Approximation Algorithms for the Multilevel Facility Location Problem with Linear/Submodular Penalties

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Abstract. We consider two multilevel facility location problems with linear and submodular penalties respectively, and propose two approximation algorithms with performance guarantee 3 and $1+\frac{2}{1-e^{-2}}(\approx 3.314)$ for these two problems.

Keywords: Multilevel facility location problem · Submodular function · Approximation algorithm

1 Introduction

Facility location is an important area in combinatorial optimization with vast applications in operation research, computer science and management science.

The k-level facility location problem (k-LFLP) is a classical and extensively investigated problem, which has many applications. For example, typical products are shipped through manufacturers, warehouses and retailers before they can reach customers. One of the problems facing the business is to minimize the cost across these multi-level facilities. Formally, we are given a set $\mathcal{D}$ of clients, and k pairwise disjoint facility sets $\mathcal{F}^\ell$ ($\ell = 1, 2, \ldots, k$). Assume that the set $\mathcal{D} \cup \left(\cup_{\ell=1}^{k} \mathcal{F}^\ell\right)$ constitutes a metric space; that is, if $i, j \in \mathcal{D} \cup \left(\cup_{\ell=1}^{k} \mathcal{F}^\ell\right)$ are connected, then we pay a connected cost c_{ij}, which is symmetric, nonnegative and satisfies triangle inequalities. Each facility has an open cost $f_{i_\ell}, i_\ell \in \mathcal{F}^\ell$. Define a facility path as a sequence of k facilities $(i_1, i_2, \ldots, i_k)$ such that $i_\ell \in \mathcal{F}^\ell$, ($\ell = 1, 2, \ldots, k$). A path is *open* if all its facilities are open. If client j is connected to a path $p = (i_1, i_2, \ldots, i_k)$, the corresponding connection cost is defined as $c_{jp} = c_{ji_1} + \sum_{\ell=1}^{k-1} c_{i_\ell i_{l+1}}$. The goal of the k-LFLP is to serve all the clients in the set $\mathcal{D}$ by connecting each of them to an open path so as to minimize the total cost, including both connection and open cost. When k is equal to 1,

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the problem is reduced to the classic uncapacitated facility location problem, which is proved to be NP-hard. The FLP has been widely studied. Shmoys et al. [14] gave the first constant 3.16-approximation algorithm using LP-rounding technique, and Li [12] proposed the current best approximation factor 1.488, close to the lower bound 1.463 of this problem given by Guha and Khuller [7]. For the k-LFLP, Aardal et al. [1] gave the current best approximation ratio 3 using the stochastic LP-rounding technique, while Ageev et al. [3] presented the current best combinatorial algorithm with performance guarantee 3.27. Based on the stochastic LP-rounding, Wu and Xu [15] proposed a bifactor $\left(\frac{\ln(1/\beta)}{1-\beta}, 1+\frac{2}{1-\beta}\right)$-approximation algorithm[1], where $\beta \in (0,1)$ is a constant. In addition, Gabor et al. [6] gave a 3-approximation algorithm by adopting a new integer programming formulation with a polynomial number of variables and constraints. Li et al. [11] gave a cross-monotonic cost sharing method for the multi-level economic lot-sizing game. On the negative side, Krishnaswamy and Sviridenko [10] proved that the lower bound for the k-LFLP is 1.61. In light of the 1.488-approximation algorithm of Li [12] for the FLP, the k-LFLP obviously is harder to approximate.

The focus of this work is on the k-LFLP with penalties, where each client is either connected to an open facility path by paying connection cost or rejected for service with a penalty cost. The goal of the problem is to open a subset of facilities from $\mathcal{F}^\ell$, $\ell = 1, 2, \ldots, k$, such that each client $j \in \mathcal{D}$ is either connected to an open facility path, or rejected with a penalty cost so as to minimize the total facility cost, connection cost and penalty cost. If each client j has a fixed penalty cost q_j, the corresponding problem is called the k-LFLP with linear penalties. On the other hand, the k-LFLP with submodular penalties treats the penalty cost as a monotone increasing submodular function $h(\cdot)$ defined on the client set $\mathcal{D}$; that is, $h(A) \leq h(B)$ and $h(A \cup \{j\}) - h(A) \geq h(B \cup \{j\}) - h(B)$ for all $A \subseteq B$, $j \notin B$. Li et al. [8] extended the primal-dual technique of Jain and Vazirani [9] to the k-LFLP with submodular penalties and obtained a combinatorial 6-approximation algorithm. Based on LP-rounding, Asadi et al. [2] gave a 4-approximation algorithm for the k-LFLP with linear penalties. For any given k, Byrka et al. [4] considered the k-LFLP with linear penalties and gave an approximation algorithm whose approximation ratio converges monotonically to three when k tends to infinity.

Our main contribution is to give two approximation algorithms for the k-LFLP with submodular/linear penalties, which are the current best approximation ratio respectively. The rest of the paper is organized as follows. We present an $\left(1+\frac{2}{1-e^{-2}}\right)$-approximation algorithm for the k-FLP with submodular penalties in Sect. 2, and a 3-approximation algorithm for the k-FLP with linear penalties in Sect. 3.

[1] Let us denote the facility cost by F^* and the connection cost by C^* in the optimal solution. If an algorithm can get an integer feasible solution to the k-LFLP with the total cost no more than $aF^* + bC^*$ in polynomial time, then this algorithm is called a bifactor (a, b)-approximate algorithm for k-LFLP.

2 Multilevel Facility Location Problem with Submodular Penalties

In this section, we consider the k-LFLP with submodular penalties. We introduce several binary decision variables as follows: x_{jp} indicates whether client j is connected to path p or not; z_S indicates whether the client set S is penalized or not; and y_{i_ℓ} indicates whether facility i_ℓ is open or not. The k-LFLP with submodular penalties can be formulated as an integer linear program:

$$\begin{aligned}
\min\ & \sum_{\ell=1}^{k}\sum_{i_\ell\in\mathcal{F}^l} f_{i_\ell}y_{i_\ell}+\sum_{p\in\mathcal{P}}\sum_{j\in\mathcal{D}} c_{jp}x_{jp}+\sum_{S\subseteq\mathcal{D}} h(S)z_S\\
\text{s. t. } & \sum_{p\in\mathcal{P}} x_{jp}+\sum_{S\subseteq\mathcal{D}:j\in S} z_S\geq 1,\ \forall j\in\mathcal{D},\\
& \sum_{p:i_\ell\in p} x_{jp}-y_{i_\ell}\leq 0,\ \forall j\in\mathcal{D}, i_\ell\in\mathcal{F}^l,\ell=1,2,\ldots,k,\\
& x_{jp}\in\{0,1\},\ \forall p\in\mathcal{P}, j\in\mathcal{D},\\
& y_{i_\ell}\in\{0,1\},\ \forall i_\ell\in\mathcal{F}^\ell,\ell=1,2,\ldots,k,\\
& z_S\in\{0,1\},\ \forall S\subseteq\mathcal{D},
\end{aligned}\tag{1}$$

where the first constraints say that client j is either connected to a facility path or penalized, and the second constraints imply that client j must be connected to an open facility path.

Now we present the LP-based approximation algorithm for k-LFLP with submodular penalties. We consider the convex relaxation of problem (1):

$$\begin{aligned}
\min\ & \sum_{\ell=1}^{k}\sum_{i_\ell\in\mathcal{F}^l} f_{i_\ell}y_{i_\ell}+\sum_{p\in\mathcal{P}}\sum_{j\in\mathcal{D}} c_{jp}x_{jp}+h'(z)\\
\text{s. t. } & \sum_{p\in\mathcal{P}} x_{jp}+z_j\geq 1,\ \forall j\in\mathcal{D},\\
& \sum_{p:i_\ell\in p} x_{jp}-y_{i_\ell}\leq 0,\ \forall j\in\mathcal{D}, i_\ell\in\mathcal{F}^l,\ell=1,2,\ldots,k,\\
& x_{jp}\geq 0,\ \forall p\in\mathcal{P}, j\in\mathcal{D},\\
& y_{i_\ell}\geq 0,\ \forall i_\ell\in\mathcal{F}^\ell,\ell=1,2,\ldots,k,\\
& z_j\geq 0,\ \forall j\in\mathcal{D},
\end{aligned}\tag{2}$$

where $h'(z)$ is the Lovász extension function of any given submodular function $h(\cdot)$(cf. [13]),

$$\begin{aligned}
h'(z)=\max\sum_{j\in\mathcal{D}}\theta_j z_j &\\
\text{s. t.}\sum_{j\in S}\theta_j\leq h(S),\ \forall S\subseteq\mathcal{D},&\\
\theta_j\geq 0,\ j\in\mathcal{D}.&
\end{aligned}$$

It follows from Li et al. [13] that the function $h'(\cdot)$ has the following properties.

Property 1. $h'(I(S)) = h(S)$, and $h'(0) = h(\emptyset)$.
Property 2. $h'(aw) = ah'(w)$, $a \geq 0$ is a number; $w \geq 0$ is a vector.
Property 3. $h'(\cdot)$ is a monotonically increasing convex function.
Property 4. Given the optimal solution (x^*, y^*, z^*) to problem (2), and let $\theta^* := \arg\max h'(z^*)$. If $j \in \mathcal{D}$, $x^*_{jp} > 0$ and $z^*_j > 0$, then we have $\theta^*_j \geq c_{jp}$.

Now we illustrate the high level idea of our algorithm. Our algorithm is motivated by Li et al. [8] for the one-level FLP (1-LFLP) with submodular penalties. By considering the Lovász extension of submodular function, we overcome the difficulties brought by submodular penalties and separate the client set into the penalized set and the serviced set according to the optimal solution to the linear programming relaxation. First, we solve the convex relaxation problem (2) to get an optimal solution (x^*, y^*, z^*). Then based on the penalized/connected proportion of each client, we partition the client set $\mathcal{D}$ into two sets: the connected set $\mathcal{D}_\gamma$ and the penalized set $\bar{\mathcal{D}}_\gamma$ (where γ is a pre-specified parameter). Next, we consider an instance of the k-LFLP with the facility sets $\mathcal{F}_1, \ldots, \mathcal{F}_k$, and the client set $\mathcal{D}_\gamma$. Then, based on the optimal solution to problem (2), we construct a feasible solution $(\hat{x}, \hat{y})$ to the linear program relaxation for the k-LFLP with client set $\mathcal{D}_\gamma$. Finally, we call the bifactor $\left(\frac{\ln(1/\beta)}{1-\beta}, 1 + \frac{2}{1-\beta}\right)$-approximation algorithm for the k-LFLP by Wu and Xu [15] as a subroutine (cf. Steps 4–6 of Algorithm 1) to get a feasible integer solution $(\bar{x}, \bar{y})$ to the k-LFLP with the client set $\mathcal{D}_\gamma$. Combining with the penalized client set $\bar{\mathcal{D}}_\gamma$, we get a feasible integer solution $(\bar{x}, \bar{y}, \bar{z})$ to our problem.

The algorithm is given below.

Algorithm 1

Step 1. *Solve the LP relaxation (2) to obtain the fractional optimal solution* (x^*, y^*, z^*).

Step 2. *According to the penalized proportion of client* j*, we partition the clients set into two sets* $\mathcal{D}_\gamma = \left\{ j \in \mathcal{D} : \sum\limits_{p \in \mathcal{P}} x^*_{pj} \geq \frac{1}{\gamma} \right\}$*, and* $\bar{\mathcal{D}}_\gamma = \mathcal{D} \setminus \mathcal{D}_\gamma$*. Penalize the clients in* $\bar{\mathcal{D}}_\gamma$ *and set* $\bar{x}_{jp} := 0,\ \forall j \in \bar{\mathcal{D}}_\gamma, p \in \mathcal{P}$,

$$\bar{z}_S := \begin{cases} 1, & \text{if } S = \bar{\mathcal{D}}_\gamma \\ 0, & \text{otherwise} \end{cases}$$

Step 3. *For the served client set* $\mathcal{D}_\gamma$*, set*

$$\hat{x}_{jp} := \frac{x^*_{jp}}{1 - z^*_j},\ \forall j \in \mathcal{D}_\gamma, p \in \mathcal{P}, \quad \hat{y}_i := \min\{\gamma y^*_i, 1\}, \forall i \in \cup_{\ell=1}^k \mathcal{F}^\ell.$$

Then $(\hat{x}, \hat{y})$ *is a feasible solution for the relaxed* k*-LFLP with the client set* $\mathcal{D}_\gamma$ *and facility sets* $\mathcal{F}_1, \ldots, \mathcal{F}_k$.

Step 4. *Given a parameter* $\beta \in (0, 1)$*, choose randomly and uniformly* $\alpha \in (\beta, 1)$.

Step 5. $\forall j \in \mathcal{D}_\gamma$*, let* $\mathcal{P}_j := \{p \in \mathcal{P} : \hat{x}_{jp} > 0\}$*. Sort the paths in* $\mathcal{P}_j$ *in an nondecreasing distance to client* j*; that is,* $c_{jp^j_1} \leq c_{jp^j_2} \leq \ldots \leq c_{jp^j_{|\mathcal{P}_j|}}$*. Let* s^*_j *be the number satisfying* $\sum_{s \leq s^*_j} \hat{x}_{jp^j_s} \geq \alpha > \sum_{s \leq s^*_j - 1} \hat{x}_{jp^j_s}$ *and define* $c_j(\alpha) = c_{jp^j_{s^*_j}}$*. Calculate the average connection cost of client* j,

$$D_{av}(j) := \frac{\sum_{s=1}^{s_j^*} c_{jp_s^j}\hat{x}_{jp_s^j}}{\sum_{s=1}^{s_j^*} \hat{x}_{jp_s^j}}.$$

Step 6. *For the served client set $\mathcal{D}_\gamma$, iteratively generate greedily cluster centers as follows.*

Step 6.1 *Set $t = 1, \mathcal{C} := \emptyset, \mathcal{C}' = \mathcal{D}_\gamma$.*

Step 6.2 *Choose cluster center $j_t := \arg\min_{j \in \mathcal{C}'} c_j(\alpha) + D_{av}(j)$.*

Step 6.3 *Set $P_t := \{p_1^{j_t}, p_2^{j_t}, \ldots, p_{s_{j_t}^*}^{j_t}\}$,*

$F_t := \cup_{\ell=1}^{k}\{i_\ell \in \mathcal{F}^\ell : \exists p \in P_t, s.t. i_\ell \in p\}$,

$\mathcal{C}_t := \{j \in \mathcal{C}' : \exists s \le s_j^* \ s.t.\ p_s^j \cap F_t \neq \emptyset\}$.

Step 6.4 *Set $\mathcal{C}' := \mathcal{C}' \setminus \mathcal{C}_t$, $\mathcal{C} := \mathcal{C} \cup \{j_t\}$.*

Step 6.5 *We randomly open a facility path from the path set P_t; namely choose a path $p \in P_t$ with probability $\hat{x}_{j_t p} / \sum_{s=1}^{s_{j_t}^*} \hat{x}_{j_t p_s^{j_t}}$; round all variables $\{\bar{y}_{i_\ell}\}_{i_\ell \in p}$ and $\{\bar{y}_{i_\ell}\}_{i_\ell \in F_t \setminus p}$ to 1 and 0 respectivly; connect the cluster center j_t to this open path; and set $\bar{x}_{j_t p} = 1$ and $\bar{x}_{j_t p'} = 0$, $\forall p' \in \mathcal{P} \setminus p$.*

Step 6.6 *If $\mathcal{C}' \neq \emptyset$, set $t = t + 1$ and go to Step 6.2.*

Step 6.7 *Connect all the clients $j \in \mathcal{D}_\gamma \setminus \mathcal{C}$ to the closest open path.*

Since Steps 4–6 of Algorithm 1 is Wu and Xu [15]'s bifactor $\left(\frac{\ln(1/\beta)}{1-\beta}, 1 + \frac{2}{1-\beta}\right)$-approximation algorithm for the k-LFLP with feasible LP solution $(\hat{x}, \hat{y})$, the cost of integer feasible solution $(\bar{x}, \bar{y})$ generated is no more than $\frac{\ln(1/\beta)}{1-\beta}\hat{F} + 1 + \frac{2}{1-\beta}\hat{C}$, where $\hat{F}, \hat{C}$ are the facility cost and connection cost respectively of solution $(\hat{x}, \hat{y})$.

The performance of Algorithm 1 is summarized in the following theorem.

Theorem 1. *When $\beta = e^{-2}, \gamma = \frac{3-e^{-2}}{2}$, Algorithm 1 is an $\left(1 + \frac{2}{1-e^{-2}}\right)$-approximation algorithm for the k-LFLP with submodular penalties.*

3 Multilevel Facility Location Problem with Linear Penalties

Analogous to the k-LFLP with submodular penalties, the k-LFLP with linear penalties can also be formulated as an integer linear program:

$$\begin{aligned}
\min\ & \sum_{\ell=1}^{k}\sum_{i_\ell \in \mathcal{F}^l} f_{i_\ell} y_{i_\ell} + \sum_{p \in \mathcal{P}}\sum_{j \in \mathcal{D}} c_{jp} x_{jp} + \sum_{j \in \mathcal{D}} q_j z_j \\
\text{s. t.}\ & \sum_{p \in \mathcal{P}} x_{jp} + z_j \ge 1,\ \forall j \in \mathcal{D}, \\
& \sum_{p: i_\ell \in p} x_{jp} - y_{i_\ell} \le 0,\ \forall j \in \mathcal{D}, i_\ell \in \mathcal{F}^l, \ell = 1, 2, \ldots, k, \\
& x_{jp} \in \{0,1\},\ \forall p \in \mathcal{P}, j \in \mathcal{D}, \\
& y_{i_\ell} \in \{0,1\},\ \forall i_\ell \in \mathcal{F}^\ell, \ell = 1, 2, \ldots, k, \\
& z_j \in \{0,1\},\ \forall j \in \mathcal{D}.
\end{aligned} \tag{3}$$

The linear programming relaxation of the above program is as follows:

$$\begin{aligned}
\min \ & \sum_{\ell=1}^{k} \sum_{i_\ell \in \mathcal{F}^l} f_{i_\ell} y_{i_\ell} + \sum_{p \in \mathcal{P}} \sum_{j \in \mathcal{D}} c_{jp} x_{jp} + \sum_{j \in \mathcal{D}} q_j z_j \\
\text{s. t. } & \sum_{p \in \mathcal{P}} x_{jp} + z_j \geq 1, \ \forall j \in \mathcal{D}, \\
& \sum_{p: i_\ell \in p} x_{jp} - y_{i_\ell} \leq 0, \ \forall j \in \mathcal{P}, i_\ell \in \mathcal{F}^l, \ell = 1, 2, \ldots, k, \\
& x_{jp} \geq 0, \ \forall p \in \mathcal{P}, j \in \mathcal{D}, \\
& y_{i_\ell} \geq 0, \ \forall i_\ell \in \mathcal{F}^\ell, \ell = 1, 2, \ldots, k, \\
& z_j \geq 0, \ \forall j \in \mathcal{D}.
\end{aligned} \tag{4}$$

The dual of the problem (4) is

$$\begin{aligned}
\max \ & \sum_{j \in \mathcal{D}} \alpha_j \\
\text{s. t. } & \alpha_j - \sum_{i_\ell \in p} \beta_{i_\ell j} \leq c_{jp}, \ \forall p \in \mathcal{P}, j \in \mathcal{D}, \\
& \sum_{j \in \mathcal{D}} \beta_{i_\ell j} \leq f_{i_\ell}, \ \forall i_\ell \in \mathcal{F}^l, \ell = 1, 2 \ldots, k, \\
& \alpha_j \leq q_j, \ \forall j \in \mathcal{D}, \\
& \alpha_j \geq 0, \ \beta_{i_\ell j} \geq 0, \ \forall j \in \mathcal{D}, i_\ell \in \mathcal{F}^\ell, \ell = 1, 2 \ldots, k.
\end{aligned} \tag{5}$$

Lemma 2. *Let (x^*, y^*, z^*) and (α^*, β^*), respectively, be the optimal solutions to the primal and dual LP. Then we have*

(1) $x^*_{jp} > 0 \Rightarrow \alpha^*_j - \sum_{i_\ell \in p} \beta^*_{i_\ell j} = c_{jp}, \ c_{jp} \leq \alpha^*_j, \ \forall p \in \mathcal{P}, j \in \mathcal{D}$;

(2) $x^*_{jp} > 0,$ *and* $z^*_j > 0 \Rightarrow c_{jp} \leq q_j = \alpha^*_j, \ \forall p \in \mathcal{P}, j \in \mathcal{D}.$

Proof. If $x^*_{jp} > 0$, it follows from the complementary slackness condition that

$$\alpha^*_j - \sum_{i_\ell \in P} \beta^*_{i_\ell j} = c_{jp}.$$

Combining with $\beta^*_{i_\ell j} \geq 0, i_\ell \in p$, we get $c_{pj} \leq \alpha^*_j$.

If $x^*_{pj} > 0$ and $z^*_j > 0$, we have $c_{jp} \leq \alpha^*_j$ from Case 1. On the other hand, we have the complementary slackness condition $z^*_j(\alpha^*_j - q_j) = 0$, implying that $q_j = \alpha^*_j$.

Our algorithm is motivated by Aardal et al. [1]. First, we solve the linear program (4) and its relaxation (5) to obtain the optimal solutions (x^*, y^*, z^*) and (α^*, β^*). Two clients j and j' are $n$eighbors if there exist two paths p and p' such that $x^*_{jp} > 0$, $x^*_{j'p'} > 0$ and $p \cap p' \neq \emptyset$. Set $(\bar{x}, \bar{y}, \bar{z}) := (x^*, y^*, z^*)$. Iteratively revise $(\bar{x}, \bar{y}, \bar{z})$ untill we obtain a feasible integer solution to problem (3). In each iterative step, we maintain the feasibility of the solution $(\bar{x}, \bar{y}, \bar{z})$. Based on the optimal solution (x^*, y^*, z^*), we partition the set $\mathcal{D}$ of clients into the penalized set $\bar{\mathcal{D}}_\gamma$ of clients and the served set $\mathcal{D}_\gamma$ of clients.

Next for the client set $\mathcal{D}_\gamma$, we construct pairwise disjoint cluster sets, each containing a client called cluster center, all its neighbors and facilities from the path of fractionally serving that cluster center in the solution (x^*, y^*, z^*).

For each cluster set, we randomly choose an open path from the path set serving cluster center fractionally, and connect cluster center to this path. Because a non-cluster-center client j is possible to be the neighbor of two or more cluster centers, we connect these clients to the closest open path finally.

Introduce the following notation:

$$D_{av}(j) = \left(\sum_{p \in \mathcal{P}} c_{pj} x^*_{pj} \right) / (1 - z^*_j).$$

Algorithm 2

Step 1. *Solve the LP relaxation (4) and its dual (5) to obtain the optimal solutions of them* (x^*, y^*, z^*) *and* (α^*, β^*). *Set* $(\bar{x}, \bar{y}, \bar{z}) := (x^*, y^*, z^*)$.

Step 2. *According to the penalized proportion of client* j, *we partition the clients set into two sets* $\mathcal{D}_\gamma = \{j \in \mathcal{D} : \sum_{p \in P} x^*_{jp} \geq \frac{1}{\gamma}\}$, *and* $\bar{\mathcal{D}}_\gamma = \mathcal{D} \backslash \mathcal{D}_\gamma$. *We penalized the clients in the set* $\bar{\mathcal{D}}_\gamma$ *and set* $\bar{z}_j := 1, \bar{x}_{jp} := 0, \forall j \in \bar{\mathcal{D}}_\gamma, p \in \mathcal{P}$.

Step 3. *For the served clients set* $\mathcal{D}_\gamma$, *generate greedily cluster center as follows.*

Step 3.1 *Set* $t := 1, \mathcal{C} := \emptyset, \mathcal{C}' := D_\gamma$.

Step 3.2 *Choose cluster center* $j_t := \arg\min_{j \in \mathcal{C}'} \alpha^*_j + D_{av}(j)$.

Step 3.3 *Set* $P_t := \{p \in \mathcal{P} : x^*_{j_t p} > 0\}$,
$F_t := \cup_{\ell=1}^{k} \{i_\ell \in \mathcal{F}^\ell : \exists p \in P_t, \text{s. t. } i_\ell \in p\}$,
$\mathcal{C}_t := \{j \in \mathcal{D} : \exists p \in \mathcal{P} \text{s. t. } x^*_{jp} > 0, p \cap F_t \neq \emptyset\}$.

Step 3.4 *Set* $\mathcal{C}' := \mathcal{C}' \setminus \mathcal{C}_t$, $\mathcal{C} := \mathcal{C} \cup \{j_t\}$.

Step 3.5 *We randomly open a path with probability* 1 *from the path set* P_t. *Formally, choose a path* $p \in P_t$ *with probability* $\frac{x^*_{j_t p}}{1 - z^*_{j_t}}$ *and perform the following operations: round all the variables* $\{\bar{y}_{i_\ell}\}_{i_\ell \in p}$ *to* 1 *and all other variables* $\{\bar{y}_{i_\ell}\}_{i_\ell \in F_t \setminus p}$ *to* 0; *connect the cluster center* j_t *to this open path; and set* $\bar{x}_{j_t p} := 1$ *and* $\bar{x}_{j_t p'} := 0$, $\forall p' \in \mathcal{P} \setminus p$.

Step 3.6 *If* $\mathcal{C}' \neq \emptyset$, *set* $t := t + 1$, *go to Step 3.2.*

Step 3.7 *Connect all the clients* $j \in \mathcal{D}_\gamma \setminus \mathcal{C}$ *to the closest open path.*

The performance of Algorithm 2 is given below.

Theorem 3. *If* $\gamma = 1$, *then the expected total cost of* $(\bar{x}, \bar{y}, \bar{z})$ *is no more than 3 times the optimum cost, which implies that Algorithm 2 is a* 3*-approximation algorithm for the k-LFLP with linear penalties.*

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