Bonds Between L-Fuzzy Contexts Over Different Structures of Truth-Degrees

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Abstract. We consider the problem of bonds between L-fuzzy contexts over different complete residuated lattices. For this purpose we define (l, k)-connection and dual (l, k)-connection – pairs of mappings between the residuated lattices based on Krupka's results on factorizations of complete residuated lattices. We show that the bonds defined using the dual (l, k)-connection have very natural properties.

Keywords: Formal concept analysis · Galois connection · Bond · Factorization · Complete residuated lattice · Fuzzy set

1 Introduction

We study the problem of bonding formal fuzzy contexts over different structures of truth-degrees. This problem was addressed in [12][1] where the authors used residuation-preserving isotone Galois connections between complete residuated lattices to define bonds. We find the definition of residuation-preserving isotone Galois connection unnecessarily strict for its purpose and we take a new look at it.

Similarly as in [12] we look for an isotone Galois connection between two complete residuated lattices. We apply Krupka's results on factorization of residuated lattices [13] to find looser and more flexible requirements for the correspondence. As a result we obtain two interrelated correspondences between complete residuated concept lattices — (l, k)-connection and dual (l, k)-connection. Both of them can be considered to be a variant of the residuation-preserving isotone Galois connection from [12]. Using the dual (l, k)-connection we define bonds between formal fuzzy contexts over different complete residuated lattices.

The paper is organized as follows. In Sect. 2, we recall fundamental notions used in the paper. Sections 3 and 4 introduce the (l, k)-connection and dual the (l, k)-connection, respectively, and describe their properties. In Sect. 5 we utilize the new connections in formal concept analysis to define bonds between formal fuzzy contexts over different residuated lattices. Finally, Sect. 6 summarizes our conclusions and ideas for future research in this area.

[1] See [12] for motivations of the present research.

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2 Preliminaries

2.1 Residuated Lattices, Fuzzy Sets, and Fuzzy Relations

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$.

0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leqslant$. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as

$$\neg a = a \rightarrow 0, \tag{1}$$

and binary operation of biresiduum $\leftrightarrow$ as

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a) \quad \text{for each } a, b \in L \tag{2}$$

An $\mathbf{L}$-set (or $\mathbf{L}$-fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all $\mathbf{L}$-sets in a universe X is denoted L^X.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in L^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. An $\mathbf{L}$-set $A \in L^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(y) = 0$, we also write

$$\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_2, \ldots, {}^{A(x_n)}/x_n\}.$$

An $\mathbf{L}$-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An $\mathbf{L}$-set $A \in L^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$.

Binary $\mathbf{L}$-relations (binary $\mathbf{L}$-fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I).

Various composition operators for binary $\mathbf{L}$-relations were extensively studied by [7]; we will use the following three composition operators, defined for relations $A \in L^{X \times F}$ and $B \in L^{F \times Y}$:

$$(A \circ B)(x,y) = \bigvee_{f \in F} A(x,f) \otimes B(f,y), \tag{3}$$

$$(A \triangleleft B)(x,y) = \bigwedge_{f \in F} A(x,f) \to B(f,y), \tag{4}$$

$$(A \triangleright B)(x,y) = \bigwedge_{f \in F} B(f,y) \to A(x,f). \tag{5}$$

All of them have natural verbal descriptions. For instance, $(A \circ B)(x,y)$ is the truth degree of the proposition "*there is a factor f such that f applies to object x and attribute y is a manifestation of f*"; $(A \triangleleft B)(x,y)$ is the truth degree of "*for every factor f, if f applies to object x then attribute y is a manifestation of f*". Note also that for $L = \{0,1\}$, $A \circ B$ coincides with the well-known composition of binary relations.

2.2 Formal Fuzzy Concept Analysis

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary nonempty) sets and $I \in L^{X \times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x,y) = a$ is read: "The object x has the attribute y to degree a."

Consider the following pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of operators ${}^{\uparrow} : L^X \to L^Y$ and ${}^{\downarrow} : L^Y \to L^X$ induced by an **L**-context $\langle X, Y, I\rangle$ as

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to I(x,y) \quad \text{and} \quad B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x,y) \tag{6}$$

for all $A \in L^X$ and $B \in L^Y$.

Furthermore, denote the set of fixed points of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ by $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$, i.e.

$$\mathcal{B}^{\uparrow\downarrow}(X,Y,I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}. \tag{7}$$

The set of fixed points endowed with $\leqslant$, defined by

$$\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle \quad \text{if } A_1 \subseteq A_2 \text{ (equivalently } B_2 \subseteq B_1)$$

is a complete lattice [2,15], called a *standard* **L**-*concept lattice* associated with I, and its elements are called *formal concepts*. In a formal concept $\langle A, B\rangle$, the A is called an *extent*, and B is called an *intent*. The set of all extents and the set of all intents are denoted by $\mathrm{Ext}^{\uparrow\downarrow}$ and $\mathrm{Int}^{\uparrow\downarrow}$, respectively. That is,

$$\begin{aligned} \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I) &= \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}^{\uparrow\downarrow}(X,Y,I) \text{ for some } B\}, \\ \mathrm{Int}^{\uparrow\downarrow}(X,Y,I) &= \{B \in L^Y \mid \langle A, B\rangle \in \mathcal{B}^{\uparrow\downarrow}(X,Y,I) \text{ for some } A\}. \end{aligned} \tag{8}$$

An **L**-relation $\beta \in L^{X_1 \times Y_2}$ is called an **L**-*bond*[2] from **L**-context $\langle X_1, Y_1, I_1\rangle$ to **L**-context $\langle X_2, Y_2, I_2\rangle$ if

$$\begin{aligned} \mathrm{Ext}^{\uparrow\downarrow}(X_1,Y_2,\beta) &\subseteq \mathrm{Ext}^{\uparrow\downarrow}(X_1,Y_1,I_1), \\ \mathrm{Int}^{\uparrow\downarrow}(X_1,Y_2,\beta) &\subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2,Y_2,I_2). \end{aligned} \tag{9}$$

[2] The notion of **L**-bond was introduced in [11]; however we adapt its definition the same way as in [8,9].

3 (l,k)-Connections Between Complete Residuated Lattices

Similarly as in [12] we look for a pair of mappings $\lambda : \mathbf{L}_1 \to \mathbf{L}_2$ and $\kappa : \mathbf{L}_2 \to \mathbf{L}_1$ which form an isotone Galois connection. Set of its fixpoints with order defined as

$$\langle a_1, a_2 \rangle \leqslant \langle b_1, b_2 \rangle \quad \text{iff} \quad a_1 \leqslant b_1 \quad (\text{or equivalently } a_2 \leqslant b_2) \tag{10}$$

is a complete lattice. We denote it as $\mathbf{L}_{\langle\lambda,\kappa\rangle}$. We need to assure that an adjoint pair exists in $\mathbf{L}_{\langle\lambda,\kappa\rangle}$ and this pair is related to adjoint pairs of both, $\mathbf{L}_1$ and $\mathbf{L}_2$. To this purpose we apply Krupka's results on factorization of residuated lattices [13]. In fact, the problem can be reformulated as finding an isomorphism between some factorizations of $\mathbf{L}_1$ and $\mathbf{L}_2$ as depicted in Fig. 1.

Let us recollect Krupka's approach to factorization of complete residuated lattices. Krupka defines the factorization by cuts of biresiduum as follows. Consider a complete residuated lattice $\mathbf{L}$, a truth degree $e \in L$, and mappings

$$a^e = \bigvee\{b \in L \mid a \leftrightarrow b \geqslant e\} = e \to a, \tag{11}$$

$$a_e = \bigwedge\{b \in L \mid a \leftrightarrow b \geqslant e\} = e \otimes a. \tag{12}$$

For each $a \in L$ define intervals

$$[a]_e = [a_e, (a_e)^e] = [e \otimes a, e \to (e \otimes a)],$$
$$[a]^e = [(a^e)_e, a^e] = [e \otimes (e \to a), e \to a].$$

Denote $L/e = \{[a]^e \mid a \in L\} (= \{[a]_e \mid a \in L\})$. Then we have the following result.

Theorem 1 ([13]). $\mathbf{L}/e = \langle L/e, \wedge, \vee, \otimes, \to, 0, 1\rangle$, *where* $\wedge$ *and* $\vee$ *are given by the order*

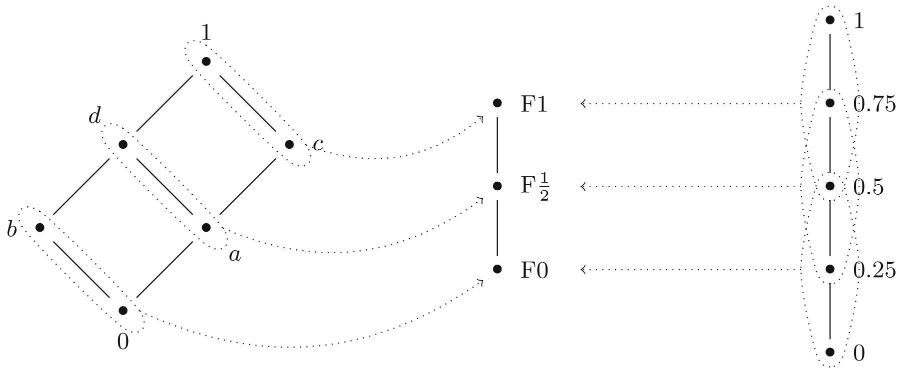
$$B_1 \leqslant B_2 \quad \textit{iff} \quad \bigvee B_1 \leqslant \bigvee B_2$$

and

$$B_1 \otimes B_2 = [\bigvee B_1 \otimes \bigvee B_2]_e,$$
$$B_1 \to B_2 = [\bigvee B_1 \to \bigvee B_2]_e,$$
$$0 = [0, e \to 0],$$
$$1 = [e, 1]$$

for each $B_1, B_2 \in L/e$, *is a complete residuated lattice.*

Following lemma shows alternative ways to define $\otimes$ and $\to$ in $\mathbf{L}/e$



$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	a	0	a
b	0	0	b	0	b	b
c	0	a	0	c	a	c
d	0	0	b	a	b	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	c	1	1
c	b	d	b	1	d	1
d	a	c	d	c	1	1
1	0	a	b	c	d	1

Fig. 1. Six-element residuated lattice, with $\otimes$ and $\rightarrow$ as showed in the bottom part (`011010:00A0B0BCAB` in [6]), factorized by c-cuts of biresiduum (left), five-element Łukasiewicz chain (`111:000AB` in [6]) factorized by 0.5-cuts of biresiduum (right), and their common lattice of factors (middle).

Lemma 1 ([13]). *For any $B_1, B_2 \in L/e$ we have*

$$\bigvee B_1 \otimes \bigwedge B_2 = \bigwedge B_1 \otimes B_2,$$
$$\bigvee B_1 \rightarrow \bigvee B_2 = \bigvee (B_1 \rightarrow B_2),$$
$$\bigwedge B_1 \rightarrow \bigwedge B_2 = \bigvee (B_1 \rightarrow B_2).$$

Note that the operators (11) and (12) form an isotone Galois connection on the complete residuated lattice $\mathbf{L}$. We extend this approach to have an isotone Galois connection between two (different) complete residuated lattices.

Definition 1. *Let $\mathbf{L}_1 = \langle L_1, \wedge_1, \vee_1, \otimes_1, \rightarrow_1, 0_1, 1_1 \rangle, \mathbf{L}_2 = \langle L_2, \wedge_2, \vee_2, \otimes_2, \rightarrow_2, 0_2, 1_2 \rangle$ be complete residuated lattices, let $l \in L_1, k \in L_2$ and let $\lambda : L_1 \rightarrow L_2, \kappa : L_2 \rightarrow L_1$ be mappings, such that*

1. *$\langle \lambda, \kappa \rangle$ is an isotone Galois connection between $\mathbf{L}_1$ and $\mathbf{L}_2$,*
2. *$\kappa\lambda(a_1) = l \rightarrow_1 (l \otimes_1 a_1)$ for each $a_1 \in L_1$,*
3. *$\lambda\kappa(a_2) = k \otimes_2 (k \rightarrow_2 a_2)$ for each $a_2 \in L_2$.*

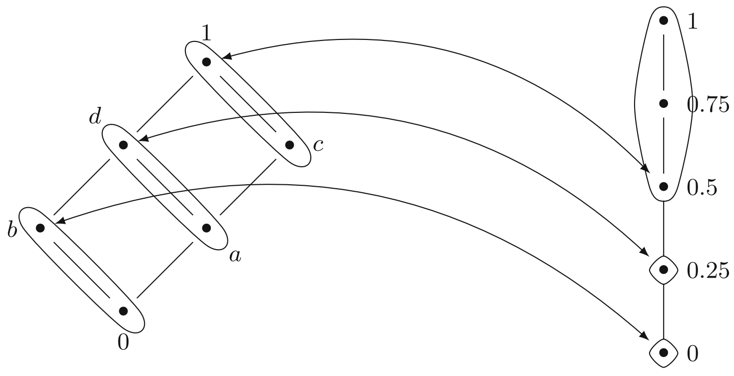


Fig. 2. $(c, 0.5)$-connection between the residuated lattices from Fig. 1.

We call $\langle \lambda, \kappa \rangle$ *an* (l, k)-connection *from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$.

Figure 2 shows an example of (l, k)-connection corresponding to the factorizations in Fig. 1.

Remark 1.

(a) A pair of identities $\langle \mathrm{id}, \mathrm{id} \rangle$ on a complete residuated lattice $\mathbf{L}$ is $(1, 1)$-connection from $\mathbf{L}$ to $\mathbf{L}$.
(b) It is worth noting that an (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$ is not uniquely given by the pair of truth degrees $l \in L_1, k \in L_2$ as more than one isomorphism between $\mathbf{L}_1/l$ and $\mathbf{L}_2/k$ can exist. For example, consider four-element complete residuated lattice $\mathbf{L}$ in Fig. 3 (left) with $\otimes = \wedge$ and $\rightarrow$ as in Fig. 3 (right) and mapping $f : L \rightarrow L$ given by $f(0) = 0, f(a) = b, f(b) = a$, and $f(1) = 1$. Then $\langle \mathrm{id}_L, \mathrm{id}_L \rangle$ and $\langle f, f \rangle$ are both $(1, 1)$-connections from $\mathbf{L}$ to $\mathbf{L}$.

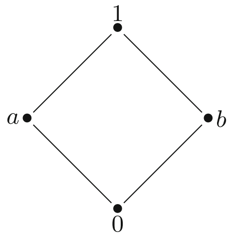
Utilizing Theorem 1 we can find particular adjoint pairs in the lattice of fixed points of $\langle \lambda, \kappa \rangle$.

Theorem 2. *Denote by* $L_{\langle \lambda, \kappa \rangle}$ *the set of all fixed points of* (l, k)*-connection* $\langle \lambda, \kappa \rangle$ *between* $\mathbf{L}_1$ *and* $\mathbf{L}_2$.

1. *The algebra* $\langle L_{\langle \lambda, \kappa \rangle}, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ *where* $\wedge$, $\vee$, 0, *and* 1 *are given by the order* (10) *and*

$$\langle a_1, a_2 \rangle \rightarrow \langle b_1, b_2 \rangle = \langle a_1 \rightarrow_1 b_1, \lambda(a_1 \rightarrow_1 b_1) \rangle,$$
$$\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle = \langle l \rightarrow (l \otimes_1 a_1 \otimes_1 b_1), \lambda(a_1 \otimes_1 b_1) \rangle$$

is a complete residuated lattice.



$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Fig. 3. Complete residuated lattice from Remark 1(b); (`0:a0b` in [6])

2. *The algebra* $\langle L_{\langle\lambda,\kappa\rangle}, \wedge, \vee, \boxtimes, \searrow, 0, 1\rangle$ *where* $\wedge$, $\vee$, 0, *and* 1 *are given by the order* (10) *and*

$$\begin{aligned}\langle a_1, a_2\rangle \searrow \langle b_1, b_2\rangle &= \langle \kappa(k \otimes_2 (a_2 \to_2 b_2)), k \otimes_2 (a_2 \to_2 b_2)\rangle \\ &= \langle \kappa((k \to_2 b_2) \to_2 (k \to_2 b_2)), (k \to_2 a_2) \to_2 (k \to_2 b_2)\rangle, \\ \langle a_1, a_2\rangle \boxtimes \langle b_1, b_2\rangle &= \langle \kappa(a_2 \otimes_2 (k \to_2 b_2)), a_2 \otimes_2 (k \to_2 b_2)\rangle \\ &= \langle \kappa((k \to_2 a_2) \otimes_2 b_2), (k \to_2 a_2) \otimes_2 b_2\rangle\end{aligned}$$

is a complete residuated lattice.

Proof. Directly from Definition 1, Theorem 1 and Lemma 1. □

Remark 2. For sake of completeness, we show how the meet, join, 0, and 1 given by the order (10) are defined in $L_{\langle\lambda,\kappa\rangle}$:

$$\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle = \langle a_1 \wedge_1 b_1, k \otimes_2 ((k \to_2 a_2) \wedge_2 (k \to_2 b_2))\rangle, \tag{13}$$

$$\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle = \langle l \to_1 ((l \otimes_1 a_1) \vee_1 (l \otimes_1 b_1)), a_2 \vee_2 b_2\rangle, \tag{14}$$

$$0 = \langle l \to 0_1, 0_2\rangle, \tag{15}$$

$$1 = \langle 1_1, k\rangle. \tag{16}$$

It is easy to see, that the two ajdoint pairs, $\langle \otimes, \to\rangle$ and $\langle \boxtimes, \searrow\rangle$, from Theorem 2 can be different. As an example consider $\mathbf{L}_1$ being three-element Łukasiewicz chain, $\mathbf{L}_2$ being three-element Gödel chain and λ and κ being identities on $L_1 = L_2$. The related factorizations, $\mathbf{L}_1/1$ and $\mathbf{L}_2/1$ are the three-element Łukasiewicz chain and the three-element Gödel chain, respectively, again. Clearly, their adjoint pairs are different.

We call the (l, k)-connections whose factorizations produce the same adjoint pair *residuation-preserving*. The following corollary shows that for residuation-preserving (l, k)-connection $\langle\lambda, \kappa\rangle$ we can specify the adjoint pair on the lattice of its fixed points without the mappings $\langle\lambda, \kappa\rangle$.

Corollary 1. *Let* $\langle\lambda, \kappa\rangle$ *be a residuation-preserving* (l, k)*-connection from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$*. The algebra* $\mathbf{L}_{\langle\lambda,\kappa\rangle} = \langle L_{\langle\lambda,\kappa\rangle}, \wedge, \vee, \otimes, \to, 0, 1\rangle$ *where* $\wedge, \vee, 0, 1$ *are given by* *the order* (10) *and*

$$\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle = \langle a_1 \to_1 b_1, k \otimes_2 (a_2 \to_2 b_2)\rangle \quad (17)$$
$$= \langle a_1 \to_1 b_1, k \otimes_2 ((k \to_2 a_2) \to_2 (k \to_2 b_2))\rangle, \quad (18)$$
$$\langle a_1, a_2\rangle \otimes \langle b_1, b_2\rangle = \langle l \to_1 (l \otimes_1 a_1 \otimes_1 b_1) a_2 \otimes_2 (k \to_2 b_2)\rangle \quad (19)$$
$$= \langle l \to_1 (l \otimes_1 a_1 \otimes_1 b_1), (k \to_2 a_2) \otimes_2 b_2\rangle \quad (20)$$

is a complete residuated lattice.

Proof. Directly from Theorem 2 and the property of residuation-preservation, that is $\otimes = \boxtimes$ and $\to = \searrow$. □

The following theorem provides more practical characterization of residuation-preserving (l, k)-connections.

Theorem 3. *Let $\langle\lambda, \kappa\rangle$ be an (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$. The following statements are equivalent*

(a) $\langle\lambda, \kappa\rangle$ is residuation-preserving.
(b) $\kappa(k \otimes_2 (\lambda(a) \to_2 \lambda(b))) = \kappa\lambda(a) \to_1 \kappa\lambda(b)$ holds true for any $a, b \in L_1$.
(c) $k \otimes_2 (\lambda\kappa(a) \to_2 \lambda\kappa(b)) = \lambda(\kappa(a) \to_1 \kappa(b))$ holds true for any $a, b \in L_2$.

Proof. (sketch) Follows from the fact, that pairs in $L_{\langle\lambda,\kappa\rangle}$ are exactly pairs $\langle\kappa\lambda(a_1), \lambda(a_1)\rangle$ for $a_1 \in L_1$ and exactly pairs $\langle\kappa(a_2), \lambda\kappa(a_2)\rangle$ for $a_2 \in L_2$. □

Note that left-hand sides of the equations in (b) and (c) of Theorem 3 contain an inconvenient multiplication by k. This leads to a quite cumbersome definition when we try to use them to define bonds between formal fuzzy context over different residuated lattices. In the next section we provide an alternative to (l, k)-connection which avoids this inconvenience.

4 Dual (l, k)-Connections Between Complete Residuated Lattices

We defined (l, k)-connections as an isotone Galois connection to assure that the set of its fixed points is a complete lattice and that it preserves order of both $\mathbf{L}_1$ and $\mathbf{L}_2$. But another property of isotone Galois connection, namely its non-duality, is undesired for our purpose, that is bonding fuzzy contexts over different residuated lattices. To fix this, we make a small trick with the (l, k)-connections. Instead of connecting upper bounds of intervals from $\mathbf{L}_1/l$ with lower bounds of intervals in $\mathbf{L}_2/k$, we simply connect upper bounds with upper bounds. To do that we need to drop the requirement of being an isotone Galois connection.

Definition 2. *Let $\mathbf{L}_1 = \langle L_1, \wedge_1, \vee_1, \otimes_1, \to_1, 0_1, 1_1\rangle$, $\mathbf{L}_2 = \langle L_2, \wedge_2, \vee_2, \otimes_2, \to_2, 0_2, 1_2\rangle$ be complete residuated lattices, let $l \in L_1, k \in L_2$ and let $\lambda' : L_1 \to L_2, \kappa' : L_2 \to L_1$ be mappings, such that*

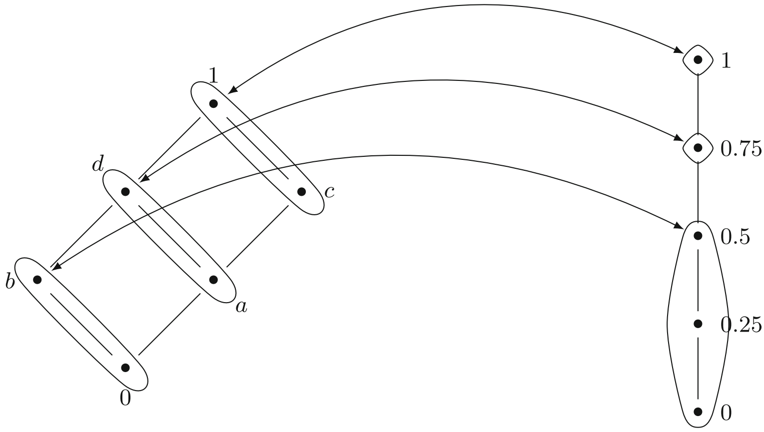


Fig. 4. Dual $(c, 0.5)$-connection between the residuated lattices from Fig. 1

– *λ' and κ' are order-preserving,*
– *$\lambda'\kappa'\lambda'(a_1) = \lambda'(a_1)$ and $\kappa'\lambda'\kappa'(a_2) = \kappa'(a_2)$ for each $a_1 \in L_1$ and $a_2 \in L_2$,*
– *$\kappa'\lambda'(a_1) = l \to_1 (l \otimes_1 a_1)$ for each $a_1 \in L_1$,*
– *$\lambda'\kappa'(a_2) = k \to_2 (k \otimes_2 a_2)$ for each $a_2 \in L_2$.*

We call the pair $\langle \lambda', \kappa' \rangle$[3] *a* dual (l, k)-connection *from $\mathbf{L}_1$ to $\mathbf{L}_2$.*

The notion of dual and non-dual (l, k)-connections are related in following way.

Theorem 4.

(a) For each (l, k)-connection $\langle \lambda, \kappa \rangle$ from $\mathbf{L}_1$ to $\mathbf{L}_2$ there is a dual (l, k)-connection $\langle \lambda', \kappa' \rangle$ from $\mathbf{L}_1$ to $\mathbf{L}_2$, such that for each $a_1 \in L_1, a_2 \in L_2$,

$$\begin{aligned} &\langle a_1, a_2 \rangle \in L_{\langle \lambda, \kappa \rangle} \textit{ implies } \langle a_1, k \to_2 a_2 \rangle \in L_{\langle \lambda', \kappa' \rangle}, \\ &\langle a_1, a_2 \rangle \in L_{\langle \lambda', \kappa' \rangle} \textit{ implies } \langle a_1, k \otimes_2 a_2 \rangle \in L_{\langle \lambda, \kappa \rangle}. \end{aligned} \tag{21}$$

(b) For each dual (l, k)-connection $\langle \lambda', \kappa' \rangle$ from $\mathbf{L}_1$ and $\mathbf{L}_2$ there is an (l, k)-connection $\langle \lambda, \kappa \rangle$ from $\mathbf{L}_1$ to $\mathbf{L}_2$ such that (21) is satisfied.

Proof.

(a) Let $\langle \lambda, \kappa \rangle$ be an (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$. We show that $\langle \lambda', \kappa' \rangle$ defined as

$$\lambda' = k \to_2 \lambda(a_1) \quad \text{and} \quad \kappa' = \kappa(k \otimes_2 a_2) \tag{22}$$

for each $a_1 \in L_1, a_2 \in L_2$ is a dual (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$ which satisfies (21). Since λ and κ are order-preserving and $\to$ and $\otimes$ are

[3] In this section, we consistently denote dual (l, k)-connections by prime, as $\langle \lambda', \kappa' \rangle$, to distinguish them from the non-dual (l, k)-connections introduced in the previous section.

both monotone in the second argument, the mapping λ' and κ' are order-preserving as well. We have for each $a_1 \in L_1, a_2 \in L_2$

$$\kappa'\lambda'(a_1) = \kappa(k \otimes_2 (k \rightarrow_2 (\lambda(a_1)))) = \kappa\lambda\kappa\lambda(a_1) = \kappa\lambda(a_1) = l \rightarrow_1 (l \otimes_1 a_1)$$

and

$$\lambda'\kappa'(a_2) = k \rightarrow_2 \lambda\kappa(k \otimes_2 a_2) = k \rightarrow_2 (k \otimes_2 (k \rightarrow_2 (k \otimes_2 a_2))) = k \rightarrow_2 (k \otimes_2 a_2).$$

Finally, we have for each $a_1 \in L_1, a_2 \in L_2$

$$\lambda'\kappa'\lambda'(a_1) = k \rightarrow_2 (k \otimes_2 (k \rightarrow_2 \lambda(a_1))) = k \rightarrow_2 \lambda(a_1) = \lambda'(a_1)$$

and

$$\kappa'\lambda'\kappa'(a_2) = \kappa'\lambda'(\kappa(k \otimes_2 a_2) = \kappa\lambda\kappa(k \otimes_2 a_2) = \kappa(k \otimes_2 a_2) = \kappa'(a_2).$$

Thus, $\langle \lambda', \kappa' \rangle$ is a dual (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$. Now we show that $\langle \lambda', \kappa' \rangle$ satisfies (21). Let $\langle a_1, a_2 \rangle \in L_{\langle \lambda, \kappa \rangle}$; from that we have

$$\begin{aligned} \langle a_1, k \rightarrow a_2 \rangle &= \langle \kappa(a_2), k \rightarrow_2 \lambda(a_1) \rangle \\ &= \langle \kappa\lambda(a_1), k \rightarrow_2 \lambda(a_1) \rangle \\ &= \langle \kappa'\lambda'(a_1), \lambda'(a_1) \rangle \end{aligned}$$

showing $\langle a_1, k \rightarrow_2 a_2 \rangle \in L_{\langle \lambda', \kappa' \rangle}$. The other part can be showed similarly.

(b) Similarly as in (a) we can show that $\langle \lambda, \kappa \rangle$ defined as

$$\lambda = k \otimes_2 \lambda'(a_1) \quad \text{and} \quad \kappa = \kappa'(k \rightarrow_2 a_2) \tag{23}$$

for each $a_1 \in L_1, a_2 \in L_2$ is a (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$ which satisfies (21).

□

What we get from this trick are more convenient operations $\wedge$ and $\rightarrow$ in the complete residuated lattice $\mathbf{L}_{\langle \lambda, \kappa \rangle}$ of fixed points of $\langle \lambda, \kappa \rangle$. That is important for definition of bonds because concept-forming operators $\langle \uparrow, \downarrow \rangle$ are defined using the operations $\wedge$ and $\rightarrow$.

Theorem 5. *The (l, k)-connections from $\mathbf{L}_1$ to $\mathbf{L}_2$ are in one-to-one correspondence with dual (l, k)-connections from $\mathbf{L}_1$ to $\mathbf{L}_2$.*

Proof. From proof of Theorem 4 we have (22) and (23) providing ways to get a dual (l, k)-connection from an (l, k)-connection and *vice versa*. We only need to show, that they are mutually inverse. Let $\langle \lambda, \kappa \rangle$ be an (l, k)-connection from $\mathbf{L}_1$ and $\mathbf{L}_2$ and let $\langle \lambda', \kappa' \rangle$ be a dual (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$ defined by (22). Applying (23) we get

$$\lambda''(a_1) = k \otimes_2 \lambda'(a_1) = k \otimes_2 (k \rightarrow_2 \lambda(a_1)) = \lambda\kappa\lambda(a_1) = \lambda(a_1)$$

for each $a_1 \in L_1$ and

$$\kappa''(a_2) = \kappa'(k \to_2 a_2) = \kappa(k \otimes_2 (k \to_2 a_2)) = \kappa\lambda\kappa(a_2) = \kappa(a_2)$$

for each $a_2 \in L_2$. Similarly, the other composition can be showed to be an identity. □

From the above one-to-one correspondence we obtain the following theorem.

Theorem 6. *Let be $\langle \lambda', \kappa' \rangle$ dual (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$.*

1. *The algebra $\langle L_{\langle \lambda', \kappa' \rangle}, \wedge, \vee, \otimes_1, \to_1, 0, 1 \rangle$ where $\wedge$ and $\vee$ are given by the order (10) and*

$$\langle a_1, a_2 \rangle \to \langle b_1, b_2 \rangle = \langle a_1 \to_1 b_1, \lambda'(a_1 \to_1 b_1) \rangle$$
$$\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle = \langle l \to_1 (l \otimes_1 a_1 \otimes_1 b_1), \lambda'(a_1 \otimes_1 b_1))) \rangle$$

is a complete residuated lattice.

2. *The algebra $\langle L_{\langle \lambda', \kappa' \rangle}, \wedge, \vee, \boxtimes, \searrow, 0, 1 \rangle$ where $\wedge$ and $\vee$ are given by the order (10) and*

$$\langle a_1, a_2 \rangle \searrow \langle b_1, b_2 \rangle = \langle \kappa'(a_2 \to_2 b_2), a_2 \to_2 b_2 \rangle$$
$$\langle a_1, a_2 \rangle \boxtimes \langle b_1, b_2 \rangle = \langle \kappa'(a_2 \otimes_2 b_2), k \to_2 (k \otimes_2 a_2 \otimes_2 b_2) \rangle$$

is a complete residuated lattice.

Proof. Directly from Theorems 2 and 4 and its proof, and Theorem 5. □

For sake of completeness, we also show how $\wedge, \vee, 0$ and 1 are defined in the complete residuated lattice from the previous theorem:

$$\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle = \langle a_1 \wedge_1 b_1, a_2 \wedge_2 b_2 \rangle, \tag{24}$$
$$\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle = \langle l \to_1 (l \otimes_1 (a_1 \vee_1 b_1)), k \to_2 (k \otimes_2 (a_2 \vee_2 b_2)) \rangle, \tag{25}$$

and $0 = \langle l \to 0_1, k \to 0_2 \rangle$, $1 = \langle 1_1, 1_2 \rangle$.

Again, we want the two adjoint pairs from Theorem 6 to be equal. We define the notion of residuation-preserving dual (l, k)-connection analogously, as in the non-dual case.

Theorem 7. *Let $\langle \lambda', \kappa' \rangle$ be a dual (l, k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$. The following statements are equivalent*

(a) $\langle \lambda', \kappa' \rangle$ *is residuation-preserving.*
(b) $\kappa'(\lambda'(a_1) \to_2 \lambda'(b_1)) = \kappa'\lambda'(a_1) \to_1 \kappa'\lambda'(b_1)$ *holds true for any* $a_1, b_1 \in L_1$.
(c) $\lambda'(\kappa'(a_2) \to_1 \kappa'(b_2)) = \lambda'\kappa'(a_2) \to_2 \lambda'\kappa'(b_2)$ *holds true for any for any* $a_2, b_2 \in L_2$.

Proof. Similar as proof of Theorem 3. □

Theorem 8. *A dual (l, k)-connection $\langle \lambda', \kappa' \rangle$ from $\mathbf{L}_1$ to $\mathbf{L}_2$ is residuation-preserving if and only if its associated (l, k)-connection is residuation-preserving.*

Proof. We have

$$\begin{aligned}\kappa'(\lambda'(a_1) \to_2 \lambda'(b_1)) &= \kappa(k \otimes_2 ((k \to_2 \lambda(a_1)) \to_2 (k \to_2 \lambda(b_1))))\\ &= \kappa(k \otimes_2 ((k \otimes_2 (k \to_2 \lambda(a_1))) \to_2 \lambda(b_1)))\\ &= \kappa(k \otimes_2 ((\lambda\kappa\lambda(a_1)) \to_2 \lambda(b_1)))\\ &= \kappa(k \otimes_2 (\lambda(a_1) \to_2 \lambda(b_1)))\end{aligned}$$

and

$$\kappa'\lambda'(a_1) \to_2 \kappa'\lambda'(b_1) = \kappa\lambda(a_1) \to_2 \kappa\lambda(b_1)$$

showing that the condition Theorem 7(b) is equivalent to Theorem 3(b). The statement of Theorem 8 then follows from Theorems 3 and 7. ☐

Remark 3. In the previous approach [12], the residuation-preserving Galois connections are defined as isotone Galois connections, as in the case of (l, k)-connections. In the same time, they have to satisfy conditions similar to Theorem 7(b) and (c), as in the case of dual (l, k)-connections. This is where we see the unnecessary strictness of the previous approach. Loosely speaking, the residuation-preserving isotone Galois connections were wanted to be both, (l, k)-connections and dual (l, k)-connections.

5 $\langle \boldsymbol{\lambda}, \boldsymbol{\kappa} \rangle$-Bonds

In this section, we define bond between formal fuzzy contexts over different complete residuated lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ and describe their properties. More specifically, we propose new bonds, called $\langle \lambda, \kappa \rangle$-bonds, which are based directly on dual (l, k)-connections[4] from $\mathbf{L}_1$ to $\mathbf{L}_2$. In this section we omit proofs due to page limit.

Below, we define the $\langle \lambda, \kappa \rangle$-bonds as a special $\mathbf{L}_{\langle \lambda, \kappa \rangle}$-relation β between X_1 and Y_2 and we define concept-forming operators $\vartriangle\colon L_1^{X_1} \to L_2^{Y_2}$ and $\triangledown\colon L_2^{Y_1} \to L_2^{X_2}$ induced by $\langle \lambda, \kappa \rangle$-bond by[5]

$$\begin{aligned} A^{\vartriangle}(y_2) &= \bigwedge\nolimits_{x_1 \in X_1}^{2} \lambda(A(x_1)) \to_2 \mathrm{proj}_2(\beta(x_1, y_2)),\\ B^{\triangledown}(x_1) &= \bigwedge\nolimits_{y_2 \in Y_2}^{1} \kappa(B(y_2)) \to_1 \mathrm{proj}_1(\beta(x_1, y_2)).\end{aligned} \tag{26}$$

Thus we can express the concept-forming operators $\langle \vartriangle, \triangledown \rangle$ using the classic ones, i.e. $\langle \uparrow, \downarrow \rangle$, as

$$A^{\vartriangle} = (\lambda(A))^{\uparrow_{\mathrm{proj}_2(\beta)}} \quad \text{and} \quad B^{\triangledown} = (\kappa(B))^{\downarrow_{\mathrm{proj}_1(\beta)}}$$

for each $A \in L_1^{X_1}$ and $B \in L_2^{Y_2}$.

[4] In this section $\langle \lambda, \kappa \rangle$ always denotes a dual (l, k)-connection.

[5] By proj_1 and proj_2 we denote projection of first and second entry of a pair, respectively; i.e. $\mathrm{proj}_1(\langle a_1, a_2 \rangle) \mapsto a_1, \mathrm{proj}_2(\langle a_1, a_2 \rangle) \mapsto a_2$.

Remark 4. The definition of concept-forming operators (26) actually follows as a corollary of particular setting in the framework of supremum-preserving aggregation structures. The framework was introduced in [3] and studied further in [4] (see also [1,5,10,14] for related works). We will bring detailed explanation in the full version of this paper.

Definition 3. *Let $\mathbf{L}_1, \mathbf{L}_2$ be complete residuated lattices, $\langle\lambda,\kappa\rangle$ be dual (l,k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$, and let $\langle X_1,Y_1,I_1\rangle$ and $\langle X_2,Y_2,I_2\rangle$ be $\mathbf{L}_1$-context and $\mathbf{L}_2$-context, respectively. We call $\beta \in L_{\langle\lambda,\kappa\rangle}^{X_1\times Y_2}$ a $\langle\lambda,\kappa\rangle$-*bond *from $\langle X_1,Y_1,I_1\rangle$ to $\langle X_2,Y_2,I_2\rangle$ if the following inclusions hold:*

$$Ext^{\Delta\nabla}(X_1,Y_2,\beta) \subseteq Ext^{\uparrow\downarrow}(X_1,Y_1,\kappa\lambda(I_1)), \tag{27}$$

$$Int^{\Delta\nabla}(X_1,Y_2,\beta) \subseteq Int^{\uparrow\downarrow}(X_2,Y_2,\lambda\kappa(I_2)). \tag{28}$$

Obviously, when $\mathbf{L}_1 = \mathbf{L}_2 = \mathbf{L}$ the pair of identities $\langle \mathrm{id},\mathrm{id}\rangle$ on L is a $(1,1)$-connection between them and the $\langle \mathrm{id},\mathrm{id}\rangle$-bonds correspond with $\mathbf{L}$-bonds. The following theorem explains the relationship of $\langle\lambda,\kappa\rangle$-bonds with the $\mathbf{L}$-bonds more generally.

Theorem 9. *Let $\beta \in L_{\langle\lambda,\kappa\rangle}^{X_1\times Y_2}$. The following statements are equivalent.*

*(a) β is a $\langle\lambda,\kappa\rangle$-*bond *from $\langle X_1,Y_1,I_1\rangle$ to $\langle X_2,Y_2,I_2\rangle$;*
(b) $\mathrm{proj}_1(\beta)$ is a $\mathbf{L}_1$-bond from $\langle X_1,Y_1,\kappa\lambda(I_1)\rangle$ to $\langle X_2,Y_2,\kappa(I_2)\rangle$;
(c) $\mathrm{proj}_2(\beta)$ is a $\mathbf{L}_2$-bond from $\langle X_1,Y_1,\lambda(I_1)\rangle$ to $\langle X_2,Y_2,\lambda\kappa(I_2)\rangle$;
(d) $\mathrm{proj}_1(\beta) = \lambda\kappa(I_1) \triangleright_1 S_{\mathrm{i}}$ and $\mathrm{proj}_2(\beta) = S_{\mathrm{e}} \triangleleft_2 \lambda\kappa(I_2)$ for some $S_{\mathrm{e}} \in L_1^{X_1\times X_2}$ and $S_{\mathrm{i}} \in L_2^{Y_1\times X_2}$.

From Theorem 9(a)⇔(d) we have the following corollary.

Corollary 2. *Set of all $\langle\lambda,\kappa\rangle$-bonds is an $\mathbf{L}_{\langle\lambda,\kappa\rangle}$-closure system.*

$\langle\boldsymbol{\lambda},\boldsymbol{\kappa}\rangle$-direct products and regular $\langle\boldsymbol{\lambda},\boldsymbol{\kappa}\rangle$-bonds

In this part, we assume that $\mathbf{L}_{\langle\lambda,\kappa\rangle}$ satisfies the *double negation law*, that is

$$(a \to 0) \to 0 = a \quad \text{for each } a \in L_{\langle\lambda,\kappa\rangle}.$$

Note that it means

$$\begin{aligned}\langle a_1,a_2\rangle &= (\langle a_1,a_2\rangle \to \langle l \to_1 0_1, k \to_2 0_2\rangle) \to \langle l \to_1 0_1, k \to_2 0_2\rangle \\ &= \langle (a_1 \to_1 (l \to_1 0)) \to_1 (l \to_1 0), (a_2 \to_2 (k \to_2 0_2)) \to_2 (k \to_2 0_2)\rangle\end{aligned}$$

for each $\langle a_1,a_2\rangle \in L_{\langle\lambda,\kappa\rangle}$.

Definition 4. *Let $\mathbb{K}_1 = \langle X_1,Y_1,I_1\rangle$ be an $\mathbf{L}_1$-context, $\mathbb{K}_2 = \langle X_2,Y_2,I_2\rangle$ be an $\mathbf{L}_2$-context, and $\langle\lambda,\kappa\rangle$ be a dual (l,k)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$. We define $\langle\lambda,\kappa\rangle$-*direct product *$\mathbb{K}_1 \boxplus_{\langle\lambda,\kappa\rangle} \mathbb{K}_2$ as $\mathbf{L}_{\langle\lambda,\kappa\rangle}$-context $\langle X_2 \times Y_1, X_1 \times Y_2, \Delta\rangle$ with*

$$\Delta(\langle x_2,y_1\rangle,\langle x_1,y_2\rangle) = \neg\langle \kappa\lambda I_1(x_1,y_1), \lambda I_1(x_1,y_1)\rangle \to \langle \kappa I_2(x_2,y_2), \lambda\kappa I_2(x_2,y_2)\rangle$$

for each $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$.

I_1	y_1	y_2	y_3
x_1	d	a	0
x_2	c	d	1

I_2	β_1	β_2	β_3
α_1	0.75	0.75	0.25
α_2	1	0.75	1

Δ	$\langle x_1, \beta_1\rangle$	$\langle x_1, \beta_2\rangle$	$\langle x_1, \beta_3\rangle$	$\langle x_2, \beta_1\rangle$	$\langle x_2, \beta_2\rangle$	$\langle x_2, \beta_3\rangle$
$\langle \alpha_1, y_1\rangle$	1	1	$\frac{1}{2}$	1	1	1
$\langle \alpha_1, y_2\rangle$	1	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$
$\langle \alpha_1, y_3\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1
$\langle \alpha_2, y_1\rangle$	1	1	1	1	1	1
$\langle \alpha_2, y_2\rangle$	1	1	1	1	1	1
$\langle \alpha_2, y_3\rangle$	1	$\frac{1}{2}$	1	1	1	1

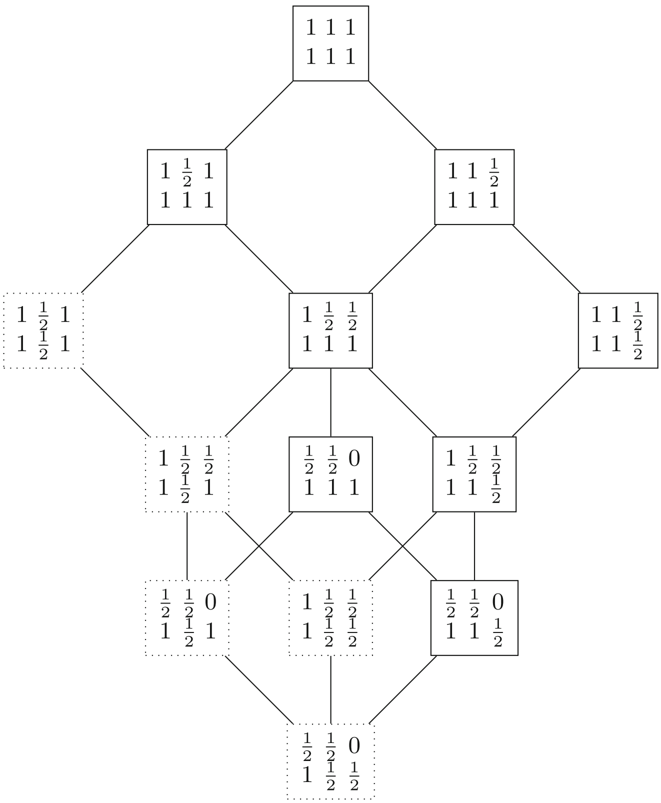


Fig. 5. A $\mathbf{L}_1$-context $\mathbb{K}_1$ and $\mathbf{L}_2$-context $\mathbb{K}_2$ (top left and top right) with $\mathbf{L}_1, \mathbf{L}_2$ as in Fig. 1; $\mathbb{K}_1 \boxplus_{\langle\lambda,\kappa\rangle} \mathbb{K}_2$ (middle) with $\langle\lambda, \kappa\rangle$ as in Fig. 4.; Lattice of all $\langle\lambda, \kappa\rangle$-bonds (bottom); the solid lined bonds are regular and the dotted lined bonds are irregular.

Extents of the $\langle \lambda, \kappa \rangle$-direct product are $\langle \lambda, \kappa \rangle$-bonds:

Theorem 10. *Let* $\mathbb{K}_1 \boxplus_{\langle \lambda, \kappa \rangle} \mathbb{K}_2 = \langle X_1 \times Y_2, X_2 \times Y_1, \Delta \rangle$ *be a* $\langle \lambda, \kappa \rangle$*-direct product. Extents in* $Ext^{\uparrow\downarrow}(X_1 \times Y_2, X_2 \times Y_1, \Delta)$ *are* $\langle \lambda, \kappa \rangle$*-bonds from* $\mathbb{K}_1$ *to* $\mathbb{K}_2$*.*

Analogously, with the **L**-bonds there exist $\langle \lambda, \kappa \rangle$-bonds which are not extents of the direct product $\mathbb{K}_1 \boxplus_{\langle \lambda, \kappa \rangle} \mathbb{K}_2$ (see Fig. 5). A $\langle \lambda, \kappa \rangle$-bond is called *regular* if it is extent of the direct product, otherwise it is called *irregular*.

6 Conclusions and Further Research

We revisited results on bonding formal fuzzy contexts in [12] and identified the main flaw: the residuation-preserving isotone Galois connections between complete residuated concept lattices had to fulfill two conflicting sets of requirements. In the present paper we studied two variants of residuation-preserving isotone Galois connections emerging by altering one of the two conflicting sets of requirements. One of the variants, namely dual (l, k)-connections, brought very convenient definition of bonds between formal fuzzy contexts with different structures of truth-degrees.

Our future research in this area includes:

- *Extension of the present results to homogeneous bonds wrt. isotone concept-forming operators and heterogeneous bonds studied in* [8,9]. We are going to generalize our previous results on bonds. Our preliminary observations show that (l, k)-connections will be useful for homogeneous bonds wrt. isotone concept-forming operators and for heterogeneous bonds.
- *Connections between complete residuated lattices based on antitone Galois connections.*

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