posts and selects those containing the ememe instances which will be used in the phase aimed at revising and enriching the original core definition of the ememe. The aim of the filtering process is to identify the reliable ememe instances among the results produced by the search process. The overall satisfaction value obtained by the aggregation operator is then used to filter the first M relevant ememe instances. The evaluation function associated with the linguistic quantifier *some* is defined by an Ordered Weighted Averaging Operator.

10 Fuzzy Performance Measures

Major factors in evaluating retrieval performance include the cost, time, and effort to retrieve relevant items, as well as user satisfaction. The standard measures of effectiveness include recall, the proportion of relevant documents retrieved (related to 1-α or 1-Type I error in classical statistics the proportion of retrieved documents that are relevant) and precision, the proportion of retrieved documents that are relevant (1-β or 1-Type II error in classical statistics). Combining recall and precision can yield

$$E = 1 - 1/\left[\alpha P^{-1} + (1 - \alpha R^{-1}\right) \text{ or } F = (1+\alpha)^{*}P^{*}R/(\alpha P + R), \quad (21)$$

where α is a user specified parameter. Other measures include G = generality = proportion of documents that are relevant (one very weak rule based on the notion that the act of retrieval should convey relevance information is that precision ≥ generality), and Fa = fallout = proportion of non relevant documents retrieved. Another measure is $S = (1/2)[1 + (S^+ + S^-)/S_{max}]$ for ordering, where S^+ = number of pairs in ranked list where order is correct; S^- = number of pairs in ranked list where order is correct; S_{max} = maximal value of S^+.

One problem with current criteria to measure the effectiveness of IR systems is the fact that recall and precision measures have been defined by assuming that relevance is a Boolean concept. In order to take into account the fact that IR systems rank the retrieved documents based on their RSVs that are interpreted either as a probabilities of relevance, similarity degrees of the documents to the query, or as degrees of relevance, Recall-Precision graphs are produced in which the values of precision are computed at standard levels of recall. Then, the average of the precision values at different recall levels is computed to produce a single estimate.

Nevertheless, these measures do not evaluate the actual values of the RSVs associated with documents and do not take into account the fact that also users can consider relevance as a gradual concept. For this reason some authors have proposed some fuzzy measure of effectiveness. [19] proposed the evaluation of fuzzy recall and fuzzy precision, defined as follows:

$$\text{Fuzzy Precision} = \frac{\sum_d \min(e_d, u_d)}{\sum_d e_d}, \tag{22}$$

$$\text{Fuzzy Recall} = \frac{\sum_d \min(e_d, u_d)}{\sum_d u_d}, \tag{23}$$

where u_d is the user's evaluation of the relevance of document d (u_d can be binary or defined in the interval [0,1]) and e_d is the RSV of document d computer by the IR system. These measures take into account the actual values of e_d and u_d, rather than the rank ordering based in descending order on e_d.

These measures can be particularly useful to evaluate the results of fuzzy clustering algorithms.

11 Experimental Results

A comparison of the results produced by using the traditional fuzzy representation of documents and the fuzzy representation of structured documents can be found in [10]. In this experiment, a collection of 2500 textual documents about descriptions of CNR research projects has been considered. The indexing module of the prototypal information retrieval system named DOMINO, used for the experiment, has been extended in order to be able to recognize in the documents any structure simply by specifying it into a definition file. In this way it is not necessary to modify the system when dealing with a new collection of documents with a different structure. The definition of the documents sections has been made before starting the archive generation phase. During this phase it was also necessary to specify the criteria by which to compute the significance degrees of the terms in each section. Two kinds of sections have been identified: the “structured” sections, i.e., the research code, title, research leader, and the “narrative” sections, containing unstructured textual descriptions, i.e., the project description and the project objective. It has been observed that while the values of precision remain unchanged in the two versions of the system, the values of recall are higher by using the structured representation than those obtained by using the traditional fuzzy representation.

We illustrate another approach which produces a weighted representation of documents written in HTML [41]. An HTML document has a specific syntactic structure in which its subparts have a given format specified by the delimiting tags. In this context, tags are seen as syntactic elements carrying an indication of the importance of the associated text. When writing a document in HTML, an author associates varying importance to each of the different subparts of a given document by delimiting them by means of appropriate tags. Since a certain tag can be employed more than once, and in different positions inside the document, the concept of document subpart is not meant as a unique, adjacent piece of text. Such a structure is subjective and carries the interpretation of the document author. It can be applied in archives, which collect heterogeneous documents, i.e. documents with possibly different “logical” structures.

An indexing function has been proposed which provides different weights for the occurrences of a given term in the document, depending on the tags by which they are delimited [41]. The overall significance degree F(d,t) of a term t in a document d is computed by first evaluating the term significance in the different document tags, and then by aggregating these contributions. With each tag, a function $F_{tag}: D \times T \rightarrow [0, 1]$ is associated together an importance weight $\mu_{tag} \in [0, 1]$. Note that the greater the emphasis of the text associated with a tag, the greater its importance weight. A possible ranking of the considered tags has been suggested [41] in decreasing order of tag importance. The definition of such a list is quite subjective, although based on objective assumptions suggested by commonsense. These rankings include notion such as a larger font, or text in boldface or italics or appearing in a list can be assumed as having a higher importance.

To simplify the hierarchy of the tags, we see that certain tags can be employed to accomplish similar aims, so one can group them into different classes. It is assumed that the members of a class have the same importance weight. Text not delimited by any tag is included into the lowest class. A simple procedure to compute numeric importance weights starting from the proposed ranking can be achieved. The definition of F_{tag} follows the same mechanism as the previous approach [10].

Once the single significance degrees of a term into the tags have been computed, these have to be aggregated in order to produce an overall significance degree of the term into the document. In the aggregation all the significance degrees should be taken into account, so as to consider the contribution of each tag, modulated by their importance weights. To this aim a weighted mean can be adopted: $A(F_{tag\,1}(d,t), \ldots F_{tag\,n}(d,t)) = \sum_{i=1..n} F_{tag\,i}(d,t) * w_i$ in which $\sum i_{=1..n} w_i = 1$. Starting from the list of tags in decreasing relative order of their importance, the numeric weights w_i are computed through a simple procedure. Assuming that tag_i is more important than tag_j iff $i < j$ (being i and j the positions of tag_i and tag_j respectively in the ordered list), the numeric importance weight w_i associated with tag_i can be computed as: $w_i = (n - i + 1)/\sum i_{=1..n} i$. In the computation of the overall significance degree F(d,t), the inverse document frequency of term t could be taken into account (the definition of $g(IDF_t)$ is given in formula (2)):

$$F(d, t) = (\sum_{i=1...n} F_{tag\,i}(d, t) * g(IDF_t), \tag{24}$$

12 Conclusions

This entry reviews the main objectives and characteristics of the fuzzy modeling of the information retrieval activity with respect to alternative approaches such as probabilistic IR and Vector space IR. The focus of the fuzzy approaches is on modeling imprecision and vagueness of the information with respect to uncertainty. The fuzzy generalizations of the Boolean Retrieval model have been discussed by describing the fuzzy indexing of structured documents, the definition of flexible query languages subsuming the Boolean language, and the definition of fuzzy associations to expand either the indexes or the queries, or to generate fuzzy clusters of documents. Fuzzy similarity and fuzzy inclusion relationships between fuzzy sets have been introduced that can help to define more evolved fuzzy IR models performing "semantic" matching of documents and queries, which is the current trend of research in Information retrieval.

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Is the World Itself Fuzzy? Physical Arguments and Unexpected Computational Consequences of Zadeh's Vision

Vladik Kreinovich and Olga Kosheleva

Abstract Fuzzy methodology has been invented to describe imprecise ("fuzzy") human statements about the world, statements that use imprecise words from natural language like "small" or "large". Usual applications of fuzzy techniques assume that the world itself is "crisp", that there are exact equations describing the world, and fuzziness of our statements is caused by the incompleteness of our knowledge. But what if the world itself is fuzzy? What if there is no perfect system of equations describing the physical world – in the sense that no matter what system of equations we try, there will always be cases when this system leads to wrong predictions? This is not just a speculation: this idea is actually supported by many physicists. At first glance, this is a pessimistic idea: no matter how much we try, we will never be able to find the the Ultimate Theory of Everything. But it turns out that this idea also has its optimistic aspects: namely, in this chapter, we show (somewhat unexpectedly), that if such a no-perfect-theory principle is true, then the use of physical data can drastically enhance computations.

1 Fuzzy Techniques: The Original Zadeh's Vision

Pre-Zadeh attitude: everything can be made precise. Scientists and engineers use both formal languages and an imprecise natural language. In engineering practice, formulas, derivations, and computations – which are described in a precise language – intertwine with explanations – which are usually described in a natural language. Even in formal mathematics, when presenting a proof, a mathematician describes part of it in precise terms and part in imprecise terms from a natural language: "one can easily see that", "since ε is small, the difference $f(x+\varepsilon) - f(x)$ is also small", etc.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_16

In formal mathematics, usually, the imprecise parts can be reformulated in precise terms; professional mathematicians can do it, mathematics students are taught how to do it – and math students do not get good grades until they are able to perform such a reformulation. In rare occasions, an attempt for such a formalization reveals a gap in the proof, but in most such cases, this gap is later filled.

Similarly, when an engineer makes an imprecise argument, it does not necessarily mean that a more precise explanation is not possible: when needed, an engineer can usually provide a precise quantitative justification of his/her original qualitative decision.

A similar precisiation is often possible beyond science and engineering. For example, instructors who grade students' work use seemingly imprecise words like "excellent", "good", "satisfactory". However, in most cases, these words have a very precise meaning. In the US grading system, we usually add up well-defined points that the students got for different problems on the test. If the resulting grade is 90 (or higher) out of 100 possible points, we assign the grade "excellent" (A). If the resulting grade is at least 80 but smaller than 90, we consider this work "good" (grade B), etc.

Similarly, in medicine, many terms that are, at first glance, imprecise, have a very precise meaning. "High blood pressure" means upper blood pressure above 140, "fever" means temperature above 37.5 C, "overweight" means that the body-mass index (body mass in kg divided by the squared height in meters) is above 25, etc. In law, a child – a seemingly informal notion, with an imprecise transition – is legally defined as someone younger than 18 years old.

These example led scientists and engineers to conclude that in principle, all the statements can be made precise. According to this belief, when a statement sounds imprecise, it is only because we have not learned the corresponding terms yet. Once we learn these terms, the statement will become very precise.

Zadeh's vision. In 1965, Lotfi Zadeh published his revolutionary paper, in which he emphasized that:

– in addition to situations when use imprecise terms but have a precise meaning in mind ("excellent test results" meaning 90+ points),
– there are also many situations when we use imprecise terms for which *no precise meaning* is known.

Moreover, he showed that such situations, in which no precise meaning is known, in which the meaning is "fuzzy", are ubiquitous in many application areas.

To deal with such situations, L. Zadeh proposed techniques – which he called *fuzzy* – that enable researchers to describe their imprecise statements in precise mathematical terms, and thus, enables computer-based systems to process such statements. These techniques has led to many successful applications; see, e.g., [3, 5, 6, 9, 16, 20, 22, 23].

2 Is the World Itself Fuzzy? and if Yes, What Are Possible Physical and Computational Consequences?

Traditional viewpoint. The traditional viewpoint in engineering and science is that the world itself is crisp, it is described by precise equations which, in principle, enable us to predict either the events themselves (in classical, pre-quantum physics) or probabilities of different events (in quantum physics). The only reason for “fuzzy” uncertainty is that we only have partial knowledge about the world.

For example, when a meteorologist makes a “fuzzy” statement that there is a good chance of rain, the meteorologist usually believes that with more information, he/she would be able to make a more definite prediction.

But what if the world itself is fuzzy? But what if there are no ultimate equations? What if, no matter what equations we formulate, no matter how accurate their predictions are so far, there will always be cases when these equations will lead to wrong predictions?

In other words, what if not only our knowledge is fuzzy, what if the world itself is fuzzy?

Somewhat surprisingly, this is what many physicists actually believe. Many physicists indeed believe that every physical theory is approximate – no matter how sophisticated a theory, no matter how accurate its current predictions, inevitably new observations will surface which would require a modification of this theory; see, e.g., [2].

This belief can be justified by the history of physics: no matter how good a physical theory, no matter how good its accordance with observations, eventually, new observations appeared which were not fully consistent with the original theory – and thus, a theory needed to be modified. For example, for several centuries, Newtonian physics seems to explain all observable facts – until later, quantum (and then relativistic) effects were discovered which required changes in physical theories.

At first glance, this belief is pessimistic. This belief sounds pessimistic: no matter how much we try, we will never find the Ultimate Theory of Everything.

But maybe there is room for optimism. But is the situation indeed so pessimistic? After all, physics is not just about finding equations. Finding equations is an important first step, but the ultimate goal of physics is not to *find* equations, but to *predict* future events – and equations are an important first step towards this prediction.

Many physical equations are very complex, solving them is a complex computational task. From this viewpoint, any possibility to enhance computations would be a great optimistic development. For example, quantum physics is clearly more pessimistic in terms of possibility of predictions, because in quantum physics, we can often only predict probabilities of future events, and not the events themselves. On the other hand, research on quantum computing has shown that the use of quantum effects can drastically enhance computations; see, e.g., [17].

How does the no-perfect-theory belief affect computations? In this chapter, we analyze how the no-perfect-theory belief affects our computational abilities.

At first glance, the fact that no theory is perfect seems to make the question of computability rather hopeless: no matter how seriously we analyze computability within a given physical theory, eventually, this theory will turn out to be, strictly speaking, false – and thus, our analysis of what is computable will have to be redone.

In this chapter, we show, however, that in spite of this seeming hopelessness, some important answers to the question of what is computable can be deduced simply from the fact no physical theory is perfect – namely, in this case, we show that computations can be enhanced in comparison with the usual (Turing machine) computability.

Comment. Some preliminary results from this chapter first appeared in [7, 8, 12, 25].

3 How to Describe, in Precise Terms, that No Physical Theory Is Perfect

Discussion. The statement that no physical theory is perfect means that no matter what physical theory we have, eventually there will be observations which violate this theory. To formalize this statement, we need to formalize what are observations and what is a theory.

What are observations? Each observation can be represented, in the computer, as a sequence of 0 s and 1s; actually, in many cases, the sensors already produce the signal in the computer-readable form, as a sequence of 0 s and 1s.

An exact description of each experiment can also be described in precise terms, and thus, it will be represented in a computer as a sequence of 0 s and 1s. An experiment should specify how long we wait for the result; in this way, we are guaranteed that we get the result. The coding should be done in such a way that the waiting time does not exceed a polynomial of the length of the code i; for example, if we want to wait for t moments of time, we should just add t copies of an appropriate wait symbol.

In each experiment, we can specify which bit of the result we are interested in; for convenience, we can consider producing different bits as different experiments.

Each such experiment is represented as a sequence of 0 s and 1s; by appending 1 at the beginning of this sequence, we can view this sequence as a binary expansion of a natural number i. This natural number will serve as the "code" describing the experiment. For example, a sequence 001 is transformed into $i = 1001_2 = 9_{10}$. (We need to append 1, because otherwise two different sequences 001 and 01 will be represented by the same integer).

For natural numbers i which correspond to experiment descriptions, let ω_i denote the bit result of the experiment described by the code i.

Let us also define ω_i for natural numbers i which do not correspond to a syntactically correct description of experiments. For example, we can take $\omega_i = 0$ for such numbers i.

In these terms, all past and future observations form a (potentially) infinite sequence $\omega = \omega_1\omega_2 \ldots$ of 0 s and 1s, $\omega_i \in \{0, 1\}$.

What is a physical theory from the viewpoint of our problem: a set of sequences. A physical theory may be very complex, but all we care about is which sequences of observations ω are consistent with this theory and which are not. In other words, for our purposes, we can identify a physical theory T with the set of all sequences ω which are consistent with this theory.

Not every set of sequences corresponds to a physical theory: the set T must be non-empty and definable. Not every set of sequences comes from a physical theory. First, a physical theory must have at least one possible sequence of observations, i.e., the set T must be *non-empty*.

Second, a theory – and thus, the corresponding set – must be described by a finite sequence of symbols in an appropriate language. Sets which are uniquely by (finite) formulas are known as *definable*. Thus, the set T must be definable.

Since at any moment of time, we only have finitely many observations, the set T must be closed. Another property of a physical theory comes from the fact that at any given moment of time, we only have finitely many observations, i.e., we only observe finitely many bits. From this viewpoint, we say that observations $\omega_1 \ldots \omega_n$ are consistent with the theory T if there is a continuing infinite sequence which is consistent with this theory, i.e., which belongs to the set T.

The only way to check whether an infinite sequence $\omega = \omega_1\omega_2 \ldots$ is consistent with the theory is to check that for every n, the sequences $\omega_1 \ldots \omega_n$ are consistent with the theory T. In other words, we require that for every infinite sequence $\omega = \omega_1\omega_2 \ldots$,

– if for every n, the sequence $\omega_1 \ldots \omega_n$ is consistent with the theory T, i.e., if for every n, there exists a sequence $\omega^{(n)} \in T$ which has the same first n bits as ω, i.e., for which $\omega_i^{(n)} = \omega_i$ for all $i = 1, \ldots, n$,
– then the sequence ω itself should be consistent with the theory, i.e., this infinite sequence should also belong to the set T.

From the mathematical viewpoint, we can say that the sequences $\omega^{(n)}$ converge to ω: $\omega^{(n)} \to \omega$ (or, equivalently, $\lim \omega^{(n)} = \omega$), where convergence is understood in terms of the usual metric on the set of all infinite sequences $d(\omega, \omega') \stackrel{\text{def}}{=} 2^{-N(\omega,\omega')}$, where $N(\omega, \omega') \stackrel{\text{def}}{=} \max\{k : \omega_1 \ldots \omega_k = \omega'_1 \ldots \omega'_k\}$.

In general, if $\omega^{(m)} \to \omega$ in the sense of this metric, this means that for every n, there exists an integer ℓ such that for every $m \geq \ell$, we have $\omega_1^{(m)} \ldots \omega_n^{(m)} = \omega_1 \ldots \omega_n$. Thus, if $\omega^{(m)} \in T$ for all m, this means that for every n, a finite sequence $\omega_1 \ldots \omega_n$ can be a part of an infinite sequence which is consistent with the theory T. In view of the above, this means that $\omega \in T$.

In other words, if $\omega^{(m)} \to \omega$ and $\omega^{(m)} \in T$ for all m, then $\omega \in T$. So, the set T must contain all the limits of all its sequences. In topological terms, this means that the set T must be *closed*.

A physical theory must be different from a fact and hence, the set T must be nowhere dense. The assumption that we are trying to formalize is that no matter how many observations we have which confirm a theory, there eventually will be a new observation which is inconsistent with this theory. In other words, for every finite sequence $\omega_1 \ldots \omega_n$ which is consistent with the set T, there exists a continuation of this sequence which does not belong to T. The opposite would be if all the sequences which start with $\omega_1 \ldots \omega_n$ belong to T; in this case, the set T will be *dense* in this set. Thus, in mathematical terms, the statement that every finite sequence which is consistent with T has a continuation which is not consistent with T means that the set T is *nowhere dense*.

Resulting definition of a theory. By combining the above properties of a set T which describes a physical theory, we arrive at the following definition.

Definition 1. *By a* physical theory*, we mean a non-empty closed nowhere dense definable set T.*

Mathematical comment. To properly define what is definable, we need to have a consistent formal definition of definability. In this chapter, we follow a natural definition from [10, 11] – which is reproduced in Appendix A.

Formalization of the principle that no physical theory is perfect. In terms of the above notations, the no-perfect-theory principle simply means that the infinite sequence ω (describing the results of actual observations) is not consistent with any physical theory, i.e., that the sequence ω does not belong to any physical theory T. Thus, we arrive at the following definition.

Definition 2. *We say that an infinite binary sequence ω is* consistent with the no-perfect-theory principle *if the sequence ω does not belong to any physical theory (in the sense of Definition 1).*

Comment. Are there such sequences in the first place? Our answer is yes. Indeed, by definition, we want a sequence which does not belong to a union of all definable physical theories. Every physical theory is closed nowhere dense set. Every definable set is defined by a finite sequence of symbols, so there are no more than countably many definable theories. Thus, the union of all definable physical theories is contained in a union of countably many closed nowhere dense sets. Such sets are knows as *meager* (or *Baire first category*); it is known that the set of all infinite binary sequences is not meager. Thus, there are sequences who do not belong to the above union – i.e., sequences which are consistent with the no-perfect-theory principle; see, e.g., [4, 18].

4 How to Describe When Access to Physical Observations Enhances Computability

How to describe general computations. Each computation is a solution to a well-defined problem. As a result, each bit in the resulting answer satisfies a well-defined mathematical property. All mathematical properties can be described, e.g., in terms of Zermelo-Fraenkel theory ZF, the standard formalization of set theory. For each resulting bit, we can formulate a property P which is true if and only if this bit is equal to 1. In this sense, each bit in each computation result can be viewed as the truth value of some statement formulated in ZF. Thus, our general ability to compute can be described as the ability to (at least partially) compute the sequence of truth values of all statements from ZF.

All well-defined statements from ZF can be numbered, e.g., in lexicographic order. Let α_n denote the truth value of the n-th ZF statement, and let $\alpha = \alpha_1 \ldots \alpha_n \ldots$ denote the infinite sequence formed by these truth values. In terms of this sequence, our ability to compute is our ability to compute the sequence α.

Kolmogorov complexity as a way to describe what is easier to compute. We want to analyze whether the use of physical observations (i.e., of the sequence ω analyzed in the previous section) can simplify computations. A natural measure of easiness-to-compute was invented by A. N. Kolmogorov, the founder of modern probability theory, when he realized that in the traditional probability theory, there is no formal way to distinguish between:

– finite sequences which come from observing from truly random processes, and
– orderly sequences like 0101 … 01.

Kolmogorov noticed that an orderly sequence 0101 … 01 can be computed by a short program, while the only way to compute a truly random sequence 0101 … is to have a print statement that prints this sequence. He suggested to describe this difference by introducing what is now known as *Kolmogorov complexity* $K(x)$ of a finite sequence x: the shortest length of a program (in some programming language) which computes the sequence x.

– For an orderly sequence x, the Kolmogorov complexity $K(x)$ is much smaller than the length $\text{len}(x)$ of this sequence: $K(x) \ll \text{len}(x)$.
– For a truly random sequence x, we have $K(x) \approx \text{len}(x)$; see, e.g., [14].

The smaller the difference $\text{len}(x) - K(x)$, the more we are sure that the sequence x is truly random.

Relative Kolmogorov complexity as a way to describe when using an auxiliary sequence simplifies computations. The usual notion of Kolmogorov complexity provides the complexity of computing x "from scratch". A similar notion of the *relative Kolmogorov complexity* $K(x\,|\,y)$ can be used to describe the complexity of computing x when a (potentially infinite) sequence y is given. This relative complexity is based on programs which are allowed to use y as a subroutine, i.e., programs which, after generating an integer n, can get the n-th bit y_n of the sequence y by simply calling y. When we compute the length of such programs, we just count the length of the parameters of this call, not the length of the auxiliary program which computes y_n – just like when we count the length of a C++ program, we do not count how many steps it takes to compute, e.g., $\sin(x)$, we just count the number of symbols in this function call. The relative Kolmogorov complexity is then defined as the shortest length of such a y-using program which computes x.

Clearly, if x and y are unrelated, having access to y does not help in computing x, so $K(x \mid y) \approx K(x)$. On the other hand, if x coincides with y, then the relative complexity $K(x \mid y)$ is very small: all we need is a simple for-loop, in which we call for each bit y_i, $i = 1, \ldots, n$, and print this bit right away.

Resulting reformulation of our question. In terms of relative Kolmogorov complexity, the question of whether observations enhance computations is translated into checking whether $K(\alpha_1 \ldots \alpha_n \mid \omega) \approx K(\alpha_1 \ldots \alpha_n)$ (in which case there is no enhancement) or whether $K(\alpha_1 \ldots \alpha_n \mid \omega) \ll K(\alpha_1 \ldots \alpha_n)$ (in which case there is a strong enhancement). The larger the difference $K(\alpha_1 \ldots \alpha_n) - K(\alpha_1 \ldots \alpha_n \mid \omega)$, the larger the enhancement.

5 First Result: No-Perfect-Theory Principle Enhances Computability

Let us show that under the no-perfect-theory principle, observations do indeed enhance computability.

Proposition 1. *Let α be a sequence of truth values of ZF statements, and let ω be an infinite binary sequence which is consistent with the no-perfect-theory principle. Then, for every integer $C > 0$, there exists an integer n for which $K(\alpha_1 \ldots \alpha_n \mid \omega) < K(\alpha_1 \ldots \alpha_n) - C$.*

In other words, in principle, we can have an arbitrary large enhancement.

Comment. For readers' convenience, all the proof are placed in a special appendix.

6 Can Access to Physical Observations Speed up Computations?

Are computations feasible? What we have shown so far is that under Zadeh-inspired no-perfect-theory belief, it is possible to compute things that are not computable in the usual physical paradigm. From the practical viewpoint, being able to compute something *in principle* is important, but even more important is *how fast* we can compute it. In many cases, computations are theoretically possible, but not practically feasible, since they require computation times which are longer than the lifetime of the Universe :-(It is therefore important to analyze which problems are feasibly computable and which are not. To perform this analysis, we need to define what is "feasible" and what is a "problem".

In computer science, "feasible" is usually interpreted as computable in polynomial time, i.e., in time t bounded by a polynomial of the length n of the input; see, e.g., [19]. This definition works in most cases:

– time 2^n is non-feasible already for $n \approx 300$, while
– time n^2 or n^3 is usually feasible.

This is not a perfect definition:

– on the one hand, time $t = 10^{400} \cdot n$ is polynomial in n but clearly not feasible;
– on the other hand, computation time $\exp(10^{-10} \cdot n)$ is not bounded by a polynomial, but it clearly corresponds to feasible computations.

However, this is the best definition we have.

By a *problem*, computer scientists usually understand a problem in which it is absolutely clear what is a solution and what is not. For example:

– finding a proof of a given mathematical statement,
– finding a formula that fits all experimental observations,
– designing a bridge under certain specifications of strength, cost, etc.,

these are all such problems – while, e.g., the problem of designing a *beautiful* bridge is *not* clearly defined.

In general, we need to find a solution that satisfies a given set of constraints – or at least check that such a solution is possible. Once we have a candidate for the solution, we can feasibly check whether this candidate indeed satisfies all the constraints.

A problem of checking whether a given set of constraints has solution is called a problem *of the class NP* if we can check, in polynomial time, whether a given candidate is a solution; see, e.g., [19].

Examples of such problem includes checking whether a given graph can be colored in 3 colors, checking whether a given propositional formula – i.e., formula of the type

$$(v_1 \vee \neg v_2 \vee v_3)\,\&\,(v_4 \vee \neg v_2 \vee \neg v_5)\,\&\ \ldots,$$

is satisfiable, i.e., whether this formula is true by some combination of the propositional variables v_i, etc.

Each problem from the class NP can be algorithmically solved by trying all possible candidates. For example, we can check whether a graph can be colored by trying all possible assignments of colors to different vertices of a graph, and we can check whether a given propositional formula is satisfiable by trying all 2^n possible combinations of true-or-false values $v_1, \ldots, v_n$. Such exhaustive search algorithms require computation time like 2^n, time that grows exponentially with n. For medium-size inputs, e.g., for $n \approx 300$, the resulting time is larger than the lifetime of the Universe. So, these exhaustive search algorithms are not practically feasible.

It is not known whether problems from the class NP can be solved feasibly (i.e., in polynomial time): this is the famous open problem $P \stackrel{?}{=} NP$. It is known, however, there are problems in the class NP which are *NP-complete* in the sense that every problem from the class NP can be reduced to this problem. Reduction means, in particular, that if we can find a way to efficiently solve one NP-complete problem, then, by reducing other problems from the class NP to this problem, we can thus efficiently solve all the problems from the class NP.

So, it is very important to be able to efficiently solve even one NP-hard problem. (By the way, both above example of NP problems – checking whether a graph can be colored in 3 colors and whether coloring a propositional formula is satisfiable – are NP-complete.)

Can the use of non-standard physics speed up the solution of NP-complete problems? NP-completeness of a problem means, crudely speaking, that the problem may take an unrealistically long time to solve – at least on computers based on the usual physical techniques. A natural question is: can the use of non-standard physics speed up the solution of these problems?

This question has been analyzed for several specific physical theories, e.g., for quantum filed theory, for cosmological solutions with wormholes and/or casual anomalies. Several possible techniques for solving NP-complete problems are described in [1, 11, 13, 15, 21].

How does the no-perfect-belief affect the speed of computations? In this chapter, we show that an important speed-up can be deduced simply from the fact no physical theory is perfect.

7 Second Result: The Use of Physical Observations Can Help in Solving NP-Complete Problems

How to represent instances of an NP-complete problem. For each NP-complete problem $\mathcal{P}$, its instances are sequences of symbols. In the computer, each such sequence is represented as a sequence of 0 s and 1s. Thus, as in the previous sections, we can append 1 in front of this sequence and interpret the resulting sequence as a binary code of a natural number i.

In principle, not all natural numbers i correspond to instances of a problem $\mathcal{P}$; we will denote the set of all natural numbers which correspond to such instances by $S_{\mathcal{P}}$.

For each $i \in S_{\mathcal{P}}$, the correct answer (true or false) to the i-th instance of the problem $\mathcal{P}$ will be denoted by $s_{\mathcal{P},i}$.

What we mean by using physical observations in computations. In addition to performing computations, our computational device can produce a scheme i for an experiment, and then use the result ω_i of this experiment in future computations. In other words, given an integer i, we can produce ω_i.

In precise theory-of-computation terms, the use of physical observations in computations thus means computations that use the sequence ω as an *oracle*; see, e.g., [19].

Definition 3. *By a* ph-algorithm $\mathcal{A}$*, we mean an algorithm which uses, as an oracle, a sequence* ω *which is consistent with the no-perfect-theory principle.*

Notation. The result of applying an algorithm $\mathcal{A}$ using ω to an input i will be denoted by $\mathcal{A}(\omega, i)$.

Definition 4. *Let* $\mathcal{P}$ *be an NP-complete problem. We say that a feasible ph-algorithm* $\mathcal{A}$ solves almost all instances of $\mathcal{P}$ *if for every* $\varepsilon > 0$, *and for every natural number* n*, there exists an integer* $N \geq n$ *for which the proportion of the instances* $i \leq N$ *of the problem* $\mathcal{P}$ *which are correctly solved by* $\mathcal{A}$ *is greater than* $1 - \varepsilon$:

$$\forall \varepsilon > 0\, \forall n\, \exists N \left(N \geq n \,\&\, \frac{\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \leq N : i \in S_{\mathcal{P}}\}} > 1 - \varepsilon \right).$$

Comment. The restriction to sufficiently long inputs $N \geq n$ makes perfect sense: for short inputs, NP-completeness is not an issue: we can perform exhaustive search of all possible bit sequences of length 10, 20, and even 30. The challenge starts when the length of the input is high.

Proposition 2. *For every NP-complete problem* $\mathcal{P}$*, there exists a feasible ph-algorithm* $\mathcal{A}$ *that solves almost all instances of* $\mathcal{P}$.

In other words, we show that the use of physical observations makes all NP-complete problems easier-to-solve (in the above-described sense).

Acknowledgments This work was supported in part by the National Science Foundation grants HRD-0734825, HRD-124212, and DUE-0926721. The authors are thankful to the anonymous referees for valuable suggestions.

A A Formal Definition of Definable Sets

Definition A1. *Let* $\mathcal{L}$ *be a theory, and let* $P(x)$ *be a formula from the language of the theory* $\mathcal{L}$*, with one free variable* x *for which the set* $\{x \mid P(x)\}$ *is defined in the theory* $\mathcal{L}$*. We will then call the set* $\{x \mid P(x)\}$ $\mathcal{L}$-definable.

Crudely speaking, a set is $\mathcal{L}$-definable if we can explicitly *define* it in $\mathcal{L}$. The set of all real numbers, the set of all solutions of a well-defined equation, every set that we can describe in mathematical terms: all these sets are $\mathcal{L}$-definable.

This does not mean, however, that *every* set is $\mathcal{L}$-definable: indeed, every $\mathcal{L}$-definable set is uniquely determined by formula $P(x)$, i.e., by a text in the language of set theory. There are only denumerably many words and therefore, there are only denumerably many $\mathcal{L}$-definable sets. Since, e.g., in a standard model of set theory ZF, there are more than denumerably many sets of integers, some of them are thus not $\mathcal{L}$-definable.

Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets. Therefore, in addition to the theory $\mathcal{L}$, we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a set – and it is a countable set.

Denotation. *For every formula F from the theory $\mathcal{L}$, we denote its Gödel number by* $\lfloor F \rfloor$.

Comment. A Gödel number of a formula is an integer that uniquely determines this formula. For example, we can define a Gödel number by describing what this formula will look like in a computer. Specifically, we write this formula in LaTeX, interpret every LaTeX symbol as its ASCII code (as computers do), add 1 at the beginning of the resulting sequence of 0 s and 1s, and interpret the resulting binary sequence as an integer in binary code.

Definition A2. *We say that a theory $\mathcal{M}$ is* stronger *than $\mathcal{L}$ if it contains all formulas, all axioms, and all deduction rules from $\mathcal{L}$, and also contains a special predicate* $\mathrm{def}(n,x)$ *such that for every formula $P(x)$ from $\mathcal{L}$ with one free variable, the formula* $\forall y\,(\mathrm{def}(\lfloor P(x)\rfloor, y) \leftrightarrow P(y))$ *is provable in $\mathcal{M}$.*

The existence of a stronger theory can be easily proven: indeed, for $\mathcal{L}$=ZF, there exists a stronger theory $\mathcal{M}$. As an example of such a stronger theory, we can simply take the theory $\mathcal{L}$ plus all countably many equivalence formulas as described in Definition A2 (corresponding to all possible formulas $P(x)$ with one free variable). This theory clearly contains $\mathcal{L}$ and all the desired equivalence formulas, so all we need to prove is that the resulting theory $\mathcal{M}$ is consistent (provided that $\mathcal{L}$ is consistent, of course). Due to compactness principle, it is sufficient to prove that for an arbitrary finite set of formulas $P_1(x), \ldots, P_m(x)$, the theory $\mathcal{L}$ is consistent with the above reflection-principle-type formulas corresponding to these properties $P_1(x), \ldots, P_m(x)$.

This auxiliary consistency follows from the fact that for such a finite set, we can take

$$\mathrm{def}(n,y) \leftrightarrow (n = \lfloor P_1(x)\rfloor \;\&\; P_1(y)) \vee \ldots \vee (n = \lfloor P_m(x)\rfloor \;\&\; P_m(y)).$$

This formula is definable in $\mathcal{L}$ and satisfies all m equivalence properties. The statement is proven.

Important comments. In the main text, we will assume that a theory $\mathcal{M}$ that is stronger than $\mathcal{L}$ has been fixed; proofs will mean proofs in this selected theory $\mathcal{M}$.

An important feature of a stronger theory $\mathcal{M}$ is that the notion of an $\mathcal{L}$-definable set can be expressed within the theory $\mathcal{M}$: a set S is $\mathcal{L}$-definable if and only if $\exists n \in \mathbb{N}\, \forall y(\mathrm{def}(n,y) \leftrightarrow y \in S)$.

In the chapter, when we talk about definability, we will mean this property expressed in the theory $\mathcal{M}$. So, all the statements involving definability become statements from the theory $\mathcal{M}$ itself, *not* statements from metalanguage.

B Proofs

Proof of Proposition 1. Let us fix an integer C. To prove the desired property for this C, let us prove that the set T of all the sequences which do not satisfy this property, i.e., for which $K(\alpha_1 \dots \alpha_n \,|\, \omega) \geq K(\alpha_1 \dots \alpha_n) - C$ for all n, is a physical theory in the sense of Definition 1. For this, we need to prove that this set T is non-empty, closed, nowhere dense, and definable. Then, from Definition 2, it will follow that the sequence ω does not belong to this set and thus, that the conclusion of Proposition 1 is true.

The set T is clearly non-empty: it contains, e.g., a sequence $\omega = 00\dots0\dots$ which does not affect computations. The set T is also clearly definable: we have just defined it.

Let us prove that the set T is closed. For that, let us assume that $\omega^{(m)} \to \omega$ and $\omega^{(m)} \in T$ for all m. We then need to prove that $\omega \in T$. Indeed, let us fix n, and let us prove that $K(\alpha_1 \dots \alpha_n \,|\, \omega) \geq K(\alpha_1 \dots \alpha_n) - C$. We will prove this by contradiction. Let us assume that $K(\alpha_1 \dots \alpha_n \,|\, \omega) < K(\alpha_1 \dots \alpha_n) - C$. This means that there exists a program p of length $\text{len}(p) < K(\alpha_1 \dots \alpha_n) - C$ which uses ω to compute $\alpha_1 \dots \alpha_n$. This program uses only finitely many bits of ω; let B be the largest index of these bits. Due to $\omega^{(m)} \to \omega$, there exists M for which, for all $m \geq M$, the first B bits of $\omega^{(m)}$ coincide with the first B bits of the sequence ω. Thus, the same program p will work exactly the same way – and generate the same sequence $\alpha_1 \dots \alpha_n$ – if we use $\omega^{(m)}$ instead of ω. But since $\text{len}(p) < K(\alpha_1 \dots \alpha_n) - C$, this would means that the shortest length $K(\alpha_1 \dots \alpha_n \,|\, \omega^{(m)})$ of all the programs which use $\omega^{(m)}$ to compute $\alpha_1 \dots \alpha_n$ also satisfies the inequality $K(\alpha_1 \dots \alpha_n \,|\, \omega^{(m)}) < K(\alpha_1 \dots \alpha_n) - C$. This inequality contradicts to our assumption that $\omega^{(m)} \in T$ and thus, that $K(\alpha_1 \dots \alpha_n \,|\, \omega^{(m)}) \geq K(\alpha_1 \dots \alpha_n) - C$. The contradiction proves that the set T is indeed closed.

Let us now prove that the set T is nowhere dense, i.e., that for every finite sequence $\omega_1 \dots \omega_m$, there exists a continuation ω which does not belong to the set T. Indeed, as such a continuation, we can simply take a sequence $\omega = \omega_1 \dots \omega_m \alpha_1 \alpha_2 \dots$ obtained by appending α at the end. For this new sequence, computing $\alpha_1 \dots \alpha_n$ is straightforward: we just copy the values α_i from the corresponding places of the new sequence ω. Here, the relative Kolmogorov complexity $K(\alpha_1 \dots \alpha_n \,|\, \omega)$ is very small and is, thus, much smaller than the complexity $K(\alpha_1 \dots \alpha_n)$ which – since ZF is not decidable – grows with n.

The proposition is proven.

Proof of Proposition 2.

1°. As the desired ph-algorithm, we will, given an instance i, simply produce the result ω_i of the i-th experiment. Let us prove, by contradiction, that this algorithm satisfies the desired property.

2°. We want to prove that for every $\varepsilon > 0$ and for every n, there exists an integer $N \geq n$ for which

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} > (1 - \varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\}.$$

The assumption that this property is not satisfied means that for some $\varepsilon > 0$ and for some integer n, we have

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} \le (1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge n. \quad (1)$$

Let T denote the set of all the sequences x that satisfy the property (1), i.e., let

$$T \stackrel{\text{def}}{=}$$

$$\{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le (1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge n\}.$$

We will prove that this set T is a physical theory in the sense of Definition 1.

Then, due to Definition 2 and the fact that the sequence ω satisfies the no-perfect-theory principle, we will be able to conclude that $\omega \notin T$, and thus, that the property (1) is not satisfied for the given sequence ω. This will conclude the proof by contradiction.

3°. By definition of a physical theory T, it is a set which is non-empty, closed, nowhere dense, and definable. Let us prove these four properties one by one.

3.1°. Non-emptiness comes from the fact that the sequence x_i for which $x_i = \neg s_{\mathcal{P},i}$ for $i \in S_{\mathcal{P}}$ and $x_i = 0$ otherwise clearly belongs to this set: for this sequence, for every N, we have $\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} = 0$ and thus, the desired property is satisfied.

3.2°. Let us prove that the set T is closed, i.e., that if we have a family of sequences $x^{(m)} \in T$ for which $x^{(m)} \to x$, then $x \in T$.

Indeed, let us take any $N \ne n$, and let us prove that

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le (1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\}$$

for this N. Due to $x^{(m)} \to x$, there exists M for which, for all $m \ge M$, the first N bits of $x^{(m)}$ coincide with the first N bits of the sequence x: $x_i^{(m)} = \omega_i$ for all $i \le N$. Thus,

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i^{(m)} = s_{\mathcal{P},i}\}.$$

Since $x^{(m)} \in T$, we have

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i^{(m)} = s_{\mathcal{P},i}\} \le (1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\},$$

thus

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le (1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\}.$$

So, the set T is indeed closed.

3.3°. Let us now prove that the set T is nowhere dense, i.e., that for every finite sequence $x_1 \ldots x_m$, there exists a continuation x which does not belong to the set T.

Indeed, as such a continuation, we can simply take a sequence

$$x = x_1 \ldots x_m x_{m+1} x_{m+2} \cdots$$

where for $i > m$, we take $x_i = s_{\mathcal{P},i}$ if $i \in S_{\mathcal{P}}$ and $x_i = 0$ otherwise. For this new sequence, for every N, at most m first instances may lead to results different from $s_{\mathcal{P},i}$, so we have

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \ge \#\{i \le N : i \in S_{\mathcal{P}}\} - m.$$

When $N \to \infty$, then $\#\{i \le N : i \in S_{\mathcal{P}}\} \to \infty$, so for sufficiently large N, we have

$$\#\{i \le N : i \in S_{\mathcal{P}}\} - m > (1 - \varepsilon) \cdot \#\{i \le N : i \in S_{\mathcal{P}}\},$$

thus,

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} > (1 - \varepsilon) \cdot \#\{i \le N : i \in S_{\mathcal{P}}\},$$

and we cannot have

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le (1 - \varepsilon) \cdot \#\{i \le N : i \in S_{\mathcal{P}}\}.$$

Therefore, this continuation does not belong to the set T.

3.4°. Finally, since the formula (1) explicitly defines the set T, this set T is clearly definable.

So, T is a physical theory, hence $\omega \notin T$, and the proposition is proven.

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Authors Biography

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Handling Noise and Outliers in Fuzzy Clustering

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Abstract Since it is an unsupervised data analysis approach, clustering relies solely on the location of the data points in the data space or, alternatively, on their relative distances or similarities. As a consequence, clustering can suffer from the presence of noisy data points and outliers, which can obscure the structure of the clusters in the data and thus may drive clustering algorithms to yield suboptimal or even misleading results. Fuzzy clustering is no exception in this respect, although it features an aspect of robustness, due to which outliers and generally data points that are atypical for the clusters in the data have a lesser influence on the cluster parameters. Starting from this aspect, we provide in this paper an overview of different approaches with which fuzzy clustering can be made less sensitive to noise and outliers and categorize them according to the component of standard fuzzy clustering they modify.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_17

1 Introduction

In general, *clustering* [1–4] is a data analysis method that tries to group the records, cases or generally data points of a data set in such a way that points in the same group (or *cluster*) are as similar as possible, while points in different groups are as dissimilar as possible. There is no predefined target attribute (like a class label) that guides the analysis process and hence clustering belongs to the so-called *unsupervised* methods (in contrast to supervised methods like, for example, classifier construction): it relies solely on the location of the data points in the data space or, alternatively, on their relative distance or similarity.

Unfortunately, due to this exclusive dependence on location and/or distance information, clustering algorithms can suffer from noisy data points and outliers that are present in the data. Such data points, which we may define informally as points that do not conform (well) to the actual cluster structure of the data, can obscure the true cluster structure and thus may lead clustering algorithms to produce results that are far from optimal or even misleading.

Fuzzy clustering [4–7] is no exception in this respect, although it features an aspect of robustness, due to which outliers and generally data points that are atypical for the clusters in the data have a lesser influence on the cluster parameters (like, for instance, the location of the cluster centers as well as shape and size parameters that may be present). We emphasize this aspect in the next section (Sect. 2), in which we briefly review standard fuzzy clustering.

Afterward we turn to methods that try to make fuzzy clustering (even more) robust w.r.t. noise and outliers. Such approaches can be roughly categorized into two classes: (1) approaches that modify the "(influence) weight" of the (atypical) data points, either by changing how membership degrees are computed from the (relative) data point distances to the clusters or by introducing and adapting an explicit data point weight, and (2) approaches that rely on other distance measures than the usually employed squared Euclidean distance or transform the distance measure before computing membership degrees.

Among the approaches in the first class are the popular noise cluster approach [8–10] (Sect. 3), introducing and adapting an explicit data point weight (outlier clustering) [11] (Sect. 4), possibilistic fuzzy clustering [12, 13] and its variants that combine it with standard fuzzy clustering [14–18] (Sect. 5), as well as using an alternative transformation of the membership degrees [19] (Sect. 6). In the second class we find approaches based on squared and particularly unsquared Minkowski distances [20–22] (Sect. 7) or transformed or otherwise modified distance measures [23–26] (Sect. 8).

2 Fuzzy Clustering

In the clustering approaches we study in this paper, the similarity of data points is formalized by a *distance measure* on the data space and the clusters are described by *prototypes* that capture the location and possibly also the shape and size of the clusters in the data space. With such an approach the general objective of clustering can be reformulated as the task to find a set of cluster prototypes together with an assignment of the data points to them, so that the data points are as close as possible to their assigned prototypes. By formalizing this approach, and using for the prototypes only points in the data space that represent the *cluster centers*, one obtains immediately the objective function of classical c-means clustering [27–29]: simply sum the squared distances of the data points to the center of the cluster to which they are assigned. The c-means clustering algorithm then strives to minimize this objective function.

Unfortunately, c-means clustering always partitions the data, that is, each data point is assigned to one cluster and one cluster only. This is often inappropriate, as it can lead to somewhat arbitrary cluster boundaries and certainly does not treat points properly that lie between two (or more) clusters without belonging to any of them unambiguously. A solution to this problem consists in employing one of the different "fuzzifications" of the classical *crisp* (or *hard*) scheme (see, for instance, [4–7, 26, 30]), which modify the objective function of classical c-means clustering in order to obtain graded cluster memberships. In principle, there are two ways to do this, namely (1) by membership transformation, which maps the memberships with a convex function, and (2) by membership regularization, which adds a regularization term, usually derived from an entropy measure, to the objective function to prevent crisp assignments (see, for instance, [31] for a discussion). Here we focus on the first approach (membership transformation), because it exhibits a certain robustness property we are interested in. However, most of the approaches we study in Sects. 3 to 8 can equally well be applied to fuzzy clustering by membership regularization.

Formally, we are given a data set $\mathbf{X} = \{\vec{x}_1, \dots, \vec{x}_n\}$ with n data points, each of which is an m-dimensional real-valued vector, that is, $\forall j; 1 \le j \le n : \vec{x}_j = (x_{j1}, \dots, x_{jm}) \in \mathbb{R}^m$. These data points are to be grouped into c clusters, each of which is described by a prototype $\vec{c}_i$, $i = 1, \dots, c$. The set of all prototypes is denoted by $\mathbf{C} = \{\vec{c}_1, \dots, \vec{c}_c\}$. We confine ourselves here to cluster prototypes that consist only of a cluster center, that is, $\forall i; 1 \le i \le c : \vec{c}_i = (c_{i1}, \dots, c_{im}) \in \mathbb{R}^m$, although (most of) the approaches we study below may just as well be applied if the cluster prototypes comprise shape and size parameters (like, for instance, in [32, 33]). The assignment of the data points to the cluster centers is encoded as a $c \times n$ matrix $\mathbf{U} = (u_{ij})_{1 \le i \le c; 1 \le j \le n}$, which is often called the *partition matrix*. In the crisp case, a matrix element $u_{ij} \in \{0, 1\}$ states whether data point $\vec{x}_j$ belongs to cluster $\vec{c}_i$ ($u_{ij} = 1$) or not ($u_{ij} = 0$). In the fuzzy case, $u_{ij} \in [0, 1]$ states the degree to which $\vec{x}_j$ belongs to $\vec{c}_i$ (*degree of membership*).

Since we do not obtain graded memberships by merely allowing $u_{ij} \in [0, 1]$ (see, for example, [19, 31]), the membership degrees are transformed with a convex mapping $h : [0, 1] \to [0, 1]$. This yields an objective function of the form [19]

$$J(\mathbf{X}, \mathbf{C}, \mathbf{U}) = \sum_{i=1}^{c} \sum_{j=1}^{n} h(u_{ij})\, d_{ij}^2.$$

The clustering task now consists in finding for a given data set $\mathbf{X}$ and a user-specified number of clusters c, cluster prototypes $\mathbf{C}$ and a partition matrix $\mathbf{U}$ such that $J(\mathbf{X}, \mathbf{C}, \mathbf{U})$ is minimized under the constraints

$$\forall j; 1 \leq j \leq n : \sum_{i=1}^{c} u_{ij} = 1 \qquad \text{and} \qquad \forall i; 1 \leq i \leq c : \sum_{j=1}^{n} u_{ij} > 0.$$

Unfortunately, cluster prototypes $\mathbf{C}$ and a partition matrix $\mathbf{U}$ that minimize J are difficult to find by analytic means. Therefore one takes refuge to an *alternating optimization* scheme: starting from randomly chosen cluster centers (for example, sampled from the data set $\mathbf{X}$), one iterates (1) updating the partition matrix for fixed cluster prototypes and (2) updating the cluster prototypes for a fixed partition matrix until convergence. Convergence may be checked with a limit for the change of the cluster parameters (e.g. center coordinates) or a limit for the change of the membership degrees from one iteration to the next.

In order to derive the update rule for the partition matrix (and thus for the membership degrees u_{ij}) we need to know the exact form of the function h. The most common choice is $h(u_{ij}) = u_{ij}^2$, which leads to the standard objective function of fuzzy clustering [30]. The more general form $h(u_{ij}) = u_{ij}^w$ was introduced in [6]. The exponent w, $w > 1$, is called the *fuzzifier*, since it controls the "fuzziness" of the data point assignments: the higher w, the softer the boundaries between the clusters, while a crisp partition results in the limit for $w \to 1$. This leads to the commonly used objective function [4, 6, 7, 26]

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}^w \, d_{ij}^2.$$

The update rule for the membership degrees is now derived by incorporating the constraints $\forall j; 1 \leq j \leq n : \sum_{i=1}^{c} u_{ij} = 1$ with Lagrange multipliers into the objective function. (The second set of constraints, that is, $\forall i; 1 \leq i \leq c : \sum_{j=1}^{n} u_{ij} > 0$ can usually be neglected, because it is satisfied by the clustering result anyway.) This yields the Lagrange function

$$L(\mathbf{X}, \mathbf{U}, \mathbf{C}, \Lambda) = \underbrace{\sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}^w \, d_{ij}^2}_{=J(\mathbf{X},\mathbf{U},\mathbf{C})} + \sum_{j=1}^{n} \lambda_j \left(1 - \sum_{i=1}^{c} u_{ij}\right),$$

where $\Lambda = (\lambda_1, \dots, \lambda_n)$ are the Lagrange multipliers, one per constraint.

Since a necessary condition for a minimum of the Lagrange function is that the partial derivatives w.r.t. the membership degrees vanish, we obtain

$$\frac{\partial}{\partial u_{kl}} L(\mathbf{X}, \mathbf{U}, \mathbf{C}, \Lambda) = w\, u_{kl}^{w-1}\, d_{kl}^2 - \lambda_l \stackrel{!}{=} 0 \qquad \text{and thus} \qquad u_{kl} = \left(\frac{\lambda_l}{w\, d_{kl}^2}\right)^{\frac{1}{w-1}}.$$

Summing these equations over the clusters (in order to be able to exploit the corresponding constraints on the membership degrees, which are recovered from the fact that it is a necessary condition for a minimum that the partial derivatives of the Lagrange function w.r.t. the Lagrange multipliers vanish), we get

$$1 = \sum_{i=1}^{c} u_{ij} = \sum_{i=1}^{c} \left(\frac{\lambda_j}{w\, d_{ij}^2}\right)^{\frac{1}{w-1}} \qquad \text{and thus} \qquad \lambda_j = \left(\sum_{i=1}^{c} \left(w\, d_{ij}^2\right)^{\frac{1}{1-w}}\right)^{1-w}.$$

Therefore we finally have for the membership degrees $\forall i; 1 \leq i \leq c$: $\forall j; 1 \leq j \leq n$:

$$u_{ij} = \frac{d_{ij}^{\frac{2}{1-w}}}{\sum_{k=1}^{c} d_{kj}^{\frac{2}{1-w}}} \qquad \text{and thus for } w = 2\text{:} \qquad u_{ij} = \frac{d_{ij}^{-2}}{\sum_{k=1}^{c} d_{kj}^{-2}}.$$

This rule is fairly intuitive, as it updates the membership degrees according to the relative inverse squared distances of the data points to the cluster centers.

In order to derive the update rule for the cluster centers, we need to know the (squared) distances d_{ij}^2. The most common choice is the (squared) Euclidean distance, that is, $d_{ij}^2 = (\vec{x}_j - \vec{c}_i)^\top (\vec{x}_j - \vec{c}_i)$. With this choice, we can easily derive the update rule for the cluster centers, namely by exploiting that a necessary condition for a minimum of the objective function J is that the partial derivatives w.r.t. the cluster centers vanish. Therefore we have $\forall k; 1 \leq k \leq c$:

$$\begin{aligned} \nabla_{\vec{c}_k} J(\mathbf{X}, \mathbf{C}, \mathbf{U}) &= \nabla_{\vec{c}_k} \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}^{w}\, (\vec{x}_j - \vec{c}_i)^\top (\vec{x}_j - \vec{c}_i) \\ &= -2 \sum_{j=1}^{n} u_{ij}^{w} (\vec{x}_j - \vec{c}_i) \stackrel{!}{=} 0. \end{aligned}$$

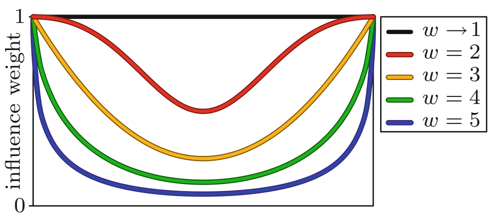

Fig. 1 "Influence weight" of a data point between two cluster centers for different values of the fuzzifier w. The two cluster centers are at the left and the right border of the diagram

It follows immediately $\forall i; 1 \le i \le c$:

$$\vec{c}_i = \frac{\sum_{j=1}^{n} u_{ij}^{w} \vec{x}_j}{\sum_{j=1}^{n} u_{ij}^{w}}.$$

For the topic of this paper it is important to note that this update rule draws on the *transformed* membership degrees u_{ij}^w rather than on u_{ij} directly. As a consequence the effective "influence weight" of a data point on the cluster parameters is not 1 (as one may be led to believe by the constraints $\forall j; 1 \le j \le n : \sum_{i=1}^{c} u_{ij} = 1$), but rather $\alpha_j = \sum_{i=1}^{c} u_{ij}^w$. It is $\alpha_j = 1$ only if the data point $\vec{x}_j$ coincides with a cluster center (or if $w \to 1$); otherwise it is $\alpha_j < 1$.

As an illustration, Fig. 1 shows, for $c = 2$ clusters, the influence weight of a data point lying on a straight line connecting the two cluster centers: one cluster center is at the left border of the diagram, the other at the right border. Clearly, for a fuzzifier $w > 1$ the total influence weight $\alpha_j = \sum_{i=1}^{c} u_{ij}^w$ of a data point with a less ambiguous assignment (that is, close to the left or right border of the diagram) is higher than that of a more ambiguously assigned data point (in the middle of the diagram). Also, this influence weight is the lower, the larger the fuzzifier. The minimum influence weight is always obtained for equal distances (and thus equal membership degrees $u_{ij} = 1/c$) to all c clusters. In this case the influence weight of the data point is $\alpha = \sum_{i=1}^{c} (1/c)^w = c \cdot c^{-w} = c^{1-w}$.

Note that a unit data point weight is obtained only at the cluster centers or for the limiting case of crisp clustering (that is, for $w \to 1$). This distinguishes fuzzy clustering from classical (crisp) clustering, where each data point has a unit influence (on exactly one cluster). It also distinguishes the membership transformation approach to fuzzy clustering from an approach that relies on membership regularization, since in the latter the update rule for the cluster centers refers to *untransformed* membership degrees (see, for instance, [31]), thus endowing each data point with a unit effective influence weight.

Due to the reduced influence weight that ambiguously assigned data points receive in the membership transformation approach, the locations of the cluster centers

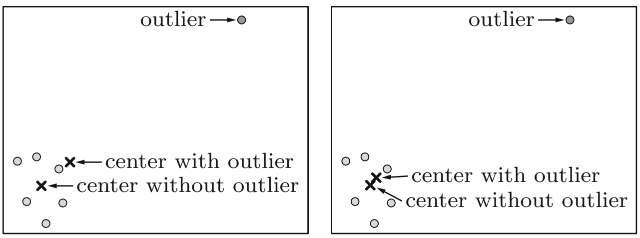

Fig. 2 Effect of an outlier on the location of a cluster center that minimizes the sum of the squared (left) and unsquared Euclidean distances (right)

depend more strongly on those data points that are "typical" for the clusters. This effect can be desirable and is very much in the spirit of, for instance, robust regression techniques, in which data points also receive a lower weight if they are not fitted well by the regression function. This connection to robust statistical methods was explored in more detail, for example, in [34, 35].

Despite this inherent robustness of fuzzy clustering, the influence of noisy data points and outliers on the clustering result can still be too strong to yield sufficiently good clustering results. A core reason for this is that the standard objective function is defined in terms of sums of *squared* Euclidean distances. Due to this squaring of distances, outliers can have an overly strong influence on the cluster parameters. This is illustrated in Fig. 2 on the left, which shows six data points forming a cluster (light gray circles at the left bottom) and one outlier (dark gray circle at the top right). Computing the mean vector of the six data points forming the cluster—that is, computing the point that minimizes the sum of the *squared* Euclidean distances to the data points—yields a center vector that lies, as one would expect, in the middle of this group of data points (lower left cross). However, if the outlier is included in this mean computation, the cluster center is strongly pulled out of the cloud of the six data points towards the outlier. The reason is, of course, that the large distance to the outlier becomes even bigger by squaring and thus dominates the smaller (squared) distances to the other six data points, producing an undesirable result.

Summarizing our discussion, we see that we can try to tackle noise and outliers in fuzzy clustering in essentially two ways: (1) we can try to reduce the influence weight of atypical data points even further than the membership transformation already does (approaches based on this idea are studied in Sects. 3 to 6) or (2) we can change or transform the distance measure in the objective function to reduce or eliminate the deteriorating effect of the squared distances (approaches based on this idea are studied in Sects. 7 and 8).

3 Noise Clustering

The best known and most popular approach to handle noise and outliers in fuzzy clustering is so-called *noise clustering*, which was first proposed in [8], but received attention only after it was independently developed again in [9]. The core idea of this method is to introduce a pseudo-cluster, called the *noise cluster*, that has no specific location, but rather the same distance δ from all data points in $\mathbf{X}$. Thus data points that are far away from the actual clusters (in particular: farther away than the *noise distance* δ), receive a high degree of membership to the noise cluster. As a consequence, the influence of noisy data points and outliers on the parameters of the actual clusters is reduced, since the membership degrees to the actual clusters now sum to a value that is the smaller, the higher the degree of membership to the noise cluster.

Formally, this leads to the objective function [8, 9]

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}^{w}\, d_{ij}^{2} + \delta^2 \sum_{j=1}^{n} u_{0j}^{w},$$

where the index $i = 0$ refers to the noise cluster. Of course, the first set of constraints now includes the noise cluster in the sum, that is, $\forall j; 1 \leq j \leq n : \sum_{i=0}^{c} u_{ij} = 1$. As a consequence, even the untransformed membership degrees to the actual clusters do not sum to 1 anymore, but only to $1 - u_{0j}$, where

$$\forall j; 1 \leq j \leq n : \quad u_{0j} = \frac{\delta^{\frac{2}{1-w}}}{\delta^{\frac{2}{1-w}} + \sum_{i=1}^{c} d_{ij}^{\frac{2}{1-w}}}$$

is the degree of membership of the data point $\vec{x}_j$ to the noise cluster. Clearly, this reduces the influence weight (in the sense of Sect. 2) of data points that are atypical for the actual clusters and thus renders the result much more robust.

Of course, introducing a noise clusters raises the question of how to choose the noise distance δ. If δ is (too) small, a large portion of the data set will receive a high degree of membership to the noise cluster, possibly rendering the majority of the data points noise and outliers. On the other hand, if δ is chosen (too) large, membership degrees to the noise cluster will remain small, possibly rendering its influence negligible [36]. A proper choice depends on many aspects [37]: the amount of noise present in the data set, the employed distance measure, the size of the feature space (in terms of the range of possible values for the distance measure), the number c of clusters to the found etc. In [9] it was suggested to compute the noise distance (in each iteration) from the (unweighted) average distance of the data points to the cluster prototypes as

$$\delta^2 = \frac{\kappa}{nc} \sum_{i=1}^{c} \sum_{j=1}^{n} d_{ij}^{2},$$

where κ is user-specified factor that becomes the actual parameter.

An alternative to this basic approach consist in choosing the noise distance as the (average) "cluster radius" that is derived from the requirement that the sum of the hypervolumes of the clusters (as computed with this cluster radius) should equal the size of the feature space (derived, for example, from the extreme data points) [37]. A good value of the noise distance may also be determined by trying multiple values for δ (starting at a large value and halving δ in each step), computing the fraction p of data points that have their highest degree of membership to the noise cluster (and thus may be considered as being assigned to the noise cluster), fitting the resulting points (δ, p) with a Pareto-curve $p = q\delta^{-s}$ and finding the point of this curve at which its slope is -1 [38]. Finally, a term may be added to the objective function that controls what fraction of the data points can be expected to be noise or outliers, thus rendering the method more robust against bad choices of the noise distance δ [36].

4 Data Point Weights

As we have seen in the preceding section, noise clustering relaxes the constraints $\forall j; 1 \leq j \leq n : \sum_{i=1}^{c} u_{ij} = 1$ somewhat by including the membership degree to the noise cluster, due to which the membership degrees to the actual clusters can sum to values less than 1. Alternatively, one may introduce an explicit *data point weight* and adapt this weight in the optimization process [11], an approach which is also referred to as *outlier clustering*. It permits that the membership degrees effectively sum to values less than 1 (namely to the data point weights) for atypical data points, while for very typical data points they may even sum to values larger than 1, endowing them with a greater influence on the clusters.

Outlier clustering is based on the objective function [11]

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} \frac{u_{ij}^{w}}{v_j^{\theta}} d_{ij}^2,$$

where v_j is the weight of the data point $\vec{x}_j$ and θ is a constant that acts on the data point weights in an analogous way as the fuzzifier w acts on the membership degrees. A typical choice is therefore $\theta = 2$ (in analogy to the fuzzifier w).

To avoid the trivial solution in which all data point weights go to infinity and thus the value of the objective function becomes zero, a constraint analogous to the constraints of the membership degrees is introduced, namely [11]

$$\sum_{j=1}^{n} v_j = v.$$

With the natural choice $v = n$, the total weight n of the n data points is redistributed to capture the typicality of the data points for the clusters. As an equally natural alternative, one may choose $v = n(1 - \rho)$, where ρ is a user estimate of the fraction of data points that are noise or outliers.

Note that the objective function contains (a function of) the reciprocal values $1/v_j$ of the data point weights, which produces exactly the desired effect: in order to minimize the objective function, large membership degrees will have to be combined with large data point weights and small membership degrees with small data point weights. Note also that with this approach the optimization scheme has three steps: (1) optimize the data point weights for fixed membership degrees and cluster prototypes, (2) optimize the membership degrees for fixed data point weights and cluster prototypes, and finally (3) optimize the cluster prototypes for fixed data point weights and membership degrees. Finally, note that for the last step the membership degrees u_{ij} and the data point weights v_j can be combined into membership degrees $\tilde{u}_{ij}^m = u_{ij}^m / v_j^\theta$, since both values are fixed in this step. As a consequence, the update rules for the cluster parameters are not affected by using outlier clustering and hence it can also be used, for example, with shape and size parameters for the clusters (like, for instance, in [32, 33]) or other modifications of the cluster prototypes.

In order to derive the update rule for the data point weights v_j, the same approach is employed as it was demonstrated in Sect. 2 for the membership degrees. The constraint $\sum_{j=1}^n v_j = v$ is incorporated into the objective function with the help of a Lagrange multiplier. Then the fact is exploited that at the minimum of the objective function the partial derivatives w.r.t. the data point weights v_j must vanish. In this way we easily obtain [11] $\forall j; i \le j \le n$:

$$v_j = v \cdot \frac{\left(\sum_{i=1}^c u_{ij}^w d_{ij}^2\right)^{\frac{1}{\theta+1}}}{\sum_{k=1}^n \left(\sum_{i=1}^c u_{ik}^w d_{ik}^2\right)^{\frac{1}{\theta+1}}},$$

which vanishes only if all clusters collapse to a single point. Using a threshold for the data point weights v_j one may finally identify data points as outliers.

5 Possibilistic Clustering

While the two approaches studied in Sects. 3 and 4 merely relax the constraints $\forall j; 1 \le j \le n : \sum_{i=1}^c u_{ij} = 1$, by (implicitly or explicitly) allowing the membership degrees to sum to values less than 1 (because the membership degree to the noise cluster is deducted or an adaptable data point weight is introduced), *possibilistic (fuzzy) clustering* [12, 13] is more radical and abandons these constraints altogether, allowing the membership degrees to sum to arbitrary values. However, this permits the trivial solution $\forall i; 1 \le i \le c : \forall j; 1 \le j \le n : u_{ij} = 0$, which obviously minimizes the objective function $J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^c \sum_{j=1}^n u_{ij}^w d_{ij}^2$, but, as is equally obvious, is entirely useless.

To fix this problem, a term is added to the objective function that drives the membership degrees away from zero, leading to [12]

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^c \sum_{j=1}^n u_{ij}^w d_{ij}^2 + \sum_{i=1}^c \eta_i \sum_{j=1}^n (1 - u_{ij})^w.$$

Here the η_i are suitable positive numbers (one per cluster $\vec{c}_i$, $1 \le i \le c$) that determine the (squared) distance at which the membership degree of a point to a cluster is 0.5.

They are usually initialized, based on the result of a preceding run of standard fuzzy clustering, as the average fuzzy intra-cluster distance

$$\eta_i = \frac{\sum_{j=1}^{n} u_{ij}^{w} d_{ij}^{2}}{\sum_{j=1}^{n} u_{ij}^{w}}$$

and may or may not be updated in each iteration of the optimization process [12]. The membership degrees are then computed as

$$u_{ij} = \left(1 + \left(d_{ij}^{2}/\eta_i\right)^{\frac{1}{w-1}}\right)^{-1}.$$

It should be noted that the above objective function is truly optimized only if all clusters are identical [39], because the missing constraints decouple the clusters (as can be seen from the computation of the membership degrees). Possibilistic clustering thus actually *requires* that the optimization process gets stuck in a local optimum in order to yield useful results, which is a somewhat strange property. Although the missing constraints certainly help dealing with outliers, this property limits the usefulness of a pure possibilistic approach, although the problem may be mitigated by introducing cluster repulsion [39].

A first solution to this problem was suggested in [14], which combined possibilistic and standard fuzzy clustering, where the latter is sometimes also called *probabilistic fuzzy clustering*, because of the formal resemblance of the membership degrees of a data point to probabilities, due to the constraints $\forall j; 1 \le j \le n : \sum_{i=1}^{c} u_{ij} = 1$. This approach works with the objective function

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} (u_{ij}^{w} + v_{ij}^{\kappa})\, d_{ij}^{2},$$

with the usual constraint for the membership degrees u_{ij}, but the constraints $\forall i; 1 \le i \le c : \sum_{j=1}^{n} v_{ij} = 1$ for the possibilistic *typicality values* v_{ij}. However, it turns out that the membership degrees dominate this approach and since the typicality values depend on the number n of data points, they become very small for large data sets. As an improvement, in [15, 16] the objective function

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} (a u_{ij}^{w} + b v_{ij}^{\kappa})\, d_{ij}^{2} + \sum_{i=1}^{c} \eta_i \sum_{j=1}^{n} (1 - v_{ij})^{\kappa}$$

was proposed, which contains a second term that is characteristic for possibilistic clustering. This leads to the usual (probabilistic) update rule for the membership degrees u_{ij}, while the possibilistic typicality values are updated with

$$v_{ij} = \left(1 + \left(bd_{ij}^2/\eta_i\right)^{\frac{1}{\kappa-1}}\right)^{-1},$$

that is, like the membership degrees in possibilistic fuzzy clustering.

A fundamentally different solution is the graded possibilistic approach presented in [17], which allows for a smooth transition between possibilistic and probabilistic fuzzy clustering. By drawing on an adequately relaxed form of the constraints $\forall j; 1 \leq j \leq n : \sum_{i=1}^{c} u_{ij} = 1$, data points can have a lower total influence weight (in the sense of Sect. 2), but the cluster prototypes are still coupled and (thus) the trivial solution (that is, $\forall i,j : u_{ij} = 0$) is avoided.

The class of constraints suggested in [17] is $\forall j; 1 \leq j \leq n : \sum_{i=1}^{c} u_{ij}^{[\xi]} = 1$, where $[\xi] = [\xi_*, \xi^*]$ is an interval variable, with the natural restrictions $0 \leq \xi_* \leq 1$ and $1 \leq \xi^*$. These generalized constraints are satisfied if for each j there exists a value $\xi_j \in [\xi]$ such that $\sum_{i=1}^{c} u_{ij}^{\xi_j} = 1$. Note that standard probabilistic fuzzy clustering results as a special case of this scheme for $[\xi] = [1, 1]$ and possibilistic fuzzy clustering for $[\xi] = [0, \infty]$. Note also that we may choose $\xi_* = \alpha$ and $\xi^* = \frac{1}{\alpha}$ with a single parameter $\alpha \in [0, 1]$ as a natural simplification.

With this approach the membership degrees are computed as $u_{ij} = u_{ij,\circ}/\kappa_j$, where $u_{ij,\circ}$ is a "free" or "raw" or unnormalized membership degree, as it results from standard possibilistic fuzzy clustering (see above) and [17]

$$\kappa_j = \begin{cases} \left(\sum_{i=1}^{c} u_{ij,\circ}^{1/\alpha}\right)^{\alpha} & \text{if } \sum_{i=1}^{c} u_{ij,\circ}^{1/\alpha} > 1, \\ \left(\sum_{i=1}^{c} u_{ij,\circ}^{\alpha}\right)^{1/\alpha} & \text{if } \sum_{i=1}^{c} u_{ij,\circ}^{\alpha} < 1, \\ 1 & \text{otherwise.} \end{cases}$$

An extensive discussion of several formulations of this soft transition or graded possibilistic approach to fuzzy clustering can be found in [18].

6 Alternative Transformation

A disadvantage of the standard membership transformation approach to fuzzy clustering, which relies on $h(u_{ij}) = u_{ij}^{w}$ (see Sect. 2), is that it *always* produces membership degrees. That is, regardless of how far away a data point is from a cluster center, its membership degree never vanishes. This is one of the core reasons for the negative influence of noise and outliers on fuzzy clustering results.

In order to allow some membership degrees to be zero, an alternative membership transformation was suggested in [19]: $h(u_{ij}) = \alpha u_{ij}^2 + (1-\alpha)u_{ij}$, $\alpha \in (0, 1]$, or, with a more easily interpretable parametrization, $h(u_{ij}) = \frac{1-\beta}{1+\beta}u_{ij}^2 + \frac{2\beta}{1+\beta}u_{ij}$, $\beta \in [0, 1)$. It relies on the standard transformation $h(u_{ij}) = u_{ij}^2$ and mixes it with the identity to avoid a vanishing derivative at zero. The parameter β is, for two clusters, the ratio of the smaller to the larger squared distance, at and below which we get a crisp assignment [19]. It therefore takes the place of the fuzzifier w: the smaller β, the softer the boundaries between the clusters.

The update rule for the membership degrees is derived in essentially the same way as for $h(u_{ij}) = u_{ij}^w$, although one has to pay attention to the fact that crisp assignments are now possible and thus some membership degrees may vanish. The detailed derivation, which we omit here, can be found in [19, 26]. It yields

$$u_{ij} = \frac{u'_{ij}}{\sum_{k=1}^{c} u'_{kj}} \qquad \text{with} \qquad u'_{ij} = \max\left\{0,\ d_{ij}^{-2} - \frac{\beta}{1+\beta(c_j-1)} \sum_{k=1}^{c_j} d_{\varsigma(k)j}^{-2}\right\},$$

where $\varsigma : \{1,\ldots,c\} \to \{1,\ldots c\}$ is a mapping function for the cluster indices such that $\forall i; 1 \le i < c : d_{\varsigma(i)j} \le d_{\varsigma(i+1)j}$ (that is, ς sorts the distances) and

$$c_j = \max\left\{k \,\middle|\, d_{\varsigma(k)j}^{-2} > \frac{\beta}{1+\beta(k-1)} \sum_{i=1}^{k} d_{\varsigma(i)j}^{-2}\right\}$$

is the number of clusters to which the data point x_j has a non-vanishing membership. This update rule is fairly interpretable, as it still assigns membership degrees essentially according to the relative inverse squared distances to the clusters, but subtracts an offset from them, which makes crisp assignments possible.

7 Unsquared Distances

Up to now we considered how fuzzy clustering can be made more robust by changing the way in which data points are assigned to the clusters. Now we turn to the more fundamental approach of changing how distances between the data points and the clusters are measured.[1] As explained in Sect. 2, one of the core reasons for outliers having a strong influence on the cluster parameters is the use of *squared* Euclidean distances. If we used *unsquared* Euclidean distances instead, the clustering algorithm would become much more robust w.r.t. noise and outliers. This can be seen clearly in the right diagram of Fig. 2: the outlier in the top right of the diagram has a much weaker influence on the cluster center if it is computed as the point that minimizes the sum of the *unsquared* Euclidean distances to the data points. Although the center moves if the outlier is included in the computations, it stays much closer to the center computed without the outlier and remains inside the group of data points forming the cluster.

[1] Note that this approach is not restricted to fuzzy clustering, but can be applied for any clustering scheme, including classical c-means clustering.

However, a disadvantage of unsquared Euclidean distances is that the standard approach of finding the cluster update rules as it was reviewed in Sect. 2 becomes problematic, since the square is essential for obtaining (simple) derivatives. Several solutions have been suggested to solve or circumvent this problem. In the first place, one may rely on a scheme as it was introduced for hard clustering with the c-medoids algorithm [40]: instead of computing c cluster centers in the data space that minimize the sum of the distances, one selects those c data points that have this property. This is achieved by starting with a random selection of c data points as the initial cluster centers and assigning, as in c-means clustering, each data point to the center that is closest to it. Then it is tried to improve each cluster center in turn by replacing it with a data point that is not currently a cluster center. The best replacement is chosen and then another replacement is sought for improvement. The process stops if no replacement of a cluster center reduces the sum of unsquared distances to the data points.

This c-medoids approach has been transferred to fuzzy clustering, for example, in [41] under the name "relational fuzzy c-means clustering" (RFCM) and in [3] under the name FANNY (Fuzzy Analysis). The difference between the two approaches consists merely in the fuzzifier used, which is fixed to 2 in FANNY, but can take any value greater than 1 in RFCM. An efficient version for large data sets was proposed in [42]. A combination of this scheme with the noise clustering approach studied in Sect. 3 was presented in [43].

The restriction that in the c-medoids approach only data points can become cluster centers can be removed by using so-called c-medians clustering [2]. Again, however, the problem consists in finding the c (geometric) medians that minimize the sum of the distances to the data points. This is easy only if instead of the Euclidean distance another member of the Minkowski family of distance functions, namely the L_1 distance, is used: $d_{ij} = \sum_{k=1}^{m} |c_{ik} - x_{jk}|$. In this case the medians can be determined separately in each of the m dimensions of the data space, reducing the problem to trivial statistics in one dimension. [2]

For any other member L_p, $p \geq 1$, of the Minkowski family, an iterative majorization scheme was suggested in [22]. This extends the core idea of [21], which introduced an iterative majorization scheme for squared Minkowski distances with $1 \leq p \leq 2$, after the special cases of the L_1 distance and the L_∞ distance had been studied, for example, in [20] and [44]. The approach in [22] is even more general than merely allowing unsquared distances from the Minkowski family. Rather it defines the objective function as [22]

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}^{w} \, d_{ij,p}^{2\lambda} \qquad \text{with} \qquad d_{ij,p}^{2\lambda} = \Big(\sum_{k=1}^{m} |c_{ik} - x_{jk}|^{p} \Big)^{\frac{2\lambda}{p}},$$

[2]Note that computing the membership degrees remains unchanged, regardless of the distance measure and whether it is squared or not, because for this computation the cluster prototypes are fixed and thus the distances are effectively constants.

where $p \geq 1$ is the parameter that selects the member of the Minkowski family of distance functions and the parameter λ, $0 \leq \lambda \leq 1$, allows to make the clustering algorithm robust by choosing a small value for λ. For example, $p = 2$ and $\lambda = \frac{1}{2}$ specify the most interesting case of unsquared Euclidean distances.

Intuitively, the iterative majorization procedure consists in finding, for the current state of the cluster prototypes, a sufficiently simple auxiliary function majorizing the actual objective function. That is, this auxiliary function touches the objective function at the current cluster prototypes and is nowhere smaller than the objective function. Furthermore, it should be easy to find the optimum of this majorizing function, so that one can jump to this optimum in a single step, obtaining new cluster prototypes. Then a new majorizing function is constructed for the new prototypes and the process is iterated until convergence. Presenting mathematical details of this scheme is beyond of the scope of this paper, though. An interested reader is referred to [22], which provides an extensive treatment.

8 Transformed Distances

Instead of using one of the approaches discussed in the preceding section, one may also stick with the (squared or unsquared) Euclidean distance and modify the distance computation or transform the distance measure before computing the membership degrees to increase robustness. One of the most straightforward approaches in this direction is to use an ε-insensitive distance function [24]. It contains the c-medians approach that was mentioned in the preceding section as a special case (for $\varepsilon = 0$), because it employs the objective function [24]

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}^{w}\, d_{ij,\varepsilon} \qquad \text{with} \qquad d_{ij,\varepsilon} = \sum_{k=1}^{p} \max\{0, |x_{jk} - c_{ik}| - \varepsilon\},$$

where ε is the user-specified insensitivity parameter. The update rules for the membership degrees and the cluster centers can be derived in a fairly standard fashion from this objective function (using Lagrange multipliers to incorporate the constraints and partial derivatives), but as the result is mathematically somewhat involved, we do not reproduce it here, but refer an interested reader to [24].

Note that the idea of an ε-insensitive distance function is essentially to give a larger weight to typical data points, since the points in the ε-vicinity of a cluster center are assigned crisply (i.e. $u_{ij} = 1$) to this cluster center, unless such a data point has a vanishing ε-insensitive distance from multiple cluster centers, in which case equal membership degrees to all of these clusters are chosen. Together with the employed unsquared Manhattan distance, this considerably increases the robustness of the algorithm w.r.t. noise and outliers. This effect is particularly pronounced if a larger fuzzifier is employed (compare Fig. 1, even though this figure refers to squared Euclidean distances).

A more general alternative consist in exploiting the idea of robust estimators (especially M-estimators, cf. [45]) as in [25], which uses the objective function

$$J(\mathbf{X}, \mathbf{U}, \mathbf{C}) = \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}^{w} \, \rho_i(d_{ij}),$$

where the ρ_i, $1 \leq i \leq c$, are robust symmetric positive definite functions having their minimum at 0 (with $\rho_i(d_{ij}) = d_{ij}^{w}$ as a special case). In [25] the same function ρ is used for all clusters, which is derived from Tukey's bisquare function [45]. This leads to update rules for the membership degrees, in which merely the distances are replaced by $\rho(d_{ij})$, while the cluster centers are updated with

$$\vec{c}_i = \frac{\sum_{j=1}^{n} u_{ij}^{w} f_{ij} \vec{x}_j}{\sum_{j=1}^{n} u_{ij}^{w} f_{ij}} \qquad \text{where} \qquad f_{ij} = \frac{\mathrm{d}\rho(d_{ij})}{\mathrm{d}\, d_{ij}}.$$

An even more general, but closely related approach is the *alternating cluster estimation (ACE)* scheme that was proposed in [23] (see also [4]). The idea of this approach is to abandon the requirement of an objective function that is to be optimized and from which the update rules can be derived. Rather the alternating optimization scheme is taken as the core algorithmic component, for which plausible update rules are chosen for the two steps of recomputing the membership degrees and recomputing the cluster parameters.

In its most common form, such an approach first transform the distances d_{ij} with a radial function $r : \mathbb{R} \to [0, 1]$ to obtain "free" or "raw" membership degrees $r(d_{ij})$ to the clusters. These raw membership degrees may then be normalized using, for instance, the constraints $\forall j; 1 \leq j \leq n : \sum_{i=1}^{c} u_{ij} = 1$. Typical choices for the radial functions (the name of which stems from the fact that they are defined on a ray—latin: *radius*—from the cluster center), are shown in Fig. 3. Especially those radial functions that have a finite support (that is, for which exists $x_0 \in \mathbb{R}_+$ with $\forall x > x_0 : r(x_0) = 0$) are well suited for handling noise and outliers, because data points with a distance outside the support of the radial function have a vanishing influence on the corresponding cluster.

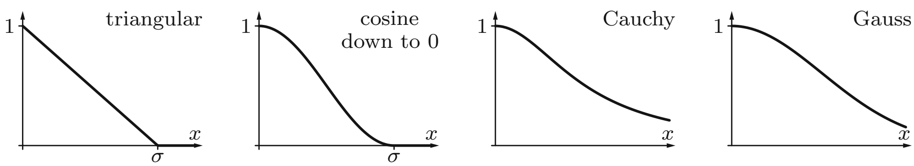

Fig. 3 Radial functions that may be used in alternating cluster estimation (ACE) [23]

The update rules for this scheme are simply (assuming merely cluster centers)

$$u_{ij} = \frac{r(d_{ij})}{\sum_{k=1}^{c} r(d_{kj})} \qquad \text{and} \qquad \vec{c}_i = \frac{\sum_{j=1}^{n} u_{ij}^{w} \vec{x}_j}{\sum_{j=1}^{n} u_{ij}^{w}}.$$

Generally, these update rules cannot be derived from an objective function (as shown in Sect. 2 for the standard case), but are merely transferred from the standard approach. It should be noted, though, that for certain radial functions, for example, the (generalized) Gaussian and the Cauchy function

$$r_{\text{Gauss}}(x) = e^{-\frac{1}{2}r^{a}} \qquad \text{and} \qquad r_{\text{Cauchy}}(x) = \frac{1}{x^{a} + b},$$

where a and b are parameters to be specified by a user, a formulation with the help of an objective function is possible, so that the needed update rules can be obtained in the usual way (using Lagrange multipliers to incorporate the constraints and setting partial derivatives equal to 0, see [26] for details).

A noteworthy alternative, which also relies on an ACE scheme instead of deriving the update equations for the membership degrees and cluster parameters from an objective function, is to compute the membership degrees as $u_{ij} = ((\max_k d_{ik}) - d_{ij}) / \max_k d_{ik}$ [46]. In this way the degree of membership of the data point that is farthest from a cluster center always vanishes, which has the additional advantage that it renders the membership degrees independent of the scale of the data set. Note that it is closely related to a possibilistic approach, because it is usually not $\forall j; 1 \leq j \leq n : \sum_{i=1}^{c} u_{ij} = 1$.

9 Summary

In this paper we reviewed several approaches to make fuzzy clustering (even) more robust against noise and outliers. The studied approaches fall into two categories: (1) reduce the "influence weight" of atypical data points and outliers on the cluster parameters by changing how membership degrees are computed from the distances, and (2) change the distance function or transform it before the membership computation in order to reduce the degrees of memberships of atypical data points and outliers. Approaches in the former category are usually easier to handle, because in them the update rules are fairly easily obtained from an objective function using standard tools. Changing the distance measure causes more problems in this respect and thus often either a majorization approach has to be called upon or the rooting in an objective function is abandoned as in alternating cluster estimation (ACE). However, all of these approaches have the desired effect of making fuzzy clustering (even) more robust. Our personal favorites are noise clustering (see Sect. 3) and using an alternative membership transformation (see Sect. 6). However, this should not be interpreted as a recommendation against any of the other approaches.

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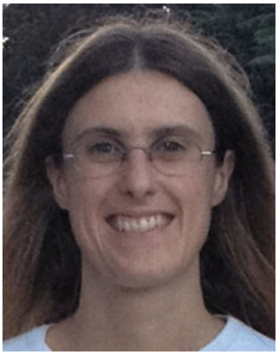

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A Fuzzy-Based Approach to Survival Data Mining

Mark Last and Hezi Halpert

Abstract Traditional data mining algorithms assume that all data on a given object becomes available simultaneously (e.g., by accessing the object record in a database). However, certain real-world applications, known as *survival analysis*, or *event history analysis (EHA)*, deal with monitoring specific objects, such as medical patients, in the course of their lifetime. The data streams produced by such applications contain various events related to the monitored objects. When we observe an infinite stream of events, at each point in time (the "cut-off point"), some of the monitored entities are "right-censored", since they have not experienced the event of interest yet and we do not know when the event will occur in the future. In *snapshot monitoring*, the data stream is observed as a sequence of periodic snapshots. Given each snapshot, we are interested to estimate the probability of a critical event (e.g., patient death or equipment failure) as a function of time for every monitored object. In this research, we use *fuzzy class label adjustment* so that standard classification algorithms can seamlessly handle a snapshot stream of both censored and non-censored data. The objective is to provide reasonably accurate predictions after observing relatively few snapshots of the data stream and to improve the classification performance with additional information obtained from each incoming snapshot. The proposed fuzzy-based methodology is evaluated on real-world snapshot streams from two different domains of survival analysis.

1 Introduction

Survival analysis, or *event history analysis (EHA)*, deals with monitoring specific objects/entities in the course of their lifetime [1]. The data streams produced by the monitored objects contain various events that occur to those objects. In survival analysis, we are usually interested in estimating the amount of time before the

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_18

occurrence of an event or in the probability of event occurrence during certain period. The following are some common applications of survival/event history analysis:

- *Health Care*. The objects are patients who underwent a medical procedure (like surgery) and the event of interest may be disease recurrence or patient death within a certain follow-up period.
- *Warranty Management*. The objects are products and the event of interest is a product failure during its warranty period.
- *Customer Retention*. The objects are service customers and the event of interest is customer churn.

Since each monitored object is typically described by a set of features (e.g., clinical variables in case of a patient), the problem of estimating the time to event occurrence may be defined as a *regression task*. Similarly, the problem of predicting the object survival (non-occurrence of an event) within a pre-defined follow-up period (e.g., product warranty period) may be considered a *classification task* with two possible outcomes – "survived" and "failed". To induce a regression or a classification model, we need a training set of objects with a known value of the dependent variable or the class label, respectively. In *snapshot monitoring*, such a training set may be obtained every time we take a "snapshot" of a data stream produced by a set of monitored objects, which may be continuously updated by new objects entering the observation process. The problem is that in each snapshot, some objects may not have experienced the event of interest or completed the follow-up period until the time of the snapshot (the "cut-off point"). These incompletely observed objects are termed "right-censored", since their time to event is longer than the time to the current snapshot but we do not know whether they will experience the event before the end of their follow-up period. On the other hand, we do know how long these unlabeled objects have survived from the beginning of their follow-up period (called the "birth event") to the time of the current snapshot.

In this chapter, we focus on the classification task in snapshot streams, where each snapshot may contain both censored and non-censored observations. The proposed CENSMINER (CENsored data Stream MINing) algorithm uses a *fuzzy class label adjustment* [2] so that standard classification algorithms can be trained on *all* objects observed in each snapshot rather than just the objects with a known label. The CENSMINER objective is to provide reasonably accurate predictions after observing the first few snapshots of a data stream and to improve the classification performance with additional labeled objects observed in each incoming snapshot.

The rest of this chapter is organized as follows. Section 2 discusses the related work on survival analysis and mining censored data. The CENSMINER methodology for mining censored data streams is introduced in Sect. 3. In Sect. 4, we evaluate the effectiveness of the proposed methodology on two real-world snapshot streams from the warranty and the medical domains, respectively. Section 5 presents some conclusions and directions for future research.

2 Related Work

2.1 Survival Analysis

Moeschberger and Klein in [3] review the field of survival analysis and its relationship to censored data. Traditionally, examples of survival data come from the medical domain, where the event of interest is usually a disease recurrence or a patient's death. An event of interest may also be a positive event, such as a full recovery from some disease, conception, smoking cessation, and so forth. Usually, the event of interest is related to a specific object (e.g., a patient).

The opposite of the death event of an object in survival analysis is the *birth event*. The birth event of an object denotes the time point when the object starts being monitored. For example, in the medical field, the birth event may represent the starting point of a particular treatment. In warranty data, the birth event is usually the sale event, which indicates the beginning of the product usage.

The term *follow-up period*, or the *prediction period*, is used to denote the period for which the survival probability is of interest, such as the warranty period in product warranty management. The term *cut-off point* represents the time point when the monitoring of all objects has been stopped. Usually, this point is the end of the experiment. A continuous data stream may be sampled periodically, resulting in multiple cut-off points, each representing a new snapshot.

The *snapshot monitoring* approach is more suitable for cases where the change rate of the monitored stream is relatively slow. This allows periodically taking discrete samples ("snapshots") of the data stream and producing a prediction model from each incoming snapshot. Whenever there is no high rate of changes in the set of monitored objects, the periodic, snapshot monitoring is preferable over the continuous, event-driven monitoring, which requires more computational resources.

Additionally, for each sampled object, the time elapsed from its birth event to the event of death or to the cut-off point is defined as the object's *lifetime* (or its *age*). For example, in warranty data, the lifetime of a product is the usage time elapsed since it has been sold.

Figure 1 describes these concepts and their relations.

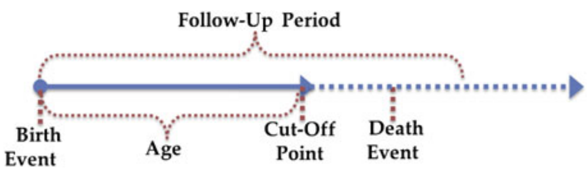

Fig. 1 Terms in survival analysis

Two basic concepts in the survival analysis are the *survival function* and the *hazard function*. Assuming X is a nonnegative random variable from a homogeneous population that represents the age of an individual, the survival function is the probability of an individual to survive beyond time x. The *hazard rate* (or hazard/risk function) is the conditional probability that an individual of age x will experience the event in the next instant after x, given that the event has not occurred until x. When X is a discrete random variable, the following equation describes the relationship between these two functions:

$$S(x) = \prod_{x_j \le x} \left[1 - \lambda\left(x_j\right)\right]$$

where $S(x)$ is the *survival function*, the probability of an ndividual of age x to survive, and $\lambda\left(x_j\right)$ is the *hazard function*, the "approximate" probability of an individual of age x_j to experience the event in the next instant.

2.2 Survival Probability Estimation from Censored Data

Censored data is very common in the survival analysis, especially when inter-snapshot times tend to be significantly shorter than the overall follow-up period. Thus, a problem of censored data frequently arises in clinical trials where a substantial fraction of surviving participants may be removed from the study to reduce the monitoring costs. Patients with incomplete outcome information are considered censored or, more precisely, right-censored, since we only know that the actual time-to-event for that participant exceeds the duration of their follow-up [4].

2.2.1 The Kaplan–Meier Estimator

The Kaplan-Meier method (also called the *product-limit estimator*) is aimed at estimating the survival function, i.e., the probability for an individual in a population to survive beyond a specific time. This method was first proposed by Böhmer in 1912 and rediscovered by Kaplan and Meier in 1958 [5]. The Kaplan-Meier method provides a non-parametric estimation from incomplete observations. The term *observed lifetime* refers to the time interval from the birth event of each object (e.g., a surgery or a product purchase) to its failure or "death" event. The survival probability is estimated separately for each observed lifetime value in the dataset. It is also assumed that the probability of surviving each lifetime t_i is statistically independent of every other lifetime. The Kaplan-Meier estimate of the survival curve $\hat{S}(T)$ is then:

$$\hat{S}(T) \approx \prod_{t_i \le T} \frac{n(t_i) - d_i}{n(t_i)}$$

where $n(t_i)$ is the number of objects "at risk" (i.e., not lost through censoring or failure) at time t_i and d_i is the number of objects which failed at time t_i. Usually $d_i = 1$, since t_i is chosen as the observed time to failure.

Costella [6] points out some of the disadvantages of the Kaplan-Meier method:

1. The "danger times" defined by the failure occurrence times and represented by vertical drops in the Kaplan-Meier estimation seem "unnatural" compared to the original survival curve. The actual survival curve is probably continuous rather than discrete and is unlikely to decrease locally at those particular failure occurrence times.
2. With the advance of the monitoring time, there are fewer and fewer remaining objects at risk (due to censoring and failures). As a result, the impact of a single failure seems unjustifiably magnified if it occurs at a later time.
3. If the last remaining object at risk fails, the Kaplan-Meier estimate of *S(t)* drops to zero and does not generally reflect the original survival function.
4. For each censored record, the amount of survival time from the previous failure to the time of censoring is ignored.

In [6], Costella suggests an alternative method to the Kaplan-Meier method, which offers a better visual representation of survival curves. Another solution for the fourth problem above (i.e., ignoring the amount of the survival time from the previous failure to the time of censoring) is to weigh the censored records according to their relative time of survival. This way the information is not discarded and can be taken into account when calculating the survival (or failure) probability. This approach was used in our previous work with warranty data [7].

2.2.2 The Cox Proportional Hazard Models

The Kaplan-Meier estimation is a simple concept: estimate the survival curve when there are no other covariates than the usage factor (e.g., patient age or vehicle mileage). Other models are designed to handle covariates as well. D. R. Cox [8] introduced the semi-parametric proportional hazard model, a continuous statistical model for time-to-event data. In his approach, the model's covariates are multiplicatively related to the hazard function such that the effect of a unit increase in a covariate is multiplicative with respect to the hazard rate.

The Cox model contains two different parts: the underlying hazard function denoted $h_0(\mathrm{t})$ (also called the baseline hazard function), which describes how the hazard (e.g., failure rate) changes over time when setting the covariate values to zero and the other covariate parameters, which describe how the hazard varies in response to explanatory covariates. The general form of the Cox model is as follows: $h(t|x) = h_0(t) * \exp(x\beta)$, where β is the parameter vector. The Cox model is semi-parametric, since its baseline hazard function $h_0(t)$ can be modeled by any probability distribution function, such as the Gompertz and Weibull distributions (the nonparametric part), whereas the covariate coefficients must be estimated using the *partial maximum likelihood* (the parametric part).

2.3 Mining Censored Data

2.3.1 Static Datasets

Segal [9] introduced a new approach to splitting criterion in regression trees induced from datasets with censored target variables. In each node, the Kaplan-Meier (i.e., survival curve estimation for censored data, see Sect. 2.2.1 above) and its median are first stored. Then, all predictive attributes and all splitting points within an attribute are considered, subject to a minimal ratio of uncensored to censored records in each child node. The best split is the attribute and the splitting point, which maximize the between-node separation. In addition to the regression tree induction itself, this method also has the advantage of presenting informative characteristics for each terminal node, such as the estimated Kaplan-Meier survival curve and the Kaplan-Meier median [9].

Zupan et al. [10] deal with predicting the probability of prostate cancer recurrence in patients after prostate removal. It is assumed that if a prostate cancer patient who has undergone a prostatectomy remains disease free for at least 7 years then the cancer has been successfully cured. In this case, the censored data comes from non-recurring patients who have been monitored for less than 7 years and whose outcome at the end of the 7-year period remains uncertain.

Zupan's suggested approach is to split the prostate cancer survival data into three groups: patients who experienced a recurrence of cancer (i.e., the outcome is known to be "recurrence"), patients who did not experience a recurrence and were monitored for more than 7 years after their operation (i.e., the outcome is known to be "non-recurrence"), and patients who have not experienced a recurrence but have been monitored for less than 7 years (the outcome is censored). For the third group, instead of a single outcome, two outcomes are considered: recurrence and non-recurrence. Each outcome is weighted using the Kaplan-Meier method, which estimates the probability of non-recurrence at a particular follow-up time based on all patient groups. For a patient's follow-up time T_f from the third group, the weight of the non-recurrence outcome is calculated by the following formula, where $P(non-recurrence(\mathrm{t}))$ is the Kaplan-Meier estimation for non-recurrence probability at a time t:

$$P_{rf}\left(T_f\right) = \frac{P(non-recurrence(7\,years))}{P\left(non-recurrence\left(T_f\right)\right)}$$

The weight of the recurrence outcome is: $P_r = 1 - P_{rf}$.

Since most machine-learning techniques cannot be trained on probabilistic outcomes, two copies of patient's record are created, one weighted with P_r values and the other one with P_{rf} values. The probability of class C (recurrence or non-recurrence) is then denoted by: $P(C) = \frac{\sum_{E \in C} weight(E)}{\sum_E weight(E)}$, where E is an individual example from the "new" patient's record. A recurrence probability of higher than 0.5 is considered as a prediction for a patient to recur.

Zupan et al. have used three methods for constructing predictive models: the Naive Bayes Classifier, the decision tree induction, and the Cox proportional hazards model (see Sect. 2.2.2 above). The evaluation was performed using stratified 10-fold cross-validation on the basis of classification accuracy, specificity and sensitivity, correlation of predicted probability and probability estimated by the Kaplan-Meier method, and the concordance index (i.e., the area under the Receiver Operating Characteristic (ROC) curve). The non-recurring patients with incomplete follow-ups in the test set were weighted by the Kaplan–Meier estimates calculated from the training sets. The Naive Bayes and the Cox proportional hazards models seemed to perform better than decision trees, although the differences were not significant. The results did not include any comparison of the proposed weighting technique to training on non-censored records only. Testing results on the actual labels of non-censored records were not reported either.

2.3.2 Data Streams

A recent paper by [11] considers event history analysis in the data stream setting. They assume a fixed set of data streams; each corresponding to an object characterized by a feature vector (a vector of covariates). Each stream produces a sequence of recurrent events related to a specific object. The authors of [11] introduce an incremental, adaptive version of the Cox proportional hazard model (see Sect. 2.2.2), which re-estimates the covariate coefficients of the hazard function at every discrete point in time by averaging over all sliding time windows covering that point. The sliding window has a fixed length and it is shifted by one unit at a time. In each window, the maximum likelihood estimation of the covariate coefficients in each sequence is based on the events observed in that sequence and thus it ignores the right censoring of the next event, which has not been observed yet.

The continuous monitoring approach of [11] may be appropriate for high frequency data streams of recurrent events, such as earthquakes or tweets. In [12], we have proposed a snapshot monitoring approach for an infinite stream of "birth" and "failure" (non-recurrent) events related to a set of objects. In each snapshot, some of the monitored objects may be right-censored, since their "failure" event has not been observed yet. Given a snapshot, we are interested to induce a classification model from both censored and non-censored observations for predicting the survival of each object to the end of a pre-defined follow-up period. In the present chapter, we describe in detail the survival analysis methodology initially introduced by us in [12] and evaluate it on two real-world streams of survival data.

3 The CENSMINER Methodology

3.1 Overview

The CENSMINER algorithm deals with a sequence of periodical snapshots from an infinite stream of "birth" and "death/failure" events related to a set of monitored objects. The snapshot frequency does not have to be fixed and the set of objects under study may be dynamic as well. Two consecutive snapshots may differ in the following aspects:

(1) New objects have entered the observation process (i.e., their "birth event" took place between the snapshots). The new objects remain censored (more precisely, right-censored) as long as they cannot be labeled (cases 2 and 3 below).
(2) Some objects have survived the follow-up period (i.e., the end of their follow-up period took place between the snapshots *and* no "failure event" was observed before that time point). Such objects are assigned the crisp "survived" label and become uncensored.
(3) Some objects have experienced a "failure event" between the snapshots. If the event occurred before the end of their follow-up period, the object is assigned the crisp "failed" label and becomes uncensored as well.

In each snapshot, the algorithm is aimed at predicting (or estimating the probability of) a "death event", such as a disease recurrence in a patient or a technical failure in a car, within a pre-defined follow-up period for all monitored objects (entities), which are still censored and thus cannot be labeled. The classification/probability estimation model induced in each snapshot can only be evaluated on all objects that become uncensored between the current snapshot and the next one (cases 2 and 3 above). We assume that the focus on these object records as a testing set represents many real-world scenarios, such as evaluating the outcome of a new clinical procedure or estimating the reliability of a new car model, where we are interested in estimating the failure probability for the unlabeled (i.e., censored) records after as few data snapshots as possible. The model performance can be evaluated using various measures, such as the Area under ROC curve (AUC), classification accuracy, classification sensitivity, etc.

Our fuzzy class label adjustment pre-processing strategy works as follows. Upon arrival of a new data snapshot, the algorithm identifies all censored objects and calculates their age value, i.e., the time since their "birth" event. Each censored object creates two weighted records representing the two fuzzy labels ("survived" and "failed"), respectively. The age value of the censored object is used to calculate the weight of each record based on the Kaplan-Meier estimate [10]. Another procedure compares the current and the previous data snapshots, finds the records that have become uncensored since the previous snapshot, and labels them with their true (crisp) labels.

In general, any classification algorithm can be integrated with the proposed methodology as long as it can process weighted data instances. The problem of learning from data instances, which are assigned fuzzy class labels, can be handled by assuming different "label proportions" for each instance. This problem setting is close to the tasks of unsupervised and semi-supervised learning, where only unlabeled or partially labelled instances are given. The adaptation of a classification algorithm to training instances with label proportions is not always trivial and it has not gained much attention in the machine learning community. A similar problem of learning a classifier from group probabilities is known as Multiple-Instance Learning with Label Proportions (MIL-LP). Rueping, in [13], proposed an MIL-LP algorithm based on SVR, a version of SVM for regression. Hernández, in [14], proposed an adaptation of the Naive Bayes Classifier to the MIL-LP problem. In some classification methods, the way of treating the weighted instances is straightforward, namely, by replacing any reference to an instance, such as summing, multiplication, etc., by its weight. For example, in the C4.5 decision-tree induction method, the entropy calculations regarding the splitting criteria of each node include the summation of probabilities of a class C in the dataset. Usually, the probabilities are calculated as the number of instances assigned the C label divided by the number of all instances. However, in the presence of weighted instances, these probabilities are calculated as the sum of weights of instances with the C label divided by the sum of all instance weights.

Although the CENSMINER algorithm focuses on a classification task, it may be easily adapted to the probability estimation task, as demonstrated in the warranty use case below. Additionally, instead of a simple classification model, the algorithm may use the probability estimation tree (PET) model [15], which can compute the probability of each outcome. The decision whether to induce a classification model or a probability estimation model is domain-dependent. In imbalanced domains, such as warranty data of high quality products, a classification model would be useless, as it will always predict a "survival". A probability estimation model would be more appropriate for such cases.

3.2 The CENSMINER Algorithm

3.2.1 The Notation

The CENSMINER algorithm uses the following notations:

$i \in [1, \infty)$ – a discrete index of each snapshot.
t_i – The timing of a snapshot i.
D_i – The set of records in a snapshot received at time t_i. Each record $x \in D_i$ contains the information known at the time of t_i about the individual (object) under study, such as a vehicle, a patient or a customer.
D_i^S, D_i^F – Two sets of weighted records representing the "survival" and the "failure" outcomes, respectively. Each weighted record is stored as a pair $< x, w(x) >$,

where *w(x)* is the class membership value of an object record $x \in D_i$ in the respective outcome.
$C_i \subseteq D_i$ – The subset of censored records in D_i, i.e., the records with unknown outcome at the time t_i.
$NC_i \subseteq D_i$ – The set of new uncensored records in snapshot i, i.e., the records that were censored at time t_{i-1} and are no longer censored at time t_i ($i > 1$).

In each snapshot i, every object record is either censored or non-censored. Consequently, if we refer to the set of non-censored records in the first snapshot as NC_1, we get the following formula which mathematically defines the relationship between the sets above:

$$D_i = \left(\bigcup_{j \leq i} NC_j \right) \bigcup C_i$$

This formula emphasizes the fact that every object record is represented in any snapshot either by a censored record or by a record, which became uncensored in one of the previous snapshots (including the latest one). The records that became uncensored in the latest snapshot are used as a testing set for evaluating the performance of the last model. It also means that every monitored object, excluding the records that became uncensored in the first snapshot NC_1, is used exactly once during its lifetime for testing a classification model.

x_τ – The censoring value (age) of $x \in D_i$. For example, in warranty data, this is the time elapsed (or the usage value) since vehicle x was sold. In churn prediction, x_τ is the time elapsed since customer x signed up for the service. In medical survival analysis, x_τ is the time elapsed since patient x experienced a medical procedure, such as a surgery.

event(x) – The age or the usage value of x when it experienced the death/failure event. For example, in vehicle warranty data, *event(x)* may contain the mileage of a vehicle x at its first claim. In churn prediction, *event(x)* may contain the time until the customer x churned. In medical survival analysis, *event(x)* may contain the time until a patient x was diagnosed with a re-infection. For individuals who have not experienced the event of interest yet, the *event(x)* value is null.

M_i – The classification/probability estimation model induced from the snapshot i.
Ac_i – The testing performance of the model M_i. The testing performance value (e.g., classification accuracy, AUC, etc.) is calculated on the next snapshot (i.e., at the time t_{i+1}) by testing the M_i model on the NC_{i+1} records described above.
W – The duration of the follow-up (prediction) period defined in Sect. 2.1 above. We assume that this duration is fixed for all monitored objects disregarding the time of their birth event.
A – The classification/probability estimation algorithm (e.g., C4.5 Decision Tree) for inducing a classification model from each snapshot.
E – The evaluation measure (e.g., AUC) for testing the induced model on each incoming snapshot.

3.2.2 The Model Induction Process

The general input of the CENSMINER algorithm includes a "snapshot stream" – a potentially infinite sequence of data snapshots. Each snapshot i consists of the record set D_i defined above where some of the records may be censored. A record contains a list of attributes, which may be used as predictive features. In addition, each record includes the x_τ (the "age" of x) and the *event*(x) attributes defined above. The value of *event*(x) may be null.

In addition, the algorithm has the following global parameters:

- W - the duration of the follow-up period
- A - the classification algorithm
- E - the evaluation measure

The algorithm flow is described below:
For each snapshot i:
Input:

D_i – The new snapshot i.
D_{i-1} – The previous snapshot (for $i > 1$).
M_{i-1} – The classification/probability estimation model induced from the previous snapshot D_{i-1} (for $i > 1$).

Output:

Ac_{i-1} – The testing performance of the model M_{i-1} on the new snapshot i
M_i – The classification model induced from the new snapshot i.

The algorithm:

1. Identify the subset of censored records C_i using *censored_identification_procedure*(D_i,W)
2. Identify the subset of new uncensored records NC_i using *new_uncensored_identification_procedure*(D_i,D_{i-1},W)
3. If ($i > 1$)then:
   3.1. Calculate Ac_{i-1} by applying M_{i-1} on NC_i
4. Induce a new prediction model M_i:
   4.1. Calculate S_i function, the survival curve estimation given the snapshot i, using the Kaplan Meier estimate [5]:

$$S_i(T) = \prod_{t_j < T} \frac{n(t_j) - d_j}{n(t_j)}$$

Where $S_i(T)$ is the estimated probability of a monitored object to exceed the lifetime T, $n(t_j)$ is the number of objects "at risk" (not lost through censoring or failure) at time t_j and d_j is the number of objects which failed at time t_j. Usually $d_j = 1$, since t_j is chosen as the observed time to failure.

4.2. Set $SW = S_\mathrm{i}(W)$, the probability of an individual to survive for the duration of the follow-up period W

4.3. Calculate the probabilities $st(x), \overline{st}(x)$ of the two outcome classes for each record $x \in D_i$:

4.3.1. if $x \in C_i$ (i.e., x is censored):

4.3.1.1. Find $S_i(x)$, the probability of an object x to survive beyond its age x_τ

4.3.2. Set the total probability of the survival $st(x)$ of an object x up to the end of the follow-up period:

$$st(x) = \begin{cases} \frac{SW}{S_i(x)} & x \in C_i \\ 0 & x \notin C_i \; and \; event(x) \neq null \\ 1 & else \end{cases}$$

4.3.3. Set the total probability of failure $\overline{st}(x)$ of record x up to the end of the prediction period: $\overline{st}(x) = 1 - st(x)$

4.4. Initialize the two sets of weighted records: $D_i^S = \varnothing$, $D_i^F = \varnothing$. D_i^S and D_i^F store the "survival" and the "failure" records with their class membership weights, respectively.

4.5. For each record $x \in D_i$ do:

4.5.1. Append x to the set of "survival" records: $D_i^S{:} = D_i^S \cup \{ < x, st(x) > \}$

4.5.2. Append x to the set of "failure" records: $D_j^F : = D_j^F \cup \{ < x, \widehat{st}\}(x) > \}$

4.6. Combine D_i^S and D_i^F into one weighted training dataset $D_i^{'} : D_i^{'} = D_i^S \cup D_i^F$

4.7. Induce a new model M_i by applying the learning algorithm A to the set $D_i^{'}$

5. Return M_i – the classification/probability estimation model induced from the records in snapshot i

<u>*censored_identification_procedure*</u> (D_i, W)
Input:

D_i – The set of records in a snapshot received at time t_i. Some of the records may be censored.
W – The duration of the follow-up (prediction) period.

Algorithm:

1. Initialize the set of censored records: $C_i = \varnothing$
2. For each record $x \in D_i$ do:

   2.1. If *event(x)* is null and $x_\tau <$ W;
   Append x to the set of censored records: $C_i{:} = C_i \cup \{x\}$
3. return C_i

<u>*new_uncensored_identification_procedure*</u> (D_i, D_{i-1})
Input:

D_i – The set of records in a snapshot i.
D_{i-1} – The set of records in the previous snapshot $i-1$.

Algorithm:

1. If D_{i-1} is null (the first snapshot):
   1.1. Return $\varnothing$
2. Otherwise:
   2.1. Initialize the set of new uncensored records: $NC_i = \varnothing$.
   2.2. For each record $x \in D_i$ do:
      2.2.1. If $x \in C_{i-1}$ and $x \notin C_i$;
      Append x to the set of new uncensored records: $NC_i := NC_i \cup \{x\}$
   2.3. Return NC_i

4 Empirical Evaluation

4.1 Experimental Settings

Our empirical evaluation is aimed at comparing the performance of the CENS-MINER methodology, which involves the Kaplan-Meier estimation for the outcome label of censored records, to a baseline approach, where the censored records are completely discarded during the model induction process. Both approaches are evaluated using a variety of standard classification algorithms.

4.1.1 Data Sources

In this chapter, we present the results from two real-world survival datasets representing two different domains: warranty management and health care. The first data set, the Vehicle Warranty Dataset, contains the warranty data of more than 200,000 vehicles of a given model sold in North America between the years 2008 and 2011 by a major automotive company. The second data set, the STD Dataset, contains data collected from STD (Sexually Transmitted Diseases) patients, some of whom were re-infected after a period of time. The dataset was downloaded from the University of North Texas website.

Table 1 displays some characteristics of the datasets used in our experiments.

4.1.2 Snapshot Simulation

Both datasets were originally obtained in the form of a single snapshot observed at the end of the data collection period. In the pre-processing stage, we created several snapshots from each dataset to emulate a real-world situation where data arrives as a sequence of consecutive snapshots. For this purpose, we set fictitious snapshot dates for each dataset. Then we created multiple samples of the original dataset, each representing the event information, which was available on the date of a particular snapshot. That is, an object ‘born’ after the snapshot date was removed from the snapshot sample and each ‘failure’ event occurred after that date was ignored. All object records associated with a given snapshot (both censored and uncensored) were copied from the original dataset along with their attributes. In the case of a ‘failed’ record, the event date was copied as well.

The snapshot arrival rate was set for each simulated data stream in accordance to the nature of the relevant domain. For example, in the Vehicle Warranty Dataset, the snapshots were created every 90 days (about three months), a reasonable period to sample a dataset of 200,000 vehicles, which resulted in the total of 15 snapshots. We assumed that three months is a sufficiently long period to expect a change in the existing classification model. On the other hand, in the STD dataset, we simulated a snapshot arrival rate of 305 days, which resulted in the total of 6 snapshots.

4.1.3 The Follow-up Period Duration

The duration of the follow-up (prediction) period was also chosen in order to generate a real-world scenario of the relevant data stream. For the Vehicle Warranty Dataset, four prediction periods were simulated: 365, 730, 1095, and 550 days. The first three values represent one, two, and three years out of the three-year warranty period, respectively, and they are reasonable choices to estimate the survival probability. The value of 550 was chosen because it represents the median time-to-failure in the Vehicle Warranty Dataset. For the STD Dataset, we simulated a prediction period of 300 days, which was close to the average time to re-infection in this data.

4.1.4 Performance Measures

The choice of the most appropriate performance measures for each data stream depends on its class imbalance ratio. The Vehicle Warranty Dataset (with ~4 % of failures) is extremely imbalanced whereas the STD Dataset (with 45 % of re-infected patients) is relatively balanced. The standard performance metrics (e.g., classification accuracy) are ineffective in the case of imbalanced classes because they favor “majority rule” models like “a car under warranty never fails”. Therefore, in our experiments with the Vehicle Warranty dataset, we use the AUC (Area Under ROC curve) value instead of the standard accuracy metric.

Table 1 Dataset characteristics

Characteristic	Vehicle warranty data	STD data
# of snapshots	15	6
Duration of the follow-up period (days)	365, 550, 730, 1095	300
Evaluation method	AUC	Accuracy
Imbalance ratio (%)	4/96	45/55
Missing values	11.70 %	0 %
Data size (total # of observed objects)	204,708	877
# of features (numeric, date, nominal)	3 (1, 1,1)	21 (3, 0, 18)

4.1.5 Experimental Environment

The experiments were run in the following environment: Windows7 Enterprise SP1, Intel i5-2400 CPU 3.10 GHz with 4 GB RAM, and 32-bit architecture. The classification models were induced and tested using a local extension of the MOA environment, Version 2012.08.31 [16]. The MOA (Massive Online Analysis) environment allows for the implementation and evaluation of learning algorithms on evolving data streams. In particular, it allows for implementing and running various classification algorithms. Some of these classification algorithms have been implemented within the MOA environment, while others are available by interfacing with WEKA, the Waikato Environment for Knowledge Analysis [17]. The interface with WEKA is particularly important for running standard classification algorithms on the static training datasets extracted from each snapshot.

4.2 Vehicle Warranty Dataset Results

4.2.1 Data Description

The original dataset contains 204,708 records of vehicles of a given model sold in North America (i.e., U.S.A and Canada) between April 2008 and December 2011. Some of the vehicles had one or more warranty claims (e.g., failures) during this time. We focused on 8,147 vehicles (~4 % of the entire dataset) that have experienced a specific (battery) failure during their warranty period. The first claim was recorded in September of 2008 and the last one in June 2011. Thus, the vehicle sale date is considered the “birth event” and the first battery failure is considered the “death event”. Recurrent battery failures in the same under-warranty vehicle were ignored due to their extremely low incidence.

Each vehicle record is associated with a unique identification number and has values for the following attributes:

1. Manufacturing date – The date when the vehicle was produced (a continuous variable).

2. Sale gap – A continuous non-negative variable representing the number of days elapsed from the manufacturing time to the sale date.
3. Area of sale – A nominal variable representing the geographic area where the vehicle was sold. This variable can take one of the following five values: MIDWEST, SOUTH, WEST, NORTHEAST or CANADA. About 23 % of vehicles are missing this value.

Each warranty claim record contains the following information recorded by the service dealer:

1. Vehicle ID number.
2. Labor code (i.e., failure ID).
3. Job card date – The date when the vehicle failure was identified by the dealer.
4. Vehicle odometer mileage – The mileage value of the vehicle's odometer at the time of failure.

The pre-processing stage included the creation of snapshots from the initial database in order to emulate a real-world situation where the data arrives in several snapshots. The snapshot frequency was set to 90 days (about 3 months) beginning on April 2008 (the first sale date in the database). In addition, the final snapshot date was set to December 2011, the date of the last recorded claim in the dataset. We have created 15 snapshots this way. Table 2 shows the characteristics of the 15 simulated snapshots. One can see that the total number of observed vehicles increases over time as more vehicles are "born", i.e., sold to the customers. In the first few snapshots, the records of nearly all monitored vehicles are censored, since

Table 2 Characteristics of simulated snapshots in the vehicle warranty data stream

Snapshot ID	Number of days since the beginning of the stream	Total number of observed vehicles	Number of censored vehicles (prediction period = 365 days)
0	91	26	26
1	182	38,788	38,783
2	273	85,963	85,905
3	365	114,312	114,134
4	456	165,978	165,560
5	547	188,346	149,019
6	638	199,300	112,749
7	730	203,216	88,144
8	821	204,181	37,702
9	912	204,515	15,906
10	1,003	204,595	5,189
11	1,095	204,632	1,385
12	1,186	204,701	507
13	1,277	204,703	185
14	1,352	204,704	117

there are still too new to experience a failure or to reach the end of the follow-up period. However, after Snapshot 4, the number of censored records starts decreasing as less vehicles of the specific model are sold whereas more and more vehicles leave the monitoring process due to a "survival" or a "failure" outcome.

4.2.2 Summary of Results

Figures 2, 3 and 4 present the testing AUC results in each snapshot for prediction periods of 365, 550, and 730 days, respectively. The results for the longest prediction period of 1095 days (three years) are not shown here, since for this value, it took almost the entire monitoring period to obtain a sufficient number of surviving records in the testing set. All classification models were induced by the AdaBoost with Decision Stump algorithm (a one-level decision tree presented in [18] and wrapped by the AdaBoost [19]), which provided in this dataset higher AUC values than several other popular classifiers such as Decision Tree, Decision Stump, and Naïve Bayes Classifier (NBC). Each chart compares the baseline approach (using uncensored records only) to the CENSMINER methodology, which utilizes the Kaplan-Meier outcome estimation for censored records. Each chart also shows the

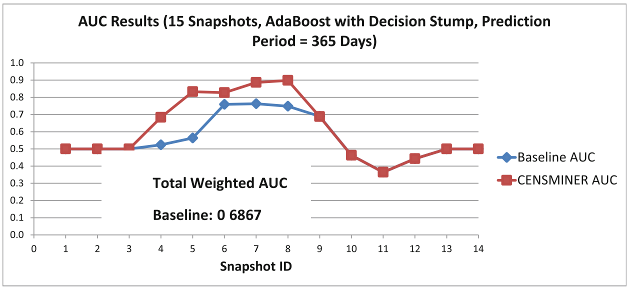

Fig. 2 Vehicle warranty data: prediction period = 365 Days

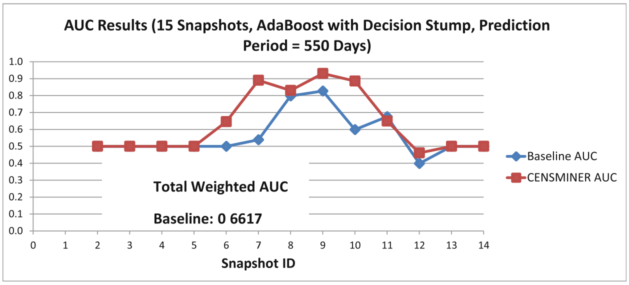

Fig. 3 Vehicle warranty data: prediction period = 550 Days

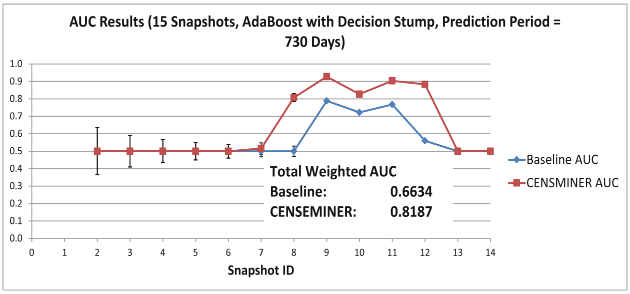

Fig. 4 Vehicle warranty data: prediction period = 730 Days

weighted AUC values of both approaches, where the testing AUC of every snapshot is weighted by the amount of testing (new uncensored) records in that snapshot.

For the three different prediction periods, the Total Weighted AUC values of the proposed CENSMINER methodology are significantly higher than the results of the baseline approach. More specifically, the CENSMINER has an advantage over the baseline as long as the percentage of censored records in a snapshot is high. When this percentage goes down, the difference between the two approaches diminishes, since the Kaplan-Meier estimate is applied to much fewer records. In addition, there is no difference between the approaches in the first few snapshots, which still have not accumulated any "survived" records to induce a reliable classification model. The number of such snapshots increases with an increase in the duration of the follow-up period (compare Figs. 2 to 3 and Figs. 3 to 4), since it takes more time for surviving vehicles to get a "survived" label. The low AUC values in the last few snapshots, disregarding the follow-up duration, are explained by the lack of testing records with the "failed" label in those snapshots. Thus, we may conclude that the CENSMINER tends to reach better classification performance than the baseline in snapshots having a significant amount of censored records along with some uncensored records from both classes. In other snapshots, it usually performs the same as the baseline.

4.3 STD Re-Infection Dataset Results

4.3.1 Data Description

The STD (Sexually Transmitted Diseases) dataset was taken from a collection of survival datasets discussed in [3]. The collection is called KMsurv and it was downloaded from the University of North Texas website for the purposes of survival analysis research. The direct link is http://rss.acs.unt.edu/Rdoc/library/

KMsurv/data/std.txt. Unfortunately, this link has stopped being active at the time of writing this chapter.

The raw data contains records of 877 patients diagnosed with STD. Each record is characterized by 21 different features. Three features are continuous and the rest are nominal, mostly binary. Each record has also a value for the binary target variable "REINFECTION" which indicates whether the patient was eventually diagnosed with re-infection of STD or not. In addition, each record has a value for the numeric variable "TIME_TO_REINFECTION", which contains the number of days elapsed since the initial diagnosis. The actual dates of the initial diagnosis and the re-infection (if occurred) are not included in the dataset.

To simulate the snapshot monitoring process, we randomly set the dates of the initial diagnosis and the re-infection diagnosis (if occurred). We assumed the date of 01/01/2013 to be the CUT_OFF_DATE, which means that patients were no longer observed after that date. Consequently, for patients who had not been diagnosed with re-infection yet, the date of the initial diagnosis was set to the CUT_OFF_DATE (01/01/2013) minus the amount of days elapsed since the initial diagnosis (given by the TIME_TO_REINFECTION variable). Then, the earliest date of the initial diagnosis (25/10/2008) was marked as MIN_START_DATE (the start date of the monitoring process). For all other patients diagnosed with re-infection, the date of the initial diagnosis was chosen randomly between the MIN_START_DATE and the last date possible: CUT_OFF_DATE - TIME_TO_REINFECTION. Accordingly, the re-infection diagnosis date of these patients was set as the (random) initial diagnosis date plus TIME_TO_REINFECTION. The randomization process was repeated three times to verify the consistency of the results. Similar to the case of the Vehicle Warranty data, all snapshot dates were simulated for the period between the MIN_START_DATE and the CUT_OFF_DATE. In each snapshot and for every patient, only the event information available on the snapshot date was taken into account (e.g., the initial diagnosis date and the re-infection diagnosis date).

Table 3 shows the characteristics of the six simulated snapshots with the first randomization scheme. The total number of observed patients increases over time as more patients are "born", i.e., diagnosed with STD for the first time. In the first snapshot (Snapshot 0), the percentage of censored patient records is higher than in

Table 3 Characteristics of simulated snapshots in the STD Re-infection dataset

Snapshot ID	Number of days since the beginning of the stream	Total number of observed patients	Number of censored records (prediction period = 300 days)
0	304	128	92
1	608	257	96
2	912	403	117
3	1,216	571	133
4	1,520	836	233
5	1,529	876	262

the subsequent snapshots, since it has very few patients that reached the end of the follow-up period (300 days since the initial diagnosis).

4.3.2 Summary of Results

Figures 5, 6 and 7 present the testing accuracy results in each snapshot for the prediction period of 300 days and three different randomization schemes. All classification models were induced by the J48 Decision Tree algorithm, which provided in this dataset higher accuracy values than several other popular classifiers such as Naïve Bayes Classifier (NBC) and Support Vector Machine (SVM). Each chart compares the baseline approach (using uncensored records only) to the CENSMINER methodology, which utilizes the Kaplan-Meier outcome estimation for censored records. Each chart also shows the weighted accuracy values of both approaches, where the testing accuracy of every snapshot is weighted by the amount of testing (new uncensored) records in that snapshot.

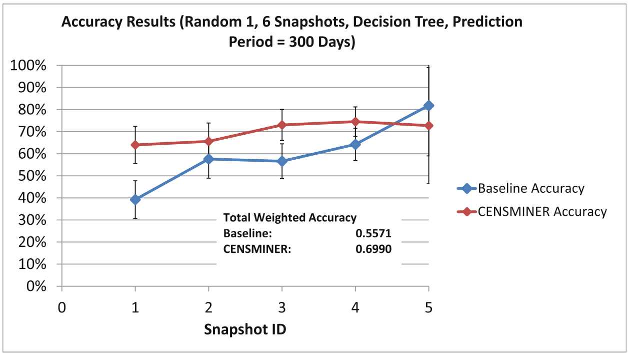

Fig. 5 STD re-infection data: random scheme 1, prediction period = 300 days

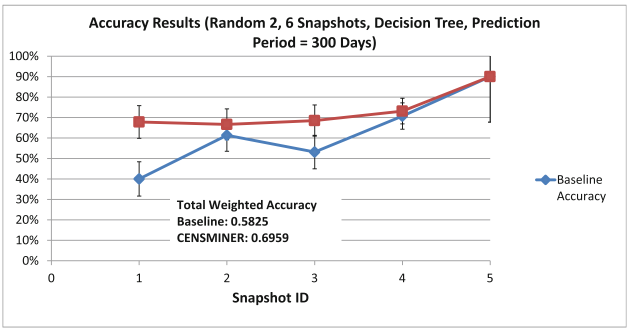

Fig. 6 STD re-infection data: random scheme 2, prediction period = 300 days

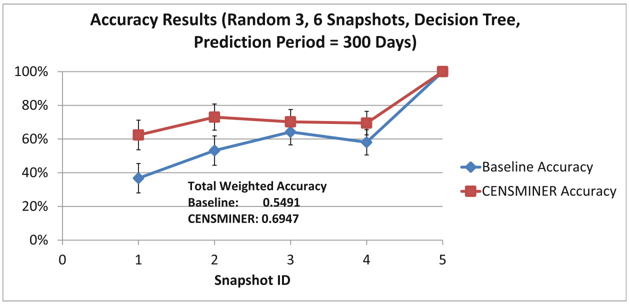

Fig. 7 STD re-infection data: random scheme 3, prediction period = 300 days

For the three different randomization schemes, the Total Weighted Accuracy values of the proposed CENSMINER methodology are significantly higher than the results of the baseline approach. Similar to the Vehicle Warranty results, in this dataset, the CENSMINER has also a greater advantage over the baseline when the percentage of censored records in a snapshot is higher. These results confirm our previous conclusion that the CENSMINER tends to reach better classification performance than the baseline in snapshots having a significant amount of censored records along with some uncensored records from both classes. In other snapshots (like Snapshot 5 in the STD Re-infection Data), it usually performs almost the same as the baseline.

5 Conclusions

In this chapter, we presented the CENSMINER fuzzy-based methodology, which allows the standard probability estimation and classification algorithms to handle censored data streams. We consider the domains where the data stream arrives in the form of consecutive snapshots. In each new snapshot, the performance of the last classification model is tested on the new uncensored records. Then a new model is induced from on all available data, including the censored records. The censored records are used in the model induction process by weighting their outcome with the Kaplan-Meier survival curve estimation. The proposed methodology shows a significant improvement over the baseline approach, which completely discards the censored data. The CENSMINER methodology tends to be most useful in the snapshots containing a relatively high percentage of censored records along with some amount of uncensored records from both classes.

In this study, the Kaplan-Meier estimation was applied to all snapshot records without any distinction between different object groups that might exist in the data. In future research, it would be worthwhile to examine the effect of clustering the monitored objects into relatively homogeneous groups and then calculating the

Kaplan-Meier estimation for each group separately. One can also evaluate additional parametric and non-parametric approaches to the survival curve estimation.

Since we are dealing with a data stream, it would be natural to examine the use of incremental classification algorithms that are designed to handle this kind of data. Such algorithms can save the computational effort required for inducing a new model from every snapshot by reusing or updating the existing one.

Acknowledgments. This work was supported in part by the General Motors Global Research & Development - India Science Lab.

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Knowledge Extraction from Support Vector Machines: A Fuzzy Logic Approach

Shahaf Duenyas and Michael Margaliot

Abstract Support vector machines (SVMs) proved to be highly efficient computational tools in various classification tasks. However, SVMs are nonlinear classifiers and the knowledge learned by an SVM is encoded in a long list of parameter values, making it difficult to comprehend what the SVM is actually computing. We show that certain types of SVMs are mathematically equivalent to a specific fuzzy–rule base, called the *fuzzy all–permutations rule base* (FARB). The equivalent FARB provides a symbolic representation of the SVM functioning. This leads to a new approach for knowledge extraction from SVMs. An important advantage of this approach is that the number of extracted fuzzy rules depends on the number of support vectors in the SVM. Several simple examples demonstrate the effectiveness of this approach.

Keywords Support vector machines · Knowledge extraction · Artificial neural network models · Fuzzy rule–base · Neurofuzzy systems

1 Introduction

Support vector machines (SVMs) are a popular tool in machine learning combining a solid theoretical background with efficient learning–from–examples algorithms (see, e.g., [1–5]). An important advantage of SVMs is that their classification decision is based on a subset of the training examples, referred to as the *support vectors*. The number of support vectors may be much smaller than the number of training examples.

A drawback of SVMs, that is shared by many other artificial neural network models, is their *black–box* nature. SVMs usually employ nonlinear kernel functions, and the knowledge learnt by the SVM is represented as a set of numerical parameter values. This makes it difficult to understand what the SVM is actually computing.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_19

This black–box character hinders a wider acceptance of SVMs, especially in safety–critical applications (e.g. life–support systems).

In this chapter, we show that the input–output (IO) mapping of certain classes of SVMs is identical to the IO mapping of a specific type of a fuzzy rule–base, referred to as the *fuzzy all–permutations rule base* (FARB). The knowledge embedded in the FARB is represented in a symbolic form, so this equivalence yields a new approach for understanding the functioning of SVMs. We demonstrate the usefulness of this new approach using several simple examples.

The remainder of this chapter is organized as follows. Section 2 reviews knowledge extraction from SVMs. Section 3 reviews the FARB, and Sect. 4 defines the mathematical equivalence between a certain class of SVMs and a corresponding FARB. This leads to a new approach for knowledge extraction from SVMs. Section 5 demonstrates our knowledge extraction approach using several simple examples. The final section concludes and describes possible directions for further research. An abridged version of this chapter has appeared in [6].

2 Knowledge Extraction from SVMs

Extracting the knowledge learned by a black–box classifier and representing it in a comprehensible form is referred to as *knowledge extraction* (KE). KE from SVMs and other artificial neural network (ANN) models has received considerable attention in the literature. Indeed, potential benefits of successful KE include [7–10]:

Validation. Explaining how the network actually works in an intelligible fashion can validate its suitability for a specific application or, alternatively, help the human expert to identify errors in the conclusion reached by the system. This is crucial in safety–critical applications [11]. For example, medical decision aids require approval by government agencies. Gaining such an approval is much easier for transparent systems that include adequate explanation capabilities [12].

Feature extraction. During training, classifiers learn to identify the *relevant* features in immense data sets. Understanding what these features are, and how they should be integrated to yield a correct classification, may help in gaining a deeper understanding of the problem. Efficient feature extraction may also assist in improving the accuracy and generalization capabilities of a classifier.

Knowledge refinement and improvement. An intelligible model can be examined by human experts and potentially improved in precision and efficiency.

Knowledge acquisition for symbolic AI systems. The most difficult and time consuming task in the construction of symbolic AI systems is knowledge acquisition [13]. Efficient KE from trained black–box classifiers may help to overcome this problem.

Scientific discovery. Classifiers that learn from examples, such as SVMs, may be able to solve problems for which no other solution is known (or demonstrate better performance than any other solution). Understanding the way the classifier works may thus lead to new discoveries [14].

The term *rule extraction* (RE) is used when the extracted knowledge is stated in the form of rules. Methods for RE from ANNs have been classified into three categories: *decompositional*, *pedagogical*, and *eclectic* [15]. The decompositional approach focuses on symbolic representation of the hidden and output associations in the network. It aims to explain what individual components in the ANN are computing. Pedagogical approaches use the trained classifier as an oracle to produce a set of input–output (IO) examples. This set is the input to the KE algorithm. Pedagogical methods can thus extract knowledge from any classifier, but cannot explain the specific structure and parameter values of the classifier. Eclectic methods combine both the pedagogical and the decompositional approaches. We now briefly review several known approaches for KE from SVMs. A detailed overview may be found in [16].

Pedagogical methods are based on querying the SVM to generate IO examples and then applying standard algorithms for building decision trees [17, 18]. The *Trees Parroting Networks* (TREPAN) algorithm introduced by Craven [9, 19] extracts decision trees from trained neural networks. Martens et al. [17] used it to extract rules from SVMs. The *Genetic Rule Extraction* (G–REX) algorithm [20] for KE from ANNs was also modified to handle SVMs [17]. This method uses genetic programming [21] to evolve an optimal set of rules. The extracted rules may be Boolean, fuzzy, or M–of–N type rules.

Eclectic approaches use some of the characteristics of the trained SVM, namely, the support vectors or the separating hyperplane. Fung et al. [22] proposed a KE approach for hyperplane classifiers (including *linear* SVMs) that extracts rules in the form:

$$\text{If } \ell_1 \le x_1 \le u_1 \text{ and } \dots \text{ and } \ell_n \le x_n \le u_n \text{ Then class is } C.$$

Here x_i is the ith coordinate of the input, ℓ_i, u_i are scalars, and $C \in \{-1, 1\}$. Thus each rule describes a hypercube in the n–dimensional space with edges parallel to the axes (see also [23]). Rules are extracted by solving a constrained optimization problem. Nunez et al. [24] introduced the *SVM+Prototypes rule extraction method*. The extracted rules describe ellipsoids or hypercubes in the input space, and are derived by combining information from prototype vectors (obtained via clustering) and support vectors. Martens et al. [25] suggest enriching the training set by generating more classified examples near the support vectors, and then applying an algorithm for building decision trees.

Castro et al. [26] developed a decompositional approach for KE based on the equivalence between a special fuzzy rule–base (FRB) and an SVM. This approach is closely related to our work, so we review it in more detail. An SVM is usually trained using a set of classified patterns $\{\boldsymbol{x}^i, y_i\}$, where $\boldsymbol{x}^i \in \mathbb{R}^n$ is the feature vector of example i, and $y_i \in \{-1, 1\}$ is the classification of example i. The IO mapping of the trained SVM $f : \mathbb{R}^n \to \{-1, 1\}$ is given by

$$f(\boldsymbol{x}) = \text{sgn}(h(\boldsymbol{x})), \tag{1}$$

where

$$h(\boldsymbol{x}) = b^* + \sum_{i=1}^{N_{SV}} \alpha_i^* y_i K(\boldsymbol{x}, \boldsymbol{s}^i). \tag{2}$$

Here $\{\boldsymbol{s}^i, y_i\}_{i=1}^{N_{SV}}$ is a subset of the classified training examples, with the $\boldsymbol{s}^i$s known as the *support vectors* (SVs), $\alpha_i^*, b^* \in \mathbb{R}$, and $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the kernel function. The superscript $*$ is used as these values are determined via solving a quadratic optimization problem characterizing classification with the best possible error margin (see, e.g. [3, 4]).

Castro et al. showed that an equivalent mapping is produced by inferring the set of rules:

$$\begin{aligned} &R_1\text{: If } h(\boldsymbol{x}) \text{ is } \textit{positive} \text{ Then } f = 1, \\ &R_2\text{: If } h(\boldsymbol{x}) \text{ is } \textit{negative} \text{ Then } f = -1, \end{aligned} \tag{3}$$

where the linguistic terms *positive* and *negative* are modeled using the membership functions $\mu_{pos}(y) = \sigma(\lambda y)$, $\mu_{neg}(y) = \sigma^*(\lambda y)$, with $\sigma(y) = 1/(1 + e^{-y})$, $\sigma^*(y) = 1 - \sigma(y) = \sigma(-y)$, and $\lambda \to \infty$. In other words, inferencing the rule–base (3) yields an input–output mapping $\boldsymbol{x} \to f(\boldsymbol{x})$ that is identical to (1). Obviously, (3) provides no verbal interpretation of the function $h(\cdot)$. To overcome this, Castro et al. noted that $h(\boldsymbol{x}) = b^* + \sum_{i=1}^{N_{SV}} h_i(\boldsymbol{x})$, where $h_i(\boldsymbol{x}) = \alpha_i^* y_i K(\boldsymbol{x},\boldsymbol{s}^i)$, and introduced the operator $*: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by:

$$a * b = \frac{ab}{ab + (1-a)(1-b)}.$$

It is straightforward to verify that: $\sigma(a(x+y)) = \sigma(ax) * \sigma(ay)$, and $\sigma^*(a(x+y)) = \sigma^*(ax) * \sigma^*(ay)$. Therefore, (3) may be rewritten as:

$$\begin{aligned} R_1 \ : \ &\text{If } \{h_1(\boldsymbol{x}) \text{ is } \textit{positive}\} * \{h_2(\boldsymbol{x}) \text{ is } \textit{positive}\} * \ldots \\ &* \{h_{N_{SV}}(\boldsymbol{x}) \text{ is } \textit{positive}\} * \{b^* \text{ is } \textit{positive}\} \\ &\text{Then } f = 1, \\ R_2 \ : \ &\text{If } \{h_1(\boldsymbol{x}) \text{ is } \textit{negative}\} * \{h_2(\boldsymbol{x}) \text{ is } \textit{negative}\} * \ldots \\ &* \{h_{N_{SV}}(\boldsymbol{x}) \text{ is } \textit{negative}\} * \{b^* \text{ is } \textit{negative}\} \\ &\text{Then } f = -1. \end{aligned}$$

This set of two fuzzy rules thus provides a symbolic representation of the SVM functioning. However, it seems that the extracted FRB may be cumbersome and that the use of the special operator $*$ makes the FRB less comprehensible. For other approaches for KE from SVMs, see [27, 28] and the references therein.

In this chapter, we describe a new decompositional method for KE from SVMs that also relies on the mathematical equivalence between SVMs and a specific FRB, namely, the FARB. The FARB was successfully used for KE and knowledge–based design of ANNs [8, 29–32]. It is based on the hyperbolic FRB first suggested in [33], and further developed in [34–36]. We show that SVMs with either a *Multi Layer Perceptron* (MLP) or *Radial Basis Function* (RBF) kernels are mathematically equivalent to a suitable FARB.

3 The FARB

To motivate the definition of the FARB, we begin with a simple example adapted from [35] (see also [34]).

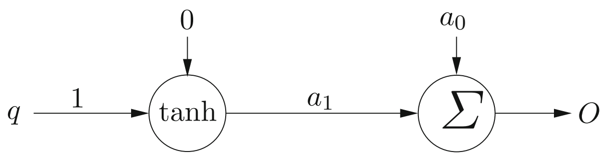


Fig. 1 Graphical representation of the FRB IO mapping

Example 1 Consider an FRB with input $q \in \mathbb{R}$, output $O \in \mathbb{R}$, and rules:

$$R_1\text{: If } q \textit{ is equal to } k \text{ Then } O = a_0 + a_1,$$
$$R_2\text{: If } q \textit{ is equal to } -k \text{ Then } O = a_0 - a_1,$$

where $a_0, a_1, k \in \mathbb{R}$, with $k > 0$. Assume that the linguistic terms *equal to* k and *equal to* $-k$ are modeled using the Gaussian membership functions:

$$\mu_{=k}(q) = \exp\left(-\frac{(q-k)^2}{2k}\right), \qquad \mu_{=-k}(q) = \exp\left(-\frac{(q+k)^2}{2k}\right), \tag{4}$$

respectively. Note that these functions satisfy

$$\begin{aligned}\frac{\mu_{=k}(q) - \mu_{=-k}(q)}{\mu_{=k}(q) + \mu_{=-k}(q)} &= \frac{\exp(\frac{-(q-k)^2}{2k}) - \exp(\frac{-(q+k)^2}{2k})}{\exp(\frac{-(q-k)^2}{2k}) + \exp(\frac{-(q+k)^2}{2k})} \\ &= \frac{\exp(q) - \exp(-q)}{\exp(q) + \exp(-q)} \\ &= \tanh(q). \end{aligned} \tag{5}$$

Applying the singleton fuzzifier and the center of gravity defuzzifier [37] to the rule base yields:

$$O(q) = \frac{(a_0 + a_1)\mu_{=k}(q) + (a_0 - a_1)\mu_{=-k}(q)}{\mu_{=k}(q) + \mu_{=-k}(q)}$$
$$= a_0 + a_1 \tanh(q).$$

Hence, this FRB is mathematically equivalent to a feedforward ANN with a single neuron employing the activation function $\tanh(\cdot)$ (see Fig. 1). □

This example motivates the search for an FRB whose IO mapping is mathematically equivalent to that of a feedforward ANN.

Definition 1 (FARB) *A fuzzy rule base with input $\boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{R}^m$ and output $O \in \mathbb{R}$ is called a FARB if the following conditions hold:*

1. *Every input variable q_i is characterized by two linguistic terms: $term^i_-$ and $term^i_+$, modeled using the membership functions $\mu^i_-(\cdot)$ and $\mu^i_+(\cdot)$. These membership functions (MFs) satisfy the following property. There exist $z_i, r_i, u_i, v_i \in \mathbb{R}$ and a sigmoid[1] function $g_i : \mathbb{R} \to \mathbb{R}$ such that*

$$\frac{\mu^i_+(q) - \mu^i_-(q)}{\mu^i_+(q) + \mu^i_-(q)} = z_i g_i(u_i q - v_i) + r_i, \text{ for all } q \in \mathbb{R}. \quad (6)$$

2. *The form of every rule is:*

$$\text{If } q_1 \text{ is } term^1_\pm \text{ and } \ldots \text{ and } q_m \text{ is } term^m_\pm$$
$$\text{Then } O = a_0 \pm a_1 \pm a_2 \pm \cdots \pm a_m, \quad (7)$$

where $term^i_\pm$ stands for either $term^i_+$ or $term^i_-$, $\pm$ stands for either the plus or the minus sign, and $a_i \in \mathbb{R}$, $i = 1, \ldots, m$. The actual signs in the Then–part satisfy the following property. If the term characterizing q_i in the If–part is $term^i_+$, then in the Then–part, a_i appears with a plus sign, otherwise a_i appears with a minus sign.

3. *The rule–base contains exactly 2^m rules spanning in their If–part all the possible assignment combinations of $q_1, ..., q_m$.*

This definition guarantees that the IO mapping of the FARB admits a simple closed–form expression.

Theorem 1 *[8] Applying the product–inference rule, singleton fuzzifier, and the center of gravity defuzzifier to a FARB yields:*

[1] We say that a function $g : \mathbb{R} \to \mathbb{R}$ is a *sigmoid* if g is continuous and the limits $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$ exist.

$$O(\boldsymbol{q}) = a_0 + \sum_{i=1}^{m} r_i a_i + \sum_{i=1}^{m} z_i a_i g_i(u_i q_i - v_i). \tag{8}$$

This IO mapping is depicted in Fig. 2.

Several commonly used MFs satisfy the constraint (6) [8, Chap. 2]. We consider three specific examples. First, it follows from (5) that the pair of Gaussian membership functions $\{\mu_{=k}, \mu_{=-k}\}$ in (4) satisfy (6) with $z_i = u_i = 1$, $v_i = r_i = 0$, and $g_i(x) = \tanh(x)$. Thus we can use the linguistic terms *equal to* k, and *equal to* $-k$ in the FARB. As a second example, consider the linguistic terms *larger than* k and *smaller than* k modeled using the Logistic MFs:

$$\begin{aligned} \mu_{>k}(q) &= \frac{1}{1 + \exp(-\tau(q-k))}, \\ \mu_{<k}(q) &= \frac{1}{1 + \exp(\tau(q-k))}, \end{aligned} \tag{9}$$

respectively (see Fig. 3). It is straightforward to verify that

$$\frac{\mu_{>k}(q) - \mu_{<k}(q)}{\mu_{>k}(q) + \mu_{<k}(q)} = \tanh((q-k)\tau/2),$$

so (6) holds for $g_i(x) = \tanh(x)$, $z_i = 1$, $u_i = \tau/2$, $v_i = k\tau/2$, and $r_i = 0$. Third, if we model the linguistic terms *equal to* k and *not equal to* k using the MFs:

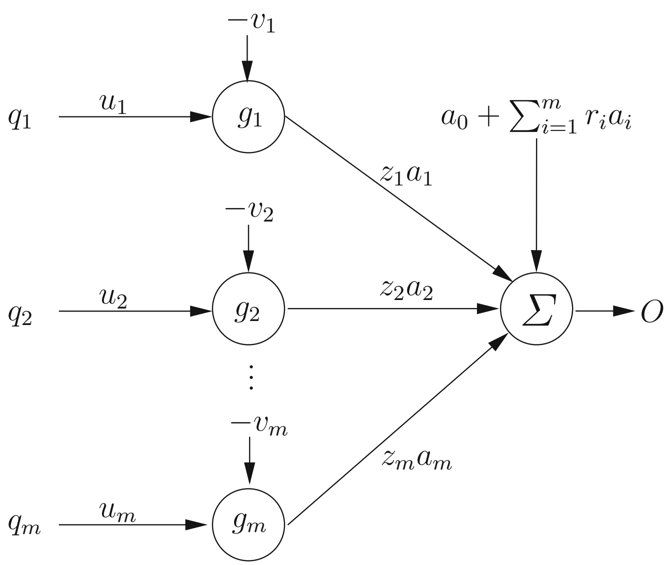


Fig. 2 Graphical representation of the FARB IO mapping

$$\mu_{=k}(q) = \exp\left(\frac{-(q-k)^2}{2\sigma^2}\right), \qquad \mu_{\neq k}(q) = 1 - \mu_{=k}(q), \tag{10}$$

then

$$\frac{\mu_{=k}(q) - \mu_{\neq k}(q)}{\mu_{=k}(q) + \mu_{\neq k}(q)} = 2\exp\left(\frac{-(q-k)^2}{2\sigma^2}\right) - 1,$$

so for $k \geq 0$, (6) holds for $g_i(x) = \exp(-x^2)$, $z_i = 2$, $r_i = -1$, $u_i = \sqrt{1/(2\sigma^2)}$, and $v_i = \sqrt{k^2/(2\sigma^2)}$.

Example 2 Consider a FARB with $m = 2$ inputs, linguistic terms: $term_-^1$ = *smaller than* 5, $term_+^1$ = *larger than* 5 (modeled using (9)), and $term_-^2$ = *equal to* -7, $term_+^2$ = *equal to* 7 (modeled using (4)), and parameters $a_0 = 1$, $a_1 = 1/3$, and $a_2 = 2/5$. Thus, the rules are:

R_1 : If q_1 is *smaller than* 5 and q_2 is *equal to* -7
Then $O = a_0 - a_1 - a_2 = 4/15$,
R_2 : If q_1 is *larger than* 5 and q_2 is *equal to* -7
Then $O = a_0 + a_1 - a_2 = 14/15$,
R_3 : If q_1 is *smaller than* 5 and q_2 is *equal to* 7
Then $O = a_0 - a_1 + a_2 = 16/15$,
R_4 : If q_1 is *larger than* 5 and q_2 is *equal to* 7
Then $O = a_0 + a_1 + a_2 = 26/15$.

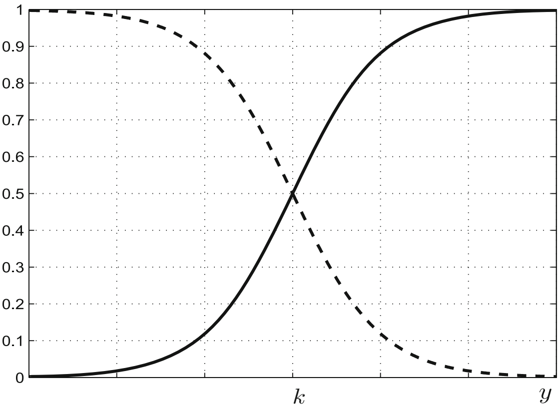


Fig. 3 MFs $\mu_{>k}(y)$ (solid) and $\mu_{<k}(y)$ (dashed) as a function of y

The IO mapping of this FARB is: $O(\boldsymbol{q}) = F(\boldsymbol{q})/D(\boldsymbol{q})$, where $\boldsymbol{q} = [q_1, q_2]'$,

$$F(\boldsymbol{q}) = \frac{4}{15}\mu_{<5}(q_1)\mu_{=-7}(q_2) + \frac{14}{15}\mu_{>5}(q_1)\mu_{=-7}(q_2) + \frac{16}{15}\mu_{<5}(q_1)\mu_{=7}(q_2) + \frac{26}{15}\mu_{>5}(q_1)\mu_{=7}(q_2),$$

and

$$D(\boldsymbol{q}) = \mu_{<5}(q_1)\mu_{=-7}(q_2) + \mu_{>5}(q_1)\mu_{=-7}(q_2) + \mu_{<5}(q_1)\mu_{=7}(q_2) + \mu_{>5}(q_1)\mu_{=7}(q_2).$$

It is straightforward to verify that:

$$\frac{F(\boldsymbol{q})}{D(\boldsymbol{q})} = 1 + \frac{1}{3}\tanh\left(\frac{\tau}{2}(q_1 - 5)\right) + \frac{2}{5}\tanh(q_2). \tag{11}$$

and this agrees with (8). □

The fact that the two atoms q_i *is* $term^i_-$ and q_i *is* $term^i_+$ in (7) usually contradict, and the use of the logical *and* operator suggest that only very few rules will have a substantial degree of firing for any given input. Hence, the fuzzy rules in the FARB actually cover the entire input domain, with each rule corresponding to a different region in $\mathbb{R}^m$. Thus, it is usually possible to determine which rule yielded a specific output. This is useful in explaining how the FARB reached a certain conclusion.

Kolman and Margaliot [8, 29] showed that every standard ANN has a corresponding FARB with an identical IO mapping. More precisely, there exists an explicit transformation T such that

$$T(ANN) = FARB \text{ and } T^{-1}(FARB) = ANN. \tag{12}$$

This provides a symbolic representation of the knowledge embedded in the ANN.

In the next section, we extend this idea to develop an equivalence between a class of SVMs and a FARB. Given an SVM in this class, there exists a transformation P such that:

$$P(SVM) = ANN. \tag{13}$$

Combining this with (12) yields:

$$T(P(SVM)) = T(ANN) = FARB. \tag{14}$$

Thus, the transformation $S = T \circ P$ transforms an SVM into a FARB with an *identical IO mapping*. This provides a representation of the knowledge embedded in the SVM as a set of If-Then fuzzy rules. Note that this method uses no approximations, and that the extracted FARB is based on standard fuzzy logic tools.

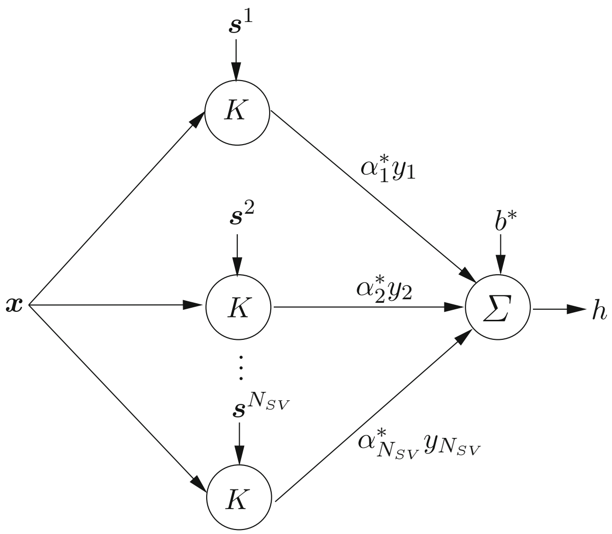


Fig. 4 Schematic description of the IO mapping (2) of an SVM

4 The SVM–FARB Equivalence

The computation of the SVM IO mapping h in (2) is depicted graphically in Fig. 4. It is clear that we can view this as a kind of a feedforward ANN with a hidden–layer of neurons employing $K(\cdot,\cdot)$ as an activation function. Since any standard ANN is equivalent to a FARB, we may expect this SVM to have a corresponding FARB with an identical IO mapping. Indeed, comparing (8) with (2) immediately yields our main result in this section.

Theorem 2 ***(SVM–FARB equivalence)***
Consider the SVM function h given in (2). Suppose that it is possible to find a FARB with: $m = N_{SV}$ inputs q_i, parameters a_i, and membership functions μ_-^i, μ_+^i, so that the following conditions hold

$$\begin{aligned} a_0 + \sum_{i=1}^{m} r_i a_i &= b^*, \\ z_i a_i &= \alpha_i^* y_i, \\ g_i(u_i q_i - v_i) &= K(\boldsymbol{x}, \boldsymbol{s}^i). \end{aligned} \tag{15}$$

Then the IO mapping of the FARB is identical to that of the function h, i.e. $O(\boldsymbol{q})=h(\boldsymbol{x})$.

One measure for the quality of rule extraction methods is their *fidelity* [15], that is, the percentage of examples on which the original black–box and the extracted rule–base agree. The FARB provides perfect fidelity, as its IO mapping is identical to that of the SVM.

Recall that the number of SVs may be much smaller than the total number of training examples. This is a manifestation of the *sparseness* property of SVMs [2], i.e. the fact that only a relatively small part of the training examples actually influence the decision boundary. Since the FARB has $2^m = 2^{N_{SV}}$ rules, this implies that the FARB may be relatively small, and thus simple to understand even for complex learning problems. In other words, the spareness property carries over to the equivalent FARB.

Recall that popular kernel functions include [17]:

$$\begin{aligned}
&K(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^T\boldsymbol{y}, \text{ (linear kernel)}\\
&K(\boldsymbol{x},\boldsymbol{y}) = (1+\boldsymbol{x}^T\boldsymbol{y}/c)^d, \ c \in \mathbb{R}, \ d \in \mathbb{N}, \text{ (polynomial kernel)}\\
&K(\boldsymbol{x},\boldsymbol{y}) = \tanh(\rho\boldsymbol{x}^T\boldsymbol{y}+\eta), \ \rho > 0, \ \eta < 0, \text{ (MLP kernel)}\\
&K(\boldsymbol{x},\boldsymbol{y}) = \exp(-\|\boldsymbol{x}-\boldsymbol{y}\|^2/(2\hat{\sigma}^2)), \ \hat{\sigma} \in \mathbb{R}, \text{ (RBF or Gaussian kernel).}
\end{aligned} \tag{16}$$

The next two corollaries show that for SVMs with MLP or RBF kernel functions, the equivalence in Theorem 2 indeed holds.

Corollary 1 *Consider the SVM function h given in* (2) *with an MLP kernel, i.e.*

$$h(\boldsymbol{x}) = \sum_{i=1}^{N_{SV}} \alpha_i^* y_i \tanh(\rho\boldsymbol{x}^T\boldsymbol{s}^i + \eta) + b^*. \tag{17}$$

Consider a FARB with inputs $q_i = \boldsymbol{x}^T\boldsymbol{s}^i$, $i = 1, \dots, N_{SV}$, *membership functions* $\mu_+^i = \mu_{>k_i}$, $\mu_-^i = \mu_{<k_i}$ *modeled using the Logistic MFs in* (9) *with* $\tau_i = 2\rho$, $k_i = -\eta/\rho$, *and parameters* $a_0 = b^*$ *and* $a_i = \alpha_i^* y_i$, $i = 1, \dots, N_{SV}$. *Then the IO mapping of the FARB is identical to that of the function h.*

Proof. For Logistic MFs, (6) holds with $z_i = 1$, $u_i = \tau_i/2 = \rho$, $v_i = k_i\tau_i/2 = -\eta$, $r_i = 0$, and $g_i(x) = \tanh(x)$, for $i = 1, \dots, N_{SV}$. It is straightforward to verify that this implies that the three conditions in (15) hold. □

Note that the q_is that appear in the FARB If–part have a clear geometric meaning. Every $q_i = \boldsymbol{x}^T\boldsymbol{s}^i$ is the projection of the input $\boldsymbol{x}$ on an SV. If all the vectors are normalized, then

$$\begin{aligned}
q_i &= \boldsymbol{x}^T\boldsymbol{s}^i\\
&= ||\boldsymbol{x}||||\boldsymbol{s}^i|| \cos(\varphi)\\
&= \cos(\varphi),
\end{aligned}$$

where φ is the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{s}^i$. Thus, we can think of q_i as a measure of the "resemblance" between the current input $\boldsymbol{x}$ and the ith SV.

Specializing Theorem 2 to the case of an SVM with an RBF kernel yields the following.

Corollary 2 *Consider an SVM with an RBF kernel, i.e.*

$$h(\boldsymbol{x}) = b^* + \sum_{i=1}^{N_{SV}} \alpha_i^* y_i \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{s}^i\|^2}{2\hat{\sigma}^2}\right). \tag{18}$$

Consider a FARB with inputs $q_i = \|\boldsymbol{x}-\boldsymbol{s}^i\|$, $i = 1,\dots,N_{SV}$, *linguistic terms equal to* 0 *and not equal to* 0 *modeled using the MFs* $\mu_+^i = \mu_{=0}$, $\mu_-^i = \mu_{\neq 0}$ *in* (10), *and parameters* $a_0 = b^* + \sum_{i=1}^{N_{SV}} \alpha_i^* y_i/2$, $a_i = \alpha_i^* y_i/2$, $i = 1,\dots,N_{SV}$, $\sigma = \hat{\sigma}$. *Then the IO mapping of the FARB is identical to that of the function h.*

Proof. For the MFs in (10) with $k = 0$, (6) holds with $z_i = 2$, $u_i = 1/\sqrt{2\sigma^2}$, $v_i = 0$, $r_i = -1$, and $g_i(x) = \exp(-x^2)$. It is straightforward to verify that this implies that the three conditions in (15) hold. □

Summarizing, Corollaries 1 and 2 show that MLP– or RBF–kernel SVMs are equivalent to a FARB, and explicitly define the transformation $S = T \circ P$ in (14).

5 Examples

The SVM–FARB equivalence yields a symbolic representation of the IO mapping of an SVM. The equivalent FARB rules are determined by the SVM structure and parameters. We now demonstrate this using several simple examples.

Example 3 Consider the training set $\{\boldsymbol{x}^i, y_i\}_{i=1}^2$ given by:

$$\begin{aligned} \boldsymbol{x}^1 &= [1,0]^T, \ y_1 = 1, \\ \boldsymbol{x}^2 &= [2,0]^T, \ y_2 = -1. \end{aligned} \tag{19}$$

Training an SVM with kernel function $K(\boldsymbol{x},\boldsymbol{y}) = \tanh(2\boldsymbol{x}^T\boldsymbol{y} - 5)$ (i.e. an MLP kernel with $\rho = 2$ and $\eta = -5$) yields $N_{SV} = 2$, $\alpha_1^* = \alpha_2^* = 1.313 = \alpha^*$ and $b^* = 1.3065$ (all the SVMs described in this section were derived using the C–support vector classification algorithm [38]). The SVM IO mapping is thus $f(\boldsymbol{x}) = \text{sgn}(h(\boldsymbol{x}))$, where

$$\begin{aligned} h(\boldsymbol{x}) &= b^* + \alpha^* \left[\tanh(2\boldsymbol{x}^T\boldsymbol{x}^1 - 5) - \tanh(2\boldsymbol{x}^T\boldsymbol{x}^2 - 5)\right] \\ &= 1.3065 + 1.313 \left[\tanh(2x_1 - 5) - \tanh(4x_1 - 5)\right]. \end{aligned} \tag{20}$$

Corollary 1 implies that this mapping is also the IO mapping of a FARB with: two inputs $q_1 = \boldsymbol{x}^T\boldsymbol{x}^1 = x_1$ and $q_2 = \boldsymbol{x}^T\boldsymbol{x}^2 = 2x_1$, Logistic MFs $\mu_{>k_i}$, $\mu_{<k_i}$ with $k_i = 2.5$, and $\tau_i = 4$, and parameters $a_0 = 1.3065$, $a_1 = \alpha_1^* y_1 = 1.313$, and $a_2 = \alpha_2^* y_2 = -1.313$. This yields the four rule FARB:

R_1 : If $q_1 > 2.5$ & $q_2 > 2.5$ Then $O = a_0 + a_1 + a_2 \cong 1.3$,
R_2 : If $q_1 > 2.5$ & $q_2 < 2.5$ Then $O = a_0 + a_1 - a_2 \cong 3.9$,
R_3 : If $q_1 < 2.5$ & $q_2 > 2.5$ Then $O = a_0 - a_1 + a_2 \cong -1.3$,

R_4 : If $q_1 < 2.5$ & $q_2 < 2.5$ Then $O = a_0 - a_1 - a_2 \cong 1.3$.
Here and below & denotes 'and', $q_i < 2.5$ stands for q_i *is smaller than* 2.5, and $q_i > 2.5$ for q_i *is larger than* 2.5. In other words, we use a "crisp" notation such as >, but this is just a shorthand notation for a set modeled using a fuzzy MF. This rule–base may be written as:
R_1 : If $x_1 > 2.5$ & $x_1 > 1.25$ Then $O \cong 1.3$,
R_2 : If $x_1 > 2.5$ & $x_1 < 1.25$ Then $O \cong 3.9$,
R_3 : If $x_1 < 2.5$ & $x_1 > 1.25$ Then $O \cong -1.3$,
R_4 : If $x_1 < 2.5$ & $x_1 < 1.25$ Then $O \cong 1.3$.
Rule R_2 is self–contradicting, so we delete it. The other three rules may be summarized as:
If $x_1 > 2.5$ or $x_1 < 1.25$ Then $O \cong 1.3$,
Else $O \cong -1.3$.
If we define the FARB decision to be $\text{sgn}(O)$ (as in the SVM), then this becomes
If $x_1 > 2.5$ or $x_1 < 1.25$. Then $\text{sgn}(O) = 1$,
Else $\text{sgn}(O) = -1$.
This rule indeed correctly classifies the two examples in (19). Figure 5 depicts the decision region of the SVM classifier (20). It is clear that this agrees well with the verbal description given by the above rule. □

The training set in the Example 3 is clearly linearly separable. The next example describes KE from an SVM trained on a non–linearly separable training set.

Example 4 Consider the training set:

$$\begin{aligned} \boldsymbol{x}^1 &= [1, 1]^T, & y_1 &= 1, \\ \boldsymbol{x}^2 &= [-1, -1]^T, & y_2 &= 1, \\ \boldsymbol{x}^3 &= [-1, 1]^T, & y_3 &= -1, \\ \boldsymbol{x}^4 &= [1, -1]^T, & y_4 &= -1. \end{aligned}$$

This corresponds to the classic symmetric *XOR* problem.

Applying the training algorithm (with the same kernel function as in Example 3) yields an SVM with four SVs (i.e. all the training examples), and parameters $\alpha_i^* = 4.1977 \cong 4.2, i = 1, \ldots, 4$, and $b^* = 0$. Thus, the SVM IO mapping is $f(\boldsymbol{x}) = \text{sgn}(h(\boldsymbol{x}))$, with

$$\begin{aligned} h(\boldsymbol{x}) = 4.2(&\tanh(2\boldsymbol{x}^T\boldsymbol{x}^1 - 5) + \tanh(2\boldsymbol{x}^T\boldsymbol{x}^2 - 5) \\ &- \tanh(2\boldsymbol{x}^T\boldsymbol{x}^3 - 5) - \tanh(2\boldsymbol{x}^T\boldsymbol{x}^4 - 5)). \end{aligned} \tag{21}$$

Figure 6 depicts the equal height contours $h(\boldsymbol{x}) = c$, for $c = -1, 0, 1$. Note that $\{\boldsymbol{x} \in \mathbb{R}^2 : h(\boldsymbol{x}) = 0\}$ is the decision boundary.

By Corollary 1, (21) is also the IO mapping of a FARB with: four inputs

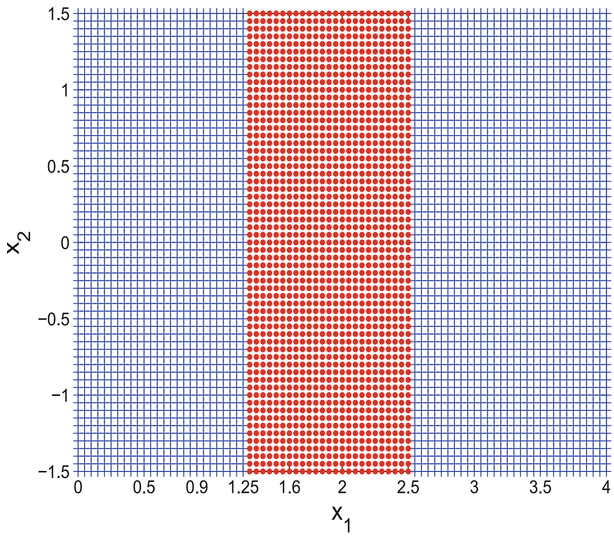


Fig. 5 IO mapping $f(\boldsymbol{x}) = \text{sgn}(h(\boldsymbol{x}))$ of the SVM (20): + (•) denotes points for which $f(\boldsymbol{x}) = 1$ ($f(\boldsymbol{x}) = -1$)

$$q_1 = \boldsymbol{x}^T\boldsymbol{x}^1 = x_1 + x_2, \qquad q_2 = \boldsymbol{x}^T\boldsymbol{x}^2 = -x_1 - x_2,$$
$$q_3 = \boldsymbol{x}^T\boldsymbol{x}^3 = -x_1 + x_2, \qquad q_4 = \boldsymbol{x}^T\boldsymbol{x}^4 = x_1 - x_2,$$

and MFs $\mu_{>k_i}$, $\mu_{<k_i}$ defined in (9) with $\tau_i = 2\rho = 4$, and $k_i = -\eta/\rho = 2.5$. The parameters in the Then–part of the rules are $a_0 = b^* = 0$, $a_1 = a_2 = 4.2$, and $a_3 = a_4 = -4.2$.

Out of the sixteen rules in this FARB, seven are self–contradicting. After deleting them we are left with a nine rule FRB. The IO mapping of this FRB is shown in Fig. 7. Comparing this with Fig. 6 suggests that the FRB IO mapping is still quite close to that of the SVM.

To obtain a more comprehensible representation, we use the fact that five of the nine rules include $O = 0$ in their Then part. Since the final classification decision is based on whether the output is positive or negative, we delete these rules. We are left with the following FRB:

$$\begin{aligned}
R_1 :& \text{If } q_1 > 2.5 \,\&\, q_2 < 2.5 \,\&\, q_3 < 2.5 \,\&\, q_4 < 2.5 \\
& \text{Then } O = 8.4, \\
R_2 :& \text{If } q_1 < 2.5 \,\&\, q_2 > 2.5 \,\&\, q_3 < 2.5 \,\&\, q_4 < 2.5 \\
& \text{Then } O = 8.4, \\
R_3 :& \text{If } q_1 < 2.5 \,\&\, q_2 < 2.5 \,\&\, q_3 > 2.5 \,\&\, q_4 < 2.5 \\
& \text{Then } O = -8.4, \\
R_4 :& \text{If } q_1 < 2.5 \,\&\, q_2 < 2.5 \,\&\, q_3 < 2.5 \,\&\, q_4 > 2.5 \\
& \text{Then } O = -8.4.
\end{aligned}$$

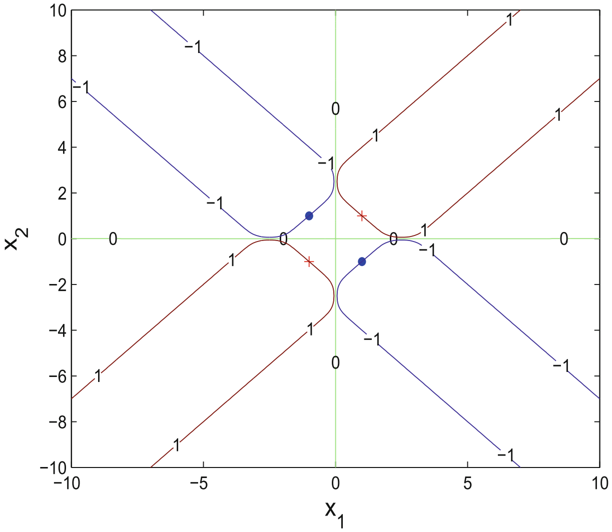


Fig. 6 Equal height contours $h(\boldsymbol{x}) = c$, for $c = -1, 0, 1$. The points $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ ($\boldsymbol{x}^3$, $\boldsymbol{x}^4$) are marked by + (•)

Rewriting this in terms of the x_is yields

$$\begin{aligned}
R_1 :& \text{If } (x_1 + x_2 > 2.5) \,\&\, (-2.5 < x_2 - x_1 < 2.5) \\
& \text{Then } O = 8.4, \\
R_2 :& \text{If } (x_1 + x_2 < -2.5) \,\&\, (-2.5 < x_2 - x_1 < 2.5) \\
& \text{Then } O = 8.4, \\
R_3 :& \text{If } (-2.5 < x_1 + x_2 < 2.5) \,\&\, (x_2 - x_1 > 2.5) \\
& \text{Then } O = -8.4, \\
R_4 :& \text{If } (-2.5 < x_1 + x_2 < 2.5) \,\&\, (x_2 - x_1 < -2.5) \\
& \text{Then } O = -8.4.
\end{aligned}$$

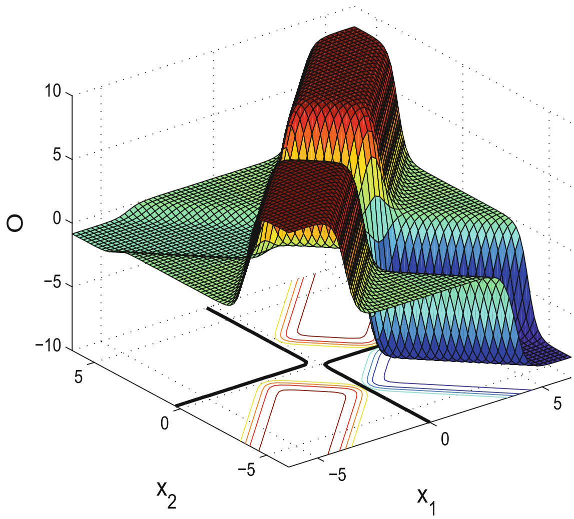
To simplify this, consider rule R_1. The If–part here is

$$(x_1 + x_2 > 2.5) \;\&\; (-2.5 < x_2 - x_1 < 2.5).$$

It is straightforward to verify that this condition implies that

$$(x_1 > 0) \;\&\; (x_2 > 0).$$

Fig. 7 IO mapping $O(x_1, x_2)$ of the nine–rule FRB, and several equal height contours $O(x_1, x_2) = c_i$. The bold contour corresponds to $O(x_1, x_2) = 0$



Similarly, the If–part in Rules R_2, R_3, and R_4 imply

$$(x_1 < 0) \,\&\, (x_2 < 0),\ (x_1 < 0) \,\&\, (x_2 > 0),\ (x_1 > 0) \,\&\, (x_2 < 0),$$

respectively. The FRB can thus be written as

$$\begin{aligned} R_1 &: \text{If } (x_1 > 0) \,\&\, (x_2 > 0) \text{ Then } O = 8.4, \\ R_2 &: \text{If } (x_1 < 0) \,\&\, (x_2 < 0) \text{ Then } O = 8.4, \\ R_3 &: \text{If } (x_1 < 0) \,\&\, (x_2 > 0) \text{ Then } O = -8.4, \\ R_4 &: \text{If } (x_1 > 0) \,\&\, (x_2 < 0) \text{ Then } O = -8.4. \end{aligned}$$

This can be summarized as

$$\begin{aligned} &\text{If } \operatorname{sgn}(x_1) = \operatorname{sgn}(x_2) \text{ Then } \operatorname{sgn}(O) = 1, \\ &\text{Else } \operatorname{sgn}(O) = -1, \end{aligned}$$

and this is indeed a succinct verbal description of the XOR function.

We note that in this example the trained SVM turned out to be symmetric in the sense that $b^* = 0$, and $\alpha_i^* = \alpha_j^*$, for all i, j. This led a to symmetric FARB where many of the rules could be deleted without affecting the fidelity.

The next example describes KE from a hierarchical SVM trained to classify the Iris data set.

Example 5 The Iris classification problem is a common benchmark for demonstrating KE methods from various machine learning algorithms. The data consists of 150 examples of irises classified into three classes. Each example is in the form (z_1, z_2, z_3, z_4, c) where the z_is are physical parameters of the flower (sepal

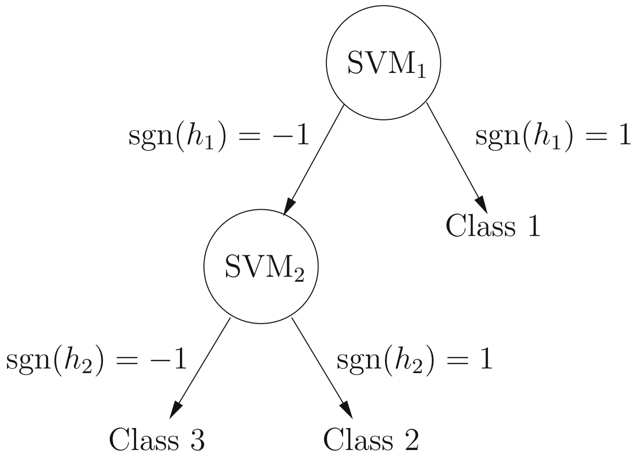


Fig. 8 Graphical representation of the hierarchical classification method in Example 5. Class 1 is Setosa, class 2 is Versicolor and class 3 is Virginica

length, sepal width, petal length and petal width, respectively), and $c \in \{1, 2, 3\}$ is the classification in one of three classes: Setosa, Versicolor, and Virginica, respectively. Each class includes 50 examples.

We trained the hierarchical SVM depicted in Fig. 8. If the output of SVM$_1$ satisfies $f_1 = \text{sgn}(h_1) = 1$, then the classification decision is Setosa; otherwise the example is fed into SVM$_2$ that separates between Virginica and Versicolor.

Preprocessing included scaling all the examples via $x_i = \frac{z_i}{m_i} - 1$, $i = 1, \ldots, 4$, where m_i is the maximal value of the ith feature over the 150 examples. This guarantees that each scaled feature satisfies $x_i \in [-1, 1]$.

Knowledge Extraction from SVM$_1$

We used the kernel function $K(\boldsymbol{x}, \boldsymbol{y}) = \tanh(\rho \boldsymbol{x}^T \boldsymbol{y} + \eta)$, with $\rho = 1.5$, $\eta = -0.75$. The trained SVM$_1$ includes two SVs, and parameter values $y_1\alpha_1^* = 3.32$, $y_2\alpha_2^* = -3.32$, $b^* = -2.178$, so its IO mapping is $f_1(\boldsymbol{x}) = \text{sgn}(h_1(\boldsymbol{x}))$ where:

$$h_1(\boldsymbol{x}) = 3.32(\tanh(1.5\boldsymbol{x}^T\boldsymbol{s}^1 - 0.75) - \tanh(1.5\boldsymbol{x}^T\boldsymbol{s}^2 - 0.75)) - 2.178.$$

This mapping is depicted in Fig. 9. Substituting the SVs yields:

$$h_1(\boldsymbol{x}) = 3.32(\tanh(1.5q_1 - 0.75) - \tanh(1.5q_2 - 0.75)) - 2.178, \tag{22}$$

where $q_1 = 0.468x_1 + 0.818x_2 - 0.652x_3 - 0.84x_4$, and $q_2 = 0.443x_1 + 0.181x_2 + 0.014x_3 - 0.2x_4$.

By Corollary 1, the mapping (22) is also the IO mapping of a FARB with inputs q_i, $i = 1, 2$, MFs $\mu_{>k}$, $\mu_{<k}$, and parameter values: $\tau = 2\rho = 3$, $k = -\eta/\rho = 0.5$, $a_0 = b^* = -2.178$, and $a_1 = 3.32$, $a_2 = -3.32$, i.e. the four–rule FARB:

R_1 : If $q_1 > 0.5$ & $q_2 > 0.5$ Then $O = -2.178$,

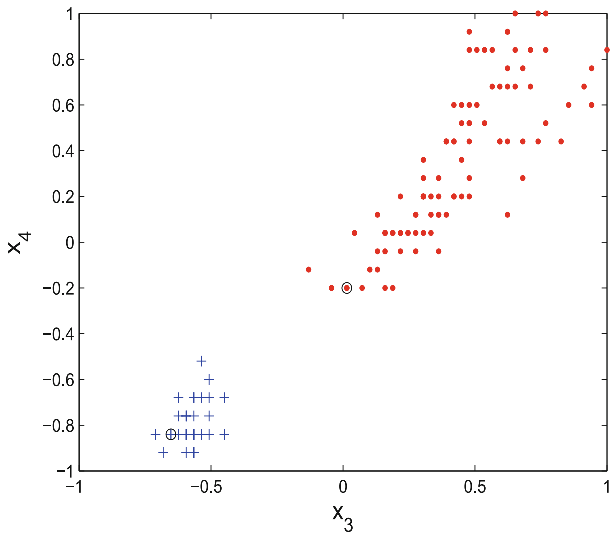


Fig. 9 Classification of the data set by SVM_1 projected to the $x_3 - x_4$ plane. $+$ $(\bullet)$ denotes points for which $\text{sgn}(h_1(\boldsymbol{x})) = 1$ $(\text{sgn}(h_1(\boldsymbol{x})) = -1)$. SVs are encircled

R_2 : If $q_1 > 0.5$ & $q_2 < 0.5$ Then $O = 4.462$,
R_3 : If $q_1 < 0.5$ & $q_2 > 0.5$ Then $O = -8.818$,
R_4 : If $q_1 < 0.5$ & $q_2 < 0.5$ Then $O = -2.178$.
This may be summarized as:

$$\text{If } q_1 > 0.5 \,\&\, q_2 < 0.5 \text{ Then } \text{sgn}(O) = 1,$$
$$\text{Else } \text{sgn}(O) = -1.$$

Examination of the training set shows that $q_2 < 0.5$ for 148 of the 150 examples, so we delete this condition from the rule. This yields

$$\text{If } q_1 > 0.5 \text{ Then } \text{sgn}(O) = 1,$$
$$\text{Else } \text{sgn}(O) = -1.$$

Recall that $q_1 = 0.468x_1 + 0.818x_2 - 0.652x_3 - 0.84x_4$ Thus, the simple rule above includes a single linear function of the x_is and, therefore, a linear function of the flower features (i.e., the z_is). To further simplify this rule, rewrite $q_1 > 0.5$ as

$$-0.652x_3 - 0.84x_4 > 0.5 - 0.468x_1 - 0.818x_2. \tag{23}$$

Examination of the data set shows that $x_1 \geq 0.0886$ and $x_2 \geq -0.0909$ for all the examples. Hence, we replace the right–hand side of (23) by

$$0.5 - 0.468 \cdot 0.0886 - 0.818 \cdot (-0.0909) = 0.5329.$$

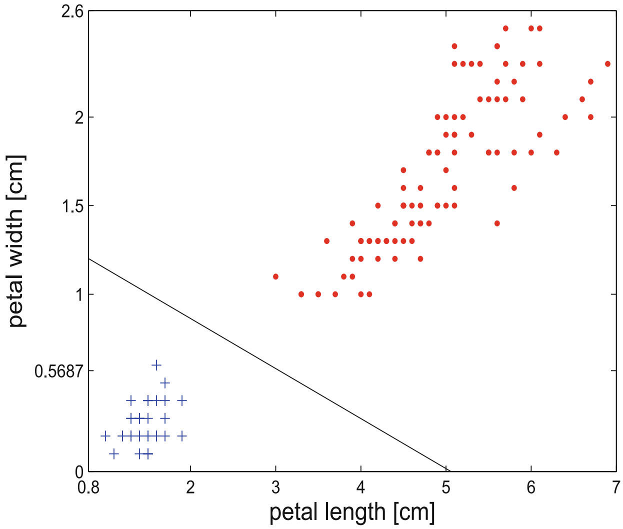


Fig. 10 Classification of the training set using the fuzzy rule extracted from SVM_1: + (•) denotes examples classified as Setosa (Versicolor or Virginica). The line is the classification threshold

This yields the condition

$$-0.652x_3 - 0.84x_4 > 0.5329. \tag{24}$$

Restating (24) in terms of the original (unnormalized) features via $z_i = m_i(1 + x_1)$ yields:

$$-0.19z_3 - 0.672z_4 + 1.492 > 0.5329. \tag{25}$$

Recalling that z_3 is petal length (PL), and z_4 is petal width (PW) provides the final rule:

If (PW $< 1.427 - 0.282 \cdot$ PL) Then class is Setosa,
Else class is either Versicolor or Virginica.

This rule provides a correct classification for all the 150 examples in the training set (see Fig. 10). Furthermore, note that the decision line depicted in this figure suggests that this fuzzy rule is not too far from an optimal margin classifier.

Knowledge Extraction from SVM_2

SVM_2 separates between the classes Versicolor and Virginica. These two classes are not linearly separable. Training is based on the 100 examples in the iris data set that belong to either class Versicolor or Virginica, and the kernel function $K(\boldsymbol{x}, \boldsymbol{y}) = \tanh(\rho \boldsymbol{x}^T \boldsymbol{y} + \eta)$, with $\rho = 0.22$, $\eta = -0.352$. This yields an SVM with 14 SVs. Six of these have very small α_i^* values, so we set $\alpha_i^* = 0$ for these SVs, leaving eight SVs with parameter values:

$$\boldsymbol{\alpha}^* = [10^4, 2719.1, 3708.8, 10^4, 4417.2, 2010.7, 10^4, 10^4],$$

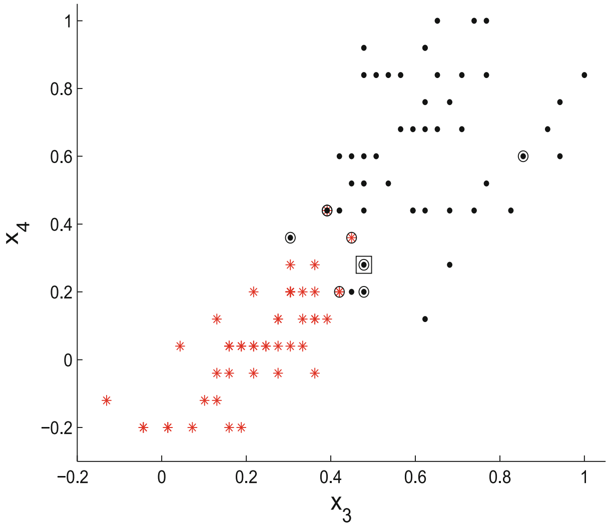


Fig. 11 Classification of SVM_2 projected to the $x_3 - x_4$ plane: • (∗) denotes points for which $\text{sgn}(h_2) = 1$ ($\text{sgn}(h_2) = -1$). Errors are squared. The SVs are encircled

and $b^* = 18.936$. Since $b^* \ll \alpha_i^*$ for all i, we also set $b^* = 0$. Therefore, the IO mapping of the (simplified) SVM is $f_2(\boldsymbol{x}) = \text{sgn}(h_2(\boldsymbol{x}))$ where:

$$h_2(\boldsymbol{x}) = 10^4 k_1(\boldsymbol{x}) + 2719.1 k_2(\boldsymbol{x}) + 3708.8 k_3(\boldsymbol{x}) + 10^4 k_4(\boldsymbol{x}) \\ -4417.2 k_5(\boldsymbol{x}) - 2010.7 k_6(\boldsymbol{x}) - 10^4 k_7(\boldsymbol{x}) - 10^4 k_8(\boldsymbol{x}), \tag{26}$$

with $k_i(\boldsymbol{x}) = \tanh(0.22\boldsymbol{x}^T\boldsymbol{s}^i - 0.352)$. This SVM correctly classifies 99 of the 100 examples in the training set (see Fig. 11).

By Corollary 1, the mapping in (26) is also the IO mapping of a FARB with: inputs $q_i = \boldsymbol{x}^T\boldsymbol{s}^i$, MFs $\mu_{>k_i}$, $\mu_{<k_i}$, and parameter values $k_i = -\eta/\rho = 1.6$, $\tau_i = 2\rho = 0.44$, $a_0 = 0$, and

$$[a_1, \dots, a_8] = [10^4, 2719.1, 3708.8, 10^4, -4417.2, -2010.7, -10^4, -10^4].$$

This yields a FARB with 256 rules. A calculation shows that for all the examples in the training set, only q_3 and q_6 ever exceed the value $k = 1.6$. Thus, all the rules that contain the linguistic term q_i *is larger than* k for $i \in \{1, 2, 4, 5, 7, 8\}$ are deleted from the rule–base. We are left with an FRB with just four rules:

R_1 : If $q_3 > 1.6$ & $q_6 > 1.6$ Then $O = 3396.2$,
R_2 : If $q_3 > 1.6$ & $q_6 < 1.6$ Then $O = 7417.6$,
R_3 : If $q_3 < 1.6$ & $q_6 > 1.6$ Then $O = -4021.4$,
R_4 : If $q_3 < 1.6$ & $q_6 < 1.6$ Then $O = 0$.

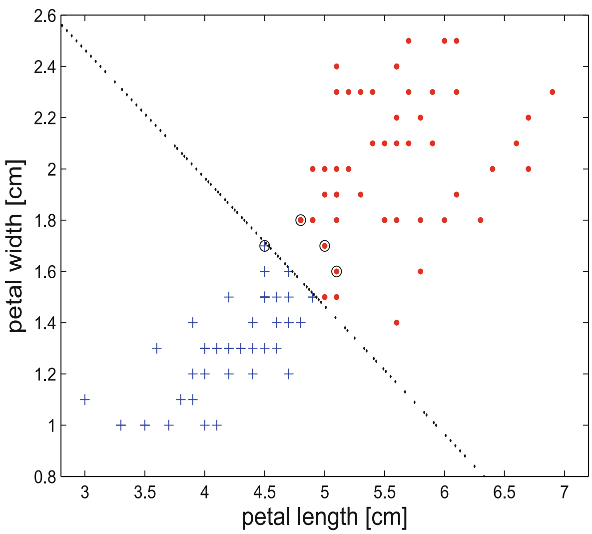


Fig. 12 Classification of the FRB extracted from SVM_2: + (•) denotes points of class Versicolor (Virginica). Errors are encircled. The threshold is doted

We may summarize this using the single rule:

If $q_3 < 1.6$ & $q_6 > 1.6$ Then $\text{sgn}(O) = -1$,

Else $\text{sgn}(O) = 1$.

Substituting q_3 and q_6 yields:

If $0.696x_1 + 0.363x_2 + 0.449x_3 + 0.36x_4 < 1.6$

and $x_1 + 0.727x_2 + 0.855x_3 + 0.6x_4 > 1.6$

Then $\text{sgn}(O) = -1$, Else $\text{sgn}(O) = 1$.

Examining the training set, we find that the average value of x_1 is $\bar{x}_1 = 0.5853$, and that $x_2 \leq 0.7272$. Replacing x_1 by $\bar{x}_1$ and x_2 by 0.7272 in the FRB yields:

If $0.449x_3 + 0.36x_4 < 0.9287$ and $0.855x_3 + 0.6x_4 > 0.48$

Then $\text{sgn}(O) = -1$, Else $\text{sgn}(O) = 1$.

Examining the two parts of this rule, we find that the first condition in the If–part holds for all the training example, so we delete this part. This yields:

If $0.855x_3 + 0.6x_4 > 0.48$ Then $\text{sgn}(O) = -1$,

Else $\text{sgn}(O) = 1$.

Converting back to the original feature values yields:

If $(7.93 - 2 \cdot \text{PW} < \text{PL})$ Then class is Virginica,

Else class is Versicolor,

where PL is petal length, and PW is petal width (in centimeters). This simple rule classifies the training set with an 4 % error (see Fig. 12).

Figure 13 summarizes the hierarchical FRB obtained by KE from SVM_1 and SVM_2. This FRB classifies the data set with an 2.66 % error. Thus for this example the KE approach provides a highly transparent classifier with a negligible degradation in performance. □

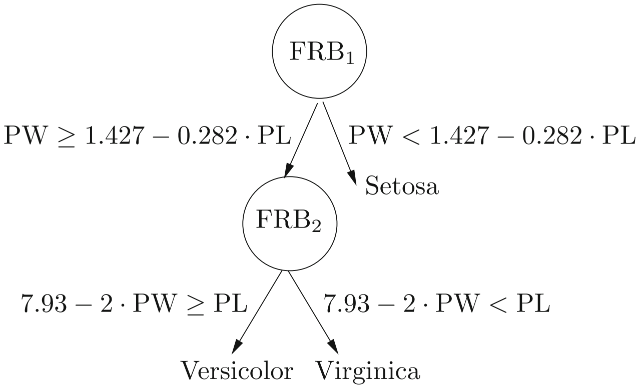


Fig. 13 Graphical representation of the extracted hierarchical FRB

6 Discussion

Certain classes of SVMs produce an input–output mapping that is mathematically equivalent to that of a corresponding FARB. This FARB then provides a symbolic representation of the SVM functioning stated in the form of fuzzy rules. This yields a new approach for KE from SVMs. Unlike previous approaches, the FARB uses only standard tools from fuzzy logic theory. Furthermore, the size of the extracted FARB depends on the number of SVs, that may be much smaller than the number of training examples.

We demonstrated this KE approach using several examples of SVMs trained to solve classification tasks. We deliberately used relatively simple tasks, as these allow a clear presentation of the rule extraction and simplification process. Application of our approach for larger problems is left for future work.

One drawback of the FARB is that the number of rules increases exponentially with the number of inputs. In our context of KE from SVMs, the number of rules increases exponentially with the number of SVs. An important direction for further research is thus developing a systematic approach for simplifying the extracted FARB in order to increase transparency. One possible approach is to first simplify the SVM itself, and then transform the *simplified* SVM into a corresponding FARB. It is important to note that several studies have already developed methods for simplifying SVMs and, in particular, for reducing the number of SVs (see, e.g., [39–46]). This is motivated by the need to reduce the run time of the training and classification algorithms, but is of course highly relevant in the context of KE using the FARB.

An interesting question is how to modify the training algorithm *beforehand* so that the resulting SVM will correspond to a simple as possible FARB. For example, setting certain parameters (e.g. ρ, η) properly may lead to a FARB with MFs $\mu_{>k_i}$, with the k_i values relatively large. Then it may be possible to delete all the rules that contain the term: x_i *is larger than* k_i, as their degree of firing is close to zero.

Acknowledgments This research was partially supported by a research grant from the Israel Science Foundation (ISF).

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Authors Biography

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On Type-Reduction Versus Direct Defuzzification for Type-2 Fuzzy Logic Systems

Jerry M. Mendel

Abstract This chapter examines type-reduction and direct defuzzification for interval type-2 fuzzy logic systems. It provides critiques of type-reduction as an end to itself as well as of direct defuzzification, and concludes that: (1) a good way to categorize type-reduction/direct defuzzification algorithm papers is as papers that either focus on algorithms that lead to a type-reduced set, or directly to a defuzzified value; (2) research on type-reduction as an end to itself has led to results that are arguably of very little value; and, (3) the practice of base-lining an IT2 FLS that uses direct defuzzification against one that uses type-reduction followed by defuzzification is unnecessary.

1 Introduction

A *type-2 fuzzy set* (Fig. 1) (T2 FS) can be thought of as a fuzzy-fuzzy set. Its membership function (MF) no longer has a single value at each value of the primary variable, but instead is a blurred version of that function, i.e., at each value of the primary variable the membership is itself a function, called a *secondary MF*. When the secondary MF is a constant equal to 1, the T2 FS is called an *interval type-2 fuzzy set* (IT2 FS) or an *interval-valued fuzzy set*; otherwise, it is called a *general type-2 fuzzy set* (GT2 FS).

The MF of a T2 FS is three-dimensional, with *x-axis* called the *primary variable*, *y-axis* called the *secondary variable* and *z-axis* called the *MF value* (or *secondary grade*). A *vertical slice* is a plane that is parallel to the MF-value *z*-axis. The *footprint of uncertainty* (*FOU*) of a T2 FS lies on the *x*-*y plane* and includes the closure of all points on that plane for which the MF value is non-zero; it is the 2D-domain on which

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_20

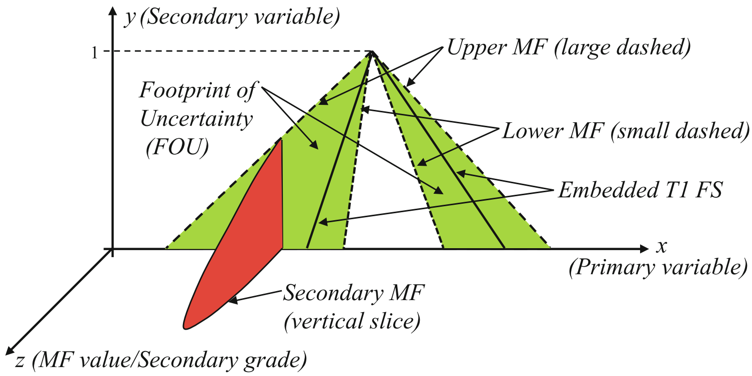

Fig. 1 Components of a General type-2 fuzzy set

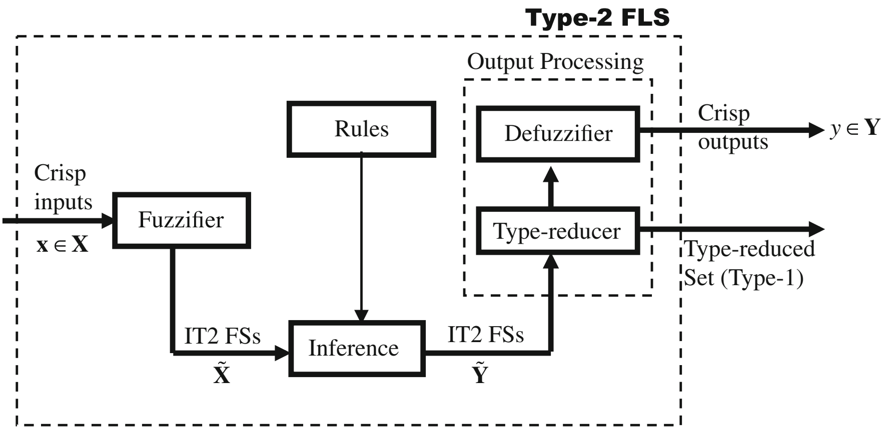

Fig. 2 Type-2 fuzzy logic system [23]. When all T2 FSs are IT2 FSs, the GT2 FLS becomes an IT2 FLS

sit the secondary grades. The FOU is lower and upper bounded by a *lower membership function* (LMF) and an *upper membership function* (UMF), both of which are type-1 fuzzy sets (T1 FSs). The FOU can be completely covered by T1 FSs that are called *embedded T1 FSs*.

T2 FSs are used in type-2 fuzzy logic systems (T2 FLSs) (Fig. 2). A general T2 FLS (GT2 FLS) was originally called a T2 FLS; however, because most of the works about T2 FLSs have focused on IT2 FSs, and only more recently on GT2 FSs, we now view the field of T2 FLSs as the union of the sub-fields of IT2 FLSs and GT2 FLSs.

Practical applications of T2 FLSs (e.g., fuzzy logic control [26]) require a number at the output of the FLS and not a FS. Readers of this book know that for a T1 FLS such a number is obtained by a process called *defuzzification*, which can be interpreted as the projection of a T1 FS into a type-0 FS. Many kinds of defuzzifiers are possible, including center of gravity (centroid), height and center of sets.

So, how does one go from a T2 FS to a number? Nilesh Karnik and I struggled with how to do this for months and finally we came up with a *two-stage approach* [14, 23]: type-reduction (TR)—a new concept for FSs—followed by defuzzification.

TR projects a T2 FS into a T1 FS, after which that T1 FS is projected into a type-0-FS by defuzzification to obtain a number. Because a type-reduced set is a T1 FS, defuzzification is achieved by computing the centroid of that set. For an IT2 FLS the type-reduced set is an interval set whose center of gravity is the average value of its two end-points; so, defuzzification for an IT2 FLS is trivial.

Just as there can be different kinds of defuzzification methods for a T1 FLS, there can be different kinds of TR methods for a T2 FLS. It is generally agreed that all TR methods must satisfy Karnik and Mendel's [23] *fundamental design requirement* for a T2 FLS, namely: *When all sources of [membership function] uncertainty disappear, a T2 FLS must reduce to a comparable T1 FLS*. This design requirement seems as reasonable today (in 2015) as it did to Karnik and Mendel in 2001, and is analogous to what happens to a probability density function when random uncertainties disappear. In that case, the variance of the pdf goes to zero, and a probability analysis reduces to a deterministic analysis. So, just as the capability for a deterministic analysis is embedded within a probability analysis, the capability for a T1 FLS is embedded within a T2 FLS.

As is stated in [25]: "TR became burned into the architecture of a T2 FLS because Karnik and Mendel first developed all of their T2 concepts and calculations for a general T2 FS and a GT2 FLS. Although TR was originally developed for a GT2 FLS, because it was so simple to perform for an IT2 FLS it was kept in the architecture of an IT2 FLS. There is nothing wrong with doing this; however, in retrospect we may have been blind-sided by the need for TR in a GT2 FLS from asking the question 'Is TR really needed in an IT2 FLS?'

"In fact, there are many ways to go from an IT2 FLS to a number that bypass TR and still satisfy the fundamental design requirement."

"A student in a class that I taught some years ago asked: 'Instead of performing TR, why can't we just use a combination of two T1 FLSs, one that uses only the lower membership functions and the other that uses only the upper membership functions?' My answer at that time was: 'You can't do this because each end-point of the type-reduced set uses a mixture of lower and upper membership function information.' While my answer was technically correct, it was predicated on using type-reduction, rather than on what the student had suggested. My answer today would be: 'You can do what you are suggesting, and this can be done in different ways; however, by bypassing TR you may not be able to provide a measure of the uncertainties that have flowed through all of the IT2 FLS computations (analogous to a standard deviation).' For example, you could begin with the architecture of an IT2 FLS as a linear combination of two T1 FLSs, as in [1], or as the centroid of the average of the lower and upper membership functions of the aggregated rule fired sets, as in [29]. All of these IT2 FLSs go directly to the defuzzified output value and they all satisfy the fundamental design requirement."

Unfortunately, TR cannot be performed using a closed-form mathematical formula. Its calculations led to two simple iterative algorithms that have been and continue to be called *KM-Algorithms* (or Karnik-Mendel Algorithms) [14, 23]. Because the computational complexity of using GT2 FSs was so great in the late 1990s, until around 2008 we as well as all others focused on TR for IT2 FLSs. It is fair to say that all thinking about type-reduction was in the context of IT2 FLSs.

Karnik and Mendel's two-step approach for going from a T2 FS to a number spawned a plethora of research and subsequent publications that fall into two categories: (1) type-reduction as an end to itself (i.e., TR viewed as a mathematical problem but with no application in mind, focusing on finding improved ways to perform TR); and, (2) direct defuzzification (i.e., methods for bypassing TR that project a T2 FS directly into a number). Unfortunately, some authors use the phrase "type-reduction" in the titles of papers that are about direct defuzzification, which is (in the opinion of this author) arguably incorrect.

Two journal articles that cover all aspects of this appeared in 2013, [24, 33]. The article by Wu [33] categorizes type-reduction and direct defuzzification algorithms into the following two categories:

- Enhancements to the KM TR algorithms (W1) (these are about type-reduction) and
- Alternative TR algorithms (W2) (these are almost all about direct defuzzification)

The article by Mendel [24] categorizes type-reduction algorithms into the following four categories:

- Improved KM algorithms (M1) (these are about type-reduction)
- Understanding the KM/EKM algorithms leading to further improved algorithms (M2) (these are about type-reduction)
- Non-KM algorithms that preserve the ability to approximate the centroid or type-reduced set (M3) (these are about type-reduction), and,
- Non-KM algorithms that do not preserve the ability to approximate the centroid or type-reduced set (M4) (these are about direct defuzzification)

While both articles contain a wealth of information about type-reduction and direct defuzzification algorithms, it requires a considerable effort by readers to relate an algorithm to its different categorizations across these two papers. In order to help with this, I will now classify the TR algorithms differently from both of these articles, as:

- Algorithms that lead to a type-reduced set (Table 1)
- Algorithms that lead directly to a defuzzified value (Table 2)

The connections between these two classes of algorithms and the classifications in [24, 33] are given in these tables.

Table 1 Algorithms that lead to a type-reduced set

Authors/references	Year	Wu Class [33]	Mendel Class [24]	Emphasizes		FLS Application?	Baselines against
				Speedup	Accuracy		
Wu and Mendel [37]	2002	W2	M3	No	No	Time-series forecasting	Nothing
Niewiadomski, Ocheleska and Szczepaniak [30]	2006	None[a]	M3	No	No	Linguistic summarization	Nothing
Melgarejo [22]	2007	W1	M3	Yes	No[b]	None	KM
Duran, Bernal and Melgarejo [5]	2008	W1	M3	Yes	No[c]	None	EKM
Li, Yi and Zhao [17]	2008	W2	M3[d]	Yes	Yes	Fuzzy logic control	COS TR
Wu and Mendel [34]	2009	W1	M1	Yes	No	None	KM
Hu, Zhao and Yang [13]	2010	W1	M3	Yes	Yes	None	EKM
Wu and Nie [35]	2011	W1	M1	Yes	No	None	KM and EKM
Liu andMendel [19]	2011	W1[e]	M2	No	Yes	None	Nothing
Liu, Qin and Wu [21]	2012	W1[e]	M2	Yes	No	None	Continuous EKM
Liu, Mendel and Wu [20]	2012	F[f]	M2	Yes	Yes	None	Nothing
Hu, Wang and Cai [12]	2012	W1	M3	Yes	No	None	EKM
Ulu, Guzelkaya and Eksin [32]	2013	W2	M3	Yes	Yes	None	EKM

[a] Wu includes this reference but does not include its algorithm in the main body of his paper
[b] Same accuracy achieved for their algorithms and KM algorithms
[c] Same accuracy achieved for their algorithms and EKM algorithms
[d] Although this paper is not listed in [24], if it had been it would be in M3
[e] Although this paper is not listed in [33], if it had been it would be in W1
[f] Mentioned only in a footnote in [33]

Table 2 Algorithms that lead directly to a defuzzified value

Authors/references	Year	Wu Class [33]	Mendel Class [24]	Emphasizes		FLS Application?	Baselines against
				Speedup	Accuracy		
Dziech and Gorzalczany [6]	1987	W2[a]	M4	No	No	Signal transmission[b]	Nothing
Gorzalczany [7]	1988	W2	M4	No	No	Fuzzy logic control[c]	Nothing
Liang and Mendel [18]	2000	W2	M4[d]	No	No	Equalization	Nothing
Wu and Tan [36]	2005	W2	M4	Yes	No[e]	PI Control	WM uncertainty bounds + defuzzification
Coupland and John [3]	2007	W2	M4[d]	Yes	No	None	Nothing
Nie andTan [29]	2008	W2	M4	Yes	Yes[f]	Fuzzy logic control	COS TR + defuzzification and WM uncertainty bounds + defuzzification
Greenfield et al. [9]	2009	W2	M4	Yes	Yes	None	KM
Greenfield, Chiclana and John [10]	2009	W2	M4	No	Yes	None	Exhaustive defuzzification
Greenfield, Chiclana and John [11]	2009	W2	M4	Yes	Yes	None	Exhaustive defuzzification
Du and Ying [4]	2010	W2	M4[d]	No	No	Fuzzy logic control	COS TR + defuzzification
Biglarbegian, Melek and Mendel [1]	2010	W2	M4	No	No	Fuzzy logic control	Nothing
Biglarbegian, Melek and Mendel [2]	2011	W2	M4	No	No	Function approx., identification	T1 FLS
Greenfield and Chiclana [8][g]	2011	W2	M4	No	Yes	None	Exhaustive defuzzification
Mendel and Liu [27][a, d]	2012	W2	M4	Yes	Yes	None	Nie-Tan
Tau, et al. [31]	2012	W2	M4[d]	Yes	No	Fuzzy logic control	Other controllers
Mendel and Liu [28][a, d]	2013	W2	M4	Yes	Yes	None	Nie-Tan
Khosravi, Nahavandi & Khosravi [15]	2013	W2[a]	M4[d]	Yes	Yes	Time-series prediction, identification	Five other methods
Khosravi & Nahavandi [16]	2014	W2[a]	M4[d]	Yes	Yes	Load forecasting	Five other methods

[a] Although this paper is not listed in [33], if it had been it would be in W2
[b] Not much detail is provided for this application
[c] This application is only suggested, but it is not examined in the paper
[d] Although this paper is not listed in [24], if it had been in would be in M4
[e] Commonly used control metrics, IAE, ISE and ITAE are compared
[f] Uses ITAE to do this
[g] Although "type-reduction" appears in the title of this paper, no type-reduced set is obtained in it

2 Explanations of the Tables

Tables 1 and 2 organize the papers chronologically, so that one can see some sort of continuity in thought as time progresses. Unfortunately, no standards existed when these studies were performed (nor do they exist today, in 2015) for how to compare TR or direct defuzzification algorithms, so it is very difficult to draw statistically meaningful conclusions from the papers that are in these two tables. Most of the time the authors used different IT2 FSs (FOUs) (some of which are sampled versions of continuous FOUs), or randomly chosen samples for the LMFs and UMFs of discrete FOUs, or FOUs for which mathematical formulas are provided for their LMF and UMF.

Fortunately, [33] provides statistically meaningful comparisons for two kinds of FOUs: (1) randomly chosen samples for the LMFs and UMFs of discrete FOUs, and (2) evolutionary fuzzy logic controller design. If one is interested in which of the algorithms are the fastest or most accurate, then [33] is the paper to go to.

3 Critique of Type-Reduction as an End to Itself

Research on *type-reduction as an end to itself*, as summarized in Table 1, uses the (oft unstated) assumption that one begins with an IT2 FS to perform TR on. It then focuses on how to improve (or approximate) the KM Algorithms to perform type-reduction on that given IT2 FS. One or both of two performance measures are used to evaluate improvements: computing time and accuracy. All of this research (except for [30, 37]) is done outside of the original context of a T2 FLS, and is therefore open to some critical examinations.

For starters, the assumption that *one begins with an IT2 FS to perform TR on* needs to be questioned. I have explained in [24] that there are two distinctly different situations that can occur in an IT2 FLS:

1. Fired-rule IT2 FSs are first aggregated by means of the union operation resulting in an aggregated IT2 FS (so that the assumption is valid), after which this aggregated IT2 FS is type-reduced (this is called *centroid type-reduction*); and,
2. Fired-rule IT2 FSs are aggregated as part of type-reduction (this is called *center-of sets type-reduction*), in which case one does not begin with an aggregated IT2 FS, and so the assumption is invalid.

Obviously the people who are researching *type-reduction as an end to itself* are focusing on Situation 1; however, in real-world applications of FLSs it is Situation 2 that is more often used than Situation 1, because (this is also true for T1 FLSs) performing the aggregation of the fired-rule IT2 FSs is too time consuming. Consequently, research on *type-reduction as an end to itself* is about things that arguably will be of very little direct value or use in a T2 FLS.

One is then led to question whether or not the two metrics that are used in this kind of research for Situation 1 are important, namely computing time and accuracy.

Consider first the metric of *computing time*. If the results of this research are not going to be used in a real-time FLS then does it really matter if the algorithms that perform type-reduction take 10^{-3}, 10^{-4}, 10^{-5}, etc. sec? KM or EKM [34] algorithms are already quite fast; they converge quadratically [19] and have been observed to take 10^{-5} to 10^{-3} s to converge (see, e.g., [34]). The EIASC algorithms [35] are even faster and they are very easy to understand. So, focusing on convergence time for Situation 1 is to me arguably a red herring because it makes no perceptual difference to a human when it is already quite small.

Consider next the metric of *accuracy*. To me an important question is: How much accuracy is required by the type-reduced set? Unfortunately, this question is never asked in the papers that use the metric of accuracy. Instead, one is left with an impression that higher accuracy is always better. Unless one knows what the type-reduced set will be used for then, I would like to ask: "What is higher accuracy better for?" To answer this question there are again two different situations/cases that need to be examined:

- The LMF and UMF are given by mathematical formulas and so they can be sampled at any desired rate (C1), and
- The LMF and UMF are given only by a collection of samples (C2).

Consider C1, in which the LMF and UMF are given by mathematical formulas, and so they can be sampled at any desired rate. Then KM algorithms will give very accurate results. To counter this some authors perform studies in which they reduce the sampling rate to see how more accurate their algorithms are than the KM algorithms. The Catch-22 to this work is that they are comparing the results from their algorithms and the KM algorithms both of which use the reduced sampled data with so-called *true results*, where the latter use the highly sampled data. To me this is circular reasoning, because if one has access to highly sampled data and is performing the type-reduction computations off-line then why not use all of the highly sampled data to perform the type-reduction?

Consider next C2 in which the LMF and UMF are given only by a collection of samples, as would occur if perchance fired rule output sets were aggregated by using the union (although, as explained above, this is usually avoided). The reason that the resulting aggregated IT2 FS is given only by a collection of samples is that it has been computed using an algorithm for the union that only uses sampled values. So, what exactly does accuracy mean in this situation? In this case the KM algorithms compute the true values of the type-reduced set and so they are 100 % accurate. To get around this fact, the authors pretend that they have access to the LMF and UMF given by mathematical formulas and have created a set of sampled values for the LMF and the UMF from those formulas. This throws us back to the previous case, but now that case is reached by violating the assumption that the LMF and UMF are given only by a collection of samples. This again is circular reasoning.

Regrettably, the conclusion I am led to about research on *type-reduction as an end to itself* is that it has led to results that are arguably of very little value.

4 Critique of Direct Defuzzification

All research on direct defuzzification should be applauded as long as the direct defuzzification method obeys the already-mentioned *fundamental design requirement* for a T2 FLS, namely: *When all sources of [membership function] uncertainty disappear, a T2 FLS must reduce to a comparable T1 FLS*. It should be applauded because its researchers have had the courage and conviction to challenge the need for type-reduction in an IT2 FLS. To me, this is how real progress occurs. These authors (although they may not have actually said it this way or at all) asked: "Why is type reduction needed in an IT2 FLS? Why can't we go directly to a defuzzified output?"

As is demonstrated by the large number of papers in Table 2, there are many ways to go directly to a defuzzified output; however, many of the authors of direct defuzzification papers compare their numerical results with the ones obtained from using type-reduction followed by defuzzification, as though the latter was a baseline approach. So another natural question to ask is:

- Should an IT2 FLS that uses type-reduction followed by defuzzification be the baseline IT2 FLS against which all others must be compared?

My answer to this question is "No," and my reason is that maybe it should be the other way around, i.e. maybe an IT2 FLS that uses type-reduction followed by defuzzification should be compared against an IT2 FLS that uses direct defuzzification. This may sound like double talk or the chicken before the egg parable, but I am quite serious about my answer. Let me explain.

Real-world FLSs are designed with application-dependent performance specifications in mind. Unfortunately, authors (myself included) do not usually state what those specs are. Instead they assume that the goal is to optimize the performance metrics rather than meet those metrics, something that is not done in the business world.

Many years ago I attended a one-day seminar on entrepreneurship given by world-famous Peter Drucker. It just so happened that we were sitting at the same table for lunch. Everyone around the table introduced themselves. When it came to my turn and I told everyone that I was an engineer, Drucker shook his head. I asked him why he was doing that, to which he replied:

> You engineers are always trying to optimize something, which is why you make such poor businessmen. A business person develops a product with a certain level of performance as quickly as possible, manufactures it, sells it and makes a profit. He or she then improves the product a bit and sells the next version, making even more profit. In the meantime, you engineers have not gotten off of the block; you are still trying to optimize your product, and have nothing to show for it.

So if an application's performance metrics can be met by an IT2 FLS that uses direct defuzzification, that should be the end of the story. If, however, those performance metrics cannot be met by such an IT2 FLS then one could try either a different IT2 FLS that uses direct defuzzification, or, perhaps as a last resort, an IT2

FLS that uses type reduction followed by defuzzification. There is no a priori guarantee, however, that even the latter will be able to meet the performance metrics. If it cannot, then one may need to use a GT2 FLS.

5 Conclusions

Ever since Karnik and Mendel introduced the concept of type-reduction, which leads to solving two optimization problems, one for the left-end and one for the right end of the type-reduced set, a cottage industry has emerged that has focused on better ways to perform TR or to bypass it entirely. This paper has tried to provide a critical examination of the research that has been performed on these two topics.

It has explained that:

1. A good way to categorize type-reduction/direct defuzzification algorithm papers is, as: papers that focus on algorithms that lead to
   a. A type-reduced set, or
   b. Directly to a defuzzified value.
2. Research on type-reduction as an end to itself has led to results that are arguably of very little value.
3. The practice of base-lining an IT2 FLS that uses direct defuzzification against one that uses type-reduction followed by defuzzification is unnecessary.

Note that if Karnik and Mendel had never introduced type-reduction followed by defuzzification, and instead had achieved their design requirement by using direct defuzzification, then we would not be having this conversation, and people would be comparing results obtained by using one kind of direct defuzzification method against those obtained by using at least one other kind of direct defuzzification method.

As a final thought, note that type reduction was invented because Karnik and Mendel thought that the type-reduced set would itself be of value, in that it would provide a valuable measure of the flow of MF uncertainties through the FLS. Although it does do this, regrettably, to the best of knowledge of this author, there is not one application paper that makes use of the type-reduced set in this way. On the other hand, type-reduction did lead to the KM Algorithms (as well as others) which are very useful for solving other non-FLS problems, as is explained in [24].

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On Fuzzy Theory for Econometrics

Hung T. Nguyen and Songsak Sriboonchitta

Abstract This paper aims mainly at informing statisticians and econometricians of relevant concepts and methods in fuzzy theory that are useful in addressing economic problems. We emphasize three recent significant contributions of fuzzy theory to economics, namely fuzzy games for capital risk allocations, fuzzy rule bases and compositional rule of inference for causal inference, and a statistical setting for fuzzy data based on continuous lattices.

1 Introduction

Lotfi Zadeh advocated fuzzy sets as mathematical modeling of fuzzy concepts in natural language in 1965 [1]. Ever since, while fuzzy theory found significant applications in engineering and technology fields from its own concepts and methods, it was Zadeh himself who emphasized that fuzziness, as a distinct concept of uncertainty, should be a complement to randomness, i.e., when facing uncertainty in real-world complex systems, we should handle both fuzziness and randomness together. This paper is about a concrete and important situation where random fuzzy sets appear naturally and should be studied simultaneously.

This paper is organized as follows. In Sect. 2, biased by our own experience, we recall two anecdotes concerning fuzzy theory, as a "contribution" to the celebration of 50 years of fuzzy sets. The rest of the paper is devoted to elaborating on some significant contributions of fuzzy theory to a specific area of social science, namely economics, again biased by our own current research interests. In Sect. 3, within the nice

Dedicated to Lotfi Zadeh.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_21

connection of Von Neumann's view of economics and [2], we bring out the necessity to take fuzzy games into account in order to investigate coherent risk allocations in financial risk management. In Sect. 4, we illustrate the appearance of "observed" time series of fuzzy data in econometric regression with *seemingly unobservable variables* (SUV), as well as discussing the necessity of using fuzzy technology in both *statistical and econometric causal inference.* To put fuzzy data on a firm basis for statistical treatment, we outline, in Sect. 5, our recent development of random fuzzy sets using topologies from continuous lattices.

2 Two Anecdotes of Fuzzy Theory

Professor Dennis Lindley gave a thought-provoking lecture in Zadeh's seminar at the University of California, Berkeley, in 1981, in which he showed that, in the framework of scoring rules, fuzzy sets, belief functions and even confidence intervals are all inadmissible measures of uncertainty. His message is that "you cannot avoid probability". His lecture was published in 1982 (see [3]). In a followed-up debate organized by a statistical journal, all invited discussants felt strongly that Lindley was wrong, but Lindley replied something like "everybody said that I am wrong, but nobody was able to show me where I am wrong, mathematically".

I. R. Goodman and H. T. Nguyen spent a year or so, starring at Lindley's paper. And, finally, they saw what was wrong with Lindley's result. There is nothing wrong with Lindley's result, only his implications! His result is this. If an uncertainty measure is admissible, then it must be a function of a probability measure. But then, that uncertainty measure need not be additive, such as the square of a probability measure which is precisely a belief function. So, the matter is clear and settled! We can go on to use fuzziness without contradicting Lindley's message. Our findings were published in 1991 (see [4]). It is interesting to note that the philosophical side of the above issue seems to lie between (semantic) information and probability. In 1982, A. N. Kolmogorov said "Information theory must precede probability and not be based on it" (See [5]). This is a quote in the book *Information Theory and The Central Limit Theorem* by [6].

The second anecdote is even more "threatening"! C. Elkan presented a paper entitled "The paradoxical success of fuzzy logic" at a Conference on Artificial Intelligence in 1993 in which he said that all successes attributed to fuzzy logic are due to something else, and in fact, fuzzy logic does not exist, using a short mathematical theorem showing that, from his set of axioms for fuzzy logic, fuzzy sets just collapse to ordinary (crisp) sets! See [7]. Again, a journal debate was organized with a familiar outcome: discussants "argued" that Elkan was wrong, but nobody was able to point out where he was wrong mathematically! It was Vladik Kreinovich who found a way to get around Elkan's theorem by examining logics of experts (See [8]). It was a good step but not quite an answer to Elkan's result. Upon seeing that paper by Nguyen et al., three logicians at New Mexico State University (Las Cruces, USA), Mai Gehrke, Carol and Elbert Walker, quickly saw the flaw in Elkan's theorem. It is his own system of axioms which was wrong. Their findings were published in 1997 (See [9]). So, again, we can go on!

3 Fuzzy Games for Financial Risk Management

This section is devoted to the explanation of a significant contribution of fuzzy set theory to economics, in the sense that its generalization of crisp sets is necessary to solve important problems, and not just a generalization per se. In a sense, this is somewhat similar to the invention of complex numbers!

Extensions of concepts are familiar phenomena in mathematics. In the applied sciences, mathematical generalizations of concepts are *immediately* appreciated when not only they are motivated by concrete needs in applications, but also since they help to solve problems that otherwise we cannot come up with solutions. The striking example is the generalization of (pure) strategies in non-cooperative games to mixed strategies, leading to the existence of Nash equilibria in behavioral economics. Now, as we will see, Von Neumann's cooperative (coalitional) games need to be generalized to solve a problem in *econometrics*. It is interesting to mention that R. J. Aumann and L. S. Shapley (1974), in their famous book *Values of Non-Atomic Games*, needed to generalize ordinary sets to obtain what they called "evenly spread sets" to formulate their notion of values for games with large masses of players. As they put down in a footnote [10], their "ideal sets" are formally Zadeh's fuzzy sets.

Recent literature on the capital risk allocation (CRA) problem revealed a nice marriage between two seemingly "disjoint" ingredients in studying economics, namely von Neumann's game theory, and Haavelmo's econometrics. The basic reference is [11] which triggered current intensive research on *fuzzy games* for practical applications.

As its name indicates, the CRA problem consists of finding ("fair") ways to allocate capital risk of a firm to its (business) units. Suppose there are n units in a firm, each unit has a risk of X_i (a random variable). The risk X_i is quantified by using a risk measure $r(.)$. The "cost" of a group of units $A \subseteq I = \{1, 2, ..., n\}$, is $c(A) = r(\sum_{i \in A} X_i)$. To capture diversification effect, the risk measure $r(.)$ should be chosen to be "coherent", i.e., subadditive, such as the Tail Value-At-Risk. For such a risk measure, the set function $v : 2^I \to \mathbb{R}$, defined by $v(A) = \sum_{i \in A} c(\{i\}) - c(A)$, is super additive, so that, formally, (I, v) is a coalitional (cooperative) game, in which units play the role of "players", subsets of I are coalitions, and $v(.)$ gives the "payoffs" of coalitions. Thus, the CRA problem can be reformulated as a coalitional game which is more "convenient", in the sense that game solution concepts will be used to derive allocation principles for CRA problem. Specifically, a cost allocation is a vector $(x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n x_i = c(I)$ which is obtained from an allocation $(y_1, y_2, ..., y_n) \in \mathbb{R}^n_+$ of the game (I, v) by taking $x_i = c(\{i\}) - y_i$.

Specifically, the Shapley value S of (I, v) is a "coherent" allocation (i.e., a solution for the CRA problem) when it belongs to the core $\mathscr{C}(v)$ of the game (which is non empty here by the choice of a coherent risk measure). Note that, if a game (I, v) can be modeled arbitrarily, e.g., as a *convex* game, then it is well-known that not only the core is non empty, and equals to the unique stable set, but also contains the Shapley value. The problem here is that, we cannot choose the associated characteristic function $v(.)$ arbitrarily, since it is defined in terms of a (coherent) risk measure $r(.)$, via the cost function $c(.)$, namely

$$v(A) = \sum_{i \in A} c(\{i\}) - c(A) = \sum_{i \in A} r(X_i) - r(\sum_{j \in A} X_j)$$

If (I, v) is a convex game, then necessarily the risk measure $r(.)$ is linear, which is not acceptable since such a risk measure eliminates the diversification effect. Thus, we are facing a *non convex game* (I, v), i.e., lacking all the nice properties of convex games, in particular, a coherent allocation principle for the original CRA problem. Hence, the question is: for a non convex game, but with non empty core, when the Shapley value belongs to the core? As [11] put it "We thus fall short of a convincing proof of the existence of coherent allocations". The "way out" is *fuzzy games!*

So let's us just spell out what are fuzzy games, how they appear, and how they provide solutions to CRA problem.

In standard coalitional games (I, v), a coalition is a subset A of the set $I = \{1, 2, ..., n\}$ of players. It was Aubin (see e.g., [12]) who made the following realistic observation. When joining a coalition, players might not necessarily commit their full resources (e.g., money, time) to it. They could choose their levels of participation to a coalition, and not just whether they decide to participate or not. As such, degrees of participation of players should be taken into account when considering coalitions (e.g., here, for a "fair" capital risk allocation). A coalition A when all of its members committing fully is just an ordinary coalition, i.e., a crisp subset of I, whereas, a "partial" coalition can only be characterized by degrees of participation of players, i.e., by a function $f : I \to [0, 1]$ with $f(i)$ represents the degree of participation of player i in the coalition. Thus, a partial coalition is not a set unless $f : I \to \{0, 1\}$, but a kind of a "generalized set", a *fuzzy set.* It should be noted that $f(.) < 1$ is possible: it corresponds to a coalition where no player wants to commit fully to it.

Remark. Using indicator functions of sets, we identify the power set of I with the subset $\{0, 1\}^n$ of $\mathbb{R}^n$, denoted as 2^I. The convex hull of $\{0, 1\}^n$ is $[0, 1]^n$ which is identified as the set of functions $I \to [0, 1]$. Thus, each "membership function" $f : I \to [0, 1]$ is written as $f(.) = \sum_{A \subseteq I} \beta(A) 1_A(.)$ with $\beta(.) \geq 0$ and $\sum_{A \subseteq I} \beta(A) = 1$. As a result, $f(i) = \sum_{i \in A} \beta(A)$ which can be *interpreted* as the (one-point) coverage function of the random set $S : (\Omega, \mathscr{A}, P) \to 2^I$, with $P(S = A) = \beta(A)$, i.e., $f(i) = P(i \in S)$. It is important to note that this interpretation is by no means an indication that fuzziness is subsumed by probability! First, the above connection exhibits only a weak form of randomness. Membership functions (defining fuzzy sets) are not probability distributions. Fuzziness is a matter of degree. Let's elaborate on these two points. The function $A(.) : \mathbb{R}^+ \to [0, 1]$ given by

$$A(u) = \begin{cases} 0 & \text{if } u < 20 \\ \frac{u-20}{55} & \text{if } 20 \le u \le 75 \\ 1 & \text{if } u > 75 \end{cases}$$

is a membership function for the fuzzy concept "high income" (where, e.g., 20 means \$20,000, annually). The value $\frac{30}{55} \in [0, 1]$ is the degree to which the income of 50 is compatible with the meaning of "high income". It is not the probability that an income of 50 is high! Membership functions on general domains (not necessarily finite domains) exhibit a weak form of randomness as follows. Let $f : U \to [0, 1]$, then there exists a random set (see Sect. 5) $S : (\Omega, \mathscr{A}, P) \to 2^U$, such that, for each $u \in U, f(u) = P(u \in S)$. Indeed, it suffices to consider a random variable $\alpha : (\Omega, \mathscr{A}, P) \to [0, 1]$, uniformly distributed, and take $S(\omega) = \{u \in U : f(u) \ge \alpha(\omega)\}$. The adjective "weak" refers to the knowledge of the coverage function $u \to P(u \in S)$ of a random set (as in the theory of sampling in finite populations) which is weaker than the knowledge of the "distribution" of the random set. This is illustrated by a previous work of Robbins (1944). Historically, while random sets appeared naturally in many places, such as stochastic geometry, its formal theory was not rigorously established until 1975 (by [13]). When estimating the "size" (area, volume) of a random set, [14] did not really consider a formal concept of a random set. This so since the size of a random set $\mu(S)$ (where μ the Lebesgue measure on $\mathbb{R}^d$) is in fact a numerical random variable, although it depends on the random set S. Without a formal concept of random sets, it is not possible to find the distribution of the non-negative random variable $\mu(S)$ which is a function of S. The clever result of Robbins is this. As far as the expected value of $\mu(S)$ is concerned, we need much less than the distribution of $\mu(S)$. Specifically, the knowledge on the coverage function of the informal random set S is sufficient to determine $E\mu(S)$, a "weaker" form of information. The computation of the expected length of a random set of the form $S = [0, X]$ where $X \ge 0$ is a random variable is simple: the length of S is X, so that

$$E\mu(S) = EX = \int_0^\infty P(X > x)dx = \int_0^\infty \pi(x)dx$$

where

$$\pi(x) = P(x \in S) = P(x \in [0, X]) = P(X > x)$$

is the coverage function of the random set S. The Robbins' formula says that the above formula is in fact general: For any random set S on $\mathbb{R}^d$, we have

$$E\mu(S) = \int_{\mathbb{R}^d} \pi(x)d\mu(x)$$

where $\pi(.) : \mathbb{R}^d \to [0, 1]$ is the coverage function of S.

Coalitional games in which partial coalitions are considered are referred to as fuzzy games. Just like fuzzy logic (which really means "a logic of fuzzy concepts" and not a logic which is fuzzy!), maybe we should rename fuzzy games by *games with fuzzy coalitions* to be specific?

A fuzzy game is a pair (I, v^*) where $I = \{1, 2, ..., n\}$ and the generalized "characteristic function" (of coalitional games) is $v^* : [0,1]^n$(fuzzy subsets of I)$\rightarrow \mathbb{R}$, with $v^*(0) = 0$, noting that a fuzzy coalition is characterized by a vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in [0,1]^n$ with λ_i being the participation level of player i in that fuzzy set. An allocation for a fuzzy game is a vector $x \in \mathbb{R}^n$ such that $\sum_{i=1}^n x_i = v^*(I)$ and, for any fuzzy coalition λ, $\sum_{i=1}^n \lambda_i x_i \geq v^*(\lambda)$ (Noting that, the attainable outcome of a fuzzy coalition depends obviously on the degrees of commitment of the players). This is the generalization of the core of a standard coalitional game. However, while the game is fuzzy, its "fuzzy core" is not.

Remark. The "cost function" of a fuzzy coalition λ is defined in terms of a risk measure $r(.)$ as $c(\lambda) = r(\sum_{i=1}^n \lambda_i X_i)$.

In the context of the CRA problem, the associated fuzzy game is this. Let $\Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_n) \in \mathbb{R}_+^n$ be the vector of "business volumes" of units in a firm. Suppose each unit can commit some portion of its volume to a coalition. Let denote by $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ a "resource commitment" of the units to a coalition, where, of course, $0 \leq \lambda \leq \Lambda$ (componentwise). Then the degree of membership (level of participation) of unit i in that coalition is $\frac{\lambda_i}{\Lambda_i} \in [0,1]$. The fuzzy coalition is determined by the membership function $f : I \rightarrow [0,1]$, given by $f(i) = \frac{\lambda_i}{\Lambda_i}$.

When taking into account of partial participation in coalitions, such as this important situation in financial risk management, we arrive as a natural concept of fuzzy games which turn out to be a "good" example of Aumann and Shapley's *non-atomic games* [10]. Indeed, since each player (business unit) can choose to joint a coalition by declaring a portion $f(i) = \frac{\lambda_i}{\Lambda_i} \in [0,1]$ of its total business volume, the "real" players in $I = \{1, 2, ..., n\}$ become "players" in the infinite set $[0,1]$. In other words, *a fuzzy game is a game with infinite number of players.* Note that, in this view, coalitions of the game $([0,1], w)$ are taken as Borel sets of $[0,1]$, i.e., $w(.) : \mathscr{B}_1 \rightarrow \mathbb{R}$, but noting also that $([0,1], \mathscr{B}_1, w)$ is not a measure space since the set function $w(.)$ is not additive.

Without entering into more technical details, let's just mention how the extension to fuzzy games leads to solutions for the CRA problem. Extending basic concepts of crisp coalitonal games to fuzzy games, such as core of fuzzy games, (Shapley) fuzzy values, as well as associated coherent risk measures, the solution to the CRA can be taken as an allocation (fuzzy value map) with values in the core (non fuzzy) of the fuzzy game. It turns out that the Aumann-Shapley value (extending the usual Shapley value) of non-atomic games is a coherent CRA solution, when the underlying risk measure is chosen to be coherent (in an extended sense), and under mild and possible conditions. See [11] for details.

The advantage of considering fuzzy games in the CRA problem is this. With coherent risk measures, it is not clear whether the Shapley value is a coherent allocation, but, the Aumann-Shapley value (when it exists) is a coherent allocation (i.e., solution of the CRA problem) in the context of fuzzy games.

4 Fuzzy Methods for Econometric Causal Inference

We turn now to another significant contribution of fuzzy theory to econometrics, namely using fuzzy rule bases to investigating causal inference (specifically, in counterfactual problem: how to use a knowledge of a causal structure to "reason" to unobservable outcomes?), leading to econometric predictions with seemingly unobservable variables.

But first, while fuzzy technology is well-known in engineering circle, it is not so for econometric community. Here is a tutorial on the basis of fuzzy technology that we will employ in this paper. According to [1], fuzzy concepts in a natural language, such as "low" (temperature), "efficiency", "happiness", can be modeled mathematically for information processing purposes. As we will see, fuzzy concepts are usually values of qualitative (linguistic) variables of interest in decision-making (e.g., not disabled, partially disabled, fully disabled are "values"/outcomes of the linguistic variable "disability status"). More importantly, fuzzy concepts are used as *coarsening schemes* in human intelligent behavior (e.g., in intelligent control). Note that qualitative/latent models and regression are also in the practical statistical tool box.

Using a familiar procedure in mathematics, namely, as far as generalizations are concerned, some equivalences of a concept are more suitable for the purpose than others (e.g., an equivalent framework for deriving the Black-Scholes option pricing formula from PDE is martingales, allowing extensions to other markets), Zadeh defined fuzzy sets (i.e., mathematical objects representing fuzzy concepts) by extending the range of ordinary (crisp) set indicator functions (for a complete theory of fuzzy sets and logics, see e.g., [15]). Let U be a set. A subset $A \subseteq U$ is characterized by it indicator function $A(.) : U \to \{0, 1\}$ (note that we use the same notation A for the set and its indicator function, the context will tell us clearly which is which!), where $A(u) = 1$ or 0 according to $u \in A \in$ or $u \notin A$. This equivalent way to describe a set brings out the notion of membership of elements of U to subsets of U: when $A(u) = 1$, the element u is a member of A, whereas when $A(u) = 0$, u is not a member of A. Here, membership degrees are only 1 and 0 (there is no partial membership). Extending the range $\{0, 1\}$ to the whole unit interval $[0, 1]$ lead to a generalization of crisp sets to fuzzy sets. Specifically, a fuzzy subset of U is a function $A(.) : U \to [0, 1]$ where $A(u) \in [0, 1]$ is the membership of an elements $u \in U$ in the underlying fuzzy concept.

Having fuzzy data modeled as fuzzy sets, we can proceed to manipulate/process them by extending operations on the underlying set U. This is achieved by the well-known extension principle. If $\varphi : U \times V \to W$ and A, B are fuzzy subsets of U, V, respectively, then φ is extended to fuzzy subsets as

$$\varphi(A, B)(w) = \max_{\{(u,v):\varphi(u,v)=w\}} (A(u) \wedge B(v))$$

where $\wedge$ denotes minimum (and $\vee$ denotes maximum).

Note that, in one hand, since any function $A(.) : U \to [0, 1]$ can be recovered from its level sets $A_\alpha = \{u \in U : A(u) \geq \alpha\}$, $\alpha \in [0, 1]$ by $A(u) = \int_0^1 A_\alpha(u) d\alpha$, and on the other hand, manipulations with level sets are simpler, the following well-known result, known in the literature as Nguyen's Theorem (See, [16–18]) is useful. A necessary and sufficient condition for

$$\varphi(A_\alpha^{(1)}, A_\alpha^{(2)}, ..., A_\alpha^{(n)}) = [\varphi(A^{(1)}, A^{(2)}, ..., A^{(n)})]_\alpha$$

where $\varphi : U_1 \times U_2 \times ... \times U_n \to V$, and $A^{(i)}$ is a fuzzy subset on U_i, $i = 1, 2, ..., n$ is that for each $v \in V$,

$$\max_{\{(u_1, u_2, ..., u_n) \in \varphi^{-1}(v)\}} [\wedge_{i=1}^n A^{(i)}(u_i)]$$

is attained.

While fuzzy logic connectives, including "implication operator" $\Longrightarrow$ (representing "If... Then..." rules in fuzzy technology), are familiar with engineers, they appear as tools to suggest (nonlinear) models for statistics.

This can be seen as follows. A statistical model, such as a linear regression model $Y_i = \theta X_i + \varepsilon_i$, $i = 1, 2, ..., n$, is in fact a collection of "If...Then..." rules, since what they mean is that, for each i, the model reads "If X is X_i (and ε is ε_i), then Y is Y_i" (where "is" stands for "equal"). This observation allows an extension to fuzzy data: when (X_i, Y_i), $i = 1, 2, ..., n$ are fuzzy data (linguistic labels), the "rules" become R_i : "If X is X_i, then Y is Y_i" or "$X_i \Longrightarrow Y_i$" where the connective "If...Then..." is the implication $\Longrightarrow$ in fuzzy logic, which is a fuzzy relation on the Cartesian product, say, $U \times V$, i.e. $f_{\Longrightarrow}(u, v)$ is the degree to which u "implies" v. In the simplest fuzzy logic system, $f_{\Longrightarrow}(u, v)$ in "$X_i \Longrightarrow Y_i$" can be taken as $X_i(u) \wedge Y_i(v)$.

In the statistical linear regression, the goal is to arrive at a "prediction formula" from, say, given numerical data (X_i, Y_i), $i = 1, 2, ..., n$. This is achieved by combining the "rules" $Y_i = \theta X_i + \varepsilon_i$ by using some method of estimation (e.g., least squares, when random variables have finite variances) to estimate the parameter θ to arrive as $Y = \hat{\theta}_n X$. A counterpart of such a procedure when the data (X_i, Y_i), $i = 1, 2, ..., n$ are fuzzy is combining the rules "$X_i \Longrightarrow Y_i$" by the compositional rule of inference $\vee_{i=1}^n [X_i(u) \wedge Y_i(v)]$ to obtain the combined membership function, where $\vee_{i=1}^n [X_i(u) \wedge Y_i(v)]$ is the degree to which u implies v given the rule base. Given a value u, the implied consequence is a fuzzy subset of V given by $v \to \vee_{i=1}^n [X_i(u) \wedge Y_i(v)]$. The important point is this. Fuzzy rule bases play the role of statistical models in the presence of fuzzy data. We will elaborate on this important ingredient in the problem of "estimating" unobservables using nonparametric causal "structures".

Remark. With respect to the "conditional" implication "$A \Longrightarrow B$" and its degree of compatibility, used in the previous context, it is of course tempting to ask whether they can be given a probabilistic flavor? For example, for crisp events A, B on some probability space $(\Omega, \mathscr{A}, P)$, can we view the degree of "$A \Longrightarrow B$" as $P(B|A)$?

If $A \Longrightarrow B = A^c \cup B$ (material implication), then

$$P(A \Longrightarrow B) = P(A^c \cup B) = P(A^c \cup (A \cap B))$$

$$= P(A^c) + P(A \cap B) = P(A^c) + P(B|A)P(A) = P(A^c) + P(B|A)[1 - P(A^c)]$$

$$= P(B|A) + P(A^c)[1 - P(B|A)] = P(B|A) + P(A^c)P(B^c|A) \geq P(B|A)$$

with equality holding if and only if $P(B^c \cap A) = 0$ or $P(A) = 1$, a rather trivial case. It is known that there is no operation $\bigtriangledown$ on the $\sigma-$ field $\mathscr{A}$ such that $P(A \bigtriangledown B) = P(B|A)$ (see Goodman, Nguyen and Walker [19]). Thus, the answer is no, even for crisp events let alone for fuzzy events (a fuzzy event is a fuzzy subset of Ω whose membership function is measurable). Therefore, if we insist on $P(A|B)$ as the degree for $A \implies B$ to be true, we have to represent mathematically the rule $A \implies B$ differently. Since $A \implies B \notin \mathscr{A}$, it could be an object lying outside of $\mathscr{A}$ for the equation $P(A \implies B) = P(B|A)$ to hold. This is similar to complex numbers. For the solution to this problem, see [19]. See also [20–23].

Fuzzy rules and their fusion are very important for reasoning in intelligent systems. Each rule reflects common sense knowledge, and will be used (see later) to provide models for knowledge acquisition. Just like the case of default reasoning in computer science, rules could have exceptions (for reasoning with such rules, see [24]).

We turn now to the question: how can fuzzy technology help statistics? In a sense, we will point out some *significant contributions* of fuzzy theory, outside the engineering fields, to statistics in particular, and to decision theory in general.

Regression analysis is the main tool of statistics for investigating relationships between economic variables. As statistical theory, based upon probability theory, seems to leave no stone unturned in its path, it addresses also the important situations where variables (response variables, regressors and covariates) could be latent or qualitative. However, in this context, there is one stone unturned. It is the case where regressors are *seemingly unobservable* and need to be "estimated" to run the regression. A typical situation is the study of the effect of underground economy (u.e.) on national economy, recently investigated by [25], and [26]. Let Y denote the GDP of a country and X denote the size of the u.e. of that country. A simple linear regression model is $Y = aX + b + \varepsilon$. While Y is observable (say, yearly), X is not. It is not realistic to make further assumptions to proceed as in standard practice! To run such a regression, we need to "create" a time series of X. How? It is precisely here that fuzzy technology could help! But before elaborating on this possibility, let's pause and says few words about this interesting demand. Although X is unobservable, we might be able to identify some main *causes* of it. Then we need an ingredient to infer X, *in some fashion*, from these observable causes. This sounds somewhat like we are in the context of *causal inference?* As emphasized by [27]: "*Causal inference requires two additional ingredients: a scientific language for articulating causal knowledge, and a mathematical machinery for processing that knowledge, combining it with data and drawing new causal conclusions about the phenomenon*". While our problem here is not about causal inference, since we assume that some causes of X are identified, and we proceed to "estimate" X from them. However, fuzzy theory seems to provide the two additional ingredients mentioned by Pearl: the "scientific language for articulating knowledge" is fuzzy logic in the form of fuzzy "if...then..." rules, and "a mathematical machinery for processing knowledge" is the compositional rule of inference.

Remark. When considering causality, we face two problems: finding the causes of an "effect", and studying the effects of causes of an effect. See Holland [28] for an excellent paper on causal inference from a statistical viewpoint. The above fuzzy procedure offers a more realistic way for assessing the effects of given causes on an "effect" variable, even for the case where the *effect variable is not observable.* It does so by using (linguistic) coarsening schemes, resulting in assessing (rather than "measuring") causal effects from a common sense knowledge, without imposing untested statistical assumptions. This is a very important issue to explore for future research and applications, where fuzzy technology could provide a realistic alternative to statistical approach, somewhat in line with Pearl's view. It should be remembered that when we offer an *alternative* (to existing approaches) we need to explain why another alternative (e.g., what are its advantages?). In view of the state-of-the-art of causal inference, an alternative to carry out causal inference by fuzzy theory should be welcome and well-justified. Indeed, as [29] put it "*I would advise against regarding any one approach or blending as a complete solution or algorithm for problems of causal inference; the area remains one rich with open problems and opportunities for innovation*".

In the specific case of u.e., causes of u.e. could be "taxation", "unemployment rate" and "corruption index". A typical fuzzy rule is of the form "If taxation is high, unemployment rate is low, and corruption index is medium, then the size of u.e. is medium". Given a fuzzy rule base (consisting of a finite set of fuzzy rules), the compositional rule of inference allows the derivation, from observed causes, the "estimated" size of u.e., expressed as fuzzy sets. With such "fuzzy estimates" of the regressor "size of u.e.", we then face a linear regression model $Y = aX + b + \varepsilon$ with fuzzy data (X_i, Y_i), $i = 1, 2, ..., n$. Note that (statistical) *regression with fuzzy data* is different than conventional "fuzzy regression" in engineering literature, in which the model is $Y = aX + b$ with coefficients a, b being fuzzy sets, X non fuzzy, resulting in fuzzy output Y.

In order to carry out statistical regression with fuzzy data, we need to treat fuzzy data as realizations of bona fide random elements in probability theory. These will be called *random fuzzy sets*, and will be discussed in the next section.

5 Incorporating Fuzzy Data into Statistics

In order to view fuzzy data as bona fide statistical observations, it is necessary to define the formal concept of a random element whose values are fuzzy sets. The standard process consists of specifying a measurable space $(U, \mathscr{U})$ where U is the set of "oucomes" of the random element X that we wish to define, and $\mathscr{U}$ is an appropriate $\sigma-$field of subsets of U, representing events. Also, as usual, the $\sigma-$field $\mathscr{U}$ is Borel, i.e., constructed from some topology on U. Thus, the main task is to topologize U. For concreteness, we take U to be the set $\mathscr{F}$ of *closed fuzzy subsets* of $\mathbb{R}^d$, i.e., fuzzy subsets whose membership functions $f : \mathbb{R}^d \rightarrow [0, 1]$, are upper semi continuous, i.e., their level sets $A_\alpha(f) = \{x \in \mathbb{R}^d : f(x) \geq \alpha\}$, $\alpha \in [0, 1]$, are closed subsets of $\mathbb{R}^d$. It is not obvious how to come up with some metric in order to topologize the space $\mathscr{F}$. Fortunately, we are standing on the shoulders of giants (!): the theory of *continuous lattices* [30] offers the topology we need here. Note that, continuous lattices, or "domains" are known also in computer science, see e.g., [31].

Topologies generated by (partial) order relations in general spaces are well-known. What is interesting is that depending upon how we choose an order relation, we can obtain an interesting topology. Now, fuzzy sets not only represent the semantic meaning of fuzzy concepts, they are used also as "coarse data", exhibiting a form of localization information (just like crisp sets), in the sense that the fuzzy set A is "more informative" than a fuzzy set B if A is contained in B, in symbol $A \subseteq B$ i.e., $A(u) \leq B(u)$ for all $u \in U$. Thus, an (partial) order relation on $\mathscr{F}$ could be taken as $A \gtrsim B$ (A is "greater" than B) if $A \subseteq B$. In other words, we take as an order relation on $\mathscr{F}$ the reverse order of set inclusion. It is this order $\gtrsim$ which makes $\mathscr{F}$ a continuous lattice. Indeed, first, the poset $(\mathscr{F}, \gtrsim)$ is a complete lattice. Moreover, it is a continuous lattice, i.e., for every $F \in \mathscr{F}$, we have $F = \vee\{G \in \mathscr{F} : G \Supset F\}$ where the finer relation $\Supset$ ("much less informative than") or "way below" is defined as follows. $G \Supset F$ if for every collection D of elements of $\mathscr{F}$ for which $F \gtrsim \vee D$, there is a $A \in D$ such that $G \gtrsim A$.

On a continuous lattice $(\mathscr{F}, \succcurlyeq, \Supset)$, there is a canonical topology, called the *Lawson topology*, which is the topology τ with a subbase consisting of sets $\{G \in \mathscr{F} : G \Supset F\}$ and $\{G \in \mathscr{F} : G \not\gtrsim F\}$ for all $F \in \mathscr{F}$, i.e., open sets are taken as arbitrary unions of finite intersections of these sets. Note that, the Lawson topology on the set of closed (crisp) subsets coincides with the Matheron (1975) hit-or-miss topology.

Let $\mathscr{B}(\tau)$ be the Borel $\sigma-$ field generated by the Lawson topology τ on $\mathscr{F}$. Let $(\Omega, \mathscr{A}, P)$ be a probability space. Then by a *random fuzzy (closed) set*, we mean a map $X : \Omega \to \mathscr{F}$, $\mathscr{A} - \mathscr{B}(\tau)-$ measurable, i.e., $X^{-1}[\mathscr{B}(\tau)] \subseteq \mathscr{A}$.

Remark. Our approach here is to place fuzzy sets as values of random elements within probability theory, as opposed to "the fuzzy approach to statistical analysis", see, e.g., [32].

In the special case of *random (crisp) closed sets* [13], the counterpart of Lebesgue-Stieltjes theorem is the Choquet theorem characterizing probability measures by capacity functionals. Specifically, let $\mathscr{F}, \mathscr{K}$ denote the classes of closed and compact subsets of $\mathbb{R}^d$, respectively, and define $T : \mathscr{K} \to [0, 1]$ by

$$T(K) = P(F \in \mathscr{F} : F \cap K \neq \varnothing\}$$

then T satisfies the following axioms

(1) $T(\varnothing) = 0$

(2) T is alternating of infinite order, i.e., for any $n \geq 2$ and $K_1, K_2, ..., K_n$ in $\mathscr{K}$,

$$T(\cap_{i=1}^n K_i) \leq \sum_{\varnothing \neq I \subseteq \{1,2,...,n\}} T(\cup_{i \in I} K_i)$$

(3) If $K_n \searrow K$ in $\mathscr{K}$, then $T(K_n) \searrow T(K)$

Any function $T : \mathscr{K} \to [0, 1]$ satisfying the above three axioms is called a *capacity functional.* Capacity functionals play the role of distribution functions of random variables. The collection of closed sets $\mathscr{F}_K = \{F \in \mathscr{F} : F \cap K \neq \varnothing\}$ plays the role of intervals $(-\infty, y]$ on the real line, in the determination of the distribution function of a real-valued random variable Y: $F_Y(y) = P(Y \leq y) = P_Y((-\infty, y])$.

Like Lebesgue-Stieltjes theorem, the following result simplifies the search for probability laws governing random evolution of random sets.

Choquet Theorem. If $T : \mathscr{K} \to [0, 1]$ is a capacity functional, then there exists a unique probability P on $\mathscr{B}(\mathscr{F})$ such that $P(\mathscr{F}_K) = T(K)$ for all $K \in \mathscr{K}$.

For more details on the above, see Nguyen and Tran [33]. For the extension of Choquet theorem to random fuzzy sets, see [34]. Finally, for representing fuzzy information on a computer system, see, e.g., [35].

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On Z-numbers and the Machine-Mind for Natural Language Comprehension

Romi Banerjee and Sankar K. Pal

Abstract This article is centred on two themes. The first is the extension of Zadeh's basic Z-numbers into a tool for level-2 Computing With Words (CWW) and consequently subjective natural language understanding. We describe an algorithm and new operators (leading to complex or spectral Z-numbers), use them to simulate differential diagnosis, and highlight the inherent strengths and challenges of the Z-numbers. The second theme deals with the design of a, Minsky's Society of Mind based, natural language comprehending machine-mind architecture. We enumerate its macro-components (function modules and memory units) and illustrate its working mechanism through simulation of metaphor understanding; validating system outputs against human-comprehension responses. The framework uses the aforementioned new Z-number paradigm to precisiate knowledge-frames. The completeness of the conceptualized architecture is analyzed through its coverage of mind-layers (Minsky) and cerebral cortex regions. The research described here draws from multiple disciplines and seeks to contribute to the cognitive-systems design initiatives for man-machine symbiosis.

Keywords Artificial general intelligence (AGI) · Cognitive machines · Emotion machine · Machine subjectivity · Machine understanding · Man-machine symbiosis · Qualia · Society of mind · Thinking machines

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_22

1 Introduction

"Language is, at its core, a system that is both digital and infinite. To my knowledge, there is no other biological system with these properties…" - [1]

Languages are built from finite sets of primitive symbols or alphabets, which are combined into complex elements like words, phrases, clauses, sentences, etc., to express thoughts and the world in general. Natural language understanding begins with decoding the context-sensitive perceptions of words, followed by incremental computations on these interpretations across levels of language units (phrases, sentences, etc.). A sentient machine capable of natural language comprehension should typically possess high Machine Intelligence Quotient (MIQ) [2], and can be envisioned as an intelligent human aid.

Coined by Zadeh in 1996, Computing With Words (CWW) [3] is inspired by the cognitive process of comprehension. It aspires to capacitate a machine to 'learn', 'think' and 'respond' to words akin to human beings. CWW heralds a paradigm shift from definite, numeric processing to imprecise word-sense computations. Realization of the paradigm requires the machine to learn words – both existing and new, apply them to form semantically, and ideally syntactically, correct natural language statements. Meanings of words, thus, require to be translated into some symbolic form. CWW is founded on the concepts of fuzzy logic [2–4], fuzzy linguistics [4], test score semantics [5] and Precisiated Natural Language (PNL) [6], and is the precursor to Computing With Perceptions (CTP) [7].

The Z-number [8], proposed by Zadeh, is a fairly new addition to the CTP family. Besides unifying fundamentals of CWW, it incorporates an indicator of the information-reliability of a statement. Z-numbers, thus, can precisiate the information (factual and subjective) in a 'statement'. This article describes our efforts – through an algorithm and perception-operators (forming complex or spectral Z-valuations) – towards extending it to CWW in 'sentences'. Our defined methods are used to simulate a real-life differential diagnosis scenario. The section on Z-numbers ends with a study of its strengths and practical challenges. The issue that interests us profoundly is the endogenous instantiation of machine-subjectivity towards the formation of bespoke elements of comprehension (will we ever be able to emulate a naturally functioning brain?).

"The world isn't just the way it is. It is how we understand it, no? And in understanding something, we bring something to it, no?" – Yann Martel in *Life of Pi*, 2001

Encouraged by the capabilities of the Z-numbers, we direct our efforts to the design of machine-mind architecture – a framework of function modules and memory constructs that realize the machine-self, emulate thinking and the autogenous arousal of affects and qualia, for natural language comprehension. The framework is based on Minsky's Society of Mind [9] and Emotion Machine [10] theories of 'productive' (intuitive, commonsense-based) and 'reproductive' (learned, deliberative, reflective, self-reflective, self-conscious) thinking. It utilizes the Z-numbers to represent knowledge-frames, facilitating constituent perception computations.

The role of the cerebral cortex regions in language comprehension [11, 12] serves as our reference of design 'completeness'.

Besides Zadeh and Minsky's ideas, our work draws from key natural language understander design efforts spanning over the last four decades [13–19]. Turing's landmark paper [20] and Vannevar Bush's phenomenal article [21] are primary stimuli behind the research.

In addition to the expansion of the basic Z-numbers, the novelty in our work lies in using Minsky's theories of the mind to design a framework for cognitive language understanding. The system aims to formulate procedures of comprehension customized to the problem being processed, learn from mistakes and improvise as well. While 'existing' language understanders [19, 22–24], either do not 'reflect', are not 'self-reflective' or 'self-conscious', or do not possess intuition and commonsense, our framework conceptually includes each of these elements.

The design is currently in its very early stages and is subject to evolution with recurrent knowledge gain on the brain-processes.

The article is divided into the following sections: Sect. 2 – presents an overview of the fundamental theories (CWW, Z-numbers and the Society of Mind) underlying our work; Sect. 3 – describes our work (extension of the Z-numbers and enumeration of the computational-mind architecture); and Sect. 4 – summarizes the key elements of this article.

2 Fundamentals

This section is dedicated to a brief discussion of the theories underlying the synthesis of intelligent or thinking machines. The discussion begins with an outline of the Computing With Words (CWW) paradigm, followed by highlights on the Z-numbers and design principles of a machine-mind framework.

2.1 CWW

> *"In the coming years, computing with words is likely to evolve into a basic methodology in its own right with wide-ranging ramifications on both the basic and the applied levels."* - [3]

The Computing With Words (CWW) [3, 25] paradigm deals with processing rhetoric perceptions encoded in the building blocks (words and phrases) of natural language expressions. CWW is imperative when:

i. *Do not know rationale* – The available information is imprecise and cannot be represented as numbers. E.g., a person being called 'young' when the exact age is unknown but is perceived to be within a certain age-group.

ii. *Do not need rationale* – The tolerance to imprecision can be exploited to devise "tractable, robust and low cost solutions". E.g., putting up a picture on the wall based on the directions of another person.
iii. *Cannot solve rationale* – The problems cannot be solved via numerical computing processes. E.g., automation of driving in traffic.
iv. *Cannot define rationale* – Words express more than numbers. E.g., a patient describing his illness.

The foundations of CWW lie in fuzzy logic [2–4], fuzzy linguistics and information granulation [4], test-score semantics for mapping natural language statements to degrees of constraint satisfaction [5], precisiation of natural language [6], and the computational theory of perceptions [7]. CWW can be divided into levels [26] – 'level-1' being the quantification of perceptions of adjectives and adverbs (words and phrases), while 'level-2' focuses on the precisiation of entire natural language statements. Refer to [27, 28] for a discussion on the generic architecture and algorithm for CWW.

Since its coinage, the last decade has been a witness to phenomenal research on CWW – particularly 'level-1' CWW. Some significant endeavours are: an effort towards formalization of fuzzy number arithmetic [29]; application of mass assignment theory on fuzzy sets to realize semantic interpretations of membership functions [30]; design-concepts of fuzzy Finite State Machine (FSM) that generate linguistic descriptions of complex phenomenon [31], and simulate emotions [32]; studies on the use of Interval Type-2 Fuzzy Sets (IT2-FS) to represent levels of ambiguities in word perceptions [33]; amalgamation of concepts of fuzzy sets and ontology [34]; formalization of the Generalized Constraint Language (GCL) into a toolkit for CWW [35, 36].

CWW can be intuitively envisioned as capable of supporting the emulation of "subjective-comprehension" of natural language; thereby contributing to research initiatives in Man-machine symbiosis [37], Artificial General Intelligence (AGI) [38] and the Intelligent Systems Revolution [20, 39]. The primary challenges in the construction of a system capable of CWW are:

i. Representation of the richness and inherent ambiguity of natural languages – perceptions (meanings and qualia) of words and phrases:
    a. Accommodate polysemes, homonyms, capitonyms, synonyms, metaphors and other figures of speech.
    b. Provide for simple, complex, compound, declarative, interrogative, imperative, exclamatory and conditional sentences.
    c. Model changes in perceptions and encode reasons thereof – emulate procedural (*predestined*) learning.
    d. Simulate endogenous arousal of subjective associations to events.
    e. Synthesize an evolving context-sensitive rule-base and system lexicon to guide comprehension computations.
ii. Encoding real-world knowledge and 'common-sense'.
iii. Construction of concept granules underlying natural language expressions:

a. Translate natural language inputs to some precisiated form for machine-processing.
b. Identify contexts and test subsequent sentence relevance [40].
c. Design rules of computations for word and sentence perceptions; integration of inter-sentence perceptions towards construction of concept granules.
d. Comprehend semantics despite syntactical errors or incomplete sentences.
e. Encode system inputs and results of computation into machine mentalese [41] and human-understandable forms, respectively.

iv. Processing time requirements to be of the order of average human cognition [~150–300 ms].
v. Natural language comprehension involves processing multi-modal sensory affects (facial expressions, voice intonations, eye-gaze and brightness, response times, etc.). Such systems, thus, require components that can accept, interpret and process these non-verbal or non-textual elements [this has been conceptually handled in our machine-mind architecture discussed in Sect. 3.2], resulting in the integration of CWW, Natural Language Processing (NLP) and affective computing processes.

2.2 Z-Numbers

> *"You cannot see what I see because you see what you see. You cannot know what I know because you know what you know. What I see and what I know cannot be added to what you see and what you know because they are not of the same kind. Neither can it replace what you see and what you know, because that would be to replace you yourself."* – Douglas Adams in *Mostly Harmless*, 1992.

Voluntary actions are the result of decision-making processes, and decisions depend on information. Thus, greater the reliability of the information, stronger is the decision made. The Z-number [8] philosophy aims at encoding the reliability or the confidence in the information conveyed by natural language statements.
The Z-number draws on the concepts in [2–7, 25, 42]. The novelty of the Z-numbers lies in the fact that it not only considers perceptions of individual words, but also the perception of an entire statement. It is a manifestation of level-2 CWW. Consequently, if it were possible to simulate the endogenous arousal of beliefs on information, the Z-numbers would prove to be an effective means of representations of machine-subjectivity.

Given a natural language statement, Y, the 'Z-number' of Y is a 2-tuple $Z = <A, B>$, where A is the restriction (constraint) on the values of X (a real-valued uncertain variable, interpreted as the subject of Y) and B is a measure of the reliability (certainty) of A. Typically, A and B are expressed as words or clauses, and are both fuzzy numbers. Some examples of Z-numbers are:

i. Y_1 = This book is absolutely excellent.
   Therefore, *X* = Quality of the book, and *Z* = <excellent, absolutely>.
ii. Y_2 = It takes me about half an hour to reach point A.
   Therefore, *X* = Time to reach point A, and *Z* = <about half an hour, usually>.

Understandably, *A* is context-dependent while *B* summarizes the certainty or belief in the applicability of *A* given *X* within the purview of the designated context. The value of *B* could be explicitly quoted in the statement (as in example (i)) or it could be implicit (as in example (ii)).

The ordered 3-tuple <*X*, *A*, *B*> is referred to as a 'Z-valuation'. A Z-valuation is equivalent to an assignment statement '*X* is <A, B>'. As for example:

i. The Z-valuation of Y_1 is <Quality of the book, excellent, absolutely>.
   Implication: [Quality of the book] is <excellent, absolutely>.
ii. The Z-valuation of Y_2 is <Time to reach point A, about half an hour, usually>.
   Implication: [Time to reach point A] is <about half an hour, usually>.

A collection of Z-valuations is referred to as 'Z-information' and is the stimulus to a decision-making process.

Preliminary rules of Z-number computations [8] are:

i. For the purpose of computation, the values of *A* and *B* need to be precisiated through association with membership functions, μ_A, μ_B respectively.
ii. *X* and *A* together define a random event in *R*, and the probability of this event, *p*, may be expressed as:

$$p = \int_R \mu_A(u) p_X(u) du \tag{1}$$

   where, *u* is a real-valued generic value of *X* and p_X is the underlying (hidden) probability density of *X*.
iii. The Z-valuation <*X*, *A*, *B*> is viewed as a generalized constraint on *X*, and is defined by:

$$\begin{aligned} &\text{Probability } (X \text{ is } A) \text{ is } B \\ &\text{or, } p = \int_R \mu_A(u) p_X(u) du \text{ is } B \end{aligned} \tag{2}$$

   (2) is mathematically equivalent to the expression:

$$\begin{aligned} &p = \mu_B\left(\int_R \mu_A(u) p_X(u) du\right) \\ &\text{subject to, } \int_R p_X(u) du = 1 \end{aligned} \tag{3}$$

iv. Computation using the Z-numbers is based on the 'Principle of Extension'.

   For example, considering a problem statement of the form:

   "It is probable that Mr. Smith is old. What is the probability that he is not?"

   Let, *X* = Mr. Smith's age, *A* = old, *B* = probable, *C* = not old, *D* = degree of certainty;

μ_A, μ_B, μ_C, μ_D are the membership functions associated with A, B, C and D respectively;

p_X is the underlying (hidden) probability density of X; u is a real-valued generic value of X.

We therefore have: Probability (X is A) is B; and

We need to evaluate: Probability (X is C) is *? D*.

Thus, using the Principle of Extension and expressions (1), (2) and (3),

$$\frac{<X,A,B>}{<X,C,?D>} = \frac{\text{Probability}(X \text{ is } A) \text{ is } B}{\text{Probability}(X \text{ is } C)\text{is}?D} = \frac{\mu_B(\int_R \mu_A\ (u)p_X(u)du)}{\left(\int_R \mu_C\ (u)p_X(u)du\right)\ is?\ D}$$

Implying,

$$\begin{aligned} &\mu_D(w) = sup_{p_X}(\mu_B(\int_R \mu_A(u)p_X(u)du)) \\ &\text{subject to, } w = \int_R \mu_C(u)p_X(u)du \text{ and } \int_R p_X(u)du = 1 \end{aligned} \tag{4}$$

Refer to [43] for a comprehensive discussion on the work done on Z-numbers. While most of these initiatives concentrate on the definition of Z-calculi, our focus is on its use in the embodiment of semantic-sense comprehension of natural language sentences. Major challenges in the implementation of the Z-numbers lie in the lack of: (a) Representation of meanings of natural language elements in some form of machine-language, and (b) the emulation of endogenous arousal of certainty values and subsequent metacognition. These issues call for synergistic research across multiple disciplines: philosophy, psychology, neuroscience and computer science towards understanding the neural processes of comprehension in human beings and defining their computational equivalents.

The capabilities of the methodologies elucidated in the preceding sections, encourage machine-mind synthesis – the design of a machine-mind framework that is self-evolving and self-organizing, autogenously attaches subjectivity to comprehension of the world and possesses common-sense. The following section is dedicated to a discussion of the essential properties of such a cognitive architecture.

2.3 Machine-Mind

"You end up with a tremendous respect for a human being if you're a roboticist" – Joseph Engelberger, 1985

The human brain is a continually evolving, self-organizing computing system that acquires, constructs, stores and processes symbols. A cognitive system, analogous to the human brain, therefore, must think, improve by learning, adapt to the environment, and find structure in ever-growing amounts of real-world data. Such systems require analyzing problems from multiple perspectives and ascertain viewpoints that are in harmony with the context, identify objectives, weigh multiple solution strategies and activate scheme(s) that predictably lead the system to some goal state in reasonable time. These solution schemes include commonsense reasoning [44–48] and improvisation. Cognitive systems aim at man-machine symbiosis [37], e.g., intelligent sentient aids for the unwell and elderly, library cataloguers, supports for children with learning disorders.

The Cognitive Machine or 'Thinking Machines' were pioneered by Turing. In [49], he describes the design of random, self-organized, self-evolving structures for the construction of intelligent machines. The pleasure-pain system outlined here is perhaps the earliest work on 'understanders' built on the philosophy of CWW [3, 25]. The Turing test described in [20] yet stands as a test of machine think-ability. The last four decades or so has witnessed sporadic work in this area, with major milestones [19] being achieved over the last decade. [Refer to [50] for a broad comparison between some well known computational-mind theories.]

Our efforts in the thinking-systems initiative, is based on Minsky's phenomenal compilation on the Society of Mind [9] and Emotion Machine [10] theories. These theories are ultimate culminations of a computational theory of the human mind, consequential catalysts for 'thinking' on 'thinking', and are yet to be entirely realized. While the 'Society of Mind' has been widely used [22–24, 51–54], the 'Emotion Machine' has seen sparse implementation initiatives [55–57].

Drawing from [58–63], the design prerogatives of a cognitive machine are:

i. Possess a finite alphabet set, which are used to form complex components like words, sentences, etc.
ii. Have a finite, substantial memory unit that can store a large number of independently variable symbols.
    a. These symbols assume values from elements in the alphabet set; values represent data and instructions.
    b. These values can be generated, stored, searched for, manipulated upon and deleted. The system thus should include a large and adaptive repository of information or knowledge.
    c. Symbols interpreted as instructions control system behaviour (internal, external, self-modification, intuition, etc.).
    d. Symbols that represent information flowing into the system through sensors and other input devices represent beliefs about the world.
iii. System-knowledge to be inclusive of commonsense [45] and acquired run-time concepts. Knowledge-handling mechanisms require strategies that can realize cross-contextual associations [21].
iv. An adaptive system is reflective [64] or history-sensitive [58] and self-conscious [65, 66]. It incorporates structures that represent the self, and questions its own actions towards robustness and fault-tolerance improvement.

These necessitate maintaining performance statistics for debugging [67], stepping and tracing facilities, interfacing with the external world, predicting future computations, self-optimisation, self-modification and self-activation.

v. Data structures to represent the complexity, variety, unpredictability and degrees of familiarity of an environment.

vi. Emulate neurogenesis [68] by being part of a social system – acquire new forms of knowledge (e.g. new concepts and language skills), and adapt to changing goals, principles, ideals, preferences, likes, dislikes. This demands motive-comparators ('critics and selectors' [10, 69, 70],) to trade-off between competing alternatives, analyze long-term or short term objectives, ignore or suppress some motives or needs in the light of others and form new goals.

vii. The system must be comparable to average human processing [51] – conscious processing (of the order of 100 ms) and unconscious processing (at the speed of neural firing which is 40–1000 times per second).

2.3.1 Building Blocks of the Society of Mind Theory

The Society of Mind [9] theory construes the mind as a hierarchical, modular computing mechanism, assembled of the following:

Agents: Building blocks of a computational mind; a component of a cognitive process that is simple enough to 'understand'.
Agency: Societies of *agents* which in unison perform functions more complex than a single agent. The mind is a society of agencies.
K-lines: Agents (analogous to data and control structures in system architectures) that turn on designated sets of agents. Types of K-lines:

Nemes: Agents responsible for the representation of an idea (context) or a state of the mind.
Nomes: Agents that control the manipulation of representations and effect agencies in a predetermined manner.

Frames [71]: Data structures representing the depiction of events and its constituent properties. These structures connect into hierarchical connected graphs of nodes and relations, where 'top-level' frames (or 'frame-headers') depict designated abstractions of a situation, while the 'lower-level' frames have terminal slots (or 'sub-frames') instantiated to event-specific data. Data entry into the terminal slots is directed by assignment conditions like 'name of a person', 'pointer to another sub-frame', 'relation to another sub-frame', etc. Types of frames:

Surface Syntactic Frames: For verb and noun structures, prepositional and word-order indicator conventions.
Surface Semantic Frames: For action-centred meanings of words, qualifiers and relations involving participants, instruments, trajectories and strategies, goals, consequences and side-effects.

Thematic Frames: For scenarios concerned with topics, activities, portraits, setting, outstanding problems and strategies commonly linked to a topic.
Narrative Frames: For scaffoldings of typical stories, explanations, and arguments, conventions about foci, protagonists, plot forms, development, etc.; assists the construction of new, instantiated *thematic frame* in the mind.

Difference-Engines: Problem solvers based on the identification of the dissimilarities between the current state of the mind and some goal state.
Censors: Restrain mental activity that precedes unproductive or dangerous actions.
Suppressors: Suppress unproductive or dangerous actions.
Protospecialists: Highly evolved *agencies* that yield initial behavioural solutions to basic problems like locomotion, defence mechanisms etc. These develop with time.
Types of Learning:

Accumulating: Remember every experience as a separate case.
Unframing: Find a general description for multiple examples.
Transframing: Form an analogy or mapping between two representations.
Reformulation: Find new schemes of representing existing knowledge.
Predestined Learning: Learning that develops under sufficient internal and external constraints such that the goal is assured, like learning a language or learning to walk.
Learning from Attachment Figures: Learning how and when to adopt a particular goal and prioritize it, based on reinforcement of knowledge by 'attachment figures'- acquaintances who profoundly influence our minds. E.g., 'praise' and 'censure' from parents and teachers contribute significantly to goal learning.

2.3.2 Layers of the Mind

'Thinking' is a complex phenomenon entailing the analysis of a given situation across multiple causal perspectives, consideration of valid propositions and solution methods, and to apply or improvise upon them towards appropriate solution(s). This involves recall, manipulation and organization of a vast repertoire of knowledge, and powerful automated reasoning processes. Thinking operates across a diverse array of mental realms [70, 72] some of which are: (a) **Physical:** Where object behaviour is predicted; (b) **Social:** Dealing with inter-personal relationships; and, (c) **Mental:** Reflections upon mistakes, failures and successes.

In [10] Minsky describes the mind as 'thinking' in terms of a 'layered-critic-selector-reflective' [48, 69, 70, 72] framework. The six-layered structure in Fig. 1 illustrates this model. Each of the layers incorporates 'critics' that consider the external world and the internal system states, and activate 'selectors' to initiate 'thinking' on the interpretation strategies. The lower levels of the model handle and represent 'instinctive reactions' to the external world, while the higher levels control the reactions of the lower levels in accordance with the system's model of itself. The basic functions [10, 72] of the layers in the model are:

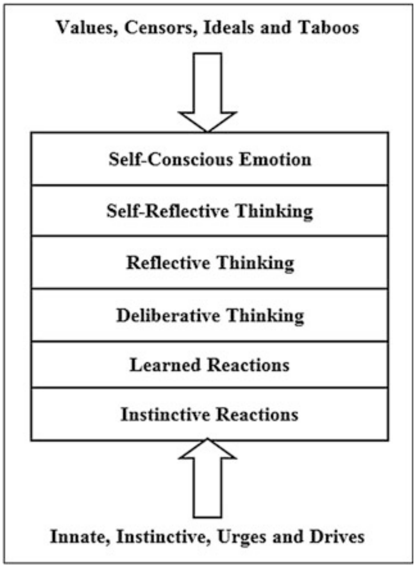


Fig. 1 The layers of the computational mind, as defined by Minsky in [10]

Instinctive or inborn reactions: An implicit database of '***if*** *situation and* ***goal****, then* ***do*** *action*' reaction-rules like: '***if*** *there is a seat and* ***you are tired****, then* ***sit***'; often instrumental in predicting outcomes to situations.

Learned reactions: A database of <*problem_descriptors*, *action*, *result, reason*> tuples ranked in the decreasing order of reinforcement; greater the reinforcement, higher is the probability of the *action* being recalled. E.g: *I am far from something I need immediately → Run towards it; I feel scared → Run quickly to a safe place.*

Deliberative thinking: Consideration of several alternative solution approaches, and choosing the best; logic and commonsense reasoning to select solution paths. E.g: *Action A did not quite achieve my goal → Try harder, or find out why; Action A worked but had adverse effects → Try some variant of that action.*

Reflective thinking: Introspection over mental activities that went into decision-making, rank inference methods, representation selection, etc. E.g.: *The search has become too extensive → Find methods that yield fewer alternatives; Overlooked some critical feature → Revise problem description.*

Self-reflective thinking: While the reflective layer considers only recent thoughts that went into some decision-making, the self-reflective layer focuses on the entity that 'thought'. E.g: *I missed an opportunity by not acting quickly enough → Activate a mental alarm that alerts me whenever I am procrastinating; I can never get this right → Spend time practicing required skills.*

Self-conscious emotion: Verification of agreement of decisions with ideals; self-appraisal. E.g: *I think I am good at this task → Can I do it as well as the best people I know? How is it that other people can solve this problem? → Spend time with those who are good at it.*

With this brief overview of the theories underlying our efforts, the article now progresses to a discussion of the work done. In the following section, we illustrate the very basic of our initiatives towards the design on a Machine-mind or a computational-mind for man-machine symbiosis. The section begins with a discussion on the primary features of the Z-number approach to CWW – a mechanism for processing machine-subjectivity, followed by a presentation of a generic machine-mind framework for natural language comprehension.

3 Proposed Work: Theory and Results

3.1 The Z-Number Approach to CWW

As has been emphasized in almost each of the preceding sections, the emulation of subjectivity is a prime property of a sentient system; the Z-number is our principal tool towards representing and processing machine subjectivity. In this section we present an algorithm for Z-number based CWW, primitive operators for processing Z-valuations or Z-information sets and an analysis of the advantages and associated issues with Z-number realizations.

3.1.1 Z-Number Based Algorithm for CWW [27, 28]

Natural language comprehension, intuitively, follows an incremental-developmental strategy (described and illustrated in Sect. 3.2) – building upon existing knowledge and previous comprehension granules. The following algorithm works through a primitive developmental methodology of understanding complex and compound sentences (*S*) by decomposing them into their simple sentence constituents, evaluating and processing their Z-valuation equivalents, and integrating these results into a comprehension granule of *S*. Section 3.1.3 presents a snapshot of a real-life example of execution of the following algorithm for differential diagnosis.

Input: Natural language sentence (*I*).
Output: Context-dependent response (*O*) to *I*.
Assumptions:

i. The system is capable of identifying irrelevant sentences.
ii. The system grasps the perception of a complex or a compound sentence (*Y*) by –
    a. Extracting the simple sentence components of *Y*.
    b. Individually comprehending each of these simple sentence components.
    c. Integrating these component perceptions.

Steps:

1. If I is irrelevant
   Then
   Goto step 10
   Else
   Goto step 2
2. If I is a simple sentence
   Then
   Goto step 3
   Else
   a. Extract the simple sentence component set (I') of I
   b. Repeat steps 3 through 4 for each sentence in I'
   c. Goto step 5.
3. Extract the values of X, A and B in I to evaluate the Z-valuation (Z_I)
4. Convert Z_I into equivalent mathematical expression (Z_E) (based on Eqs. (3) and (4))
5. Assemble all Z_E to form the logical expression (E) guided by the conjunctions or connectives in I
6. Convert E to the mathematical expression (M)
7. Evaluate M to receive a set of Z-valuations (Z_O) in response
8. Translate Z_O into simple sentences (S)
9. If step 8 results in more than one simple sentence
   Then
   If some or all the sentences in S can be compiled into a single sentence
   Then
   a. Assimilate all compatible simple sentences into a complex/compound sentence (S')
   b. If S' does not include all the sentences in S
   Then
   b.1. $S'' = S - S'$
   b.2. $O = S' \cup S''$
   Else
   $O = S'$
   Else
   $O = S$
   Else
   $O = S$
10. Stop

3.1.2 Primitive Z-Number Based Operators for Perception Manipulation

The operators described here are basic Z-number based operators for perception intersection and union, and process Z-information sets towards comprehension granule formations.

The simulation of differential diagnosis described in Sect. 3.1.3, utilizes the following operators – in conjunction with the algorithm described in Sect. 3.1.1, to arrive at results that coincide with natural intuitive responses of human beings.

Let S_1 and S_2 be two natural language statements, and $Z_1 = <X_1, A_1, B_1>$ and $Z_2 = <X_2, A_2, B_2>$ be the Z-valuations of S_1 and S_2 respectively

- **Operator for the intersection of perceptions** $(\cap_P)$ **[27, 28]**

If S_1 defines a requirement and S_2 describes an event from the perspective of S_1, i.e., $X_1 = X_2 \neq \emptyset$, $A_1 = A_2 \neq \emptyset$, and $B_1 \in \{\text{words that are synonymous to 'expect'}\}$,

The Intersection of Perceptions $(Z_1 \cap_P Z_2)$ is defined as,

$$(Z_1 \cap_P Z_2) = <X_1, A_1, B_2> \tag{5}$$

This expression describes the certainty with which Z_2 complies with Z_1, which in turn is a measure of the certainty with which S_2 is in accordance with S_1.

For example:

i. Requirement: S_1 = I would like my book to be a Miss Marple mystery; Z_1 = <detective, Marple, expected>
   Statement in given mystery book: S_2 = Miss Marple was investigating into whereabouts; Z_2 = <investigator, Marple, appears_to_be> => Z_2 = <detective, Marple, appears_to_be>
   Thus, $(Z_1 \cap_P Z_2)$ = <detective, Marple, appears_to_be> => the book appears to be a Miss Marple mystery.
ii. Requirement: S_1 = I prefer my soup piping hot; Z_1 = <soup temperature, piping hot, expected>
   Situation: S_2 = The soup served is lukewarm; Z_2 = <soup temperature, lukewarm, certainly> => Z_2 = <soup temperature, piping hot, false>
   Thus, $(Z_1 \cap_P Z_2)$ = <soup temperature, piping hot, false> => the soup served is not piping hot.

The perception-intersection operator defined above could predictably come of use in scenarios where it is imperative to verify the confidence with which the current situation satisfies a given requirement.

- **Operator for the union of perceptions** $(\cup_P)$

If S_1 and S_2 depict progressions in perceptions of the same event,

Where, S_1 represents an earlier perception, and S_2 is interpreted from the perspective of S_1 i.e., $A_1 = A_2 \neq \emptyset$,

The Union of Perceptions $(Z_1 \cup_P Z_2)$ is defined as,

$$\begin{aligned} &(Z_1 \cup_P Z_2) = <X, A_1, B> \\ &\text{Where, } X = \text{the event and } B = (B_1, B_2) \\ &\text{i.e., } B = \text{an ordered set of certainty progressions} \end{aligned} \tag{6}$$

This expression leads to complex or a spectral Z-number (drawing from the basic philosophy of 'spectral fuzzy sets' [73]) or a spectral Z-valuation that describes the updated certainty perception of the event. [Details on the spectral Z-valuations are out of the scope of the current article.]

For example:

i. S_1 = Miss Marple made certain inquiries; Z_1 = <detective, Marple, appears_to_be>
   S_2 = Miss Marple solved the mystery; Z_2 = <detective, Marple, certainly>
   Thus, $(Z_1 \cup_P Z_2)$ = <detective, Marple, (appears_to_be, certainly) => I wasn't sure that Miss Marple was the detective, but I now am.
ii. S_1 = It took me about an hour to reach point A yesterday; Z_1 = <time to reach point A yesterday, an hour, probably>
   S_2 = It took me about an hour to reach point today as well; Z_2 = <time to reach point A today, an hour, probably>
   Thus, $(Z_1 \cup_P Z_2)$ = <time to reach point A, an hour, (probably, probably) > => It probably takes me an hour to reach point A.
iii. S_1 = It took me about an hour to reach point A yesterday; Z_1 = <time to reach point A yesterday, an hour, probably>
   S_2 = It took me about half-an-hour to reach point today; Z_2 = <time to reach point A today, half-an-hour, supposed> => Z_2 = <time to reach point A today, hour, false>
   Thus, $(Z_1 \cup_P Z_2)$ = <time to reach point A, an hour, (probably, false)> => I am not very sure as to how long it would take me to reach point A, close to an hour perhaps.

The perception-union operator defined above could be of use in the study of modulations in the confidence in a concept and reasons thereof. Intuitively, lesser the modulation in certainty, greater is the stability of the association between X and A in the current context. It might thus be possible to locate abrupt changes in certainty patterns and identify consequent reasons – a step towards emulation of intuitive (or *predestined*) learning by a machine.

Observations:

i. These primitive operators do not claim to lead to definitive conceptual summaries, but do initiate the formation of instantaneous summaries at time t – with respect to the information experienced till t.
ii. These operators depend on the 'meanings' of the constituent parameters of the Z-numbers.
iii. The interpretation of S_2 from the perspective of S_1, as demonstrated in the examples above, involves loss in information. This, however, is countered by the substantiation of perception summarization.

The use of these operators involves measures of information-loss and summarization-degree, following which the interpretations may be modelled to be inclusive of high-information facts and yet arrive at a reasonable balance between the measures. E.g., in differential diagnosis, it would be imperative to include symptom-specifics in the interpretations (refer to Sect. 3.1.3).

3.1.3 An Example of Z-Number Based Comprehension

The experiment described here, simulates an instance of differential diagnosis. We consider here, a sample discourse (Table 1) between a doctor (D) and patient (P) as the basis of the simulation.

Assumptions:

i. The Machine (Mc) is aware of the probability distribution of the symptoms and symptom details per disease.
ii. Mc is proficient in syntactic analysis, and can resolve anaphoric and cataphoric dependencies.
iii. Mc can categorize symptom descriptions into information granules like 'fever-presence-symptom', 'fever-fall-symptom'.
iv. Mc comprehends 'meanings' of words in natural language expressions.

Execution:

Mc should ideally behave like the doctor (D) in the conversation shown in Table 1.

Table 1 Sample discourse between a patient (P) and a doctor (D)

Natural language statement	Equivalent Z-valuations
D: Please tell me about your problems	Z_1 = <problem, exists, expectedly>
P: It's been two days since I've been suffering from high fever!	Z_{11} = <problem, fever, certainly> => Z_{11} = <problem, exists, certainly> Z_{111} = <fever intensity, high, certainly> Z_{112} = <fever duration, 2 days, probably>
D: Did you note the temperature? Did you notice if it comes at specific times, is accompanied by sensations of nausea?	Z_2 = <fever temperature, above 100, expectedly> Z_3 = <fever arrival time, specific, uncertain> Z_4 = <fever accompaniment, nausea, uncertain>
P: The fever's been ranging at around 102–104° and it's been coming in the mornings and evenings. The temperature falls after a bout of intense sweating and is accompanied by shivering, nausea, and aches all over	Z_{21} = <fever temperature, 102–104, certainly> => Z_{21} = <fever temperature, above 100, certainly> Z_{31} = <fever arrival time, morning and evening, certainly> => Z_{31} = <fever arrival time, morning and evening, certainly> Z_{41} = <fever accompaniment, shivering, certainly> Z_{42} = <fever accompaniment, nausea, certainly> => Z_{42} = <fever accompaniment, nausea, certainly> Z_{43} = <fever accompaniment, aches, certainly> Z_{51} = <fever release, sweating certainly> Z_{511} = <sweating pattern, intense, certainly>

(continued)

Table 1 (continued)

Natural language statement	Equivalent Z-valuations
D: What kind of aches?	$\boldsymbol{Z_6}$ = <aches, specific, expectedly>
P: A blinding headache and stinging muscle-aches as well	$\boldsymbol{Z_{61}}$ = <ache, head, certainly> => $\boldsymbol{Z_{61}}$ = <aches, specific, certainly> $\boldsymbol{Z_{611}}$ = <headache intensity, blinding, certainly> $\boldsymbol{Z_{62}}$ = <ache, muscles, certainly> => $\boldsymbol{Z_{62}}$ = <aches, specific, certainly> $\boldsymbol{Z_{621}}$ = <muscle-ache, stinging, certainly>
D: Do you have any problems in your appetite and sleep patterns?	$\boldsymbol{Z_7}$ = <appetite problem, exists, expectedly> $\boldsymbol{Z_8}$ = <sleep problem, exists, expectedly>
P: I do not have an appetite and sleep's rather disturbed with the headaches	$\boldsymbol{Z_{71}}$ = <appetite problem, appetite absent, certainly> => $\boldsymbol{Z_{71}}$ = <appetite problem, exists, certainly> $\boldsymbol{Z_{81}}$ = <sleep problem, sleep disturbed, certainly> => $\boldsymbol{Z_{81}}$ = <sleep problem, exists, certainly>

Notes:

i. For an expression symbolized by Z_{ijk}, i = the Z_i^{th} enquiry, j = the jth response to Z_i, k = the kth detail of Z_{ij}.

ii. $\cap_P$ is used from the perspective of the doctor on the expression pairs (Z_1, Z_{11}), (Z_2, Z_{21}), (Z_3, Z_{31}), (Z_4, Z_{42}), (Z_6, Z_{61}), (Z_6, Z_{62}), (Z_7, Z_{71}), (Z_8, Z_{81}).

iii. On relevance analysis of the Z-valuations, expressions symbolized Z_{11}, Z_{43}, Z_{61}, Z_{62} are inconsequential or redundant.

iv. The doctor, intuitively, uses $\cup_P$ to unite patient symptom-persistence intensity certainty after a course of medication. For example, after a certain course of medication, from observations in Table 2, we have:

$$(Z_{611} \ \cup_P \ Z_{91}) = <\text{headache intensity, blinding, (certainly, rarely)}>$$
$$=> Z'_{91} = \ <\text{headache, improvement, certainly}>; \text{ and}$$
$$(Z_9 \ \cap_P \ Z'_{91}) = <\text{headache, improvement, certainly}>$$

v. The Z-valuations in the example depict top-level perceptions. Each of the natural language statements incorporate micro-aspects of subjective experiences (e.g. the subtleties that lead to a headache being attributed 'blinding'), which have not been considered.

Table 2 Sample discourse between patient (P) and doctor (D), after a course of medication

Natural Language Statement	Equivalent Z-valuations
D: How is the headache?	$\boldsymbol{Z_9}$ = <headache, improvement, expectedly>
P: There is just a mild throbbing every now and then	$\boldsymbol{Z_{91}}$ = <headache intensity, mild, certainly> => $\boldsymbol{Z_{91}}$ = <headache intensity, blinding, rarely>

Thus, assuming the probability distributions of disease-symptoms is well-defined, after precisiating the memberships of the presence and intensity of symptoms from the Z-information set acquired from the patient, the system computes the rank of the membership of the probable diseases with respect to the expression:

$$E = ((Z_{111} \wedge Z_{112})(Z_{21} \wedge Z_{31} \wedge Z_{41} \wedge Z_{42} \wedge (Z_{51} \wedge Z_{511})) \wedge (Z_{611} \wedge Z_{621}) \wedge (Z_{71} \wedge Z_{81})) \quad (7)$$

$$\text{Or, E} = ((\mu_{certainly}\int_R \mu_{high}(u)P_{feverintensity}(u)du \wedge \mu_{probably}\int_R \mu_{2days}(u)P_{feverduration}(u)du) \wedge$$
$$(\mu_{certainly}\int_R \mu_{above100}(u)P_{fevertemperature}(u)du \wedge \mu_{certainly}\int_R \mu_{morningandevening}(u)P_{feverarivaltime}(u)du) \wedge$$
$$\mu_{certainly}\int_R \mu_{shivering}(u)P_{feveraccompaniment}(u)du \wedge \mu_{certainly}\int_R \mu_{nausea}(u)P_{feveraccompaniment}(u)du \wedge$$
$$(\mu_{certainly}\int_R \mu_{sweating}(u)P_{feverrelease}(u)du \wedge \mu_{certainly}\int_R \mu_{intense}(u)P_{sweatingpattern}(u)du \wedge$$
$$(\mu_{certainly}\int_R \mu_{blinding}(u)P_{headacheintensity}(u)du \wedge \mu_{certainly}\int_R \mu_{stinging}(u)P_{muscleache}(u)du) \wedge$$
$$(\mu_{certainly}\int_R \mu_{exists}(u)P_{appetiteproblem}(u)du \wedge \mu_{certainly}\int_R \mu_{exists}(u)P_{sleepproblem}(u)du)) \tag{8}$$

i.e., the system uses Eq. (8) to evaluate and rank the probability of all diseases in its repertoire, as per Eq. (4)), ideally leading to the Z-valuation expression: (<Disease, malaria, most likely> ^ <Blood test, required, definitely>).

3.1.4 Analysis of the Z-Number Approach to CWW

- **Pros**
  The potential of the Z-numbers is illustrated through the experiment described in Sect. 3.1.3 [refer to [27, 28] for other simulation examples]. Features of interest are:

i. Though originally a model for the precisiation of natural language statements, the Z-number methodology can be envisioned to precisiate any natural language sentence.
ii. A Z-number captures the perception of a single natural language sentence, while the Z-information does the same for a group of sentences.
iii. A Z-number summarizes simple sentences only. Thus, if a complex or a compound sentence (*S*) were decomposed into its simple sentence constituents, assimilating these constituent Z-numbers would lead to the Z-information equivalent of *S*.
iv. Z-numbers support the identification of the context (e.g., 'fever-occurrence', 'fever-presence' etc. with respect to Table 1) of a statement in the universe of discourse.
v. Z-information allows grouping statements into context-sensitive granules of information. This mimics the natural data-compression and subsequent data-comprehension by the human brain. The mechanism, intuitively, should help in knowledge-extraction from a sentence.

vi. Z-numbers represent the inherent uncertainty associated with machine behaviour. Moreover, implicit certainty levels of sentences involve affective computing principles. Z-numbers consequentially integrate affective computing and CWW, thereby adding a new dimension or 'level' to CWW.
vii. Parameters of Z-numbers are context-independent.
viii. Translation from Z-numbers to simple sentences is straightforward.

- **Challenges in the implementation of Z-number based natural language processing**
  There are indeed quite some challenges that need to be overcome before the concept finds emulation on real systems. Some of these issues are:

i. Construction of a self-evolving system lexicon – with words and phrases granulated and self-organized into semantic nets and synonym clusters; these clusters and nets form the value pool for parameter *X*, given a concept.
ii. Identification of an evolutionary fuzzy set model that represents perceptions of words in *A* and *B*.
iii. Identification of the probability distribution of the events defined by the words in *A* – is it practically possible to evaluate the probability distribution for all events in a context, and how can the probability distributions be updated? Could the probability distribution be replaced by better statistical parameters? Encoding of reasons for modulations in probability.
iv. Creation of a dynamic rule-base or an explanatory database – akin to an Answer-Library (described in Sect. 3.2.1).
v. Formulation of well-defined rules of computation – towards a grammar of Z-numbers or perceptions
vi. Identification of the category of the input statement – simple, complex or compound, declarative, exclamatory, interrogative, conditional and imperative.
vii. Decomposition of complex or compound sentences (*S*) into simple sentence constituents, comprehension of individual component perceptions and integration of perceptions – based on the connectives used in *S*.
viii. Evaluation of the relevance [40] of the sentence with respect to the text corpus.
ix. Identification of parameters *X*, *A* and *B* from a sentence. The major challenge lies in extraction of an implicit affect and translating it into parameter *B*.
x. Formulation of the logical expression based on conjunctions linking Z-valuations of the input sentence.
xi. Conversion of the Z-valuation logical expression into an equivalent context-sensitive mathematical expression.
xii. Translation of the components of resultant Z-valuations into simple sentences and integration into concept granules.

xiii. Definition of algorithms for knowledge-extraction – new words and reasons of comprehension successes and failures – resulting in text corpus and rule base updating.
xiv. Formulation of methods that simulate the endogenous arousal of emotions and qualia.

3.2 The Machine-Mind Framework for Natural Language Understanding

In this section we present the macro-components and working principle of a machine-mind framework – built on the philosophy of the Society of Mind – followed by an analysis of its analogy with the human brain. The framework uses the Z-number equivalents of knowledge in the machine mind as the operands for processing. Elaborations on agent-structures and algorithms, and detailed memory data structure formats, are out of the scope of this article.

Here it might be prudent to mention that though the machine-mind constructs have been predominantly studied in their capacities as text-understanders, the principles are applicable to the general concept of natural language understanding.

3.2.1 Macro-Components of the Machine-Mind Architecture: Agencies and Memory Constructs [62]

Comprehension or understanding involves the execution of a number of complex conscious and omniscient unconscious cognitive processes that ideally lead to the following mind-activities:

Prediction – Envisage a future action – causally relate the present to past experiences and judge expectations on the basis of intuition, commonsense, reinforced learning and reflection.
Visualization – Conjure mind-images (real or intentional [74]) of language components depicting people, places, events, etc.
Connection – Build factual or conceptual associations between: (a) all frames recalled and those created for the current language processing event, and (b) real-world or domain knowledge and new information.
Question and **Clarification** – Reflect upon, and test the strength, completeness, correctness and relevance of knowledge associations; re-organization and rectification of associations.

Evaluation – Test coherence between perception granules, measure relevance of each and prune the insignificant; attach notions of subjectivity or 'self consciousness' (emotions, degrees of interest, summarize, biases, etc.).

Intuitively,

i. Comprehension 'iterates [75, 76]' through the above stages – working incrementally on micro-granules of information to form coherent networks of information and a macro-granule summary of the language unit (a text sample, for example) being understood. Figure 3 illustrates this point.
ii. These processes are complex, mostly concurrent, and co-operative.
iii. The 'meaning' of a word or a phrase implies the manner in which the sense of the language unit is encoded in the mind. These encodings could be in the form of precise symbols in the native language of the system or as metaphors, synonyms or associations with other words. A single word or phrase may have multiple sensory (visual, auditory, etc.) implications as well.
iv. *Prediction* and *visualization* involves all but the topmost two layers of the mind; *connection* – the four lower layers; *question* and *clarification* – the learned, deliberative and reflective thinking layers; and *evaluation* involves the top three layers.
These functions straddle multiple layers of thinking and involve bi-directional information percolation. The information that is transferred to the higher layers relies on the extracted language-sample while that from the higher layers is conceptual and relates to the reader's sensibilities acquired through learning, experience and commonsense reasoning.
v. The above list is not an exhaustive enumeration of the broad mechanisms leading to comprehension. We hope to add to it in the process of understanding how the brain 'understands' the real world.

- **Mind-agencies**

A computational mind typically processes, concurrently arriving multi-modal sensory inputs with existing knowledge about the real-world and the problem domain, through the above steps to produce granules of understanding.

We categorize the agencies of such a computational mind into **super-agencies**, each representing a complex cognitive functionality like 'reasoning' or 'processing', and **sub-agencies**. A super-agency comprises of a cluster of sub-agencies that work in health and harmony to achieve the super-agency functionality. The sub-agencies are again built of agents each representing an atomic sub-process of the sub-agency purpose. Figure 2 is a pictorial representation of the mind-agency framework.

The super-agencies and constituent sub-agencies in a computational mind are as follows:

Sensory_Gateway (**SG**): At any instant, **SG** serves as the receiver of sensory information, based on the nature of which, 'sensory' sub-agencies [*Vision* (**V**), *Audition* (**A**), *Olfaction* (**O**), *Tactile* (**Tc**), *Taste* (**Ta**), *Balance* (**B**) [77], *Temperature* (**Te**) [77], *Pain* (**P**) [77] and *Kinesthetic* (**K**) [77]] activate other framework components for further processing. **SG** transports system results to the external world as well.

Sub-agencies like **A**, **O** and **Te** continually receive stimuli from the environment and process these unconsciously; **Tc** and **K** are activated in the 'turning pages', 'scrolling over text' activities. However, none of these contribute significantly to the 'comprehension' phenomenon and have thus not been elaborated upon, in this article.

Vision (**V**): The 'eyes' of the system – leads to textual symbol-extraction, symbol-interpretation (numbers, alphabets, punctuation, etc.), and symbol-granulation into complex language elements (words, sentences, etc.).

Deducer (**De**): The 'brain' of the system; is responsible for the emulation of each of the comprehension processes named above. It receives outputs (data) of **SG** to formulate units (frames) of comprehension – utilizing syntax and semantic analysis mechanisms, relevance-evaluation, affect-extraction, comprehension-evaluation and error-handling processes; sends out instructions (activation, re-evaluation, error signals, inhibition) to the other super-agencies. The sub-agencies of **De** are:

Syntax (**Sy**): Syntax-resolution and consequent generation and manipulation of surface syntactic frames.
Semantic (**Se**): Semantic-resolution, generation and updating of surface semantic, narrative and thematic frames.
Self (**Sf**): Seasons comprehension granules with components of subjectivity (affects, biases, ideals, etc.) based on the system personality; multiple mental-realm activations.
Recall (**Re**): Thin-slices a problem into sub-problems, maps problems to memories and retrieves them from long-term into the working memory for processing in the current context.
Creative (**Cr**): Projects and suggests solutions for 'new' problems; the hub of reflection, imagination, creativity and system IQ [2].
Summary (**Su**): Analyzes the distance between the current state of the system and the projected goal through relevance, affect and comprehension progression evaluation; can activate or inhibit agencies (under **De** and **SG**) based on summary results; consolidation of memories.

Manager (**M**): The global administrator of the system; runs in the background and is responsible for the activation and execution of 'involuntary' functions (system-time management, memory handling, process synchronization, K-line management, frame encoding/decoding, job scheduling, etc.) that support the functioning of all the other agencies; continual self-evaluation of system processes and subsequent updating towards optimal (cost effective and robust) system performance.

- **Memory Constructs**

The *long-term* memory stores of knowledge, that support the functioning of the agencies, are:

Lexicon (**L**): System vocabulary; a resource of language units – words, phrases, idioms – and their meanings encoded in machine 'understandable' form; includes meanings of words 'learnt on the fly' and jargon; meanings may be encoded as precise statements or exist in a number of data types (sounds, images, metaphors) – indicating the different ways the machine 'understands' or 'remembers' an element.
Answer-Library (**AL**): Resource of <*solution_strategy, result, reasons*> for a given <*context_parameters*, *problem*> query.
Concept-Network (**CoN**): Hypergraph of inter-contextual frame associations; elements are incorporated consciously or unconsciously, but are retrieved consciously.
Commonsense-Network (**ComN**): Network of networks of commonsense and intuitive (automatic) behaviours; components are retrieved 'unconsciously'; elements of **L**, **CoN** and **AL** are incorporated into **ComN** after prolonged periods of reinforcement.

The basic *working-memory* data-structures are as follows; these are referenced by all the agencies and support the deliberative and reflective actions of the system:
Log: A global record of time-stamped agency-activity entries; indicates the instantaneous state of the system, analyzing which – a number of agencies may be autogenously or exogenously activated, mechanisms like intelligent backtracking [78] initiated, error signals generated, etc.; serves as an indicator of solution strategy results and reasons thereof for the system to 'reflect' upon.
Frame-Associations (**FA**): A blackboard or scratchpad for frame manipulations during the process of understanding; categorized into *global* and *local* (per sub-agency); all frame recollections are placed in the *global* **FA** space, while sections of the *global* **FA** are transferred into *local* **FA** for deliberations by sub-agencies; sub-agencies under **De** use their *local* **FA** workspace to reason through the appropriateness of multiple solution-perspectives before globally 'advocating (a <*problem*, *solution*, *reason*> tuple)' frame manipulation processes through **Log**; sub-agencies under **M**, use their *local* **FA** to reason through system optimization mechanisms; each sub-agency can share sections or all of its *local* **FA** with other agencies; globally approved suggestions (by **Su**) are implemented in the *global* **FA**, and all updates to existing networks of information are reflected across the long-term memory networks; all local sub-agency trials are annotated in *local* **FA**; trial-results are annotated in **Log** and *global* **FA** for deliberation and reflection by other agencies.

The *system memory-management* constructs, used by **M**, are:
Working-Set (**WS**): Set of pointers to frame-networks in **FA** being referenced within a narrow time-window (intuitively, of the order of seconds).
Active-Frames (**AF**): Set of pointers to frame-networks in **FA** being referenced within a broad time-window (intuitively of the order of minutes); **WS** is a subset of **AF**.
Passive-Frames (**PF**): Set of pointers to frame-networks in **FA** that were members of **AF** but were pruned away due to irrelevance or lack of use; instead of consolidating them back to the long-term memory, these frames remain available during the entire span of the processing of the current text for quick 'on-demand' placement into **FA** for re-processing.

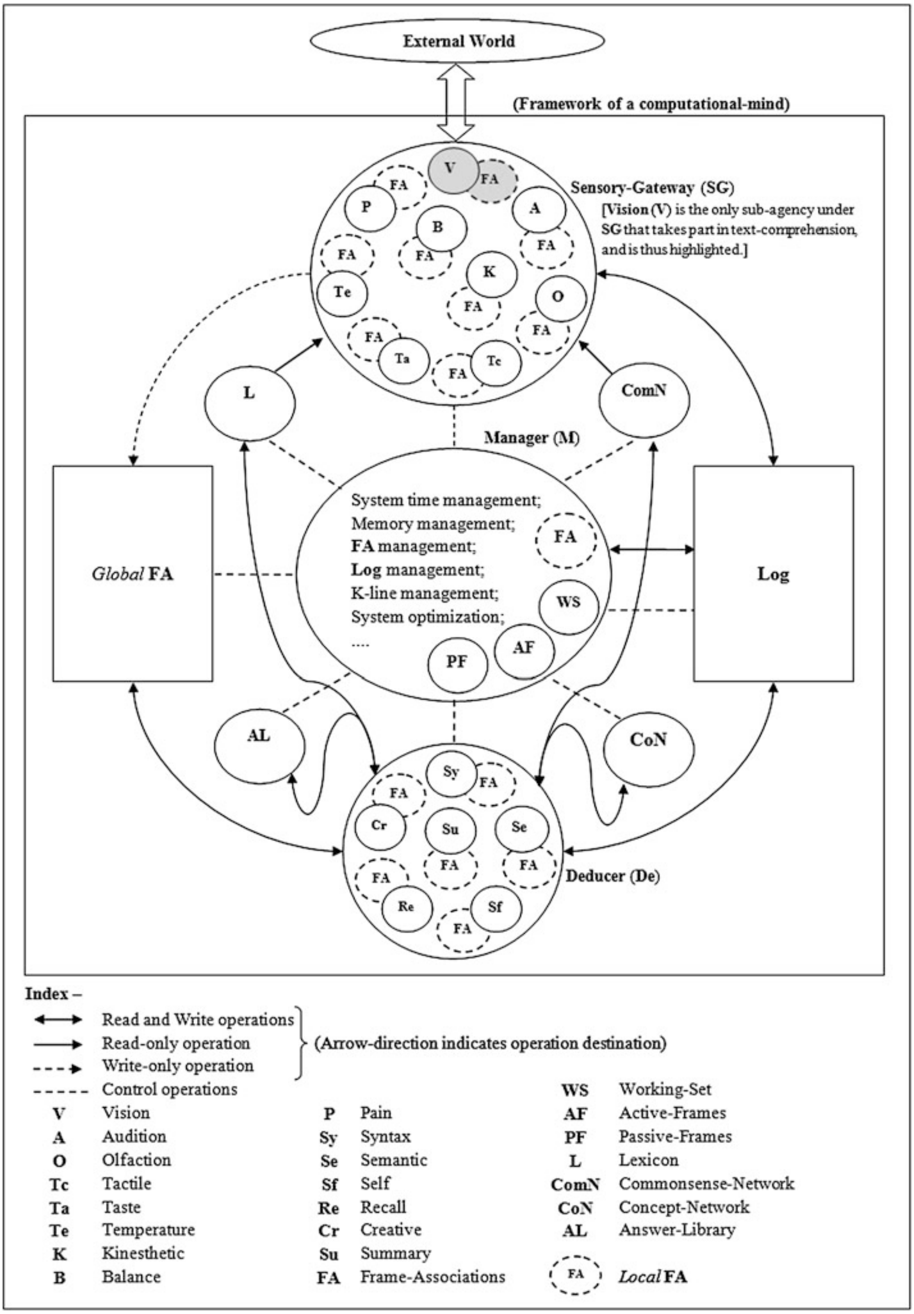
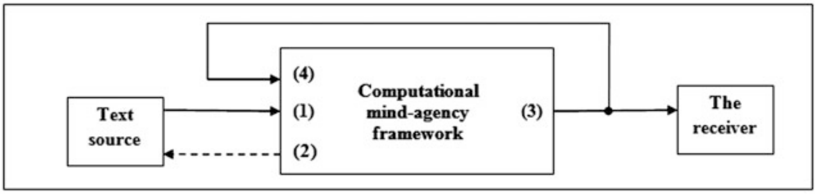


Fig. 2 Macro-components (agencies and memory constructs) of the machine-mind framework [62]



Index –

(1) – Input to the system = the text-elements.

(2) – **Vision** might be instructed, by the **Deducer,** to re-read sections of the current text-source.

(3) – Output of the system = a granule of comprehension = deep-semantics (cross-contextual information associations) + reflections + emotions + new-knowledge + updated old-knowledge.

(4) – Granules of comprehension, produced at time (t_i), are re-introduced into the system along with the current text-inputs at time (t_{i-1}). The elements along the feed-back path influence the current comprehension – set the context of interpretation, facilitate anaphora/cataphora resolution, etc.

At time (t_1), when the input to the system is the first text-element of the current session, values in the feedback path symbolize the state of the computational mind – cognitive biases arising out of comprehension processes handled in the recent past. Ideally, the system is robust and is able to override these exogenous influences during the course of current processing activities.

Fig. 3 The iterative incremental-developmental execution schematic of comprehension [62]

- **Working Mechanism**

Referring to the functionalities of the components defined in the preceding section, the basic working mechanism of the framework (illustrated in Fig. 4), is as follows:

Comprehension is an iterative process, where complex granules of comprehension (inclusive of surface and deep semantics) are incrementally developed from primitive knowledge elements and simpler information granules. Given a sample of natural language to understand, e.g., a text to read, **V** is activated and it makes corresponding **Log** and *global* **FA** entries – indicating the symbols extracted, granulated and interpreted. These interpretations could include annotations like (*author_name*, *text_name*, *title*, *chapter_name*, *starting words, word meanings*, etc.), based on the **L** and **ComN** memories (frames) retrieved. Once initiated, **V** extracts text in saccadic-granules [76] until reading and subsequent comprehension is complete. **De** regulates the length of the saccadic-granules and the location thereof (initiating re-reads on insufficient comprehension or endogenous enquiries, etc.).

All retrievals by **V** are visible, through working-memory entries, for all the other agencies to deliberate upon. The sub-agencies under **De** assess the status (familiarity, syntax, semantics, context, affects, interests, relevance, etc.) of the problem (words, clauses, sentences, paragraphs, frame-systems, etc.) and opportunistically 'suggest' interpretation mechanisms and consequent results. These involve decomposing the problem into sub-problems and incremental-developmental iterations – through long-term to working-memory frame transfers, agency-specific local frame-manipulation trials and broadcasting of predictable success-rendering schemes, signals to improvise or construct new solutions from scratch, alignment of interpretations with self-interests, and information consolidation – towards the formation of a granule of comprehension of the entire text sample. **M** works seamlessly in the background to support agency activities.

Every hypothesis, agency operation, information retrieval, or change in the working-memory is corroborated by a **Log** entry. This allows **Su** to constantly monitor (predict, visualize, question, clarify and evaluate) the convergence of solutions suggested by the sub-agencies, and accordingly activate or inhibit operations (e.g. **Sy** and **Se** might be requested to re-process an incoherent granule). Ideally, an inhibited agency possesses the right to 'question' **Su**'s directions, and thereby all of **Su**'s instructions are annotated with encoded-reasons for evaluation and reflection. In the current version of the system, though no agency can override **Su**'s commands, none of its possible partial processing results are lost. All partially processed frames or inhibited processing vestiges can be retrieved from **PF,** on requirement, for re-analysis. [Refer to [62] for a detailed description of the functions of each of the sub-agencies for comprehending the world and text-understanding in particular.]

An algorithmic or effective procedural view of the working principle necessitates detailed elucidation of the working-memory formats and definition of frame structures of the architecture, time and space complexity analyses, and correctness and completeness verifications. This article clearly focuses on the higher-level elements of the framework. Subtle hints towards parameters and tuples of these macro-constructs have been provided across this article, but we deliberately refrain from discussions on their fine-grained components.

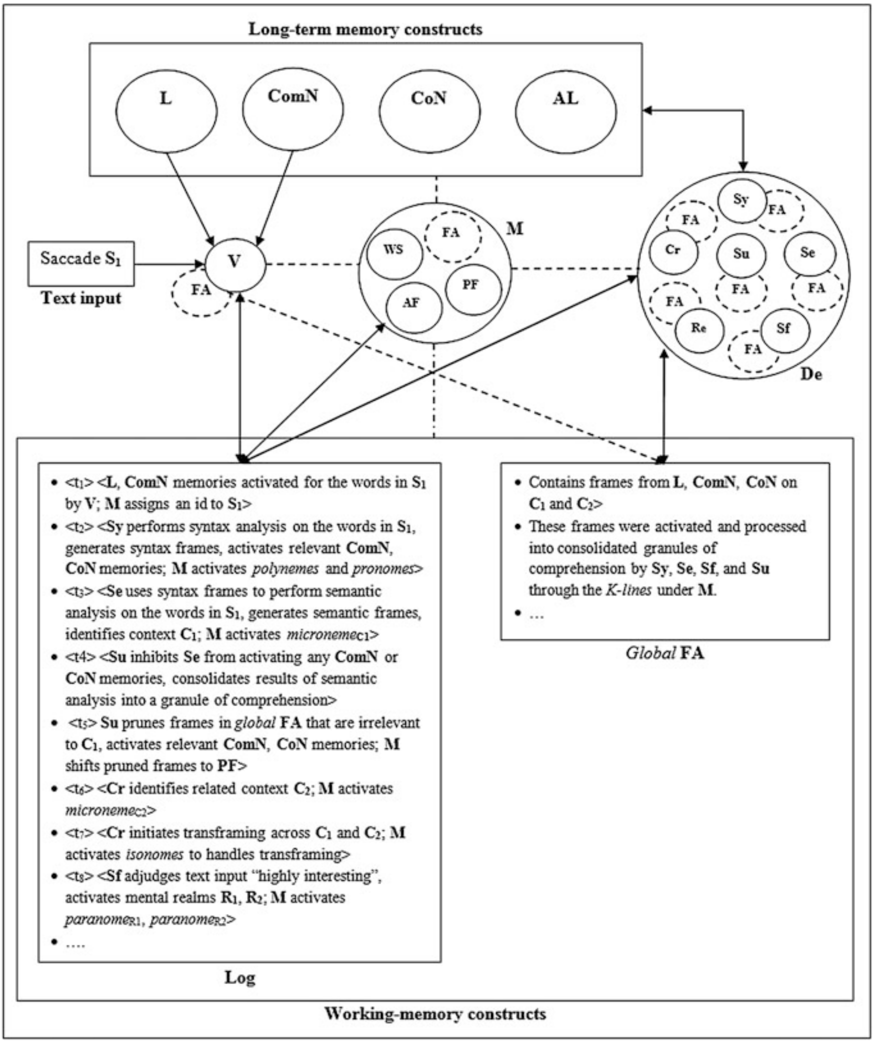


Fig. 4 The working mechanism of the machine-mind framework [62]. [Refer to [9] for definitions of the *nemes* and *nomes* mentioned here]

- **Properties of the Framework**

Essential properties of the machine-mind framework include:

i. The design of the framework is bound to evolve as we learn more about how the human brain functions. The primary advantage that function-designated agencies provides is the ease with which an agency may be modified without affecting the design of the entire framework; introduction of new agencies or framework components would however involve amendments to every level of the design.

ii. The agencies are interconnected such that they form a causal system. This is roughly demonstrated in the feed-back schematic of the system in Fig. 2.
iii. **SG** depicts instinctive and learned behaviour, while all the other agencies transverse all the layers of thinking.
iv. **SG** sub-agencies reference **L** and **ComN**, and **De** references **CoN**, **ComN** and **AL**.
v. **CoN** and **ComN** draws inspiration from ConceptNet [79], while **AL** from Hacker [17].
vi. Information storage and retrieval from each of the long-term knowledge databases involves encoding/decoding processes across frame-types and data-types.
vii. **M** is responsible for arbitrating multiple log-access requests from a number of agencies.
viii. **Re**, **Cr**, and **Su** constitute the *Difference-Engines* of the machine-mind framework. These are built on the functional programming [58] paradigm – where function-modules are combined into bespoke algorithm fitting the current interpretation problem.
ix. **Su** is the control shell of the architecture – coordinating inter-agency activities via heuristics and approximation schemes, to handle combinatorial explosions of thoughts and solution strategies; ensures tractability of the comprehension problem. Acts like the *Censor* and *Suppressor* of the framework.
It annotates solutions with <*problem*, *process*, *result*, *reason*> for storage in **AL**, and annotates memories by <*environment descriptors*, *problem*, *solution*, *result*, *reason*, *affects*, *beliefs*, etc.> for storage in **CoN**.
x. Agencies possess local *critic-selector agents* that gauge the effectiveness of different algorithms to reason and choose the best option. **Su** monitors global appropriateness of solution-strategies.
xi. *Global* **FA** and **Log** resemble the global workspace [51, 80, 81] construct of blackboard architectures [59].
xii. **Log** serves as the basis of inter-agency communication, thereby grossly reducing these costs – any message on **Log** is equivalent to broadcasting it across all the agencies for reflection or deliberation. This instinctively implies the use of standard formats for **Log**-messages for uniform comprehension across the system.

   a. Each agency has at least one *critic-selector agent* dedicated to the analysis of **Log** entries and subsequent agency self-activation.
   b. **Su** through **Log** messages - <*agency*, *operation completed*, *frame-systems handled*, *terminal values before operation*, *terminal values after operation*, *questions in the mind, probable future operations, reasons*> tuples - broadcasts the current status of the interpretation

- The <*probable future operations*> symbolize hypotheses by **Cr**, sub-problems identified by **Re**, or suggestions by **Su**, **Se** and **Sy**.
- The <*questions in the mind*, *probable future operations*> parameters indicate frame-terminals with uncertain, missing slot values or incoherent granules of comprehension, and exogenously or endogenously activate specific sub-agencies.

  These activated agencies, execute innate algorithm trials in their *local* **FA** space, and then through **Log**, 'suggest': (a) strategies towards the resolution of the <*probable future operations*>, or (b) new operations altogether. **Su** analyses this candidate solution space for the effective mix of partial solutions for the problem.

c. Status updates and records of partial-solution in **Log**, allows **Su** to back-track, `deliberate and reflect upon' schemes in case of erroneous or cost-ineffective choices made.

xiii. Operations activated by agencies, depends on the encoding of meanings into frames. The local *critic-selector* analyses of agency-operations, as well as global agency-suggestions are analogous to mentalese [41] in the computational mind. While **Log** is a manifestation of the mentalese of the computational mind, frames represent the components of system-mentalese.

xiv. Frame manipulation schemes operate seamlessly across multiple data-types representing different sensory memories.

xv. Besides parameters that describe a fact or an event, frames include parameters to denote the system's belief of the world and itself. The *Z-number* philosophy, described in the earlier sections, is an effective scheme for subjective-belief symbolization, and thus is the macro-knowledge processing element of the system.

System inputs are translated into Z-number or Z-information sets that are processed as per the algorithm described in Sect. 3.1.1. Given a Z-valuation <*X*, *A*, *B*>, the parameters are synonymous to frame-terminal, slot-value, and the strength in the terminal-slot_value connectivity, respectively. Z-valuations summarize the information in a frame while Z-information summarizes that in a frame-network. Frame-manipulations are typically Z-number calculi.

xvi. The sub-agencies under **De** can be categorized into the following, based on the levels of information-granules they deal with:

a. Tier 1 – Acknowledge system 'self'; subjective decisions – **Sf**
b. Tier 2 – Conjecture abstract or well-defined procedures for text interpretation – **Re**, **Cr**, **Su**
c. Tier 3 – Hypothesize steps of abstract procedures; procedure-step execution – **Se**, **Sy**

3.2.2 Demonstration of the Operation of the Machine-Mind Architecture

Table 3 presents an explicit run through the framework – depicting the stages of comprehension and the roles of the mind-agencies. A unit of text is being comprehended. Components in the table abide by the following schematics:

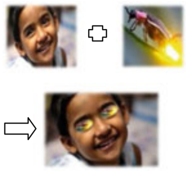
i. <**bold**> => ‘frame_header’
ii. <*italics*> => ‘terminal_slot’
iii. <***bold and italics***> => ‘slot_value’
iv. () => ‘frame-terminal_slot’ relation
v. Arrow heads => connectivity destinations; destinations could be ‘terminal_slots’ or ‘slot_values’

Assumptions: Each of the mind-agencies and memory constructs is functional, as per the descriptions in Sect. 3.2.1.
Input text: *Smita’s eyes were fireflies.*
Expected output: Narrative(s) of comprehension.

Observations:

i. Inference results and data of one stage percolate across parallel threads of operation (co-operating threads) as well as down to the next stage of comprehension.
ii. Entries across time units indicate **Log** as well as *global* **FA** values.
iii. Activities of **M** have not been highlighted, as our focus was exclusively on the emulation of comprehension – unhindered by consideration of system optimization issues.
iv. The progression of comprehension depicted in Table 3 was validated by the thought processes of twenty random individuals. These individuals were asked to list – over a time period of two days – all that their minds processed in relation to the given text input, and in the order that their thoughts were activated. Synopsis of the survey results are illustrated in Table 4 and Fig. 5. The Venn diagram (Fig. 5) of data in Table 4 (where U = universe of discourse), clearly depicts the system outputs to be in line with majority human interpretations.
We are currently in the process of performing more such experiments, so as to understand better the average thought processes given random text instances. [Refer to [62] for other examples of system execution.]
v. Formats of Z-valuations, with respect to agency-operations, are indicated in Table 3.
vi. The framework is conceptually a cognitive model of text comprehension, as it demonstrates: (a) multiple-realm ‘thinking’, (b) ambiguity resolution, (c) recollection and reflection, and (d) subjective decision-making.
vii. The question of interest here is how can a machine prove to be ‘thinking’ or behaving ‘intelligently’? Ryle states in [82], that the procedure a system uses

Table 3 The action dynamics of comprehension by the computational mind framework

Time	System-action threads			
T_0	**V** – Extracts: ‘Smita’, ‘’s’, ‘eyes’, ‘were’, ‘fireflies’, ‘.’			
T_1	**V** – Granulates: <**Smita, ‘’s’, eyes, were, fireflies, ‘.’**>	**M** – Assigns: granule id $\mathbf{S_1}$	**V** – References: **L**, **ComN** and **ConN** to retrieve knowledge or frames on: <**Smita**>, <**’s**>, <**eyes**>, <**were**>, <**fireflies**>	
T_2	**Sy** – Activates: syntax frames for $\mathbf{S_1}$: [format of corresponding Z-valuations: <word, syntax, confidence>] <**Smita**> → <***noun***> <**’s**> → <***is***> <***was***> <***has***> <**eyes**> → <***noun***> <***verb***> <**were**> → <***verb***> <**fireflies**> → <***noun***> → <***common noun***>	**Re** – Recognizes: from $\mathbf{S_1}$: [format of corresponding Z-valuations: <word, recognized fact, confidence>] <**Smita**> → <*proper noun*> → <*name*> → (of) → <*human being*> → (gender) → <***girl***>		
T_3	**Se** – Prunes: irrelevant syntax frames for $\mathbf{S_1}$: <**Smita**> → <***noun***> <**’s**> → <***has***> <**eyes**> → <***noun***>			
	Se – Constructs: semantic frames for $\mathbf{S_1}$: [format of corresponding Z-valuations: <word, semantics, confidence>] **Frame1**$_{\mathbf{S1}}$ – <**Smita**> → (what) → <*a girl*> → <**has**> → (what) →<***eyes***> **Frame2**$_{\mathbf{S1}}$ – <**eyes**> → <**were**> → (what) → <***fireflies***>	**Cr** – Construes: [format of corresponding Z-valuations: <word, interpretation, confidence>] <**eyes**> → (possessed by) → <***living beings***> <**Smita**> → <*proper noun*> → <*name*> → (of) → <***living being***> <**living being**> → [***human*** | ***non-human***]		
T_4	**Su** – Consolidates (after a series of Z-number calculii operations on frames instantiated by **Se** and **Cr**): frames for $\mathbf{S_1}$ into a narrative: <***Smita is a living being with eyes that are fireflies. Smita could be a girl or a non-human***>	**Su** – Conjures: visual for $\mathbf{S_1}$: <***a girl with fireflies for eyes***> 	**Su –** Senses: incomprehensibility and initiates **Cr** intervention	**Sf** – Assigns: [format of corresponding Z-valuations: <parameter, value, confidence>] <**system affect**> → <*confused*> → (why) → <***incomprehensible***>
T_5	**Cr –** Activates: study of similarities between <**eyes**> and <**fireflies**> **–** analyzes respective frames transferred from **L**, **ComN** and **ConN** into *Global* **FA** by **V** at T_1: <**eyes**> → [***beautiful*** | ***bright*** | ***shine*** | ***sparkle*** | ***glow*** | ***twinkle***...] <**fireflies**> → [***beautiful*** | ***bright*** | ***shine*** | ***glow*** | ***twinkle***]	**Sf** – Contemplates: i. What does the system regard as “beautiful” eyes? ii. Is the system’s sense of **V** beautiful? Recollect acquaintances who remarked as such. iii. What does the system regard as [*bright* | *shine* | *sparkle* | *glow* | *twinkle*] eyes? iv. When was the last time the system was very [happy | excited]? v. Were the system’s sense of **V** then [*bright* | *shine* | *sparkle* | *glow* | *twinkle*]?	**Re** – Recognizes: [format of corresponding Z-valuations: <word, recognized fact, confidence>] 1a. <**Smita**> → <*proper noun*> → <*name of a human being*> → <***name of a girl***> 1b. <**eyes**> → (of what) → <*human beings*> → (are / do) → [*bright* | *shine* | *sparkle* | *glow* | *twinkle*] → (when) → [*happy* | *excited*] → (degree of affect) → <***high***> 2a. <**Smita**> → <*proper noun*> → <*name of a non human living being*> → <***name of an animal***> 2b. <**eyes**> → (of what) → <*animal* – [*cats* | *dogs* | *deer* |...> → (do) → [*glow*] → (when) → <***in the dark***> **Re** – Recollects (in sync with **Sf**): i. ‘Beautiful eyes’ remark memories ii. Happy memories	
T_6	**Su** – Consolidates (after a series of Z-number calculii operations on frames instantiated by **Cr**, **Sf** and **Re**): frames for $\mathbf{S_1}$ into narratives: i. <***Smita – a girl – has beautiful eyes***> ii. <***Smita – a girl – has*** [***bright*** | ***shining*** | ***sparkling*** | ***glowing*** | ***twinkling***] ***eyes because she is*** **very** [***happy*** | ***excited***] iii. <***Smita – an animal –*** [***cats*** | ***dogs*** | ***deer*** | ...] ***– has eyes that glow in the dark***>	**Su** – Conjures: visuals per interpretation: i. <***a girl with beautiful eyes***>  ii. <***a happy girl with bright eyes***>  iii. <***an animal with eyes that glow in the dark***> 	**Se** – Identifies: [format of corresponding Z-valuations: <word, semantics, confidence>] <**fireflies**> → <*adjective*> → <***metaphor***>	**Sf** – Assigns: [format of corresponding Z-valuations: <parameter, value, confidence>] <**system affect**> → <*excited*> → (why) → <***interesting expression***> **Sf** – Contemplates: i. Why would Smita be very [happy | excited]? ii. Projects machine-self onto Smita – would the reasons that made the system [happy | excited] apply to Smita?
T_7	**Su** – Consolidates: new knowledge on **fireflies** into **AL**, **L** and **ConN**			

to arrive at solutions is an indication of its intelligence; the procedures being an amalgamation of its knowledge, intuition, commonsense, and experience. Thus, a measure of think-ability or intelligence of a machine (MIQ [2]) requires parameters that represent the strength of the system in each of these areas.

Considering the implication of the self-conscious facet of thinking, [83, 84] indicate the need for effective indicators of 'conscious' thoughts and states of the system. These need to incorporate both objective and subjective machine responses. (Would a 'conscious', 'thinking', 'understanding' machine be immune to consciousness disorders leading to psychiatric or neurologic disorders or minimal conscious states?)

viii. The ultimate test of understanding is if the machine is able to utilize this newly learnt expression in appropriate situations.

ix. As stated in [59], the key requirements of knowledge based language understander systems are:

a. Representation and structuring of the problem in a way that permits decomposition.
b. Total interpretation is to be broken down into hypotheses and modularized into different types of knowledge that can operate independently and co-operatively.

The framework supports the conceptual acknowledgement of these requirements, in the following ways:

a. We have factored the problem of comprehension into its component functions and assigned their execution to mind-agencies (Sect. 3.2.1). What remains to be done is factoring the agency-functions into individual agents – which represent and operate on natural language samples to construct modules of comprehension.
b. Decomposition of interpretations into hypotheses and knowledge modularization is executed by:

- **Re** disintegrates an interpretation problem into "similar" sub-problems and recalls known solutions.
- **Cr** conjectures new answers.
- **Su** periodically summarizes comprehension statuses that consequently activate solution suggestions by different agencies.
- *Critic-selector agents* critically analyze multiple approaches towards the realization of an agency-function.
- *Global* **FA** and **Log** serve as global workspaces for the agencies to co-operate towards solutions.
- *Local* **FA** supports independent agency-activity trials, moderated by *critic-selector agents*.
- The **De** agencies (Sect. 3.2.1), operate across a number of information-granular levels – beginning with the syntax of a word to its subjective perception.

- Analyses across agencies form pools of candidate partial solutions, which **Su** combines into effective global solutions.
- Knowledge, modularized into facts, concepts, intuition, commonsense and procedures, is referenced by agencies relative to the demands of the status of comprehension.

Table 5 maps the processing activities, during comprehension of the given input, to the layers (Sect. 2.3.2) of the mind (the machine-mind) they pertain to, thereby demonstrating thinking across all the layers. [Refer to [62] for other depictions of natural language comprehension by the defined machine-mind framework.]

Table 4 Summary of human subject responses to metaphor – *Smita's eyes were fireflies*

System input – text – Smita's eyes were fireflies				
Number of human subjects in survey – 21				
No. of survey subjects who stated…	System interpretation: 'a' only	System interpretation: 'b' only	System interpretations: 'a' & 'b'	System interpretation: 'c' only
	1	14	3	0

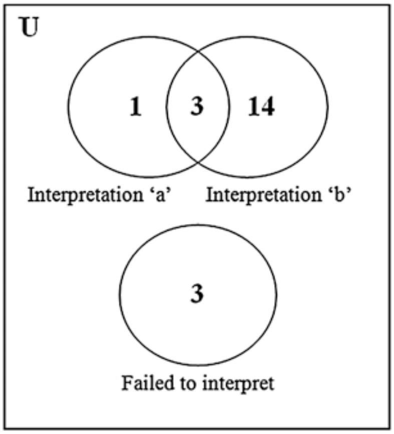


Fig. 5 Venn diagram summarizing human responses to metaphor - *Smita's eyes were fireflies*

3.2.3 Validation of Completeness of Machine-Mind Conceptualization

- **Correspondence between machine-mind agencies and the layers of the mind**

Table 6 maps the layers of the mind and the agencies that emulate them. The agency functionalities seamlessly traverse the layers (refer to Table 5), cover all of them and overlap multiple levels. A (*) in a cell indicates coverage of the layer-functionality (row) by the mind-agency (column).

Table 5 Correspondence between comprehension activity, for the given input, and layer of mind functionality

<table>
<tr><th>Layer of the mind</th><th>Processing activity</th></tr>
<tr><td>Instinctive</td><td>• Read given text
• Updating of long-term memory constructs in response to new metaphorical interpretation of “fireflies”</td></tr>
<tr><td>Learned</td><td>• Syntax analysis by Sy: Smita → proper noun, ‘s → [is | was | has], eyes → [common noun | verb], were → verb, fireflies → common noun
• Surface semantic analysis by Se of the words in the sentence
• Re recalls properties of ‘eyes’ and ‘fireflies’
• Re recalls Smita has eyes → Smita living (human | non-human)</td></tr>
<tr><td>Deliberative</td><td rowspan="2">• Su simulates visuals of “eyes were fireflies” based on surface interpretation as well as metaphorical sense
• Cr and Re map properties of fireflies with that of eyes and the subjective meaning as well => [bright | glow | shining | beautiful |…] => [happy | excited | …]
• Se prunes irrelevant non-contextual semantic frames
• Se identifies “fireflies” being used as an adjective or metaphor</td></tr>
<tr><td>Reflective</td></tr>
<tr><td>Self-reflective</td><td>• Sf awakens affect – incomprehension (confusion); interest (new knowledge gained)
• Sf contemplating on what could make Smita [happy | excited]? – projection of self onto Smita</td></tr>
<tr><td>Self-conscious</td><td>Sf contemplating on what would make the system [happy | excited], what the system considers “beautiful”, whether acquaintances remarked on the system’s beautiful eyes, …</td></tr>
</table>

- **Correspondence between machine-mind agencies and parts of the human brain**

The human brain serves as our reference. In Table 7 and Table 8 we summarize the correspondence between the different parts of the brain – that play significant roles in natural language comprehension and agency functionalities. Table 7 depicts the equivalence between the human memory categories and the framework memory constructs. Table 8 illustrates the analogy between parts of the cerebral cortex of the human brain and the framework agencies.

The conceptualized architecture-agencies cover all known memory categories and essential language processing centres of the human brain.

Having described our efforts towards the emulation of natural language understanding, the questions that loom up as major design issues are as follows. It is in the search for answers to these issues that our research is currently directed:

i. How does a natural organism encode real-world information? Could it provide clues to encoding the ‘meaning’ of natural language elements in machine-understandable form?

Table 6 Correspondence between computational mind-agencies and layer of mind functionalities

Layers of the mind \ Computational mind-agencies	V	Sy	Se	Sf	Re	Cr	Su	M
Instinctive reactions: Accept sensory input; procedural memory updating	*							*
Learned reactions: Assign meaning to language elements (alphabets, digits, special symbols, white-spaces, punctuation, etc.); syntax and semantic analysis;	*	*	*		*		*	*
Deliberative thinking: Context identification; disambiguation of word and sentence meanings; rhetoric and prosodic analysis; analyze relevance and coherence of flow of understanding; consolidate individually understood elements into concepts; visualize scenes; deliberative memory updating		*	*		*	*	*	*
Reflective thinking: Reason and optimize deliberative thinking processes; generate curiosity (questions in the computational mind) and activate schemes to gratify the same; build cross-contextual associations		*	*		*	*	*	*
Self-reflective thinking: Evaluate interest and comprehension progression; overcome cognitive biases and subsequent concept reformation – identify new knowledge gained, clarify misconceptions				*			*	*
Self-conscious emotions: Attachment of emotions or levels of interest and perceptions; awareness of social norms and ideals				*			*	*

Table 7 Correspondence between computational mind memory constructs and human memory categories

Human memory categories	Framework memory constructs
Working (temporary representations of information on current task)	*global* **FA**; *local* **FA** of **De** and **M** sub-agencies; **AF**; **PF** [**WS** $\subseteq$ **AF** and is therefore not explicitly mentioned]
Declarative (represent explicitly stored and recalled memories)	**CoN**
Procedural (represent implicitly stored and recalled memories of automatic behaviours)	**ComN**
Long-term (semantically encoded declarative and procedural memories)	**L**; **CoN**; **ComN**; **AL**
Short-term (memories recalled, without repetition, for a duration of the order of seconds; are acoustically encoded)	First set of entries into *global* **FA** by **SG**
Sensory (memories of sensory stimulus after it has ceased; is of order of milliseconds)	*local* **FA** of **SG** sub-agencies
Visual (visual experiences) *Olfactory* (olfactory experiences) *Haptic* (tactile experiences) *Taste* (taste experiences) *Auditory* (auditory experiences)	Memories annotated by the senses they represent – indicated by their data-types in **ComN** and **CoN**
Autobiographic (personal episodic experiences)	Subset of **CoN**
Retrospective (action of remembering the past)	Constructed out of **ComN** and **CoN** [**PF** supports the emulation of these memories]
Prospective (memories activated in the future based on time and event cues)	

ii. Could scientific definitions of the 'self' and 'consciousness', provide for the machine-self – resulting in the emulation of self-reflection and self-consciousness?
iii. Encoding meta-cognition, i.e., machine-awareness of comprehension – the autogenous arousal of the 'Aha!' moment of understanding
iv. The emulation of endogenous instantiation of information-certainty and event-affects
v. Tests for machine-thinking and MIQ
vi. Devise measures of information-loss on Z-valuation compaction of frames and degrees of comprehension

Table 8 Correspondence between computational mind-agencies and cerebral cortex region functionalities

Lobe	Parts /Functions of the lobe in relation to language understanding	Framework agencies
Occipital	Processes visual information	**V**
Frontal	*Broca's area* (syntax and morphology analysis)	**Sy**
	Self definition, attention, social behaviour	**Sf**
	Reasoning, judgment, strategic thinking	**Re**, **Cr**, **Su**
Parietal	*Angular gyrus* (language and number processing, spatial cognition, memory retrieval, attention mediation and metaphor comprehension [85, 86])	**Se**, **Sf**, **Su**, **Re**, **Cr**
Temporal	*Wernicke's area* (semantic resolution)	**Se**
	Amygdala (affective processing and memory consolidation)	**Sf**, **Su**
	Hippocampus (storage and consolidation of semantic and episodic memories)	**Su**
	Basal Ganglia (reinforcement learning, procedural memory – priming and automatic behaviours or habits, eye movements [87] and cognition [88])	**Sf**, **Su**, **Re**
	Recognition	**Re**

4 Conclusion

This article is an elucidation on our study of the Z-number [8] approach to CWW [3, 25] and the enumeration of high-level Society of Mind [9, 10] elements of a machine-mind framework for natural language understanding.

The Z-numbers, as defined by Zadeh, provide a basis for the precisiation of the meaning (both objective and subjective) of natural language statements. We extend this capability to that for sentences – define an algorithm for Z-number based CWW and perception-operators that form complex Z-numbers. These have been used to simulate an instance of differential diagnosis – thereby demonstrating the possible strengths and challenges of the paradigm.

Work on the Z-numbers prompted our focusing on the synthesis of mechanisms for the endogenous instantiation of system-affects and emulation of system-mentalese [41]. Thus, utilizing the primary philosophy of Minsky's Society of Mind theories of the human mind and using the human brain as a reference, we turned to the design of a machine-framework for natural language understanding. This framework, currently in its early stages, is a co-operating association of function-modules and memory elements. It conceptually emulates every aspect of cognitive comprehension – intuition, commonsense, thinking, adapting and learning. The modified Z-numbers precisiate knowledge in the framework and allow system-perception manipulation. The working of the architecture has been demonstrated through a detailed example of metaphor comprehension. System results have been validated against human responses. System-conceptualization completeness has been studied through correspondence analysis with the layers of the mind (Minsky) and cerebral cortex regions.

We do not claim that the proposed design mimics the vast repertoire of mind functions nor have we defined every psychological process in its computational equivalents, but we have here a set of very basic agencies and methodologies that work in unison and harmony to realize language understanding. The concepts here serve as a blueprint for our endeavours. Our current research initiatives are primarily concentrated in defining parameters that represent the dynamic machine-self, encoding 'meanings' (objective and subjective) in machine-understandable form, and the emulation of metacognition. The designed computational-mind is envisaged to define its own self, be self-organized, dynamic, adaptable, and social – behave like an 'intelligent' object [65, 66].

A cognitive system, we believe, applies to the development of 'intelligent' and 'symbiotic' man-machine interactive systems – plagiarism-checkers, library cataloguing systems, text summarizers, differential diagnosis systems, educational aids for children with reading disorders, etc.

Acknowledgments This project is being carried out under the guidance of Professor Sankar K. Pal who is an INAE Chair Professor and J.C. Bose Fellow of the Government of India.

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Evolutionary Reduction of Fuzzy Rule-Based Models

Witold Pedrycz, Kuwen Li and Marek Reformat

Abstract In the design of fuzzy rule-based models we strive to develop models that are both accurate and interpretable (transparent). The approach proposed here is aimed at the enhancement of transparency of the fuzzy model already constructed with the accuracy criterion in mind by proposing two modifications to the rules. First, we introduce a mechanism of reduction of the input space by eliminating some less essential input variables. This results in rules with the reduced subspaces of input variables making the rules more transparent. The second approach is concerned with an isolation of input variables: fuzzy sets defined in the *n*-dimensional input space and forming the condition part of the rules are subject to a decomposition process in which some variables are isolated and interpreted separately. The reduced dimensionality of the input subspaces in the first approach and the number of isolated input variables in the second one are the essential parameters controlling impact of enhanced transparency on the accuracy of the obtained fuzzy model. The two problems identified above are of combinatorial character and the optimization tasks emerging there are handled with the use of Genetic Algorithms (GAs). A series of numeric experiments is reported where we demonstrate the effectiveness of the two approaches and quantify the relationships between the criterion of accuracy and interpretability.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_23

Keywords Rule-based models • Interpretability • Reduction • Isolation of variables • Fuzzy clustering • Curse of dimensionality

1 Introductory Notes

In the construction of fuzzy rule based systems, we have been witnessing a wealth of design strategies and detailed algorithms involving the technology of Evolutionary Computing and neurocomputing. Just recent developments reported in this realm can be found in a series of studies [1–3, 5, 6, 8]. Predominantly, the development of fuzzy models is guided by the criterion of accuracy. Another fundamental criterion being at the heart of fuzzy modeling is interpretability (transparency) of resulting fuzzy models. This criterion is central to fuzzy models however its multifaceted nature requires a thorough formulation and a detailed quantification of essential aspects of interpretability. Subsequently, it calls for engaging advanced optimization techniques supporting the realization of the ensuing design.

The concept of interpretability of fuzzy rule-based models has been around for several decades and attracted a significant deal of attention. The transparency of fuzzy models is one of the outstanding and important features of fuzzy models. In contrast to the criterion of accuracy, whose quantification is relatively straightforward and easy to come up with performance indexes, transparency of fuzzy rules is more difficult to describe. What makes the fuzzy rule-based easier to interpret and comprehend is still an open issue. It is quite subjective to assess and in one way or another invokes a factor of subjective judgment given that a human user is ultimately involved in the evaluation process. What also becomes apparent, is a multifaceted nature of the problem and a multitude of various approaches supported by various optimization technologies including evolutionary optimization. When it comes to the main factors worth considering when discussing a concept of interpretability, we can enumerate a list of factors that may be involved in the reduction process:

- number of rules forming a rule base of the model,
- number of sub-conditions (input variables) forming a condition part of a given rule,
- number or rules and the number of input variables,
- complexity of local regression models forming the conclusion part of the rules (in case of Takagi-Sugeno model)
- interpretability of a family of fuzzy sets formed in the input space for individual variables.

As a result, given this diversity of possible ways of reduction of rules, it is difficult to quantify the effect of reduction. For instance, it is not always clear if it would be better to have a larger number of simple rules (whose condition parts are linear functions) or a smaller number of rules of a more complex conclusion parts (say, those of polynomial form).

The reader may refer to a large body of studies devoted to the issue of interpretability, cf. [3, 10]. Quite often, given the combinatorial nature of the reduction problems, Genetic Algorithms are used in the optimization process, see [9]. There are also more specialized algorithmic vehicles to support the reduction process such as e.g., singular value decomposition [14] however one has to be cognizant as to the interpretability of the reduced rules.

As to the overall strategy of rule reduction and interpretability enhancement, there are two general design strategies: either the reduction is completed once the model has been constructed or the reduction mechanism of rule-based modeling is incorporated into the design process from its very beginning.

The objective of the study is to pursue a fundamental issue of building more transparent and user-centric fuzzy rule-based models by starting from an already designed fuzzy model. The intent is to make the rules more readable in two different ways: (a) by reducing the input space (the number of antecedents) of the individual rules, and (b) by isolating input variables completed for the input variables treated *en block* in the condition parts of the rules.

In both these fundamental scenarios we can establish and quantify a tradeoff between a gradual reduction of accuracy (which is inevitable when realizing any of the reduction mechanisms of the rules and enhancing its intensity) and the increased interpretability of the rules. The presentation of the material is structured in the following way. We briefly revisit the essentials of Takagi-Sugeno rule- based systems, stressing the proposed design approach in which we use fuzzy clustering (Sect. 2). The reduction of input space and a formation of input subspaces for each rule is introduced in Sect. 3; in the same section we also present an optimization process realized with the use of genetic algorithm. The second approach to the enhancement of the interpretability of the rule – an isolation of input variables is discussed in Sect. 5. Detailed experimental studies are reported in Sects. 3 and 6.

In the study, we experiment with a number of publicly available datasets coming from eight datasets from UCI Machine learning repository and DELVE repository; their main characteristics are listed in Table 1.

Table 1 Main characteristics of datasets used in the experiments (number of data and dimensionality –number of input variables)

Dataset	Number of input variables	Number of data	Origin of the data
Abalone	8	4177	UCI Machine learning repository (http://archive.ics.uci.edu/ml/)
Auto MPG	7	392	UCI Machine learning repository
Boston Housing	13	506	UCI Machine learning repository
Computer Activity	21	8,192	DELVE repository (http://www.dcc.fc.up.pt/~ltorgo/Regression/DataSets.html)
Concrete strength	8	1030	UCI Machine learning repository
Forest fires	12	517	UCI Machine learning repository
Red wine quality	11	1599	UCI Machine learning repository
White wine quality	11	4898	UCI Machine learning repository

2 Fuzzy Rule-Based Models: An Overview and Main Design Issues

Our point of departure of the reduction processes of the rules is a “standard” Takagi-Sugeno fuzzy model comprising c rules coming in the following form

$$-\,if\ \boldsymbol{x}\ is\ \mathrm{A}_i\ then\ y = f_i(\boldsymbol{x}, \boldsymbol{a}_i) \tag{1}$$

for $i = 1, 2, .., c$ where $\boldsymbol{x} \in \boldsymbol{R}^n$. A_i is a fuzzy set defined in the n-dimensional space while f_i is the corresponding local model (linear or nonlinear) endowed with its parameters $\boldsymbol{a}_i$ forming the conclusion part of the i^{th} rule. The design of such models is well reported in the literature and as usual consists of the two main steps, namely (a) a construction of condition parts (through clustering of input data done in the input space) and (b) estimating parameters of the linear models (which leads to the problem of linear regression). The number of rules (c) is determined by monitoring the behavior of the model on the training and testing data. The rules in the form (1) come with several essential properties. The condition part is a fuzzy set expressed in $\boldsymbol{R}^n$ making the rules concise, which helps avoid a curse of dimensionality we are commonly faced with in rule based systems with a higher number of input variables. In the case of treating all input variables at the same time, the number of rules becomes small (c) and the rule base itself is compact. Unfortunately, the interpretability could be negatively impacted as no individual variables in the condition part are treated and visualized separately.

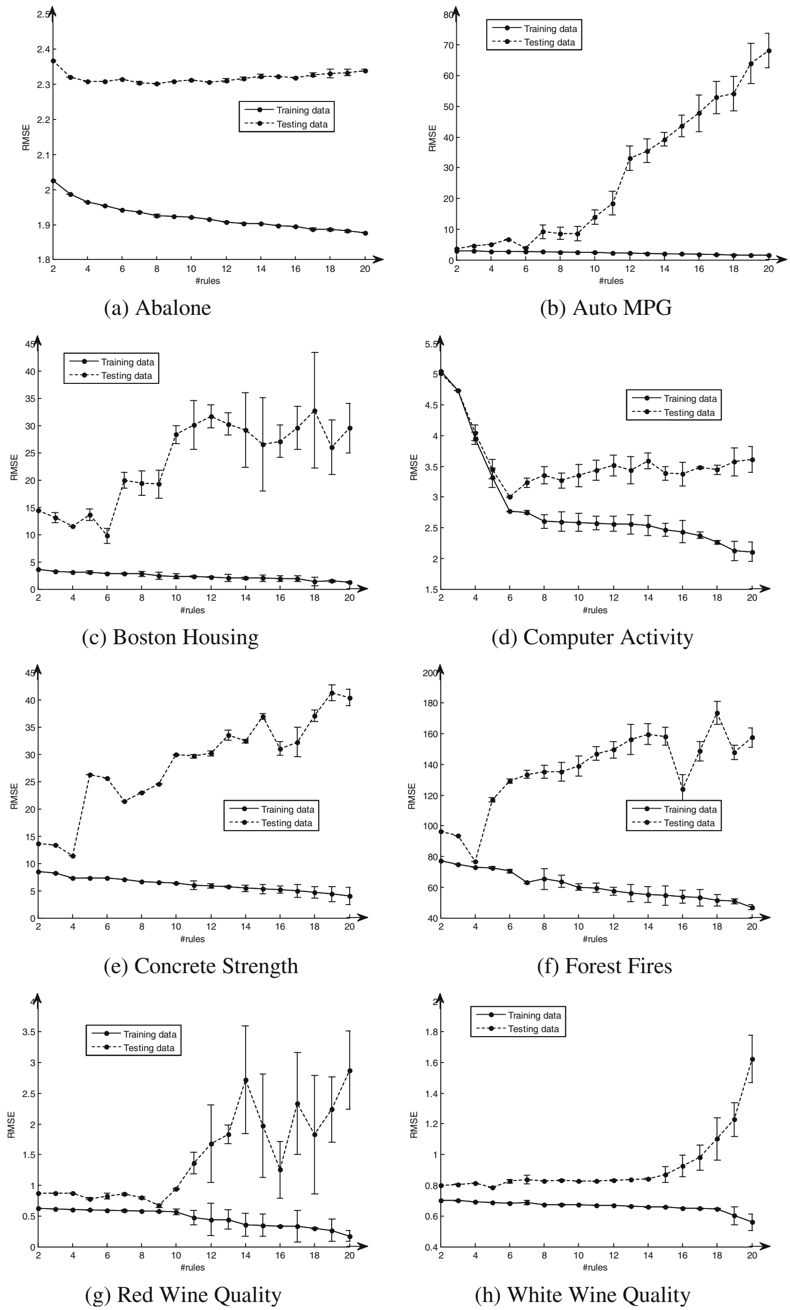
When it comes to the quantification of the accuracy of the fuzzy model (1) its performance is commonly expressed by the RMSE index computed for the training set

$$\sqrt{\frac{1}{N}\sum\nolimits_{k=1}^{N}(\mathrm{FM}(\mathbf{x}_k) - target_k)^2} \tag{2}$$

where N denotes the number of data in the training set. In the same way, quantified is the performance of the constructed model on the testing data (consisting of M data points)

$$\sqrt{\frac{1}{\mathrm{M}}\sum\nolimits_{\mathrm{k}=1}^{\mathrm{M}}(\mathrm{FM}(\mathbf{x}_\mathrm{k}) - \mathrm{target}_\mathrm{k})^2} \tag{3}$$

Proceeding with the data summarized in Table 1, the performance of the corresponding models visualized versus the number of rules is illustrated in a series of figures shown below, Fig. 1. The results are reported both for the training and testing data. In the design, we use a standard version of the Fuzzy C-Means (FCM) [4] with the fuzzification coefficient (m) set to 2.0. The algorithm was run for 10 iterations (more specifically, we completed 10 runs with different splits of data into training/testing data)



(a) Abalone (b) Auto MPG

(c) Boston Housing (d) Computer Activity

(e) Concrete Strength (f) Forest Fires

(g) Red Wine Quality (h) White Wine Quality

Fig. 1 Performance of fuzzy models versus the number of rules reported for the training and testing data for datasets (Table 1)

For the training sets, there is a general tendency of having lower values of the RMSE with the increase of the number of rules. The performance reported on the testing sets points at the memorization effect where the some models tend to lose their generalization capabilities. By eyeballing these plots, we choose a suitable number of rules (clusters) where sound approximation abilities come hand in hand with the generalization of the models. The values selected in this way are collected in Table 2.

Table 2 Number of rules of fuzzy models constructed for the corresponding data

Data name	Selected number of rules
Abalone	7
Auto MPG	6
Boston Housing	6
Computer Activity	6
Concrete strength	4
Forest fires	4
Red wine quality	9
White wine quality	5

3 Reduction of Input Subspaces in Rule-Based Models

The essence of the enhancement of interpretability of the rules is accomplished by reducing the number of input variables standing in the condition parts of the rules. The reduced rules are concisely described in the form

$$-\, if\ \boldsymbol{x}\ is\ [A_i]_{Xi}\ then\ y = f_i(\boldsymbol{x}, \boldsymbol{a}_i) \tag{4}$$

where the symbol $[]_{Xi}$ stresses the fact that the fuzzy set A_i is now effectively confined to the reduced input space $\boldsymbol{X}_i \subset \boldsymbol{R}^n$ where some original input variables have been removed. In other words, dim $(\boldsymbol{X}_i) = n_i < n$.

The computing of the activation level of A_i positioned in this new reduced space is realized as follows

$$[\mathrm{A_i}]_{\mathrm{Xi}}(\mathbf{x}) = 1/\sum_{\mathrm{j}=1}^{\mathrm{c}} \left(\frac{\cdot \left\|\underline{\mathbf{x}} - \underline{\mathbf{v}_{\mathrm{i}}}\right\|\Big|_{\aleph_{\mathrm{i}}}}{\cdot \left\|\underline{\mathbf{x}} - \underline{\mathbf{v}_{\mathrm{j}}}\right\|\Big|_{\aleph_{\mathrm{i}}}}\right)^{\frac{2}{m-1}}, \tag{5}$$

for i = 1,2,.., c; $m > 1$ where the computations of the distance are realized in the reduced input space $\boldsymbol{X}_i$.

The reduction of the rules can be quantified in terms of a reduction factor ν, which relates with the number $n*c$ (expressing an overall number of variables across all the rules) in the following way

$$p = n(n^*c). \tag{6}$$

where p represents a reduced number of input variables used in c rules.

The reduction of the input variables existing in the new, more interpretable rules is done via engaging the optimization capabilities of Genetic Algorithms (GAs). More specifically, we optimize a matrix of allocation of input variables $W = [w_{ij}]$ with c rows and n input variables. The p largest entries of W are selected giving rise to a binary 0–1 matrix. The p entries with the largest values are set to 1 while the remaining ones are suppressed to zero. Each row of the matrix formed in this way identifies the variables to be used in the corresponding reduced rule. If all entries of the i^{th} row of W are equal to zero, this entails that the corresponding rule does not exist in the reduced set of rules.

The process of identifying which input variables should be kept in the antecedents of rules is translated into an optimization problem. Its solution is obtained through evolutionary optimization, namely Genetic Algorithm (GA) [7]. GA uses elements of natural selection to determine the best solution to a problem by minimizing a certain fitness function capturing the essence of the optimization problem. The best solution is obtained via selecting and modifying a population of potential solutions (chromosomes). A main flow of GA computing is outlined in Table 3.

Table 3 A flow of optimization realized by genetic algorithm

Parameters of genetic algorithm:
iter – number of generations
Z – size of population
p_c – crossover rate
p_m – mutation rate
Algorithm
1 *Initialization:*
2 Random generation of chromosomes c_k using uniform distribution in [0,1]), for $k = 1, 2, \ldots, Z$
3 *Iterate (*
4 Calculate the fitness value for each chromosome c_k
5 Select the candidates for the next generation based on their fitness values
6 Retain the overall best chromosome so far, say c_{best}
7 Apply the crossover and mutation operations based on the crossover (p_c) and mutation (p_m) rates
8 *Until* the number of iterations does not exceed the predetermined limit, iter*

The two aspects of GA that require special attention and directly impact the quality of results of the optimization process are: 1) construction of a suitable fitness function, and 2) mapping a solution to the problem onto a structure of the chromosome.

A fitness function is used to evaluate possible solutions. Fitness values associated with solutions (chromosomes) represent their ability to solve a given problem. The values are used during selection of chromosomes for further processing (Table 3, line 5). The fitness function has to reflect the objective of optimization process and ensure that the obtained fitness values are adequate for mechanisms of selection, i.e., the fitness values should be such that a selection process leads to a diversified set of potential solutions [11]. In the case of our problem of selecting input attributes that should be kept in the rules, the fitness function evaluates quality of fuzzy rules that model a given dataset. The form of the fitness function used in the optimization is given by (2).

As noted earlier, each chromosome represents a solution to the considered problem. It consists of simple elements, called genes, that can assume values 1 or 0 (Binary Coded GA), or any real numbers from a specified range (Real Coded GA). A chromosome with such a simple structure, called genotype, has to represent a solution to the problem, i.e., it should be mapped into a form called phenotype that is adequate for a given problem domain.

For the optimization problem considered in the paper, we use a Real Coded GA (RCGA). The size of chromosome, i.e., a number of genes *gsize*, is equal to the product of a number of rules c and a number of input attributes n of a given dataset: $gsize = cn$. Therefore, each chromosome contains information which input attributes are present in each rule. A single gene is a real number between 0 and 1. The translation of such a chromosome (genotype) into a solution to our problem (phenotype) occurs in the following way. The user has to determine a number p of input attributes or isolated attributes (Sect. 5) that should be kept in all c rules. This means that among all genes of a chromosome only p of them should be used to construct c rules. The selection of these genes is done via sorting all genes based on their values (between 0 and 1) and identifying the first p of them. The process is presented in Fig. 2.

Once an initial population of chromosomes c_k has been generated, RCGA starts an iteration process. A single iteration consists of calculating fitness values (Table 3, line 4), selecting best chromosomes (line 6), and performing crossover and mutation operations on selected chromosomes (line 7). All this constitutes a single generation. After executing a number of generations, GA provides an optimal solution in a form of a chromosome that gives the highest value of the fitness function.

The purpose of crossover is to exchange information between chromosomes. In its simplest form it means exchange of genes between a pair of randomly selected, with the probability p_c given by the user, chromosomes. For our RCGA, a linear crossover is applied [13]. The linear crossover generates three offspring (new chromosomes): $O_k = (o_1^k, \ldots, o_i^k, \ldots o_n^k), k = 1, 2, 3,$ where o_i^k represents the ith gene of the kth chromosome. The genes of offspring are built from the genes of two parents. Let us assume that the parents are: $C_k = (c_1^k, \ldots, c_i^k, \ldots c_n^k),\ k = 1, 2$. The values of genes of the offspring are calculated in the following way:

$$o_i^1 = \frac{1}{2}c_i^1 + \frac{1}{2}c_i^2,\ o_i^2 = \frac{3}{2}c_i^1 - \frac{1}{2}c_i^2 \text{ and } o_i^3 = -\frac{1}{2}c_i^1 + \frac{3}{2}c_i^2 \tag{7}$$

Each of the offspring is evaluated, i.e., a fitness value is determined for each of them. The two most promising offspring are selected to substitute their two parents in the population.

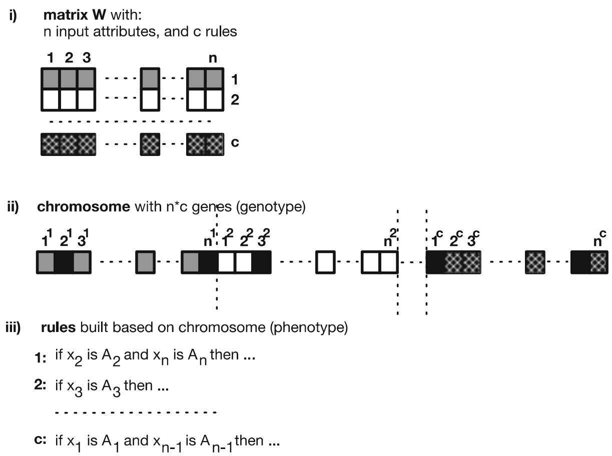


Fig. 2 GA chromosome: (i) original matrix of rules, (ii) a single chromosome built from the whole matrix, (iii) rules built based on this chromosome (there are p black boxes representing attributes selected for c rules)

The mutation operator modifies genes of a single chromosome in order to introduce diversity to the population. The frequency of modifications is controlled by the mutation probability p_m provided by the user. Mutation ensures that a search process covers the whole space of possible solutions. In the case of RCGA, the

Muhlenbein's mutation is adopted [12]. For a given parent chromosome $C_k = (c_1, \ldots, c_i, \ldots c_n)$, the gene values of a new – mutated – chromosome are calculated as $c_i^{'} = c_i \pm rang_i \bullet g$, where $rang_i$ defines the mutation range, and it is normally set to $0.1*(b_i - a_i)$, where b_i and a_i are the maximum and the minimum of c_i, respectively. The sign + or − is chosen with a probability of 0.5, while $g = \sum_{k=0}^{15} a_k 2^{-k}, a_k \in \{0, 1\}$ is randomly generated with $p(a_i = 1) = \frac{1}{16}, i = 0.15$.

The operations on a single chromosome are implemented in such a way that at least one input attribute is kept in each rule.

For reduction of input spaces, if p is a number of input attributes to be kept, the top p genes will be used to calculate the model output. The rest *gsize-p* attributes will be omitted in the calculation.

For isolation of attributes (Sect. 5), if p is a number of isolated attributes in each rule, the top p attributes in each rule (row of the matrix W) will be marked as isolated. The rest $n - p$ attributes will be treated as the remaining group of attributes. Thus, the total number of isolated attributes is pc.

4 Experimental Studies

In the following experiments we present how the reduction of the rules proceeds and how the reduced, more interpretable rules perform. We use different values of ν and report the corresponding values of the RMSE for the training and the testing data. The GA used a population of 100 individuals and was run for 100 generations. The crossover rate was set to 0.8 while the mutation rate was equal to 0.1. The choice of these numeric values was a result of some preliminary experimentation. The results are quantified by reporting the RMSE values obtained for different values of the reduction index; refer to Fig. 3.

It becomes apparent (and intuitively anticipated) that lower values of ν result in higher RMSE values. The detailed behavior varies across data with regard to how far the rules can be reduced and how the differences shape up for the training and testing data. For example, the reduction could be made quite substantial not compromising the performance of the model as this becomes present in case of abalone, auto, concrete, and white wine. In some case, we witness a phenomenon of increased generalization abilities of the model (lower differences of the RMSE for the training and testing data for lower values of ν). Figure 4 illustrates the performance of the GA for some selected data; most of the improvement is visible at the beginning of the optimization (first 20–30 generations).

The detailed results of reduction of the number of variables in the rules are contained in Fig. 5. The shaded regions identify the input variables being retained in the corresponding rules. This offers a better view as to which input variables can be dropped and points at a sequence of the variables, which have been eliminated.

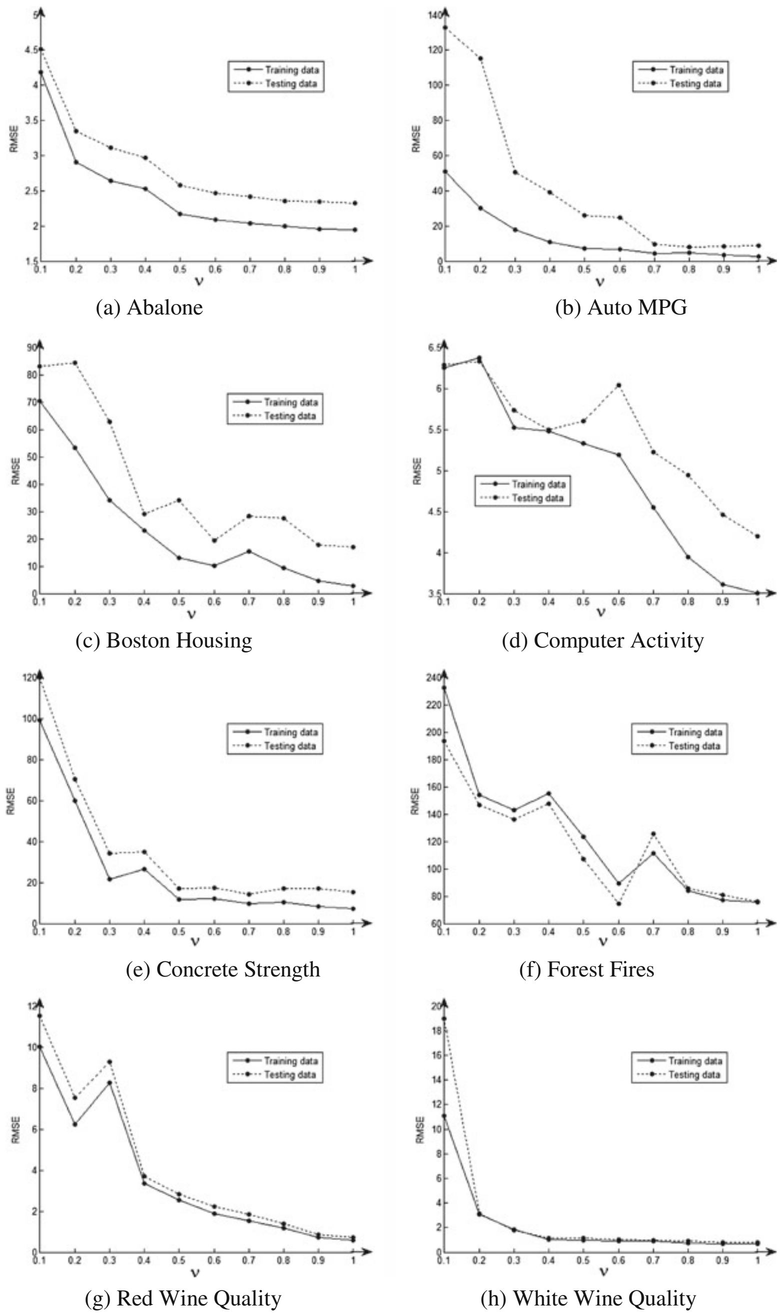
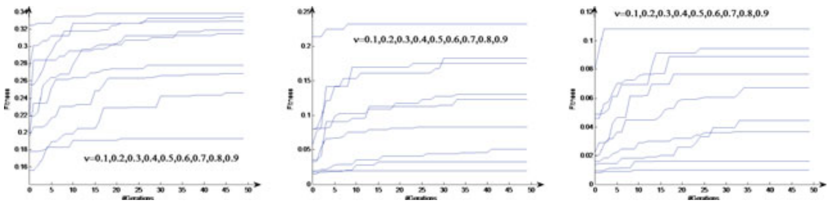


Fig. 3 RMSE values of the reduced rule-based models versus reduction level ν



(a) Abalone (b) Auto MPG (c) Concrete Strength

Fig. 4 Values of fitness function reported in successive GA generations for ν ranging from 0.1 to 0.9 with step 0.1 show direction of ν

5 Isolation of Input Variables

To enhance the transparency of the rules, we express the fuzzy set A_i as a Cartesian product of a single *isolated* fuzzy set defined in $\boldsymbol{R}$ and a *relational* remainder expressed in $\boldsymbol{R}^{n-1}$. In other words, we form the expression describing the condition part as follows

$$A_i^{\wedge}(x_j) \times A_i^{\wedge} \sim (x^{\sim}) \tag{8}$$

where $A_i^{\wedge}$ is a fuzzy sets defined in $\boldsymbol{R}$ and $A_i^{\sim}$ is expressed in $\boldsymbol{R}^{n-1}$. Then the rules of the form read as follows

$$-\, if\; x_j\; is\, A_i^{\wedge}(x_j)\; and\; x^{\sim} A_i^{\sim}(x^{\sim})\; then\; y\; is\, f_i(x, a_i) \tag{9}$$

Here a certain input variable (jth one) has been selected to be isolated. The term isolation pertains to the fact that a certain variable has been chosen and subsequently a fuzzy set isolated from the fuzzy set A_i is treated separately and thus becomes more visible and interpretable.

Obviously, the above Cartesian product is not identical to the original A_i that is the following holds

$$A_i \neq (A_i^{\wedge} \times A_i^{\sim}) \tag{10}$$

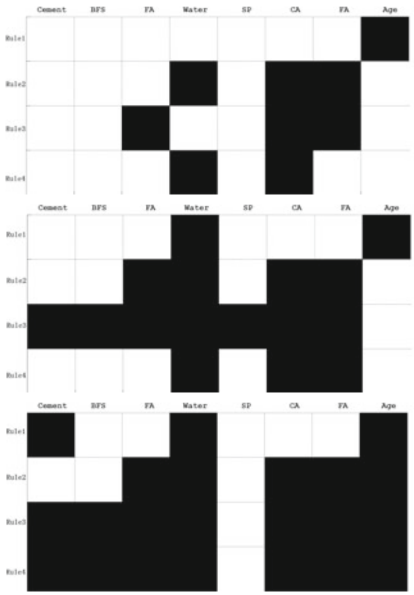
In terms of membership functions this means that the following relationship holds

$$A_i(\boldsymbol{x}) \neq \min(Ai^{\wedge}(xj), Ai^{\sim}(x^{\sim})) \tag{11}$$

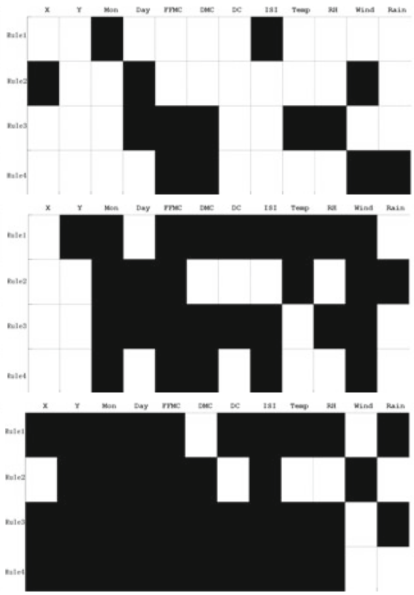
(a) Abalone; ν=0.2, 0.4, 0.5 (b)Auto MPG; ν=0.3, 0.5, 0.7

(c) Boston Housing; ν=0.4, 0.6, 0.9 (d) Computer Activity; ν=0.3, 0.4, 0.9

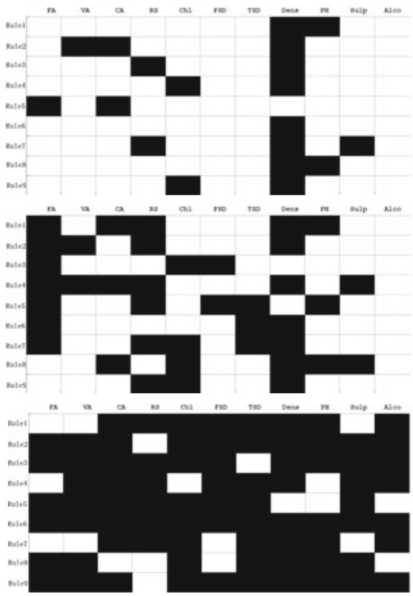
Fig. 5 Visualization of reduced rules: shaded regions identify the input variables being retained



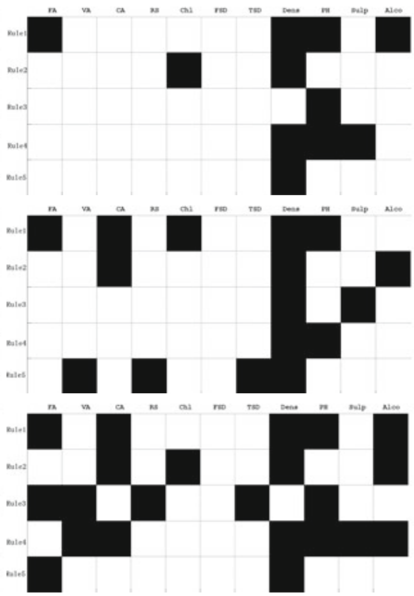
(e) Concrete Strength ν=0.3, 0.5, 0.7



(f) Forest Fires ν=0.3, 0.6, 0.8



(g) Red Wine Quality; ν=0.2, 0.4, 0.8



(h) White Wine Quality; ν=0.2, 0.3, 0.4

Fig. 5 (continued)

Note that the corresponding membership functions of $A_i^{\wedge}(x_j)$ and $A_i^{\sim}(\boldsymbol{x}^{\sim})$ are computed as follows

$$A_i^{\wedge}(x_j) = \frac{1}{\sum_{l=1}^{c}\left(\frac{x_j - v_{ij}}{x_j - v_{il}}\right)^{2/(m-1)}} \tag{12}$$

$$A_i^{\sim}(x^{\sim}) = \frac{1}{\sum_{l=1}^{c}\left(\frac{||x^{\sim} - v_i^{\sim}||}{||x^{\sim} - v_l^{\sim}||}\right)^{2/(m-1)}} \tag{13}$$

The consequence is that if $A_i^{\wedge}(x_j) \times A_i^{\sim}(\boldsymbol{x}^{\sim})$ is used as the condition part of the *i*th rule, the output of the model is going to be different than the original rule. It is likely that the accuracy of the model could be reduced as a result of the increased interpretability of the rules because of the isolation of the input variables. In the above formulation, one is interested in choosing an individual variable (*j*th one) for which the results provided by the rule-based model are as close as possible to those formed by the original fuzzy model. The selection of the input variable is quite straightforward through a direct enumeration.

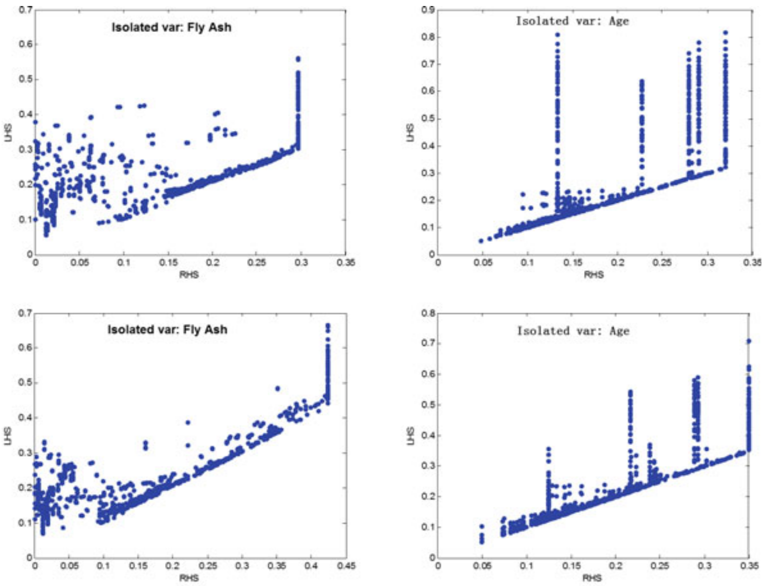
The plots showing the LHS and the RHS relationship is shown for concrete strength with one or five isolated input variables.

From the plots above, several observations of a general character can be drawn. First, the values of the LHS are higher than the corresponding ones for the RHS. This is reflective of the fact that the separation of the variable(s) leads to the higher activation levels of the rules with eventual reduction of the specificity of the results of reasoning. It is also apparent that with the increase of the variables being isolated – compare Fig. 6a, b, the differences between the values produced by the LHS and RHS of the expression (10) are more profound. Again, this is not surprising as by isolating more variables we depart from the RHS more vigorously.

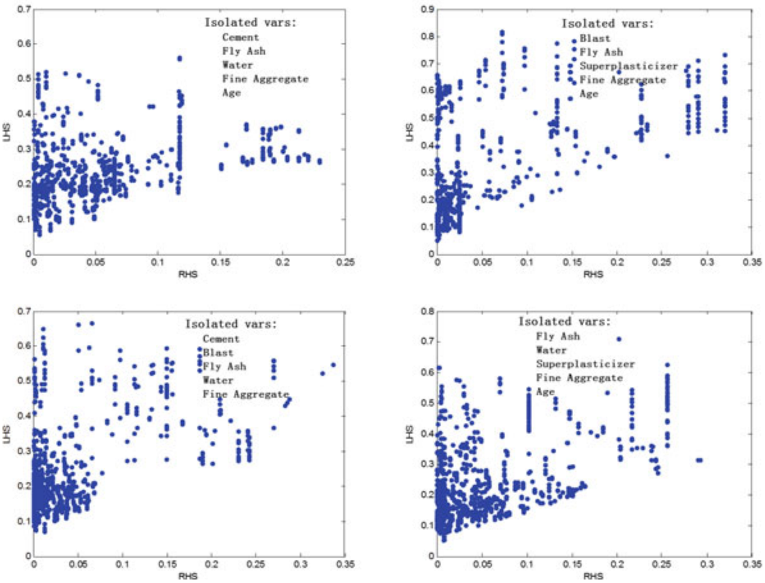
In a general setting, one can realize an isolation of L input variables, which as a result leads to the rules in the form

$$\begin{aligned} &- \text{if } x_{j1} \text{ is } A_i^{\wedge} x_{j1} \textit{ and } x_{j2} \textit{ is } A_i^{\wedge}(x_{j2}) \\ &\textit{and} \cdots \textit{and } x_{jL} \text{ is } A_i^{\wedge}(x_{jL}) \textit{ and } x^{\sim}\ A_i^{\sim}(\mathbf{x}^{\sim}) \\ &\qquad\qquad \text{then y is } f_i(\mathbf{x}, \mathbf{a}_i) \end{aligned} \tag{14}$$

note that in this case $\boldsymbol{x}^{\sim}$ is defined in $\boldsymbol{R}^{n-L}$.



(a) single input variable isolated



(b) 5 isolated input variables

Fig. 6 Values of the original activation of the rules value versus the one with isolated input variables - concrete strength data set, 4 rules in the rulebase

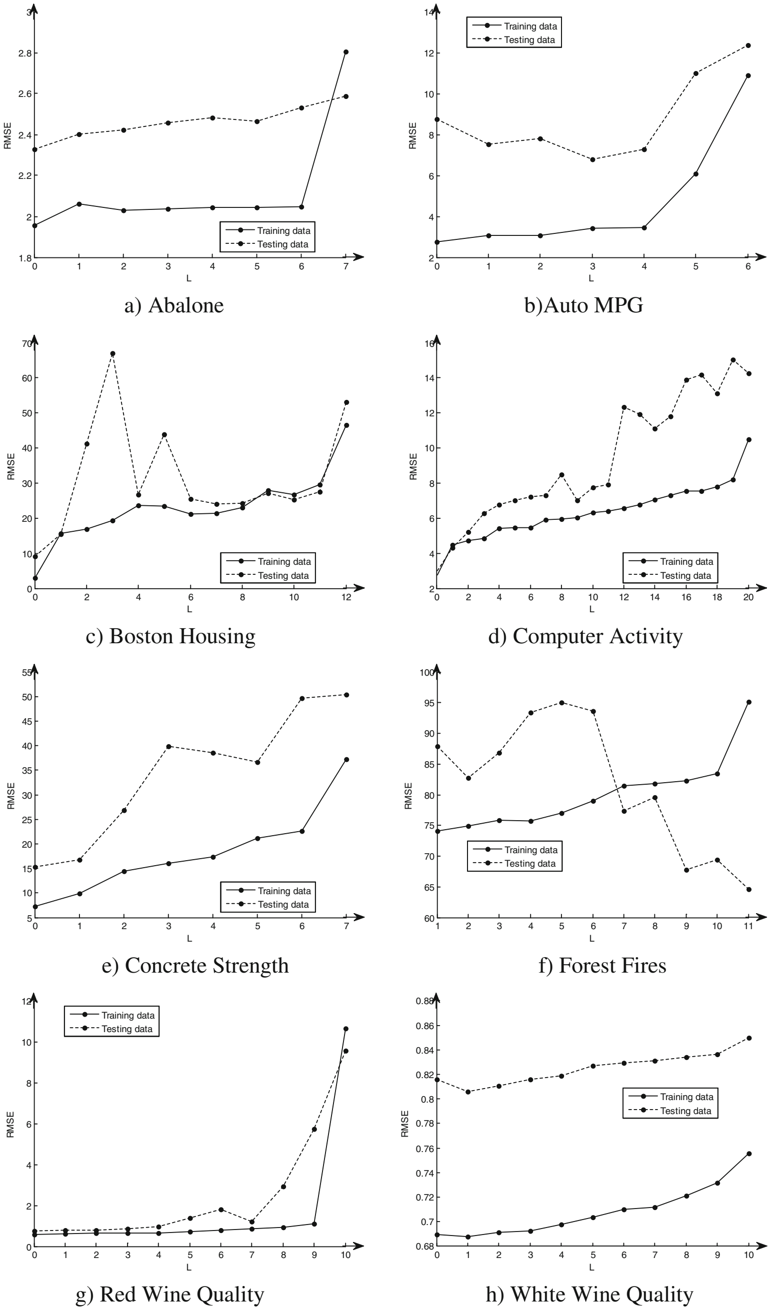
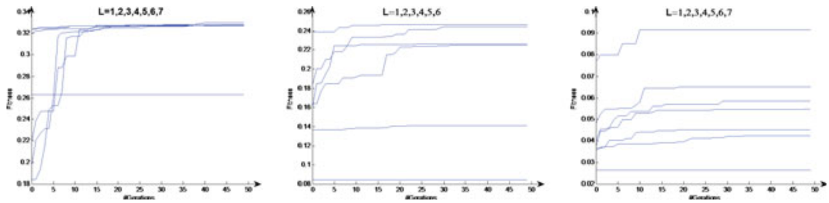


Fig. 7 Performance of the fuzzy model versus the number of isolated input variables



(a) Abalone (b) Auto MPG (c) Concrete Strength

Fig. 8 Fitness function reported in successive GA generations

Here an optimal choice of L variables gives rise to a combinatorial optimization problem. This could be solved by GA optimization. Considering that the value of L is specified in advance, GA forms an optimal isolation matrix $\mathbf{I}$ consisting of c rows (number of rules) and n rows (number of input variables) where in each row there are L 1 s indicating the variables which are isolated in the rule. For instance, for $L = 3$ the matrix with the entries

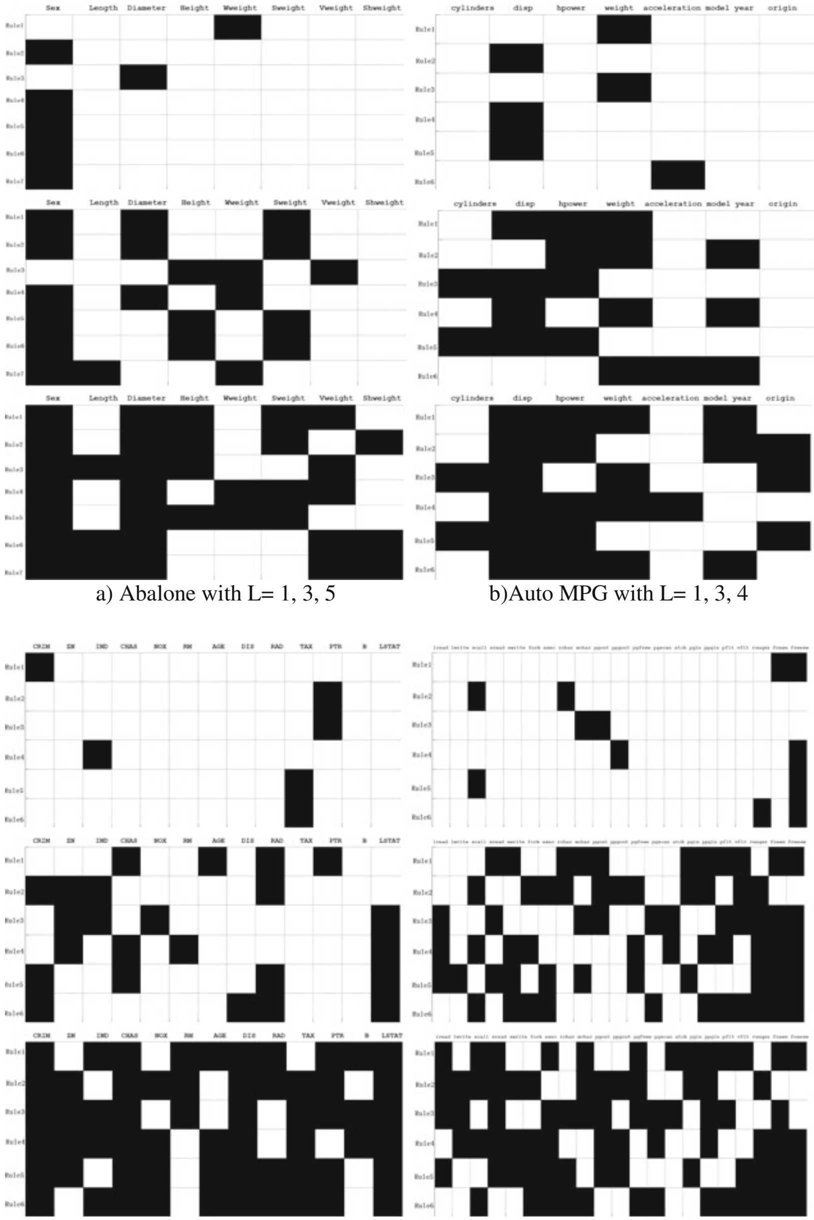
$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ \ldots \end{bmatrix} \tag{15}$$

states that in the first rule isolated are variables 1, 4, and 5; in the second rule we isolate variables 2, 3, and 4, and so on.

6 Experiments

The GA was carried out with 100 populations and maximum 50 generations. Preliminary experiments with GA indicated lack of improvement before 50th generation, so the maximum generation is set to 50. The population is not large, so relative large values of moderate crossover and mutation rates are used. The crossover rate is set as 0.8 and mutation rate is 0.1.

We present the results in a similar way as before by focusing on the presentation of the rules with isolated variables and showing how the families of isolated variables impact the performance of the model (Figs. 7, 8 and 9).



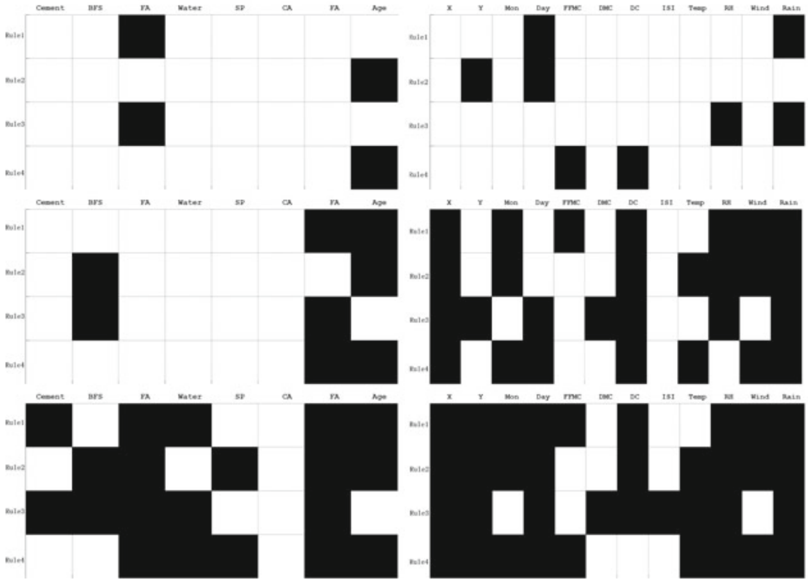
a) Abalone with L= 1, 3, 5

b)Auto MPG with L= 1, 3, 4

c) Boston Housing with L= 1, 4, 10

d) Computer Activity with L= 2, 11, 13

Fig. 9 Isolated variables (shaded) obtained for selected values of L



e) Concrete Strength with L= 1, 2, 5

f) Forest Fires with L= 2, 7, 9

(g) Red Wine Quality with L= 4, 5, 7

(h) White Wine Quality with L= 1, 3, 5

Fig. 9 (continued)

7 Conclusions

The two approaches enhancing the interpretability of rule-based models are directly applied to the already constructed Takagi-Sugeno fuzzy models realized with the use of fuzzy clustering. Both the formation of the input subspace of conditions as well as the isolation of the input variables are the methods refining multivariable fuzzy sets produced through fuzzy clustering. The proposed approaches are quantifiable in terms of the level of the interpretability abilities offered by them (expressed either in terms of the number of variables eliminated or the variables isolated). This aspect is helpful in determining how much the interpretability could be enhanced without any significant sacrifice of accuracy of the model. Furthermore in this way one could reveal input variables (or their combinations) that are essential in rule-based modeling.

The approach offers a certain new view at the enhancement of fuzzy rule-based models. There could be several avenues worth pursuing in the future including (a) development of a hybrid arrangement of the formation of subspaces of conditions of the rules associated with some further isolation of variables from such subspaces, (b) use of other techniques of Evolutionary Optimization in the entire process, (c) construction of interpretability measures quantifying various facets of the interpretation mechanisms.

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Witold Pedrycz is a Professor and Canada Research Chair (CRC) in Computational Intelligence in the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada. He is also with the Systems Research Institute of the Polish Academy of Sciences, Warsaw, Poland. He also holds an appointment of special professorship in the School of Computer Science, University of Nottingham, UK. In 2009 Dr. Pedrycz was elected a foreign member of the Polish Academy of Sciences. In 2012 he was elected a Fellow of the Royal Society of Canada. Witold Pedrycz has been a member of numerous program committees of IEEE conferences in the area of fuzzy sets and neurocomputing. In 2007 he received a prestigious Norbert Wiener award from the IEEE Systems, Man, and Cybernetics Council. He is a recipient of the IEEE Canada Computer Engineering Medal 2008. In 2009 he has received a Cajastur Prize for Soft Computing from the European Centre for Soft Computing for “pioneering and multifaceted contributions to Granular Computing”. In 2013 has was awarded a Killam Prize. In the same year he received a Fuzzy Pioneer Award 2013 from the IEEE Computational Intelligence Society. His main research directions involve Computational Intelligence, fuzzy modeling and Granular Computing, knowledge discovery and data mining, fuzzy control, pattern recognition, knowledge-based neural networks, relational computing, and Software Engineering. He has published numerous papers in this area. He is also an author of 15 research monographs covering various aspects of Computational Intelligence, data mining, and Software Engineering. Dr. Pedrycz is intensively involved in editorial activities. He is an Editor-in-Chief of Information Sciences and Editor-in-Chief of WIREs Data Mining and Knowledge Discovery (Wiley). He currently serves as an Associate Editor of IEEE Transactions on Fuzzy Systems and is a member of a number of editorial boards of other international journals.

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Dr. Reformat is a past president of the North American Fuzzy Information Processing Society, and a vice president of the International Fuzzy Systems Association. He has been a member of program committees of multiple international conferences related to Computational Intelligence and Software Engineering.

Geospatial Uncertainty Representation: Fuzzy and Rough Set Approaches

Frederick Petry and Paul Elmore

Abstract Uncertainty in geospatial data is often considered in the context of geographical information systems which enable a variety of operations and manipulation of spatial data. Here we consider how both fuzzy set and rough set theory has been used to represent geospatial data with uncertainty. Terrain modeling and triangulated irregular networks techniques utilizing fuzzy sets are presented. Rough set theory is overviewed and its application to spatial data is described. Issues of uncertainty in the representation of spatial relationships such as topological and directional relationships are discussed.

Keywords Geographic information systems • Spatial database • Fuzzy sets • Rough sets • Triangulated irregular networks • Indiscernibility relation • Spatial relations • Upper and lower approximations

1 Introduction

Representation of uncertainty in geospatial data can viewed in context of the most important use of such spatial data in geographic information systems (GIS). GIS are employed extensively throughout governmental and industrial organization for planning and decision making [1]. Typically underlying a GIS is a spatial database in which many of the representation issues arise [2, 3].

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_24

Of particular interest for geographical information systems are the ongoing efforts in the area known as “big data” [4, 5]. Typically with respect to big data concerns, a major issue involves how to efficiently utilize the variety of sources providing a vast amount of heterogeneous data [6]. Special emphasis is on environmental data that has spatial aspects or with a spatial basis [7]. Sensor and instrumentation advances have tremendously increased the variety and amount of remote sensed data available. In particular at the NASA EOSDIS (Earth Observing System Data and Information System) the imagery data archived is greater than 3 PB (Petabytes). Furthermore this system is creating each day an additional 5 TB (Terabytes) of data. To effectively utilize such volumes of data, data mining techniques are very critical [8]. One factor that must be considered in particular is how to deal with the inherent uncertainty involved with the huge amount of such spatial data in databases.

Uncertainty has been widely accepted as an implicit factor in geographical data [9, 10]. Varsi [11, 12] has stated that vagueness is one of the principal issues in geography, since concepts such a river’s length or a peak’s height in a specific area are uncertain as the specification of a river or peak are vague concepts. Couclelis [13] posits that uncertainty is an intrinsic property of complex geospatial knowledge and as such should be managed properly. It is not simply a problem or flaw to be reduced or eliminated. Many of the problems associated with data are prevalent in all types of database systems. Spatial databases and GIS contain descriptive as well as positional data. The various forms of uncertainty occur in both types of data, so many of the issues to be discussed apply to ordinary databases as well. These same techniques, including integration of data from multiple sources [14], time-variant data, uncertain data, imprecision in measurement, inconsistent wording of descriptive data, and “binning” or grouping of data into fixed categories, also are employed in spatial contexts [15].

Figure 1 illustrates the complexity that can be observed in a real world environment. This figure is an image of the Louisiana gulf coastal region in the area of the Atchafalaya Bay and illustrates the difficulty of specifying the characteristics of the spatial features. The boundary between the coastline and the Gulf of Mexico, the relationship of the various waterways and their characterization are difficult to specify as they exhibit both spatial and temporal uncertainty.

In this chapter we will examine some of the approaches that have been taken to represent various aspect of geospatial data with fuzzy set and rough set techniques. Here we consider how both fuzzy set and rough set theory has been used to represent geospatial data with uncertainty. Terrain modeling and triangulated irregular networks techniques utilizing fuzzy sets are presented. Rough set theory is overviewed and its application to spatial data is described. Issues of uncertainty in the representation of spatial relationships such as topological and directional relationships are discussed.

Fig. 1 Gulf of Mexico coastal region: Atchafalaya Bay area

2 Fuzzy Set Representations

2.1 Background

In general, the idea of implementing fuzzy set theory as a way to model uncertainty in spatial databases has a long history. Geographical research investigations of the use of fuzzy set approaches [16] involved areas such as geographical decision-making and behavioral geography [17, 18]. However, the most consistent considerations of the utilization of fuzzy set theory in applications to geographic information systems was developed initially by Robinson [19]. Several models appropriate to this situation were considered —fuzzy database representations using simple membership values in relations, and a similarity–based approach for geospatial features. In an application for which both the data as well as spatial relationships are imprecise, he modeled this situation as one that entails imprecision intrinsic to natural language which is possibilistic in nature.

In the following period of time there were a number of efforts using fuzzy set approaches for spatial databases developed. These included among others: querying spatial information [20], representing spatial relationships [21], and object-oriented modeling [22, 23]. Models have been proposed as well that allow for enhancing database models to manage uncertain geospatial data [24]. A major motivation for many approaches are the uncertain boundaries associated with geographic features and objects. Fuzzy sets provide a natural representation for such forms of uncertainty

2.2 Terrain Modeling

Digital elevation models (DEM) have been used to produce fuzzy representations of terrain features. In Skidmore [25] a formulation of Euclidean distances from the nearest streamline and ridgeline was used to represent the relative position, of a specified location. However to represent finer local morphological characteristics of a location, a Euclidean distance may not always be adequate.

K-means approach. Another approach [26] used fuzzy k-mean classifiers to provide a continuous classification of terrain features. Since fuzzy k-means is predominately an unsupervised classification, it may have a problem to produce results that provide a compatible matching to a landscape with the views provided by domain experts such as soil scientists.

Rule-based approach. MacMillan et al. [27] developed a sophisticated fuzzy classification of terrain features using a comprehensive rule-based approach. This method is particularly suitable for problems requiring intensive terrain analysis operations and needs extensive information of local landforms from users..

Similarity approach. Locational similarity is another approach to determining a specific area's degree of membership in a spatial category. This is done by computing a similarity of a location of interest to a typical location for the terrain morphology [28]. For example special terrain features can have a very specific interpretations by soil-landscape analysts where unique conditions of soil types occur in locations of interest. Both a knowledge-based and definition-based approach are presented as ways to specify the needed proto-typical locations. Straightforward rules based on definitions can be utilized to determine the proto-typical locations when there is a clear geomorphology. For example there are algorithms for determining ridgelines and streamlines that can be used. However, if a terrain feature has only has a local or regional meaning, finding the typical location may require knowledge from local experts. This may be captured through manual delineation using a GIS visualization tool.

The similarities of any other location to those specified typical locations can be evaluated based on a set of selected terrain attributes such as elevation, slope gradient, curvatures, etc. The process of assigning fuzzy membership value to a location then consists of three steps:

1. A similarity evaluation of a location of interest and a typical location based on the individual terrain attribute level is calculated.
2. The similarities are then aggregated based on individual terrain attributes. This produces an overall similarity between a test location of interest and a typical location.
3. Finally the similarities of the location of concern with all typical locations are integrated. This results in the test location's final fuzzy membership relative to the terrain feature of interest.

2.3 Fuzzy Triangulated Irregular Networks

A common approach to represent field data in contrast to object-based spatial data is the Triangulated Irregular Networks (TINs). A TIN represents a surface in a GIS by a partitioning of the 2-dimensional space into a network of non-overlapping triangles. Applications using TINs in a GIS have motivated TIN extensions (ETINs) [29] by using fuzzy memberships, fuzzy numbers and type-2 fuzzy sets. The ETIN structure uses a mapping function S that specifies a geographic area's property.

For example consider the problem of selection and acquisition of a site for a manufacturing facility which is "Close to" Seattle, Washington. So S = 1.0 for the function indicates a location "near " Seattle and S = 0.0 is interpreted as "far" from Seattle. An intermediate value such as S = 0.5 could be interpreted as being "more or less" close to the city.

Another ETIN alternative extension uses the common, easily implemented representation by triangular membership function for fuzzy numbers. So here it is necessary to extend the fuzzy number associated data types with the associated data value for a point from a simple (crisp) value to a fuzzy set. By associating a triangular membership function, this can be accomplished at every point of the region of interest. There are then three characterizing points of importance: two points where the membership grade equals 0 which delimit the membership function, and the intermediate point for which the membership grade equals 1.

Lastly an ETIN extension can be based on type-2 fuzzy sets, which are a generalization of regular fuzzy sets. This type permits imprecision as well as the modeling of uncertainty regarding the membership grades. Again in our example we can consider how to capture the certainty about the extent to which a site can be characterized as "close to" Seattle. If we wish to describe the location of some person, there could be doubt as to where they are exactly located. The individual could be in a location close to Seattle, but also near Tacoma Washington. This sort of imprecision or doubt can be modeled by using a type-2 fuzzy set: the membership grade on every location is extended to a "fuzzy" membership grade. As a result, every point will now have an associated fuzzy set over [0,1].

3 Rough Set Approaches to Spatial Data Representation

Another approach for uncertainty representation uses the rough set concept of indiscernibility of values. We first give a background of rough sets and spatial data. Then we provide the overview needed of rough set theory and discuss the rough set model for imprecise spatial data.

3.1 Background

A number of geospatial research investigations have considered the applications of rough set approaches. The ROSE system [30], proposed a description of spatial data using rough sets. This approach focused on a formal modeling framework for realm-based spatial data types in general. Based on the resolution at which data is represented, Worboys [31] developed a model for imprecision of spatial data and applied it to issues related to the integration of such data. This approach relies on the use of indiscernibility – a central concept in rough sets. However it does not carry over the entire framework and is just described as "reminiscent of the theory of rough sets" [32]. The definition of a rough classification of spatial data and representation of inexact spatial locations using rough sets was developed by Ahlqvist et al. [33]. This work also developed a quality metric of a rough classification as compared to a crisp classification. This technique was evaluated on ground truth data from vegetation map layers. In it combinations of rough and fuzzy set approaches were utilized for reclassification as needed to obtain the integration of geographic data.

Other research investigators in a geographical information systems and mapping environments [34] have developed an approach for the field representation of a spatial entity using a rough raster space which was evaluated for remote sensing images in a classification case study. Using K-labeled partitions, which can represent maps, Bittner and Stell [35] developed their relationship to rough sets in order to approximate map objects with vague boundaries. Also they investigated stratified partitions and extended this approach using the concepts of stratified rough sets. For example in the case of map scale transformations, this approach can obtain more refined levels of details or granularity in many applications.

3.2 Rough Set Theory

Rough set theory [36] provides another mathematical formalism for capturing uncertainty. The partitioning of some universe U into equivalence classes is known as an approximation region in rough set theory. Such a partitioning can be modified by increasing or decreasing its granularity, used to group elements together that, for a particular application, are considered indiscernible, or to "bin" ordered domains into range groups. The following is a set of common terminology and notation for rough sets:

U is the universe, which cannot be empty,

R : indiscernibility relation, or equivalence relation,

$A = (U, R)$, an ordered pair, called an approximation space,

[x]R denotes the equivalence class of R containing x, for any element x of U,

elementary sets in A - the equivalence classes of R.

Any finite union of these elementary sets in A is called a definable set. A particular rough set $X \subseteq U$, however, is defined in terms of the definable sets by specifying its lower $(\underline{R}X)$ and upper $(\overline{R}X)$ approximation regions:

$$\underline{R}X = \{x \in U | [x].R \subseteq X\}$$

and

$$\overline{R}X = \{x \in U | [x].R \cap X \neq \emptyset\}$$

$\underline{R}X$ is the R-positive region, $U - \overline{R}X$ is the R-negative region, and $\overline{R}X - \underline{R}X$ is the R-boundary or R-borderline region of the rough set X. This allows for the distinction between certain and possible inclusion in a rough set. The set approximation regions provide a mechanism for determining whether something certainly belongs to the rough set, may belong to the rough set, or certainly does not belong to the rough set. X is called R-definable if and only if $\underline{R}X = \overline{R}X$. Otherwise, $\underline{R}X \neq \overline{R}X$ and X is rough with respect to R. In Fig. 2 the universe U is partitioned into equivalence classes denoted by the rectangles. Those elements in the lower approximation of X, $\underline{R}X$, are denoted with the letter "p" and elements in the R-negative region by the letter "n". All other classes belong to the boundary region of the upper approximation.

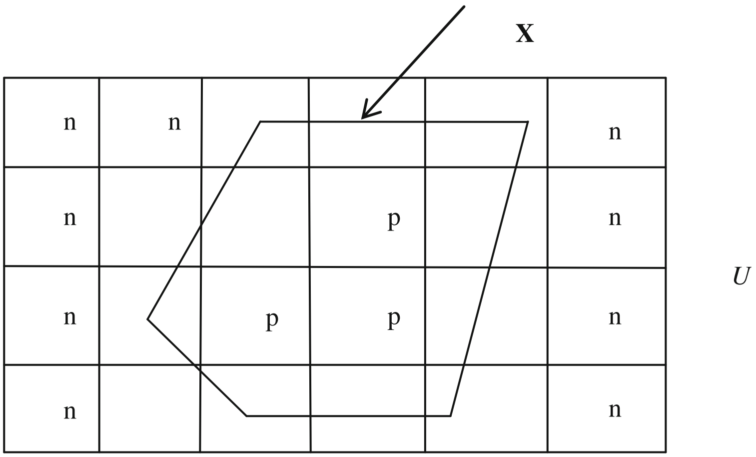

Fig. 2 Illustration of the concept of a rough set X

Rough sets use an indiscernibility relation to partition domains into equivalence classes, and lower and upper approximation regions for distinguishing between certain and possible (or partial) inclusion in a rough set. The indiscernibility relation allows us to group items based on some definition of 'equivalence', which basically depends on the application domain. We may use this partitioning to increase or decrease the granularity of a domain, to group items together that are considered indiscernible for a given application, or to "bin" ordered domains into range groups. For example, for data mining applications one may need to vary the partitioning of domains in systematic ways in the process of discovering rules and relationships in the data. The indiscernibility relation is fundamental to rough sets. The approximation regions can only be defined in terms of the indiscernibility relation and the equivalence classes it creates.

To obtain possible results, in addition to the obvious, certain results encountered in querying an ordinary spatial database system, we may employ the use of the boundary region information in addition to that of the lower approximation region. The results in the lower approximation region are certain. They correspond to exact matches. The boundary region of the upper approximation contains those results that are possible, but not certain.

3.3 Rough Set Spatial Example

Let us assume we have a spatial area with a particular set of spatial objects comprising the universe U:

$$U = \{\text{mast, brook, bayou, river, woods, thicket, pasture, field}\}.$$

The particular equivalence relation R* for this specific area might be defined as follows:

$$R^* = \{[\text{mast}],\ [\text{brook, bayou, river}],\ [\text{woods, thicket}],\ [\text{pasture, field}]\}$$

Then given some subset X of spatial objects

$$X = \{\text{mast, brook, bayou, river, woods, pasture}\},$$

we would like to define X in terms of its lower and upper approximations:

$$\underline{R}X = \{\text{mast, brook, bayou, river}\},$$

$$\overline{R}X = \{\text{mast, brook, bayou, river, woods, thicket, pasture, field}\}$$

The equivalence classes that are totally included in the subset X form the lower approximation. Then the upper approximation contains the lower approximation and the classes that are only partially contained in X. For this example all the values in the classes [mast] and [brook, bayou, river] are included in X. Hence these belong to the lower approximation region. Note that the class [woods, thicket] is not completely contained in X as X does not contain "thicket." But, [woods, thicket] is part of the upper approximation since "woods" $\in$ X. In the example given, the rough set is:

$$
\begin{aligned}
&\{\{\text{mast, brook, bayou, river, woods, pasture}\}.\\
&\{\text{mast, brook, bayou, river, woods, field}\}\\
&\{\text{mast, brook, bayou, river, thicket, pasture}\}\\
&.\{\text{mast, brook, bayou, river, thicket, field}\}\}
\end{aligned}
$$

Although rough set theory defines the set in its entirety this way, for most applications one typically will be dealing with only certain parts of this set at any given time.

3.4 Spatial Data Uncertainty and Rough Sets

The approximation regions of rough sets are useful when information related to spatial data regions is queried. Consider a region such as a woodland. One can reasonably conclude that any grid point labeled as "woods" which on all sides is surrounded by grid points also classified as "woods" is, indeed a point characterized by the feature "woods". But we may also be interested in grid points labeled as "woods" that adjoin points identified as "field". Here it is possible that such points represent field areas as well as forest areas but were identified as "woods" during the classification. Likewise, points identified as "field" but adjacent to "woods" points may represent areas that contain part of the forest.

This uncertainty maps naturally to the use of the approximation regions of rough set theory, where the lower approximation region represents certain data and the boundary region of the upper approximation represents uncertain data. It applies to spatial database querying and spatial database mining operations.

If we force a finer granulation of the partitioning, a smaller boundary region results. This occurs when the resolution is increased. As the partitioning becomes finer and finer, finally a point is reached where the boundary region is non-existent. Then the upper and lower approximation regions are the same and there is no uncertainty in the spatial data as can be determined by the representation of the model.

3.5 Rough Sets for Gridded Data

For spatial data with a raster data or other non-vector type formats as used for continuous variables such as temperatures, soil types and population densities, such data is often mapped with a particular grid representation. A regular matrix-like structure organizes the spatial coordinates and the specific data is associated with point locations on the grid. Clearly for improved data representation, a finer resolution or scale of a grid is desired. However there is a tradeoff between the resolution and the processing required as higher resolutions provide more information, but at a cost of execution complexity and memory space needed.

It is clear that from the rough set data viewpoint, the process of rasterizing or gridding data. represents an inherent indiscernibility. Consider for example grid locations representing various object /feature classifications such as forested, agricultural, residential and industrial areas. We find that certain grid points may directly correspond to one of these classifications but some are in between one or more of them. A data item at a particular grid point in essence may represent data near the point as well. This is due to the fact that often point data must be mapped to the grid using techniques such as nearest-neighbor, averaging, or statistics. We may set up our rough set indiscernibility relation so that the entire spatial area is partitioned into equivalence classes where each point on the grid belongs to an equivalence class. If we change the resolution of the grid, we are in fact, changing the granularity of the partitioning, resulting in fewer, but larger classes.

4 Representation of Spatial Relations

Relationships among spatial objects can generally be classified in three types:

1. Topological - Touches, Disjoint, Overlap,... The border of Vietnam touches the Laotian border
2. Directional - West, South-East, ... Chicago is West of New York
3. Metric – Distance San Francisco is about 50 km from San Jose

The nine-intersection model allows many topological relations between two objects C and D to be specified. This model uses the intersections between the interior, exterior and boundary of C and D [37]. This section will describe a variety of approaches introducing uncertainty into these relationships.

4.1 Spatial Relations

Papadias et al. [38] describe a process for determining configuration similarity for spatial constraints involving distance, direction and topology. Their approach uses the centroids of objects for distance and extended objects for direction and topology. Uncertainty in the areas of fuzzy relations such as objects satisfying multiple directional constraints are handled in their approach. Additionally fuzziness related to linguistic relationship terms is supported as well. To allow comparison of alternative conceptualizations of direction, the concept of graded sections was developed by another research effort [39]. Section bundles are introduced in order to describe graded sections. This technique provides a formal means to:

(1) compare alternative candidates related, for example by a direction relation like “south-west” or “east,”
(2) characterize the candidates relative to their quality - between “satisfactory” and “not as satisfactory” candidates, and
(3) select the optimal candidate from the set of possibilities.

Vazirgiannis [40] also considers the problem of representing uncertain topological, directional, and distance relationships on the assumption of crisply bounded objects. All relationship definitions for this approach are centroid-based. A minimal set of topological relations, overlapping and adjacency, are defined based on Egenhofer’s boundary/interior model [37]. This model is enhanced by providing degrees of relationship satisfaction. Direction relations are defined by a sinusoidal function based on the angle between two objects’ centroids. The linguistic terms corresponding to “near” and “distant” are used for to provide characterizations of distances, where the ratio of the distance to a maximum application-dependent distance determines the degree of membership in one of these categories. The three relationships are combined for query retrieval. Afterward, a similarity measure is computed for each relationship and then combined into a single, overall similarity measure.

Another approach to spatial relations uses the histogram of forces [41] to provide a fuzzy qualitative representation of the relative position between two-dimensional objects. This can also be used in scene description where relative positions are represented by fuzzy linguistic expressions. In Guesgen [42] we see the introduction of several approaches for reasoning about fuzzy spatial relations, including an extension of Allen’s algorithm and additionally methods for fuzzy constraint satisfaction. Also relevant is [43] which presents a unified framework for approximate spatial and temporal reasoning using topological constraints as the representation schema and fuzzy logic for representing imprecision and uncertainty. The application of the resulting fuzzy representation to each of Allen’s interval relationships [44] is developed as the possibility of the occurrence of the conditions of the original definition.

Another approach using Allen’s relationships and based on minimum bounding rectangles (MBRs) was developed by Cobb and Petry [45, 46]. A minimum bounding rectangle is determined by the smallest X-Y parallel rectangle which completely encloses a spatial object and so can provide an effective approximation of the geometry of such objects. The wide use of MBRs in geographic information systems and spatial databases is because they provide a computationally efficient way of locating and accessing objects in a spatial realm. Extending Allen’s temporal relationships [46] into the spatial domain permits the representation of any relationship that can occur between two one-dimensional (temporal) intervals including: before, equal, meets, overlaps, during, starts, and finishes, along with their inverses.

The binary relationship between objects in both the vertical and horizontal direction can be entirely defined by a tuple, [rx, ry] as determined by the minimum bounding rectangles of two objects. Here rx is the one of the Allen’s temporal relations described above that defines the interaction of the object MBRs in the x direction, and ry represents the same for the y direction. For example, for the case of the relationship, A [finishes, starts] B, the definition is given as:

$$\{B_{xl} < A_{xl} < B_{x2}, A_{x2} = B_{x2}, B_{yl} < A_{y2} < B_{y2}, A_{yl} = B_{yl}\}$$

where {x1,y1} and {x2, y2} represent the lower left and upper right corners, respectively, of the minimum bounding rectangles.

In Fig. 3 is an example set of four object MBRs, {A,B,C,D}. A subset of the existing relationships between them consists of:

$$\{A[\text{before, overlaps}]B;\ B[\text{before, overlaps} - 1]C; D[\text{during, meets}]C\}.$$

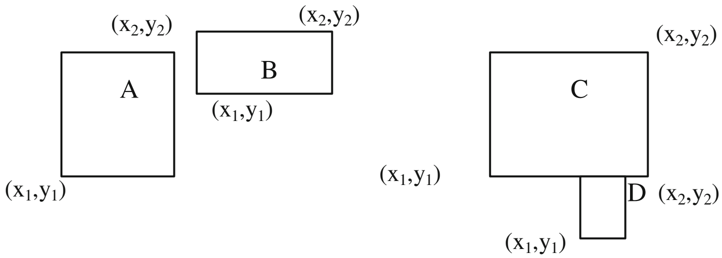

Fig. 3 MBR relationships: A [before, overlaps] B; B [before, overlaps $^{-1}$] C; D [during, meets] C

5 Conclusion

We have described a number of approaches using fuzzy set or rough set theory to describe various aspects of spatial as might be utilized in geographic information systems. To provide modeling of finer details of spatial data type 2 fuzzy sets [47] have been considered [48, 49] and other possible approaches include intuitionistic fuzzy sets [50].

Acknowledgments We would like to thank the Naval Research Laboratory's Base Program, Program Element No. 0602435 N for sponsoring this research.

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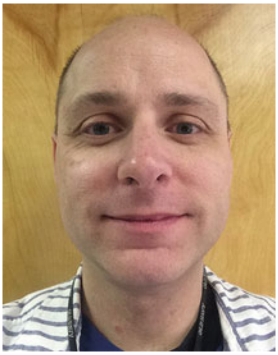

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How to Efficiently Diagnose and Repair Fuzzy Database Queries that Fail

Olivier Pivert and Grégory Smits

Abstract Telling the user that there is no result for his/her query is very poorly informative and corresponds to the kind of situation cooperative systems try to avoid. Cooperative systems should rather explain the reason(s) of the failure, materialized by Minimal Failing Subqueries (MFS), and build alternative succeeding queries, called maXimal Succeeding Subqueries (XSS), that are as close as possible to the original query. In the particular context of fuzzy querying, we propose an efficient unified approach to the computation of gradual MFSs and XSSs that relies on a fuzzy-cardinality-based summary of the relevant part of the database.

1 Introduction

The paradigm of cooperative answering is originated from the works in the context of natural-language question-answering done by Kaplan [1] at the end of the seventies. Cooperative responses to a query are indirect responses or instructions that are more helpful to the user than direct, literal responses would be. Interest in cooperative responses in the database field arose in the middle of the eighties [2–6]. A major objective of cooperative answering systems is to avoid producing "there is no result" when a query fails. Cooperative systems should rather explain the reason(s) of the failure and suggest succeeding queries that are as close as possible to the original one. Whereas most cooperative techniques from the literature deal with the "empty answer set" problem in a classical (Boolean) query setting, in this paper, we rather consider *fuzzy* queries that express user *preferences*. We address the situation where a fuzzy query "fails" — we will see below that this term may take different meanings — and we propose an explain-and-repair strategy relying on a summary of a part of the database. This summary provides information about the distribution of the data

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_25

over the different predicates involved in the failing query. Moreover, the complexity of the summarization process only linearly depends on the size of the database, and this fuzzy-cardinality-based summary is then efficiently used to detect the original reasons of the query failure and to identify succeeding subqueries.

With respect to Boolean queries, fuzzy queries reduce the risk of obtaining an empty set of answers since the use of a finer discrimination scale — [0, 1] instead of {0, 1} — increases the chance for an element to be considered somewhat satisfactory. Nevertheless, the situation may occur where none of the elements of the target database satisfies the query even to a low degree. In the context of fuzzy queries, beside the empty answer set (EAS) problem *stricto sensu*, another situation may be considered as a failure: that where the answer set is not empty but only contains elements which satisfy to a *low degree* the preferences specified in the user query. We show in this paper that a generic type of approach that leverages fuzzy cardinalities may be employed to (i) provide explanations for both types of situations (empty or unsatisfactory answer set), and (ii) suggest relevant non-failing alternative queries. As we will see, minimal failing subqueries [7] constitute useful explanations about the conflicts in a failing query, that may be used to set up a semi-automatic and targeted relaxation strategy. This relaxation strategy consists in identifying the maximal succeeding subqueries of an initial failing query and to let users decide which relaxation they are ready to accept.

The remainder of the paper is structured as follows. Section 2 provides a refresher about fuzzy queries and fuzzy cardinalities. Section 3 presents how such a summary is used in a unified approach to the computation of minimal failing subqueries and maximal succeeding subqueries, that respectively explains and repairs the initial failing fuzzy query. Before discussing related works in Sect. 5, results of a first experimentation are presented in Sect. 4. Finally, Sect. 6 recalls the main contributions and outlines perspectives for future work.

1.1 Principle

Figure 1 graphically illustrates the principle of the approach and the three step process used by the proposed cooperative system. Faced with a failing or poorly satisfied fuzzy query, the first step consists in summarizing the useful part of the database, this summary being then used to explain the reasons of the failure and to detect succeeding subqueries as well.

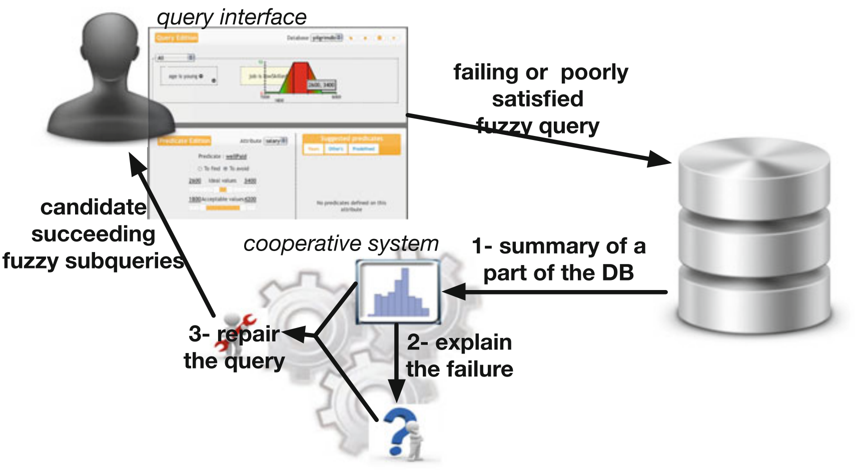


Fig. 1 Overview of the cooperative system

2 Preliminaries

2.1 Fuzzy Sets and Fuzzy Preference Queries

Fuzzy set theory was introduced by L.A. Zadeh [8] for modeling classes or sets whose boundaries are not clear-cut. For such objects, the transition between full membership and full mismatch is gradual rather than crisp. Typical examples of such fuzzy classes are those described using adjectives of the natural language, such as *young, cheap, fast*, etc. Formally, a fuzzy set E on a referential U is characterized by a membership function $\mu_E : U \to [0, 1]$ where $\mu_E(u)$ denotes the grade of membership of u in E. In particular, $\mu_E(u) = 1$ reflects full membership of u in E, while $\mu_E(u) = 0$ expresses absolute non-membership. When $0 < \mu_E(u) < 1$, one speaks of partial membership. Two crisp sets are of particular interest when defining a fuzzy set E:

- the core $C(E) = \{u \in U \mid \mu_E(u) = 1\}$, which gathers the *prototypes* of E,
- the support $S(E) = \{u \in U \mid \mu_E(u) > 0\}$.

The notion of an α-cut encompasses both these concepts. The α-cut E_α of a fuzzy set E is defined as the set of elements from the referential which have a degree of membership to E at least equal to α. Straightforwardly, one has: $C(E) = E_1$ and $S(E) = E_{0^+}$.

The complement of E, denoted by E^c, is defined by $\mu_{E^c}(u) = 1 - \mu_E(u)$. Furthermore, $E \cap G$ (resp. $E \cup G$) is defined the following way:

- $\mu_{E \cap G}(u) = min(\mu_E(u), \mu_G(u))$
- $\mu_{E \cup G}(u) = max(\mu_E(u), \mu_G(u))$.

As usual, the logical counterparts of the theoretical set operators ∩, ∪ and the complementation operator correspond respectively to the conjunction ∧, disjunction ∨ and negation ¬. See [9] for more details.

In a database context, fuzzy sets make it possible to rank-order the items that somewhat satisfy a user query involving gradual preference criteria. The language called SQLf described in [10, 11] extends SQL so as to support fuzzy queries. The general principle consists in introducing gradual predicates wherever it makes sense. The three clauses *select*, *from* and *where* of the base block of SQL are kept in SQLf and the *from* clause remains unchanged. The principal differences concern mainly two aspects:

- the calibration of the result since it is made with discriminated elements, which can be achieved through a number of desired answers (k), a minimal level of satisfaction (α), or both, and
- the nature of the authorized conditions as mentioned previously.

Therefore, the base block is expressed as:

select [**distinct**] [k | α | k, α] *attributes*
from *relations* **where** *fuzzy-condition*

where *fuzzy-condition* may involve both Boolean and fuzzy predicates.

The operations from relational algebra — on which SQLf is based — are extended to fuzzy relations by considering fuzzy relations as fuzzy sets on the one hand and by introducing gradual predicates in the appropriate operations (selections and joins especially) on the other hand. The definitions of these extended relational operators can be found in [11, 12]. As an illustration, we give the definition of the fuzzy selection hereafter, where ϕ denotes a fuzzy predicate and r is a fuzzy (gradual) relation:

$$\mu_{\sigma_\phi(r)}(t) = \min(\mu_r(t),\ \mu_\phi(t)).$$

2.2 Fuzzy Cardinalities

In the context of flexible querying, fuzzy cardinalities appear to be a convenient formalism to represent how many tuples from a relation satisfy a fuzzy predicate (or a logical combination of fuzzy predicates) to various degrees. We assume in the following that these various membership degrees are defined by a finite scale $\mathscr{S} : \alpha_1 = 1 > \alpha_2 > ... > \alpha_f > 0$. Fuzzy cardinalities can be incrementally computed and maintained for each predicate involved in the user query and for all possible conjunctions of predicates as well. Fuzzy cardinalities are represented by means of a possibility distribution [13] or, without loss of information, with a more compact representation:

$$F_P = 1/c_1 + \alpha_2/c_2 + ... + \alpha_f/c_f,$$

where $c_i, i = 1..f$ is the number of tuples in the concerned relation that are P to a degree at least equal to α_i. For the computation of cardinalities concerning a conjunction of q fuzzy predicates, like $F_{P^1 \wedge P^2 \wedge \ldots \wedge P^q}$, one takes into account the minimum of the satisfaction degrees obtained by each tuple t for the concerned predicates, $min(\mu_{P^1}(t), \mu_{P^2}(t), ..., \mu_{P^q}(t))$.

2.3 Gradual Minimal Failing Subqueries

An empty set of answers associated with a fuzzy query $Q = P^1 \wedge P^2 \wedge \ldots \wedge P^q$ is necessarily due to an empty support (w.r.t. the current state of the database) for a least one of the *subqueries* of Q. The notion of an unsatisfactory set of answers generalizes this problem by considering an empty α-cut of Q where α is a user-defined qualitative threshold. As explained in Sect. 2, the support and the core of a fuzzy set are particular cases of α-cuts where α is respectively equal to 0^+ and 1. In the rest of the paper we only use the notion of an empty α-cut to refer to failing queries as well as poorly satisfied ones.

Thus, an extreme case of a failing query corresponds to an empty 0^+-cut for Q only. The opposite extreme is when one or several predicates have an empty 0^+-cut. Between these two situations, it is of interest to detect the subqueries composed of more than one predicate and less than n predicates, that have an empty 0^+-cut. From an empty to an unsatisfactory set of answers, the problem defined above just has to be slightly revisited, transposing the condition of an empty 0^+-cut to α-cuts, where α is taken from the predefined scale of membership degrees $\mathscr{S} : 1 = \alpha_1 > \alpha_2 > ... > \alpha_f = 0^+$.

Definition 1 Let us consider a query $Q = P^1 \wedge P^2 \wedge \ldots \wedge P^q$, and let S and S' be two subsets of predicates such that $S' \subset S \subseteq \{P^1, P^2, \ldots, P^q\}$. A conjunction of elements from S (resp. S') is a *subquery* (resp. *strict subquery*) of Q.

If one wants to explain why the result of the initial query is empty (resp. unsatisfactory), one must naturally require that such subqueries be minimal: a subquery Q' of a query Q constitutes a minimal explanation if the considered α-cut is empty and if no (strict) subquery of Q' has an empty α-cut. This corresponds to a generalization of the concept of a *Minimal Failing Subquery* (MFS) [14].

Let us denote by α_Q^α the set of answers that corresponds to the α-cut of a query Q against a given database D: $\alpha_Q^\alpha = \{t \in D \mid \mu_Q(t) \geq \alpha\}$.

Definition 2 A *Minimal Failing Subquery* of a query $Q = P^1 \wedge P^2 \wedge \ldots \wedge P^q$ for a given α is any subquery Q' of Q such that $\alpha_{Q'}^\alpha = \emptyset$ and for all strict subquery Q'' of Q', $\alpha_{Q''}^\alpha \neq \emptyset$.

Property 1 Let $Q' \in Q$ be an MFS of Q for a given α. As we consider convex fuzzy sets only, Q' also fails for higher satisfaction degrees $\alpha' > \alpha$. However, the property of minimality of the MFS Q is not guaranteed for higher satisfaction degrees. A subquery Q' may indeed be an MFS of Q for a given α without being minimal for higher satisfaction degrees $\alpha' > \alpha$ as a strict subquery of Q', say Q'', may fail for α' and not for α.

The set of minimal failing subqueries of an initial failing fuzzy query Q for a degree α is denoted by MFS_{α}^{Q}.

Example 1 To illustrate this property, let us consider a conjunctive query $Q = P^1 \wedge P^2 \wedge P^3 \wedge P^4 \wedge P^5$ and five tuples whose satisfaction degrees on these five predicates are given in Table 1. One may notice that no tuple satisfies the subquery $Q' = P^1 \wedge P^2 \wedge P^4$ to the degree 0.2, whereas at least one tuple satisfies each strict subquery of Q' to this degree. Thus, Q' is an MFS of Q for the degree 0.2 and using Property 1, one also knows that Q' fails for more demanding satisfaction degrees ($\alpha \geq 0.2$). However, despite the fact that Q' also fails for $\alpha = 0.4$, Q' is not an MFS for that degree as there exists a strict subquery of Q', namely $P^1 \wedge P^2$, whose 0.4$-$*cut* is empty. Thus, Q' fails for $\alpha = 0.4$ but is not a minimal failing subquery as $Q'' = P^1 \wedge P^2 \subset Q'$ also fails. This is also the case for $P^1 \wedge P^3 \wedge P^4$ that is an MFS for $\alpha = 0.2$ but not for $\alpha = 0.4$, as $P^1 \wedge P^3$ fails for $\alpha = 0.4$ making $P^1 \wedge P^3 \wedge P^4$ not minimal.⋄

Table 1 Satisfaction of five tuples wrt. the predicates of Q

tuple	μ_{P^1}	μ_{P^2}	μ_{P^3}	μ_{P^4}	μ_{P^5}
t_1	0.2	0.5	0	0	0
t_2	0.5	0.1	0	0.5	0
t_3	0	0.2	1	1	0
t_4	0.2	0.5	0.4	0	0
t_5	0	0.8	0.7	1	0

2.4 Gradual Maximal Succeeding Subqueries

For a fuzzy query Q that fails at a degree α, MFSs detected for this degree thus constitute the explanations of the failure. Fixing Q so as to obtain a non empty set of answers at this degree α may be achieved by a relaxation of Q, where one or more predicates appearing in the MFSs are discarded. This (these) predicate(s) is (are) chosen in such a way that (i) the obtained subquery does not contain any of the detected MFS, and (ii) the subquery is as close as possible to the initial query, i.e. with a minimal number of removed predicates. Obviously, if the subquery Q' of Q does not contain any MFS of Q for a given threshold α, then one has the guarantee that Q' returns a non-empty set of answers. Repairing an initial query Q that fails at a degree α consists in identifying the maximal succeeding subqueries of Q for α. As several distinct maximal succeeding subqueries may be envisaged for a given threshold, candidate relaxations may be ordered according to two criteria: their closeness wrt. the initial query Q and the number of answers that they return, which has to be as close as possible to the quantitative threshold k that may be defined by the user in his/her query (Sect. 2.1).

Definition 3 Consider a query Q that fails at a given qualitative threshold α. A *maximal succeeding subquery* (XSS for short) at a degree $\alpha' \leq \alpha$ is a query $Q' \subseteq Q$ such that $|\alpha_{Q'}^{\alpha'}| > 0$, and such that there exists no other query Q'', $Q' \subset Q''$ that succeeds at the degree α'.

Property 2 Symmetrically to Property 1, a subquery Q' of a failing query Q that is an XSS at a degree α also succeeds for lower satisfaction degrees $\alpha' < \alpha$. However, the property of maximality of such an XSS is not guaranteed for lower satisfaction degrees. A subquery Q' may indeed be an XSS of Q for a given α without being maximal for lower satisfaction degrees $\alpha' < \alpha$ as a query $Q'', Q' \subset Q''$ may succeed for α' and not for α.

The set of maximal succeeding subqueries of an initial failing fuzzy query Q for a degree α is denoted by XSS_{α}^{Q}. Obviously, an XSS of XSS_{α}^{Q} cannot contain any predicate (or conjunction of predicates) from MFS_{α}^{Q}.

3 A Unified Approach to Failing Queries

Faced with a failing conjunctive query $Q = P^1 \wedge P^2 \wedge ... \wedge P^q$, the cooperative system presented in this paper provides explanations for the vacuity of the answer set using layered MFSs, and also suggests layered and ordered relaxations materialized by XSSs. The identification of these MFSs and XSSs relies on a fuzzy-cardinality-based summary of a part of the searchable database, whose construction constitutes the first step of our unified approach.

3.1 Summarizing Step

Considering an initial user query $Q = P^1 \wedge P^2 \wedge ... \wedge P^q$ that fails, the first step of our approach relies on the construction of a fuzzy-cardinality-based summary of the useful part of the database (one focuses on the predicates from Q). Thus, the summary is not constructed on the whole database but on the set of items that somewhat satisfy at least a subquery of Q. This summary tells us how many tuples from the database satisfy each conjunctive subquery of Q for the different satisfaction degrees involved in the predefined scale $\mathscr{S}$ (Sect. 2.2).

A fuzzy-cardinality-based summary wrt. a set of predicates $\{P^1, P^2, ..., P^q\}$ is composed of up to $2^q - 2$ fuzzy cardinalities, i.e. one for each nonempty strict subquery of Q. Obviously, it is not necessary to store a fuzzy cardinality for a conjunction of predicates that is not satisfied at all ($\alpha^f = 0^+$) by at least one tuple of the database.

As said previously, the fuzzy-cardinality-based summary is not computed on the whole database but on the set of tuples that somewhat satisfy at least one predicate of the initial query. Thus, considering a conjunctive fuzzy query $Q = P^1 \wedge P^2 \wedge ... \wedge P^q$, one first builds a disjunctive query $P^1_{0+} \vee P^2_{0+} \vee ... \vee P^q_{0+}$, where P^i_{0+} is a crisp set corresponding to the 0^+-cut (i.e., the support) of the fuzzy predicate P^i. Processing this Boolean disjunctive query returns a relation denoted by r, on which the fuzzy-cardinality-based summary is then computed using the following algorithm:

for every tuple t of r **do**
  compute $\mu_{P^1}(t)$, $\mu_{P^2}(t)$; ... and $\mu_{P^q}(t)$;
  $V \leftarrow \langle \mu_{P^1}(t), ..., \mu_{P^q}(t) \rangle$;
  update the fuzzy cardinalities for all strict subqueries of Q using V;
done.

Ordered by the inclusion relation, the set of all subqueries of a conjunctive query, say $Q = P^1 \wedge P^2 \wedge P^3 \wedge P4$ forms a lattice as illustrated by Fig. 2, with Q as the upper bound and ∅ as the lower bound. For a given satisfaction vector V, updating the fuzzy cardinalities consists in traversing this lattice using a bottom-up breadth-first strategy. The exploration of a branch of the lattice stops when the current tuple doest not satisfy at all the corresponding conjunction.

It is worth recalling that the computation of the satisfaction degree $\mu_{Q'}(t)$ of a nonatomic subquery Q' consists in taking the minimum of the individual satisfaction degrees of t on the different predicates involved. Internally, at the end of the process, each node of the lattice is associated with its fuzzy cardinality.

Moreover, thanks to Property 1, one knows that if $Q \subset Q'$, $\mu_Q(t) = 0 \Rightarrow \mu_{Q'}(t) = 0$. This property is used to prune branches of the lattice during the update of the fuzzy cardinalities for a given tuple t.

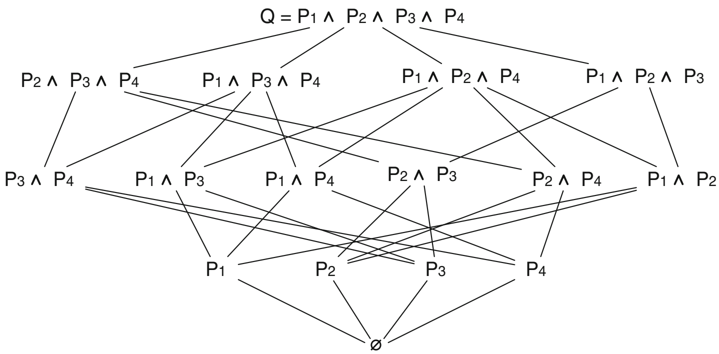


Fig. 2 Lattice of subqueries of $Q = P^1 \wedge P^2 \wedge P^3 \wedge P^4$

Remark 1 A possible heuristic for optimizing the computation of the fuzzy cardinalities is to consider the different predicates in increasing order of the size of their support as branches of the exploration tree are cut as soon as no item satisfies the conjunction associated with the current node. Indeed, one may assume, in the absence of further information about data distribution, that the smaller the support of a predicate is, the more likely the number of items satisfying it is low.

Example 2 Let us consider a fuzzy query $Q = P^1 \wedge P^2 \wedge P^3 \wedge P^4 \wedge P^5$ that fails even for the lowest satisfaction threshold $\alpha = 0^+$, where $\{P^1, P^2, P^3, P^4, P^5\}$ are fuzzy predicates. Table 1 gives the exact satisfaction degrees of five tuples wrt. the predicates involved in Q, and Table 2 is an extract of the fuzzy-cardinality-based summary where exact satisfaction degrees are projected on the scale $\mathscr{S} = \langle 0^+, 0.2, 0.4, 0.6, 0.8, 1\rangle$.⋄

Table 2 Extract of the fuzzy-cardinality-based summary

$Q' \in \mathscr{P}(Q)$	$\lvert Q'_1\rvert$	$\lvert Q'_{0.8}\rvert$	$\lvert Q'_{0.6}\rvert$	$\lvert Q'_{0.4}\rvert$	$\lvert Q'_{0.2}\rvert$	$\lvert Q'_{0.0+}\rvert$
P^1	0	0	0	1	3	3
...	...	...	...	...	...	...
P^4	2	2	2	3	3	3
$P^1 \wedge P^2$	0	0	0	0	2	3
...	...	...	...	...	...	...
$P^3 \wedge P^4$	1	1	2	2	2	2
$P^1 \wedge P^2 \wedge P^3$	0	0	0	0	1	1
...	...	...	...	...	...	...
$P^2 \wedge P^3 \wedge P^4$	0	0	1	1	2	2

3.2 Explaining the Failure

In the presence of an empty or unsatisfactory set of answers for a query Q, the conflict detection process described hereafter generates layered MFSs for the different satisfaction degrees in $\mathscr{S}$.

The algorithm used to identify the layered MFSs investigates the different subqueries of the initial failing query Q for the different satisfaction degrees in $\mathscr{S}$. To determine if a subquery Q' is a failing subquery of Q for a satisfaction degree α_i, one just has to check the table storing the fuzzy cardinalities.

As illustrated by Fig. 3, detecting layered MFSs relies on a bottom-up breadth-first traversal of the lattice, where for each subquery, one checks the emptiness of the α-cut, using the fuzzy cardinalities, in an increasing order of α from α_f to α_1. Property 1 is used to propagate failing subqueries to more demanding satisfaction degrees. The worst case in terms of complexity is when only Q is an MFS for the highest satisfaction degree considered in $\mathscr{S}$, i.e. $\alpha_1 = 1$.

In the manner of Apriori [7], Algorithm 1 traverses the lattice of subqueries of Q (Fig. 2) starting with atomic predicates and the less demanding satisfaction threshold α_f. If no tuple satisfies an atomic subquery, say P^1, to the degree α_i then P^1, as an

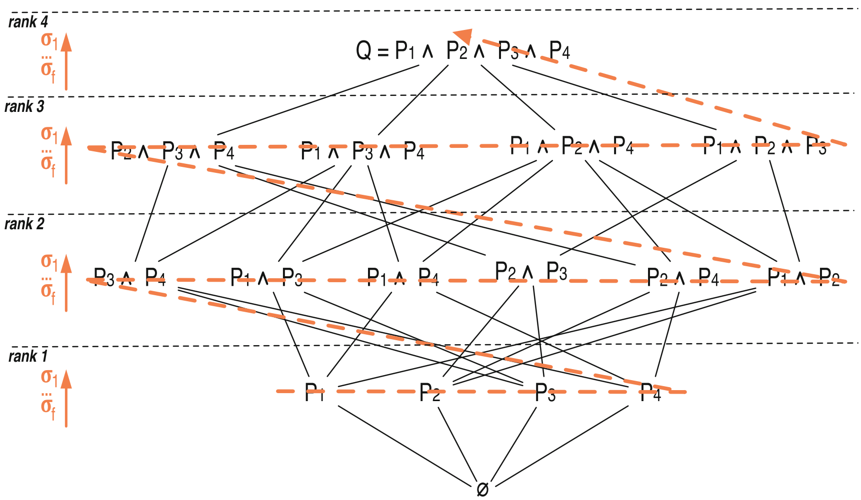


Fig. 3 Bottom-up breadth-first traversal of the lattice to detect MFSs

atomic subquery, is by definition an MFS of Q and is also an MFS for $\alpha_j \geq \alpha_i$. Obviously, no subquery containing P^1 will be considered as a candidate MFS during the rest of the process. In the case where P^1 is satisfied for the current satisfaction threshold, say α_i, one checks for more demanding thresholds $\alpha_j > \alpha_i$ if P^1 is still satisfied. These tests are performed for each possible atomic subquery of Q.

Then, one goes to the second round of the loop (line 1.7 of Algorithm 1) where conjunctions containing two nonfailing predicates are generated for the different satisfaction degrees of $\mathscr{S}$. For each subquery (line 1.11), one checks the fuzzy cardinalities so as to determine if it is an MFS. A subquery c is identified as an MFS for the degree α_i if the function that checks the summary $card(c, \alpha_i)$ returns 0. If one of these conjunctions, let say $P^2 \wedge P^3$, is an MFS for a degree α_i one tries to propagate it to higher satisfaction degrees (see Algorithm 2 where $isMinimal(L, MFS_{\alpha_j}(Q))$ returns $true$ if $\forall M \in MFS_{\alpha_j}(Q), M \not\subset L$, $false$ otherwise). It is worth recalling that this last test is necessary as the MFS property is not monotonic with respect to α-cuts. One indeed has to check, using Algorithm 2, that for each $\alpha_j > \alpha_i$, no subquery of $P^2 \wedge P^3$ has already been detected as an MFS for degree α_j. If it is not the case, $P^2 \wedge P^3$ is stored as an MFS of Q for α_j. Obviously, an atomic query that fails is by definition an MFS for all α-cuts. Then, the algorithm goes back to the loop (line 1.7) and conjunctions containing three predicates are generated for each considered satisfaction degree (line 1.8) taking care that these conjunctions do not contain an already identified MFS. This recursive process goes on until candidate conjunctions cannot be generated anymore.

Remark 2 Even if a qualitative threshold α_u is specified by the user in the initial query Q that fails, MFSs are detected for the different satisfaction degrees defined in the scale $\mathscr{S}$. It is indeed important to detect that it is useless to expect any result if an atomic subquery of Q fails for the least restrictive satisfaction degree α_f. In the case of Boolean queries, Algorithm 1 can also be used: one just needs to change the scale of satisfaction degrees into the singleton $\{1\}$.

Input: a failing query $Q = P^1 \wedge \ldots \wedge P^q$; a scale of degrees $\mathscr{S} = \alpha_f < ... < \alpha_2 < (\alpha_1 = 1)$;
Output: $MFS^Q_{\alpha_i}, i \in \mathscr{S}$ layered MFS's of Q;

begin
  foreach $\alpha_i \in \mathscr{S}$ **do**
    $MFS_{\alpha_i}(Q) \leftarrow \emptyset$; $E_{\alpha_i} \leftarrow \{P^1, \ldots, P^n\}$;
    $Cand_{\alpha_i} \leftarrow E_{\alpha_i}$;
  end
  $nbPred \leftarrow 1$;
  while $Cand_{\alpha_1} \neq \emptyset$ **do**
    foreach $\alpha_i \in \mathscr{S}$ *from* α_f *to* α_1 **do**
      // generation of the candidates of size *nbPred*
      $Cand_{\alpha_i} \leftarrow \{$*subqueries* Q' composed of *nbPred* predicates from E_{α_i} such that $\forall Q'' \subset Q', Q'' \notin MFS_{\alpha_i}(Q)\}$;
      foreach *c in* $Cand_{\alpha_i}$ **do**
        if $card(c, \alpha_i) = 0$ **then**
          $MFS_{\alpha_i}(Q) \leftarrow MFS_{\alpha_i}(Q) \cup \{c\}$;
          //E_c is the set of predicates involved in c
          //thus remove E_c from the list of predicates used to generate
          //candidate subqueries of size $nbPred + 1$
          $E_{\alpha_i} \leftarrow E_{\alpha_i} - E_c$;
          //Propagate c to higher satisfaction degrees
          $MFS = \cup_i MFS_{\alpha_i}(Q)$
          propagate(α_i, A, c, MFS, $E_{\alpha_1} ... E_{\alpha_f}$);
        end
      end
    end
    $nbPred \rightarrow nbPred + 1$;
  end
end

Algorithm 1: Gradual MFS computation

Input: α_i is a satisfaction degree; $\mathscr{S}$ is the scale of degrees; c is the MFS detected for α_i to propagate; $MFS^{Q}_{\alpha_i}, i \in \mathscr{S}$ layered MFS's of Q; $E_{\alpha_i}, i\mathscr{S}$ arrays of predicates to use for the generation of candidates, one array per degree;

procedure propagate(α_i, A, c, MFS, $E_{\alpha_1}...E_{\alpha_f}$) **begin**
  foreach $\alpha_j \in \mathscr{S} \mid \alpha_j \geq \alpha_i$ **do**
    if *isAtomic(c) or isMinimal(c,*$MFS_{\alpha_j}(Q)$*)* **then**
      $MFS_{\alpha_j}(Q) \leftarrow MFS_{\alpha_j}(Q) \cup \{c\}$;
      $E_{\alpha_j} \leftarrow E_{\alpha_j} - E_c$;
    else
      break;
    end
  end
 end
end

Algorithm 2: Procedure that propagates an MFS to higher satisfaction degrees

Example 3 Applied to the summary of Table 2, Table 3 gives the layered MFSs that explain the failure of the query Q introduced in Example 1.◇

Table 3 Layered MFSs of Q

α	MFS^{Q}_{α}	α	MFS^{Q}_{α}
$\alpha_1 = 1$	$\{P^1, P^2, P^5\}$	$\alpha_4 = 0.4$	$\{P^1 \wedge P^2, P^1 \wedge P^3, P^5\}$
$\alpha_2 = 0.8$	$\{P^1, P^2 \wedge P^3, P^5\}$	$\alpha_5 = 0.2$	$\{P^5, P^1 \wedge P^2 \wedge P^4,$ $P^1 \wedge P^3 \wedge P^4\}$
$\alpha_3 = 0.6$	$\{P^1, P^5\}$	$\alpha_{f=6} = 0^+$	$\{P^5\}$

3.3 Repairing the Failure

Repairing a failing fuzzy query consists in suggesting XSSs for the different satisfaction degrees considered in the scale $\mathscr{S}$. Obviously, the identification of the maximal succeeding subqueries for a given degree α_i depends on the minimal failing subqueries found for this degree, this (these) failing subquery(ies) being denoted by $MFS^{Q}_{\alpha_i}$. Indeed, a subquery Q' of an initial failing query Q succeeds at a degree α_i if it does not contain any MFS from $MFS^{Q}_{\alpha_i}$.

The algorithm that is used to identify layered XSSs is pretty much the same as the one used for detecting MFSs (Alg. 1). As illustrated by Fig. 4, it consists in performing a top-down breadth-first traversal of the lattice of candidate XSSs, this time starting with the largest subqueries, and considering also the satisfaction degrees of $\mathscr{S}$ in a decreasing order. For a given node, i.e. a subquery $Q' \in Q$, of the lattice and a given satisfaction degree α_i, one just has to check whether Q' does not contain any MFS from $MFS^{Q}_{\alpha_i}$. If Q' succeeds, i.e. returns a nonempty answer set whose cardinality is stored in the fuzzy-cardinality-based summary, then Property 2 is used to propagate Q' to lower satisfaction degrees $\alpha_j, \alpha_j < \alpha_i$, taking care that the maximality of the identified succeeding subquery is preserved for less demanding satisfaction thresholds. As soon as a subquery $Q' \subseteq Q$ is identified as an XSS of Q for a degree α_i then, by definition of an XSS, one can prune the part of the lattice composed of strict subqueries of Q'. The worst case in terms of complexity is when the XSSs correspond to atomic subqueries for the least restrictive considered satisfaction degree α_f.

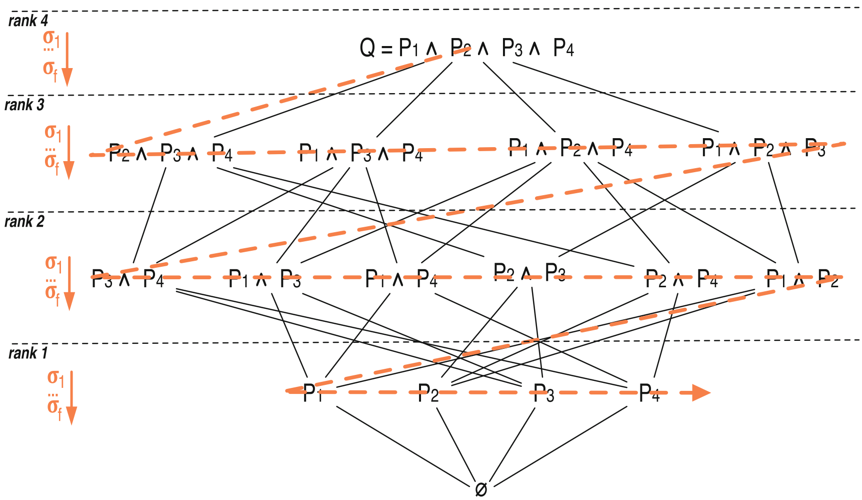


Fig. 4 Top-down depth-first traversal of the lattice to detect XSSs

Example 4 For the failing fuzzy query Q introduced in Example 1, Table 4 gives the layered repaired queries returned to the user (who may then choose the one that best fits his/her need).⋄

Table 4 Layered XSSs of Q

α	MFS^{Q}_{α}	XSS^{Q}_{α}
$\alpha_1 = 1$	$\{P^1, P^2, E\}$	$\{P^3 \wedge P^4\}$
$\alpha_2 = 0.8$	$\{P^1, P^2 \wedge P^3, P^5\}$	$\{P^2 \wedge P^4, P^3 \wedge P^4\}$
$\alpha_3 = 0.6$	$\{P^1, P^5\}$	$\{P^2 \wedge P^3 \wedge P^4\}$
$\alpha_4 = 0.4$	$\{P^1 \wedge P^2, P^1 \wedge P^3, P^5\}$	$\{P^2 \wedge P^3 \wedge P^4\}$
$\alpha_5 = 0.2$	$\{P^1 \wedge P^2 \wedge P^4, P^1 \wedge P^3 \wedge P^4, P^5\}$	$\{P^2 \wedge P^3 \wedge P^4\}$
$\alpha_{f=6} = 0^+$	$\{P^5\}$	$\{P^1 \wedge P^2 \wedge P^3 \wedge P^4\}$

3.4 Human Readable Explanations

Explanations of the failure of an initial failing query are given to the user, as well as succeeding subqueries, in order to let him/her choose the predicate(s) to relax. The graduality of the detected MFSs and suggested XSSs significantly improves the informativeness of this cooperative feedback, as users can precisely adjust their expectations. Moreover, thanks to the fuzzy-cardinality-based summary, it is possible to inform the users about the number of answers returned by each candidate XSS. This latter piece of information may strongly influence the user about the predicates he/she is ready to relax. Example 4 illustrates the structure of the explanations and suggestions addressed to the user in case of a failing query.

Example 5 Considering the query introduced in Example 1, its MFSs presented in Example 2 and the subsequent XSSs given in Example 3, the following human readable explanations are generated:

No item satisfies simultaneously P^1, P^2, P^3, P^4 and P^5 to the degree α_u because ...

- no item satisfies P^5,
- no item satisfies P^1 and P^4, with P^2 or P^3 to a degree ≥ 0.2,
- no item satisfies P^1 with P^2 or P^3 to a degree ≥ 0.4,
- no item satisfies P^1 to a degree ≥ 0.6;
- no item satisfies P^2 with P^3 to a degree ≥ 0.8,
- no item fully satisfies P^2.

... but the database contains:

- k_1 items that fully satisfy P^3 and P^4,
- k_2 items that satisfy P^4, with P^2 or P^3 to a degree ≥ 0.8,
- k_3 items that satisfy P^2, P^3 and P^4 to a degree ≥ 0.6,
- k_4 items that satisfy P^1, P^2, P^3 and P^4 to a degree > 0.⋄

4 Experimentation

In this section, we present experimental results obtained using a prototype that implements the approach described above. The goal of this experimentation is to assess the efficiency of the approach with real data, and more precisely with a relation containing ads about 92,178 second hand cars. The relation schema is {*price*, *make*, *mileage*, *year*, *length*, *height*, *nbseats*, *acceleration*, *consumption*, *co2emission*}. The results presented in this section have been obtained using an Intel Core 2 Duo 2.53GHz computer with 4Go 1067 MHz of DDR3 ram.

In a first experimentation, we have observed the time needed to compute the fuzzy-cardinality-based summary on preloaded datasets of various sizes. It is worth recalling that, faced with a failing query, the fuzzy-cardinality-based summary is not computed on the whole database but on the result of the crisp and disjunctive version of the initial query.

Figure 5 shows the time needed to compute the summary on datasets of various sizes for queries involving a fixed number of predicates (here 6). This result empirically confirms that the complexity of the summarization step linearly depends on the size of the dataset. Fuzzy cardinalities are stored in main memory in a hashtable, keys being conjunctions of predicates whereas values are the cardinalities themselves. To complete the experimentation on the fuzzy-cardinality-based summarization of a dataset, we have observed the space taken in memory by this hashtable according to different dataset sizes. Figure 6 shows an interesting though predictable phenomenon: the convergence of the size of the summary. This convergence can be indeed explained by the fact that, whatever the number of tuples, the possible combinations of properties that make sense to describe them is finite and can be quickly enumerated. This last result has been obtained considering 10 fuzzy predicates for the summarization, one for each attribute on which a second hand car is described in our database. Thus, even for very large databases, fuzzy-cardinality-based summaries easily fit in main memory. This phenomenon is all the more pronounced as there exist correlations between attributes of the considered relation.

Based on the summary stored in main memory, we have applied the cooperative approach introduced in this paper to identify the MFSs and XSSs of failing queries and observed the time needed to perform these tasks. Figure 7 shows that for different sizes of failing queries, i.e. from one to ten predicates, the time needed to compute the MFSs and XSSs is negligible compared with the computation of the summary which is obviously the most costly task.

As said in Sect. 3, a single disjunctive query has to be submitted to the database in order to explain and repair a failing fuzzy query. Even with efficient indexes defined on the different searchable attributes, it is not conceivable to apply a naive strategy where each subquery of the initial failing subquery is submitted to the database. To confirm the relevance of a summary-based approach that relies on a single query to the database, Fig. 8 compares the time needed to identify the MFSs of a failing query Q using our approach and a naive one that runs each subquery of Q (note that the y-axis corresponds to a logarithmic scale).

These experimentations show that the proposed approach is not very sensitive to the size of the dataset and remains tractable even for very large databases as long as the size of the query is reasonable. This is why one can conclude that this approach may be efficiently integrated in many applicative contexts, for instance e-business as online shops generally propose query interfaces where at most half a dozen of predicates can be specified.

Fig. 5 Computation time of the summarization step wrt. the size of the DB

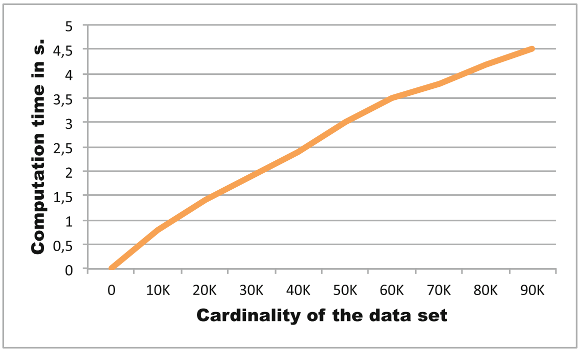


Fig. 6 Size of the summary wrt. the number of considered predicates

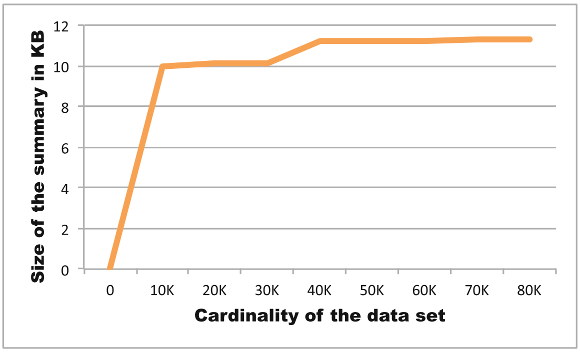
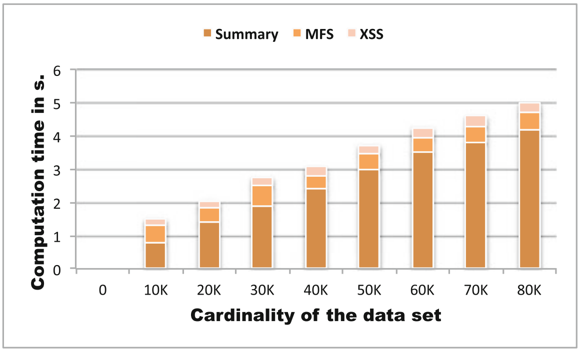


Fig. 7 Computation time of the different steps: summarization, explanation, repair



5 Related Works and a Few Words About Complexity

Providing explanations and repairs for a failing query is a crucial issue if one wants to devise genuinely cooperative information systems. In [14], Godfrey studied the computational complexity entailed by the identification of all of the MFSs and XSSs for a failing conjunctive query. Even though it is easy to find one MFS, Godfrey

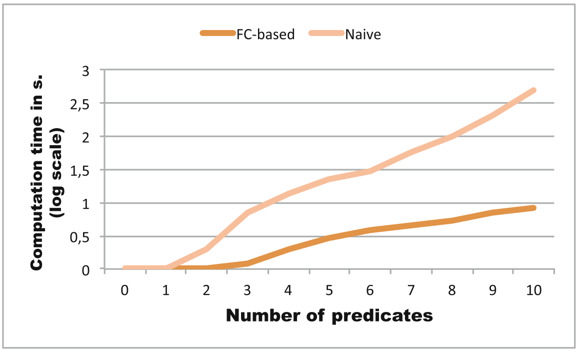


Fig. 8 fuzzy-cardinality-based approach vs. naive approach (logarithmic scale)

showed that finding all the minimal causes of the emptiness of the answer set is NP-Hard and implies to run an exponential number of subqueries. In the context of recommendation systems, McSherry introduced in [15] a heuristic called *coverage* for removing first the predicates that are the most likely to be responsible for the failure. The coverage of a predicate is the number of times this predicate is involved in the discovered MFSs. In [16], Bidault et al. also studied the problem of identifying the minimal explanations of a failing query in the particular context of a mediation system and suggesting possible query repairs.

As databases are getting larger and larger, it is not realistic to run an exponential number of queries, even if such a strategy would produce very informative explanations. This is why Jannach proposed in [17] an MFS and XSS detection technique that relies on a single scan of the database. During this full scan of the database, a binary matrix is built, that contains as many rows as there are tuples, and as many columns as there are Boolean predicates in the query. A 1 bit is set to a cell if the concerned tuple satisfies the concerned predicate, 0 otherwise. This binary matrix is then used to identify the MFSs and suggest XSSs. Even though the strategy proposed by Jannach relies on a single query to the database, the size of its binary matrix linearly depends on the size of the database, which can be highly problematic in practice.

We show in this paper that a unified approach based on fuzzy cardinalities may be used to detect the layered MFSs and to repair the initial query as well. The summarization step is obviously the most costly step of these two approaches. Indeed, the construction of the fuzzy-cardinality-based summary linearly depends on the number of tuples considered when building this summary. So, one can say that the proposed approach, as the one introduced in [17], linearly depends on the size of the database, which is a very positive point, but exponentially depends on the number of predicates involved in the initial failing query, where the worst case corresponds to a single MFS Q for the maximal satisfaction degree of 1. This is not a serious limitation in practice as the number of predicates generally defined by users in their queries is rather low (≤ 8) in most application contexts. Thus, even for a medium-sized database, a fuzzy-cardinality-based summary is significantly smaller than the binary matrix used in [17].

Note also that the works mentioned above all deal exclusively with *Boolean* failing queries, whereas our approach is suitable both for Boolean and fuzzy queries.

6 Conclusion

There is a consensus inside the community working on cooperative approaches that clear explanations have to be given when users are faced with failing queries. These explanations have to focus on the minimal causes of the failure and should come along with succeeding queries corresponding to relaxations of the original one. In this paper, we have proposed an efficient unified approach based on a summary of the database composed of fuzzy cardinalities. The main advantage of this approach is that a single scan of a part of the database is needed in order to identify the causes of the failure and suggest succeeding queries that are as close as possible to the initial failing query.

As perspectives for future works, we are currently investigating the computation of fuzzy cardinalities in a massively distributed system in a context of big data. We also intend to use correlation measures between attributes or a workload of succeeding queries to define heuristics that could be used during the XSSs detection step for influencing the choice of the predicates to discard first.

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The Web, Similarity, and Fuzziness

Parisa D. Hossein Zadeh and Marek Z. Reformat

Abstract The Internet is perceived as a source of multiple types of information, a large shopping mall, a social forum and an entertainment hub. The users constantly browse and search the web in order to find things of interest. The keyword-based search becomes less and less efficient in the case of more refined searches where details of items become important. The introduction of Resource Description Framework (RDF) as a relation-based format for data representation allows us to propose a different way of performing a relevancy-based search. The approach proposed is based on representation of items as sets of features. This means that evaluation of items' relevance is based a feature-based comparison. A more realistic relevancy determination can be been achieved via categorization and ranking of features. The calculated relevancy measures for individual categories of features are aggregated using a fuzzy-based approach.

1 Introduction

Internet is perceived as a repository of information. Everyday, millions of users search and browse the web. Besides news and information, they also look for items of possible interest: books, movies, hotels, travel destinations, and many more. It is anticipated that these items possess specific or similar features (Tversky 1977). Additionally, not all of these features are equally important, some of them are significant with a high selective power, while some are entirely negligible.

The improvement of the users' searching activities depends on development of web applications that are able to support the users in finding relevant entities. So far, identification of data satisfying user's request is realized by matching the query keywords to pieces of information. Most of the web search engines utilize this approach and its variations.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_26

A novel representation of information on the web, introduced by the concept of Semantic Web (Berners-Lee et al. 2001), changes the way how individual pieces of information are stored and accessed on Internet. The fundamental data format is called Resource Description Framework (RDF) (Lassila et al. 2014) and it relies on a simple concept of a triple: <subject-property-object>. In other words, a triple can be perceived as a relation existing between two entities: one of them – subject – is the main entity that is in relation embodied by a property with another item – subject. It means that any piece of information can be represented as a set of triples where multiple items are linked to each other being subjects and/or objects in different triples.

An evaluation of relevancy between two items is associated with determining similarity between them. Therefore, similarity assessment between two items is an important and fundamental step in processes and applications related to information extraction and retrieval, web search, automatic annotation, etc. Application of RDF as data representation format allows us to propose a different approach to evaluate items' relevancy. The approach is based on the fact that an item is represented as a set of triples while all of them are "tied" by the fact that a subject of these triples is the same, i.e., it is an item under consideration. In such representation of a single item, each triple is treated as its feature. This allows us to apply a feature-based comparison of items and to take advantage of its flexibility and adaptability in a process of evaluating relatedness between items.

Our approach is based on the idea that properties/features of an entity can be categorized and ranked based on their importance in describing the entity. The calculated similarity measures for these categories of features are aggregated using fuzzy weights associated with the importance of these categories.

This chapter introduces a novel approach suitable for identification of related items. A number of important aspects of the proposed approach are presented here:

- The evaluation of relatedness of two items is performed using features of the items. The RDF representation allows us to compare items on a feature-by-feature (triple-by-triple) basis. The principles of the approach are explained in Sect. 2.
- The importance of features is recognized as essential characteristics of similarity evaluation process. Different features contribute to the overall similarity in different ways. It is important to automatically determine importance of features, as well as to apply a proper aggregation process to combine similarities of individual features. The process of determining importance of features is based on Wikipedia Infoboxes[1] as explained in Sect. 3.1. Further, a fuzzy-based method of aggregating evaluated similarities of single features and taking into account different importance levels of the features are fully explained in Sect. 3.2.

[1]http://en.wikipedia.org/wiki/Help:Infobox

- The proposed method is applied to a real-life scenario of finding relevant books. The results obtained using the proposed approach are compared with the suggestions provided by Google. The case study is presented in Sect. 4.

2 Feature-based Similarity in Linked Data

Similarity is essential for finding relevant things. This can be further explored by a need to dig a bit "deeper" and look not only on items as whole units but also at individual features of these items. Potentially, this can lead to a more refined similarity estimation process and better results. The introduction of Resource Description Framework (RDF) provides an opportunity to "see" items as sets of features and built simple procedures for evaluating similarity of items.

2.1 Linked Data and RDF

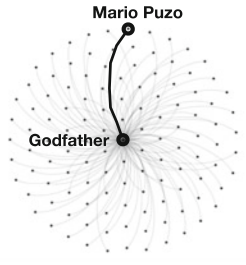
The goal of the Semantic Web as an enhancement to the current web is to provide a meaning and structure to the web content. The Semantic Web's road map points to ontology as a way of accomplishing this. Ontology defines a structural organization and relations between concepts,[2] properties and instances. It also adds semantic richness and reasoning capabilities. Any type of information expressed with a means of ontologies can be semantically analyzed and processed leading to more comprehensive results.

The Semantic Web concept introduces RDF as a way of representing information including ontologies and their instances. The fundamental idea is to represent each piece of data as a triple: <subject-property-object>, where the subject is an entity being described, the object is an entity describing the subject, and the property is a "connection" between the subject and object. For example, *Godfather is book* is a triple with *Godfather* as its subject, *is* its property, and *book* its object. In general, a subject of one triple can be an object of another triple, and vice versa. The growing presence of RDF as a data representation format on the web brings opportunity to develop new ways how data is processed, and what type of information is generated from data.

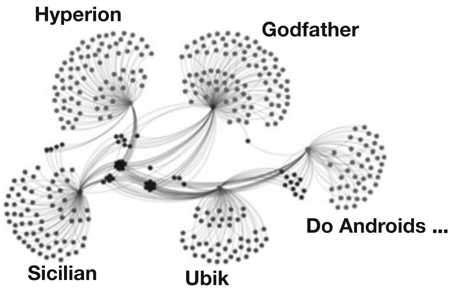
A single RDF-triple <subject-property-object> can be perceived as a feature of an entity identified by the subject. In other words, each single triple is a feature of its subject. Multiple triples with the same subject constitute a definition of a given entity. A simple illustration of this is shown in Fig. 1(a). It is a definition of Godfather. If we visualize it, definition of the entity resembles a star with the defined objects as its core. We can refer to it as an RDF-star.

[2]Throughout this chapter, two terms "concept" and "entity" are used interchangeably.

Fig. 1 (a) RDF-stars: a definition of Godfather with one of its features enhanced, (b) interconnected RDF-stars representing: Godfather, Hyperion, The Sicilian, Ubik and Do Androids Dream of Electric Sheep



(a)



(b)

Quite often a subject and object of one triple can be involved in multiple other triples, i.e., they can be objects or subjects of other triples. In such a case, multiple definitions – RDF-stars – can share features, or some of the features can be centres of another RDF-stars. Such interconnected triples constitute a network of interleaving definition of entities, Fig. 1(b).

In general, Uniform Resource Identifiers (URI) are used to uniquely identify subjects and properties. Objects, on the other hand, are either URIs or literals such as numbers or strings.

Due to the fact that everything is interconnected, we can state that numerous entities share common features. In such a case, comparison of entities is equivalent to comparison of RDF-stars. This idea is a pivotal aspect of the proposed approach for determining relevance of items.

Based on the Semantic Web vision, several knowledge bases have been created including DBpedia,[3] Geonames,[4] YAGO,[5] and FOAF[6]. The collection of interrelated datasets on the Web is referred to as Linked Data (LD) (Bizer and Berners-Lee 2009). DBpedia is a large linked dataset, which contains Wikipedia data translated

[3]http://dbpedia.org/About

[4]http://www.geonames.org/

[5]http://www.mpi-inf.mpg.de/yago-naga/yago/

[6]http://www.foaf-project.org/

into RDF triples. Even though DBpedia is a large dataset that has over one billion triples from Wikipedia, it also provides RDF links to other datasets on the Web such as Geonames and Freebase[7]. With a growing number of RDF triples on the web – more than 62 billions[8] triples– processing data in RDF format is gaining a special attention. There are multiple works focusing on RDF data storage and querying strategies using a specialized query language SPARQL (Levandoski and Mokbel 2009)(Schmidt et al. 2008).

2.2 Similarity in RDF Data

The underlying idea of the proposed approach for relevancy evaluation is to determine number of common and relevant features. In the case of RDF defined concepts, this nicely converts into checking how many features they share as presented in Fig. 2. Defined entities are books "The Godfather" and "The Sicilian" that share number of features. Some of these features are identical – the same property and the same subject (black circles in Fig. 2), while some have the same object but different properties.

Basically, number of different comparison scenarios can be identified. It depends on interpretation of the term "entities that they share". The possible scenarios are (for details see Hossein Zadeh and Reformat 2013a, b):

- identical properties and identical objects
- identical properties and similar objects
- similar properties and identical objects
- similar properties and similar objects

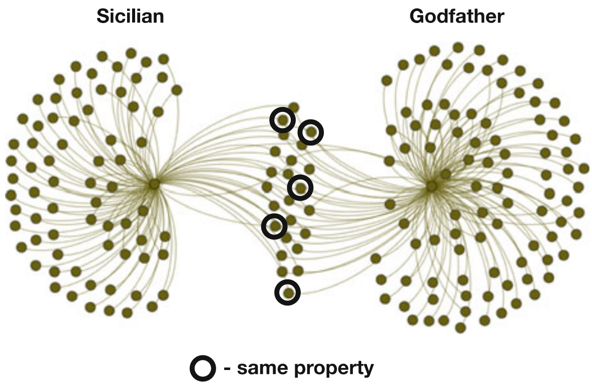


Fig. 2 Similarity of RDF defined concepts: based on shared objects connected to the defined entities with the same properties

[7]http://www.freebase.com/

[8]http://stats.lod2.eu

To better understand the proposed methodology, Fig. 3 shows similarity assessment between two entities x and y. As can be seen, similarity is evaluated by taking into account two layers, **Layer 1** and **Layer 2**. Similarity of **Layer 1** is assessed via two components: common objects of the two entities, i.e., the common object $\{z\}$ in Fig. 3; and the pair of unique objects $\{(w, u)\}$. Similarity of **Layer 2** is to estimate similarity between unique pairs – $\{(w, u)\}$. It is evaluated based on all permutations of objects that are connected to the elements, i.e., $\{(s, g), (s, f), (s, r), (t, g), (t, f)$, and $(t, r)\}$.

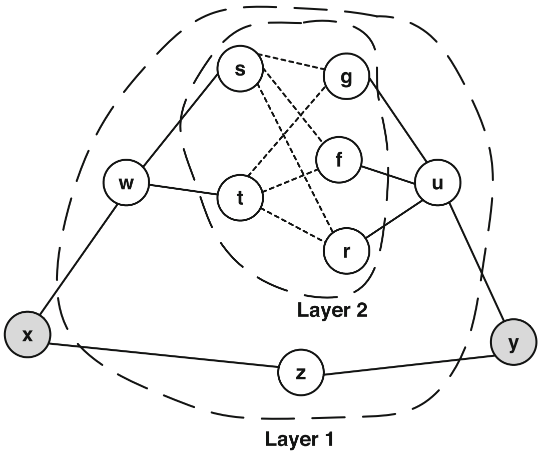


Fig. 3 Similarity evaluation process for Layer 1 and Layer 2

3 Similarity Evaluation

The principle idea presented here relies on the assertion that properties of an entity should have different importance values in similarity assessment between that entity and other entities. These importance values reflect their influence in describing that entity. Thus, similarity between entities cannot be ideally calculated with properties having equal weights. In fact, human judgment of similarity considers relative importance values for properties of an item. As an example, in a problem of finding books similar to a particular book, properties such as “author”, “genre”, and “subject” are more dominant compared to such properties as “country”, “number of pages”, and “cover artist”. It is worth noting that the present study should not be confused with similarity assessment within a context defined via specific properties. For example, a context similarity evaluation can be applied in the question, “How much these two books are similar in the context of their *topic*?” For similarity assessment in a context in Linked Data (LD), see (Hossein Zadeh and Reformat 2013a, b). The solution presented in this chapter determines the semantic similarity between entities expressed in RDF triples while recognizing and dealing with the importance of each property associated with the entities under study.

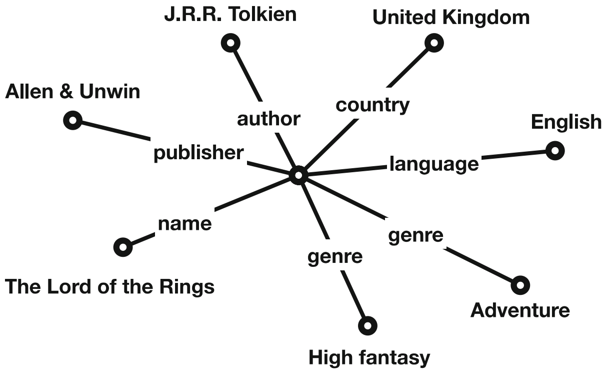


Fig. 4 Book "The Lord of the Rings" with its features

A fundamental step in similarity assessment of two entities is comparing features associated with the entities. Having RDF triples for representing information in LD, features are represented with properties and their values with objects (Sect. 2.2). For example, in LD a book can be represented with multiple features (properties) such as name, author, country, language, genre, and publisher. An example of a real entity (from DBpedia) described with multiple features is shown in Fig. 4.

The problem of handling importance of features is composed of two sub-problems: (1). How to recognize dominant properties? (2). How to deal with them? In real life, it is intuitive to distinguish very important properties of an entity from less important and not important ones. A computer program requires a well-defined approach to do the same task since every piece of information in LD is represented as RDF triples, and all the triples are equally important. Additionally, importance values of properties for all entities are constantly modified on the web. Therefore, a process of determining the importance values is a very time-consuming and impractical task.

Section 3.1 presents an answer to the first sub-problem of how to detect and distinguish the properties that vary in their importance. We present a solution to this problem by obtaining this information from the most popular online encyclopedia, Wikipedia. The next important question is how to use the obtained properties and their significance values in similarity evaluation. The answer to this question, usage of fuzzy membership functions, is explained in Section 3.2.

3.1 Properties and their Importance

Once the properties and their importance values are defined, we create a fuzzy set (Zadeh 1965) L of n subsets that categorize the properties with respect to their importance. Each subset has an assigned linguistic label that corresponds to its importance degree, such as:

$$L = \{l_1 = critical, l_2 = very\ important,\ \ldots, l_n = not\ important\} \quad (1)$$

In other words, properties with equal importance describing an entity are classified in the same subset. Each subset l_i may contain any number of properties:

$$l_i = \{p_1, p_2, \ldots, p_m\} \quad (2)$$

A total number of subsets, n, is a user-defined constant. This helps us to categorize a property in a proper subset that indicates its importance.

We developed an approach using Wikipedia Infoboxes to find key properties of an entity and to classify them into the suitable importance subset. Infobox is summarized information represented in a table on the top right-hand side of a Wikipedia page. It provides information about a particular entity. An example of Infobox for the book "The Lord of the Rings" is shown in Fig. 5.

First, we obtain the Infobox template[9] corresponding to the category of a considered entity. Properties included in the Infobox template are selected as the characteristic properties of the entity that we keep. We discard the rest of properties that the entity has. This step reduces the amount of data to be processed in a similarity evaluation method. Next, we classify the properties into proper subsets of L based on their importance in describing the entity.

The main idea proposed here is to exploit the information in the *domain* of a property. Domain of a property is a class of the subject in the <subject-property-object> RDF triple. Basically, the class refers to an item located in a hierarchical taxonomy. We argue that this information plays a critical role in identifying the importance of a property. Therefore, we categorize the properties based on the location of their domains in taxonomy of domains. This approach is justified because classes located in higher levels of taxonomy are more abstract than the ones in lower levels (Hossein Zadeh and Reformat 2013a, b). In general, abstract classes carry paramount description of an entity compared to less abstract and specific classes. Thus, properties with more abstract domains are more important. For example, considering an entity "book" properties such as {subject, genre, name} carry important information, intuitively, and they belong to the most abstract class, "thing", in DBpedia ontology[10]. In Fig. 6, a fragment of DBpedia taxonomy related to an entity "book" is depicted.

In a situation of comparing entities that belong to different Infobox templates, e.g., a book and a car, same process is followed. However, the obtained similarity will be very low as it lacks existence of common properties.

In LD, information is represented as a set of triples:

$$LD = \{ <s, p, o> | s \in C, p \in P, o \in C \cup D. \} \quad (3)$$

[9] http://en.wikipedia.org/wiki/Template:Infobox_book

[10] http://mappings.dbpedia.org/server/ontology/classes/

Fig. 5 Wikipedia Infobox for the book "The Lord of the Rings"

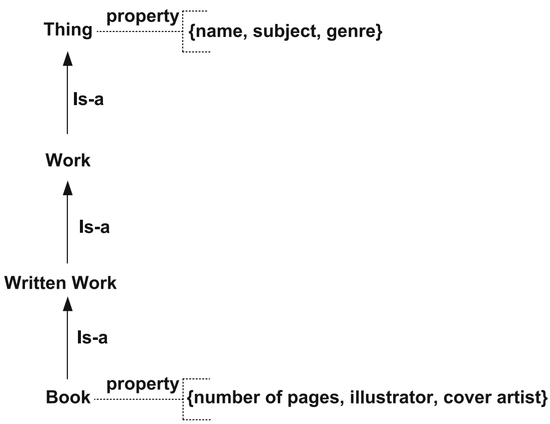


Fig. 6 Small fragment of DBpedia taxonomy for an entity "book"

where C, D and $P=\{p_1,p_2,\ldots,p_m\}$ are sets of entities, data values and properties, respectively. Each entity, c, is a subject in a number of triples connected to it via properties, p. Therefore, we represent an entity as a set of triples defining it.

Our proposed algorithm for similarity calculation is shown in details in Pseudo codes 1 (Table 1) and 2 (Table 2). In line 5 (Table 1), a set of common triples between two entities x and y, O_{common}, is obtained. O_x and O_y are two sets of objects unique to each entity x and y, respectively. They are initialized in line 6. For all permutations of elements in sets O_x and O_y, a sub-function *Sim_Second_Layer* is

Table 1 Pseudo code for similarity calculation

Similarity_final(x,y)

$L = \{l_1, l_2, ..., l_n\}$ */* initializing set L */*
for $\forall l_i \in L$
  $l_i = \{p_1, p_2, ..., p_m\}$ */* initializing set l_i */*
  for $\forall p_j \in l_i$
    O_{common} */* initialize set of objects common to x and y attached via property p_j */*
    O_x, O_y */*initialize set of objects unique to x and y attached via property p_j */*
    sim_layer1=| O_{common} |
    sim_layer2=0
    for $\forall (a,b) \mid a \in O_x \ \& \ b \in O_y$
      *sim_layer2=sim_layer2+**Sim_Second_Layer(a,b)***
    end
    $Sim_{p_j}^{l_i} = \dfrac{sim\_layer1 + sim\_layer2}{|\mathrm{O_{common}}| + |\mathrm{O_x}|.|\mathrm{O_y}|}$
  end
  $Sim^{l_i} = avg\left(Sim_{p_1}^{l_i}, Sim_{p_2}^{l_i}, \ldots, Sim_{p_m}^{l_i}\right)$
end
$Sim^{final} = aggr\left(Sim^{l_1}, Sim^{l_2}, \ldots, Sim^{l_n}\right)$

Table 2 Pseudo code of *Sim_Second_Layer* (a, b)

Sim_Second_Layer(a,b)

O_a, O_b */* initializing sets of objects attached to a and b visa property rdf:type */*
$k = 0$
for all pairs (*c*, *d*) ***such that*** $c \in O_a$, $d \in O_b$
  $k = k + 1$
  if $c \neq d$
    $\mathrm{Sim}_{[k]}(c,d) = \dfrac{2 * \mathrm{depth}(\mathrm{LCS}(c,d))}{\mathrm{depth}(c) + \mathrm{depth}(d)}$ */* LCS: least common subsumer of c and d in ontology */*
  else if
    $\mathrm{Sim}_{[k]}(c,d) = 1 - \dfrac{1}{\mathrm{depth}(c)}$
  end
end
return *max_over_k(* $\mathrm{Sim}_{[k]}(c,d)$ *)*

called (line 10). Similarites calculated for all pairs (a, b) are obtained and combined in line 12. Similarity between two entities x and y is calculated in line 16. It should be noted that to avoid division by zero, which happens in the case if there are no common objects, the value of similarity, line 8, is set to zero. This situation may happen when different entities are compared.

In sub-function *Sim_Second_Layer*, triples of the pair of entities a and b are extracted, Table 2. Two sets of O_a and O_b are initialized each containing the objects attached to entities a and b respectively via the property "rdf:type" (line 1). Note that, only triples having the property "rdf:type" are obtained and the rest are discarded. This is because, the description of an entity provided by the property "rdf: type" is of a special importance in Linked Data. "rdf:type" is used to say that things are of certain types. It is worth noting that a similar procedure can be repeated for the property "rdf:subject". Results from these two properties may be combined depending on how the information is expressed in a data set. For simplicity, we only consider the property "rdf:type" in the proposed similarity computation process. Similarity between c and d as the permutations of elements in sets O_a and O_b is calculated in lines 5–8. If c and d are different, their similarity is calculated using (Wu and Palmer 1994). Otherwise, their similarity is calculated based on the depth of the ontology that c or d belongs to. Finally, the maximum of similarities of all pairs is returned to the main function *Similarity_final(x,y)* (line 11).

3.2 Similarity and Fuzziness

Similarity between two entities x and y is defined as the aggregated similarity values computed for every subset l_i:

$$Sim^{final}(x,y) = aggr(Sim^{l_1}(x,y), Sim^{l_2}(x,y), \ldots, Sim^{l_n}(x,y)) \tag{4}$$

where *aggr(.)* is an aggregation operator, described later. Similarity values related to each subset l_i is obtained as the average of similarities for all properties in that subset:

$$Sim^{l_i}(x,y) = avg(Sim^{l_1}_{p_1}(x,y), Sim^{l_1}_{p_2}(x,y), \ldots, Sim^{l_1}_{p_m}(x,y)) \tag{5}$$

The *Sim(.)* function calculates the similarity value between two entities as defined below. Due to the nature of LD, entities along with their properties are distributed over the Web in a form of connected RDF triples. Considering an entity in LD, values of its properties may be a subject of another triple and so on. Those triples provide further information that can be used in similarity evaluation of an entity to another. For this reason, we include in similarity evaluation not only the triples that are directly connected to the entity (Layer 1) but also the triples connected one layer further away from the entity (triples describing the objects of that entity), see Fig. 3.

In (4), the *aggr(.)* operation can be any process that takes the weights of the similarity values into consideration. The main idea is that similarity measures calculated for each subset l_i contribute differently to the final similarity according to their importance. Here, it is defined as the normalized weighted sum of the similarity measures in which weights are the membership degrees obtained in (7). The final similarity is calculated as:

$$Sim^{final}(x,y) = w_1 . Sim^{l_1}(x,y) + w_2 . Sim^{l_2}(x,y) + \ldots + w_n . Sim^{l_n}(x,y) \quad (6)$$

where,

$$w_i = \frac{\mu(Sim^{l_i}(x,y))}{\sum_i \mu(Sim^{l_i}(x,y))} \quad (7)$$

$\mu(.)$ gives the membership degree for each $Sim^{l_i}(x,y)$ and is obtained as follows:

$$\mu(Sim^{l_i}(x,y)) = \left(Sim^{l_i}(x,y)\right)^{\psi_i} \quad (8)$$

Here, we know that $Sim^{l_i}(x,y) \in [0,1]$, therefore larger values of ψ_i leads to smaller values of membership degrees. ψ is a significance power and is calculated as:

$$\psi_i = i - f(l_i, c) \quad (9)$$

where, i - index of the subset l_i representing importance of a property, and $f(l_i, c)$ is a ratio of a number of properties of a subset l_i for a given entity to the total number of properties over all l_i ‘s of that particular entity. To justify (9) it should be noted that more important properties have smaller values of i and their $\mu(Sim^{l_i}(x,y))$ are larger. Also, $f(l_i, c)$ adjusts $\mu(Sim^{l_i}(x,y))$ such that if the subset l_i constitutes a substantial part of all properties, $f(l_i, c)$ is larger, $i - f(l_i, c)$ becomes smaller, and $\mu(Sim^{l_i}(x,y))$ increases. This shows higher influence of more representative properties.

All the steps in our approach of calculating similarity between any two entities are shown in Fig. 7. As it can be seen, triples in Layer 1 and Layer 2 of entities under similarity assessment are extracted from datasets in LD. After detecting the category of the entity by examining the related property, the corresponding Infobox template from Wikipedia is obtained. The Infobox contains key properties of that entity. We categorize them and rank the subsets of properties $L = \{l_1, l_2, \ldots, l_n\}$. Similarity values for all subsets are calculated separately, and further they are aggregated to compute the final similarity.

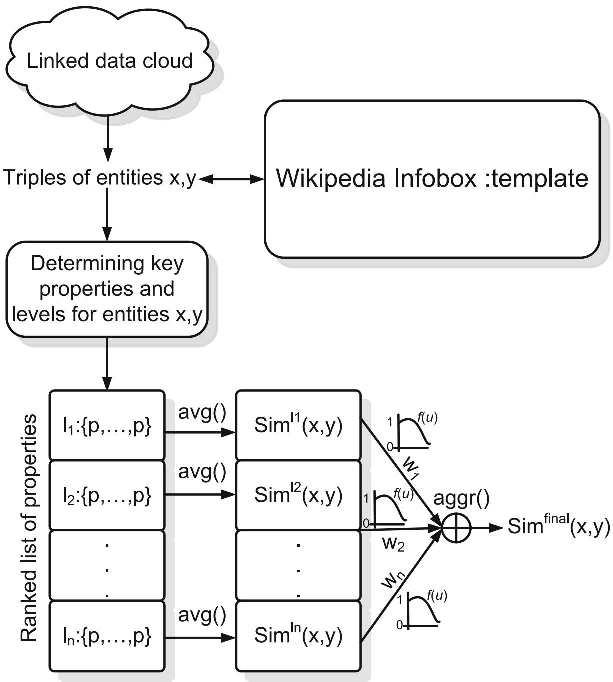


Fig. 7 Schematic of the similarity evaluation approach

4 Experiments

To evaluate the above approach, a set of entities is selected from a real-world dataset DBpedia and their associated RDF triples were extracted. The entities are instances of a concept "book" in DBpedia. As discussed previously, the Infobox template representing the concept "book" is extracted. This helps to detect characteristic properties and to group them. Next, a list of classes, {book, thing, work, written work}, are obtained from this template representing domains of the Infobox properties. Accordingly, we define four subsets of properties that group the properties within the same domain. Based on the location of each domain in the DBpedia ontology, we assign importance ranking to the created subsets. Table 3 shows the formed subsets for the entity "book" for this experiment.

The last subset in Table 3, null, contains properties related to an entity "book" that are labeled as non-important and are ignored in the similarity evaluation process. It is worth noting that discarding the non-important properties may cause losing some information. However, this will reduce the time and increase the speed of the similarity evaluation process especially when large number of entities are described with large numbers of properties. In addition, selection of these subsets and assigning each property to a subset can be customized to users' preferences or an application context.

Table 3 Different subsets of properties and their importance

Subset	Properties
l_1	Name, caption, title, country, language, series, subject, genre, publication date
l_2	Author, translator, publisher, preceded by, followed by
l_3	oclc[11], lcc[12]
l_4	Illustrator, cover artist, media type, number of pages, isbn, dewey[13]
null	First publication date, last publication date, number of volumes, based on, completion date, license, description, abstract, rights, editor, format, sales, etc

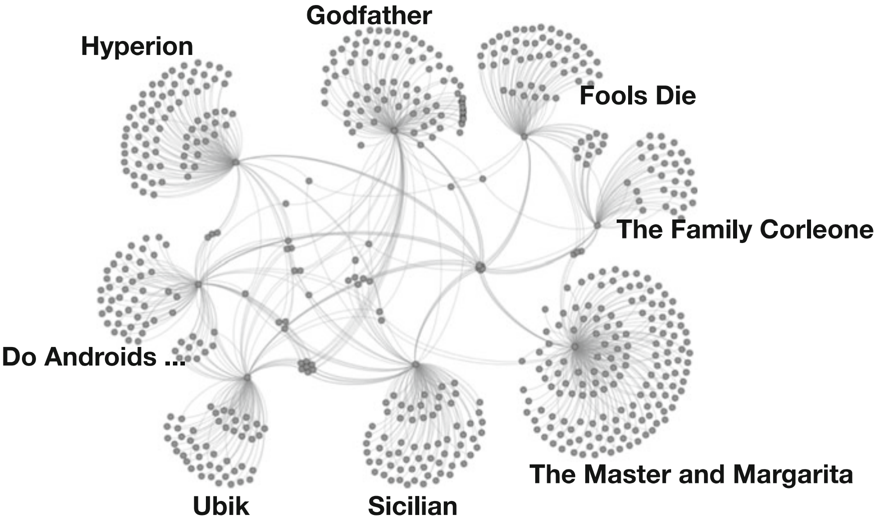


Fig. 8 Relationship of the given entities in LD

Eight instances of the entity "book" are selected: "The Godfather", "The Sicilian", "Do Androids Dream of Electric Sheep?", "Hyperion", "Ubik", "The Master and Margarita", "Fools Die" and "The Family Corleone". Figure 8 illustrates entities and their features, as well as relationships between. As it can be seen, entities may be connected directly via common objects or through subsequent connections.

Figure 9 shows similarity results between the entities based on the approach presented in Sect. 3.

According to the obtained values of similarity between any pair of books, the books "The Sicilian", "The Family Corleone" and "Fools die" are ranked as top three matches. Table 4, compares this result to the Google Knowledge Graph and

[11]Online Computer Library Center number

[12]Library of Congress Classification

[13]Dewey Decimal System Classification

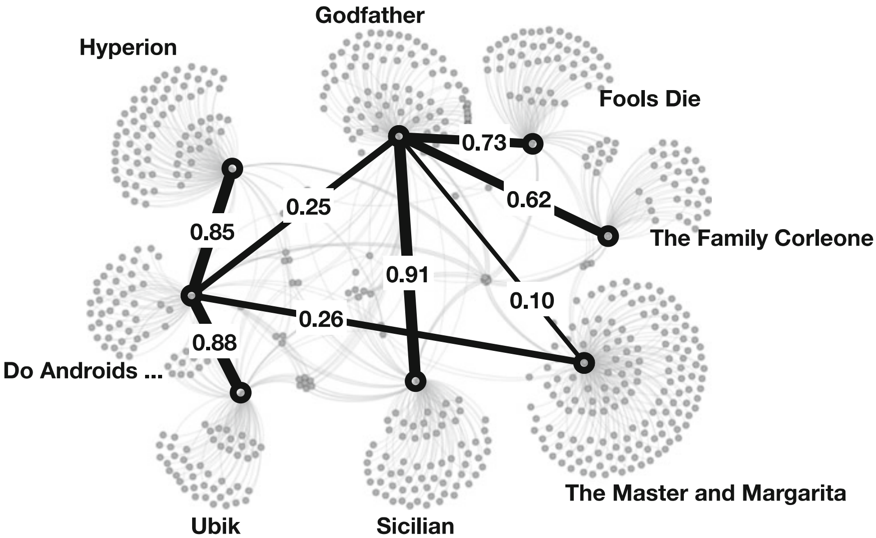


Fig. 9 Similarity values between entities

Table 4 Results of searching for the book "The Godfather"

Our approach	"The Sicilian", "Fools Die", "The Family Corleone"
Google Knowledge Graph	"**The Sicilian**", "The Godfather returns", "**The Family Corleone**", "The Last Don", "**Fools Die**", "The Fortunate Pilgrim", …
Amazon	"The Sicilian", "The Family Corleone"

the suggestions provided by Amazon. For the Google Knowledge Graph, the list is compiled based on searches performed by the users, and the position of books reflects the frequency searches performed after "The Godfather" was searched for. It should be noted, that this list contains three books from our experiment. The swapped position of "The Family Corleone" and "Fools Die" is due to the fact that our approach assigns high importance to an author. In the Google Graph the importance of features does not exist. Also, the Amazon website suggests "The Sicilian" and "The Family Corleone" to the buyers of "The Godfather". This emphasizes the high similarity of these books.

Table 5 compares the obtained similarity values of the proposed approach in the case when similarity values are averaged and weighted. In the averaged case, properties are considered to have equal importance, thus final similarity is averaged over the similarities of all properties regardless of their dominance in defining an entity. The weighted approach is the weighted sum of the similarities (6). The influence of recognizing importance of properties and the application of fuzzy membership functions can be easily observed.

Table 5 Comparison of similarity measures for averaged and weighted aggregations

Pairs of Concepts	Sim^{final} – averaged	Sim^{final} – weighted
(The Godfather, The Sicilian)	0.88	0.91
(The Godfather, Fools Die)	0.73	0.73
(The Godfather, The Family Corleone)	0.53	0.62
(The Godfather, The Master and Margarita)	0.16	0.01
(Do Androids Dream, Hyperion)	0.65	0.85
(Do Androids Dream, The Godfather)	0.12	0.25
(Do Andoirds Dream, Ubik)	0.79	0.88
(Do Andoirds Dream, The Master and Margarita)	0.19	0.26

5 Conclusion

The work presented here addresses the problem of relevance assessment between concepts within the environment of Linked Data (LD). In LD, data is represented in a form of RDF triples. RDF triples that share the same subject are perceived as features describing an entity identified by this subject. In other words, a given concept is defined via a set of features, and these features can be used to compare two entities. Such a simple idea is applied here to perform similarity assessment between concepts.

It is intuitively obvious, that not all features of concepts are equally important. Therefore, we propose a novel approach to calculate the semantic similarity by taking into account the importance levels of properties defining concepts. A method is presented to identify these properties and their degree of importance using the information included in Wikipedia Infoboxes. Based on these importance levels, fuzzy membership functions are developed. They are used to determine weights associated with levels of properties. These weights are further used to aggregate similarity measures assessed for each group of properties with the same importance.

The proposed approach is deployed to estimate similarities between several books. The RDF-based definitions of these books have been obtained from dBpedia. The results are very encouraging. They are comparable with the lists of books identified by Google Knowledge Graph that is composed based on the users' searches, and by Amazon suggestions that are determined based on the users' purchases.

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Authors Biography

Parisa D. Hossein Zadeh received her B.Sc. degree in Computer Engineering from University of Science and Technology, Tehran, Iran in 2006. She worked as a Computer Engineer for DPI Iran Company (former IBM Co.) from 2005 to 2007, where she led multiple projects in Network and Software fields. She received her M.Sc. degree in Electrical and Computer Engineering from University of Alberta, Edmonton, Canada. She is currently a Ph.D. candidate in Computer Engineering at University of Alberta. Her research interests include Semantic Web, similarity assessment, big data, search engines, fuzzy sets, data analysis, and machine learning.

Marek Reformat received his M.Sc. degree (with honors) from Technical University of Poznan, Poland, and his Ph.D. from University of Manitoba, Canada. Presently, he is a professor with the Department of Electrical and Computer Engineering, University of Alberta.

The goal of his research activities is to develop methods and techniques for intelligent data modeling and analysis leading to translation of data into knowledge, as well as to design systems that possess abilities to imitate different aspects of human behavior. He recognizes the concepts of Computational Intelligence – with fuzzy computing and possibility theory in particular – are key elements necessary for capturing relationships between pieces of data and knowledge, and for mimicking human ways of reasoning about opinions and facts. Dr. Reformat applies elements of fuzzy sets to social networks, Linked Open Data, and Semantic Web in order to handle inherently imprecise information, and provide users with unique facts retrieved from the data.

Dr. Reformat is a past president of the North American Fuzzy Information Processing Society, and a vice president of the International Fuzzy Systems Association. He has been a member of program committees of multiple international conferences related to Computational Intelligence and Software Engineering.

The Genesis of Fuzzy Sets and Systems – Aspects in Science and Philosophy

Rudolf Seising

Abstract In 1965 Lotfi A. Zadeh founded the theory of Fuzzy Sets and Systems. This chapter deals with developments in the history of philosophy, logic, and mathematics during the time before and up to the beginning of fuzzy logic and it also gives a view of its first application in control theory. Regarding the term "fuzzy" we note that older concepts of "vagueness" and "haziness" had previously been discussed in philosophy, logic, mathematics. This chapter delineates some specific paths through the history of the use of these "loose concepts ". Haziness and fuzziness were concepts of interest in mathematics and philosophy during the second half of the 20th century. The logico-philosophical history presented here covers the work of Russell, Black, Hertz, Wittgenstein and others. The mathematical-technical history deals with the theories founded by Menger and Zadeh. Menger's concepts of probabilistic metrics, hazy sets (ensembles flous) and micro-geometry as well as Zadeh's theory of Fuzzy Sets paved the way for the establishment of Soft Computing methods. In the first decade of Fuzzy Sets and Systems, nobody thought that this theory would be successful in the field of applied sciences and technology. Zadeh expected that his theory would have a role in the future of computer systems as well as Humanities and Social Sciences. When Mamdani and Assilian picked up the idea of Fuzzy Algorithms to establish a first Fuzzy Control system for a small steam engine, this was the Kick-off for the "Fuzzy Boom" and Zadehs primary intention trailed away for years. Then in the new millennium a new movement for Fuzzy Sets in Social Sciences and Humanities was launched.

1 Introduction

In the summer of 2015 we will commemorate the 50th anniversary of the theory of Fuzzy Sets and Systems (FSS). This is a fuzzy jubilee because the exact time when Lotfi A. Zadeh (born 1921, Fig. 1 (a)) discovered the concept of fuzzy sets is

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_27

unknown. The event is roughly assigned to the summer of 1965. This chapter reviews the genesis of this new mathematical theory following my original (historical) research work that encompasses inspections of scientific articles, newspapers, letters, and – most importantly – interviews with Lotfi A. Zadeh and other protagonists of the early years of the theory of FSS. For details see [1].

In Sects. 2 and 3 we briefly follow Zadeh's development as an electrical engineer in the 1950s and 1960s from concrete Network theory and Filter theory to abstract System theory and then to the theory of Fuzzy sets and Systems as a generalized System theory. In Sect. 4, we present short views on historical paths of modern logic and philosophy of mathematics in the 20th century, the logical analysis of vagueness, the concepts of statistical metrics and ensembles flous and we consider Wittgenstein's late philosophy in relation to our subject. Section 5 gives a review of Zadehs works on Fuzzy Sets in language and meaning that appeared before the "Fuzzy Boom" with real-world application systems that is the subject of Sect. 6. Section 7 gives an outlook on Zadeh's later theories: *Computing with words* and *with perceptions*. Finally the bibliogrpahy includes some comments to most of the references to this book contribution.

2 From Electrical Engineering to System Theory

Fuzzy Sets and Systems can look back upon an eventful story in the scientific environment of electrical engineering, including the initial system theory and computer sciences known during this time, which were part of Zadeh's training as a student in Tehran, Iran. Following his immigration to the USA in 1942, Zadeh continued his studies at the *Massachusetts Institute of Technology* (MIT) in Cambridge, Massachusetts. He moved to New York in 1946, where he was awarded a Ph.D by *Columbia University* in 1949. Since 1958, he has been a Professor of Electrical Engineering at the *University of California* at Berkeley. When he established the theory of fuzzy sets in the mid-1960s, he was already a well-known protagonist of the system theoretical approach in electrical engineering, which was a new scientific trend from the 1950s onward. Together with Charles A. Desoer (1926–2011, Fig. 1 (b)) he published in 1962 *Linear System Theory: The State Space Approach* [2], which became a standard textbook. In 1963, together with Desoer's former Ph.D. student Elijah Polak (born 1931, Fig. 1 (c)) he edited the volume *System Theory* [3]. In his own contribution to this volume, which was entitled "The Concepts of System, Aggregate, and State in System Theory" [4], Zadeh presented his state space approach. Two years later, when he introduced fuzzy sets, he construed his new theory as a "general system theory".

In 1954 – Zadeh was then an instructor at Columbia University in New York - he wrote for the *Columbia Engineering Quarterly* an article, named "System Theory" [5] that begins as follows: "If you never heard of system theory, you need not feel like an ignoramus. It is not one of the well-established branches of science. In fact, it has not yet been officially recognized as a scientific discipline. It does not appear on programmes of meetings of scientific societies nor in indices to scientific publications. It does not have well-defined boundaries, nor does it have settled objectives." [5, p. 34]. Zadeh emphasized that all scientific disciplines are concerned with systems, but the new branch, named system theory, considers systems as mathematical constructs rather than physical objects: "The distinguishing characteristic of system theory is its abstractness." [5, p. 16] In this and later papers Zadeh quoted the definition of a "system" from Webster's Dictionary: A system is "an aggregation or assemblage of objects united by some form of interaction or interdependence." (Fig. 2)

System theorists deal with abstract systems, "that is, systems whose elements have no particular physical identity" [5, p. 16]; they deal with "black boxes". Fig. 3 reproduces the illustration to this article – actually a "black box"!

Communication systems are a special type of systems that have been of interest since the 1950s, when information and communication theory emerged as successful scientific and technological disciplines. Zadeh was deeply involved in the development of this new communication theory and its techniques when he delivered a lecture on "Some Basic Problems in Communication of Information" at the meeting of the Section of Mathematics and Engineering of the *New York Academy of Sciences* in March 1952 [6]. He represented signals as ordered pairs $(x(t), y(t))$ of points in a signal space Σ, which is imbedded in a function space with a delta-function basis. This analogy between projection in a function space and filtration by an ideal filter led Zadeh to postulate a function symbolism of filters in the early 1950s [7]. Thus, $N = N_1 + N_2$ represents a filter consisting of two filters connected by addition, $N = N_1 N_2$ represents their tandem (sequential) combination and $N = N_1 \left| N_2 \right.$ the separation process (Fig. 4).

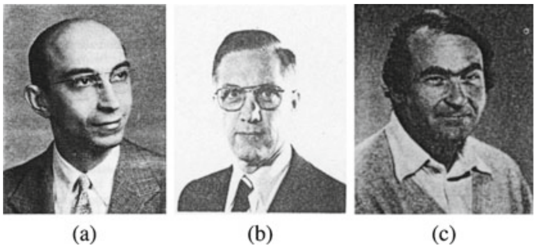


Fig. 1 Lotfi A. Zadeh, Charles A. Desoer, and Elijah Polak: colleagues at the University of California, Berkeley

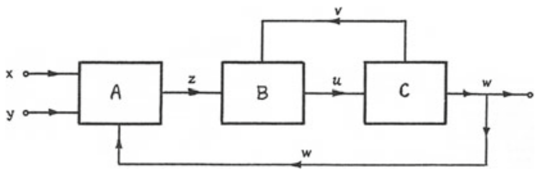


Fig. 2 Block diagram of interconnected objects, [5, p. 17]

Later (in [8]), Zadeh discussed the concept of optimal filters as opposed to ideal filters following Norbert Wiener's work. Ideal filters are defined as filters that achieve a perfect separation of signal and noise. However, in reality there are no ideal filters, e.g., an ideal low-pass filter should retain all frequency components until a certain threshold until which all components should be fully suppressed. In practice, we cannot have such a step-shaped separation. Zadeh knew that we get a smooth transition from 1 to 0 transmission coefficient as the frequency decreases. These transitions are similar to the well-known membership functions of fuzzy sets. However, in the 1950s the time was not ripe for this new mathematical theory. Zadeh defined optimal filters as those that give the "best approximation" of a signal, and he noted that "best approximations" depend on reasonable criteria. At this time he formulated these criteria in statistical terms.

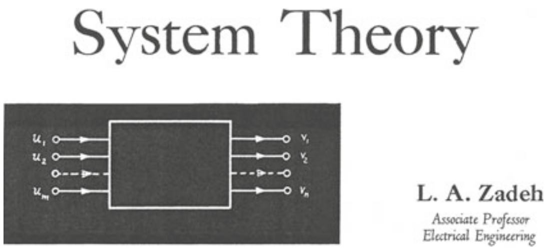


Fig. 3 Illustration from Zadeh's article [5]

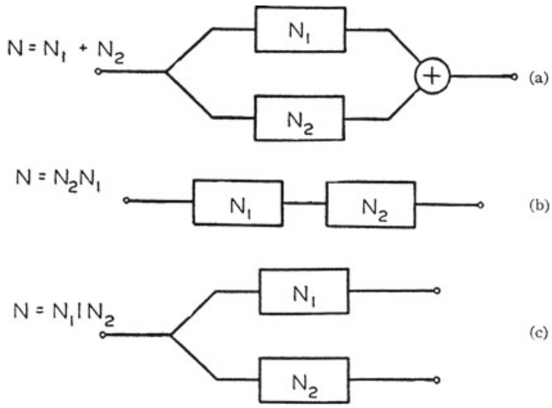


Fig. 4 Functional symbolism of ideal filters, [7, p. 225]

But starting in the next decade he wrote the landmark article "From Circuit Theory to System" for the anniversary edition of the *Proceedings of the IRE* that appeared in May 1962 to mark the 50th year of the *Institute of Radio Engineers* (IRE) [9]. Here, he could outline problems and applications of system theory and its relations to network theory, control theory, and information theory. Furthermore, he pointed out "that the same abstract 'systems' notions are operating in various guises in many unrelated fields of science is a relatively recent development. It has been brought about, largely within the past two decades, by the great progress in our understanding of the behaviour of both inanimate and animate systems—progress which resulted on the one hand from a vast expansion in the scientific and technological activities directed toward the development of highly complex systems for such purposes as automatic control, pattern recognition, data-processing, communication, and machine computation, and, on the other hand, by attempts at quantitative analyses of the extremely complex animate and man-machine systems which are encountered in biology, neurophysiology, econometrics, operations research and other fields" [9, p. 856f].

In this article Zadeh used for the very first time the word "fuzzy" and he wrote it down to characterize his vision of new mathematics:

> In fact, there is a fairly wide gap between what might be regarded as 'animate' system theorists and 'inanimate' system theorists at the present time, and it is not at all certain that this gap will be narrowed, much less closed, in the near future. There are some who feel that this gap reflects the fundamental inadequacy of the conventional mathematics – the mathematics of precisely-defined points, functions, sets, probability measures, etc. – for coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitude more complex than man-made systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions. [9, p. 857f].

However, when Zadeh published these notions, he did not know what this mathematics of fuzzy quantities would look like.

Another method to deal with imperfect or noisy signals in communication systems was introduced in the 1950s by Richard E. Bellman (1920–1984, Fig. 5 (a)), a young mathematician working at the RAND Corporation, United States Air Force Project in Santa Monica, California. Bellman was the founder of the method of *Dynamic Programming* [10], and tried to apply his "principle of optimality" in communication theory.

In the late 1950s, Bellman met Zadeh in New York, where Zadeh worked at *Columbia University*. Their friendship lasted until Bellman's death in 1984. Even though they considered diverse mathematical aspects of electrical engineering, system theory and, later, computer science, they met each other very often and discussed several aspects of their scientific work. Bellman was the first and most important critic of Zadeh's new theory of fuzzy sets in 1965.

Fig. 5 (a): Richard E. Bellman and (b): Robert E. Kalaba, (c): Lotfi A. Zadeh

3 New View on System Theory: Fuzzy Sets and Systems

Zadeh and Bellman planned to work together at RAND in Santa Monica, California, in the summer of 1964. Prior to the summer of 1964, Zadeh gave a talk on pattern recognition at the *Wright-Patterson Air Force Base*, Dayton, Ohio. It may have been on this occasion that he started thinking about the use of grades of membership for pattern classification and that he conceived the first example of fuzzy mathematics, which he wrote in one of his first papers on the subject: “For example, suppose that we are concerned with devising a test for differentiating between handwritten letters O and D. One approach to this problem would be to give a set of handwritten letters and indicate their grades of membership in the fuzzy sets O and D. On performing abstraction on these samples, one obtains the estimates $\tilde{\mu}_O$ and $\tilde{\mu}_D$ of μ_O and μ_D, respectively. Then given a letter x which is not one of the given samples, one can calculate its grades of membership in O and D; and, if O and D have no overlap, classify x in O or D.” [11, p. 30]

In a few days he extended this concept to the theory of fuzzy sets and a few weeks later, he discussed this preliminary version of the theory of fuzzy sets with Bellman. Then he wrote his manuscript on “Fuzzy Sets” [12] and submitted it to the journal *Information and Control* in November 1964. “Fuzzy Sets” appeared in June 1965 and was the first article on fuzzy sets in a scientific journal. However, Lotfi Zadeh also wrote other papers at the time. According to common practice at the department of electrical engineering in Berkeley, the article “Fuzzy Sets” was preprinted as a report of the *Electronics Research Laboratory* in November 1964 [13]. As a result of his talk in Dayton, Ohio, he wrote a paper which he sent to Bellman who was the editor of the *Journal of Mathematical Analysis and Applications*. Bellman agreed to publish the paper in said journal but the publication appeared late, in 1966, under the title “Abstraction and Pattern Classification” [14]. The authors of the article were Bellman, his associate Robert E. Kalaba (1926–2004, Fig. 5 (b)) and Zadeh. The text and the authors’ names are identical to those of the RAND memorandum RM-4307-PR, which appeared as early as October 1964. This memo was written by Zadeh alone. Here he defined fuzzy sets for the first time in a scientific paper, establishing a general framework for the treatment of pattern recognition problems [15].

In April 1965 the *Symposium on System Theory* was held at *Polytechnic Institute* in Brooklyn. At this meeting Zadeh presented “A New View on System Theory”: a view that deals with the concepts of fuzzy sets, “which provide a way of treating fuzziness in a quantitative manner.” In the subsequent publication of the proceedings of this symposium we find a shortened manuscript version of the talk. His contribution was entitled “Fuzzy Sets and Systems” in this publication: [11, p. 29]. In this lecture and in the paper, Zadeh first defined “fuzzy systems” as follows:

> A system S is a fuzzy system if (input) $u(t)$, output $y(t)$, or state $s(t)$ of S or any combination of them ranges over fuzzy sets, [11, p. 33].

In “Fuzzy Sets and Systems” Zadeh explained that “these concepts relate to situations in which the source of imprecision is not a random variable or a stochastic process but rather a class or classes which do not possess sharply defined boundaries." [11, p. 29] His "simple" examples in this brief summary of his new "way of dealing with classes in which there may be intermediate grades of membership" were "the 'class' of real numbers which are much larger than, say, 10, and "the 'class' of 'bald man', and also the 'class' of adaptive systems." [11, p. 29] For further details on the roots of fuzzy systems in system theory, the reader is referred to the author's book [1].

In "Fuzzy Sets" [12], Zadeh introduced the new mathematical entities "fuzzy sets": "Such classes are not classes or sets in the usual sense of these terms, since they do not dichotomize all objects into those that belong to the class and those that do not." He introduced "the concept of a fuzzy set, that is a class in which there may be a continuous infinity of grades of membership, with the grade of membership of an object x in a fuzzy set A represented by a number $f_A(x)$ in the interval $[0, 1]$."[1] Zadeh maintained that these new concepts provide a "convenient way of defining abstraction — a process which plays a basic role in human thinking and communication." [11, p. 29] The question was how to generalize various concepts, union of sets, intersection of sets, and so forth. Zadeh defined *equality*, *containment*, *complementation*, *intersection* and *union* relating to fuzzy sets A, B in any universe of discourse X as follows (for all $x \in X$; see Fig. 6):

- $A = B$ if and only if $f_A(x) = f_B(x)$,
- $A \subseteq B$ if and only if $f_A(x) \leq f_B(x)$,
- $\neg A$ is the complement of A if and only if $f_{\neg A}(x) = 1 - f_A(x)$,
- $A = \cup B$ if and only if $f_{A \cap B}(x) = max(f_A(x), f_B(x))$,
- $A = \cap B$ if and only if $f_{A \cap B}(x) = min(f_A(x), f_B(x))$,

For his interpretation of fuzzy unions and intersections he wrote a separate paragraph, which shows a very important analogy to sieves, because Zadeh wrote: "Specifically, let $f_i(x), i = 1, \ldots, n$, denote the value of the membership function of A_i at x. Associate with $f_i(x)$ a sieve $S_i(x)$ whose meshes are of size $f_i(x)$. Then,

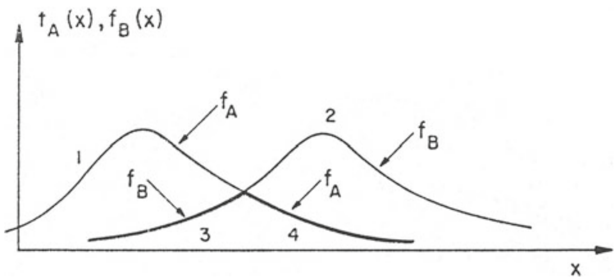


Fig. 6 Illustration of the union as maximum of membership functions f_A and f_B (1, 2) and the intersection as minimum of membership functions f_A and f_B (3, 4), [12]

[1]Later Zadeh and also the whole "fuzzy community" made use of the greek letter μ to mark the membership function in Fuzzy Set Theory.

$f_i(x) \vee f_j(x)$ and $f_i(x) \wedge f_j(x)$ correspond, respectively, to parallel and series combinations of $S_i(x)$ and $S_j(x)$," as shown in Fig. 7.

More generally, a well-formed expression involving $A_i, \ldots, A_n$ $\cup$ and $\cap$ corresponds to a network of sieves $S_i(x), \ldots, S_n(x)$ which can be found by the conventional synthesis techniques for switching circuits. As a very simple example,

$$C = \left[(A_1 \cup A_2) \cap A_3\right] \cup A_4 \tag{1}$$

corresponds to the network shown in Fig. 8." [12, p. 344]

If the reader takes into account the fact that the term "sieve" denotes a filter, he will comprehend the analogy of fuzzy sets and electrical filters as outlined in the first section.

In the decade that followed the first publications on Fuzzy Sets and Systems [11–15] Zadeh expected that they would have a role in the future of computer systems as well as humanities and social sciences. At that time nobody thought that this theory would be successful in the field of applied sciences and technology. Quite the contrary, he remarked that he did not expect the incorporation of FSS into the fields of sciences and engineering: "What we still lack, and lack rather acutely, are methods for dealing with systems which are too complex or too ill-defined to admit of precise analysis. Such systems pervade life sciences, social sciences, philosophy, economics, psychology and many other 'soft' fields." [16]

After "Fuzzy Sets" had appeared in print, Zadeh received many requests for offprints. Philosopher Max Black (1909–1988) who had published a paper entitled

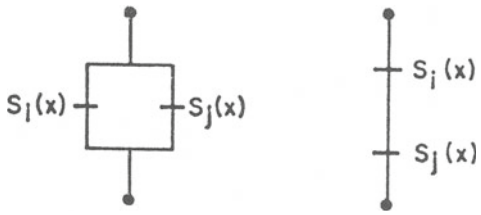


Fig. 7 Parallel and serial combination of sieves illustrating the fuzzy union, ∪, (maximum) and intersection, ∩, (minimum), [12]

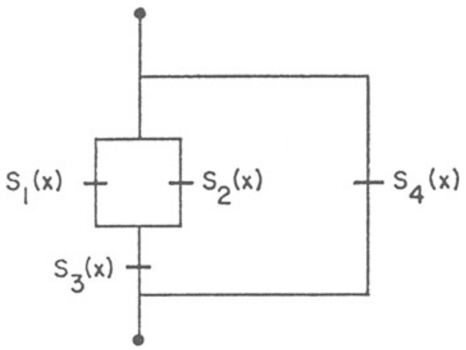


Fig. 8 A network of sieves simulating $\{(f_1(x) \vee f_2(x)) \wedge f_3(x)\}$ $\vee f_4(x)$, [12]

"Vagueness – An Exercise in Logical Analysis" [17] back in 1937 anticipated a vague idea from Zadeh's theory , for he wrote:

> The vagueness of the word chair is typical of all terms whose application involves the use of the senses. In all such cases, 'borderline case' and 'doubtful objects' are easily found to which we are unable to say either that the class name does or does not apply. [17, p. 434]

He now told Zadeh in a letter:

> You were good enough to send me, some time ago, some of your recent papers on topics connected with 'Fuzzy Sets'. If I have not written before, the reason has not been lack of interests, but an inescapable press of other duties. Now that I have had a chance, at least, to study your work, I want to express my admiration and interest. I believe that your ingenious construction promises to provide intellectual tools of great value. In case you have not come across it, I might draw your attention to an early article of mine ... [18]

He referred to his article [17], that was also already reprinted in his book, *Language and Philosophy* [19] and also to his "more recent article on similar topics ...": "Reasoning with Loose Concepts" [20]. To understand the history of the logical analysis of vagueness let's go back to the history of modern science and its philosophy!

4 Philosophy of Science, Vagueness and Fuzzy Sets

Beginning as early as the 17th century, a primary quality factor in scientific work has been a maximal level of exactness. Galileo Galilei (1564–1642) and René Descartes (1596–1650) started the process of giving modern science its preciseness through the use of the tools of logic and mathematics. The language of mathematics has served as a basis for the definition of theorems, axioms, and proofs. The works of Isaac Newton (1643–1727), Gottfried Wilhelm Leibniz (1646–1769, Pierre-Simon Laplace (1749–1827), and many others led to the ascendancy of modern science, fostering the impression that scientists were able to represent – completely and exactly – all the facts and processes that people observe in the world. But this optimism has gradually begun to seem somewhat nave in view of the discrepancies between the exactness of theories and what scientists observe in the real world. From the empiricist point of view the source of our knowledge is sense experience. John Locke (1632–1704) used the analogy of the mind of a newborn baby as a "tabula rasa" that would be written by the sensual perceptions the child has later. In Locke's opinion these perceptions provide information about the physical world. Locke's view is called "material empiricism" whereas so-called idealistic empiricism was the position of George Berkeley (1685–1753) and David Hume (1711–1776): the material world does not exist; only perceptions are real. Immanuel Kant (1724–1804, Fig. 9 (a)) achieved a synthesis of rationalism and empiricism in his magnum opus *Critique of Pure Reason*, published in 1781 [21]). Kant argued that human experience of a world is only possible if the mind provides a systematic structuring of its representations that is logically prior to the mental representations that were analyzed by empiricists and rationalists. With these philosophical views alone, we would not be able to explain the nature of our experience because these views only considered the results of the interaction between our mind and the world, but not the contribution made by the mind. Kant concluded that it must be the minds structuring that makes experience possible.

4.1 Heinrich Hertz' Philosophy of Science

In his book *The Principles of Mechanics Presented in a New Form* the German physicist Heinrich Hertz (1857–1894, Fig. 9 (b)) had established his theory of knowledge: he viewed physical theories as "pictures" of reality. He began his introduction with the following words:

> The most direct, and in a sense the most important, problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation. As a basis for the solution of this problem we always make use of our knowledge of events which have already occurred, obtained by chance observation or by pre-arranged experiment. In endeavoring thus to draw inferences as to the future from the past, we always adopt the following process. We form for ourselves images or symbols of external objects; and the form which we give them is such that the necessary consequents of the images in thought are always the images of the necessary consequents in nature of the things pictured. [...]
>
> The images which we here speak of are our conceptions of things. With the things themselves they are in conformity in one important respect, namely, in satisfying the above-mentioned requirement. For our purpose it is not necessary that they should be in conformity with the things in any other respect whatever. As a matter of fact, we do not know, nor have we any means of knowing, whether our conceptions of things are in conformity with them in any other than this one fundamental respect. [22, p. 1]

We know from experience the conformity between nature and our mind that is necessary for that: (logically) inadmissible images are "all images which implicitly contradict the laws of our thought." Although images are logically admissible, they can be incorrect "if their essential relations contradict the relations of external things."

For one external object there can exist more than one correct image, differing in respect to appropriateness:

> Of two images of the same object that is the more appropriate which pictures more of the essential relations of the object, the one which we may call the more distinct. Of two images of equal distinctness the more appropriate is the one which contains, in addition to the essential characteristics, the smaller number of superfluous or empty relations, the simpler of the two. [22, p. 2]

Hertz's epistemology and his view of scientific theories as mind-created "images", based on the scientist's experience, was contrary to the dominant view at his time. Most scientists during the years around the turn of the 20th century regarded empirical theories as objective, and in particular, most of them believed in the existence of one unique theory. On the other hand, Hertz knew from the experience he had gathered in the genesis of electrodynamics that various theories with different systems of concepts are possible, and that one theory may eventually become accepted. In his “Language of images”, he wrote:

> What enters into the images for the sake of correctness is contained in the results of experience, from which the images are built up. What enters into the images, in order that they may be permissible, is given by the nature of our mind. To the question whether an image is permissible or not, we can without ambiguity answer yes or no; and our decision will hold good for all time. And equally without ambiguity we can decide whether an image is correct or not; but only according to the state of our present experience, and permitting an appeal to later and riper experience. But we cannot decide without ambiguity whether an image is appropriate or not; as to this differences of opinion may arise. One image may be more suitable for one purpose, another for another; only by gradually testing many images can we finally succeed in obtaining the most appropriate. [22, p. 3]

Hertz spoke about “images” or “symbols” of external objects, because they are replacements for concepts in physical theories (e.g., mechanics, electricity and magnetism, and electrodynamics) that are not accessible to our sensory perceptions.

4.2 Wittgenstein’s Tractatus logico-philosophicus

“A picture is a model of reality.”, “We picture facts to ourselves.”, “A picture is a fact.” These are three consecutive propositions in Ludwig Wittgenstein’s *Tractatus logico-philosophicus* [23, prop. 2.1, 2.12, 2.141]. They demonstrate the influence of Heinrich Hertz’s *Principles of Mechanics* on his thinking – a debt that Wittgenstein himself acknowledged when he referred to Hertz in his diary [24, p. 476] and explicitly in another part of the *Tractatus*: “In the proposition there must be exactly as many things distinguishable as there are in the state of affairs which it represents. They must both possess the same logical (mathematical) multiplicity (cf. Hertz’s Mechanics, on Dynamic Models).” [23, prop. 4.04]

The first two propositions in Wittgenstein’s Tractatus are:

1. The world is everything that is the case.
2. The world is the totality of facts, not of things. [23]

Fig. 9 (a): Immanuel Kant (b): Heinrich Hertz and (c): Ludwig Wittgenstein

Then, in the Tractatus, Wittgenstein (1889–1951, Fig. 9 (c)) wrote that the world consists of facts. Facts may or may not contain smaller parts. If a fact has no smaller parts, he calls it an "atomic fact." If we know all atomic facts, we can describe the world completely by corresponding "atomic propositions." – Propositions 3 and 4 in the Tractatus are:

3. The logical picture of the facts is the thought.
4. The thought is the significant pro position. [23]

"The totality of propositions is language." [23, prop. 4.001] Wittgenstein argued that sentences in colloquial language are very complex. He conceded that there is a "silent adjustment to understand colloquial language" but it is "enormously complicated." Therefore it is "humanly impossible to gather immediately the logic of language." [23, prop. 4.002] This is the task of philosophy: "All philosophy is 'Critique of language'." [23, prop. 4.0031] Wittgenstein knew that common linguistic usage is vague, but at the time when he wrote the *Tractatus*, he tried to solve this problem by constructing a precise language – an exact logical language that gives a unique picture of the real world. Wittgenstein thought that the *Tractatus* solved all philosophical problems.

But the *Tractatus* spared many problems for the future! One of these philosophical problems concerns the vagueness in our language and also two founders of modern logic, Gottlob Frege (1848–1925) and Bertrand Russell (1872–1970), focused attention on and analyzed this problem. A separate and isolated development took place at the Lvov-Warsaw School of logicians; one of them was Jan Łukasiewicz (1878–1956) who introduced in 1920 a three-valued logic and later other multi-valued logics. Simultaneously the American mathematician, Emil L. Post (1897–1954), also introduced a logic of additional truth degrees with $n \geq 2$, where n are the truth values.

The important contributions of these Polish mathematicians and logicians to modern logic were recognized when Alfred Tarski (1902–1983) followed an invitation of the Viennese mathematician Karl Menger to give a lecture in Vienna. All these thinkers had been influenced by Frege's studies. This was especially true of Tadeusz J. S. Kortabiński (1866–1938), who argued that a concept for a property is vague (Polish: *chwiejne*) if the property may be the case by grades [25] and Kazimierz Ajdukiewicz (1890–1963), who stated the definition that "a term is vague if and only if its use in a decidable context … will make the context undecidable in virtue of those [language] rules" [26]. The Polish characterization of "vagueness" was therefore the existence of fluid boundaries.

4.3 Vagueness and Logic

The German philosopher and mathematician Gottlob Frege confronted the problem of vagueness when formalizing the mathematical principle of complete induction: he saw that some predicates are not inductive, viz. they have been defined for all natural numbers, but they result in false conclusions, e.g. the predicate "heap" cannot be evaluated for all natural numbers [27]. When he revised the basics of his *Begriffsschrift* for a lecture to the *Society of Medicine and Science* in Jena, Germany, of the beginning year 1891, Frege reinterpreted concept functions and subsequently he introduced these functions of concepts everywhere. He stated: If "$x + 1$" is meaningless for all arguments x, then the function $x + 1 = 10$ has no value and no truth value either. Thus, the concept "that which when increased by 1 yields 10" would have no sharp boundaries. Accordingly, for functions the demand on sharp boundaries entails that they must have a value for every argument [28]. This is a mathematical verbalization of what is called the classical sorites paradox that can be traced back to the old Greek word $\sigma o \rho \bar{\omega} \varsigma$ (for "heap") used by Eubulid of Alexandria (4th century BC). In his *Grundgesetze der Arithmetik* (*Foundations of Arithmetic*) that appeared in the years 1893–1903, Frege called for concepts with sharp boundaries, because otherwise we could break logical rules and, moreover, the conclusions we draw could be false [29]. Frege's specification of vagueness as a particular phenomenon influenced other scholars, notably his British contemporary and counterpart the philosopher and mathematician Bertrand Russell (1872–1970, Fig. 9 (b)), who published his article on "Vagueness" in 1923 [30].

Russell quoted the sorites—in fact, he did not use mathematical language in this article, but, for example, discussed colours and "bald men" (Greek: $\varphi \alpha \lambda \alpha \kappa \rho o \varsigma$, *falakros*, English: fallacy, false conclusion):

> Let us consider the various ways in which common words are vague, and let us begin with such a word as red. It is perfectly obvious, since colours form a continuum, that there are shades of color concerning which we shall be in doubt whether to call them red or not, not because we are ignorant of the meaning of the word red, but because it is a word the extent of whose application is essentially doubtful. This, of course, is the answer to the old puzzle about the man who went bald. It is supposed that at first he was not bald, that he lost his hairs one-by-one, and that in the end he was bald; therefore, it is argued, there must have been one hair the loss of which converted him into a bald man. This, of course, is absurd. Baldness is a vague conception; some men are certainly bald, some are certainly not bald, while between them there are men of whom it is not true to say they must either be bald or not bald. [30, p. 85].

Russell also argued that a proper name – and here we can take as an example the name "Lotfi Zadeh" – cannot be considered to be an unambiguous symbol even if we believe that there is only one person with this name. Lotfi Zadeh "was born, and being born is a gradual process. It would seem natural to suppose that the name was not attributable before birth; if so, there was doubt, while birth was taking place, whether the name was attributable or not. If it be said that the name was attributable before birth, the ambiguity is even more obvious, since no one can decide how long before birth the name become attributable." [30, p. 86]

Russell reasoned "that all words are attributable without doubt over a certain area, but become questionable within a penumbra, outside which they are again certainly not attributable." [30, p. 86f] Then he generalized that words of pure logic also have no precise meanings, e.g. in classical logic the composed proposition "p or q" is false only when p and q are false and true elsewhere. He went on to claim that the truth values " 'true' and 'false' can only have a precise meaning when the symbols employed – words, perceptions, images … – are themselves precise". As we have seen above, this is not possible in practice, so he concludes "that every proposition that can be framed in practice has a certain degree of vagueness; that is to say, there is not one definite fact necessary and sufficient for its truth, but certain region of possible facts, any one of which would make it true. And this region is itself ill-defined: we cannot assign to it a definite boundary." Russell emphasized that there is a difference between what we can imagine in theory and what we can observe with our senses in reality: "All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial existence." [30, p. 88f]. He proposed the following definition of accurate representations:

> One system of terms related in various ways is an accurate representation of another system of terms related in various other ways if there is a oneone relation of the terms of the one to the terms of the other, and likewise a oneone relation of the relations of the one to the relations of the other, such that, when two or more terms in the one system have a relation belonging to that system, the corresponding terms of the other system have the corresponding relation belonging to the other system." And in contrast to this, he stated that "a representation is vague when the relation of the representing system to the represented system is not oneone, but onemany. [30, p. 89]

He concluded that "Vagueness, clearly, is a matter of degree, depending upon the extent of the possible differences between different systems represented by the same representation. Accuracy, on the contrary, is an ideal limit." [30, p. 90].

The Cambridge philosopher and mathematician Max Black (1909–1988, Fig. 9 (b)) responded to Russell's article in his already mentioned article of 1937 [17]. He differentiated vagueness from ambiguity, generality, and indeterminacy. He emphasized

> that the most highly developed and useful scientific theories are ostensibly expressed in terms of objects never encountered in experience. The line traced by a draughtsman, no matter how accurate, is seen beneath the microscope as a kind of corrugated trench, far removed from the ideal line of pure geometry. And the 'point-planet' of astronomy, the 'perfect gas' of thermodynamics, and the 'pure species' of genetics are equally remote from exact realization." [17, p. 427]

Black proposed a new method to symbolize vagueness: "a quantitative differentiation, admitting of degrees, and correlated with the indeterminacy in the divisions

Fig. 10 (a): Bertrand Russell, (b): Max Black and (c): Karl Menger

made by a group of observers." [17, p. 441] He assumed that the vagueness of a word involves variations in its application by different users of a language and that these variations fulfill systematic and statistical rules when one symbol has to be discriminated from another. He referred to situations in which a user of the language makes a decision whether to apply L or $\neg L$ to an object x. Black exemplified: "Such a situation arises, for instance, when an engine driver on a foggy night is trying to decide whether the light in the signal box is really a red or a green light" [17, p. 442] He defined this discrimination of a symbol x with respect to a symbol L by DxL. (We obtain $DxL = Dx\neg L$ by definition.) Most speakers of a language and the same observer in most situations will determine that either L or $\neg L$ is used. In both cases, among competent observers there is a certain unanimity, a preponderance of correct decisions. For all DxL with the same x but not necessarily the same observer, m is the number of L uses and n the number of $\neg L$ uses. On this basis, Black stated the following definition (see Fig. 11):

> We define the consistency of application of L to x as the limit to which the ratio $\frac{m}{n}$ tends when the number of DxL and the number of observers increase indefinitely. [...] Since the consistency of the application, C, is clearly a function of both L and x, it can be written in the form $C(L, x)$." [17, p. 442]

In 1963, Black labeled concepts without precise boundaries as "loose concepts" rather than "vague" ones, in order to avoid misleading and pejorative implications [20]. Once again he expressly rejected Russell's assertion that traditional logic is "not applicable" as a method of conclusion for vague concepts: "Now, if all empirical

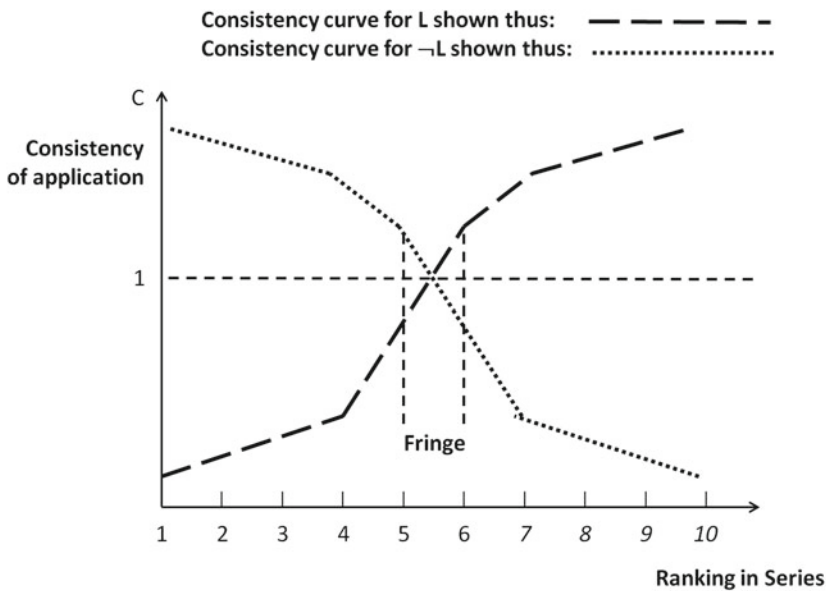


Fig. 11 Consistency of application of a typical vague symbol, [17, p. 443]

concepts are loose, as I think they are, the policy becomes one of abstention from any reasoning from empirical premises. If this is a cure, it is one that kills the patient. If it is always wrong to reason with loose concepts, it will, of course, be wrong to derive any conclusion, paradoxical or not, from premises in which such concepts are used. A policy of prohibiting reasoning with loose concepts would destroy ordinary language – and, for that matter, any improvement upon ordinary language that we can imagine." [20, p. 7]

4.4 Menger's Ensembles Flous

In Vienna in the 1920s and 1930s, Karl Menger (1902–1985, Fig 10 (c)) evolved into a specialist in topology and geometry, particularly with regard to the theories of curves, dimensions, and general metrics. After he immigrated to the USA, he continued his work on these subjects. In 1942, with the intention of generalizing the theory of metric spaces more in the direction of probabilistic concepts, he introduced the term "statistical metric":

A *statistical metric* is "a set S such that with each two elements ('points') p and q of S, a probability function $\Pi(x; p, q)$ (The probability that the distance between p and q is x) is associated satisfying the following conditions:

1. $\Pi(0; p, p) = 1$ (The probability is 1 that the distance between p and q is 0.)
2. If $p \neq q$, then $\Pi(0; p, p) < 1$.
3. $\Pi(x; p, q) = \Pi(x; q, p)$.
4. $T(\Pi(x; p, q), \Pi(y; q, r) \leq \Pi(x + y; p, r)$.

where $T(\alpha, \beta)$ is a function defined for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$, such that

(a) $0 \leq T(\alpha, \beta) \leq 1$.
(b) T is non-decreasing in either variable.
(c) $T(\alpha, \beta) = T(\beta, \alpha)$.
(d) $T(1, 1) = 1$.
(e) If $\alpha > 0$ then $T(\alpha, 1) > 0$. [31, p. 535f]

Condition 4., the "triangular inequality" of the statistical metric S implies the following inequality for all points q and all numbers x between 0 and z:

$$\Pi(z; p, r) \geq Max; T(\Pi(x; p, q), \Pi(z - x; q, r)) \quad (2)$$

Here, Menger introduced the term *triangular norm* (*t-norm*) to indicate the function T.

In 1951 Menger introduced a new notation Δ_{ab} for the non-decreasing cumulative distribution function, associated with every ordered pair (a, b) of elements of a set S and he wrote: "The value $\Delta_{ab}(x)$ may be interpreted as the probability that the distance from a to b be $< x$." [32, p. 226] Much more interesting is the following text passage: "We call a and b *certainly-indistinguishable* if $\Delta_{ab} = 1$ for each $x > 0$.

Uniting all elements which are certainly indistinguishable from each other into identity sets, we decompose the space into disjoint sets $A, B, \ldots$. We may define for any a belonging to A and b belonging to B. (The number is independent of the choice of a and b.) The identity sets form a perfect analog of an ordinary metric space since they satisfy the condition[2]:

If $A \neq B$, then there exists a positive x with $\Delta_{ab}(x) < 1$."

In the same year Menger addressed the difference between the mathematical continuum and the physical continuum. Regarding A, B, and C as elements of a continuum, he referred to a claim of the French mathematician and philosopher Henri Poincaré, "that only in the mathematical continuum do the equalities $A = B$ and $B = C$ imply the equality $A = C$. In the observable physical continuum, 'equal' means 'indistinguishable', and $A = B$ and $B = C$ by no means imply $A = C$. 'The raw result of experience may be expressed by the relation $A = B$, $B = C$, $A < C$ which may be regarded as the formula for the physical continuum.' According to Poincaré, physical equality is a non-transitive relation." [33, p. 178]

Menger suggested a realistic description of the equality of elements in the physical continuum by associating with each pair (A, B) of these elements the probability that A and B will be found to be indistinguishable. He argued:

> For it is only very likely that A and B are equal, and very likely that B and C are equal – why should it not be less likely that A and C are equal? In fact, why should the equality of A and C not be less likely than the inequality of A and C?" [33, p. 178]

To solve "Poincaré's paradox" Menger used his concept of probabilistic relations and geometry: For the probability $E(a, b)$ that a and b would be equal he postulated:

- $E(a, a) = 1$, for every a;
- $E(a, a) = E(b, a)$, for every a and b;
- $E(a, b) \cdot E(b, c) \leq E(a, c)$ for every a, b, c.

If $E(a, a) = 1$, then he called a and b *certainly equal*. (In this case we obtain the ordinary equality relation.) "All the elements which are certainly equal to a may be united to an 'equality set', A. Any two such sets are disjoint unless they are identical." [33, p. 179]

In addition to studies of well-defined sets, he called for a theory to be developed in which the relationship between elements and sets is replaced by the probability that an element belongs to a set; in contrast to ordinary sets, he called these entities "ensembles flous" [34, p. 226]. Later, Menger used the English term "hazy set" and to elucidate the contrast he referred to conventional sets as "rigid sets." [35].

Menger never envisaged a mathematical theory of loose concepts that differs from probability theory. At a symposium of the *American Association for the Advancement of Science* organized in 1966 to commemorate the 50th anniversary of Ernst

[2] In the original paper Menger wrote ">". The present author thanks Erich Peter Klement for this correction.

Mach's death, he spoke about "Positivistic Geometry". When he compared his "micro geometry" with the theory of fuzzy sets – he wrote: "In a slightly different terminology, this idea was recently expressed by Bellman, Kalaba and Zadeh under the name fuzzy set." [35, p. 232] – He did not see that the "slight difference" between "degrees" (fuzziness) and "probabilities" is a difference not just in terminology but in the meaning of the concepts.

4.5 Wittgenstein's Family Resemblances

In his later years (after 1947), Wittgenstein turned away from the epistemological system of the *Tractatus* with its ideal mapping between the objects of reality and a logically precise language. This philosophy of his later years is completely different from that of the *Tractatus* years. It seems as though the two philosophical systems were created by different men because this new view said: If we are not able to find such an exact logical language, then we have to accept the fact that there is vague linguistic usage in all languages. Then the images, models, and theories that we build with the words and propositions of our languages to communicate with them are and will also be vague.

His second main work, the *Philosophical Investigations* epitomize Wittgenstein's late philosophy: "Instead of producing something common to all that we call language, I am saying that these phenomena have no one thing in common which makes us use the same word for all, but that they are related to one another in many different ways. And it is because of this relationship, or these relationships, that we call them all 'language'. I will try to explain this." [36, §65] We find the following explanation in the next paragraph of this book, in keeping with the concept of a game:

> Consider for example the proceedings that we call "games". I mean board-games, card-games, ball-games, Olympic games, and so on. What is common to them all? Don't say: "There must be something common, or they would not be called 'games' " but look and see whether there is anything common to all. For if you look at them you will not see something that is common to all, but similarities, relationships, and a whole series of them at that. To repeat: dont think, but look! Look for example at board-games, with their multifarious relationships.
>
> Now pass to card-games; here you find many correspondences with the first group, but many common features drop out, and others appear. When we pass next to ball-games, much that is common is retained, but much is lost. Are they all "amusing"? Compare chess with noughts and crosses. Or is there always winning and losing, or competition between players? Think of patience. In ball games there is winning and losing; but when a child throws his ball at the wall and catches it again, this feature has disappeared. Look at the parts played by skill and luck; and at the difference between skill in chess and skill in tennis. Think now of games like ring-a-ring-a-roses; here is the element of amusement, but how many other characteristic features have disappeared! sometimes similarities of detail. And we can go through the many, many other groups of games in the same way; can see how similarities crop up and disappear. And the result of this examination is: we see a complicated network of similarities overlapping and crisscrossing: sometimes overall similarities. [36, §66]

In the next paragraph Wittgenstein creates a new concept to describe this new epistemological system:

> I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. etc. overlap and crisscross in the same way. And I shall say: "games" form a family. [36, §67]

Concepts and their families have no sharp boundaries, as he also wrote in [36, §71]:

> One might say that the concept "game" is a concept with blurred edges. "But is a blurred concept a concept at all?" Is an indistinct photograph a picture of a person at all? Is it even always an advantage to replace an indistinct picture by a sharp one? Isn't the indistinct one often exactly what we need? Frege compares a concept to an area and says that an area with vague boundaries cannot be called an area at all. This presumably means that we cannot do anything with it. But is it senseless to say: "Stand roughly there"? [36, §71]

And in a later paragraph Wittgenstein wrote: "The results of philosophy are the uncovering of one or another piece of plain nonsense and bumps that the understanding has got by running its head up against the limits of language." [36, §119] In other words, our conceptions, images, and symbols of external things or objects are entities without sharp borders. They are fuzzy entities!

5 Before the "Fuzzy Boom": Fuzziness in Language and Meaning

In the 1960s Zadeh was interested in applying fuzzy sets in linguistics. This idea led to interdisciplinary scientific exchange on the campus of the *University of California* at Berkeley between him and the mathematicians Joseph Goguen (1941–2006, Fig 14 (a)) and Hans-Joachim Bremermann, the psychologist Eleanor Rosch (Heider) and the linguist George Lakoff. Goguen generalized the fuzzy sets to so-called "*L*-sets" [37, 38]. An *L*-set is a function that maps the fuzzy set carrier *X* into a partially ordered set *L*, i.e. $L : X \to L$. The partially ordered set *L* Goguen called the "truth set" of *A*. The elements of *L* can thus be interpreted as "truth values"; in this respect, Goguen then referred to a "Logic of Inexact Concepts" [39]. His work was laid out in terms of logical algebra and category theory, and his proof of a representation theorem for *L*-sets within category theory justified fuzzy set theory as an expansion of set theory.

5.1 Fuzzy Languages and Fuzzy Algorithms

In 1970 Zadeh presented his paper "Fuzzy Languages and their Relations to Human and Machine Intelligence" at the conference *Man and Computer* in Bordeaux, France: He said: "As computers become more powerful and thus more influential in human affairs, the philosophical aspects of this question become increasingly overshadowed by the practical need to develop an operational understanding of the limitations of the machine judgment and decision making ability." [40, p. 130]

He called it a paradox that the human brain is always solving problems by manipulating "fuzzy concepts" and "multidimensional fuzzy sensory inputs" whereas "the computing power of the most powerful, the most sophisticated digital computer in existence" is not able to do this. Therefore, he stated that "in many instances, the solution to a problem need not be exact, so that a considerable measure of fuzziness in its formulation and results may be tolerable. The human brain is designed to take advantage of this tolerance for imprecision whereas a digital computer, with its need for precise data and instructions, is not." [40, p. 132] He intended to push his theory of fuzzy sets to model the imprecise concepts and directives: "Indeed, it may be argued that much, perhaps most, of human thinking and interaction with the outside world involves classes without sharp boundaries in which the transition from membership to non-membership is gradual rather than abrupt." [40, p. 131] He stated:

> Although present-day computers are not designed to accept fuzzy data or execute fuzzy instructions, they can be programmed to do so indirectly by treating a fuzzy set as a data-type which can be encoded as an array […]. Granted that this is not a fully satisfactory approach to the endowment of a computer with an ability to manipulate fuzzy concepts, it is at least a step in the direction of enhancing the ability of machines to emulate human thought processes. It is quite possible, however, that truly significant advances in artificial intelligence will have to await the development of machines that can reason in fuzzy and non-quantitative terms in much the same manner as a human being. [40, p. 132]

In August 1967, the Filipino electrical engineer William Go Wee at *Purdue University* in Indiana had submitted his dissertation "On Generalizations of Adaptive Algorithms and Application of the Fuzzy Sets Concept to Pattern Classification" [41] that he had written under King Sun Fu, one of the pioneers in the field of pattern recognition. Wee had applied the fuzzy sets to iterative learning procedures for pattern classification and had defined a finite automaton based on Zadeh's concept of the fuzzy relation as a model for nonsupervised learning systems: "The decision maker operates deterministically. The learning section is a fuzzy automaton. The performance evaluator serves as an unreliable 'teacher' who tries to teach the 'student' to make right decisions." [40, p. 101]

The fuzzy automaton representing the learning section implemented a "nonsupervised" learning fuzzy algorithm and converged monotonically. Wee showed that this fuzzy algorithm could not only be used in the area of pattern classification but could also be translated to control and regulation problems. Working with his doctoral advisor, Wee presented his findings in the article "A Formulation of Fuzzy Automata and its Applications as a Model of Learning Systems" [42].

In 1968 Zadeh presented "fuzzy algorithms" [43]. Usual algorithms depend upon precision. An algorithm must be completely unambiguous and error-free in order to result in a solution. The path to a solution amounts to a series of commands which must be executed in succession. Algorithms formulated mathematically or in a programming language are based on set theory. Each constant and variable is precisely defined; every function and procedure has a definition set and a value set. Each command builds upon them. Successfully running a series of commands requires that each result (output) of the execution of a command lies in the definition range of the following command, that it is, in other words, an element of the input set for the series. Not even the smallest inaccuracies may occur when defining these coordinated definition and value ranges. He now saw "that in real life situations people think certain things. They think like algorithms but not precisely defined algorithms [40]. Inspired by this idea, he wrote:

> Essentially, its purpose is to introduce a basic concept which, though fuzzy rather than precise in nature, may eventually prove to be of use in a wide variety of problems relating to information processing, control, pattern recognition, system identification, artificial intelligence and, more generally, decision processes involving incomplete or uncertain data. The concept in question will be called fuzzy algorithm because it may be viewed as a generalization, through the process of fuzzification, of the conventional (nonfuzzy) conception of an algorithm. [40, p. 94]

To illustrate, fuzzy algorithms may contain fuzzy instructions such as:

(a) "Set y approximately equal to 10 if x is approximately equal to 5," or
(b) "If x is large, increase y by several units," or
(c) "If x is large, increase y by several units; if x is small, decrease y by several units; otherwise keep y unchanged."

The sources of fuzziness in these instructions are fuzzy sets which are identified by their underlined names. [40, p. 94f]

All people function according to fuzzy algorithms in their daily life, Zadeh wrote – they use recipes for cooking, consult the instruction manual to fix a TV, follow prescriptions to treat illnesses or heed the appropriate guidance to park a car. Even though activities like this are not normally called algorithms: "For our point of view, however, they may be regarded as very crude forms of fuzzy algorithms." [40, p. 95]

Already in 1969 Zadeh contributed to a NATO summer school on "Architecture and Design of Digital Computers" in Grenoble with the title "Toward Fuzziness in Computer Systems. Fuzzy Algorithms and Languages" [44] The association of fuzziness and computers must have sounded surprising in the late 1960s and referring to that Zadeh said in its introduction: "At first glance, it may appear highly incongruous to mention computers and fuzziness in the same breath, since fuzziness connotes imprecision whereas precision is a major desideratum in computer design." [44, p. 9]

In the following paragraphs Zadeh justified by with arguing that future computer systems will have to perform many more complex information processing tasks than the computers that he and his contemporaries in the 1960s knew. He expected that future computers would have to process more and more imprecise information! "Fuzziness, then, is a concomitant of complexity. This implies that as the complexity of a task or a system for performing that task exceeds a certain threshold, the system must necessarily become fuzzy in nature. Thus, with the rapid increase in the complexity of the information processing tasks which the computers are called upon to perform, a point is likely to be reached perhaps within the next decade when the computers will have to be designed for processing of information in fuzzy form. In fact, it is this capability – a capability which present-day computers do not possess – that distinguishes human intelligence from machine intelligence. Without such capability we cannot build computers that can summarize written text, translate well from one natural language to another, or perform many other tasks that humans can do with ease because of their ability to manipulate fuzzy concepts." [44, p. 10] For that purpose, Zadeh asserted that "intriguing possibilities for computer systems" are offered by fuzzy algorithms and fuzzy languages!

To execute fuzzy algorithms by computers they have to get an expression in fuzzy programming languages. Consequently the next step for Zadeh was to define fuzzy languages. "All languages, whether natural of artificial, tend to evolve and rise in level through the addition of new words to their vocabulary. These new words are, in effect, names for ordered subsets of names in the vocabulary to which they are added." [44, p. 16]

Real world phenomena are very complex. To characterize or picture these phenomena in terms of our natural languages we use our vocabulary and because this set of words is restricted, Zadeh argued that this process leads to fuzziness:

> Consequently, when we are presented with a class of very high cardinality, we tend to group its elements together into subclasses in such a way as to reduce the complexity of the information processing task involved. When a point is reached where the cardinality of the class of subclasses exceeds the information handling capacity of the human brain, the boundaries of the subclasses are forced to become imprecise and fuzziness becomes a manifestation of this imprecision. This is the reason why the limited vocabulary we have for the description of colors makes it necessary that the names of colors such as red, green, bleu [sic.], purple, etc. be, in effect, names of fuzzy rather than non-fuzzy sets. This is why natural languages, which are much higher in level than programming languages, are fuzzy whereas programming languages are not. [44, p. 10]

Here, Zadeh argued explicitly for programming languages that are – because of missing rigidness and preciseness and because of their fuzziness – more like natural languages. He mentioned the concept of stochastic languages that was published by the Finnish mathematician Paavo Turakainen in *Information and Control* in the foregoing year [45], being such an approximation to our human languages using randomizations in the productions, but Zadeh preferred fuzzy productions to achieve a formal fuzzy language. Then, he presented a short sketch of his program to extend non-fuzzy formal languages to fuzzy languages which he published in elaborated form with the co-author Edward T.-Z. Lee in "Note on Fuzzy Languages" [46]. His definition in these early papers was given in the terminology of the American computer scientists John Edward Hopcroft and Jeffrey David Ullman that was published in the same year [47].

L is a fuzzy language if it is a fuzzy set in the set V_T^*, the so-called "Kleene closure of V_T, the set of all finite strings composed of elements of the finite set of terminals V_T, e.g. $V_T = \{a, b, c, \dots, z\}$ .The membership function $\mu_L(x) : V_T^* \rightarrow [0, 1]$ associates with each finite string x, composed of elements in V_T, its grade of membership in L. Here is one of the simple examples that he gave in this article [44]:

> Assume that $V_T = \{0, 1\}$, and take L to be the fuzzy set $L = \{(0, 0.9), (1, 0.2), (00, 0.8), (01, 0.3), (10, 0.7), (11, 0.3)\}$ with the understanding that all the other strings in V_T^* do not belong to L (i.e., have grade of membership equal to zero). [44, p. 16].

In general the language L has high cardinality and therefore it is not usual to define it by a listing of its elements but by a finite set of generating rules. Thus, in analogy to the case of non-fuzzy languages Zadeh defined a *fuzzy grammar* as

> a quadruple $G = (V_N, V_T, P, S)$, where V_N is a set of variables (non-terminals) disjoint from V_T, P is a set of [fuzzy] productions and S is an element of V_N. The elements of V_N (called [*fuzzy*] *syntactic categories*) and S is an abbreviation for the syntactic category 'sentence'. The elements of P define conditioned fuzzy sets in $(V_T \cup V_N)$. [44, p. 16]

5.2 Fuzzy Relations and Fuzzy Semantics

In 1971, Zadeh defined similarity relations and fuzzy orderings [48]. In doing so, he was proceeding from the concept of fuzzy relations as a fuzzification of the relation concept known in conventional set theory that he had already defined in his seminal article "Fuzzy Sets" [12]: If X and Y are conventional sets and if $X \times Y$ is their Cartesian product, let: $L(X)$ be the set of all fuzzy sets in X, $L(Y)$ be the set of all fuzzy sets in Y, and $L(X \times Y)$ be the set of all fuzzy sets in $X \times Y$.

Relations between X and Y are subsets of their Cartesian product $X \times Y$, and the composition $t = q * r$ of the relation $q \subseteq X \times Y$ with the relation $r \subseteq Y \times Z$ into the new relation $T \subseteq X \times Z$ is given by the following definition: $t = q * r = \{(x, z), \exists y : (x, y) \in q \wedge (y, z) \in r\}$.

Fuzzy relations between sets X and Y are subsets in $L(X \times Y)$. For three conventional sets X, Y and Z, the fuzzy relation Q between X and Y and the fuzzy relation R between Y and Z are defined: $Q \in L(X \times Y)$ and $R \in L(Y \times Z)$. The combination of these two fuzzy relations into a new fuzzy relation $T \in L(X \times Z)$ between X and Z can then be defined from the fuzzy relations Q and R into the fuzzy relation $T \in L(X \times Z)$ when the logical conjunctions are replaced by the corresponding ones of the membership functions.

- The above definition of the composition of conventional relations includes a logical AND ($\wedge$), which, for the "fuzzification", is replaced by the minimum operator that is applied to the corresponding membership functions.
- The above definition of the composition of conventional relations includes the expression "$\exists y$" ("there exists a y"). The existing $y \in Y$ is the first or the second or the third … (and so on); written logically: $sup_{y \in Y}$ ($\vee$). In the "fuzzifications", the logical OR conjunction is replaced by the maximum operator that is applied to the corresponding membership functions.

The fuzzy relation $T = Q * R$ is therefore defined via Zadeh's "rule of max-min combination" for membership functions: $\mu_T(x, y) = max_{y \in Y} min \left\{\mu_Q(x, y); \mu_T(y, z)\right\}$, $y \in Y$. In infinite sets the max-min composition rule is replaced with the sup-min composition rule. However, it is adequate to assume here that all of the sets are finite.

As a generalization of the concept of the equivalence relation Zadeh defined the concept of "similarity", since the similarity relation he defined is reflective, symmetrical and (max, min) transitive, i.e. for $x, y \in X$ the membership function of S has the following properties:

- reflexivity: $\mu_S(x, x) = 1$,
- symmetry: $\mu_S(x, y) = \mu_S(y, x)$,
- transitivity: $\mu_S(x, z) \geq max_{y \in Y} min \left\{ \mu_{(x,y)}, \mu_S(y, x) \right\}$.

Zadeh's *occupation* with natural and artificial languages gave rise to his studies in semantics. This intensive work let him to the question "Can the fuzziness of meaning be treated quantitatively, at least in principle?" [49, p. 160] . His 1971 article "Quantitative Fuzzy Semantics" [49] starts with this hint:

> Few concepts are as basic to human thinking and yet as elusive of precise definition as the concept of 'meaning'. Innumerable papers and books in the fields of philosophy, psychology, and linguistics have dealt at length with the question of what is the meaning of 'meaning' without coming up with any definitive answers."[3] [49, p. 159].

Zadeh started a new field of research "to point to the possibility of treating the fuzziness of meaning in a quantitative way and suggest a basis for what might be called quantitative fuzzy semantics" combining his results on fuzzy languages and fuzzy relations. In the section "Meaning" of this paper he set up the basics:

> Consider two spaces: (a) a universe of discourse, U, and (b) a set of terms, T, which play the roles of names of subsets of U. Let the generic elements of T and U be denoted by x and y, respectively. Then he started to define the *meaning* $M(x)$ of a term x as a fuzzy subset of U characterized by a membership function $\mu(y\,|x)$ which is conditioned on x. [49, p. 164f]

One of his examples was:

> Let U be the universe of objects which we can see. Let T be the set of terms *white*, *grey*, *green*, *blue*, *yellow*, *red*, *black*. Then each of these terms, e.g., *red*, may be regarded as a name for a fuzzy subset of elements of U which are red in color. Thus, the meaning of red, M(red), is a specified fuzzy subset of U. [49, p. 164f]

In the following section of this paper, that is named "Language", Zadeh regarded a language L as a "fuzzy correspondence", more explicitly, a *fuzzy binary relation*, from the term set $T = \{x\}$ to the universe of discourse $U = \{y\}$ that is characterized by the membership function $\mu_L : T \times U \rightarrow [0, 1]$. If a term x of T is given, then the membership function $\mu_L(x, y)$ defines a set $M(x)$ in U with the following membership function: $\mu_{M(x)}(y) = \mu_L(x, y)$. Zadeh called the fuzzy set $M(x)$ the *meaning* of the term x; x is thus the name of $M(x)$.

With this framework Zadeh continued in his 1970 article [40] to establish the basic aspects of a theory of fuzzy languages that is "much broader and more general than that of a formal language in its conventional sense." [40, p. 134] In the following we quote his definitions of fuzzy language, structured fuzzy language and meaning:

[3] In a footnote he named the works of 12 known philosophers, linguists or cognitive scientists.

Definition 1 A *fuzzy language* L is a quadruple $L = (U, T, E, N)$, in which U is a non-fuzzy *universe of discourse*; T (called the *term set*) is a fuzzy set of terms which serve as names of fuzzy subsets of U; E (called an *embedding set* for T) is a collection of symbols and their combinations from which the terms are drawn, i.e., T is a fuzzy subset of E; and N is a fuzzy relation from E (or more specifically, the support of $T(= supp(T) = \{x \mid \mu_A(x) > 0\})$ that is a non-fuzzy subset, to U which will be referred to as a *naming relation*.

In the case that U and T are infinite large sets, there is no table of membership values for μ_T(x) and $\mu_N(x, y)$ and therefore the values of these membership functions have to be computed. To this end, universe of discourse U and term set T have to be endowed with a structure and therefore Zadeh defined the concept of a *structured fuzzy language*.

Definition 2 A *structured fuzzy language* L is a quadruple $L = (U, S_T, E, S_N)$, in which U is a universe of discourse; E is an embedding set for term set T, S_T is a set of rules, called syntactic rules of L, which collectively provide an algorithm for computing the membership function, μ_T, of the term set T; and S_N is a set of rules, called the semantic rules of L, which collectively provide an algorithm for computing the membership function, μ_N, of the fuzzy naming relation N. The collection of syntactic and semantic rules of L constitute, respectively, the syntax and semantics of L.

To define the concept of meaning, Zadeh characterized the membership function $\mu_N : supp(T) \times U \rightarrow [0, 1]$ representing the strength of the relation between a term x in T and an object y in U. He clarified:

> A language, whether structured or unstructured, will be said to be *fuzzy* if [term set] T or [naming relation] N or both are *fuzzy*. Consequently, a non-fuzzy language is one in which both T and N are non-fuzzy. In particular, a non-fuzzy structured language is a language with both non-fuzzy syntax and non-fuzzy semantics." [40, p. 138]

Thus, natural languages have fuzzy syntax and fuzzy semantics whereas programming languages, as they were usual in the early 1970s, were non-fuzzy structured languages. The membership functions μ_T and μ_N for term set and naming relation, respectively, were two-valued and the compiler used the rules to compute these values 0 or 1. This means that the compiler decides deterministically by using the syntactic rules whether a string x is a term in T or not and it also determines by using the semantic rules whether a term x hits an object y or not. On the other hand we have natural languages, e.g. English, and it is possible that we use sentences that are not completely correct but also not completely incorrect. These sentences have a degree of grammaticality between 0 and 1. Of course, native speakers usually use correct sentences. "In most cases, however, the degree of grammaticality of a sentence is either zero or one, so that the set of terms in a natural language has a fairly sharply defined boundary between grammatical and ungrammatical sentences", Zadeh wrote [40, p. 138].

Much more fuzziness we find in semantics of natural languages: Zadeh gave the example "if the universe of discourse is identified with the set of ages from 1 to 100, then the atomic terms young and old do not correspond to sharply defined subsets of U. The same applies to composite terms such as not very young, not very young and not very old, etc. In effect, most of the terms in a natural language correspond to fuzzy rather than non-fuzzy subsets of the universe of discourse." [40, p. 139]

Zadeh now identified these fuzzy subsets of the universe of discourse that correspond to terms in natural languages with its "meaning":

Definition 3 The *meaning* of a term x in T is a fuzzy subset $M(x)$ of U in which the grade of membership of an element y of U is given by $\mu_{M(x)}(y) = \mu_N(x, y)$.

Thus, $M(x)$ is a fuzzy subset of U which is conditioned on x as a parameter and which is a section of N in the sense that its membership function, $\mu_{M(x)} : U \rightarrow [0, 1]$, is obtained by assigning a particular value, x, to the first argument in the membership function of N.

Zadeh concluded this paper mentioning that "the theory of fuzzy languages is in an embryonic stage" but he expressed his hope that based on this framework better models for natural languages will be developed than the models of the "restricted framework of the classical theory of formal languages." [40, p. 163]

Later in the 1970s he published important papers summarizing and developing the concepts we presented above: in 1973 "Outline of a new approach to the analysis of complex systems and decision processes" [50] appeared in the *IEEE Transaction on Systems, Man, and Cybernetics*, in 1975 the three-part article "The concept of a Lingustic Variable and its Application to Approximate Reasoning" [51] appeared in the journal *Information Sciences*, in the same year Zadeh published "Fuzzy Logic and Approximate Reasoning" in the philosophical journal *Synthese* [52] and in 1978

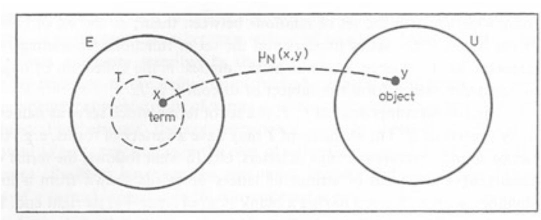
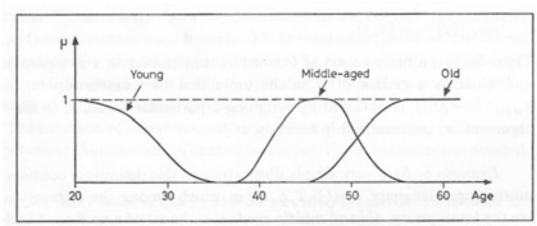


Fig. 12 The components of a fuzzy language: U = universe of discourse; T = term set; E = embedding set for T ; N = naming relation from E to U; x = term; y = object in U; $\mu_M(x, y)$ = strength of the relation between x and y; $\mu_T(x)$ = grade of membership of x in T. [40, p. 136]

Fig. 13 Membership functions of fuzzy sets M (young), M (middle-aged) and M (old), [40, p. 140]



Zadeh published "PRUF a meaning representation language for natural languages" in the *International Journal of Man-Machine Studies* [53].[4]

During the 1970's Berkeley-psychologist Eleanor Rosch developed her prototype theory on the basis of empirical studies. This theory assumes that people perceive objects in the real world by comparing them to prototypes and then ordering them accordingly. In this way, according to Rosch, word meanings are formed from prototypical details and scenes and then incorporated into lexical contexts depending on the context or situation. Rosch hypothesized that different societies process perceptions differently depending on how they go about solving problems [54]. When the linguist George Lakoff (born 1941, Fig. 14 (b)) heard about Rosch's experiments, he was working at the *Center for Advanced Study in Behavioral Sciences* at Stanford. During a discussion about prototype theory, someone there mentioned Zadeh's name and his idea of linking English words to membership functions and establishing fuzzy categories in this way. Lakoff and Zadeh met in 1971/72 at Stanford to discuss this idea and also the idea of fuzzy logic, after which Lakoff wrote his paper "Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts" [55]. In this work, Lakoff employed "hedges" (meaning barriers) to categorize linguistic expressions and he invented the term "fuzzy logic" whereas Goguen had used "logic of inexact concepts".

Based on his later research, however, Lakoff decided that fuzzy logic was not an appropriate logic for linguistics, but: "Inspired and influenced by many discussions with Professor G. Lakoff concerning the meaning of hedges and their interpretation in terms of fuzzy sets," Zadeh had also written an article in 1972 in which he contemplated "linguistic operators", which he called "hedges": "A Fuzzy Set-Theoretic Interpretation of Hedges". Here he wrote:

> A basic idea suggested in this paper is that a linguistic hedge such as *very*, *more*, *more or less*, *much*, *essentially*, *slightly* etc. may be viewed as an operator which acts on the fuzzy set representing the meaning of its operand" [56].

In the 1970s Zadeh had expected that his theory of Fuzzy Sets "provides an approximate and yet effective means of describing the behavior of systems which are too complex or too ill-defined to admit of precise mathematical analysis." [50, p. 28] He had expected that even at its present stage of development" his new fuzzy method.

Fig. 14 (a): Joseph Goguen, (b): George Lakoff and (c): Ebrahim Mamdani

[4]PRUF is an acronym for "Possibilistic Relational Universal Fuzzy."

> can be applied rather effectively to the formulation and approximate solution of a wide variety of practical problems, particularly in such fields as economics, management science, psychology, linguistics, taxonomy, artificial intelligence, information retrieval, medicine and biology. This is particularly true of those problem areas in these fields in which fuzzy algorithms can be drawn upon to provide a means of description of ill-defined concepts, relations, and decision rules. [50, p. 44]

In an interview that Zadeh gave in 1994, he mentioned his surprise that Fuzzy Logic was "embraced by engineers" and "used in industrial process controls and in 'smart' consumer products such as hand-held camcorders that cancel out jittering and microwaves that cook your food perfectly at the touch of a single button." In that interview he also said that he had "expected people in the social sciences – economics, psychology, philosophy, linguistics, politics, sociology, religion and numerous other areas to pick up on it [Fuzzy Logic]. It's been somewhat of a mystery to me, why even to this day, so few social scientists have discovered how useful it could be." [57][5]

However, it was the concept of fuzzy algorithms that fell on fertile ground first: Ebrahim H. Mamdani (1942–2010), Fig. 14 (c)),[6] a professor of electrical engineering at *Queen Mary College* in London, had read Zadeh's article [50] shortly after it was published and he directed his doctoral student Sedrak Assilian to perform a trial to realize a fuzzy system under laboratory conditions and he also pointed to this paper in the article that he published together with Assilian after he had finished his Ph. D thesis:

> The true antecedent of the work described here is an outstanding paper by Zadeh (1973) which lays the foundations of what we have termed linguistic synthesis … and which had also been described by Zadeh as approximate reasoning (AR). In the 1973 paper Zadeh shows how vague logical statements can be used to derive inferences (also vague) from vague data. The paper suggests that this method is useful in the treatment of complex humanistic systems. However, it was realized that this method could equally be applied to 'hard' systems such as industrial plant controllers." [60, p. 325]

This was the kick-off for the "Fuzzy-Boom" and Zadeh's primary intention trailed away for decades.

6 A Real-World Application Fuzzy System

The potential of the new techniques of fuzzy sets and fuzzy systems had stimulated Mamdani to attempt the implementation of a real-world fuzzy system and Sedrak Assilian designed a fuzzy algorithm to control a small steam engine (Fig. 15) within a few days. The concepts of linguistic variables and the max-min composition were suitable to establish fuzzy control rules because input, output and state of the steam engine system range over fuzzy sets. Thus, Assilian and Mamdani designed the first

[5] For Soft Computing methods Social Sciences see also [58].

[6] For more details on Abe Mamdani's work see [59].

real fuzzy application when they controlled the system with the input variables heat and throttle and the output variables pressure and speed (Fig. 16) by a fuzzy rule base system.

> It was an experimental study which became very popular. Immediately we had a steam engine, and the idea was to control the steam engine. We started working at Friday, and – I do not remember clearly – by Sunday it was working" he said in an interview in 2008 [61, p. 75].

In 1974, Assilian completed his Ph. D. thesis on this first fuzzy control system; unfortunately, no other facts about Assilian are available; he also does not appear in later literature about Fuzzy Set Theory and its applications.

The entire system consisted of the combination of a steam engine and a boiler (see Fig. 17). The steam was supposed to reach a certain predetermined pressure within the boiler; this was achieved by regulating the temperature. The engine was to run as consistently as possible at a particular piston speed, for which purpose a throttle was installed. This was therefore a system with two inputs (heat supplied to the boiler, engine throttle) and two outputs (pressure in the boiler, engine speed) (see Fig. 16). These inputs and outputs range over fuzzy sets. Thus, Assilian and Mamdani designed the first real fuzzy system and also the first real fuzzy application when they controlled this system by a fuzzy rule base system.

Sensors constantly monitored the boiler and indicated the current pressure. If the prevailing pressure corresponded to the set point value, then nothing needed be done. If it deviated from the set point, then some action had to be taken, and this task was to be assumed by an automatic fuzzy controller.

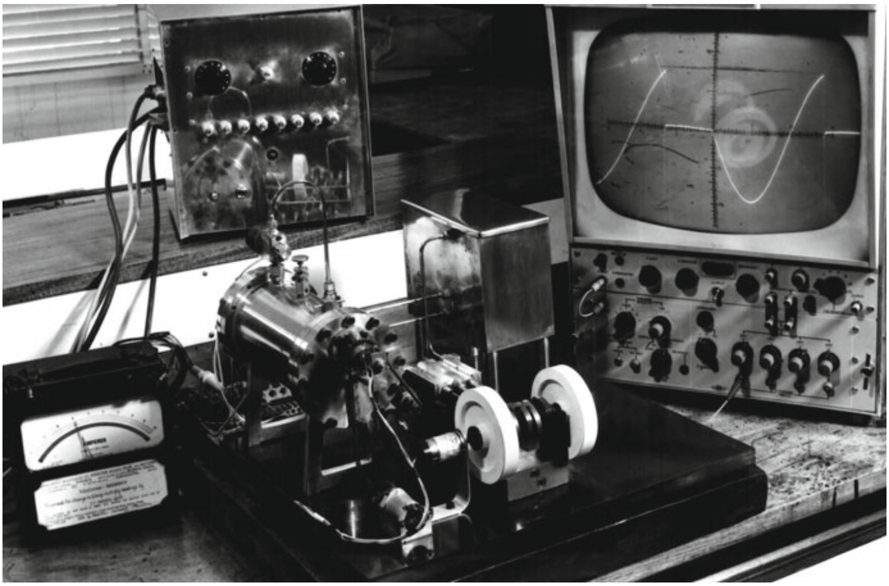


Fig. 15 Photograph of the "Fuzzy steam engine", Queen Mary College, 1974, reprint courtesy of Brian Gaines, see also: [62, p. 18]

Simple identification tests on the plant proved that it is highly nonlinear with both magnitude and polarity of the input variables. Therefore the plant possesses different characteristics at different operating points, so that the direct digital controller implemented for comparison purposes had to be returned (by trial and error) to give the best performance each time the operating point was altered. [63, p. 2]

Assilian and Mamdani defined six linguistic variables (four input and two output variables):

1. *PE* (*Pressure Error*), defined as the difference between the actual value and the set point of the pressure in the boiler.
2. *SE* (*Speed Error*), defined as the difference between the actual value and the set point of the of the piston speed
3. *CPE* (*Change in pressure error*), defined as the difference between the actual value of *PE* and its most recent value
4. *CSE* (*Change in speed error*), defined as the difference between the actual value of *SE* and its most recent value
5. *HC* (*Heat Change*) (action variable, as the result of which a command occurs).
6. *TC* (*Throttle Change*) (action variable, as the result of which a command occurs).

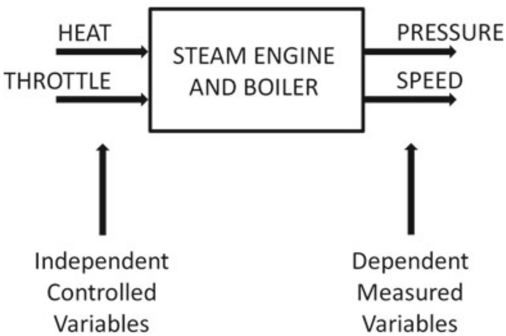


Fig. 16 The process variables of the fuzzy steam engine, [62, p. 31]

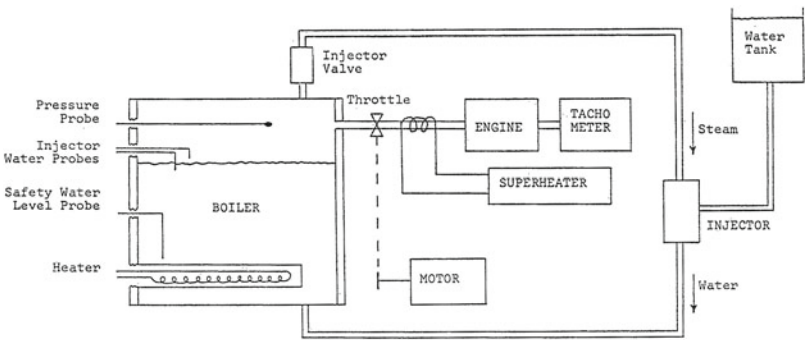


Fig. 17 The system consisting of a steam engine and a boiler, [62, p. 18]

They introduced linguistic terms for the variables: *PB* (Positive Big), *PM* (Positive medium), *PS* (Positive Small), *P0* (Positive, Zero), *N0* (Negative Zero), *NS* (Negative Small), *NM* (Negative Medium), and *NB* (Negative Big). The variables were distributed over a number of points in accordance with the universe of discourse.

- For the variables *PE* and *SE* there were 13 points, which ranged from the maximum negative error through zero to the maximum positive error, with the zero being divided into a "negative zero error" *N*0 and a "positive zero error" *P*0 ("*N*0 – just below the set point … *P*0 – just above the set point" [63, p. 7f].
- The variables *CPE* and *CSE* were similarly quantized.
- The variable *HC* was ultimately quantized over 15 points.
- Similarly, the variable *TC* was distributed over five points.

Mamdani and Assilian formed the fuzzy sets subjectively and then they defined 24 rules as IF-THEN rules. Table 1 gives three rules as examples, represented as in [63]. For the sake of simplicity, the authors of that work did not differentiate between "positive zero" and "negative zero".

Table 1 Examples of Mamdani's and Assilian's IF-THEN rules in [63]

Rule 1:		
	IF	the deviation in pressure is small and positive
	AND	the deviation in pressure does not change much
	THEN	reduce the supply of heat a little
		IF *PE* is PS AND *SE* is N, THEN *HC* is NS
Rule 2:		
	IF	the deviation in pressure is approximately zero
	AND	the deviation in pressure does not change much
	THEN	do not change the supply of heat
		IF *PE* is N AND *SE* is N, THEN *HC* is N
Rule 3:		
	IF	the deviation in pressure is small and positive
	AND	the deviation in pressure is slowly increasing
	THEN	reduce the supply of heat a little
		IF *PE* is PS AND *SE* is PS, THEN *HC* is NS

These rule relationships were implemented as fuzzy relations for which Zadeh had already indicated the max-min composition rule in his first publication on Fuzzy Sets. Additionally, a PDP 8/S digital computer [62, p. 17] calculated a corresponding fuzzy set as a value for the output variable. This method can be represented graphically in the following way (see Fig. 18):

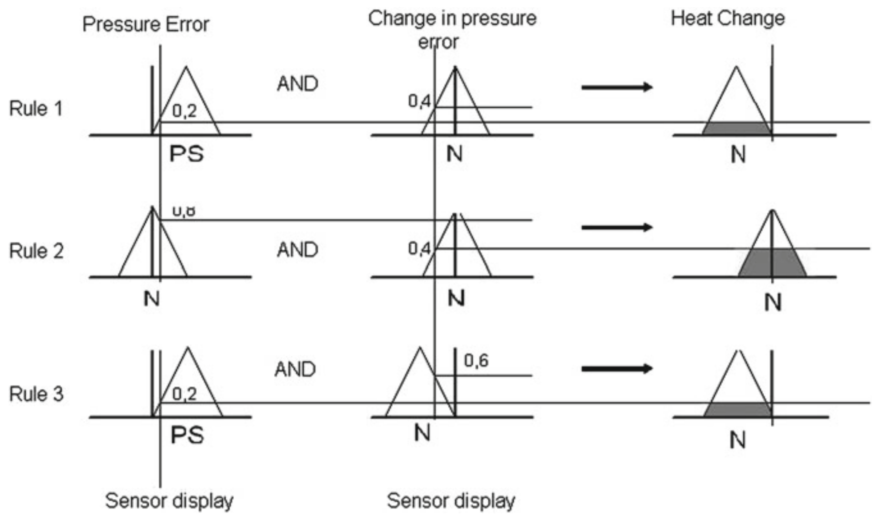


Fig. 18 Illustration of the application of the min-max rule based on [64, p. 161]

The sensors indicate sharp values for the input variables pressure deviation and its change, whose membership values with respect to the corresponding fuzzy sets can be read on the triangular membership functions. In the illustrated example for rule 1, the membership value with respect to the fuzzy set pressure deviation *PS* is 0.2 and it is 0.4 with respect to the fuzzy set change in pressure deviation *N*. Today this part of the fuzzy control process is known as “fuzzification”.

The max-min rule prescribes that the minimum of these two values is computed first. (In the example for rule 1 illustrated above, this value is 0.2). Accordingly, after executing this rule alone, the output command was “Change heat supply NS” and it had a membership value of 0.2. The result of rule 1 thus results in a triangular function that is truncated at the value 0.2 – a trapezoidal membership function. However, rule 2 and rule 3 have also fired and so they must be evaluated analogously and parallel to rule 1. The final membership function for the fuzzy set as a value of the output variable change in pressure deviation is ultimately composed of the trapezoidal membership functions of the individual rule results. This composition occurs according to the max-min rule by forming the maximum of the membership functions of all three output fuzzy sets.

Just how was the output variable change in pressure deviation supposed to be adjusted, though? For this a sharp (that is, crisp or non-fuzzy) value is required and Mamdani and Assilian decided on a simple procedure:

> Various considerations may influence the choice procedure depending on the particular application and in our case effectively that action is taken which has the largest membership grade. It is possible of course that more than one peak of a flat is obtained as illustrated below [see Fig 19]:

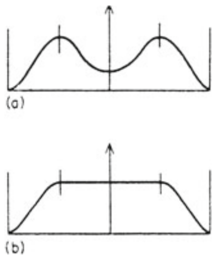


Fig. 19 Illustration of the selection of the centroid as a defuzzification method devised by Assilian and Mamdani [63, p. 627]

> Negative deviations signify a movement toward the set point, positive deviations signify a movement away from the set point [60, p. 627].

"The particular procedure in our case takes the action indicated by the arrow, which is midway between the two peaks or at the centre of the plateau." [63, p. 5]

In his dissertation thesis entitled *Artificial Intelligence in the Control of Real Dynamic Systems* that Assilian produced in response to this fuzzy control problem [62], he wrote that the control strategy they had realized was one that a human operator could use to control a steam engine.

> These control policies were established first by imagining the entire state space (*PE* × *CPE* × *SE* × *CSE*) to be divided into a number of areas, and second, writing down a control policy for each of these areas. Obviously, the first set of rules obtained in this manner does not necessarily produce the best quality of control possible … [62, p. 135]

Figure 20 shows the "Fuzzy control instructions for heat-pressure loop of steam engine" [60, p. 627]. This control algorithm was thus profoundly subjective. Not only the algorithm but also the membership function had been designed subjectively. Yet as Assilian and Mamdani managed to demonstrate, this Fuzzy Control (FC) system exceeded the performance of conventional control systems in several ways (see Fig. 21).

Fig. 20 FC commands for the steam engine designed by Assilian and Mamdani

IF PE = (NB OR NM) THEN IF CPE = NS THEN HC = PM
OR
IF PE = NS THEN IF CPE = PS THEN HC = PM
OR
IF PE = NO THEN IF CPE = (PB OR PM) THEN HC = PM
OR
IF PE = NO THEN IF CPE = (NB OR NM) THEN HC = NM
OR
IF PE = PO OR NO THEN IF CPE = NO THEN HC = NO
OR
IF PE = PO THEN IF CPE = (NB OR NM) THEN HC = PM
OR
IF PE = PO THEN IF CPE = (PB OR PM) THEN HC = NM
OR
IF PE = PS THEN IF CPE = (PS OR NO) THEN HC = NM
OR
IF PE = (PB OR PM) THEN IF CPE = NS THEN HC = NM

- Much less information is required for FC than for conventional control.
- The verbal knowledge of human experts did not have to be mathematically exact in order to be processed by the automatic control.
- Errors were reduced little by little until the set point could be reached; digital controllers "overshot" this target instead.
- The FC system worked faster than a conventional control system; the possibility of processing the parallel firing of several rules at the same time shortened the required control time.

With this fuzzy control of a steam engine – or more precisely a combination of a boiler and a steam engine – the essential principles for the construction of an entire class of fuzzy control systems were established and Mamdani went ahead. Already in January 1976 he organized – together with Brian Gaines, then a professor of computer science at *Essex university* – a Workshop on "Discrete Systems and Fuzzy Reasoning" held at London's *Queen Mary College*. At this workshop some similar projects to control technical systems using fuzzy algorithms were presented, e.g. a basic oxygen steel making process at the British Steel Corporation in Cambridge, England [65], a sinter making plant at the British Steel Corporation in Middlesborough, England [66, 67], and a pilot scale batch chemical process in the Warren Spring Laboratory in Stevenage, England in [68, 69]. Some other FC investigations of this time were a FC system to control a warm water plant in the Delft Technical High School in the Netherlands [70] and a heat exchanger at the McMaster University in Canada.

The step forward from small laboratory systems to the first large-scale commercial fuzzy controlled system was taken very soon. The first "big science" FC system was built in Denmark by Jens-Jörgen Østergaard and Lauritz Peter Holmblad who joined the company F. J. Smidth & Co. upon graduation from the *Technical University of Copenhagen*. It was a system for the automatic control of a cement kiln. Attempts

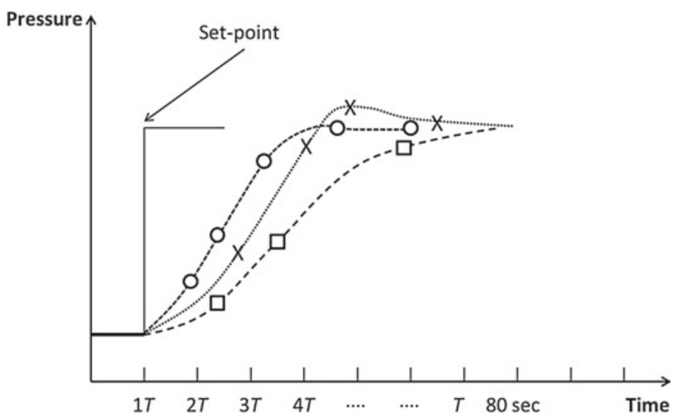


Fig. 21 The result of the Assilian-Mamdani FC (○) compared to a conventional controller. (Dynamic Divergence Caching (DDC) algorithm damped (□) and undamped (x)), [63, p. 6]

to automate cement production had always failed in the past because the process of cement burning is highly complex, ovens do not behave linearly and only a few measurements can be taken during the process [71]. The fuzzy cement kiln developed by Holmblad and stergaard functioned very successfully and reliably, however. It was the starting point of the "Fuzzy Boom" that started in the 1980s in Japan and later pervaded the Western hemisphere. Many fuzzy applications, such as domestic appliances, cameras and other devices appeared in the last two decades of the 20th century. Of greater significance, however, was the development of fuzzy process controllers and fuzzy expert systems that served as trailblazers for scientific and technological advancements of fuzzy sets and systems.

7 Fuzzy Sets in Humanities and Social Sciences

In 1969 Zadeh proposed his new theory of fuzzy sets to biologists: "The great complexity of biological systems may well prove to be an insuperable block to the achievement of a significant measure of success in the application of conventional mathematical techniques to the analysis of systems." [72] "By 'conventional mathematical techniques' in this statement, we mean mathematical approaches for which we expect precise answers to well-chosen precise questions concerning a biological system that has a high degree of relevance to its observed behaviour. Indeed, the complexity of biological systems may force us to alter in radical ways our traditional approaches to the analysis of such systems. Thus, we may have to accept as unavoidable a substantial degree of fuzziness in the description of the behaviour of biological systems as well as in their characterization." [72]

We find great complexity not only in biological systems but also in social sciences and humanities. At the end of the 1960s and for a greater audience two years later, Zadeh wrote more generally: "What we still lack, and lack rather acutely, are methods for dealing with systems which are too complex or too ill-defined to admit of precise analysis. Such systems pervade life sciences, social sciences, philosophy, economics, psychology and many other 'soft' fields." [16]

Zadeh was inspired by the remarkable human capability to perform a wide variety of physical and mental tasks without any measurements and any computations", e. g. parking a car, playing golf, deciphering sloppy handwriting, and summarizing a story. He distinguished between mechanical (or inanimate or man-made) systems at one hand and humanistic systems at the other hand and he saw the following state of the art in computer technology:

> Unquestionably, computers have proved to be highly effective in dealing with mechanistic systems, that is, with inanimate systems whose behavior is governed by the laws of mechanics, physics, chemistry and electromagnetism. Unfortunately, the same cannot be said about humanistic systems, which – so far at least – have proved to be rather impervious to mathematical analysis and computer simulation.

He defined a "humanistic system" to be

> a system whose behaviour is strongly influenced by human judgment, perception or emotions. Examples of humanistic systems are: economic systems, political systems, legal systems, educational systems, etc. A single individual and his thought processes may also be viewed as a humanistic system. [51, Part I, p. 200]

Zadeh summarized "that the use of computers has not shed much light on the basic issues arising in philosophy, literature, law, politics, sociology and other human-oriented fields. Nor have computers added significantly to our understanding of human thought processes-excerpting, perhaps, some examples to the contrary that can be drawn from artificial intelligence and related fields." [51, Part I, p. 200]

Thus, hard computing has been very successful in hard sciences but it could not be that successful in humanistic systems in the field of soft sciences. Therefore we should open the field of applications of soft computing to the soft sciences. This is what Zadeh had in mind when he proposed the notion of soft computing:

> I expected people in the social sciences-economics, psychology, philosophy, linguistics, politics, sociology, religion and numerous other areas to pick up on it. It's been somewhat of a mystery to me why even to this day, so few social scientists have discovered how useful it could be. Instead, Fuzzy Logic was first embraced by engineers and used in industrial process controls and in 'smart' consumer products such as hand-held camcorders that cancel out jittering and microwaves that cook your food perfectly at the touch of a single button. I didn't expect it to play out this way back in 1965." [57].

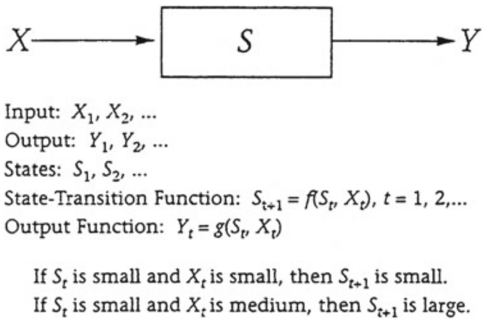
The field of Soft Computing in Humanities and Social Sciences is at a turning point. Not very long ago, the very label seemed a little bit odd. Soft Computing is a technological field while Humanities and Social Sciences fall at the other pole of the academic field. In recent years, however, this has changed. The strong distinction between "science" and "humanities" has been criticized from many fronts and, at the same time, increasing cooperation between the so-called "hard sciences" and "soft-sciences" is taking place in a wide range of scientific projects dealing with very complex and interdisciplinary topics [73].

In the last fifteen years the area of Soft Computing has also experienced a gradual rapprochement to disciplines in the Humanities and Social Sciences [58, 74].

8 Outlook: Computing with Words and Perceptions

Artificial Intelligence (AI) was born in the 1950s in the USA and spread to many scientific and technological communities throughout the world. The history of AI is a story of several successes but has lagged behind expectations. AI became a field of research to build computers and computer programs that act "intelligently" although no human being controls those systems. AI methods became logic-based to find exact solutions. However, not all problems can be resolved with these methods. On the other hand, humans are able to resolve such tasks very well, as Lotfi Zadeh mentioned in many speeches and articles over the last century. In conclusion,

Fig. 22 Perception-based system modeling, [78]



he stated that "thinking machines" do not think as humans do. From the mid-1980s he focused on "Making Computers Think like People" [75]. For this purpose, the machine's ability "to compute with numbers" was supplemented by an additional ability that was similar to human thinking. In the 1990s he established Computing with Words (CW) [76, 77] instead of exact computing with numbers, as a method for reasoning and computing with perceptions based on the theory of fuzzy sets. In his article "Fuzzy Logic = Computing with Words" in May 1996, he stated that "the main contribution of fuzzy logic is a methodology for computing with words. No other methodology serves this purpose." [75, p. 103] Three years later he wrote "From Computing with Numbers to Computing with Words – From Manipulation of Measurements to Manipulation of Perceptions", to show that a new Computational Theory of Perceptions, or CTP for short, is based on the methodology of CW. In CTP, words play the role of labels of perceptions and, more generally, perceptions are expressed as propositions in natural language. [76, p. 105].

As we said already, he was inspired by the "remarkable human capability to perform a wide variety of physical and mental tasks without any measurements and any computations. [...] Underlying this capability is the brain's crucial ability to reason with perceptions – perceptions of time, distance, speed, force, direction, shape, intent, likelihood, truth and other attributes of physical and mental objects." [77, p. 105]. Zadeh intended to establish a new dimension of artificial intelligence [78, p. 73]. He received an opportunity to propose these considerations concerning "A New Direction in AI" to the AI community at the beginning of the new millennium, when his manuscript was accepted for the *AI Magazine* issue in the spring of 2001 [78].

In this article he presented a new view on system theory, namely perception-based system modeling: In "perception-based system modeling", the input, the output and the states are assumed to be perceptions (Fig. 22).

The 50th anniversary of a scientific theory is a good opportunity to cast a retrospective look at its consequences and achievements. Many aspects of this history are a matter of course, such as definitions of the theorys entities, theorems and protagonists of important developments. However, some are facts that were unknown to most interested persons and also some specialists. The original research work on the history of the theory of fuzzy sets, as presented in this chapter, shows that its history cannot be comprehended without reflecting on the history of system theory. Moreover, fuzzy set theory must be regarded as an inherent part of the history. This deep connection is evidenced from the very beginning of Zadeh's scientific all the way up to his recent lectures and articles. With his varying views on system theory, Computing with Words and the Computational Theory of Perceptions, he postulated new directions for science and technology, in the fields of information science, computer science, and artificial intelligence. Perhaps the lesson to be learned from this history is that creating new views is one of the most effective means of keeping a scientific theory – such as fuzzy set theory – alive.

Acknowledgments The author wishes to thank Lotfi A. Zadeh for his generous help and unstinted willingness to support the author's historical project on the theory of fuzzy sets and systems during the last 15 years. Many thanks go also to Claudio Moraga and Jim Bezdek who read this text and gave important hints to improve it. Work leading to this chapter was partially supported by the *Foundation for the Advancement of Soft Computing* Mieres, Asturias (Spain).

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Rudolf Seising born 1961 in Duisburg, Germany, obtained his Ph.D. in Philosophy of Science and the German Habilitation in History of science from the *Ludwig-Maximilians-University in Munich* after studies of Mathematics, Physics and Philosophy at the Ruhr-University of Bochum (Germany). Dr. Seising has been Scientific Assistant for computer sciences at the *University of the Armed Forces in Munich* from 1988 to 1995 and for history of sciences at the same university from 1995 to 2002. From 2002 to 2008 he was with the Core unit for Medical Statistics and Informatics at the *University of Vienna* Medical School, which in 2004 became the *Medical University of Vienna*. Since 2005 he is College Lecturer at the Faculty of History and Arts, at the *Ludwig-Maximilians-University Munich*. From April to September 2008 he was acting as Professor for the History of science at the *Friedrich-Schiller-University Jena* (Germany) and from September 2009 to March 2010 at the Ludwig-Maximilians-University in Munich. He was Visiting Researcher)2008-2010) an Adjoint Researcher (2010-2014) at the *European Centre for Soft Computing* in Mieres (Asturias), Spain and he has been several times Visiting Scholar at the University of California, Berkeley. Recently he is again acting as Professor for the History of science at the *Friedrich-Schiller-University Jena* (Germany). Since 2004Dr. Seising is Chairman of the IFSA Special Interest Group "History" and since 2007, of the EUSFLAT Working Group "Philosophical Foundations".In 2011he became member of the IEEE Computational Intelligence Society (CIS) History Committee, and since 2013 also of the IEEE CIS Fuzzy Technical Committee. In 2013 he founded (with A. Sobrino, M. E. Tabacchi, and M. Pereira Fari?a the Online journal *Archives for the Philosophy and History of Soft Computing* (www.aphsc.org). His main areas of research comprise historical and philosophical foundations of science and technology. Among other books he edited *Views on Fuzzy Sets and Systems from Different Perspectives. Philosophy and Logic, Criticisims and Applications.* (Springer 2009), (with V. Sanz) *Soft Computing in Humanities and Social Sciences* (Springer 2012), (with E. Trillas, S. Termini, and C. Moraga) *On Fuzziness. A Homage to Lotfi A. Zadeh* - vols. I and II, (Springer 2013) and (with M. E. Tabacchi) and *Fuzziness and Medicine: Philosophical Reflections and Application Systems in Health Care. A Companion Volume to Sadegh-Zadeh?s "Handbook on Analytical Philosophy of Medicine*", (Springer 2013).

Fuzzy Logic in Speech Technology - Introductory and Overviewing Glimpses

Horia-Nicolai Teodorescu

Abstract The chapter critically reviews several applications of fuzzy logic and fuzzy systems in speech technology, along the main directions of the filed: speech synthesis, speech recognition, and speech analysis. A brief incursion in the use of mixed techniques, combining fuzzy logic, fuzzy classifiers and nonlinear dynamics is included. A rich list of references complements the chapter.

1 Introduction

The applications of speech technology traditionally fall in three main classes: speech recognition, speech synthesis, and speech analysis with applications in psychology – including the more recently developed emotion analysis and related sociology issues, linguistics (phonetics), and medicine – mainly speech pathology, respiratory pathology, and dentistry. At least one class of applications, namely speaker recognition, falls between the sub-domains of speech recognition and speech analysis. Automatic speech synthesis has historically been the first to be developed, much before the use of computers, with important studies between 1920 and 1960, supported chiefly by the telephony industry. However, automatic speech synthesis was successful only after the widespread introduction of microprocessors and high volume memory chips. Speech recognition became a field of systematic research only after the introduction of mini- and micro-computers (PCs), and the development was accelerated since the advent of digital signal processors (DSPs). Again one of the main beneficiaries has been the automatic response systems in telephony and later the intelligent phones. A steady development has seen speech analysis for phonetics and those with medical applications (voice and speech pathology). On the other hand, speech recognition covers numerous industrial,

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_28

human computer interaction, and communication applications, including voice command, voice data entry, and query-answer systems (Kasabov and Iliev 2000). As side products, large repositories (speech databases), corpuses (databases of annotated speech files), and ontologies were created (see for example Naphade et al. 2006).

The use of fuzzy logic (FL) in speech technology follows the same main divisions as speech technology. While one could expect that most researches would apply fuzzy logic to speech recognition, this is not precisely the case. More attention was paid to speech analysis and applications, at least by the count of papers. This may be due that good methods for speech recognition based on Markov chains were available early in the 1990 s, making fuzzy classification less attractive. The situation largely perpetuates until today. A recent search on the IEEE Xplore revealed 45 journal papers including in their abstract the words “fuzzy” and “speech” and 413 conference papers with the same key words. Out of the 45 journal papers in this database shown that the topics of most papers fall into the categories speech analysis and speech recognition. The distribution of papers per categories of topics is shown in Fig. 1.

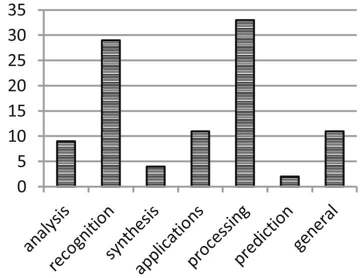

Fig. 1 Distribution of main topics of papers on speech and fuzzy published in IEEE journals, in %. Analysis category includes the evaluation of speech quality and identification of speaker. Applications category includes speech-based control, and processing category includes segmentation

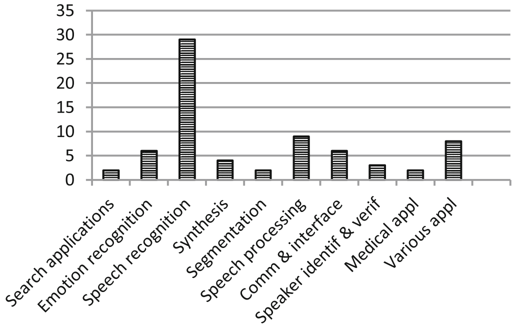

Fig. 2 Worldwide patents in the field of speech technology using FL, according to the targeted topic(s). Data based on a search on Espacenet^{TM} (74 results found in the Worldwide database for fuzzy speech in the title or abstract)

The Worldwide database on Espacenet™ contains (as at October 2014) 74 patents that include the terms “fuzzy” and “speech” in the title or abstract. Figure 2 illustrates the approximate distribution of worldwide patents using FL in speech technology; the distribution is on topics that differ from those in Fig. 1 to better reflect the categories found in patents. Some patents fall in two categories, while a few others in the search results were marginally related to speech technology and were omitted; therefore the sum is slightly smaller than 74, as obtained in the search. Interestingly, in contrast to paper literature, by far the main application of FL in speech technology is speech recognition – more than 40 % - while the topics of speech processing and various applications maintain a high position.

The division into categories in Figs. 1 and 2 is subjective and partly imposed by the topics dealt with in the papers in the database; readers may find other classifications of papers and patents more suitable for their purpose.

The number of articles that include in title and abstract both the words fuzzy and speech in the ScienceDirect database is 114. Interestingly, the yearly number of papers published in the period since 1995 in the field and reported by ScienceDirect remained essentially flat, around 4 per year. This is a very low number, taking into account that this publisher has about 20 journals that published papers in the field, including the flagship journal Fuzzy Sets and Systems. A much higher number of articles (including book chapters) is returned for the same keywords in the whole articles by Springer Link: more than 11’000, out of them more than 3’700 articles, but we estimate that about 10–30 % of them are actually related to the use of fuzzy logic in speech processing, analysis, synthesis, and recognition. On the other hand, a search using the European Patent search engine Espacenet detects 78 results found in their Worldwide Database for the words fuzzy and speech in the title or abstract. Most of these patents and patent applications are new, after 2000, and many of them come from Asian inventors (Japan, China). Also, the result of search in US Patent Collection for the terms fuzzy and speech, specifically in the claims, produces 79 patents, most of them registered after 2000.

This chapter is partly a review article with reference to a small sample of FL techniques applied to speech technology of the main papers and to some research tracks the author was involved in. The review is by no way exhaustive, however efforts have been done to illustrate a wide range of topics covered by the literature, including both journal and conference papers, moreover the patent literature, striking a balance between the two categories of articles – a feature unfortunately seldom respected in reviews.

In addition to reviewing the subject, the chapter describes in detail and generalizes an approach which, according to the author opinion, deserves further investigation in the future, namely the approach based on a generalization of fuzzy information space and related fuzzy classification based on representations in this space. While not being a review in the proper sense, because of the vast number of topics and approaches, the reviewing insights are critical in the sense that they emphasize the limits in the results obtained so far by the application of fuzzy logic to speech technology; also, the review discusses possible weaknesses and circumstances that motivated this situation.

The chapter is intended to help both experts in FL and in speech technology to understand the other domain and we hope it will help postgraduate students and young researchers to devise new research fields crossing the boundary from FL to speech technology and vice versa.

For the benefit of the audience from FL field, we recall a few notions from speech technology. Speech is either voiced or voiceless. In voiced speech, the vocal folds vibrate almost periodically; the vibration frequency is named pitch, or (number) zero formant, denoted by $F0$. The change in $F0$ value is the main contributor to the representation of prosody. The basic voice waveform (signal) produced at the level of the vocal folds is further shaped by the larynx, buccal and nasal cavities. The sound spectrum in voiced speech is due largely to this shaping, which determines the main spectral components, named formants – the main being denoted by $F1$ to $F4$. The first two formants, $F1$ and $F2$, provide the main information needed to recognize the voiced phonemes. All vowels and many consonants are voiced phonemes. Some phonemes are unvoiced, that is, they are not produced with the contribution of the vibration of the vocal folds. Examples of these unvoiced phonemes are the plosive and the fricative consonants.

Beyond the discrete Fourier power spectrum $\{S(\omega_j)\}_j$, typically reduced to the values of the pitch ($F0$) and of the formants ($F1 \cdots F4$) and of their bandwidths, speech segments are characterized by several easy to define and compute parameters, such as their energy, duration, "number of zero-crossings" (see Sect. 3) and by the values in the descriptive statistics of these parameters, typically the average value, spreading (standard deviation), minimum and maximum value, median, first and third quartiles, and range. After acquisition by a digital system, speech is originally represented by a sequence of samples, $\{s(t_1), \cdots, s(t_n), \cdots\}$, where t_n represent time moments equally spaced. In typical representations, there are between 16'000 and 96'000 such values per second, depending on the sampling frequency used. Taking into account that a phoneme lasts between a few miliseconds (ms), for plosives to several tens of ms (for sustained, long vowels), the initial processing for extracting the above parameters is performed on speech segments of around 10 ms duration. Essentially, speech representation and processing techniques belong to time series techniques – a field familiar to scientists from almost all branches of science. The computation on a 'time window' including W samples of the energy is then performed as $E = \sum_{k \in w} s_k^2$, where the (abusive) notation $k \in W$ means that the time moment t_k is in the window W. Assuming that the central sample of the time window (segment) is t_n, we denote the window by W_n and the corresponding energy by E_n. Similarly, in a second example, the range of the signal samples in the window is defined as $r_n = \max_{k \in W_n} s_k - \min_{k \in W_n} s_k$. This phase of processing represents every window by a set of values of several parameters, $E_n, r_n, \cdots$. Each value of a parameter aggregates information from all samples in the corresponding speech segment. Preserving the same order of the parameters from one window to the next, at the end of this representation phase, the segment is replaced by a vector of parameters, $v_n = (F0_n, F1_n, \cdots, E_n, r_n, \cdots)$. If the successive windows have their centers at a distance $L \gg 1, L \leq W/2$, the information in the speech segment may be significantly compressed into the vectors v_n, because there is a single vector every L samples and we assume here that the number of the components of the vector is much smaller than L. For example, if $L = W/10$, there are 10 values of the representation vector for every speech segment.

In the next steps, the sequence $\{v_n\}_n$ of vectors representing the information in the signal successive windows is used for further signal characterization. Again, one can use the descriptive statistics of the components of the vector v_n in the segment and to create a new vector for every parameter. For example, for every 3 ms, using windows of 3 ms width with the centers of successive windows every 0.1 ms, there are 30 values of the vector $v_i = (F0_i, F1_i, \cdots, E_i, r_i, \cdots)$. Then one can compute the average of the corresponding values $F0_i$ in the segment corresponding to the index n as $\overline{F0_n} = (F0(t_n - 1.5\,\text{ms}) + F0(t_n - 1.4\,\text{ms}) + \cdots + F0(t_n + 1.5\,\text{ms}))/31$. The set of these new parameters can be used as final representation of the signal in the window, for example to determine if the signal is noise, unvoiced speech, or voiced speech.

The scope of this chapter is limited to providing a general view on the topic, and a few insights that the author finds more promising or believes are good, yet easy to understand examples of applications of FL to speech technology. Some minimal knowledge of FL and rudimentary knowledge of signal processing are required, rather at the level of terminology than at that of fundamental results. For basic concepts and methods in speech technology, the reader is referred to books, such as (Benesty, Sondhi and Huang 2008).

The next sub-section describes applications of FL to voice synthesis. Applications of fuzzy logic to voice processing and analysis are dealt with in the third section, which also includes an extended sub-section on the detection of voiced, unvoiced, speech, and noise segments, moreover a subsection on medical applications of speech analysis, while the fourth section deals with speech recognition. The next (fifth) section is a special incursion into the fuzzy information space representation, which may be a good potential tool in both speech analysis and speech recognition. This section includes a subsection of speaker recognition. The sixth section overviews emotion estimation in speech, a currently important topic in the field. The final section presents conclusions and potential future directions.

2 Applications of Fuzzy Logic to Voice Synthesis

One of the early papers on fuzzy logic applied to speech synthesis is (Raptis and Carayannis 1997), who used fuzzy logic for rule-based formant speech synthesis.

In a study supported by research grants from the Romanian Academy during 1999 and 2002, Grigoras et al. (1999, 2000), Jitca et al. (2002) proposed a fuzzy system for controlling the synthesis of the phonemes that are prone to multi-definitions in rule-based speech synthesizers. The purpose was to control a Klatt-type synthesizer (Klatt 1980) with the aim of improving the intelligibility and naturalness of the speech produced by the basic Klatt synthesizer. The fuzzy system controls the values (frequency) of the formants and their dynamics. The computation of the formant values takes into account the context of a sequence of two syllables. The approach is exemplified in that paper for the liquid phoneme *l*, for the Romanian language, because of the difficulty of synthesizing this phoneme correctly. An increased intelligibility of the TtS (Text to Speech) synthesizer was demonstrated using the FL controlled synthesizer. However, this type of application fell into desuetude together with the fall of interest in Klatt synthesizers after 2000.

Also related to the topic of speech synthesis is the theme of articulatory system modeling. In his paper, Brito (2009) proposed such a model based on a fuzzy-genetic approach and confirmed its validity for the Spanish vowels. Lin et al. (2004) presented a text-to-speech (TtS) synthesis system based on FL, namely using recurrent fuzzy neural network (RFNN) and fuzzy rules for prosodic phrase structure in a concatenative synthesizer.

A few patents use FL for speech synthesizers. Tanaka and Masanobu (2000) obtained the United States Patent 6,081,781 titled “Method and apparatus for speech synthesis and program recorded medium”. Their synthesizer employs as main idea fuzzy vector quantization for coding the spectrum envelope of speech segments. Also, Coorman et al. (2007) were granted US Patent no. 7219060 for “Speech synthesis using concatenation of speech waveforms”. These inventors used only a form of characterization of the synthesized speech with linguistic degrees; they name the related representation “fuzzy table”, but there is no true use of FL in their patent.

We end this brief section signaling a recent patent that uses in a novel way the context information for improving speech synthesis in TtS (text to speech) synthesis. In the patent US2012221339, “Method, apparatus for synthesizing speech and acoustic model training method for speech synthesis” the inventors Wang et al. present a model-based speech synthesizer employing fuzzy context information for improving speech synthesis; they describe the procedure as “*generating fuzzy context feature labels based on the plurality of candidate pronunciations and probabilities thereof, and determining model parameters for the fuzzy context feature labels based on acoustic model with fuzzy decision tree*.”

Having advocated for a long time for the ambient-adaptive speech synthesis and having proposed adaptation ways (Teodorescu et al. 1988), (Teodorescu 2001a), (Teodorescu 2005) for creating stress rendering, emotion rendering, and relationship rendering, we must agree that these issues remain a largely unchartered territory. Also, little progresses were made in assessing speech quality, both synthesized (Teodorescu, Feraru and Zbancioc 2009) and natural speech.

3 Applications of Fuzzy Logic to Voice Processing and Analysis

FL for improving voice communications and general voice signal processing
There are numerous applications in speech technology, especially in speech analysis, where we see the use of fuzzy logic and fuzzy techniques as one of the primary choices. For example, detecting speech disabilities is much more difficult than detecting Parkinson for example, because, in this case, Parkinson has a few and quite specific signs, while speech disabilities have numerous causes and varied and subtle ways to manifest them, with many yet unknown processes, imprecision in knowledge, and variability. Hence, the use of FL and fuzzy classifiers may be a good choice to deal with this problem. Other similarly difficult problems are the detection of dialects and sub-dialects in phonology and of the local variants, especially in cases of small populations speaking a dialect or language variant, thus making statistical methods only partly useful (due to the small sample effect).

Already in 1994,

Ndousse (1994), in one of the first papers on FL in voice communications, introduced a FL-based controller in asynchronous transfer mode (ATM) transmission of voice messages. The motivation of using FL is that "*Typical voice cells, characterized by a high degree of burstiness, complicate any attempt to use classical control theory in the design of an ATM cell rate controller*." The FL controlled ATM management proposed by Ndousse is essentially a modified leaky bucket cell rate control. Ndouse (1998) further details the method in the book chapter on Fuzzy Expert Systems in ATM Networks.

Cheng and Chang (1996), starting from a paper by Murata et al. (reference 6 in Cheng's and Chang's paper), provide a meticulous analysis of uncertainties in ATM networks, saying that

> "it is difficult for a network to acquire complete statistics of input traffic and, as a result, it is not easy to accurately determine the effective thresholds or equivalent capacity in various bursty traffic flow conditions of ATM networks; … therefore, the decision process is full of uncertainty."

Cheng and Chang relay on the work by Ndousse, discussed above, and on other works, for their proposal of a FL-based traffic controller that models an ATM network. The FL-based traffic controller includes a fuzzy bandwidth predictor (which predicts the available equivalent capacity), a fuzzy admission controller, a fuzzy congestion estimator, and a performance estimator. The fuzzy congestion controller has three inputs, the queue length, the variation (change rate) of the queue length, and an estimate of the cell loss probability. The fuzzy admission controller inputs are the output of the fuzzy congestion controller, the estimate of the cell loss probability, and the result of network resource estimation, determined by another block. Only two linguistic degrees are used for the change rate, namely "positive" and "negative". Triangular and trapezoidal membership functions are used for the inputs and singletons for the output. The simulation results reported are impressive:

about halving of the message blocking probability for both voice and video traffic, and almost null message blocking probability for either low- and high-bit-rate data traffic.

In another line of research, aimed to characterize and better understand speech signals and processes, a specific approach in speech analysis and recognition was presented in (Rodriguez et al. 2000), combining non-linear dynamic characteristics of speech and fuzzy representation and classification methods. Precisely, the research aims to represent the dynamics of speech by temporal fuzzy sets and, based on them, to represent the trajectories of the speech production dynamical system. The application to speech analysis and potentially phoneme recognition relies on determination of similarity measures of the corresponding temporal fuzzy sets.

In a single paper, Ciota (2001) considers FL for both improving the signal-to-noise ratio (SNR) for speech signals and the decision-making process for whole-word recognition. In tackling the first issue, the author suggests a manner of sampling the noise spectrum, but we are not convinced about the efficiency.

FL in detection of voiced, unvoiced, speech, and noise segments

Speech vs. non-speech segmentation play an essential role in optimizing the communication bandwidth by suppressing transmission of un-voiced segments and in reducing the overall noise level (Bouquin-Jeannes and Faucon 1994). Also, voiced - unvoiced segmentation, that is, finding the segments of a recording where the speech is vocalic (produced with the vibration of the vocal folds) and non--vocalic helps improving the quality of communication and plays an essential role in speech analysis and recognition.

Although the tasks of segmentation into speech and non-speech (noise) segments and respectively into voiced and unvoiced segments for the speech segments look elementary, numerous uncertainties in the process still hamper the solving of these tasks by technical means. We will discuss in some more detail the topic of segmentation using FL, because the topic is of large technical interest and because it is narrower and allows us to include it in this brief chapter.

A VAD is an algorithm and the related application aimed at detecting, based on the features that differentiate speech from other sounds, the temporal boundaries of the segments of speech in a sound recording. The most obvious features relate to the Fourier spectrum of the noise and respectively speech. However, the spectrum is not a perfect indicator of the difference speech – ambient noise, because speech includes consonant sounds that are, essentially, noise. The typical examples of noise-like consonants are the unvoiced fricatives [s], [f], [sh], and the (partly) voiced fricatives, as [z]. Typically, [f] and [s] have almost-uniform (i.e., white noise) frequency spectrum. Also, plosive consonants, as [p], [d], [b] have the characteristics of impulsive noise, thus being difficult to differentiate from ambient noise. The temporal boundaries of the words starting or ending with such consonant sounds are difficult to establish, even when isolated pronounced and even in low or no noise conditions. For example, a VAD may have difficulty in detecting, under continuous speech with words pronounced with no pause between them, that the pronunciation of "suppress the sound" is a single segment of continuous speech. Even at the middle of the words, unvoiced consonants may create errors in the speech – non-speech segmentation and hence in VADs operation. Therefore, VADs need more features than the Fourier spectrum (or a set of representative spectral components) to properly operate.

Some of the features used are the amplitude of the sound and the so called "number of zero crossings", NZC, also named "zero crossing counts" or zero crossing rate (ZCR). For speech, most parameters are computed on time windows of 3 to 20 ms – time intervals that roughly correspond to the minimal duration of a phoneme. Because of the use of time windows, the term of instantaneous value (of the amplitude, for example) denotes another concept as the value of the current sample of the signal. The (instantaneous) amplitude parameter is usually defined as the average value of the rectified (absolute) value of the sound signal in a time window, where the window is centered on the time instant considered. The corresponding formula, with A_n the instantaneous amplitude at time moment n and s_{n+k} the signal sample amplitude at time moment $n + k$. is:

$$A_n = \frac{1}{2N+1} \sum_{k=-N}^{N} |s_{n+k}|$$

The NZC is defined as the number of zero-crossings per unit time (time window), taking into account the effect of the noise. When noise is not accounted for, the computation of NZC is according to

$$\begin{gathered} \textit{if}\ (s_n>0\ \textit{and}\ s_{n+1}<0)\,OR(s_n<0\ \textit{and}\ s_{n+1}>0)\ \textit{then} \\ NZC \leftarrow NZC + 1(\textit{increment the NZC counter}). \end{gathered}$$

The incrementing is performed during a specified window, then the counter is reset to 0; therefore, NZC has one computed value per window and is represented by a time series, NZC_m where m is the index of the time moment representing the center of the window. Essentially, NZC informs of the main (lower) frequency component in the signal, when such a component dominates the spectrum, or the signal is low-pass filtered. An alternative definition of NZC is (see for example Köhler, Hennig, & Orglmeister (2003),

$$NZC_n = \sum_{k \in W_n} \left| \frac{sign(s_k) - sign(s_{k-1})}{2} \right|,$$

where *sign* represents the function *signum* (sign), W_n is the window centered at time moment n and the sum is over all time moments in the window.

Noise occurs both in the recording circuits (microphone and amplifier) and in the analog to digital conversion (ADC) process, due to the finite resolution of the conversion. The ambient noise adds to these internal noises. The effect of internal noise on the NZC is a false, high increase of the NZC count. For avoiding at least partly the effect of internal noise, the computation of the NZC uses thresholds θ that are correlated with the expected internal noise level ν, according to

$$\begin{aligned} &\textit{if } (s_n > \theta(\nu) \textit{ and } s_{n+1} < -\theta(\nu)) \textit{ OR } (s_n < -\theta(\nu) \textit{ and } s_{n+1} > \theta(\nu)) \textit{ then} \\ &\qquad NZC \leftarrow NZC + 1 (\textit{increment the NZC counter}). \end{aligned}$$

On the other hand, the threshold thresholds θ cannot be larger than the voice signal at its low levels (i.e., consonants). An optimal threshold can be determined for specified noise conditions, while an adaptive threshold is needed for varying ambient noise. The main problem of current VADs is the lack of robustness: their performance strongly degrades for low signal to noise ratios (SNR), especially when noise is not stationary and several types of noise occur. The choice of the threshold could benefit of the FL techniques, when the noise is not stationary or its value is uncertain. This situation was documented by (Tian et al. 2003), who stress that

> "VAD approaches that use threshold… cannot achieve consistent accuracy since the mean-value based and the histogram based threshold estimation algorithms are not robust. They strongly depend on the percentage of voice and background noise in the estimate interval."

Tian et al. used fuzzy clustering and Bayesian information for estimating the thresholds for energy features, for VADs; they found the approach robust in non-stationary environments. Increasing the information aggregated and used in the decision making also improves the results. The so called *higher order counts*, which represent NZC of filtered signals, are also used as features in signal characterization and recognition, for example see Petrantonakis and Hadjileontiadis (2010). Again, the use of FL information aggregation and FL-based decision making for utilizing the higher order counts was not studied yet, although it may bring benefits.

In several papers, Beritelli et al. proposed various VAD algorithms involving FL. For example, Beritelli, Casale, and Cavallaro (1998, 1999) proposed a fuzzy voice activity detection (FVAD) algorithm for telephony which, according to the authors, because it is based on a new manner of pattern matching, using fuzzy logic, achieves a very good immunity to noise. The authors claim that only six fuzzy rules optimized by training using the "*FuGeNeSys hybrid learning tool*", a supervised training tool based on both NN and genetic algorithms (GA), able to generate optimized fuzzy rules.

These authors used the features as in ITU-T Recommendation G.729, namely the variation (difference) between the current temporal window of the NZC, ΔNZC, and the average value; the variation of the (full band) energy, ΔE_{tot}, the variation of the energy in a specified low frequency band, ΔE_L, and the spectral change, ΔS, where the window is 10 ms. In the newer standard ITU-T G.729.1, which supersedes and improves G.729, the lower frequency band corresponds to 50–4000 Hz, and the higher band to 4000–7000 Hz (IUT 2006); however, we are not aware of any fuzzy implementation of VADs specifically for the newer IUT standard.

The VAD proposed by Beritelli et al. (2002) uses essentially a Sugeno-type FLS (that is, crisp output values) with weighted mean (WM) defuzzification and with Gaussian input membership functions. The rules of the system have variable number of antecedents, with all antecedents connected by AND. Remarkably, Beritelli et al. succeed to reduce the knowledge base needed to a set of a few rules, to eliminate unnecessary antecedents in rules and to obtain very good results, frequently outperforming other techniques.

Correct speech /non-speech segmentation is important in improving the quality of hearing aids audition in noisy ambient by reducing noise between the non-speech duration. The solution proposed by (Ramirez et al. 2004) does not employ fuzzy logics and the authors show somewhat better results by their non-FL solution than those previously obtained by Beritelli et al. (1998, 1999), Cavallaro et al. (1998). The same paper compares results obtained by several methods, with the FL-based method comparable, but not always better (under various SNRs and noise levels) than the other solutions. However, because fuzzy logic systems (FLSs) generalize crisp systems and, according to the representation theorems, FLSs can implement any nonlinear crisp system, we can confidently say that the application of FL can bring benefit in this application too. For further researches, we consider that several other parameters should be used to increase the VAD performance and extensive use of pattern matching applied for increasing the discrimination between speech and no-speech activity under high noise levels. We also believe that nonlinear dynamics analysis, possibly combined with FL as explained in this chapter may find applications in VADs.

An interesting and powerful set of approaches mixing FL, nonlinear dynamics analysis and various statistical tools (Bayesian information criterion, BIC, among others) was recently introduced in a set of papers by Zhao et al. In the paper (Zhao et al. 2011), a robust VAD was proposed based on multi-level Lempel-Ziv complexity (MLZC) with the multiple thresholds determined by a combination of fuzzy c-means (FCM) clustering and BIC. These authors find that the VAD behaves roobustly for SNR up to 10 dB, for various types of noises.

There are numerous other contributions to the VAD topic brought under backgrounds not related to FL, including general statistical, perceptual, clustering, optimal filtering, and correlative settings. Compared to the number of papers using non-FL approaches, those using FL in VADs are very few and not always fully convincing.

FL in medical applications of speech analysis

The subtle changes in voice, compared to typical voices of healthy subjects, as well as significant and rapid changes in a subject voice may be related to a large class of pathologies. Indeed, voice production jointly and intimately relies on air flux generated by the chest (lungs and chest muscles under neural control), vocal cords, larynx, pharynx, buccal and nose cavities, neck and face muscles, teeth, and lips. Any pathology of these parts of the body, including of their neural control, may produce voice and speech changes ranging from light to dramatic. The corresponding medical fields can benefit for diagnostic purposes from tools of voice and speech analysis. For example, low air flux and volume may translate in low voice volume, inability of maintaining sustained vowels, higher voice jitter, and degraded prosody. Neurological disorders in the control of any of the parts contributig to speech may produce changes in voice and prosody quality, hoarseness and jitter. Anatomic and functional imperfections of the articulators (tongue, nose cavity, buccal cavity, teeth, lips) are well known to change speech in a dramatic way. Actually, numerous diagnostic techniques, devices and applications based on voice analysis have been proposed in the last 20 years, including entirely new sub-domain, as ‘gnathophony’ (a study of the relationship between the chewing apparatus and speech production and deficiencies), some of them based on or connected to FL. Subsequently, we exemplify some of these directions related to FL. Most uses of FL in medical applications of speech regard diagnosis, as already stated, and mainly data aggregation, classification, and decision making in diagnosis.

Stylios et al. (2008) dealt with “voice quality assessment, including nasality of speech, hoarseness, breathiness, voice tremor, strained voice, voice breaks, diplophonia”, where the voice quality was determined using a fuzzy cognitive map (FCM). These authors also fuzzified the pitch using a FCM, to render the meaning of low and high pitch, pitch breaks, and mono-pitch. The final goal was to produce an instrument for diagnosing of the dysarthria and apraxia of speech, using the aggregation of the results of the two previous FCMs in a third one. The authored argued that the results compare well to the diagnostics made by a language pathologist; however, they studied only four patients.

A partly similar research was reported by Scherer et al. (2013), who studied “fuzzy-input fuzzy-output support vector machines for robust voice quality classification”, but their aim was emotion and mood detection rather than pathologies (see Section on Emotion detection in this chapter).

In a paper on the relationship between denture state and speech quality (Teodorescu and Feraru 2008), the authors suggest the use of FL for improving diagnostic and also suggested a “high-pass” membership function for the degree of speech discrimination as a voice quality indicator for diagnostic use. The membership function was defined based on Euclidean distances between phonemes. However, these authors have left the issue open, with no checking of the capabilities of the suggested fuzzification method.

We believe that FL has a strong potential in speech-related medical applications, although it is not yet very well represented in the literature devoted to the field of speech pathology.

FL in phonetic segmentation

In the paper by Toledano, Crespo and Sardina (1998), a mixture of HMM and fuzzy logic is used for improving the phonetic segmentation (phoneme boundary detection) quality. This task does not need to recognize the phonemes, but the transitions from one to another. These authors used a two-stage segmentation algorithm, where the FL was used only in the second stage, of decision making and refining the boundaries of the phonemes, based on time marks produced by a context-dependent phonetic HMM recognizer and by a feature extraction block. The feature extraction block characterizes the speech windows employing, among others, mean energy, zero crossing rate and mean frequency. The decision on the phoneme boundaries is made using the degree of confidence produced conjointly by the feature extraction bloc and the HMM one. The results of segmentations are very good, around 97 %, which can be explained, according to the authors of that paper, by the use of context-dependent models. Despite the good results, the research track of using FL in decision making for phonetic segmentation was not followed by other papers.

4 FL in Speech and Speaker Recognition

The topic of speech and speaker recognition is too vast to deal it in any detail in a single chapter, therefore this section of the overview remains sketchy.

Speech recognition

When FL is applied in speech recognition it was complementing and improving established tools as hidden Markov models (HMMs). Some favorable results were obtained with fuzzy NNs, but also limited in extent. In an early paper, Koo and Un (1990), used a FL-based method "to smooth hidden Markov model parameters" and obtained higher recognition scores and lower computational load with their method, compared to the HMM methods using other smoothing techniques of that time.

Essentially, speech recognition is a pattern recognition process; it is approached by rule-based methods or by statistical methods, the second class of methods having higher performance and consequently being preferred nowadays (Burileanu et al. 2010). The development of statistical methods required and produced, as a side effect, large databases and annotated speech corpuses, where annotations refer to various levels, from voiced-unvoiced-no-voice segments, to phoneme (including di- and tri-phones) and prosodic levels. These speech corpuses were then used to create the so-called speech language models, essentially consisting in statistical representations of the speech, from the level of phonemes and sub-phonemes, to the level of word utterances and prosody. These pronunciation-related statistics are then combined with the so called natural language (NL) language models, which include the statistics of letters, syllables, words, etc., together with the corresponding syntax. The development of statistical tools for analyzing speech, on the other hand, allowed the automatic annotation of speech recording, which further facilitated the creation of very large speech corpuses, comprising hundreds of hours of recordings and millions of spoken words. Speech corpuses are often distinguished according to their purposes: training of speech recognition systems, training of speaker recognition or validation, emotion recognition, computer interfaces and answering machines, or general purpose "spoken language repositories", that may include tens of millions of spoken words. The particularities of the corpuses depend on their purposes. Almost any language has today several such corpuses; for example, at least three corpuses were developed for the Romanian language (Cucu et al. 2014), (Feararu et al. 2010) and a repository is currently under development.

Many if not most speech recognition systems in use today are based on hidden Markov models (HMM) for the language and speech. For example (Burileanu et al. 2010) built a sequential architecture, based on HMMs, which uses MFCC coefficients or speech signal parameterization (for both training and decoding regimes), acoustic modeling at the triphone-level, with an HMM trained for each phoneme, and with each HMM having, for each triphone, five states (including one initial state and one final state, both nonemitting, and three emitting intermediary states). Also, the system has a "left-right topology (where transitions towards remote states and backwards transitions are not allowed) and a continuous output distribution with weighted linear combination of Gaussian mixtures for each emitting state" (Burileanu et al. 2010).

In several papers, FL was proposed to advance the HMMs used in speech recognition. In an early development, Koo and Un (1990) proposed a fuzzy smoothing of the HMM parameters in speech recognition. Cheok et al. (2001), introduced novel generalized fuzzy hidden Markov model for speech recognition, based on the fuzzy integral theory (Choquet integral). According to these authors, the use of Choquet integral relaxes one of the two independence assumptions in classical HMM theory. The author claim that, although FL typically require computational-intensive algorithms, their method actually lowers the computation time.

Pal and Mitra, in their early paper in (1992), analyzed the use of multilayer perceptrons and fuzzy sets in classification applications. In this line, several research groups endeavored to use FL in speech recognition either combining FL and NNs or applying fuzzy rules and systems for decision and classification. (Kasabov and Iliev 2000), in a conference paper, deal with "adaptive speech recognition in a noisy environment (ASN)." Essentially, the authors describe "*a system based on the described method can store words and phrases spoken by the user and subsequently recognize them when they are pronounced as connected words in a noisy environment*." The system relies on a "*speech recognition module that uses evolving fuzzy neural networks* (EFuNNs)," able to assign membership degrees to noisy speech segments to non-noisy speech segments previously learned by the system. For this purpose, the input vector of features and vectors of features stored in memory are compared. The algorithm for training the "evolving fuzzy neural network" (EFuNN) is also given. These authors registered a patent application for the system, quoted in their paper as "A methodology and a system for adaptive recognition in a noisy environment based on adaptive noise cancellation and evolving fuzzy neural networks,"Preliminary Patent, University of Otago, 21 December 1999, New Zealand," (Kasabov and Iliev, 1999).

In another early paper, Mills and Bowles (1996) applied fuzzy logic to improve dynamic programming for speech recognition, for overcoming the difficulties of the crisp dynamic programming under high noise. These authors proved that the classification accuracy was increased by the use of FL-based technique.

Also an early industrial interest in speech recognition based on FL is the patent US 5,040,215, titled Speech Recognition Apparatus Using Neural Network and Fuzzy Logic, by inventors Amano et al. (1991), applied by Hitachi, Ltd., Japan. The same team has partly described the invention in the conference paper (Amano et al. 1989).

The patent claims a rule-based phoneme recognition system, where FL is used in decision making. The system employs pair-discrimination rules, that is aim to discriminate among two phonemes for every pair of phonemes. This is essentially a template matching based on FL.

In a closely connected, although different approach, (Shikano, Nakamura and Abe 1991) the speech recognition is performed by matching the voice of speakers to the speech of a reference, "standard" speaker based on codebook mapping. The approach is not related to a specified speech representation such as HMM or NN-based, but once the coding for the reference speaker is chosen, the same coding must be used in the recognition. An advantage of the algorithm is the large range of applications; for example, one performer speech can be transposed to impersonate other speakers in games and movies.

Speech recognition relying on fuzzy NNs has been also studied extensively. As a matter of example, in a book chapter and then in a paper, Melin et al. (2005, 2006) use FL (type-2 fuzzy logic), NNs and GA for pattern matching and decision making in the recognition process. However, the presented results are too few to derive statistically relevant conclusions.

Although not in the proper domain of speech, but intimately related to it, a special mention deserves the research by Temko, Macho, and Nadeu (2008), who use information fusion based on FL for dealing with highly confusable non-speech sounds. Examples of investigated sounds by these authors are cough and throat produced sounds, laughter, sneeze, sniff, and yawn. These authors find that fuzzy measures and Choquet fuzzy integrals are suitable tools for fusing several information sources, taking into account the importance of the sources. The results obtained by fusing combinations of SVM, frequency-filtered filter-bank energies (FFBE), differential ΔFFBE (where Δ indicates the time derivative), and HMM, according to combinations HMM-FFBE, HMM-ΔFFBE, HMM-FFBE+Δ FFBE, then combinations of these combinations demonstrated an improvement of several percentage points compared to the best performing of the above classifiers.

Also worth noting is the approach combining wavelets and fuzzy logic in speech technology, illustrated by the papers by Avci and Akpolat (2006), who use "a wavelet packet adaptive network based fuzzy inference system" and by Juang, Cheng and Chen (2009), who address with similar tools the speech detection in noisy conditions.

FL in speaker recognition

The speaker recognition field is well illustrated by the paper by (Chibelushi et al. 1993),

> "system that uses both acoustic speech and visual speech (motion of visible articulators). … As an initial step towards this goal, voice has been used together with still face images; this combination of vocal and facial information has resulted in better recognition accuracy than from either of the two constituents individually."

The authors, notice along with the literature that

> "the features examined to accomplish each task are not valid for the other. Specifically, the features needed to allow reliable phoneme identification must contain detailed information about the spectrum (high frequency resolution) and therefore have poor time resolution, while the features with high time resolution needed to allow precise time mark positioning have poor frequency resolution and therefore don't allow accurate phoneme identification."

These authors divide the task in the two respective tasks and perform them in parallel. At the same time, the two processes are developed as follows. On one side, the "context-dependent phonetic HMM recognizer" is applied, followed by a "context-dependent HMM training error cancellation" and "intermediate time marks" are found. On a second, parallel processing channel, a feature extraction process is applied to the voice signal, to determine in a second manner the boundaries. Then, the two time boundary sets of results are input to a fuzzy logic-based "post correction system" and the final time marks are obtained with high accuracy. With training, speaker-dependent results reach a precision of about 1 % compared to the manual segmentation, for segments longer than 35 ms, and of about 10 % for segments of duration 20 ms (see Fig. 4 in the quoted paper). Also, for speaker-independent segmentation, the results are not degraded more than about 5 % (Fig. 5 in the quoted paper). Interestingly, the fuzzy rules in that paper directly refer in their conclusions to the probabilities of transitions from one phoneme to the next.

5 FL in Emotion Estimation

Emotion detection is a relatively new sub-field in speech technology, which shows a vivid dynamic, both in papers and patents.

It is somewhat surprising that fuzzy logic has not been more intensively applied to emotion analysis in speech, and more broadly to speech prosody analysis, as speech features are reputedly fuzzy. A notable exception is the early paper by Massaro and Cohen (2000), who used a "fuzzy logical model of perception (FLMP)" to characterize emotions in speech, and (De Gelder and Vroomen 2000).

In an early paper hugely cited, Cowie et al. (2001) proposed the correlated use of clues from face image sequences and information from speech, in an intuitively natural manner, for extracting relevant features in assessing emotions. Also quite intuitively, they used a hybrid classification system combining neural network and FL to derive the emotion. It is worth mentioning that similar neuro-fuzzy approaches have been developed later for emotion recognition based solely on face expressions, see (Ioannou et al. 2005)

In a set of papers published both in conference and journal forms, Lee, Narayanan, Grimm, Kroschel, and co-workers (Lee and Narayanan 2003, 2005; Grimm, Kroschel and Narayanan 2007; Grimm et al. 2007), investigated several techniques for emotion recognition in speech, first extracting emotion primitives ("valence, activation, and dominance"), including SVM and the corresponding support vector regressions, and data-driven fuzzy inference systems. According to these researches, for German speakers,

> The results [obtained with SVMs and SVRs] were compared to a rule-based fuzzy logic classifier and a fuzzy k-nearest neighbor classifier. SVR was found to give the best results and to be suited well for emotion estimation yielding small classification errors and high correlation between estimates and reference. (from Grimm et al. 2007)

Recently, Zbancioc and Feraru (2012, 2012) have applied Fuzzy k-Nearest Neighbor (FkNN) classifiers, FCM and WKNN algorithms to the estimation of emotion in Romanian speech, on the SRoL corpus. In the approach utilizing FkNN classifiers, the features vectors consisted of 17 parameters, namely F0, F1-F4 (average, dispersion, median for all formants), jitter and shimmer, all determined at phoneme-level. These authors report an emotion recognition rate, at phoneme (vowel) level, for four emotional states (joy, sadness, furry, and normal) between 61 % and 76, with FkNN, for the Romanian language corpus SRoL described in (Feraru, Teodorescu and Zbancioc 2010). In the second paper, the same authors apply a variant of Fuzzy C-Means (FCM) classifiers, the recurrent FCM, and Weighted kNN (WkNN) classifiers to the same corpus, for emotion estimation. They obtained similar results by the two techniques (WkNN and FCM), all results being better than those obtained with kNN classifiers, for the same set of features. Notable, these authors introduced a new classifier, named FCMR algorithm (FCM recurrent), which performs better than the typical FCM algorithms.

Related to emotion estimation in speech is the topic of human voice quality estimation (Szekely et al. 2012), (Scherer et al. 2013). As (Scherer et al. 2013) put it, "*The dynamic use of voice qualities in spoken language can reveal useful information on a speakers attitude, mood and affective states*," but notice that in assessing the perceived voice quality, human experts frequently disagree. Focusing on "a voice quality feature set that is suitable for differentiating voice qualities on a tense to breathy dimension", these authors introduce a fuzzy-input fuzzy-output support vector machine (F2SVM) algorithm that was able to estimate the voice quality of speech recording better than crisp SVMs. Szekely et al. also use fuzzy SVM.

The meticulous research of Szekely et al. 2012 applied to detecting what the authors name "target voice styles" in speech combines two classifiers, "namely a fuzzy-output support vector machine (FSVM), and a Gaussian mixture model (GMM)". An agreement optimized ensemble (AOE) voting system then determines the confidence of the classification results. Confidence thresholds for the two classifiers are determined by an expression based on relative agreement relA, taking agreement. Target voice styles were automatically determined with accuracy close to a remarkable 90 %.

6 An Incursion into the Fuzzy Information Space Representation

Because of the less common approach in (Rodriguez et al. 2000) and the intricacy of the matter and method, the issue deserves a few more explanations and a general setting to potentiate further research in the area. The work was partly continued in speech analysis but with no intervention of FL (Rodriguez+Brito). The foundations of the matter are given in a paper devoted to establishing a method for signal analysis in the defined space of fuzzy information (Kosanovic, Chaparro, and Sclabassi 1996), with envisaged specific applications to biomedical (EEG) signal classification. Subsequently, we reformulate the framework of a slightly general approach based on fuzzy information.

Preliminaries. Consider a time-dependent process, as a bio-electric signal, or a set of simultaneously acquired bioelectric signals, e.g., EEG or ECG signals. The time variable is discretized, as in a sampling process. Assume that a set of features $x_j, j = 1, \ldots, m$ is derived from the acquired signal. Denote the time-dependent vector of these features by $x_i = (x_{ij}) = (x_{i1}, \ldots, x_{im})$, where the index i stands for the time moment and the second index for the component of the vector. Assume that there are n classes of signals, $C_k, k = 1, .., n$. These classes may be statistical, that is, at some specified moment of time, i, the signal has assigned a set of n probabilities to belong to these classes, $p(x_i \in C_k)$. On the other hand, the classes may be fuzzy categories, that is, signals belong to these classes with a certain degree of confidence, $\mu(x_i \in C_k)$.

The assignment of a signal to a class is time-dependent in many non-stationary processes. As a matter of example, the speech signal may represent one phoneme in a brief time interval, while in shortly after it may represent another phoneme. Therefore, we will assume that the probabilities, respectively the membership functions, according to the case in hand, varies in time. Denote the time-dependent probabilities (membership functions) by $p_{ik} = p(x_i \in C_k)$, respectively $\mu_{ik} = \mu(x_i \in C_k)$.Then, for a specified class$k$, we obtain a time-dependent probability, $p_k(i) = p(x_i \in C_k)$, respectively the membership function $\mu_k(i) = \mu(x_i \in C_k)$. With the discrete time seen as a variable, $p_k(i)$ becomes a distribution of probabilities over the space of the discrete variable i, that is, over the set of integers, Z, or on a subset of it, when the process has a finite duration. Similarly, $\mu_k(i)$ is a membership function defined on the set of integers, Z.

We name the vector of functions $\boldsymbol{\mu}$ with the components μ_k, $k = 1, \ldots, n$ a fuzzy dynamics. Recall that every component is a function, $\mu_k(i)$. We will say that the fuzzy process (dynamics) is fuzzy periodic if it has an infinite duration and the dynamic membership functions $\mu_k(i)$ are periodical. A fuzzy dynamics is chaotic when the evolution of at least one component $\mu_k(i)$ of the process is chaotic. Then, the attractor of the process can be constructed in the space of the n variables of the process.

Two finite duration processes can be compared at the level of their vectors $\boldsymbol{\mu}_1 = (\mu_1^1, \ldots, \mu_n^1)$ and $\boldsymbol{\mu}_2$. A simple method of comparison is to determine the difference of the membership functions and to compute the average distance between the respective vectors,

$$d(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) = \frac{1}{N}\sum_{i=1}^{N} d\left(\left(\mu_{1i}^1, \ldots, \mu_{ni}^1\right), \left(\mu_{1i}^2, \ldots, \mu_{ni}^2\right)\right)$$

where N is the duration of the two processes, that is, the number of time moments they are non-zero.

We will assume the classes are specified and their prototypes are known.

Example 1

Consider the EEG signal is monitored with 4 (active) electrodes and the signal is sampled. The samples acquired on the four channels are denoted as $s_{ij}, j = 1, 2, 3, 4$. Consider that the selected features to characterize the signals at every time moment are the frequency f_j and the amplitude A_j of the main spectral component in the Fourier spectrum (determined on a specified window centered on the current sample.) Consider that the EEG signals are assigned to one of the classes α, β, θ, based on the two parameters[1]. Membership functions are defined as functions of the distance from the "central" (typical, prototype) frequency and amplitude of the respective classes, namely

$$\mu_{\alpha,j}(i) = \mu_{\alpha,j}\left(f_j(t_i), A_j(t_i)\right) = \frac{1}{1 + \sqrt{\left(f_j(t_i) - f_\alpha(t_i)\right)^2 + \left(A_j(t_i) - A_\alpha(t_i)\right)^2}}.$$

The instantaneous average membership degree of the signals to the class α at time moment i is

$$\left(\mu_{\alpha,1}(i) + \mu_{\alpha,2}(i) + \mu_{\alpha,3}(i) + \mu_{\alpha,4}(i)\right)/4.$$

The overall membership degree is the average, over the recording duration, of the instantaneous average degree.

Computing as above the average membership degrees to the classes α, β, θ, the classification is produced by choosing the class with the highest degree.

Example 2

Speech signal is acquired at a sample rate of 24 kHz and acoustically Fourier analyzed. The result retains two features, the formants F1 and F2. The frequencies of the formants are determined on windows of 3 ms during the utterance of vowel-like sounds.[2] Assuming a language with 15 vowel-like sounds,[3] consider that the prototypes of these vowel-type sounds in the formant space $(F1, F2)$ denoted by π_v,

[1]In fact, only frequency is required in this case, but we pursue the two-parameter feature vector for sake of a more complete example.

[2]A vowel-like sound is detected when there is a valid pitch, that is, when the vocal folds vibrate. Pure consonants are produced without a pitch.

[3]Including [m], [n], [l], [r] etc.

where $v \in \{a, e, i, o, u, m, n, \ldots\}$. For every time moment i determine for the unknown sound its values for F1 and F2 and define its membership to a vowel class as

$$\mu_v(i) = \mu_v(F1(t_i), F2(t_i)) = \frac{1}{1 + \sqrt{(F1(t_i) - F1_v(t_i))^2 + (F2(t_i) - F2_v(t_i))^2}}.$$

For the entire duration of the sound,[4] determine the average membership to each class. The classification result is that class that to which the sound has the largest membership degree.

The above definitions of the membership functions work well for two classes and in a one-dimensional space, but have several drawbacks when several classes are involved and when their representation requires a multi-dimensional space. Considering n classes with their prototypes denoted by $\pi_k(x_{1k}, x_{2k}, \ldots, x_{rk})$, with the classes overlapping at least two by two. Assuming that the classes occupy (hyper)spheres, their intersections define (hyper)planes. Any point on such hyperplanes has the same membership degree to the respective two classes, potentially a degree close to 1. That situation is somewhat unintuitive and, when the membership degrees are close to 1, not convenient for applications (discrimination). An alternative definition of the membership function could be

$$\mu_k(Q) = 1 - d(Q, \pi_k) / \max_{\mathrm{j}} d\left(Q, \pi_j\right)$$

which satisfy the condition that $\mu_k(Q = \pi_k) = 1$, but has the drawback of arbitrary use of $\max_{\mathrm{j}} d\left(Q, \pi_j\right)$. A better definition is

$$\mu_k(Q) = 1 - d(Q, \pi_k) / \max_{\mathrm{j}} d\left(\pi_k, \pi_j\right)$$

This definition would allow for negative values, for very far apart Q points. To correct, we can use

$$\mu_k(Q) = \begin{cases} 1 - \frac{d(Q, \pi_k)}{\max_{\mathrm{j}} d\left(\pi_k, \pi_j\right)} \; for \; d(Q, \pi_k) \le \max_{\mathrm{j}} d\left(\pi_k, \pi_j\right) \\ 0 \; else \end{cases}$$

Still the definition inherits the disadvantage of the Euclidian distance, which considers all directions equal. This is convenient when the classes have spherical distributions, but not in the general case. For example, in the case of ellipsoidal classes (as corresponding to the equal probability /Galton) ellipses for Gauss distributions, the weighting along the axes of the ellipsoids is justified, see Fig. 2.

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[4]This requires a procedure for determining the boundaries of the sound.

Building the dynamic (temporal) membership functions. Consider a sampled signal $s(t_n)$ that we wish to classify, where the signal is represented by several parameters (features) computed on successive windows of specified duration. Then, the distances to the prototypes are computed in the feature space. The above procedure produces time-indexed values of the membership functions to the specified classes, $\mu_k(t_n)$. These evolving (time-dependent) membership functions are representations of the signal in the fuzzy dynamic space (also called information space). Considering frames of the same length as above or of different durations, centered on time moments t_n, the dynamics represented by the vectors of values of the membership functions can be contrasted or compared with prototype dynamics, in view of final classification of the signal, for example for phoneme recognition. Specifically, at time moment t_n, the signal representation in the fuzzy dynamic space is

$$(\mu_1(t_n), \mu_1(t_{n+1}), \ldots, \mu_1(t_{n+W})), \ldots, (\mu_m(t_n), \mu_m(t_{n+1}), \ldots, \mu_m(t_{n+W})),$$

or

$$(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_m)_n$$

where m is the number of classes in the features space. Notice that the classes used to compute the membership functions may be the same as the classes used in the final recognition process. The difference is that in the feature space, one creates the membership functions point-wise (in time), while in the final classification, in the fuzzy dynamics space, we work in matching dynamics (variations in time), not values. For example, the classification can be performed based on the minimal distance between trajectories of the membership functions,

$$\min_v \sum_h d(\boldsymbol{\mu}_h, \boldsymbol{\mu}_{hv})$$

Matching. Beyond the matching by minimal distance, methods of matching more specific to the nonlinear dynamic field are available, for example Lyapunov exponents and various fractal dimensions computed locally for a small number of windows approximately corresponding to the average duration of phonemes. These specific tools might help improve the overall recognition score and surely increases the power of speech analysis.

An example of application of the technique reviewed in this section to speech analysis is provided in (Rodriguez et al., 1997), while applications to the analysis of various bio-electrical signals are presented in a series of papers by Kosanovic et al. (cited series of papers, 1994–1996).

7 Conclusions and Potential Future Directions

The use of FL and FLSs in speech technology has seen a somewhat modest development and mixed success, which are difficult to explain else than by the large amount of computations required by both speech characterization and fuzzy systems. Also, the early development of good statistical models, as HMMs may have reduced the appetite to apply FL in speech recognition. Also, the advent of concatenative synthesizers and the transition from Klatt-type to concatenative-type speech synthesis may have hampered the use of FL in synthesis for some time.

Interestingly, several patents have been applied, possibly indicating that FL was more attractive in speech technology for industry than for academia.

Although the overall results of applying FL to speech technology are today rather limited, we are confident that further researches can successfully involve the use of FL in speech synthesis, including prosody synthesis, speech analysis with applications to emotion detection and medical conditions, and in speech and speaker recognition under noisy environment. We expect that the applications of speech analysis in medicine and psychology are likely to benefit of the use of FL and neuro-fuzzy systems. However, for achieving more significant results in speech technology, FL needs to be used in its forms merged with second order m.f.s and fuzzy probabilities.

What is striking for a reviewer who compares the processes of speech recognition in humans, as currently known, and today algorithms for speech recognition is the low parallelism (similarity) between them. In a review chapter on speech perception, based on empirical findings in the medical and psychological literature, the authors (Gierur & Pisoni 1988, pp. 253–276) say about speech perception development in humans:

> Children not only used but relied upon multiple cues in perception. Acoustic redundancies were critical to the perception of the phonemic contrast. … Children assigned relative weightings to the cues. … With development, children shift attention to more isolated, discrete segmental cues.

None of these features can be assigned today to speech recognition systems, and neither the development of these systems nor their evolving during training follows a path similar to speech recognition by humans during their development from children to adults. Notice that the relative weighting mentioned by Gierur & Pisoni may have similarities with the information aggregation in fuzzy systems, or at least may encourage the attempt to use FL to mimic human perception.

It is unclear what path the use of FL in speech technology will take in the context of the trends in data analytics, Internet of things, and cloud computing. We may expect that the extra computational effort required by the advanced and massive use of FL merged with fuzzy probabilities, which may have hampered the extensive use of FL in speech technology in the past, will be absorbed under these developments and will produce FL-based instruments.

FL use in speech technology will increase only side-by-side to new commercially-viable applications. While the application of FL in typical communication technology was partly successful, FL has potential to become a major tool in higher-intelligence applications, where man-machine interactions become ubiquitous. Speech interpretation as a representation of the state of the subjects, including emotion, intention, and mood, and of subjects' view on their relationship with the machine interlocutor [Teodorescu, 2005, Teodorescu, 2001a] might benefit of FL, and FL development may benefit of researches in this broad field. On the more research-oriented side, we believe that phonetics and language studies, as well as psychology and medical applications can tremendously profit in the future from the use of FL.

Acknowledgments The first author acknowledges the help of Prof. Corneliu Burileanu, Dr. Marius Zbancioc and Dr. Monica Feraru in reviewing a preliminary version of the chapter.

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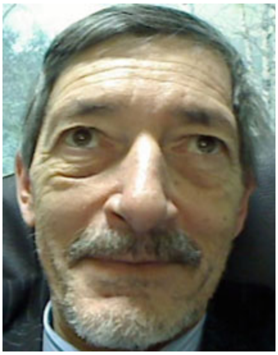

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Fuzzy Sets: Towards the Scientific Domestication of Imprecision

Enric Trillas

Abstract This paper considers fuzzy sets just as science tries to domesticate concepts once abstracted from reality: identifying them with quantities, using these quantities for building up mathematical models, escaping from just a formal logic setting, and testing the models against reality before provisionally accepting them. Its aim is that of trying to go towards a new experimental science of 'the imprecise'; to something like a 'physics of linguistic imprecision'. It contains a way for looking at fuzzy sets that, if continued, could offer a new perspective for seeing linguistic imprecision, and whose possible value lies on the idea that several forms of theorizing always can be better than a single one.

1 Introduction

From its inception by Lotfi A. Zadeh fifty years ago [1], fuzzy sets are linked with the management of imprecision in real, technological, problems. Imprecision permeates natural language and common reasoning, and hence also the linguistic description of the behavior of those systems that either cannot be described by equations, or such a description is unknown. Imprecision, necessary for transmitting ideas in not too long pieces of language, is an economic characteristic of natural language. Zadeh's paper was, after those by Max Black in 1937 and 1963 [27, 28], a true first attempt towards a domestication of linguistic imprecision, and his work from 1965 onwards constitutes a relevant 'corpus' of doctrine for it.

In the Introduction at his 1965 seminal paper [1], and to separate from the very beginning fuzzy sets from both sets and probability, Zadeh stated:

To professor Lotfi A. Zadeh, with a professional high admiration to his impressive work, and a deep personal esteem for him.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
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DOI 10.1007/978-3-319-19683-1_29

"…such a framework [that of fuzzy sets] provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables".

Of course, this 'class' approach neither means that there is a lack of relationships between sets and fuzzy sets, nor that the values of the membership function of a fuzzy set can never come from some probabilistic computation, but that the genesis of fuzzy sets neither lies in perfect classification, nor in randomness. The genesis of fuzzy sets is in language, where predicates (the words naming properties) usually designate classes or collectives [2] as, for instance, the predicate young (Y) designates the class or collective of the 'young Londoners' ([Y]) when applied to London inhabitants. The idea of a collective is well rooted in language.

By one side collectives should be mathematically represented to, for instance, translating statements into formulas, and by the other side most of them should not be confused with sets as it can be shown by using the Sorites' methodology [3]. Additionally, it should be remarked that fuzzy sets are not collectives, but only a partial representation of them depending on the directly available information on the use of the predicate and, perhaps, on some reasonable hypotheses on such use.

Were, as it is usually done, a fuzzy set just seen as a function $X \to [0, 1]$ without referring to a predicate or linguistic label, the function says nothing on any linguistic subject. Fuzzy sets do translate into functions the current meaning of predicates on X, and this paper goal is just to explain what a fuzzy set is, and how fuzzy sets allow domesticating imprecision as it is usually done in science. Most of the applications of Zadeh's fuzzy sets do concern problems involving dynamical systems whose behavior is described by means of linguistic imprecise rules, provided by an expert of the type 'If x is P and y is Q, then z is R'.

2 Measurable Predicates

Let X be a universe of discourse whose elements enjoy some property (p) named by P. What follows will proceed under the following three naïve working hypotheses of 'perceptive recognition':

- First: For all x in X it can be recognized if "x is P" or x verifies p, can be, or cannot be, stated.
- Second: For some pairs of elements in X it can be recognized if one of them enjoys p less or equal than the other.
- Third: It can be recognized that, in the language, P originates in X a 'collective', called that of the Ps.

If P verifies the three hypotheses or conditions, it will be called "measurable" [4] in X. Although it does not mean that they are for nothing, non-measurable predicates will not be considered here. The first question to answer is what it means to actually measure a measurable predicate.

Let's consider the empirically perceived binary relation $\leq_P$ defined in X [4] by,

$$x \leq_P y \Leftrightarrow \text{x shows P less or equal than y shows P},$$

provided that, at least. first and second hypotheses hold. This relation reflects that the elemental statement 'x is P' carries less or equal information than 'y is P' for what concerns property p. Hence, the use of a measurable P in X endows it with the simple mathematical structure given by the graph (X, $\leq_P$), and it will be said that such graph translates the *primary* or *qualitative* meaning of P in X [4]. In this graph there can be maximal and minimal elements, that is, elements $z \in X$ such that there are no elements $x \in X$ verifying, respectively, $z \leq_P x$, or $x \leq_P z$. Maximals can be recognized by the lack of others showing more p, and minimals by the lack of others showing less p. Provided there is only a maximal or a minimal, they are said the maximum and the minimum, respectively. If z is maximal is when, classically, is said that z is P is *true*, and if z is minimal that z is P is *false*; in the setting of this paper the classical concepts 'true' and 'false' can be avoided, they are indeed superfluous.

Notice that X does not need to be previously structured, and that the graph reflects the intuitive idea that when speaking on something, one tries to introduce 'some order' in the universe on which she/he is speaking of. This seems to be a necessary first step for reasoning. Notice that X represents a universe whatsoever; properties can be observed in any kind of universe, and not only in those endowed with some mathematical structure as it is with the predicate 'prime' on the set of natural numbers, and it is not with 'hot' in a set of cooper samples.

Notice also that the relation $\leq_P$ is not, in general, a total or linear one, that is, there are often elements x and y not-comparable under $\leq_P$, that is, neither verifying $x \leq_P y$, nor $y \leq_P x$. If, for instance, the predicate *white* were applied to snow, namely to a collection of snow's samples, it will be measurable only provided it can be always recognized that 'this sample of snow is less *white* than that sample of snow', supposed the collection of samples is actually known. It is easy to imagine that for some pair of samples it can be very difficult to state if one of them is actually less *white* than the other; if the collection of samples is large, almost surely the relation $\leq_{white}$ will not be linear. Without well enough knowing the elements in the universe of discourse, it is very difficult, if not impossible, to consider if a given predicate is or is not measurable; previous information on X is basic for capturing a qualitative use of P in X.

If the relation $\leq_P$ cannot be stated, the predicate will be said to be metaphysical in X [4], and without any contempt against them, the consideration of metaphysical predicates is not a goal of this paper. The kingdom of metaphysical predicates is philosophy; science and technology are kingdoms of the measurable. It is not to be forgotten the dictum induced from a 1883 lecture by Lord Kelvin [23], "if you cannot measure it, it is not science".

With all that, a *measure* of the extent up to which each x in X is P, verifies p, or carries the information provided by P, is a mapping [5] μ_P: X → [0, 1], such that:

(1) If x $\leq_P$ y, then μ_P (x) $\leq$ μ_P (y), in the order of the real line,
(2) If z is a maximal in (X, $\leq_P$), then μ_P (z) = 1,
(3) If z is a minimal in (X, $\leq_P$), then μ_P (z) = 0.

Hence, a measure is but a morphism between the graph (X, $\leq_P$) and the totally ordered unit interval ([0, 1], $\leq$); it translates a 'scarcely ordered' graph into a linear continuum, and it should be noticed that, in general, the three properties of a measure are not sufficient to specify a single one. This is like what happens with the definition of a measure of probability: it does not specify a single one, but more information on the case is needed for such specification. Notice that neither μ_P(x) = 1, nor μ_P(x) = 0 mean that x is, respectively, a maximal or a minimal; anyway, maximals are in the set $\mu_P^{-1}(1)$, and minimals in $\mu_P^{-1}(0)$.

For instance, provided it could be established that μ_{white} is a measure for *white*, to know a value μ_{white}(this snow's sample), either more information, or a reasonable hypothesis on the comparison of samples, will be needed. Of course, like for assigning a probability to the event 'six' in the throw of a dice, it is necessary to know in which surface the dice will land and if it is tricky, in the case of 'snow' some experimentation, even imaginary or by doing a toy-experiment, is necessary for assigning values to μ_{white}.

Example 1 *If X = [0, 1000] and P =* small*, since it can be taken* $\leq_{small}$ *coincident with the inverse* $\geq$ *of the total order* $\leq$ *of the real line (x is less small than y* $\Leftrightarrow$ *x* $\geq$ *y), with minimum 1000 and maximum 0, the measures of S =* small *are the mappings*

$$\mu_S\text{:}[0, 1000] \to [0, 1],$$

such that:

(1) μ_S *is decreasing; (2)* μ_S *(0) = 1; 3)* μ_S *(1000) = 0,*

of which there are many. For instance, provided it were known that the measure's variation should be linear, the only possible measure is $\mu_S(x) = 1 - x/1000$*, but if it were known that it should be quadratic, then among the quadratic ones it can be chosen* $\mu_S(x) = (1 - x/1000)^2$*. Provided it were known that the measure is piecewise linear, then one of them can be chosen by selecting a number, say 250, such that* $\mu_S(x) = 1$ *if* $x \in [0, 250]$*, and* $\mu_S(x) = (1000-x)/750$ *if* $x \in (250, 1000]$*. Etc.*

Hence, additional information on the variation of the measure should be known, or supposed, to specify one of them. In any case, the measure will be suitable depending on the correctness of such knowledge and/or hypotheses. Not only a good enough knowledge of X, but also of the context on which P is applied to X, should be known to capture the 'current meaning' of P in X.

Notice that if X is an interval [0, M] of the real line, and $\leq_P$ can be identified with either $\leq$, or $\geq$, there is just a single linear measure μ_P (x) = ax + b since, in the first case it is μ_P (0) = 0 and μ_P (M) = 1, with which b = 0, a = 1/M, and μ_P (x) = x/M, and in the second it is μ_P (0) = 1, μ_P (M) = 0, with which b = 1, a = -1/M, and μ_P (x) = 1 - x/M. This is the former case in which M = 1000.

Once a measure μ_P is specified, the quantity (X, $\leq_P$, μ_P) will be called a *meaning* of P in X. It represents what is known on both the qualitative use of P in X, and the context and, perhaps, also the purpose with which it is used. At the end, the concept of meaning is here interpreted following the Wittgenstein's idea that "he meaning of a word is its use in language" [31]. Different uses of the same P, given by different relations $\leq_P$, or different measures μ_P, generate different meanings.

Once a measure is given, the new relation defined by,

$$x \leq_{\mu_p} y \Leftrightarrow \mu_P(x) \leq \mu_P(y),$$

is not only reflexive, anti-symmetric and transitive (a partial order), but a total one since it is always either μ_P(x) ≤ μ_P(y), or μ_P(x)≥ μ_P(y). This new relation verifies [4],

$$\leq_P \subseteq \leq_{\mu_p},$$

since: x $\leq_P$ y ⇒ μ_P (x) ≤ μ_P (y) ⇔ x $\leq_{\mu_p}$ y. That is, the measure extends the primary meaning; after actually measuring P its meaning grows in the difference-set $\leq_{\mu_p} - \leq_P$ that, provided is non-empty, adds the information supplied by the measure to the primary or qualitative meaning. In general, it can be said that the act of measuring changes the primary meaning of the predicate.

As $\leq_P$ is not always a total, or linear, relation, both relations cannot coincide in all cases, and when $\leq_P$ is total and coincides with either $\leq_{\mu_p}$, or $\leq_{\mu_p}^{-1}$, it is said that the measure *perfectly reflects* [5] the primary meaning of P in X. This is, for instance, the case of the former example in which $\leq_{small} = \geq = \leq_{\mu_s}^{-1}$. It is also the case with the predicate *big* for which $\leq_{big} = \leq$ allows to specify a measure, for instance the only linear one μ_{big} (x) = x/1000, and then $\leq_{big} = \leq_{\mu big}$.

Remarks

(a) Provided $\leq_P$ is a preorder, that is, a reflexive and transitive relation, the relation '*equally* P *than*', defined by:

$$=_P = \leq_P \cap \leq_P^{-1},$$

under which it is [5],

$$x =_P y \Leftrightarrow x \leq_P y \;\&\; y \leq_P x,$$

is a relation of equivalence. This equivalence gives the quotient-set X/ $=_P$, whose elements are the classes

$$[x] = \{z \in X; x =_P z\}$$

consisting of those elements that are equally P than its representative. Provided there is a measure μ_P, it is constant in each equivalence class, since:

$$z \in [x] \Rightarrow x \leq_P z \,\&\, z \leq_P x \Rightarrow \mu_P(x) \leq \mu_P(z) \,\&\, \mu_P(z) \leq \mu_P(x) \Rightarrow \mu_P(z) = \mu_P(x).$$

Consequently, if the primary meaning $\leq_P$ is reflexive and transitive, once the quotient set is partially ordered by

$$[x] \leq_P^* [y] \Leftrightarrow x \leq_P y,$$

a definition not dependent on the representatives of the classes, it can be defined a measure in $X/=_P$ by

$$\mu_P^*([x]) = \mu_P(x).$$

All that gives the quantity $(X/=_P, \leq_P^*, \mu_P^*)$ that also can serve to define *a meaning* of P in X. Obviously, the number of classes equals that of the different values the measure takes. For instance, in the case of the predicate *big* in [0, 1000], it is $x =_{big} y \Leftrightarrow x = y$, and $[x] = \{x\}$, the classes are reduced to singletons. In the case of predicate *prime* in the set of positive integers, there is just one class consisting of all the prime numbers.

(b) In language predicates appear when they are applied to a first universe of discourse, possibly after some previous trials consisting in observing and recognizing some property of its elements. For instance, *large* was perhaps used by the first time applied to stones, trees, mountains, lakes, etc., and further on passed to other objects like rivers, roads, buildings, etc., and finally to numbers. When applying an old predicate to new elements, the 'history' of its former uses is a help for the new application.
In the same vein, a word P designating a property can be understood in different ways by a group of people. For instance if, in a measurable case, n persons manage P in X each one with a meaning $(X, \leq_P^k, \mu_P^k)$, $1 \leq k \leq n$, and the intersection $\leq_P = \leq_P^1 \cap \leq_P^2 \ldots \cap \leq_P^n$ is not empty [5], the group can agree on the common primary meaning $\leq_P$, and the measure $\mu_P = \min(\mu_P^1, \mu_P^2, \ldots, \mu_P^n)$, as the accepted common meaning of P in X. Provided the intersection of the n relations $\leq_P^k$ were empty, or there were no agreement on it as a common use, the group can accept either a partial non-empty intersection, or one of the individual meanings. In other case, no common understanding of P in X is possible. Possibly, it is the need of communication between humans that motivated and motivates the acceptance of common meanings like those appearing in the dictionaries. It also seems to be the starting point for negotiating.

(c) Since, in the praxis, a predicate P is usually managed in X by a measure once it is specified, only the relation $\leq_{\mu_p}$ is known by the practitioner. For this reason, $\leq_{\mu_p}$ will be called the 'working meaning' of P in X. Notice also that usually is not easy to know the full relation $\leq_P$ when, for instance, there is a big amount of links between the elements in X; for this reason, sometimes μ_P is build up only with the help of context information, and it is not well known if it verifies the properties of a measure. In this frequent praxis' situation, the function μ_P should be viewed as an approximation to a measure with an unknown error.

3 Fuzzy Sets

3.1

Once a quantity (X, $\leq_P$, μ_P) reflects what is known on the use, or meaning, of P in X, the following new terminology can be introduced [4]:

Fuzzy sets **P**, **Q**, etc., are entities mathematically created under the following rules:

1. **P** = **Q** ⇔ (X, $\leq_P$, μ_P) = (X, $\leq_Q$, μ_Q)
2. **P** ⊆ **Q** ⇔ μ_P(x) ≤ μ_Q(x), for all x in X
3. x $\in_r$ **P** ⇔ r = μ_P(x), read 'x belongs to **P** with degree of membership r ∈ [0, 1]'.
4. The predicate P is called the *linguistic label* of the fuzzy set **P**, and the measure μ_P its *membership function*. For short, **P** is called 'the fuzzy set P'.

Example 2 *If ([0, 1000], ≤, id/1000) is the quantity representing the use of the predicate* big *in [0, 1000], it specifies the fuzzy set* $\boldsymbol{B}^1$*, with membership function* $\mu^1_{big}(x) = x/1000$*, with which it is* $500 \in_{0.5} \boldsymbol{B}^1$*,* $250 \in_{0.25} \boldsymbol{B}^1$*,* $750 \in_{0.75} \boldsymbol{B}^1$*,* $0 \in_0 \boldsymbol{B}^1$*,* $1000 \in_1 \boldsymbol{B}^1$*, etc.*

The same predicate big *can also be represented by the different quantity ([0, 1000], ≤,* $(id/1000)^2$*), specifying the fuzzy set* $\boldsymbol{B}^2$*, with membership function* μ^2_{big} $(x) = x^2/10^6$*, with which* $500 \in_{0.25} \boldsymbol{B}^2$*,* $750 \in_{0.5625} \boldsymbol{B}^2$*, etc., and that from* $x^2/10^6 \leq x/10$ *shows* $\boldsymbol{B}^2 \subseteq \boldsymbol{B}^1$ *and* $\boldsymbol{B}^2 \neq \boldsymbol{B}^1$*.*

This example shows that the collective [P], linguistically generated in X by a predicate P [4], cannot be confused with those fuzzy sets specifying the predicate from the information available on its current use. In this sense, the fuzzy sets P can be seen as *states* of the collective [P] generated by the predicate; the collective consists in all the states shown by the corresponding fuzzy sets. The former predicate *big* in [0, 1000], for instance, generates a linguistic collective [*big*] consisting in states like $\mathbf{B}^1$, $\mathbf{B}^2$, etc. The collective [*big*] can be seen as defined by all the many actual quantities defined by the properties:

(1) If x ≤ y, then μ_{big} (x) ≤ μ_{big} (y);
(2) μ_{big} (0) = 0;
(3) μ_{big} (1000) = 1,

once accepted that $\leq_{big}$ is coincidental with the linear order ≤ of the real line. That is, by all the functions μ_{big}: [0, 1000] → [0, 1] verifying (1) and under the border conditions (2) and (3). The collective appears as the set of all solutions of a constrained functional inequality, that is, of all measures of a measurable predicate, once accepted that $\leq_P$ describes its qualitative use. Each measure is specified by taking into account some contextual information on the particular use of the predicate; meaning is not unique but context-dependent.

In language, measurable predicates are usually *imprecise*, or *flexible*, that is, accepting measures taking some values different from 0 and 1. Whenever a predicate is *precise, rigid,* or *not imprecise*, it only accepts measures with values in {0, 1}. In Fuzzy Logic precision is seen as a degenerate, isolate, case of imprecision.

3.2

Consider the typically rigid predicate, P = *prime* in the universe **N** of natural numbers. The use of P is known by stating the elemental statements "n is P" from the if-and-only-if definition:

$$\text{n is } \textit{prime} \Leftrightarrow \text{the only divisors of n are n and 1,}$$

with which it is just the dichotomous possibility that, given a natural number p, either p is prime, or p is not prime but composed. Set **N** is perfectly classified in the two subsets of prime and composed numbers.

Without introducing a new *iff* definition, given two natural numbers there is no way of establishing if one of them is less prime than the other. Different prime numbers are equally prime, composed numbers are not comparable by being one less prime than the other, and given a prime and a composed the second is less prime than the first. For instance, the prime numbers 5 and 103 are just equally prime, the numbers 12 and 24 are not comparable, and given 12 and 3, it is obviously 12 less prime than 3. Hence, the relation $\leq_P$ is reduced to:

n $\leq_P$ m $\Leftrightarrow$ both n and m are primes, or n is composed and m is prime. Consequently, since there is no initial way of distinguishing between prime numbers, the only admissible measure is given by.

$$\mu_P(\text{q}) = 1 \text{ if q is prime, } \mu_P(\text{q}) = 0 \text{ if q is composed,}$$

that actually verifies n $\leq_P$ m $\Rightarrow \mu_P(\text{n}) \leq \mu_P(\text{m})$, and preserves the axioms for maximals and minimals.

3.3

Mutatis mutandis, a rigid predicate P on X denotes a property p such that the elements of X either completely verify it, or not verify it at all: the use of P in X can be defined by an *iff* condition: "x is P" $\Leftrightarrow$ such and such. Hence, X is completely classified in those elements that fully satisfy p, and those that do not satisfy p at all. The rigid predicate P is completely specified by the only set

$$\mathbf{P} = \{x \in X; x \text{ fully satisfies p}\},$$

whose elements are the maximals for $\leq_P$. Consequently, there is a single quantity $(X, \leq_P, \mu_P)$ with

(a) $x \leq_P y \Leftrightarrow$ Either $x, y \in \mathbf{P}$, or $x \in \mathbf{P}^c$ and $y \in \mathbf{P}$,
(b) $\mu_P(x) = 1$ if $x \in \mathbf{P}$, and $\mu_P(x) = 0$ if $x \in \mathbf{P}^c$,

with which it follows,

$$\mu_P^{-1}(1) = \mathbf{P}, \mu_P^{-1}(0) = \mathbf{P}^c.$$

Thus, in the case of a rigid predicate P is

$$[P] = \mathbf{P} = \mu_P^{-1}(1),$$

the collective is reduced to the set specifying the predicate in X, that, at its turn, coincides with the anti-image of 1 by the membership function, now reduced to be a mapping $X \to \{0, 1\}$. Rigid predicates are specified by a single fuzzy set that is just a set, and sets are known in Fuzzy Logic as *crisp sets*.

Notwithstanding, in Fuzzy Logic there are used singular fuzzy sets whose use is also degenerate; for instance, the fuzzy sets **K** specified in X by the constant functions $\mu_k (x) = k \in [0, 1]$,

Remarks

1. The denotation of a fuzzy set **P** will be sometimes shortened by just naming them by its membership function μ_P.
2. The classical symbols $\in$ and $\notin$, do correspond to the new symbols $\in_1$ and $\in_0$, respectively.
3. The constant fuzzy sets given by the constant membership functions μ_0 and μ_1 correspond to the sets $\emptyset$ and X. The other constant fuzzy sets **K**, represented by the membership functions μ_k, with $0 < k < 1$, $\mu_k (x) = k$, are neither with, nor without points, that is, for no x in X is $x \in_1 \mathbf{K}$, or $x \in_0 \mathbf{K}$. In this sense, fuzzy sets **K** are singular fuzzy sets.
4. The only set that is self-contradictory is the empty set since it is $\mathbf{A} \subseteq \mathbf{A}^c \Leftrightarrow A = \emptyset$, but this is not the case with fuzzy sets. Once the concept of negation is introduced by associating to each fuzzy set μ_P another $\mu_P' = \mu_{notP}$, there can be many fuzzy sets such that $\mu_P \leq \mu_P'$. For instance, were $\mu_P' = 1-\mu_P$, it will be μ_P self-contradictory if and only if it is $\mu_P(x) \leq 0.5$ for all x in X, or $\mu \leq \mu_{0.5}$, and, even if there can be points x such that $\mu_P(x) = 0$, that is $x \notin \mathbf{P}$, since $0.5 < 1$ there cannot exist any point x such that $\mu_P(x) = 1$, no point x in X verifies $x \in \mathbf{P}$. As it will be shown latter on, no self-contradictory fuzzy set can have points totally belonging to it. Perhaps, that the empty set $\emptyset$ is self-contradictory, that it has no a single element, is what makes it difficult (for children) to accept it as a true set.

5. Points x such that x ∈ **P**, with degree one of membership to **P**, are those recognized as totally verifying the property named by the predicate P. They can be seen as the *prototypes* for P in X. Analogously, those x such that x ∉ **P**, with degree zero of membership to **P**, are those recognized as non verifying at all the property named by P, and can be seen as the *anti-prototypes* for P in X. Self-contradictory fuzzy sets have no prototypes.
   Both prototypes and anti-prototypes can help to evaluate, through comparison with them, the degree of membership to **P** of the other elements in X. In this sense, those fuzzy sets without prototypes or ant-prototypes can be seen, up to some extent, as "rare" fuzzy sets since they are without, respectively, maximals or minimals for the perceptive relation $\leq_P$. For instance, provided the predicate *big* were used in the set of positive real numbers $\mathbf{R}^+$, including zero, with $\leq_P = \leq$, and the non-decreasing membership function μ_{big} (x) = $x/(x^2+1)^{1/2}$, it is μ_{big} (0) = 0, accordingly with the fact that 0 is the minimum for ≤, but for no x is $\mu_{big}(x) = 1$, accordingly with the lack of maximals for ≤. In this example, *big* has no prototypes in $\mathbf{R}^+ = [0, +\infty)$. Nevertheless, since it is $\lim_{x\rightarrow+\infty} \mu_{big}(x) = 1$, there are many x ∈ $\mathbf{R}^+$ for which the corresponding value μ_{big} (x) is as closer as wished to 1; these x could be called prototypes *approximated* up to the degree $1 - \mu_{big}$ (x).
6. It should be pointed out at once that specifying a imprecise predicate is a more complex problem than that of specifying a precise one. This, of course, does not mean that for precise predicates it is easy to recognize the elements with membership one as it is, for instance, with the predicate *transcendent* in the real line as the negation of *algebraic*.

4 Opposites and Negations

4.1

Predicates are (linguistic) terms, and also its opposites are terms, but their negations are not so. For instance, the term *big* is in dictionaries, and so it is the opposite term *small*, but it is neither the negation *not big*, nor the negation *not small*. If it has sense to search for the opposites and the negations of *big*, what has no sense is to search for an opposite of *not big*, unless accepting the rule 'opposite of not P = not opposite of P', under which an opposite of *not big* will be *not small*.

If a predicate is measurable in X, are their opposites also measurable in X? Provided a predicate is specified in X by a quantity (X, $\leq_P$, μ_P), is there a quantity specifying an opposite, or antonym, aP?

In the case of *big* in [0, 10], it is $\leq_{big} = \leq$, and a (big) = *small* is with $\leq_{small} = \geq$:

x is less *big* than y ⇔ y is less *small* than x, accordingly with the same idea of opposition. Hence, since it can be always supposed that a(aP) coincides with P, it is $\leq_{aP} = \leq_P^{-1}$, the qualitative behavior of aP in X is described by the graph (X, $\leq_P^{-1}$), and aP is measurable.

To actually measure aP, and continuing with the idea that it is an opposite of P, it can be taken a symmetry [6] $\mathbf{s}_P : X \to X$, that is, a bijective mapping such that $\mathbf{s}_P^2 = \mathrm{id}_X$, verifying [$x \leq_P y \Rightarrow \mathbf{s}_P(y) \leq_P \mathbf{s}_P(x)$], and defining:

$$\mu_{aP} = \mu_P \circ \mathbf{s}_P$$

that is,

$$\mu_{aP}(x) = \mu_P(\mathbf{s}_P(x)), \text{for all x in X.}$$

This mapping verifies (shortening it by s):

(1) If $x \leq_{aP} y \Leftrightarrow x \leq_P^{-1} y \Leftrightarrow y \leq_P x \Rightarrow s(x) \leq_P s(y) \Rightarrow \mu_P(s(x)) \leq \mu_P(s(y)) \Leftrightarrow \mu_{aP}(x) \leq \mu_{aP}(y)$,
(2) Since if z is a maximal for $\leq_{aP}$, then $s(z)$ is so for $\leq_P : \mu_{aP}(z) = \mu_P(s(z)) = 1$.
(3) Since if z is a minimal for $\leq_{aP}$, then $s(z)$ is so for $\leq_P$: $\mu_{aP}(z) = \mu(s(z)) = 0$.

Hence, the quantity (X, $\leq_P^{-1}$, $\mu_P \circ s$) specifies an antonym aP of P in X.

Notice that $\mu_{a(aP)} = \mu_{aP} \circ s_{aP} = (\mu_P \circ s_P) \circ s_{aP} = \mu_P \circ (s_P \circ s_{aP}) = \mu_P$ provided $s_P = s_{aP}$. Hence, the antonym of the antonym actually coincides with the original predicate provided the symmetries allowing to specify them can be taken as coincidental.

For instance, with *big* in [0, 10], it is $\mu_{small}(x) = \mu_{big}(10-x) = (10-x)/10 = 1-x/10$, with $s(x) = 10-x$ that verifies $s^2(x) = s(10-x) = x$, and $x \leq_{small} y \Leftrightarrow y \leq x \Rightarrow 10-x \leq 10-y \Leftrightarrow s(x) \leq s(y)$. With it, the maximum 10 for big is the maximum $s(10) = 0$ for *small*, and the minimum 0 for *big* is the minimum $s(0) = 10$ for small: $\mu_{small}(0) = 1$ and $\mu_{small}(10) = 0$. Since it can be taken $s_{small} = s_{big} = 10-\mathrm{id}$, it is a(small) = a (a(*big*)) = *big*.

Of course, with each symmetry s in X the use of an opposite is reached, and the question that remains is to know if these symmetries can be freely taken. The answer is not. Examples like

'If the bottle is full, it is not empty',

with 'full' the antonym of 'empty', reveal the non reversible rule

If x is aP, then x is *not* P,

that can be translated into membership functions by

$$\mu_{ap}(x) \leq \mu_{notP}(x), \text{for all x in X,}$$

or

$$\mu_{ap} \leq \mu_{notP} \Leftrightarrow \mu_P \circ s \leq \mu_{notP},$$

showing that the symmetry s should respect this inequality. In the extreme case in which aP coincides with *not* P and the sign equal holds in last functional inequality, it is said that aP is a *non-regular* antonym.

Notice that the rule is not reversible since it is not, for instance, 'If the bottle is not empty, it is full'; in general, with the exception of a non-regular antonym, it is not "If x is *not* P, then it is x is aP". This is the beforehand case of *big* and *small*, since with μ_{notP} (x) = 1−μ_P(x), is μ_{small} (x) = 1−x/10 = 1−μ_{big} (x) = μ_{notbig} (x).

Hence, supposing that *not* P is specified by

$$\mu_{notP}\,(x) = N(\mu_P\,(x)), \text{ for all x in X [7]},$$

with a numerical function N : [0, 1] →[0,1], there should be kept the inequality

$$\mu_P \circ s \leq N \circ \mu$$

constraining both mappings s and N.

4.2

For what concerns the negation *not* P = P' of P, the only that can be presupposed in general is that [4],

$$x \ \leq_{P'} \ y \ \Rightarrow y \leq_P \ x, \text{ that is}, \leq_{P'} \subseteq \leq_P^{-1} .$$

For instance, if it seems to be "x is less *not ho*t than y" if and only if "y is less hot than x", there are more sophisticated cases in which there is not equivalence between both conditionals since not always it is *not* (*not* P) coincidental with P, as it happens, for instance, when not all that can be qualified as *not* P is completely known. This is, in Prolog, the case of the 'negation by failure'. Provided the primary meaning of P' in X can be described by a graph (X, $\leq_{P'}$), are there effective measures of it? The answer is yes, but with some conditions on the maximals and the minimals.

Consider the mappings (*negation functions*) N: [0, 1] → [0, 1], such that,

1. N (0) = 1, and N (1) = 0
2. If a ≤ b, then N (b) ≤ N (a),

and define

$$\mu_{P'} = N_P \circ \mu_P,$$

with a suitable function N. Then,

(a) $x \leq_{P'} y \Rightarrow y \leq_P x \Rightarrow \mu_P(y) \leq \mu_P(x) \Rightarrow N_P(\mu_P(x)) \leq N_P(\mu_P(y)) \Leftrightarrow \mu_{P'}(x) \leq \mu_{P'}(y)$,
(b) Provided the minimals for $\leq_{P'}$ are maximals for $\leq_P$: $\mu_{P'}(z) = N_P(\mu_P(z)) = N_P(1) = 0$,
(c) Provided the maximals for $\leq_{P'}$ are minimals for $\leq_P$: $\mu_{P'}(z) = N_P(0) = 1$,

thus, under these hypotheses $N_P \circ \mu_P$ can be taken as a measure for P'. Notice that provided it were $\leq_{P'} = \leq_P^{-1}$, (b) and (c) will hold by the same reason they hold with the opposite.

Were the 'not' involutive, that is, P coincidental with *not* (*not* P), function N_P should allow to have $\mu_{(P')'} = N_{P'} \circ \mu_{P'} = N_{P'} \circ (N_P \circ \mu_P) = (N_{P'} \circ N_P) \circ \mu_P = \mu_P$, for which it suffices to have $N_{P'} \circ N_P = \mathrm{id}_{[0,1]}$, or $N_{P'} = N_P$. Those negation functions N verifying $N^2 = N \circ N = \mathrm{id}_{[0,1]}$, are called *strong negation functions*, or, for short, *strong negations*, and can be characterized in the form [7],

$$N = \varphi^{-1} \circ (1 - \mathrm{id}_{[0,1]}) \circ \varphi,$$

or

$$N(x) = \varphi^{-1}(1 - \varphi(x)), \text{ for all x in X},$$

with φ: [0, 1] → [0, 1], strictly non-decreasing and such that $\varphi(0) = 0$, and $\varphi(1) = 1$. That is, φ is an auto-morphism of the ordered unit interval ([0, 1], ≤) of the real line; obviously, strong negations are strictly decreasing functions and, hence, continuous functions. The auto-morphism φ is not unique, but there can be several of them giving the same strong negation.

Notice that negation functions can verify,

$$N \circ N \leq \mathrm{id}_{[0,1]}, \text{ or } \mathrm{id}_{[0,1]} \leq N \circ N,$$

and that among them the only continuous are the strong negations $N \circ N = \mathrm{id}_{[0,1]}$. The first negation functions are called 'ordinary' and the second 'weak'; of course, those that are both ordinary and weak are the strong ones. Provided N^2 were not comparable with the identity $\mathrm{id}_{[0,1]}$, the negation function will be considered a rare one.

With the order auto-morphisms,

$$\varphi(x) = ln\, (1 + \tau x^{\omega})^{1/\tau}, \tau > -1, \omega > 0,$$

where *ln* shortens logarithm in the base *e*, is obtained the bi-parametric family of strong negations

$$N(x) = (1 - x^{\omega}/1 + \tau x^{\omega})^{1/\omega}.$$

With $\omega = 1$ it appears the single-parameter family

$$N_\tau(X) = 1 - x/1 + \tau x, \tau > -1,$$

consisting of the only 'rational' strong negations, and called the Sugeno's family of strong negations. The only 'linear' strong negation is that given by $\tau = 0, N_0(\mathrm{x}) = 1 - \mathrm{x}$.

It should be pointed out that in fuzzy logic only strong negations are managed, since they never add discontinuities. Because of strong negations are continuous negation functions, if μ_P is continuous it is also so $\mu_{P'} = N \circ \mu_P$, and if μ_P is discontinuous at some points, $\mu_{P'}$ is only discontinuous at the same points.

A strong negation N has a single fixed point r ∈ (0, 1), that is, such that N (r) = r. To find it, it suffices to solve the equation $\varphi^{-1}(1-\varphi(\mathrm{r})) = \mathrm{r} \Leftrightarrow 1 = 2\varphi(\mathrm{r})$, or $\mathrm{r} = \varphi^{-1}(1/2)$, and it is r ≠ 0 and r ≠ 1, since N (0) =1, and N(1)=0. For instance, in the Sugeno's family it is: $\mathrm{N}_\tau(\mathrm{r}) = \mathrm{r} \Leftrightarrow 1 - \mathrm{r} = \mathrm{r}(1 + \tau \mathrm{r})$ if $\tau \neq 0$, and $1-\mathrm{r} = \mathrm{r}$ if $\tau = 0 \Leftrightarrow \mathrm{r}((1+\tau)^{1/2} - 1)/\tau$, if $\tau \neq 0$, and r = 1/2 if $\tau = 0$.

Hence, a fuzzy set μ is self-contradictory if and only if

$$\mu(x) \leq \mu'(x) = N(\mu(x)) \Leftrightarrow \mu(x) \leq \varphi^{-1}(1/2), \text{ for all x in X} \Leftrightarrow \mu \leq \mu^{-1}_{\varphi(1/2)}.$$

It follows that no self-contradictory fuzzy sets can have prototypes, although it can have anti-prototypes. It should be pointed out that this result concerns a given strong negation N, and that the fuzzy sets that are 'absolutely' non-self-contradictory, or non-self-contradictory for all strong negation, are those μ that, for any r ∈ (0, 1), there exist points x ∈ X such that μ(x) > r; for instance, those with prototypes also called *normalized* fuzzy sets.

Notice that since it is N (0) = 1, and N (1) = 0, provided it were μ_P the membership function of a crisp set **P**, it will be $\mu_{P'}$ (x) = N (μ_P(x)) ∈ {0, 1}, changing 0 in 1, and 1 in 0. That is, $\mu_{P'}$ is also the membership function of the crisp set $1 - \mu_P$, that is, $\mathbf{P}^c$. In this case, all the strong negations collapse in N_0.

Examples in X = [0,1]

1. Define μ_P^* (x) = 1−μ_P (1−x). It is easy to check that:
   - There is no negation function N such that $\mu_P^* = N$ o μ_P. That is, the operation * is not decomposable, or functionally expressible.
   - $\mu_0^* = \mu_1, \mu_1^* = \mu_0$
   - If $\mu_P \leq \mu_Q$, then $\mu_Q^* \leq \mu_P^*$
   - $\mu_P^{**} = \mu_P$
   - $\mu_P^*(\mathrm{x}) = 1 \Leftrightarrow \mu_P(1 - \mathrm{x}) = 0$.
   Hence, the mapping given by $\mu_P \rightarrow \mu_P^*$ is a 'symmetry' for fuzzy sets that seems to be a strong negation. But it is not so since it does not preserve the complements of crisp sets: if **P** is a crisp set in [0, 1], it is x ∈ $\mathbf{P}^c$ ⇔ 1−x ∈ $\mathbf{P}^*$, hence, for instance, it is $[0, 0.8]^c$ = (0.8, 1], but $[0, 0.8]^*$ = [0, 0.2). Consequently, the mapping * cannot define a negation for fuzzy sets.
   Notice that the fixed 'points' are those fuzzy sets μ such that μ(1−x) + μ(x) = 1 as, for instance, $\mu = \mathrm{id}_{[0,1]}$, $\mu = 1 - \mathrm{id}_{[0,1]}$, [1/3, 2/3], and [0, 1].

2. The interval **P** = [0.4, 0.7] specifies the precise predicate P = *between 0.4 and 0.7*, and its negation is specified by the complement **P'** = $[0.4, 0.6]^c$ = [0, 0.4] ∪ [0.7,

1]. Nevertheless, since μ_{aP} (x) = μ_P (1−x) =1 ⇔ 0.3 ≤ x ≤ 0.6, it is **aP** = [0.3, 0.6], not verifying **aP** ⊆ **P'** since, for instance, 0.5 is in **aP** but not in **P'**. Hence, either the predicate P lacks an antonym, or it should be searched by means of a symmetry different from s (x) = 1−x, or the negation of P, *not between 0.4 and 0.7*, should be taken as the opposite. For instance, provided the Sugeno's strong negation with $\tau = 2$ could be taken as a suitable symmetry, it is

$$\mu_{aP}(\mathrm{x}) = \mu_P(1 - \mathrm{x}/1 + 2\mathrm{x}) = 1 \Leftrightarrow x \in [1/8, 1/3],$$

gives **aP** = [1/8, 1/3] ⊆ **P'**, and it can be taken aP = *between 1/8 and 1/3.*
It is neither sure that a suitable functionally expressible symmetry $s : \mathrm{X} \to \mathrm{X}$ to express the opposite, nor a suitable functionally expressible negation function $N : [0, 1] \to [0, 1]$ to express the negation, can always exist. Anyway, if existing, as it is always supposed in the applications of fuzzy logic, they should verify the 'compatibility condition'

$$\mu \circ s \leq N \circ \mu.$$

Examples like the last one with *between 0.4 and 0.7*, suggest the necessity of doing some controlled experimentation in language for knowing which pairs (s_P, N_P) are suitable at each case; of course, the case of mathematical language is the less important one.

5 Conjunction and Disjunction

5.1

Let P and Q be two measurable predicates in the same universe of discourse X:
Provided 'P *and* Q', and 'P *or* Q', are defined by:

- x is (P *and* Q) ⇔ (x is P) and (x is Q)
- x is (P *or* Q) ⇔ (x is P) or (x is Q),

are they measurable predicates?
Let's try to answer this question on the following conditions,

(1) The primary meanings of 'P *and* Q', and 'P *or* Q', based on the primary meanings of P and Q, respectively, are known. It will suffice for it to find operations F and G such that

$$\leq_{PandQ} = \mathrm{F}(\leq_P, \leq_Q), \leq_{PorQ} = \mathrm{G}(\leq_P, \leq_Q).$$

(2) Once respective measures μ_P and μ_Q are given, it will suffice to find functions f and g, such that

$$\mu_{PandQ} = \mathrm{f}(\mu_P, \mu_Q), \mu_{PorQ} = \mathrm{g}(\mu_P, \mu_Q)$$

become measures for 'P *and* Q', and 'P *or* Q', respectively.

Functions f are called conjunctions, and functions g disjunctions.

5.2

Concerning the conjunction *and*, it seems to be also acceptable that at least [5], it is,

$$\leq_{PandQ} \subseteq \leq_P \cap \leq_Q,$$

with which it is

$$\mathrm{x} \leq_{PandQ} \mathrm{y} \Rightarrow \mathrm{x} \leq_P \mathrm{y} \,\&\, \mathrm{x} \leq_Q \mathrm{y}.$$

Then, it suffices to have monotonic binary operations $\cdot$ in [0, 1] with neutral 1 and absorbent 0, for continuing last formula by

$$\mu_P(\mathrm{x}) \leq \mu_P(\mathrm{y}) \,\&\, \mu_Q(\mathrm{x}) \leq \mu_Q(\mathrm{y}) \Rightarrow \mu_P(\mathrm{x}) \cdot \mu_Q(\mathrm{x}) \leq \mu_P(\mathrm{y}) \cdot \mu_Q(\mathrm{y}) \Leftrightarrow \mu_{PandQ}(\mathrm{x}) \leq \mu_{PandQ}(\mathrm{y}),$$

and the mapping defined by

$$\mu_{PandQ} = \mu_P \cdot \mu_Q$$

verifies the first property of a measure for 'P *and* Q'. Provided if z is a maximal for $\leq_{PandQ}$, it is so for both $\leq_P$ and $\leq_Q$, follows μ_{PandQ} (z) = 1.1 = 1. Provided z is a minimal for $\leq_{PandQ}$ it is so for, at least, one of the relations $\leq_P$, $\leq_Q$, and it is analogously μ_{PandQ} (x) = 0. That is, under some reasonable hypotheses, $\mu_P \cdot \mu_Q$ is a measure for 'P *and* Q'.

Remarks

(a) The only that has been proven here is that there are ways of expressing the behavior of 'P and Q' by a quantity; that on some conditions this predicate can be measurable. Of course, it does not mean that in all cases μ_{PandQ} will be functionally expressible by a numerical operation like the $\cdot$ before considered [26].
(b) The particular case in which $\leq_{PandQ} = \leq_P \cap \leq_Q$, assures that if $\leq_P$ and $\leq_Q$ are preorders also $\leq_{PandQ}$ is so.
(c) No hypotheses on the commutativity, associativity, etc., of the conjunction *and* are necessary for just studying its measurability.
(d) Obviously, provided the fuzzy sets **P** and **Q** are crisp sets, $\mu_P \cdot \mu_Q$ gives the crisp set **P** ∩ **Q**, since the restriction of the operation $\cdot$ to {0, 1} is min.

5.3

For what concerns the disjunction *or*, in the first place it should be noticed that, in general, it is not acceptable the identity $\leq_{PorQ} = \leq_P \cup \leq_Q$, since when both relations $\leq_P$ and $\leq_Q$ are preorders it can't be assured that $\leq_{PorQ}$ is so [5]. The only acceptable is the inclusion $\leq_P \cup \leq_Q \subseteq \leq_{PorQ}$, that shows $\leq_{PorQ}$ is not empty, and seems to suggest the identification of this last relation with the minimum one containing both $\leq_P$ and $\leq_Q$, that is, to identify $\leq_{PorQ}$ with the direct-sum, $\leq_P \bigoplus \leq_Q$, or closure of both relations that, if both are preorders, is also a preorder. In this form, and for instance, if there is a P-arc joining x and y, and a Q-arc joining y and z, there is a (P *or* Q)-arc joining x and z. Were this identification accepted, with any monotonic binary operation + in [0, 1] with neutral 0, and absorbent 1, it can be said:

- $x \leq_{PorQ} y \Leftrightarrow x \leq_P y$ or $x \leq_Q y \Rightarrow \mu_P(x) \leq \mu_P(y)$ or $\mu_Q(x) \leq \mu_Q(y) \Rightarrow \mu_P(x) + \mu_Q(x) \leq \mu_P(y) + \mu_Q(y)$.
- If z is a minimal for $\leq_{PorQ}$, and it is so for both $\leq_P$ and $\leq_Q$: $\mu_P(z) + \mu_Q(z) = 0 + 0 = 0$.
- If z is a maximal for $\leq_{PorQ}$, and it is so for, at least, one of the relations $\leq_P$ and $\leq_Q$: $\mu_P(z) + \mu_Q(z) = 1$+something, or something +1, or $1 + 1 = 1$.

That is, under some reasonable hypotheses, $\mu_P + \mu_Q$ is a measure for 'P *or* Q'.

Remarks

(a) The only that is proven here is that on some conditions from the measurability of P and Q, it follows that of 'P or Q'. What is not at all proven is that the measure for 'P or Q' should be always functionally expressible [26] from those of P and Q by a numerical operation like +.
(b) No hypotheses on the commutativity, associativity, etc., of the disjunction *or* are necessary for just studying its measurability.
(c) Obviously, provided the fuzzy sets **P** and **Q** are crisp sets, $\mu_P + \mu_Q$ gives **P** ∪ **Q**, since the restriction of + to {0, 1} is max.
(d) The operations · and + are mappings

$$[0, 1]^X \times [0, 1]^X \to [0, 1]^X,$$

corresponding, respectively, to the models · = min, and + = max, but not constrained by properties like $\mu \cdot \mu = \mu$, and $\mu + \mu = \mu$, typical of these two operations. As it is well known the family of t-norms, and that of t-conorms [8], verify the basic properties of both operations but adding those of commutativity, associativity, etc. , only necessary in some particular cases; among copulas [8] some less restrictive examples can be found.
Since it is not always the case that (x is P *and* P) does coincide with (x is P), it is convenient to have operations preserving such cases, as it is with the continuous t-norms called ordinal-sums [8] that, notwithstanding, are never used in the applications since they have not too much simple mathematical expressions.

The most popular continuous t-norms are, apart from min, the product (prod), the Lukasiewicz (W), and their φ-transforms:

- prod_φ (x,y) = φ^{-1} (prod (φ(x), φ(y))= $\varphi^{-1}(\varphi(x) \cdot \varphi(y))$,
- W_φ (x,y) = φ^{-1} (W (φ(x), φ(y)) = φ^{-1} (max(0, φ(x) + φ(y)-1),

for all auto-morphism φ of the ordered unit interval in the real line, like, for instance,

- $\varphi(x) = x^r$, r a rational number, with which it is $W_\varphi(x, y) = (\max(0, x^r + y^r-1))^{1/r}$, and prod_φ = prod.
- φ(x) = 2x/1+x, with which it is prod_φ (x, y) = 2xy/(1+ y−x + xy).

Notice that prod (μ_P(x), μ_P(x)) = $\mu_P(x)^2 \neq \mu_P(x)$, except if the value of μ_P(x) is 0 or 1. Analogously, W (μ_P(x), μ_P(x)) = max(0, $2\mu_P(x)-1) \neq \mu_P(x)$, except if the value of μ_P(x) is 0 or 1.

To know something on non decomposable connectives, and particularly on negations, see [26].

(e) It lacks to develop the study of the connectives *and, or*, when the predicates are acting on different universes of discourse.

6 Constrained, Qualified and Modified Predicates

6.1

Let (X, $\leq_P$, μ_P) and (Y, $\leq_Q$, μ_Q) two quantities representing the behavior of the measurable predicates P in X, and Q in Y.

Each non-empty relation

$$R(P, Q) \subseteq X(P) \text{ x } Y(Q) : (x \text{ is } P, y \text{ is } Q) \in R(P, Q),$$

allows to define the *constrained predicate* Q/P = ‘Q if P’, on X x Y, by [5],

$$(x, y) \text{ is } Q/P \Leftrightarrow (x \text{ is } P, y \text{ is } Q) \in R(P, Q),$$

of which an example is obtained when the first term, (x, y) is Q/P, is interpreted as the conditional statement ‘If x is P, then y is Q’.

Provided Q/P is measurable by means of a quantity (X x Y, $\leq_{Q/P}$, $\mu_{Q/P}$), could it be studied how to express $\mu_{Q/P}$ by means of μ_P and μ_Q?

A measure $\mu_{Q/P}$ is said to be functionally expressible or decomposable, if there exists a function

$$J: [0, 1] \times [0, 1] \rightarrow [0, 1],$$

such that, $\mu_{Q/P}$ (x,y) = J(μ_P(x), μ_Q(y)) for all (x, y) ∈ X x Y. Let's see which properties can be required for J in several cases.

(a) Provided it were $\leq_{Q/P} = \leq_P$ x $\leq_Q$, it will suffice that J be non-decreasing in its two variables to have,

$(x_1, y_1) \leq_{Q/P} (x_2, y_2) \Leftrightarrow x_1 \leq_P x_2$ & $y_1 \leq_Q y_2 \Rightarrow \mu_P(x_1) \leq \mu_P(x_2)$ & $\mu_Q(y_1) \leq \mu_Q(y_2) \Rightarrow J(\mu_P(x_1), \mu_Q(y_1)) \leq J(\mu_P(x_2), \mu_Q(y_2)) \Leftrightarrow \mu_{Q/P}(x_1,y_1) \leq \mu_{Q/P}(x_2,y_2)$.

For what concerns the minimals and the maximals, it suffices that J also verify J(0,0) = 1, and J(1,1) = 1. Since usually Q/P ≠ P/Q, J will be non-commutative in general; for instance,

$$J(x, y) = T(x^r, y),$$

with a t-norm T (min, prod, W, …), and r > 0, is an example for this case and the most used of these J in the applications of fuzzy logic, is with either T = min, or T = prod, and r = 1. The t-norm W, as well as their φ-transforms W_φ, are never used since they have zero-divisors: W_φ (a, b) = 0 ⇔ 1 ≤ φ(a) + φ(b), and then a rule 'If x is P, then y is Q' can verify the non desirable property $\mu_{Q/P}$(x, y) = J(μ_P(x), μ_Q(y)) = 0 with μ_P(x) > 0, and μ_Q(y) > 0: that is, the degree up to which the rule holds is zero with both its antecedent and consequent with a positive degree.

(b) Provided it were $\leq_{Q/P} = \leq_P$ x $\leq_Q^{-1}$, it will suffice that J be non-decreasing in its first variable and decreasing in the second to have:

$(x_1, y_1) \leq_{Q/P} (x_2, y_2) \Leftrightarrow x_1 \leq_P x_2$ & $y_2 \leq_Q y_1 \Rightarrow \mu_P(x_1) \leq \mu_P(x_2)$ & $\mu_Q(y_2) \leq \mu_Q(y_1) \Rightarrow J(\mu_P(x_1), \mu_Q(y_1)) \leq J(\mu_P(x_2), \mu_Q(y_2)) \Leftrightarrow \mu_{Q/P}(x_1,y_1) \leq \mu_{Q/P}(x_2,y_2)$.

For what concerns the minimals and the maximals, it suffices that J also verify J(0, 1) = J(1, 0) = J(1, 1) =1. An example is given by J(a, b) = max(a, 1−b).

(c) Provided it were $\leq_{Q/P} = \leq_P^{-1}$ x $\leq_Q$, it will suffice that J be decreasing in its first variable and non-decreasing in the second, with J(0, 0) = J(0, 1) = J(1, 1) = 1 and J(1,0) = 0. Example are given by J(a, b) = max(1−a, b), and J(a, b) = 1−a(1−b), that correspond to the classically called 'implication functions'. Another example is given by the functions J(x, y) = Sup {z ∈ [0, 1]; T(x, z) ≤ y} (called *residuated* implications), that are non functionally expressible unless if T = W_φ, and that come from the identity p + q' = Sup {t; p · t ≤ q} of the so-called classical material implication, that is only valid in complete Boolean algebras.

Etc.

6.2

Let it be (X, $\leq_P$, μ_P) the quantity describing the use, or meaning, of a predicate P in X, and the range $\mu_P(X) \subseteq [0, 1]$, as well as a measurable predicate τ acting on $\mu_P(X)$ under a relation $\leq_\tau = \leq$, and a measure μ_τ. Define the qualified predicate 'P is τ' in X, by [5],

$$\text{x is (P is } \tau) \Leftrightarrow \mu_P \text{(x) is } \tau,$$

and suppose $\leq_{Pis\tau} \subseteq \leq_P$.

On these conditions, $\mu_{Pis\tau} = \mu_\tau \circ \mu_P : X \to [0, 1]$, verifies:

$$x \leq_{Pis\tau} y \Rightarrow x \leq_P y \Rightarrow \mu_P(x) \leq \mu_P(y) \Rightarrow \mu_\tau(\mu_P(x)) \leq \mu_\tau(\mu_P(y)) \Leftrightarrow (\mu_\tau \circ \mu_P)(x) \leq (\mu_\tau \circ \mu_P)(y).$$

Provided the maximals and the minimals for $\leq_{Pis\tau}$ do coincide with those of $\leq_P$, and $\mu_\tau(0)=0$, $\mu_\tau(1)=1$, it follows the same corresponding values for them under $\mu_\tau \circ \mu_P$; in another case, it only holds that $\mu_{Pis\tau}$ is non-decreasing. Hence, on this condition, the quantity (X, $\leq_{Pis\tau}$, $\mu_\tau \circ \mu_P$) mathematically describes the use of 'P is τ' in X.

Examples

1. *Let P = small in X = [0, 10], with* $\leq_P = \geq$, *and* $\mu_P(x) = 1 - x/10$, *with which* $\mu_P(X) = [0, 1]$. *Let* τ *=large in [0, 1] with* $\mu_\tau = \leq$, *and* $\mu_\tau(x) = x$, *for all x in [0,1].*

   Then, 'P is τ*' = small is large, behaves with the measure:* $\mu_\tau(1-x/10) = 1-x/10$, *and it can be said that 'small is large' coincides with small.*
2. *In the former example, change X to [0, 1] with P = large, and* $\mu_P(x) = 1 - x$, *and Y to [0, 10] with* τ *= small, and* $\mu_\tau(x) = x/10$. *Then 'P is* τ*' = 'large is small' is represented in [0, 1] by* $\mu_\tau(\mu_P(x)) = (1 - x)/10$.

6.3

A *linguistic modifier* or *hedge* (**m**), is an adverb acting on the predicate P just in the form **m**P, not to be confused with 'P is **m**'. For instance, with P = *short* and **m** = *very*, it is **m**P = *very short*, that has nothing to do with the linguistic absurd 'short is very'.

A characteristic distinguishing imprecise from precise predicates is that once P and **m** are known, **m**P is immediately understandable. Instead, if P is precise, usually **m**P needs a new definition to be understood, as it happens, for instance, with *even* and *very even* in the set of integers. In principle, adverbs **m** modify but do not abruptly change the meaning of a imprecise predicate, and this is a simple test to identify if P is either precise or imprecise.

If P in X is with $\leq_P$ and μ_P, and **m** acts in $\mu_P(X) \subseteq [0, 1]$ with $\leq \subseteq \leq_m$ and a measure μ_m, provided it is $\leq_{mP} \subseteq \leq_P$, it can be taken $\mu_{mP} = \mu_m \circ \mu_P$, since [5],

$$x \leq_{mP} y \Rightarrow x \leq_P y \Rightarrow \mu_P(x) \leq \mu_P(y) \Rightarrow \mu_P(x) \leq_m \mu_P(y) \Rightarrow \mu_m(\mu_P(x)) \leq \mu_m(\mu_P(y)) \Leftrightarrow (\mu_m \circ \mu_P)(x) \leq (\mu_m \circ \mu_P)(y).$$

Then, provided there is agreement between the corresponding maximals and minimals, **m**P is measured by this composition of functions.

Remarks

(a) Two important types of modifiers are:

- *Expansive modifiers*, those such that $id_{[0,1]} \leq \mu_m$
- *Contractive modifiers*, those such that $\mu_m \leq id_{[0,1]}$.

With the expansive it results $\mu_P \leq \mu_{mP}$, and with the contractive $\mu_{mP} \leq \mu_P$. This is what happens with the Zadeh's definitions

$$\mu_{more\ or\ less}(a) = \sqrt{a},\ \mu_{very}(a) = a^2,$$

since in [0, 1] it is $a^2 \leq a \leq \sqrt{a}$.

(b) There are again other modifiers.

- Those called *internal modifiers*, like the antonym, in which the measure of P is modified by a symmetry in the universe of discourse.
- Those called *external modifiers*, like the negation, whose measure is modified by a negation function ranging in the set [0, 1] where membership functions take their values.

7 The Algebras of Fuzzy Sets

7.1

A function $\sigma \in [0, 1]^X$ only can represent a fuzzy set provided there is a predicate P acting in X, such that the quantity $(X, \leq_P, \mu_P)$ is with $\mu_P = \sigma$. Hence, the elements in the set $[0, 1]^X$ only 'potentially are fuzzy sets', analogously that the elements in the set $\{0, 1\}^X$ are sets in the power-set $\mathbf{2}^X$, by the equivalence:

$$\mathbf{A} \in \mathbf{2}^X \Leftrightarrow \text{It exists } \vartheta \in \{0,1\}^X \text{such that } \mathbf{A} = \vartheta^{-1}(1).$$

Thus, the consideration of the (abstract) algebraic structures that can be build up from $[0, 1]^X$, has sense for theoretically studying both the 'algebras' of fuzzy sets in themselves, and for doing practical computations once fuzzy sets are combined by operations like the former ·, + , and ’, introduced in the last section. The operations to be taken into account for obtaining such structures are traditionally fixed by analogy with what happens with sets, where the axiom of specification states that all precise predicates P, Q, etc., acting in X specify single subsets **P**, **Q**, etc., such that:

- If ‘x is P *and* Q’, it is x $\in$ **P** $\cap$ **Q**
- If ‘x is P *or* Q’, it is x $\in$ **P** $\cap$ **Q**
- If ‘x is *not* P’, it is x $\in$ $\mathbf{P}^c$
- If ‘If x is P, then x is Q’, it is **P** $\subseteq$ **Q**,

with $\mu_{P\cap Q} = \min(\mu_P, \mu_Q)$, $\mu_{P\cap Q} = \max(\mu_P, \mu_Q)$, and $\mu_P^c = 1 - \mu_P$. Then with fuzzy sets in $[0, 1]^X$, it will be:

(1) **P** $\cap$ **Q** (intersection or conjunction), defined by $\mu_{P\cap Q} = \mu_P \cdot \mu_Q$, with a suitable binary operation $\cdot : [0, 1]^X \text{ x } [0, 1]^X \to [0, 1]^X$
(2) **P** $\cup$ **Q** (union or disjunction), defined by $\mu_{P\cup Q} = \mu_P + \mu_Q$, with a suitable binary operation $+ : [0, 1]^X \text{ x } [0, 1]^X \to [0, 1]^X$
(3) $\mathbf{P}^c$ (pseudo-complement or negation), defined by $\mu_P^c = \mu_P^{'}$, with a suitable unary operation ‘: $[0, 1]^X \to [0, 1]^X$
(4) **P** $\subseteq$ **Q** (inclusion), defined by [$\mu_P \leq \mu_Q \Leftrightarrow \mu_P$ (x) $\leq \mu_Q$ (x), for all x in X].

Hence, the algebras of fuzzy sets are five-tuples

$$([0, 1]^X, \leq; \cdot, +, \}),$$

restricted to verify some laws linking ≤, ·, +, and ‘. Although current fuzzy logic mainly works under the hypothesis of decomposability, or functional expressibility of the three operations ·, +, and ‘, it is not at all clear that such hypothesis can be generally acceptable. This hypothesis has, notwithstanding, the great advantage of allowing to translate into functional equations the study of the laws. For instance, denoting by μ_r the ‘constant’ fuzzy sets the possible law of non-contradiction,

$$\mu \cdot \mu' = \mu_0,$$

that with sets is **P** $\cap$ $\mathbf{P}^c = \emptyset$, is translated into the functional equation F(μ(x), μ’(x)) = 0, for all μ and x, and it suffices to solve the equation F(a, N(a)) = 0, provided there exists numerical functions F: $[0,1] \times [0, 1] \to [0, 1]$, and N: $[0,1] \to [0, 1]$ such that

$$(\mu \cdot \sigma)(\text{x}) = \text{F } (\mu(\text{x}), \sigma(\text{x})), \text{ and } (\mu')(\text{x}) = N(\mu(\text{x})),$$

for all x in X. Of course, to solve such functional equation it is necessary to count with some properties of F and N, among which continuity is essential. For instance, if F is a continuous t-norm and N a strong negation, the solutions are F = W_φ, and N $\leq N_\varphi$. Analogously, the validity of the possible distributive law $\mu \cdot (\sigma + \gamma) = \mu \cdot \sigma + \mu \cdot \gamma$, particularized with sets to **P** $\cap$ (**Q** $\cup$ **R**) = (**P** $\cup$ **Q**) $\cup$ (**P** $\cap$ **R**), can be studied by solving the functional equation F (a, G (b, c)) = G (F (a, b), F (a, c)), provided it exists a second numerical function G: [0, 1] x]0,1] → [0, 1], such that

$$(\mu + \sigma)(x) = G(\mu(x), \sigma(x)),$$

for all x ∈ X. In the case F is non-decreasing in both variables, a solution is given by G = max, regardless of which operation · is taken. The other distributive law $\mu + (\sigma \cdot \gamma) = (\mu + \sigma) \cdot (\mu + \gamma)$, holds with F = min, regardless of G but provided it is non-decreasing; of course, both laws jointly hold whenever · = min and + = max.

In the same vein, the validity of the possible laws

$$\mu \cdot \mu = \mu + \mu = \mu,$$

coming from the classical **P** ∩ **P** = **P** ∪ **P** = **P**, is reduced to solve the equations F(a, a) = a, and G(a, a) = a, with obvious solutions F = min and G = max.

In the applications it is necessary to count with operations verifying some of these laws. If, for instance, it is known that in the current universe of discourse it is always "'x is P' or 'x is not P'", the operation + should be chosen among those for which the law $\mu + \mu' = \mu_1$ holds. On the contrary, for just theorizing and obtaining results of a general validity, it is better to keep the algebra ($[0, 1]^X$, ≤; · , +, ') with a minimal number of laws, even if sometimes more laws allowing to obtain new properties should be added.

Notice that the relation ≤ inherits all the properties of the ordering of the unit interval, except linearity. It is a reflexive, anti-symmetric and transitive relation for which there are fuzzy sets that are not comparable, that cannot be linked by it; is a partial order. Hence ($[0, 1]^X$, ≤) is a poset whose minimum element is μ_0, and whose maximum is μ_1, that is $\mu_0 \leq \mu \leq \mu_1$, for every fuzzy set μ.

7.2

By definition, a **Basic Algebra** of fuzzy sets (BA) [9], $\Delta = ([0, 1]^X, \leq; \cdot , + , ')$, verifies the following laws or axioms:

(1) If $\mu \leq \sigma$, then $\sigma' \leq \mu'$; $\mu_0' = \mu_1$; $\mu_1' = \mu_0$
(2) $\mu \cdot \mu_0 = \mu_0 \cdot \mu = \mu_0$; $\mu \cdot \mu_1 = \mu_1 \cdot \mu = \mu$
$\mu + \mu_0 = \mu_0 + \mu = \mu$; $\mu + \mu_1 = \mu_1 + \mu = \mu_1$,
for all μ in $[0, 1]^X$.
(3) If $\mu \leq \sigma$, then: $\mu \cdot \delta \leq \sigma \cdot \delta$; $\delta \cdot \mu \leq \delta \cdot \sigma$; $\mu + \delta \leq \sigma + \delta$; $\delta + \mu \leq \delta + \sigma$,
for all δ in $[0, 1]^X$,
(4) In $\Delta_0 = (\{0, 1\}^X, \leq; \cdot , + , ')$ it is
$\mu \cdot \sigma = \min(\mu, \sigma)$; $\mu + \sigma = \max(\mu, \sigma)$; $\mu' = 1 - \mu$,
for all $\mu, \sigma \in \{0, 1\}^X$.

Because of the few and weak laws a BA enjoys, only the following properties can be deductively proven.

1. Δ_0 is isomorphic to the power-set $\mathbf{2}^X$ once endowed with the classical operations of intersection (∩), union (∪), and complement (^{c}), μ_0 corresponds to the empty set Ø, and μ_1 to the total X. Δ_0 is a Boolean algebra.
2. Only with $\cdot$ = min and + = max, is Δ a lattice. Provided the negation is strong (μ''= μ), Δ is a De Morgan-Kleene algebra since, in this case, it holds $\mu \cdot \mu' \leq \sigma + \sigma'$, for all μ, σ in $[0, 1]^X$, and Δ_0 consists in the Boolean elements of Δ. Hence, a BA is neither a Ortho-lattice, nor a *fortiori* is a Boolean algebra.

3. It is always $\mu \cdot \sigma \leq \min(\mu, \sigma)$, and $\max(\mu, \sigma) \leq \mu + \sigma$.

   Hence, it is:
$$\mu \cdot \sigma \leq \mu \leq \mu + \sigma; \mu \cdot \sigma \leq \sigma \leq \mu + \sigma.$$

   In particular, it holds $\mu \cdot \mu \leq \mu \leq \mu + \mu$, for all fuzzy set μ.
4. It holds:
   $\mu \leq \sigma$ & $\alpha \leq \beta \Rightarrow \mu \cdot \alpha \leq \sigma \cdot \beta$, $\mu + \alpha \leq \sigma + \beta$.
5. It holds: $\mu' \cdot \sigma' \leq (\mu \cdot \sigma)'$; $(\mu + \sigma)' \leq \mu' + \sigma'$.
6. If + = max, and regardless of $\cdot$ and ', it holds:

   $(\mu + \sigma)' \leq \mu' \cdot \sigma'$.

   If $\cdot$ = min, and regardless of + and ', it holds:

   $(\mu \cdot \sigma)' \leq \mu' + \sigma'$.

7. With $\cdot$ = min, and regardless of +, it holds:

   $\mu + \sigma \cdot \gamma = (\mu + \sigma) \cdot (\mu + \gamma)$.

8. With + = max, and regardless of $\cdot$, it holds:

   $\mu \cdot (\sigma + \gamma) = \mu \cdot \sigma + \mu \cdot \gamma$.

9. With the operations: $\mu \wedge \sigma = (\mu' + \sigma')'$, and $\mu \vee \sigma = (\mu' \cdot \sigma')'$, also $\Delta^* = ([0, 1]^X, \leq; \wedge, \vee, ')$ is a BA called the 'dual' of Δ, and that inherits all the additional properties enjoyed by Δ.

Remark Since BAs are defined by just a few and weak laws or axioms, only very simple calculations are allowed in them, but what can be proven in their framework is of a so very general character that will be preserved in case of considering operations $\cdot$, +, and ' , with more properties than those just defining the BA. Since no universal algebra of fuzzy sets can hold for all applications in all domains, as it is with classical sets, it seems suitable the name 'basic fuzzy algebra of fuzzy sets' that allow to add those properties that can be required at each case as it is, for instance, the idempotency of the conjunction.

One of the weaknesses of Boolean and De Morgan algebras, as well as of Ortho-modular lattices for the study of natural Language and ordinary reasoning, lies in the big amount of axioms they enjoy and that make such algebraic structures too rigid to afford the typical flexibility of human's language and reasoning. These structures own laws, like the commutative of ·, not always are in language where, for instance, 'time' often causes the failure of the commutative property for the conjunction 'and'. The world of human's language and reasoning is very different of those of artificial languages and formal deductive reasoning.

At this respect, the Standard algebras of fuzzy sets [11] (those in which F is a continuous t-norm, G a continuous t-conorm, and N a (continuous) strong negation), show a lot of cases in which some Aristotelian forms are valid in one but not in another algebra. For instance, the classical laws of duality,

$$(\mu \cdot \sigma)' = \mu' + \sigma', \text{ and } (\mu + \sigma)' = \mu' \cdot \sigma',$$

are broken when the t-norm giving the conjunction, and the t-conorm giving the disjunction, are not duals with respect to the strong negation giving the pseudo-complement, as it is, for instance, with $+ = \max$, and $\cdot = \text{prod}$. It is also the case that the von Neumann law of 'perfect repartition', $\mu = \mu \cdot \sigma + \mu \cdot \sigma$', only holds provided the t-norm and the t-conorm are not dual [11].

In the words of Satosi Watanabe [12], a new land can be perhaps offered to young researchers, provided "a direct contact with the world of common sense which is the mother earth of all knowledge" is not disdained at all.

8 Complex Measures and Type-Two Fuzzy Sets

8.1

A practical problem, that often appears with Zadeh's fuzzy sets $\mu \in [0, 1]^X$, is when their values μ(x) in [0, 1] only can be computed approximately. For instance, if it is known that μ(x) approaches $\sqrt{2}/2$ then, depending on the approximation needed by the problem at hand, it could be taken a value either between 0.70 and 0.71, or between 0.706 and 0.707. There is almost always a problem of uncertainty in the practical management of fuzzy sets.

This problem causes a theoretical trouble since if, for instance, two fuzzy sets μ and σ are such that $\mu(x) = \sigma(x)$ for all x in X-$\{x_0\}$, and $\mu(x_0) < \sigma(x_0)$, it is not $\mu = \sigma$, but $\mu \leq \sigma$ with $\mu \neq \sigma$. If X is with a large number of elements, X = $[0, 10^{10}]$ for instance, the same comes if the difference between μ and σ is limited to a few (say, eight hundred) points in X. It happens analogously when operating with several fuzzy sets; for instance, with X = [0, 1], μ(x)= x, $\sigma(x) = x^2$, and $\cdot$ = product, it is $(\mu \cdot \sigma)(x) = x^3$, and taking the rounding $\mu(\pi/4) = 0.7854$, and the rounding $\sigma(\pi/4) = 0.6169$, the value $(\mu \cdot \sigma)(\pi/4)$ will charge its rounding with the rounding of $\pi/4$ and

$\pi^2/4$. Of course, this is nothing strange in practical computations where the analysis of errors is well known. Nevertheless, the situation becomes worst when the value μ(x) is just appreciated empirically without a previous model for the function μ, in which case the, let's say, naked eye of the expert, only can appreciate that the value μ(x) is between values μ_1(x) and μ_2(x). For these reasons it could be suitable to change the unit interval to a set of intervals whose extremes belong to [0, 1].

There is another problem suggesting a change from Zadeh's fuzzy sets to functions whose values do not belong to the linearly ordered unit interval. If the qualitative or primary meaning of P in X, is represented by a graph (X, $\leq_P$) and the relation $\leq_P$ is not linear, as it often happens, once a measure μ_P is specified, the 'working use' given by $\leq_{\mu_P}$ cannot coincide with the primary meaning $\leq_P$, but such coincidence could be just possible provided $\leq_{\mu_P}$ were not linear. Something that only can come from re-defining fuzzy sets as functions taking their values in a non-linearly ordered set. For instance, provided $\leq_P$ is a preorder, $=_P$ is an equivalence, the quotient set inherits the partial order $\leq_P^*$ between classes, and the measure $\leq_P^*$ is that mentioned in Sect. 2, it is:

$$x \leq_P y \Leftrightarrow [x] \leq_P^* [y] \Leftrightarrow \mu_P^*(x) \leq_P^* \mu_P^*(y),$$

showing that $\leq_{\mu_P^*}$ perfectly reflects the primary meaning of P in X. This is at the cost of avoiding a 'numerical' measuring of the extent up to which 'x is P', and substituting it by a 'qualitative' type of measuring [32].

An additional view for changing the range [0, 1] by a partially ordered range, is that with the first the definition of a fuzzy set μ_P is too 'crisp', since very small variations of some few values μ_P(x) change the fuzzy set to another one, as it is shown at the beginning of this section, and of which there is no reason whatsoever for believing that it does not represent a measure of the same predicate.

8.2

Without avoiding numerical measuring, but taking care of the cases like those in Sect. 8.1, consider the complex unit square,

$$\mathbf{C} = \{a + ib; 0 \leq a \leq 1 \;\&\; 0 \leq b \leq 1\},$$

on which some well known measures used in both science and technology take their values. With the usual ordering given by

$$a + ib \leq a^* + ib^* \Leftrightarrow a \leq a^* \;\&\; b \leq b^*,$$

($\mathbf{C}$, $\leq$) is a poset with minimum $\mathbf{0} = 0 + i0$, and maximum $\mathbf{1} = 1 + i$.

If P is used on X with a primary meaning given by $\leq_P$, a *complex measure* of the extent up to which x is P [4], is a mapping

$$\mu_{P}: X \to \mathbf{C}, \mu_P(x) = \mu_P^1(x) + i\mu_P^2(x) \in \mathbf{C},$$

verifying the axioms,

1. If $x \leq_P y$, then $\mu_P(x) \leq \mu_P(y)$
2. If z is a maximal under $\leq_P$, it is $\mu_P(z) = \mathbf{1}$
3. If z is a minimal under $\leq_P$, it is $\mu_P(z) = \mathbf{0}$

and the complex quantity $(X, \leq_P, \mu_P)$ can be taken to represent the meaning of P in X. Now the 'working use' $\leq_{\mu_P}$ is given by:

$$\mathbf{x} \leq_{\mu_P} y \Leftrightarrow \mu_P(x) \leq \mu_P(y),$$

verifying $\leq_P \subseteq \leq_{\mu_P}$, and even without being sure that the equal sign can hold between both relations, but since the second is not linear, there are more possibilities of coincidence whenever the first is not linear. Of course, were $\leq_P$ linear then no coincidence is possible. Notice that μ_P^1 and μ_P^2 are in $[0, 1]^X$. With complex measures it is possible to consider fuzzy sets as functions in $\mathbf{C}^X$, and translating into it the same axioms defining a BA.

For instance, instead of the former fuzzy set μ with value $\sqrt{2}/2$ in some point, it can be taken a complex one μ whose value in the same point is the complex number $0.70 + 0.71i$. When in praxis it is not possible to determine values of μ others than with a certain approximation in the form $\mu(x) \in [\mu_1(x), \mu_2(x)]$, it can be appropriate to take a measure of the type $\mu(x) = \mu_1(x) + i\mu_2(x)$.

Notice that the set of the closed sub-intervals of [0, 1] may be endowed with the order of **C**, and with it both sets, those of the sub-intervals of [0, 1] and **C**, are posets with [0, 0] the minimum interval, and [0,1] the maximum one. Let X be the set of those sub-intervals, and P a predicate on X such that

$$[a_1, b_1] \leq_P [a_2, b_2] \Leftrightarrow a_1 \leq a_2 \,\&\, b_1 \leq b_2,$$

like, for instance, P = *to the left*, with $\leq_P$ indicating '*less to the left than*'. The function

$$\mu([a, b]) = b + i\,(a+b),$$

verifies:

(1) $[a_1, b_1] \leq_P [a_2, b_2] \Rightarrow a_1 \leq a_2 \;\&\; b_1 \leq b_2 \Rightarrow a_1 + b_1 \leq a_2 + b_2 \;\&\; b_1 \leq b_2 \Rightarrow b_1 + i(a_1 + b_1) \leq b_2 + i(a_2 + b_2) \Leftrightarrow \mu([a_1, b_1]) \leq \mu([a_2, b_1])$.
(2) $([0, 0]) = 0 + i0$
(3) $([0,1]) = 1 + i\,(0+1) = 1 + i$,

and, thus μ is a complex measure of P. Notice that if $b_2 - b_1 = r$, and also $a_2 - a_1 = s$, it is $\mu([a_2, b_2]) - \mu([a_1, b_1]) = (b_2 - b_1) + i(a_2 - a_1 + b_2 - b_1) = r + 2si$; hence, provided $r \leq e$ and $s \leq e$, it is $r + 2si \leq e(1 + 2i)$.

Remarks

(a) The similarity between interval-valued and complex-valued fuzzy sets, reveals that the more used case of the so-called 'type-two fuzzy sets' [13], the interval-valued ones, are nothing else than complex fuzzy sets.

(b) For what concerns true type-two fuzzy sets, that is those corresponding to predicates whose primary meaning only can be measured linguistically like, for instance, 'the degree up to which x is P is *high*, or $\mu_{high}(x)$', they correspond to measures valued in $[0, 1]^X$, that is, mappings μ_P: X → $[0,1]^X$. Since ($[0, 1]^X$, ≤) is a poset with minimum μ_0 and maximum μ_1, the definition of the three axioms of a measure can be immediately reproduced. Since the 'working meaning' $\leq_{\mu_P}$ will not be linear, type-two fuzzy sets also can offer more opportunities to perfectly reflect not linear primary meanings. Although the management of true type-two fuzzy sets is more complicated than that of fuzzy sets, there is in their favor the fact that fuzzy sets are, for what has been said, too crisp.

9 Conclusions

9.1

In science, and aside of its informal common use, the word 'truth', whose proper setting of study as a concept is philosophy, deserves no particular attention. What follows concerns some argumentation on why this is that.

'Truth' is an abstract concept whose mother-predicate is *true*, and that is applied to statements in the form "'x is P' is *true*". Thus, the minimal universes of discourse to be considered for beginning with an analysis of *true* are the sets X(P) = {x is P; x ∈ X}.

Provided T = *true* is measurable, that is, a graph (X(P), $\leq_T$) is known, the problem for grasping the full meaning of T in X(P), lies in obtaining a measure for it, that is, a mapping μ_T: X(P) → [0, 1] verifying the three axioms of a measure in the graph:

(1) x is P is less T than y is P ⇔ x is P $\leq_T$ y is P ⇒ μ_T(x is P) ≤ μ_T(y is P).
(2) 'z is P' is maximal under $\leq_T$ ('z is P' is completely true) ⇒ μ_T (z is P) = 1
(3) 'z is P' is minimal under $\leq_T$ ('z is P' is completely not true) ⇒ μ_T (z is P) = 0.

Since T is applied to the statements qualifying by P the elements in X, provided it were (x is P) $\leq_T$ (y is P) ⇒ x $\leq_P$ y, with the same maximals and minimals for both relations, it is sufficient to express μ_T through an order auto-morphism φ: [0, 1] → [0, 1]:

$$\mu_T \text{ (x is P)} = \varphi(\mu_P(x)),$$

since:

(1) (x is P) $\leq_T$ (y is P) $\Rightarrow$ x $\leq_P$ y $\Rightarrow$ μ_P(x) $\leq$ μ_P(y) $\Rightarrow$ $\varphi(\mu_P(\text{x})) \leq \varphi(\mu_P(\text{y}))$ $\Leftrightarrow$ μ_T(x is P) $\leq \mu_T$ (y is P)
(2) If 'z is P' is a maximal, μ_T (z is P) = φ (1)=1
(3) If 'z is P' is a minimal, μ_T (z is P) = φ (0)=0.

Of course, the specification of μ_T depends on that of φ that will come from the contextual information on the considered case. When it can be accepted $\varphi = \text{id}_{[0,1]}$, it results μ_T (x is P) = μ_P(x), that is what is classically done by accepting that the degree up to which x is P is *true* coincides with the degree up to which x is P. Nevertheless, if it should be taken, for instance, $\varphi(\text{a}) = \text{a}^2$, or $\varphi(\text{a}) = \sqrt{a}$ it will result, respectively, μ_T (x is P) $\leq$ μ_P(x), or μ_T (x is P) $\geq$ μ_P(x).

The opposite of *true* is *false*. Hence, for each symmetry s in X it can be defined μ_{false} (x is P) = φ (μ_P(s(x)). In language, the concept of truth is actually managed through the linguistic variable [29] with principal terms *true* and *false*: *{true, false, not true, not false, fairly true, fairly false, …}*.

9.2

The use of the predicate *probable* [14, 30], often appearing in the language of science and actually used in it, deserves some attention since its use is only well known in the setting of Ortho-modular lattices and, more particularly in Boolean algebras, with the Kolmogorov's type definition of a measure of probability once applied to 'events' represented by the elements in the Ortho-modular or Boolean lattice.

It should be noticed that the consideration of 'events' in a lattice, forces that the conjunction, disjunction and negation of the statements naming the events, should be represented, respectively, by a unique intersection, union and complement, provided the statements are precise. This implies a limitation on the use of 'probability' in language where imprecision and uncertainty are pervasive. For instance, this model allows to speak of the probability of extracting a ball from a urn with fifteen balls of which six are red and nine are blue, but it is not applicable when concerning a urn containing around a dozen of balls of which a few are red and several are blue, a case that can happen when the urn is observed at some distance by naked eye and it is impossible to know the exact numbers of balls.

In language, *probable* is directly applied to usual statements like 'Is probable that John is rich'. To do this, and to calculate a measure of how probable 'John is rich' is, it is necessary to grasp the meaning of P = *probable*, that is, to know a graph (X, $\leq_P$), and a measure μ_P. But, in science the universe of discourse is supposed to be an Ortho-modular or a Boolean lattice where the relation $\leq_P$ is understood as the partial order $\leq$ of the lattice [a $\leq$ b $\Leftrightarrow$ a $\cdot$ b = a $\Leftrightarrow$ a + b = b], with which it seems evident that a $\leq$ b implies a $\leq_P$ b, even if the reciprocal is not so evident, and hence the equality $\leq_P$ = $\leq$, or $\leq_P$ = $\subseteq$ in the case of a Boolean algebra of sets, is just a hypothesis.

A measure of probability is a mapping p: LE(X) → [0, 1], where LE(X) is the Ortho-modular lattice of events, verifying the axioms:

1. If x ≤ y', then p (x + y) = p(x) + p(y)
2. p(**1**) = 1,
   from which it follows,
3. p(**0**) = 0
4. p(x') = 1− p(x), for all x in LE(X)
5. If x ≤ y, then p(x) ≤ p(y),

with · , +, and ', the three lattice operations, ≤ the 'natural' order of the lattice, **1** its maximum, and **0** its minimum. Notice that just in the case of a Boolean algebra it is [x ≤ y' ⇔ x · y = **0**]. All that allows to state that p is a measure of *probable* when the relation $\leq_{probable}$ coincides with the order of the universe that is either a proper Ortho-modular lattice, or a Boolean algebra in particular, and it can be proved as follows: Since in Ortho-modular lattices hold,

$$x \leq y \Leftrightarrow y = x + x' \cdot y, \text{ with } x \leq x + y' = (x' \cdot y)',$$

provided it were x ≤ y, it will follow p(y) = p (x + x'· y) = p(x) + p(x' · y) ≥ p(x).

For weak structures, like the BA with · = min, + = max, and ' = $1-id_{[0,1]}$, there is only an almost satisfactory theory [16] with which the (numerical) probability of a fuzzy event can be computed, but until now there is no satisfactory theory for less restrictive BA. It still lacks a clear theory in which the probability can take its values in $[0, 1]^X$, a theory that is necessary for a good understanding of the cases in language in which [15] it is said, for instance, "the probability that John is rich is very low" That is, a theory of linguistic probabilities.

Analogously to probability, the concepts of possibility, necessity, uncertainty and imprecision, coming respectively from the mother-predicates possible, necessary, uncertain and imprecise, still deserve a deep theoretical consideration through its uses in language [17, 18].

9.3

As it was pointed out in Sect. 2, to specify a measure μ_P for a primary use $\leq_P$ of a predicate P in X, some additional information is needed since the three axioms defining a measure are not sufficient to individuate a single measure. This information only can come from the concrete use of P in X, that is, from the context surrounding such use in which the purpose for it often plays an important role. For instance, and even if the primary meaning can be fixed, the measure of *old* will not be the same if the contextual information is that *old* is used by people in a primary school's class, in a university class, or in an insurance company. It also will not be the same provided the purpose of using *old* is purely descriptive, or it is in a fun conversation. All this information, jointly with that furnished by the relation $\leq_P$, is actually summarized in the measure μ_P. Hence and for imprecise predicates, the quantity (X, $\leq_P$, μ_P) is context-dependent and purpose-driven. The specification of a predicate is strictly related with what surrounds the predicate's use in the universe of discourse, and except if the predicate is precise, or definable by means of 'if and only if' conditions, the specification is never unique. Up to some extent it happens analogously with the connectives, the qualified predicates and the modifiers. All of them should be specified accordingly with the contextual information available on the corresponding problem.

Hence, solving a problem of fuzzy logic requires a correct *design* of all the terms to be represented [19, 20], and specially when the problem concerns a dynamical system. If from a theoretical view, fuzzy logic is a matter of degree, from a practical point of view it is a matter of design. To wrongly design a single term, could imply to pose in fuzzy terms a system different from the current one and whose solutions rarely will keep any actual relation with it. What it usually does a designer of fuzzy systems?

In a first place, the designer should work with the information actually available on the involved terms that, more often than not, can be incomplete and, hence, neither the total relation $\leq_P$, nor a satisfactory measure μ_P will be obtained: The designed fuzzy set μ_P is no more than an approximation to the measure, with an unknown degree of approximation and giving an also approximate working meaning $\leq_{\mu_P}$.

In a second place, the designer can be forced to add some working hypotheses that can produce distortions in a model. For example, and as it is often the case, if the designer accepts a piecewise linear character for the involved membership functions, and, for the modifier *very* accepts that it squares the membership function, non piecewise linear functions are automatically introduced contrarily to the initial hypothesis. Another hypothesis that is always accepted without any kind of testing is that all the rules linguistically describing the behavior of a dynamical system are represented by the same function J of those in Sect. 6.1.

The practitioner of fuzzy logic should have a good mastering of the design of systems involving imprecision, and this requires a sufficient knowledge of the several mathematical models that can be employed. Fuzzy systems should be carefully designed.

9.4

Many questions still deserve a more deep study, and several are not just of a purely theoretical character since their possible answers require a previous checking within language and reasoning before being accepted (even in the provisory form typical of science). Provided some of these answers come from a mathematical model, can it be expected to found such model by just purely mathematical thinking?, or, should it come from abstraction on the information furnished by reality?, that is, from data captured through controlled processes of observation and experimentation within language and reasoning. The concept of a fuzzy set, once seen as a state of the collective generated in language by its predicate or linguistic label, seems to be more a scientific object than a purely logical one; and scientific objects are those typically analyzed by science. Which should be the best working methodology to follow: just theoretic thinking, like that of pure mathematics often based in second order abstractions and at once responding to theoretic interests, or mixed experimental and theoretic thinking like that of physics, where theoretic models should be tested against an observed reality?

Is it possible to conduct a mathematical analysis of language and reasoning without previously considering the nuances, dynamism and variability of the natural matter for such study? And, is not natural language plus common reasoning such natural matter? Possibly fuzzy logic needs to evolve towards an experimental science of 'language and reasoning' with a methodology analogous to that of physics; towards a kind of physics of language and reasoning, but with a working methodology based on computer science [24].

Such a scientific-like modeling of language and reasoning, jointly with that on how the human brain actually works, is one of the big challenges for science in the XXI Century, and the starting point for both the continuation of the Leibniz's 'Calculemus', and the real possibility of having machines thinking like people. Such a new way of studying imprecision and uncertainty in natural languages and common reasoning, without missing that of ambiguity, is what the author hopes it will come from the right now fifty years old fuzzy sets. At the end, it is with fuzzy logic that mathematical analysis actually begun into use for representing language and inference as people do. Like it happened before with the natural sciences where mathematics shown, in the words of Eugene Wigner, 'an unreasonable effectiveness' [21].

To end, let's say that in the author's view, Zadeh's Computing with Words [22, 25], should be theoretically re-modeled by, for instance, departing from the view of fuzzy sets presented in this paper, and being closely connected with natural language and common reasoning. It is a nice challenge for young researchers to whom the author wishes the ability of posing good questions conducting to fertile solutions like professor Zadeh's work is full of.

Acknowledgments This paper is partially funded by the Foundation for the Advancement of Soft Computing (Asturias, Spain), and by the Government of Spain Project MICIIN/TIN 2011-29827-C02-01.

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Author Biography

Enric Trillas Ph.D. in Mathematics by the University of Barcelona, got a chair in Mathematics at the Technical University of Catalonia in 1974, where he served as a full professor up to 1988 when he moved to the Department of Artificial Intelligence at the Technical University of Madrid. From 2006 is an Emeritus Researcher at the European Centre for Soft Computing, Mieres (Asturias) Spain. Formerly working in Ordered Semigroups and Generalized Metrics, from 1976 is active in Fuzzy Logic, having received several international distinctions and honors, as well as an H.C. Doctorate by the Public University of Navarre, and published over 400 papers in National/International journals, conferences and edited books. His work in Fuzzy Logic began by studying fuzzy entropies but, early in 1979, shifted to non-distributive logical connectives, like it were the characterization of strong negations, the introduction of t-norms and t-conorms, the first analysis of the 'Modus Ponens' inequality, and the introduction of T-Indistinguishabilities. After 1999 he mainly works in the analysis of conjecturing, representing the meaning of words, and the linguistic roots of fuzzy sets. Between 1983 and 1996 served in the Spanish Government, and holds several decorations from Spain and other countries.

Type 1 and Full Type 2 Fuzzy System Models

I. Burhan Türkşen

Abstract We first present a brief review of the essentials fuzzy system models: Namely (1) Zadeh's rulebase model, (2) Takagi and Sugeno's model which is partly a rule base and partly a regression function and (3) Türkşen fuzzy regression functions where a fuzzy regression function correspond to each fuzzy rule. Next we review the well known FCM algorithm which lets one to extract Type 1 membership values from a given data set for the development of Type 1 fuzzy system models as a foundation for the development of Full Type 2 fuzzy system models. For this purpose, we provide an algorithm which lets one to generate Full Type 2 membership value distributions for a development of second order fuzzy system models with our proposed second order data analysis. If required one can generate Full Type 3,…, Full Type n fuzzy system models with an iterative execution of our algorithm. We present our application results graphically for TD_Stockprice data with respect to two validity indeces, namely: (1) Çelikyılmaz-Türkşen and (2) Bezdek indeces.

1 Fuzzy System Models

Here first, historically significant fuzzy system model developments are reviewed in order to identify their unique structures and to point out how they differ from each other. Then we show the details of our FULL TYPE 2 Fuzzy System developments with a new algorithm.

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_30

2 Type 1 Fuzzy Rule Base Models

The most commonly applied fuzzy system models are fuzzy rule bases. Here, we only deal with Multi-Input Single Output (MISO) systems. Generally fuzzy system models represent relationships between the input and output variables which are expressed as a collection of IF-THEN rules that utilize linguistic labels, which are represented with fuzzy sets. The general fuzzy rule base structure which is known as Zadeh- Fuzzy Rule Base, Z-FRB, can be written as follows:

$$R\colon \underset{\mathrm{i=1}}{\overset{c*}{\mathrm{ALSO}}}(\mathrm{IF}\ antecedent_i \mathrm{THEN} consequent_i),$$

where $c*$ is the number of rules in a rule base either given by experts or it is determined by a fuzzy clustering algorithm such as FCM, Fuzzy-C-Means (Bezdek [1]) or IFC, Improved Fuzzy Clustering (Çelikyılmaz and Türkşen [2]). The fuzzy rule base structures determined by various alternatives mainly differ in the representation of the consequents. If the consequent is represented with fuzzy sets then the fuzzy rule base is known as Zadeh [13] version which is originally applied by Mamdani, et al., [3], and a modified version is proposed by Sugeno and Yasukawa, SY-FRB, [6]. Whereas, if the consequents are represented with linear equations of input variables, then the rule base structure is the Takagi-Sugeno Fuzzy Rule Base, TS-FRB, [5] structure. These are the main models amongst others which we do not review in this paper. In particular Zadeh Fuzzy Rule Bases, Z-FRB can be formalized as: $R\colon \underset{i=1}{\overset{c*}{ALSO}}(\mathrm{IF}\ x \in X \text{ isr } A_i \text{ THEN } \mathrm{y} \in Y \text{ isr } B_i)$

In general, let nv be the number of selected input variables in the system. Then, the multidimensional antecedent, x, can be defined as $x = (x_1, x_2, \ldots, x_{nv})$, where x_j is the j^{th} input variable of the antecedent and the domain of x in X, can be defined as $X = X_1 \times X_1 \times \ldots \times X_{nv},\ X_j \subseteq \mathfrak{R}$.

In particular, the Z-FRB structure can be expressed as follow, where the multi-dimensional antecedent fuzzy subset of i^{th} rule is Ai. This multi-dimensional antecedent fuzzy subset determination eliminates the search for the appropriate t-norm for the combination of antecedent fuzzy subsets with "AND".

Thus, variations of Z-FRB are Sugeno-Yasukawa, SY-FRB, and Takagi-Sugeno (TS-FRB) Fuzzy Rule Base structures:

$$(\mathrm{SY-FRB}) \qquad R\colon \underset{i=1}{\overset{c*}{ALSO}}(\mathrm{IF}\ x \in X \text{ isr } A_i \text{ THEN } \mathrm{y} \in Y \text{ isr } B_i)$$

$$(\mathrm{TS-FRB}) \qquad R\colon \underset{i=1}{\overset{c*}{ALSO}}\ \ (IF\ antecedent_i\ THEN\ y_i = a_i x^T + b_i)$$

where, antecedent$_{\mathrm{i}}$ = x $\in X$ isr A_{i}, and $a_i = (a_{i,1}, \ldots, a_{i,NV})$ is the regression coefficient vector associated with the i^{th} rule together with b_i which is the scalar associated with the i^{th} rule. For these special cases of Z-FRB, again each degree of firing, d_i, associated with the-i^{th} rule, is determined directly from the corresponding i^{th} multi-dimensional antecedent fuzzy subset A_i and applied to the consequent fuzzy subset for the SY-FRB or to the classical ordinary regression for the case of TS-FRB.

3 Fuzzy Regression Functions

There are a number of variations of the proposed Fuzzy Regression Functions. We discuss here only one alternative in this paper, namely, Fuzzy Regression Functions which we have proposed with LSE.

3.1 Fuzzy Regression Functions with LSE (FF-LSE)

In ordinary LSE (Least Square Estimation) method, the dependent variable, y, is assumed to be a linear function of input, variables, x, plus an error component:

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_{nv} x_{nv} + \epsilon$$

where y is the dependent output, x_j's are the explanatory variables input, for $j = 1, \ldots, nv$, nv is the number of selected inputs and ε is the independent error term which is typically assumed to be normally distributed. The goal of the least squares method is to obtain estimates of the unknown parameters, β_j's, $j = 0,1,\ldots, nv$, which indicate how a change in one of the independent variables affects the dependent variable.

$$\beta = \left(X^T X\right)^{-1} X^T y$$

The proposed generalization of LSE as **FF-LSE (Fuzzy Functions with LSE, more appropriately know as Fuzzy Regression Functions with LSE)**, requires that a fuzzy clustering algorithm, such as FCM, or IFC be available to determine the interactive (joint) membership values of input-output variables in each of the fuzzy clusters that can be identified for a given training data set. Let (X_k, Y_k), $k = 1,\ldots, nd$, be the set of observations in a training data set, such that $X_k = (x_{jk} \mid j = 1,\ldots, nv)$. First, one determines the optimal (m^*, c^*) pair for a particular performance measure, i.e., a cluster validity indeces such as Bezdek […], and Celikyılmaz and Türkşen […] with an iterative search and an application of FCM or IFC algorithm, where m is the level of fuzziness (in our experiments we usually take $m = 1.4, \ldots, 2.5$), Ozkan and Turksen […]) and c is the number of clusters (in our experiments we usually take $c = 2,\ldots,10$). The well known FCM (Bezdek 1973) algorithm can be stated as follows:

$$\begin{aligned} \min \quad & J(U,V)=\sum_{k=1}^{nd}\sum_{i=1}^{c}(u_{ik})^{m}(\|x_k-v_i\|)_A \\ s.t. \quad & 0\le u_{ik}\le 1, \forall i,k \\ & \sum_{i=1}^{c}u_{ik}=1, \forall k \\ & 0\le \sum_{k=1}^{nd}u_{ik}\le nd, \forall i \end{aligned},$$

where J is objective function to be minimized, $\|.\|_A$ is a norm that specifies a distance based similarity between the data vector x_k and a fuzzy cluster center v_i. In particular, *$A = I$ is the Euclidian Norm and $A = C^{-1}$ is the Mahalonobis Norm,* etc.

Once the optimal pair *(m^*, c^*)* is determined with the application of FCM algorithm, and a cluster validity index, one next identifies the

cluster centers for $m = m^*$ and $c = 1,\ldots,c^*$ *as:*

$$v_{\substack{X|Y,j \\ m^*}}=\left(x^{c}_{1,j},x^{c}_{2,j},\cdots,x^{c}_{nv,j},y^{c}_{j}\right)$$

From this, we identify the cluster centers of the input space again for $m = m^*$ *and* $c = 1,\ldots,c^*$ as: $v_{\substack{X,j \\ m^*}}=\left(x^{c}_{1,j},x^{c}_{2,j},\cdots,x^{c}_{nv,j}\right)$.

Next, one computes the normalized membership values of each vector of observations in the training data set with the use of the cluster center values determined in the previous step. There are generally two steps in this calculations:

First we determine the (local) optimum membership values u_{ik}'s and then determine μ_{ik} 's that are above an α- cut in order to eliminate harmonics generated by FCM as:

$$u_{ik}=\left(\sum_{j=1}^{c}\left(\frac{\|x_k-v_{X,i}\|}{\|x_k-v_{X,j}\|}\right)^{\frac{2}{m-1}}\right)^{-1}, \mu_{ik}\ge\alpha,$$

where μ_{ik} denotes the membership value of the k^{th} vector, $k = 1,\ldots,nd,$ in the i^{th} rule, $i = 1,\ldots,c^*$ and x_k denotes the k^{th} vector and for all the input variables $j = 1,\ldots, nv,$ in the input space. (2) Next, we normalize them as:

$$\gamma_{ij}(x_j)=\frac{\mu_{ij}(x_j)}{\sum_{i'=1}^{c}\mu_{i'j}(x_j)}$$

where γ_{ij} is the normalized membership value of x_j, $j = 1,\ldots, nv,$ in the i^{th} rule, $i = 1,\ldots,c^*$, which in turn will indicate the membership value that will constitute an new input variable in our proposed scheme of function identification for the representation of i^{th} cluster. Let $\Gamma_i = (\gamma_{ij} | i = 1, \ldots, c^*; j = 1, \ldots, nv)$ be the membership values of X in the i^{th} cluster, i.e., rule.

Next we determine a new augmented input matrix X for each of the clusters which could take on several forms depending on which *transformations* of membership values we want to or need to include in our system structure identification for our intended system analyses. Examples of these are:

$$X_i^{'} = [1, \Gamma_i, X], \qquad X_i^{''} = \left[1, \Gamma_i^2, X\right], \qquad X_i^{'''} = \left[1, \Gamma_i^2, \Gamma_i^m, \exp(\Gamma_i), X\right],$$

etc., where X_i', X_i'', X_i''' are the new input matrices to be used in least squares estimation of a new system structure identification where

$$\Gamma_i = (\gamma_{ij} | i = 1, \ldots, c^*; j = 1, \ldots, nv).$$

The choice depends on whether we want to or need to include just the membership values or some of their transformations as new input variables in order to obtain a best representation of a system behavior. In particular, this is done in order to get a higher value of R2 to show that a better model is obtained for an application. A new augmented input matrix, say X_i', would look as shown below for the special case of $X = X_j$, i.e., the matrix X is just a vector of a single variable, $X_j = (x_{jk}$ |k = 1,…,nd) for the j^{th} variable:

$$X_{ij}^{'} = [1, \Gamma_i, X_{ij}] = \begin{bmatrix} 1 & \gamma_{i1} & x_{ij1} \\ \vdots & \vdots & \vdots \\ 1 & \gamma_{ind} & x_{ijnd} \end{bmatrix}$$

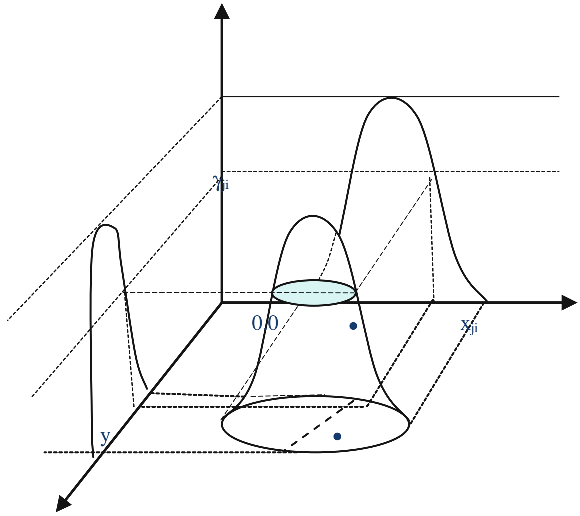
Thus the fuzzy regression function, $Y_i = \beta_{i0} + \beta_{i1}\Gamma_i + \beta_{i2}X_{ij}$, that represents the i^{th} rule corresponding to the i^{th} interactive (joint) cluster in space (Y_i, Γ_i, X_j), $\beta_i^* = (X_{ij}'^T X_{ij}')^{-1}(X_{ij}'^T Y_i)$, $\qquad X_{ij}' = \left[1, \Gamma_i, X_{ij}\right]$.

Such that $\beta_i^* = (\beta_{i0}^*, \beta_{i1}^*, \beta_{i2}^*)$ and the estimate of Y_i would be obtained as $Y_i^* = \beta_{i0}^* + \beta_{i1}^*\Gamma_i + \beta_{i2}^* X_{ij}$.

Within the proposed framework, the general form of the shape of a cluster can be conceptually captured by a second order (cone) in the space of U × X × Y which can be illustrated with a prototype shown in Fig. 1.

One usually determines Type 1 membership values with an application of FCM […] algorithm shown below:

Fig. 1 A Fuzzy cluster in U × *X* × *Y* space



ALGORITHM 3.1. FUZZY *C*-MEANS CLUSTERING ALGORITHM (FCM)

Given data vectors, $X=\{\mathbf{x}_1,..,\mathbf{x}_n\}$, number of clusters, ***c***, degree of fuzziness, *m*, and a termination constant, ε (maximum iteration number in this case). Initialize the partition matrix, *U*, randomly.

Step 1: Find initial cluster centers using via equ 1 shown below using membership values of initial partition matrix as inputs.

Step 2: Start iteration *t=1*…max-iteration value;

Step 2.1. Calculate membership values of each input data object *k* in cluster *i*, $\mu_{ik}^{(t)}$, using the membership value calculation equation in via equ 1 below, where x_k are input data objects as vectors and $\upsilon_i^{(t-1)}$ are cluster centers from $(t\text{-}1)^{th}$ iteration,

Step 2.2. Calculate cluster center of each cluster *i* at iteration *t*, $\upsilon_i^{(t)}$ using the cluster center function in equ 2 shown below, where the inputs are the input data matrix, $\mathbf{x}_k$, and the membership values of iteration *t*, $\mu_{ik}^{(t)}$.

Step 2.3. Stop if termination condition satisfied, e.g.,

$|\,\upsilon_i^{(t)} - \upsilon_i^{(t-1)}\,| \leq \varepsilon$. Otherwise go to step 1.

where Eq. 1 stated in the algorithm above is:

$$\mu_{ik}^{(t)} = \left[\sum_{j=1}^{c} \left(\frac{d\left(x_k, \upsilon_i^{(t-1)}\right)}{d\left(x_k, \upsilon_j^{(t-1)}\right)} \right)^{\frac{2}{m-1}} \right]^{-1}$$

And Eq. 2 is:

$$v_i^{(t)} = \left(\sum_{k=1}^{n} \left(\mu_{ik}^{(t)} \right)^m x_k \right) \Big/ \sum_{k=1}^{n} \left(\mu_{ik}^{(t)} \right)^m, \forall i = 1, \ldots, c$$

4 Generation of Full Type 2 Membership Values

For this purpose, we propose and hence introduce an new algorithm in order to generate Full Type 2 membership value distribution from the results obtained with an application of FCM which produce a Type 1 membership value distribution for our studies of Full Type 2 investigations.

5 Full Type 2 Fuzziness i.e., Membership of Membership

Here we want to show how one determines the second order degree of fuzziness in order to develop Full Type 2 fuzzy system models.

It should be noted that depending on where $x \in X$ is there may be more than one second order membership value distribution.

6 Full Type 2 Fuzzy Set Extraction Algorithms

We propose the following Full Type 2 fuzzy set extraction algorithm from a given data set called FT2FCM (Türkşen, 2012):

Full Type 2 Fuzzy Clustering Algorithm

$$Min j'(U'(U), W) =$$

$$= \sum_{k=1}^{nd} \sum_{i=1}^{c'} \sum_{l=0}^{1} \left(\mu_{\mu i(x_k)}(Z) \right) \left(\left\| \mu_{\mu i(x_k)}(Z_l) - \bar{\mu}_{(x_k)}(Z_l) \right\| A \right), k = 1, \ldots, nd;$$

$$i = 1, \ldots, c'$$

$$st.\ 0 \le \mu_{\mu i(xk)}(Z) \le 1$$

$$0 \le \mu_i(x_k) \le 1$$

$$0 \le \sum_{k=1}^{nd} \mu_i(x_k) \le nd$$

$$\mu_i(x_k) \ni [0,1]; \quad \mu_{\mu i(x_k)}(Z) \ni [0,1]; \quad l \ni [0,1]$$

where j' is the objective function to be minimized for a given $x_k \in X$, $\|.\|_A$ is a norm, i.e., Euclidian or Mahalanobis, that specifies a distance measure based on a membership values for a given $x_k \in X$ and its second order fuzzy cluster center $\bar{\mu}_i(x_k)$.

Next one computes the normalized membership values of these Full Type 2 membership values for each vector of membership values obtained in an initial application of the original FCM or IFC algorithm in the first stage.

There are generally two steps in these calculations:

We first determine (local) optimum membership of membership values $\mu_{\mu_i}(x_k)$'s and then apply an α-cut in order to eliminate the second order harmonics generated by an application of FT2FCM as:

$$\begin{matrix} \mu_{\mu_i}(x_k) \\ x_k \in X \end{matrix} = \left[\left(\sum_{i=1}^{c'} \frac{\|\mu_{\mu i}(x_k) - \bar{\mu}_i(x_k)\|}{\|\mu_{\mu i}(x_k) - \mu_j(x_k)\|} \right)^{\frac{2}{m-1}} \right]^{-1}$$

$$\gamma'_{\mu_{\mu_i}(x_k)} = \frac{\mu_{\mu_i}(x_k)|_{x_k \in X}}{\sum_{i=1}^{c'} \mu_{\mu_i}(x_k)}, \mu_{\mu_i}(x_k) \geq \alpha, \gamma'_{\mu_{\mu_i}(x_k)} \geq \alpha$$

where $\gamma'_{\mu_{\mu_i}(x_k)}$ denotes the membership values of the membership values of the kth vector $k = 1,..,nd$ in the ith rule, or ith fuzzy regression function (Türkşen, 2012) and $x_k \in X$ denotes the kth vector and for all the input variables, $k = 1,.., nd$ in the input space.

Recall that we are able to obtain the membership value distribution as:

$$X'_{ij} = [1, \Gamma_i, X_{ij}] = \begin{bmatrix} 1 & \gamma_{i1} & x_{i1} \\ \vdots & \vdots & \vdots \\ 1 & \gamma_{ind} & x_{ind} \end{bmatrix}$$

$$\Gamma_i = (\gamma_{ik} | i = 1, \ldots, c^*; k = 1, \ldots, nd)$$

$$\Gamma_i = \left(\gamma_{ij} | i = 1, \ldots, c^*; j = 1, \ldots, nd\right) \qquad \Gamma_i = \begin{bmatrix} \gamma_{11} & \gamma_{21} & \cdots\gamma_{c^*1} \\ \vdots & \vdots & \vdots \\ \gamma_{1nd} & \gamma_{2nd} & \cdots\gamma_{c^*nd} \end{bmatrix}$$

We process each γ_i via our Full Type 2 clustering algorithm given above, called FT2FCM, to determine Full Type 2 distribution for each cluster i, $\Gamma_i = \left(\gamma_{ij} | i = 1, \ldots, c^*; j = 1, \ldots, nd\right)$.

Thus we apply to each Γ_i, ALGORITHM 2 given below to generate Full Type 2 membership, values, i.e., membership of membership.

ALGORITHM 2. FULL TYPE 2 FUZZY *C*-MEANS CLUSTERING ALGORITHM (FT2FCM)

Given data vectors,
$\Gamma_i = (\gamma_{ij} | i = 1, \ldots, c^*; j = 1, \ldots, nd)$

, number of clusters, ***c***`, degree of fuzziness, *m*, and termination constant, ε (maximum iteration number in this case). Initialize a partition matrix, Γ, randomly.

Step 1: Find initial cluster centers using via equ 3 shown below using membership of membership values of initial partition matrix as inputs.

Step 2: Start iteration *t=1*…max-iteration value;

Step 2.1. Calculate Type 2 membership values of a given , Γ vector of each input data object *k* in cluster *i*, $\mu_{\mu_i}(x_k)$, using each Γ vector of the membership values where x_k are input data objects as vectors and $\overline{\mu}_i(x_k)$ are Type 2 cluster centers from $(t\text{-}1)^{th}$ iteration.

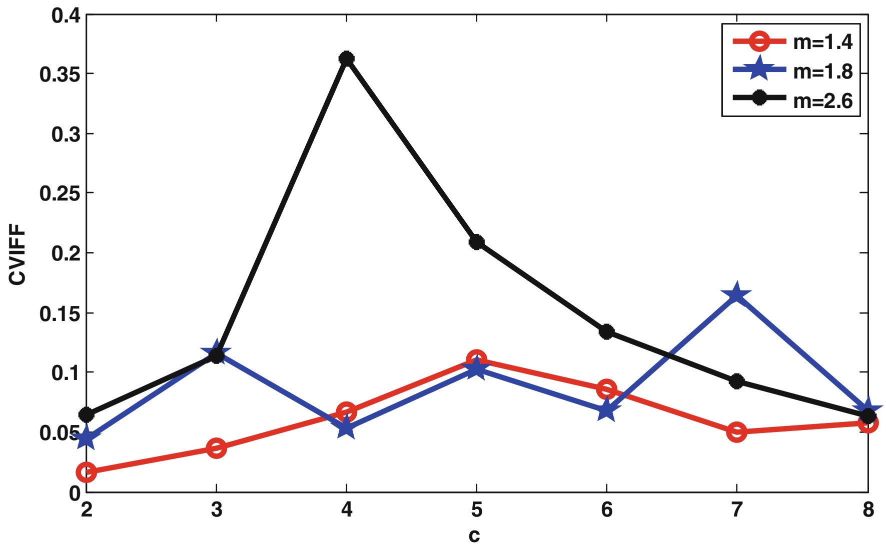
Step 2.2. Calculate Type 2 cluster center w_{ik} of each cluster *l* at iteration *t*, the t-th $\bar{\mu}_i(x_k)$ he cluster center function of Type 2 membership values in equ 4 shown below, where the inputs are the input data matrix, , , and the membership of the membership values of iteration *t*. $\mu_{\mu_i}(x_{k,t})$

Step 2.3. Stop if termination condition satisfied, e.g., $|\overline{\mu}_i(x_k)@t - \overline{\mu}_i(x_k)@t - 1| \leq \varepsilon$. Otherwise go to step 1.

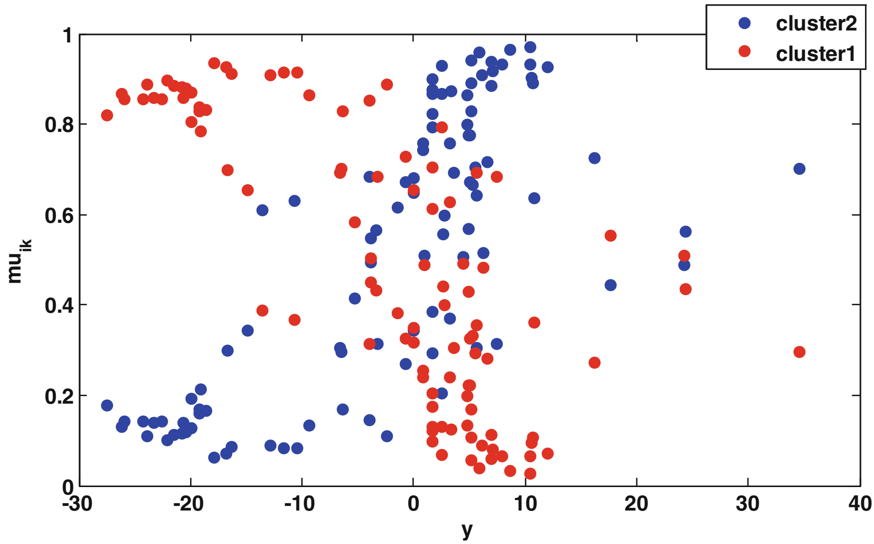
7 Experimental Results

We present here our experimental results for **TD_Stock Price Data set** that is available for all researchers on the internet.

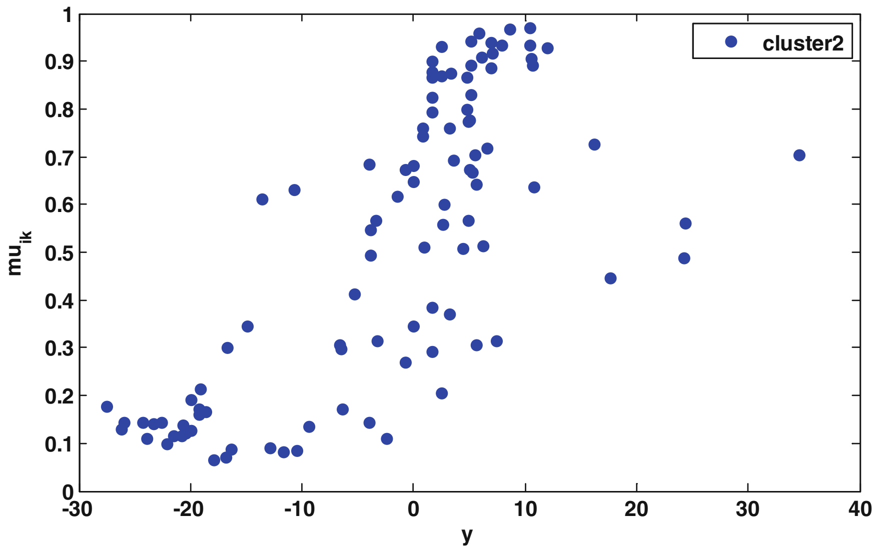
Çelikyılmaz-Türkşen's validity index results for TD_Stockprice data:



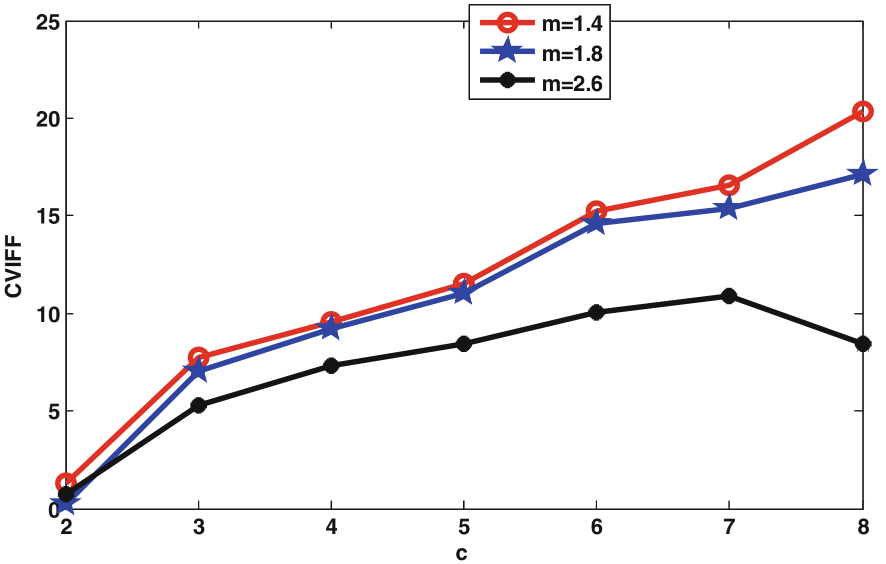
Fuzzy classification of TD_Stockprice data: ($c^* = 2, m^* = 1.8$)



Cluster-2 view for TD_Stockprice data:

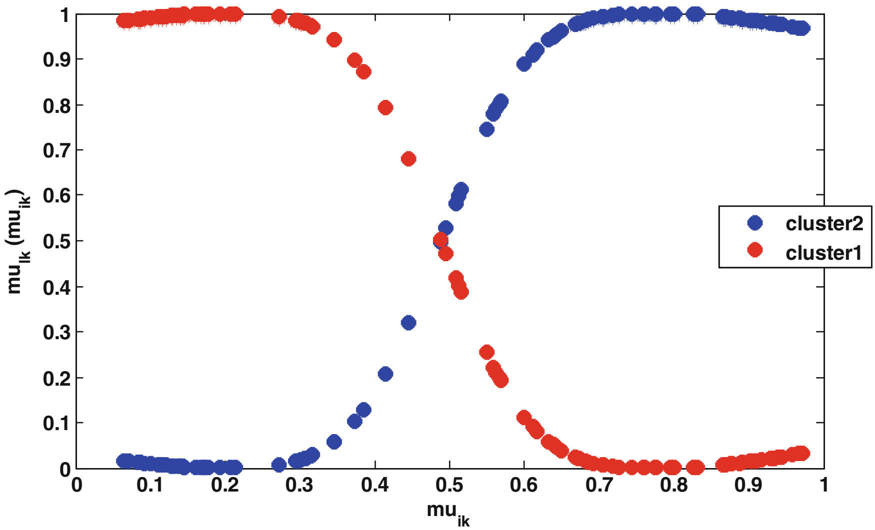


Çelikyılmaz-Türkşen's Validity Index for μ_{ik} data:



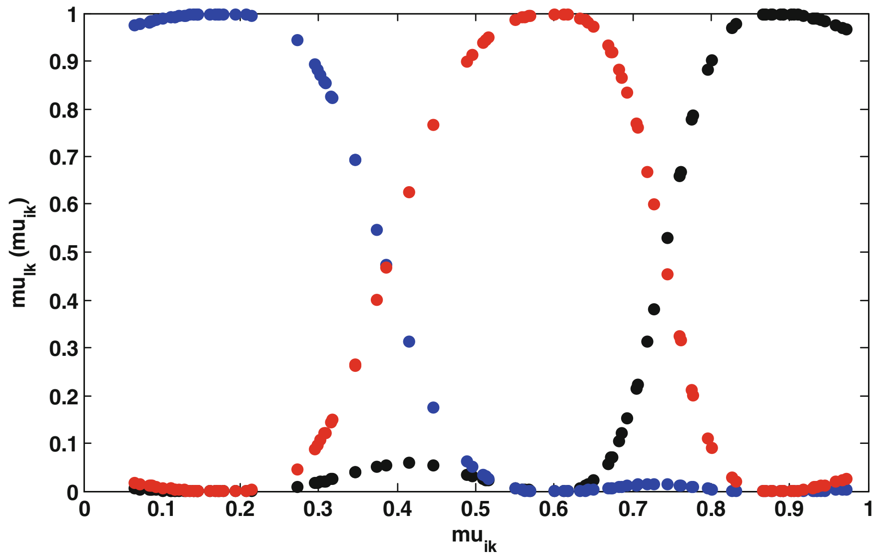
Cluster-2 results of TD-Stockprice data ($c^* = 2, m^* = 1.8$)

According Çelikyılmaz-Türkşen index, the suitable number of cluster should be chosen as c' = 2 (μ_{ik} data is the membership values of first study's cluster-2). Where c' = 2, m' = 1.8.



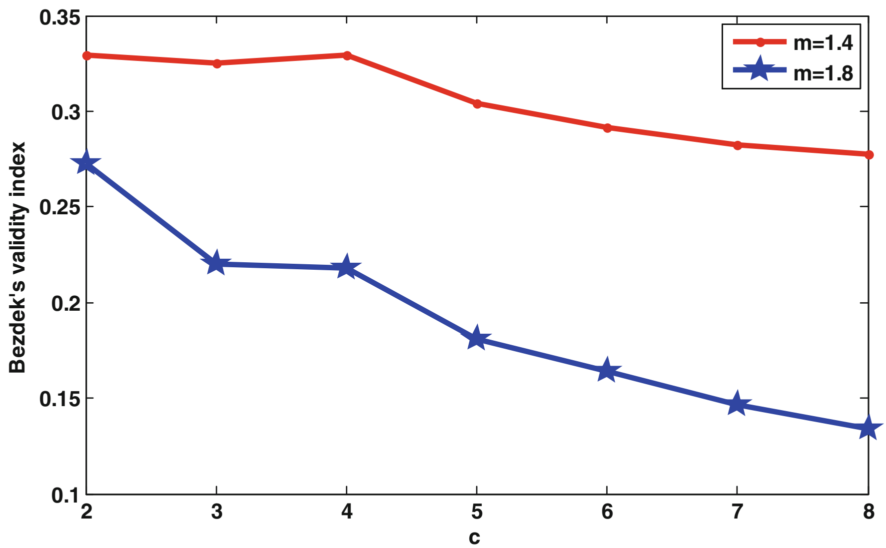
$\mu_{Ik}(\mu_{ik})$ for Cluster1 and 2 are shown above:

A possible three cluster view:

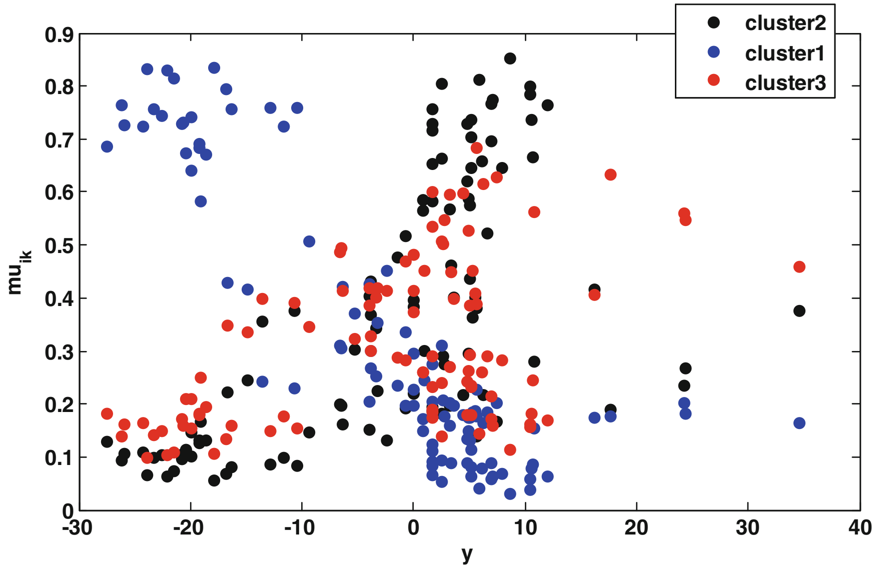


TD_Stockprice Data set:

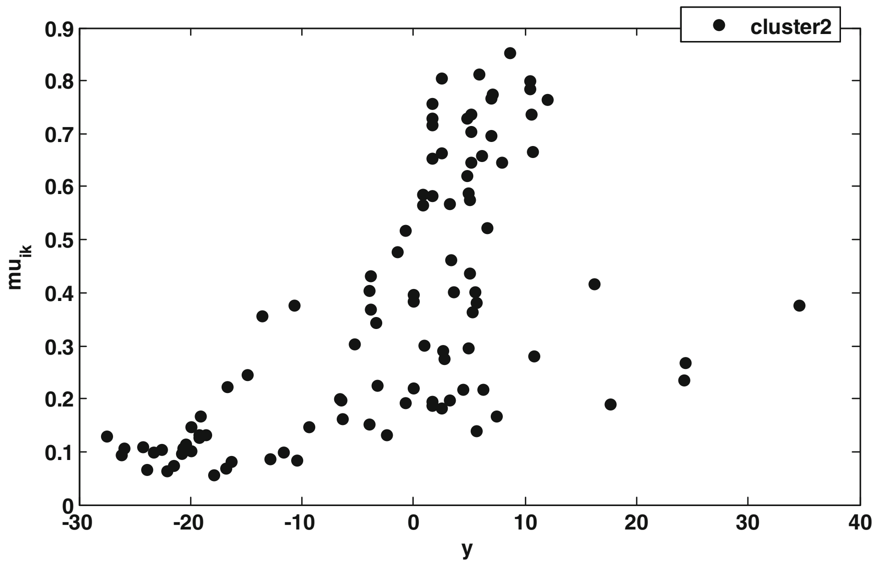
According to Bezdek's validity index results (shown as follows), the suitable number of cluster was chosen as c* = 3:



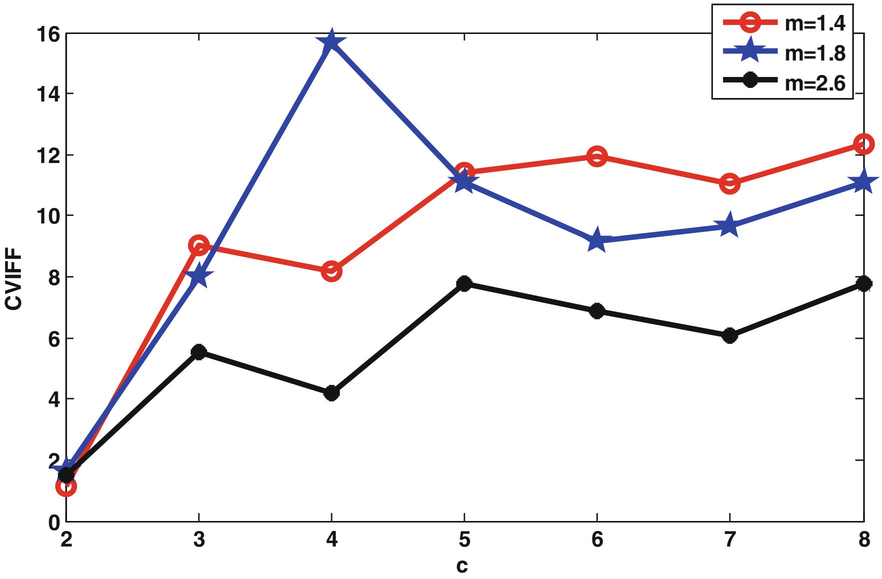
Fuzzy classification of TD_Stockprice data: (c* = 3,m* = 2.0)



Cluster-2 view for TD_Stockprice data:

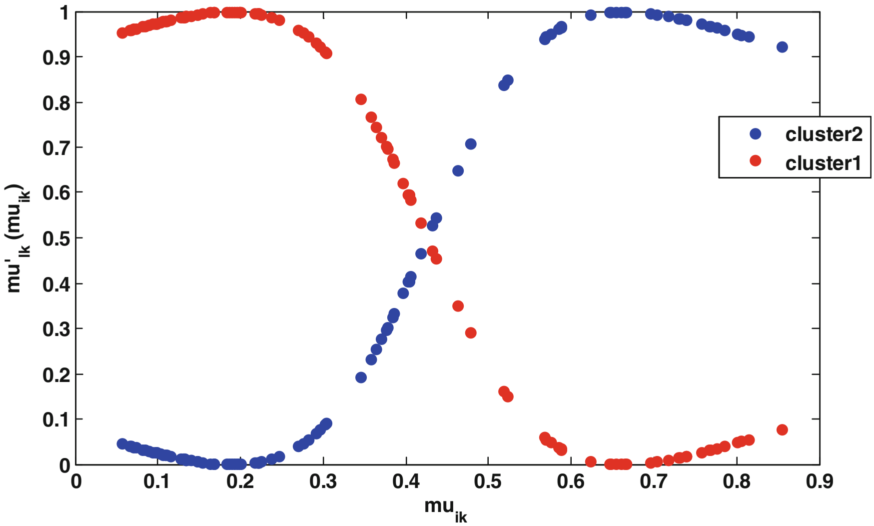


Çelikyılmaz-Türkşen's Validity Index for μ_{ik} data:

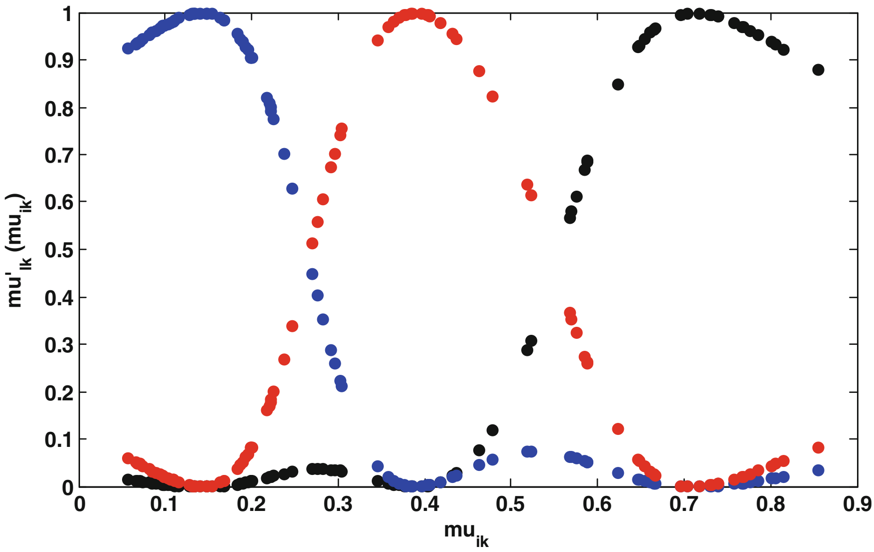


Cluster-2 results of TD-Stockprice data ($c^* = 3, m^* = 2.0$) for membership of membership.

According Çelikyılmaz-Türkşen index, the suitable number of cluster should be chosen as c' = 2 (the μ_{ik} data is the membership values of first study's cluster-2). Where c' = 2,m' = 2.0.



A possible three cluster view:



8 Conclusions

In this paper, we have first review the essentials fuzzy system models: such as (1) Zadeh's rulebase model, (2) Takagi and Sugeno's partly a rule base and partly a regression function model and (3) Türkşen's "Fuzzy Regression Functions" model where a fuzzy regression function correspond to each fuzzy rule and thus a fuzzy rule base is replaced with "Fuzzy Regression Functions" model. Next we review the well known FCM algorithm which lets one to extract Type 1 membership values from a given data set for the development of "Type 1" fuzzy system models as a foundation for the development of "Full Type 2" fuzzy system models. For this purpose, we provide an algorithm which lets one to generate Full Type 2 membership value distributions for a development of second order fuzzy system models with our proposed second order data analysis. If required one can generate Full Type 3,…, Full Type n fuzzy system models with an iterative execution of our algorithm. Finally we present our results graphically for TD_Stockprice data with respect to two validity indeces, namely: (1) Çelikyılmaz-Türkşen and (2) Bezdek indeces. Based on our development, we expect in the future new results would be obtained in "Full Type 3,…, Full Type n" fuzzy system model analyses.

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Dr. I.B. Turksen is a Professor Emeritus of Industrial Engineering in the Department of Mechanical & Industrial Engineering at the University of Toronto. He received the B.S. and M.S. degrees in Industrial Engineering and the Ph.D. degree in Systems Management and Operations Research all from the University of Pittsburgh, PA. He joined the Faculty of Applied Science and Engineering at the University of Toronto and became Full Professor in 1983. In 1984–1985 academic year, he was a Visiting Professor at the Middle East Technical University and Osaka Prefecture University. Since 1987, he has been Director of the Knowledge / Intelligence Systems Laboratory. During the 1991–1992 academic year, he was a Visiting Research Professor at LIFE, Laboratory for International Fuzzy Engineering, and the Chair of Fuzzy Theory at Tokyo Institute of Technology. During 1996 academic year, he was Visiting Research Professor at the University of South Florida, USA, and Bilkent University, Ankara, Turkey. Since December 2005, he is appointed as the Head of Department of Industrial Engineering at TOBB Economics and Technology University.

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Dr. Turksen is the founding President of CSIE. He was Vice-President of IIE, General Conference Chairman for IIE International Conference, and for NAFIPS in 1990. He served as Co-Chairman of IFES'91 and Regional Chairman of World Congress on Expert Systems, WCES'91, WCES'94, WCES'96 and WCES'98, Director of NATO-ASI'87 on Computer Integrated Manufacturing and Co-Director of NATO-ASI'96 on Soft Computing and Computational Intelligence. He was General Conference Chairman for Intelligent Manufacturing Systems, IMS'1998, IMS'2001, IMS'2003. He was the President of IFSA during 1997–2001 and Past President of IFSA, International Fuzzy Systems Association during 2001–2003. Currently, he is the President, CEO and CSO, of IIC, Information Intelligence Corporation.

He received the outstanding paper award from NAFIPS in 1986, "L.A. Zadeh Best Paper Award" from Fuzzy Theory and Technology in 1995, "Science Award" from Middle East Technical University, and an "Honorary Doctorate" from Sakarya University. He is a Foreign Member, Academy of Modern Sciences.

His current research interests centre on the foundations of fuzzy sets and logics, measurement of membership functions with experts, extraction of membership functions with fuzzy clustering and fuzzy system modeling. His contributions include, in particular, Type 2 fuzzy knowledge representation and reasoning, fuzzy truth tables, fuzzy normal forms, T-formalism which is a modified and restricted Dempster's multi-valued mapping, and system modeling applications for intelligent manufacturing and processes, as well as for management decision support and intelligent control.

He has published near 300 papers in scientific journals and conference proceedings.

His book entitled "An Ontological and Epistemological Perspective of Fuzzy Theory" was published by Elsevier, The Netherlands, in January, 2006. (ISBN-10: 0-444-51891-6, ISBN-13: 978-0-444-51891-0).

Complex Fuzzy Sets and Complex Fuzzy Logic an Overview of Theory and Applications

Dan E. Tamir, Naphtali D. Rishe and Abraham Kandel

Abstract Fuzzy Logic, introduced by Zadeh along with his introduction of fuzzy sets, is a continuous multi-valued logic system. Hence, it is a generalization of the classical logic and the classical discrete multi-valued logic (e.g. Łukasiewicz' three/many-valued logic). Throughout the years Zadeh and other researches have introduced extensions to the theory of fuzzy setts and fuzzy logic. Notable extensions include linguistic variables, type-2 fuzzy sets, complex fuzzy numbers, and Z-numbers. Another important extension to the theory, namely the concepts of complex fuzzy logic and complex fuzzy sets, has been investigated by Kandel et al. This extension provides the basis for control and inference systems relating to complex phenomena that cannot be readily formalized via type-1 or type-2 fuzzy sets. Hence, in recent years, several researchers have used the new formalism, often in the context of hybrid neuro-fuzzy systems, to develop advanced complex fuzzy logic-based inference applications. In this chapter we reintroduce the concept of complex fuzzy sets and complex fuzzy logic and survey the current state of complex fuzzy logic, complex fuzzy sets theory, and related applications.

Keywords Fuzzy set theory · Fuzzy class theory · Fuzzy logic · Complex fuzzy sets · Complex fuzzy classes · Complex fuzzy logic · Neuro-fuzzy systems

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
Studies in Fuzziness and Soft Computing 326,
DOI 10.1007/978-3-319-19683-1_31

1 Introduction

The development of computers and the related attempt to automate human reasoning and inference have posed a challenge to researchers. Humans, and in many cases machines, are not always operating under strict and well defined two-valued logic or discrete multi-valued logic. Their perception of sets and classes is not as crisp as implied by the traditional set and class theory. To capture this perception, L. A. Zadeh has introduced the theory of fuzzy sets and fuzzy logic [1–7]. The seminal paper [1],[1] published by Zadeh in 1965, ignited tremendous interest among a large number of researchers. Following the introduction of the concepts of fuzzy logic and set theory, several researchers, [8–10], have established an axiomatic framework for these concepts.

The five decades that followed Zadhe's pioneering work have produced extensive research work and applications related to control theory [11, 12], artificial intelligence [7, 13–15], inference, and reasoning [16, 17]. In recent years, fuzzy logic has been applied in many areas, including fuzzy neural networks [18], neuro-fuzzy systems and other bio-inspired fuzzy systems [19], clustering [20–22], data mining [13, 23, 24], and software testing [25, 26]. In 1975 Zadeh introduced the concept of linguistic variable and the induced concept of type-2 (type-n) fuzzy sets [3, 27–30]. Other notable extensions to the theory of fuzzy sets and fuzzy logic include complex fuzzy numbers [31], and Z-numbers [32].

Many natural phenomena are complex and cannot be modelled using one-dimensional classes and/or one-dimensional variables. For example, in pattern recognition, objects can be represented by a set of measurements and are regarded as vectors in a multidimensional space. Often, it is not practical to assume that this multidimensional information can be represented via a simple combination of variables and operators on one-dimensional clauses. Specifically, consider a set of values where each value is a member of a fuzzy set. This set, referred to as fuzzy set of type-2, cannot be compactly represented by basic operations on fuzzy sets of type-1 [3, 27–30]. This type of sets however, can be represented via complex classes presented next.

Another important extension to the theory of fuzzy logic and fuzzy sets, namely complex fuzzy logic (CFL) and complex fuzzy sets (CFS), has been developed by Kandel and his coauthors [10, 33–36]. Moses et al. introduced an aggregation of two fuzzy sets into one complex fuzzy set [33]. Next, Ramot et al. introduced the concept of a complex degree of membership represented in polar coordinates, where the amplitude is the degree of membership of an object in a CFS and the role of the phase is to add information which is generally related to spatial or temporal periodicity in the specific fuzzy set defined by the amplitude component. They used this formalism along with the theory of relations to establish the concept of CFL. Finally, Tamir et al. developed an axiomatically-based CFL system and used CFL

[1]The first documented reference by Zadeh to the concepts of Fuzzy Mathematics appeared in a 1962 paper.

to provide a new and general formalism of CFS. These formalisms significantly enhance the expressive power of type-1 and type-2 fuzzy sets [30, 37]. The successive definitions of the theory of CFL and CFS represent an evolution from a relatively naïve and restricted practice to a sound, well founded, practical, and axiomatically-based form. In recent years, several researchers have used the new formalism, often in the context of hybrid neuro-fuzzy systems to develop advanced complex fuzzy logic-based inference applications.

There is a substantial difference between the definitions of complex fuzzy numbers given by J. Buckley [31, 38–41] and the concept of complex fuzzy sets or complex fuzzy logic. Buckley is concerned with generalizing the number theory while the CFL and CFS theories are concerned with the generalization of fuzzy set theory and fuzzy logic [10, 42, 43]. Complex fuzzy numbers have been utilized in several numerical applications [44–46]. Yet, the concept of a complex fuzzy number is different from the concept of complex fuzzy sets or complex fuzzy classes. Recently, Zadeh introduced the concept of *Z-numbers*. A Z-number, $Z = (A, B)$, is an ordered pair of two fuzzy numbers. In this context A, provides a restriction on a real-valued variable X and B is a restriction on the degree of certainty that X is A [32]. Nevertheless, this concept is used to qualify the reliability of fuzzy quantities rather than to define complex fuzzy sets [10, 36].

The present chapter includes an introduction to the succession of definitions of CFL and CFS, concentrating on the axiomatic-based approach. In addition, the chapter includes a survey the current state of research into complex fuzzy logic, complex fuzzy set theory, and related applications.

The rest of the chapter is organized in the following way: Sect. 2 introduces the axiomatic-based theory of fuzzy set and fuzzy logic. Section 3 surveys the theory of complex fuzzy logic and complex fuzzy sets, concentrating on the axiomatically-based formulation of the theories. Section 4 includes a survey of recent developments in the theory and applications of CFL and CFS. Finally, Sect. 5 presents conclusions and directions for further research.

2 Fuzzy Logic and Set Theory

In 1965, L.A. Zadeh introduced the theory of fuzzy sets, where the degree of membership of an item in a set can get any value in the interval [0, 1] rather than the two values $\{\notin, \in\}$ [1]. Additionally, he introduced the notion of fuzzy logic [1–4]. Fuzzy logic is a continuous (analog) multi-valued extension of classical logic where propositions can get truth values in the interval [0, 1], and are not limited to one of the two values {True, False} (or {0, 1}) [17]. These concepts can be considered as an extension of the multi-valued logic proposed by Łukasiewicz [47]. The introduction of the concepts of fuzzy sets and fuzzy logic was followed by extensive research into fuzzy systems and their applications, related theories, and extensions of the concept [1–4, 6, 13, 17, 19, 22, 26, 48–53]. One direction of research has concentrated on the formulation of an axiomatically-based foundation of fuzzy sets and fuzzy logic [8–10, 54–63]. This is described next.

2.1 Axiomatic Fuzzy Logic

Several researchers presented an axiomatically-based formulation of fuzzy logic and fuzzy set theory [8–10, 55, 56, 58]. In this section we briefly review an axiomatic framework that is founded on the basic fuzzy propositional and predicate logic (BL), along with the fuzzy Łukasiewicz (Ł) and fuzzy product (Π) logical systems [8–10, 55, 56, 58]. We refer to the propositional logic system as ŁΠ and to the first order predicate fuzzy logic system as ŁΠ∀.

Propositional Fuzzy Logic

Several axiom-based logical systems have been investigated [8–10, 55, 56, 58]. Běhounek et al. ([8]) use the ŁΠ/ ŁΠ∀ as the basis for the definition of fuzzy class theory (FCT). Our definition of complex propositional logic presented in Sect. 3 [10, 36], closely follows ŁΠ, the system used by Běhounek et al. For clarity, we reintroduce some of the important notions, notations, and concepts from that paper.

A fuzzy proposition P can get any truth value in the real interval [0, 1], where '0' denotes "False," and '1' denotes "True". Furthermore, the relation $\leq$, over the interval $[0, 1]$ implies a monotonically increasing ordering on the truth values associated with the proposition. A fuzzy interpretation of a proposition P is an assignment of a fuzzy truth value to P. Let P, Q and R denote fuzzy propositions and let $i(R)$ denote the fuzzy interpretation of R. Table 1, includes the basic connectives of ŁΠ. Table 2 includes connectives that can be derived from the basic connectives. The constant 0 is assumed and the constant 1 can be derived from 0 and the basic connectives.

Table 1 Basic ŁΠ connectives

Operation	Interpretation
Ł-Implication	$i(P \rightarrow_L Q) = \min(1, 1 - i(P) + i(Q))$
Π-Implication	$i(P \rightarrow_\prod Q) = \min(1, i(P)/i(Q))$
Π-Conjunction	$i(P \otimes Q) = i(P) \cdot i(Q)$

Table 2 Derived ŁΠ connectives

Operation	Interpretation
Ł-Negation	$i(^{'}P) = 1 - i(P)$
Π-Delta	$\Delta(i(P)) = 1 \text{ if } i(P) = 1 \text{ else } \Delta(i(P)) = 0$
Equivalence	$i(P \leftrightarrow Q) = i(P \rightarrow_L Q) \otimes i(Q \rightarrow_L P)$
$P \ominus Q$	$i(P \ominus Q) = \max(0,\ i(P) - i(Q))$

Běhounek et al. use the basic and derived connectives along the truth constants and the following set of axioms [8]:

(1) The Łukasiewicz set of axioms
(2) The product set of axioms
(3) The Łukasiewicz Delta axiom
(4) The Product Delta axiom
(5) The axiom:

$$\mathrm{R} \otimes (P \ominus Q) \leftrightarrow_L (R \otimes P) \ominus (R \otimes Q) \tag{1}$$

The rules of inference are:

(1) Modus ponens
(2) Product necessitation.

Reference [8] includes several theorems that follow from the definition of ŁΠ propositional fuzzy logic. In the next section, we define the ŁΠ first order predicate fuzzy logic (ŁΠ∀).

First Order Predicate Fuzzy Logic

Following the classical logic, the ŁΠ first order predicate fuzzy logic, referred to as ŁΠ∀, extends the ŁΠ propositional fuzzy logic. The primitives include constants, variables, arbitrary-arity functions and arbitrary-arity predicates. Formulae are constructed using (1) the basic connectives defined in Table 1; (2) derived connectives, such as the connectives presented in Table 2; (3) the truth constants; (4) the quantifier ∀ and (5) the identity sign "=". The quantifier ∃ can be used to abbreviate formulae derived from the basic primitives and connectives. A fuzzy interpretation of a proposition $P(x_1, \ldots, x_n)$ over a domain M is a mapping that assigns a fuzzy truth value to each n-tuple of elements of M As in the case of ŁΠ, we closely follow the system used in ref. [8].

Assuming that y can be substituted for x in P and x is not free in Q the following axioms are used:

(1) Instances of the axioms of ŁΠ obtained through substitution
(2) Universal axiom I:

$$(\forall x)P(x) \rightarrow P(y) \tag{2}$$

(3) Universal axiom II:

$$(\forall x)(P \rightarrow_L Q) \rightarrow_L (P \rightarrow_L (\forall x)Q) \tag{3}$$

(4) Identity axiom I:

$$x = x \tag{4}$$

(5) Identity axiom II:

$$(x = y) \rightarrow \Delta(P(x) \leftrightarrow P(Y)) \quad (5)$$

Modus ponens, product necessitation, and generalization are used for inference. In the next section, we define propositional and first order predicate CFL.

2.2 Axiomatic Fuzzy Class Theory

The axiomatic fuzzy logic can serve as a basis for establishing an axiomatic FCT. Several variants of FCT exists, most of them use a similar approach and mainly differ in the selection of the logic base. Another difference between various approaches is the selection of class theory axioms [64]. Běhounek et al. present and analyze a few variants of FCT. Ref. [8] presents an ŁΠ∀ based FCT.

3 Complex Fuzzy Logic and Set Theory

The first formalization of complex fuzzy sets and complex fuzzy logic investigated by Kandel and his coauthors [35, 65] is a special case of the formalism presented by Tamir et al. [10]. Hence, in this section only two formalisms for complex fuzzy sets and complex fuzzy logic are considered: (1) the formal definitions provided by Ramot et al. [33], (2) the generalization of these concepts developed by Tamir et al. [10, 36, 43, 66].

3.1 Complex Fuzzy Sets (Ramot et al. [33])

This section reviews the basic concepts and operations of complex fuzzy set as defined by Ramot et al. [34, 67]. According to Ramot et al., a complex fuzzy set S on a universe of discourse U is a set defined by a complex-valued grade of membership function $\mu_s(x)$ [33, 34]:

$$\mu_s(x) = r_s(x)e^{j\omega s(x)} \quad (6)$$

where $j = \sqrt{-1}$. The function $\mu_s(x)$ maps U into the unit disc of the complex plane. This definition utilizes polar representation of complex numbers along with conventional fuzzy set definition; where $r_s(x)$, the amplitude part of the grade of member-ship, is a fuzzy function defined in the interval [0, 1]. On the other hand, $\omega_s(x)$ is a real valued function standing for the phase part of the grade of membership.

In the definition provided by Ramot, the absolute value, or the amplitude part of the membership grade, behaves in the same way as in traditional fuzzy sets. Its value is mapped into the interval [0, 1]. On the other hand, the phase component of the expression is not a fuzzy function; it is a real valued function that can get any real value. Furthermore, the grade of membership is not influenced by the phase. The phase role is to add information which is generally related to spatial or temporal periodicity in the specific fuzzy set defined by the amplitude component. For example, fuzzy information related to solar activity along with crisp information that relates to the date of measurement of the solar activity [33]. Another example where complex fuzzy set has an intuitive appeal comes from the stock market. Intuitively, the periodicity of the stock market along with fuzzy set based estimate of the current values of stocks can be represented by a complex grade of membership such as the one proposed by Ramot. The amplitude conveys the information contained in a fuzzy set such as "strong stock" while the phase conveys a crisp information about the current phase in the presumed stock market cycle.

Following the basic definition of complex-valued grade of membership function Ramot et al. define the basic set operations such as complement, union, and intersection. Each of these operations is defined via a set of theorems [42].

3.2 Complex Fuzzy Logic (Ramot et al. [34])

There are several ways to define fuzzy logic, fuzzy inference, and fuzzy logic system (FLS). One of these ways is to use fuzzy set theory to define fuzzy relations, and then define logical operations, such as implication and negation, as well as inference rules, as special types of relations on fuzzy sets. Alternatively, fuzzy logic can be formalized as a direct generalization of classical logic. Under this "traditional" approach, notions that relate to the syntax and semantics of classical logic, such as propositions, interpretation, and inference are used to define fuzzy logic. Although the relations-based definition can be carefully formalized, it is generally less rigorous than the traditional approach.

Ramot et al. use the first approach [34]. They use the definition of complex fuzzy relations to define complex fuzzy logic via the definition of logical operations. Additionally, Ramot et al. restrict complex fuzzy logic to propositions of the form '*X is A*' , where *X* is a variable that receives values *x* from a universal set *U* and *A* is a complex fuzzy set on *U*. They use this type of propositions to introduce implications of the form '*if X is A then Y is B*' . Finally, they use modus ponens to produce a complex fuzzy inference system. Clearly their approach is limited due to two facts: (1) they rely on complex fuzzy sets and relations to define CFL and (2) their fuzzy inference system is limited to propositions on complex fuzzy sets. These limitations are resolved via the axiomatically-based approach presented in the next section.

3.3 Generalized Complex Fuzzy Logic (Tamir et al. [10])

This section presents the generalized form of complex fuzzy logic investigated by Tamir et al. [10].

Propositional and First Order Predicate Complex Fuzzy Logic

A complex fuzzy proposition P is a composition of two propositions each of which can accept a truth value in the interval $[0, 1]$. In other words, the interpretation of a complex fuzzy proposition is a pair of truth values from the Cartesian interval $[0, 1] \times [0, 1]$. Alternatively, the interpretation can be formulated as a mapping to the unit circle. Formally a fuzzy interpretation of a complex fuzzy proposition P is an assignment of fuzzy truth value of the form $i(p_r) + j \cdot i(p_i)$ or of the form $i(r(p))e^{j\sigma i(\theta(p))}$, where σ is a scaling factor in the interval $(0, 2\pi]$, to P.

For example, consider a proposition of the form "$x \ldots A \ldots B \ldots$," along with the definition of a linguistic variables and constants. Namely, a *linguistic variable* is a variable whose domain of values is comprised of formal or natural language words [3]. Generally, a linguistic variable is related to a fuzzy set such as {*very young male*, *young male*, *old male*, *very old male*} and can get any value from the set. A linguistic constant has a fixed and unmodified linguistic value, i.e. a single word or phrase from a formal or natural language.

Thus, in a proposition of the form "$x \ldots A \ldots B \ldots$," where A and B are linguistic variables, $i(p_r)$ $(i(r(p)))$ can be assigned to the term A and $i(p_i)$ $(i(\theta(p)))$ can be assigned to term B .

Propositional CFL extends the definition of propositional fuzzy logic and first order predicate CFL extends the notion of first order predicate fuzzy logic. Nevertheless, since propositional CFL is a special case of first order predicate CFL, we only present the formalism for first order predicates CFL here.

Tables 3 and 4 present the basic and derived connectives of ŁΠ∀ CFL. In essence, the connectives are symmetric with respect to the real and imaginary parts of the predicates.

Table 3 Basic ŁΠ∀ CFL connectives

Operation	Interpretation
L-Implication	$i(P \rightarrow_L Q) = \min(1, 1 - i(p_r) + i(q_r)) + j \cdot \min(1, 1 - i(p_i) + i(q_i))$
Π-Implication	$i(P \rightarrow_{\prod} Q) = \min(1, i(p_r)/i(q_r) + j \cdot \min(1, i(p_i)/i(q_i))$
Π-Conjunction	$i(P \otimes Q) = i(p_r) \cdot i(q_r)) + j \cdot (i(p_i) \cdot i(q_i))$

Table 4 Derived ŁΠ∀ CFL connectives

Operation	Interpretation
L-Negation	$i(' P) = 1 + j1 - i(P)$
Π-Delta	$\Delta(i(P)) = \text{if } (\text{i}(P)) = 1 + j1 \text{ else } \Delta\,(\text{i}(P)) = 0 + j0$
Equivalence	$i(P \leftrightarrow Q) = i(P_r \rightarrow_L Q_r) \otimes i(Q_r \rightarrow_L P_r) + j \cdot i(P_i \rightarrow_L Q_i) \otimes i(Q_i \rightarrow_L P_i)$
$P \ominus Q$	$i(P \ominus Q) = \max(0, i(p_r) - i(q_r)) + j \cdot \max(0, i(p_i) - i(q_i))$

Following classical logic, ŁΠ∀ CFL extends, ŁΠ CFL. The primitives include constants, variables, arbitrary-arity functions and arbitrary-arity predicates. Formulae are constructed using the basic connectives defined in Table 3, derived connectives such as the connectives presented in Table 4, the truth constants, the quantifier ∀ and the identity sign = The quantifier ∃ can be used to abbreviate formulae derived from the basic primitives and connectives. A fuzzy interpretation of a proposition $P(x_1, \ldots, x_n) = P_r(x_1, \ldots, x_n) + j.\ P_i(x_1, \ldots, x_m)$ over a domain M is a mapping that assigns a fuzzy truth value to each (n-tuple) × (m-tuple) of elements of M. As in the case of ŁΠ fuzzy logic, we closely follow the system used in ref [8].

The same axioms used for first order predicate fuzzy logic are used for first order predicate complex fuzzy logic; Modus ponens as well as product necessitation, and generalization are the rules of inference.

Complex Fuzzy Propositions and Inference Examples

Consider the following propositions:

1. P(x) ≡ "x is a *destructive hurricane with high surge*"
2. Q(x) ≡ "x is a destructive hurricane with fast moving center"

Let A be the term "*destructive hurricane*." Let B be the term "*high surge*," and let C be the term "*fast moving center*." Hence, P is of the form: "x is a A with B" and Q is of the form "x is A with C" In this case, the terms "*destructve hurricane*," "*high surge*," and "*fast moving center*," are values assigned to the linguistic variables $\{A, B, C\}$. Furthermore, the term "*destructve hurricane*" can get fuzzy truth values (between 0 and 1) or fuzzy linguistic values such as: "*catastriphic*," "*devastating*," and" *disastrous*." Assume that the complex fuzzy interpretation (i.e., the degree of confidence or complex fuzzy truth value) of P is $p_r + jp_i$, while the complex fuzzy interpretation of Q is $q_r + jq_i$. Thus, the truth value of "*x is a devastating hurricane*" is p_R, the truth value of "*x is in a high surge*" is p_i, the truth value of " x is a *catastriphic huricane*" is q_r, and the truth value of "x is a *fast moving center*" is q_i, Suppose that the term "*moderate*" stands for "*non* – destructive" which stands for "*NOT destructive*," the term "*low*" stands for "*NOT high*,", and the term "*slow*" stands for "*NOT fast*." In this context, *NOT* is interpreted as the fuzzy negation operation. Note that this is not the only way to define these linguistic terms and it is used to exemplify the expressive power and the inference power of the logic. Then, the complex fuzzy interpretation of the noted composite propositions is:

(1) $f({}'P) = (1 - p_r) + j(1 - p_I)$

That is, ${}'P$ denotes the proposition:

"*x is a moderate hurricane with a low surge*." The confidence level in ${}'P$ is $(1 - p_r) + j(1 - p_i)$; where the fuzzy truth value of the term "*x is a non – destructive hurricane*," is $(1 - p_r)$ and the fuzzy truth value of the term "low surge," is $(1 - p_i)$.

(2) $'P \rightarrow 'Q = \min(1, q_r - p_r) + j \times \min(1, q_i - p_I)$
Thus, $('P \rightarrow 'Q)$ denotes the proposition: *If* "x is a *moderate* hurricane with a low surge"
THEN x is a *moderate hurricane with low moving center*." The truth values of individual terms, as well as the truth value of $'P \rightarrow 'Q$ are calculated according to Table 1.

(3) $f(P \oplus 'Q) = \max(p_r, 1 - q_r) + j \times \max(p_i, 1 - q_i)$.
That is, $(P \oplus 'Q)$ denotes a proposition such as: "x is a *destructive hurricane with high surge*" OR
"x is a *moderate huricane with slow moving center*" The truth values of individual terms, as well as the truth value of $P \oplus 'Q$ are calculated according to Table 1.

(4) $f('P \otimes Q) = \min(1 - p_r, q_r) + j \times \min(1 - p_i, q_i)$
That is, $('P \otimes Q)$ denotes the proposition "*x is a moderate hurricane with low surge*" AND "*x is a destructive huricane with fast moving center*."
The truth values of individual terms, as well as the truth value of $'P \otimes Q$ are calculated according to Table 1.

Complex Fuzzy Inference Example

Assume that the degree of confidence in the proposition $R = 'P$ defined above is $r_r + jr_i$. Let $S = 'Q$ and assume that the degree of confidence in the fuzzy implication $T = R \rightarrow S$ is $t_r + jt_i$. Then, using Modus ponens

$$R \quad \frac{R \rightarrow S}{S}$$

one can infer S with a degree of confidence $\min(r_r, t_r) + j \times \min(r_i, t_i)$.
In other words if one is using:
"*x is a non − destructive hurricane with a low surge*"
$\underline{IF}$ "$\underline{x}$ $\underline{is}$ $\underline{a\ non}$ $\underline{-}$ $\underline{destructive}$ $\underline{hurricane}$ $\underline{with}$ $\underline{a}$ $\underline{low}$ $\underline{surge}$" $\underline{THEN}$
"$\underline{x}$ $\underline{is\ non}$ $\underline{-}$ $\underline{destructive}$ $\underline{huricane}$ $\underline{with}$ $\underline{slow}$ $\underline{moving}$ $\underline{center}$"
"*x is non − destructive huricane with slow moving center.*"

Hence, using Modus ponens one can infer:
"*x is moderate hurricane with slow moving center*." with a degree of confidence of $\min(r_r, t_r) + j \times \min(r_i, t_i)$.

3.4 Generalized Complex Fuzzy Class Theory (Tamir et al. [10])

The axiomatic fuzzy logic can serve as a basis for formal FCT. Similarly, axiomatic based complex fuzzy logic can serve as the basis for formal definition of complex fuzzy classes. In this section we provide a formulation of complex fuzzy class theory (CFCT) that is based on the logic theory presented in Sect. 3.3.

The main components of FCT are:

(1) Variables

  (a) Variables denoting objects (potentially complex objects)
  (b) Variables denoting crisp sets, i.e. a universe of discourse and its subsets
  (c) Variables denoting complex fuzzy classes of order 1
  (d) Variables denoting complex fuzzy classes of order n, that is, complex fuzzy classes of complex fuzzy classes of order n-1.

(2) The LΠ∀ CFL system along with its variables, connectives, predicates, and axioms as defined in Sect. 3.3.
(3) Additional predicates

  (a) A binary predicate $\in(x, \Gamma)$ denoting membership of objects in complex fuzzy classes and/or in crisp sets

(4) Additional Axioms

  (a) Instances of the comprehension schema (further explained below)

$$(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow P(x)) \tag{7}$$

  Where x is a complex fuzzy object, Γ is a complex fuzzy class, and $P()$ is a complex fuzzy predicate.
  (b) The axiom of extensionality

$$(\forall x)\Delta(x \in \Gamma \leftrightarrow x \in \Psi) \rightarrow \Gamma = \Psi \tag{8}$$

Where, x is a complex fuzzy object, Γ is a complex fuzzy class, and $P()$ is a complex fuzzy predicate.

Note that a grade of membership is not a part of the above specified terms; yet it can be derived or defined using these terms.

The comprehension schema is used to "construct" classes. It has the basic form of: $(\forall x)(x \in \Gamma \leftrightarrow P(x))$. Intuitively, this schema refers to the class Γ of all the objects x that satisfy the predicate $P()$. Instances of this schema have the generic form:$(\exists\Gamma)(\forall x)(x \in \Gamma \leftrightarrow P(x))$. Associated with this schema are comprehension terms of the form: $\in \{\mathrm{x}|P(x) \leftrightarrow P(y)\}$. The Δ operation introduced in Eq. 8 is used to produce precise instances of the extensionality schema and ensure the conservatism of comprehension terms.

Fixing a standard model over the CFCT enables the definition of commonly used terms, set operations, and definitions, as well as proving CFCT theorems. Some of these elements are listed here:

(1) The complex characteristic function $\chi_{x\in\Gamma} \equiv \chi_\Gamma$ and the grade of membership function $\mu_{x\in\Gamma} \equiv \mu_\Gamma$

(2) Complex class constants, α-cuts, iterated complements, and primitive binary operations, such as union, intersection etc. These operations are constructed using the schema $O_P(\Gamma) \equiv \{x | P(x \in \Gamma)\}$. Table 5 lists some of these elements.
(3) Uniform and supreme relations defined in ref. [8] enable the definition of fuzzy class relations such as inclusion
(4) Theorems, primitive fuzzy class operations, and fuzzy class relations [8].

Following the axiomatically-based definition of grade of membership, Eqs. (9-11) can be used as a basis for the definition of "membership grade based" complement, union, and intersection.

Table 5 Derived primitive class operations

Term	Symbol	P	Comments
Empty complex class	Θ	0	
Universal complex class	Φ	1	
Strict complement	\Γ	~	~ stands for Gödel (G) negation
Complex class intersection	∩	⊕	⊕ stands for a G, Ł, or Π conjunction T-norm
Complex class union	∪	∨	∨ stands for a G, Ł, or Π disjunction

Complex Fuzzy Classes and Connectives Examples

In order to provide a concrete example, we define the following complex fuzzy classes using the comprehension schema. Let the universe of discourse be the set of all the stocks that were available for trading on the opening of the New York stock exchange (NYSE) market on January 5, 2015 along with a set of attributes related to historical price performance of each of these stocks.

Consider the following complex propositions:
P(x) ≡ "x is a *volatile stock* in a *strong portfolio*"
Q(x) ≡ "x is a stock in a *decline trend* in a *strong portfolio*"

Then, the proposition: $(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow (P(x) \otimes Q(x))$, where x is any member of the universe of discourse, defines a complex fuzzy class Γ that can be "described" as the class of "*volatile stocks in a decline trend in strong portfolios*." On the other hand, the proposition $(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow ({}'P(x) \vee Q(x))$, where x is any member of the universe of discourse, defines a complex fuzzy class Γ that can be "described" as the class of "*non – volatile stocks in a decline trend in strong portfolios*."

3.5 Pure Complex Fuzzy Classes

Often it is useful to define complex fuzzy sets via membership functions rather than through axioms. To this end, Tamir et al. have introduced the concept of pure complex fuzzy sets [42]. This concept is reviewed in this section.

The Cartesian representation of the pure complex grade of membership is given in the following way:

$$\mu(V,x) = \mu_r(V) + j\mu_i(z) \tag{9}$$

Where $\mu_r(V)$ and $\mu_i(z)$, the real and imaginary components of the pure complex fuzzy grade of membership, are real value fuzzy grades of membership. That is, $\mu_r(V)$ and $\mu_i(z)$ can get any value in the interval $[0,1]$. The polar representation of the pure complex grade of membership is given by:

$$\mu(V,x) = r(V)e^{j\sigma\phi(z)} \tag{10}$$

Where $r(V)$ and $\phi(z)$, the amplitude and phase components of the pure complex fuzzy grade of membership, are real value fuzzy grades of membership. That is, they can get any value in the interval $[0,1]$. The scaling factor σ is in the interval $(0,2\pi]$. It is used to control the behavior of the phase within the unit circle according to the specific application. Typical values of σ are $\{1, \frac{\pi}{2}, \pi, 2\pi\}$.

The main difference between pure complex fuzzy grades of membership and the complex fuzzy grade of membership proposed by Ramot et al. [33, 34] is that both components of the membership grade are fuzzy functions that convey information about a fuzzy set.

4 Recent Developments in the Theory and Applications of CFL and CFS

In this section we review recent literature on complex fuzzy logic and complex fuzzy sets. First we review papers that enhance the theoretical basis of CFL/CFS. Next, we outline some of the recent reports on CFL/CFS related applications.

4.1 Advances in the Theoretical Foundations of CFL/CFS

Yager et al. have presented the idea of Pythagorean membership grades and the related idea of Pythagorean fuzzy subsets [68]. They have focused on the negation operation and its relationship to the Pythagorean Theorem. Additionally, they examined the basic set operations for the case of Pythagorean fuzzy subsets. Yager et al. further note that the idea of Pythagorean membership grades can provide an interesting semantics for complex number-valued membership grades used in complex fuzzy sets.

Greenfield et al. ([69]) have compared and contrasted the FCS formalism proposed by Ramot et al. ([33]) as well as the "innovation of pure complex fuzzy sets, proposed by Tamir et al. ([42])" with type-2 fuzzy sets [30, 37]. They have concentrated on the rationales, applications, definitions, and structures of these constructs. In addition, they have compared pure complex fuzzy sets with type-2 fuzzy sets in relation to inference operations. They have concluded that complex fuzzy sets and type-2 fuzzy sets differ in their roles and applications. They have identified similarities between pure complex fuzzy sets and type-2 fuzzy sets; but concluded that type-2 fuzzy sets were isomorphic to pure complex fuzzy sets.

Apolloni et al. propose to define and manage a complex fuzzy set by computing its membership function using a few variables quantized into a few elementary granules and elementary functions connecting the variables [70].

Guosheng et al. have introduced three complex fuzzy reasoning schemes: Principal Axis, Phase Parameters, and Concurrence Reasoning Scheme [49]. They have demonstrated that a variety of conjunction operators and implication operators can be selected to compose the corresponding instances of complex reasoning schemes.

Guangquan et al. have investigated various operation properties of complex fuzzy relations [71]. They have defined a distance measure for evaluating the differences between the grades as well as the phases of two complex fuzzy relations. Furthermore, they have used the distance measure to define δ-equalities of complex fuzzy relations. Finally, they have examined fuzzy inference in the framework of δ-equalities of complex fuzzy relations.

Tamir et al. have proposed a complex fuzzy logic (CFL) system that is based on the extended Post multi-valued logic system (EPS) of order $p > 2$, and have demonstrated its utility for reasoning with fuzzy facts and rules. The advantage of this formalism is that it is discrete. Hence, it better fits real time applications, digital signal processing, and embedded systems that use integer processing units.

4.2 Applications of CFL/CFS

A group of researchers working along with Dick have developed the concept of Adaptive Neuro Fuzzy Complex Inference System (ANCFIS) and explored related applications by integrating complex fuzzy logic into Adaptive Neuro Fuzzy Complex Inference System (ANFIS) [72].

Man et al. have extended the concept of ANFIS and introduced ACNFIS [73]. They have applied ANCFIS in time series forecasting. They compared ACNFIS to three commonly cited time series datasets and demonstrated that ACNFIS was able to accurately model relatively periodic data.

An extension of this work, including synthetic time series and several real-world forecasting problems, is presented by Zhifei et al. [74]. They have found that ANCFIS performs well on these problems and is also very parsimonious. Their work demonstrates the utility of complex fuzzy logic on real-world problems.

Aghakhani et al. have developed an online learning algorithm for ACNFIS and applied it to time series prediction [75]. Their experimental results show that the online technique is comparable to existing results, although slightly inferior to the off-line ANCFIS results.

Yazdanbaksh et al. applied ANCFIS to the problem of short-term forecasts of Photovoltaic power generation [76]. They compared ANFIS and radial basis function networks against ANCFIS. Their experimental results have demonstrated that the ANCFIS based approach was more accurate in predicting power output on a simulated solar cell. Additionally, in a recent paper Yazdanbaksh et al. presented a recommended approach to determining input windows that balances the accuracy and computation time [77].

Another group that is active in exploring CFL/CFS applications is led by Li [14, 78]. Li et al. have proposed a novel complex neuro-fuzzy autoregressive integrated moving average (ARIMA) computing approach and applied it to the problem of time-series forecasting [79]. They have found that their new formalism, referred to as CNFS-ARIMA, has excellent nonlinear mapping capability for time-series forecasting.

Additionally, Li et al. have presented a neuro-fuzzy approach using complex fuzzy sets (CNFS) for the problem of knowledge discovery [80]. They have devised a hybrid learning algorithm to evolve the CNFS for modeling accuracy, combining artificial bee colony algorithm and recursive least squares estimator method. They have tested the CNFS based approach in knowledge discovery through experimentation, and concluded that the proposed approach outperforms comparable approaches.

Another application of CNFS presented by Li et al. is adaptive image noise cancelling [81]. Two cases of image restoration have been used to test the proposed approach and have shown a good restoration quality. Additionally, Li et al. have presented a hybrid learning method that enables efficient and quick CNFS convergence procedure and applied the hybrid learning based CNFS to the problem of function approximation [14, 81]. They have concluded that the CNFS shows much better performance than its traditional neuro-fuzzy counterpart and other compared approaches.

Ma et al. applied complex fuzzy sets to the problem of multiple periodic factor prediction (MPFP) [46]. They have developed a product-sum aggregation operator (PSAO), which is a set of complex fuzzy sets. PSAO has been used to integrate information with uncertainty and periodicity. Next, they have developed a PSAO-based prediction (PSAOP) method to generate solutions for MPFP problems. The experimental results indicate that the proposed PSAOP method effectively handles the uncertainty and periodicity in the information of multiple periodic factors simultaneously and can generate accurate predictions for MPFP problems.

Tamir et al. have considered numerous applications of CFL [24, 36, 43, 66, 82, 83]. They have introduced several soft computing based methods and tools for disaster mitigation [24] and epidemic crises prediction [83]. Additionally, they have demonstrated the potential use of complex fuzzy graphs as well as incremental fuzzy clustering in the context of complex and high order fuzzy logic systems. Additionally, they have developed an axiomatic based framework for discrete complex fuzzy logic and set theory [66].

In [82] Tamir et al. have presented a complex fuzzy logic based inference system used to account for the intricate relations between software engineering constraints such as quality, software features, and development effort. The new model concentrates on the requirements specifications part of the software engineering process. Moreover, the new model significantly improves the expressive power and inference capability of the soft computing component in a soft computing based quantitative software engineering paradigm.

5 Conclusion

We have reviewed the theoretical basis of complex fuzzy logic and complex fuzzy sets and the current state of related applications. We have surveyed the research related to the underlying theory as well as recent applications of the theory in complex fuzzy based algorithms and complex fuzzy inference systems.

The concepts of complex fuzzy logic and complex fuzzy sets have undergone an evolutionary process since they were first introduced [35]. The initial definitions were practical but somewhat naïve and limited [33, 34, 65, 67]. The introduction of axiomatically based approach ([10, 36, 43]) has enabled extending the concepts, maintaining practicality, and providing a solid foundation for further theoretical development. Several applications of the new theories have emerged; based on recent reports this area of applications is gaining momentum.

There are numerous fields where fuzzy concepts interact in intricate ways, which can be effectively captured by the semantics of FCL FCS, pure complex fuzzy classes, and discrete signals. We plan to investigate some of these concepts in the near future. Of particular interest are multimedia signals. Often, these signals are represented using complex functions. On the other hand, due to noise, the processing of such signals might require using complex fuzzy logic. We plan to assess the utility of CFS, CFL, and complex fuzzy sets to the processing of signals in certain noisy situations.

Acknowledgment This work is based in part upon work supported by the National Science Foundation under grants I/UCRC IIP-1338922, AIR IIP-1237818, SBIR IIP-1330943, III-Large IIS-1213026, MRI (CNS-1429345, CNS-0821345, CNS-1126619), and CREST HRD-0833093 and by DHS S&T at TerraFly (http://terrafly.com) and the NSF CAKE Center (http://cake.fiu.edu).

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D.E. Tamir et al. (eds.), *Fifty Years of Fuzzy Logic and its Applications*,
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DOI 10.1007/978-3-319-19683-1