Chapter 7
Homomorphism Groups

Abstract The fact that the homomorphisms of a group into another group form an abelian group has proved extraordinarily profound not only in abelian group theory, but also in Homological Algebra where the functor Hom is one of the cornerstones of the theory. Our first aim is to find relevant properties of Hom both as a bifunctor and as a group.

It is rather surprising that in some significant cases $\mathrm{Hom}(A, C)$ is algebraically compact; for instance, when A is a torsion group, or when C is algebraically compact. In the special situation when C is the additive group $\mathbb{T}$ of the reals mod 1, in which case $\mathrm{Hom}(A, \mathbb{T})$, furnished with the compact-open topology, will be the character group of A, our description leads to a complete characterization of compact abelian groups by cardinal invariants. An analogous result deals with the linearly compact abelian groups.

The final section discusses special types of homomorphisms that play important roles in the theory of torsion groups.

1 Groups of Homomorphisms

Homomorphism Groups We have already noticed earlier that, if α and β are homomorphisms of A into C, then their sum $\alpha + \beta$, defined as

$$(\alpha + \beta)a = \alpha a + \beta a \qquad (a \in A),$$

is again a homomorphism $A \to C$. It is now routine to check that the homomorphisms of A into C form an abelian group under addition. This group is called the **homomorphism group** of A into C and is denoted by $\mathrm{Hom}(A, C)$. The zero in this group is the trivial homomorphism mapping A to $0 \in C$, and the inverse $-\alpha$ of $\alpha : A \to C$ maps $a \in A$ upon $-(\alpha a) \in C$.

If $A = C$, then the elements of $\mathrm{Hom}(A, A)$ are the endomorphisms of A, and the group $\mathrm{Hom}(A, A) = \mathrm{End}\, A$ is called the **endomorphism group** of A. This group carries a ring structure where the product $\alpha\beta$ of $\alpha, \beta \in \mathrm{End}\, A$ is defined by $(\alpha\beta)a = \alpha(\beta a)$ for $a \in A$ (observe the order of maps). The ring identity is the identity automorphism of A.

Next we list some simple facts on homomorphism groups.

(A) There are two important necessary conditions to satisfy when we are looking for homomorphisms $\alpha : A \to C$: one is that if $a \in A$ is annihilated by $n \in \mathbb{N}$, then also $n(\alpha a) = 0 \in C$, and the other is that we must have $h_p(\alpha a) \geq h_p(a)$.

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(B) $\text{Hom}(A, C) = 0$ *in the following cases:* (i) *A is torsion and C is torsion-free;* (ii) *A is a p-group and C is a q-group, for primes $p \neq q$;* (iii) *A is divisible and C is reduced.*
(C) *If $C[n] = 0$ for some $n \in \mathbb{N}$, then* $\text{Hom}(A, C)[n] = 0$ *for every group A.* Indeed, if $\alpha : A \to C$ and $n\alpha = 0$, then for $a \in A$ we have $n(\alpha a) = (n\alpha)a = 0$ whence $C[n] = 0$ implies $\alpha a = 0$, i.e. $\alpha = 0$.
(D) $\text{Hom}(A, C)$ *is torsion-free whenever C is torsion-free.*
(E) *If C is torsion-free and divisible, then so is* $\text{Hom}(A, C)$ *for every A.* In order to show that $\text{Hom}(A, C)$ is now also divisible, pick an $\alpha \in \text{Hom}(A, C)$ and an $n \in \mathbb{N}$. For $a \in A$, there exists a unique $c \in C$ with $nc = \alpha a$, and thus we may define a map $\beta : A \to C$ via $\beta a = c$. It follows readily that β is a homomorphism $A \to C$ satisfying $n\beta = \alpha$.
(F) *If $nA = A$ for some $n \in \mathbb{N}$, then* $\text{Hom}(A, C)[n] = 0$. Indeed, let $\alpha \in \text{Hom}(A, C)$ with $n\alpha = 0$. Write $a \in A$ as $a = nb$ for some $b \in A$. Then $\alpha a = \alpha(nb) = (n\alpha)b = 0$ shows that $\alpha = 0$.
(G) *If A is divisible, then* $\text{Hom}(A, C)$ *is torsion-free.*
(H) *If A is torsion-free and divisible, then the same holds for* $\text{Hom}(A, C)$, *for any C.* The proof is similar to the one in (E).

Example 1.1. If $A = \mathbb{Z}$, then every $\alpha : \mathbb{Z} \to C$ is completely determined by $\alpha(1) = c \in C$. Moreover, evidently, for every $c \in C$ there is a homomorphism $\gamma : \mathbb{Z} \to C$ such that $\gamma(1) = c$. Since $\alpha(1) = c_1$ and $\beta(1) = c_2$ imply $(\alpha + \beta)(1) = c_1 + c_2$, the correspondence $\gamma \mapsto c$ given by $\gamma(1) = c$ is a natural isomorphism between $\text{Hom}(\mathbb{Z}, C)$ and C,

$$\text{Hom}(\mathbb{Z}, C) \cong C \qquad \text{for all groups } C.$$

Example 1.2. If $A = \mathbb{Z}(m)$ with $m \in \mathbb{N}$, then again, every homomorphism $\alpha : \mathbb{Z}(m) \to C$ is determined by the image $\alpha(\bar{1}) = c$ of the coset $\bar{1} = 1 + m\mathbb{Z}$, but here $mc = 0$ must hold, i.e. $c \in C[m]$. Conversely, each such c gives rise to a homomorphism $\gamma : \bar{1} \mapsto c$, and as in the preceding example, the correspondence $\gamma \mapsto c$ given by $\gamma(\bar{1}) = c$ is a natural isomorphism

$$\text{Hom}(\mathbb{Z}(m), C) \cong C[m] \qquad \text{for all groups } C.$$

Example 1.3. From the preceding example we obtain

$$\text{Hom}(\mathbb{Z}(p^k), \mathbb{Z}(p^n)) \cong \mathbb{Z}(p^\ell) \quad \text{where } \ell = \min\{k, n\}.$$

Example 1.4. Next, let C be quasi-cyclic, say, $C = \langle c_1, \dots, c_n, \dots \rangle$ with the defining relations $pc_1 = 0, pc_{n+1} = c_n (n \geq 1)$. If η is an endomorphism of C, then write $\eta c_n = k_n c_n$ with an integer k_n $(0 \leq k_n < p^n)$ for every n. Now $k_n c_n = \eta c_n = \eta(pc_{n+1}) = p\eta c_{n+1} = pk_{n+1}c_{n+1} = k_{n+1}c_n$ implies $k_n \equiv k_{n+1} \bmod p^n$. This means that the sequence of the k_n is a Cauchy sequence in J_p, so it has a limit, say $\pi \in J_p$ is the limit. The correspondence $\eta \mapsto \pi$ between the endomorphisms η of C and the p-adic integers π is evidently additive. If the endomorphisms η_1 and η_2 define the same π, then $\eta_1 - \eta_2$ maps every c_n to 0, i.e. $\eta_1 = \eta_2$. On the other hand, if $\pi = s_0 + s_1 p + \dots + s_n p^n + \dots$ is any p-adic integer, then the correspondence $c_n \mapsto (s_0 + s_1 p + \dots + s_n p^n)c_n$ for all n (which we may write simply as πc_n) extends uniquely to an endomorphism η of C such that $\eta \mapsto \pi$. We conclude:

$$\text{End}\,\mathbb{Z}(p^\infty) \cong J_p.$$

Example 1.5. Consider $\mathbb{Q}^{(p)}$, the group of rational numbers with powers of p as denominators, and $C = \mathbb{Z}(p^\infty)$. Suppose $\mathbb{Q}^{(p)} = \langle 1, p^{-1}, \dots, p^{-n}, \dots \rangle$ and C as in the preceding example (and $c_n = 0$ for $n \leq 0$). A p-adic number $\rho = p^k\pi$ (with a p-adic unit π and $k \in \mathbb{Z}$) induces a homomorphism $\eta\colon \mathbb{Q}^{(p)} \to \mathbb{Z}(p^\infty)$ such that $p^{-n} \mapsto \pi c_{n-k}$ for all n. As in the preceding example, we can convince ourselves that different p-adic numbers ρ give rise to different homomorphisms, and every homomorphism $\eta\colon \mathbb{Q}^{(p)} \to \mathbb{Z}(p^\infty)$ arises in this way. Consequently, $\mathrm{Hom}(\mathbb{Q}^{(p)}, \mathbb{Z}(p^\infty))$ is isomorphic to the additive group of all p-adic numbers, i.e., we have

$$\mathrm{Hom}(\mathbb{Q}^{(p)}, \mathbb{Z}(p^\infty)) \cong \oplus_\kappa \mathbb{Q} \qquad \text{with } \kappa = 2^{\aleph_0}.$$

Example 1.6. If $A = C = J_p$, then it is evident that multiplication by a fixed p-adic integer π is an endomorphism of J_p (which we denote by $\dot{\pi}$), and different p-adic integers yield different endomorphisms of J_p, since they map $1 \in J_p$ differently. Let $\xi \in \mathrm{End}\, J_p$ such that $\xi(1) = \pi$. Then ξ and $\dot{\pi}$ are identical on $\mathbb{Z} \leq J_p$, so $\mathbb{Z} \leq \mathrm{Ker}(\xi - \dot{\pi})$. But $J_p/\mathbb{Z}$ is divisible, while J_p is reduced, so $J_p/\mathrm{Ker}(\xi - \dot{\pi}) = 0$. It follows that $\xi = \dot{\pi}$, and we have

$$\mathrm{End}\, J_p \cong J_p.$$

Hom and Direct Sums and Products Our next concern is the behavior of Hom towards direct sums and direct products. The following theorem is fundamental.

Theorem 1.7. *For an arbitrary index set I, there are natural isomorphisms*

$$\mathrm{Hom}(\oplus_{i\in I} A_i, C) \cong \prod_{i\in I} \mathrm{Hom}(A_i, C) \tag{7.1}$$

and

$$\mathrm{Hom}(A, \prod_{i\in I} C_i) \cong \prod_{i\in I} \mathrm{Hom}(A, C_i). \tag{7.2}$$

Proof. In order to prove (7.1), let $\rho_i : A_i \to \oplus A_i$ and $\pi_i : \oplus A_i \to A_i$ denote the injection and the projection maps, respectively. We map the left side of (7.1) to the right side by sending $\alpha : \oplus A_i \to C$ to $(\dots, \alpha\rho_i, \dots)$ where $\alpha\rho_i : A_i \to C$. This is evidently a homomorphism ϕ from the left to the right side. It is clear that ϕ maps α to 0 only if $\alpha = 0$. Since every $(\dots, \alpha_i, \dots) \in \prod \mathrm{Hom}(A_i, C)$ defines an $\alpha \in \mathrm{Hom}(\oplus A_i, C)$ *via* $\alpha = \oplus(\alpha_i\pi_i)$, ϕ is epic as well.

For the proof of (7.2), let $\sigma_i : C_i \to \prod C_i$ and $\tau_i : \prod C_i \to C_i$ denote the injection and the projection maps, respectively. Every $\beta \in \mathrm{Hom}(A, \prod C_i)$ defines a homomorphism $\tau_i\beta \in \mathrm{Hom}(A, C_i)$ for each i. As in the preceding paragraph, we conclude that the correspondence $\beta \mapsto (\dots, \tau_i\beta, \dots)$ is an isomorphism of the left-hand side of (7.2) with its right-hand side. $\square$

We can now derive the following corollary.

Corollary 1.8. *Assume A is a torsion group with p-components A_p, and C is a group with p-components C_p. Then*

$$\mathrm{Hom}(A, C) \cong \prod_p \mathrm{Hom}(A_p, C_p).$$

Proof. Apply (7.1) and observe that $\mathrm{Hom}(A_p, C) = \mathrm{Hom}(A_p, C_p)$. □

Example 1.9. For any group A,

$$\mathrm{Hom}(A, \mathbb{Q}) \cong \prod_{\mathrm{rk}_0(A)} \mathbb{Q}.$$

Because of (E), the description of $\mathrm{Hom}(A, \mathbb{Q})$ becomes a simple calculation in cardinal arithmetics. If F is a free subgroup of A generated by a maximal independent system of elements of infinite order only, then every $\phi: F \to \mathbb{Q}$ extends uniquely to a map $\alpha: A \to \mathbb{Q}$. This amounts to saying that $\mathrm{Hom}(F, \mathbb{Q}) \cong \mathrm{Hom}(A, \mathbb{Q})$ naturally. The former Hom is evaluated by using (7.1).

Hom As Bifunctor The correct way of viewing Hom is as a functor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$ associating the group $\mathrm{Hom}(A, C)$ with the ordered pair $(A, C) \in \mathcal{A}b \times \mathcal{A}b$. In the balance of this section we investigate the functorial behavior of Hom.

Let $\alpha : A' \to A$ and $\gamma : C \to C'$ be fixed homomorphisms. An $\eta \in \mathrm{Hom}(A, C)$ defines a homomorphism $A' \to C'$ as the composite $A' \xrightarrow{\alpha} A \xrightarrow{\eta} C \xrightarrow{\gamma} C'$. The correspondence $\eta \mapsto \gamma\eta\alpha$ is a homomorphism

$$\mathrm{Hom}(\alpha, \gamma)\colon\ \mathrm{Hom}(A, C) \to \mathrm{Hom}(A', C'),$$

called the **homomorphism induced by** α **and** γ. Clearly, $\mathrm{Hom}(\mathbf{1}_A, \mathbf{1}_C) = \mathbf{1}_{\mathrm{Hom}(A,C)}$. Furthermore, if $A'' \xrightarrow{\alpha'} A' \xrightarrow{\alpha} A$ and $C \xrightarrow{\gamma} C' \xrightarrow{\gamma'} C''$, then

$$\mathrm{Hom}(\alpha\alpha', \gamma'\gamma) = \mathrm{Hom}(\alpha', \gamma')\,\mathrm{Hom}(\alpha, \gamma).$$

Evidently, $\mathrm{Hom}(\alpha, \gamma)$ is additive in both arguments. Therefore, we can conclude:

Theorem 1.10. Hom *is an additive bifunctor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$, contravariant in the first and covariant in the second argument.* □

It is often convenient to use abbreviated notations (provided there is no danger of confusion):

$$\alpha^* = \mathrm{Hom}(\alpha, \mathbf{1}_C) \qquad \text{and} \qquad \gamma_* = \mathrm{Hom}(\mathbf{1}_A, \gamma).$$

The following result describes the behavior of Hom towards direct and inverse limits.

Theorem 1.11 (Cartan–Eilenberg [CE]). *Assume*

$$\mathfrak{A} = \{A_i\ (i \in I); \pi_i^k\} \qquad \textit{and} \qquad \mathfrak{C} = \{C_j\ (j \in J); \rho_j^\ell\}$$

are a direct and an inverse system of groups, respectively, and let $A = \varinjlim A_i$, $C = \varprojlim C_j$ *with canonical maps* $\pi_i: A_i \to A$ *and* $\rho_j: C \to C_j$. *Then*

$$\mathfrak{H} = \{\mathrm{Hom}(A_i, C_j)\ ((i,j) \in I \times J; \mathrm{Hom}(\pi_i^k, \rho_j^\ell)\}$$

is an inverse system of groups whose inverse limit is $\operatorname{Hom}(A, C)$ *with* $\operatorname{Hom}(\pi_i, \rho_j)$ *as canonical maps.*

Proof. It is straightforward to check that $\mathfrak{H}$ is an inverse system; let H denote its inverse limit. From the required commutativity of the triangles we can conclude that there exists a unique map ξ rendering all triangles

$$\begin{array}{ccc} \operatorname{Hom}(A,C) & \stackrel{\xi}{\dashrightarrow} & H \\ & {\scriptstyle \operatorname{Hom}(\pi_i,\rho_j)} \searrow \quad \swarrow {\scriptstyle \xi_{ij}} & \\ & \operatorname{Hom}(A_i, C_j) & \end{array}$$

commutative, where the ξ_{ij} are the canonical maps. To show that ξ is monic, let $\eta \in \operatorname{Ker} \xi$. Then $\xi_{ij}\xi\eta = 0$, that is, $\rho_j\eta\pi_i = \operatorname{Hom}(\pi_i, \rho_j)\eta = 0$ for all i, j. Thus the map $\eta\pi_i \colon A_i \to C$ is 0, because all of its jth coordinates are 0, and since $\cup_i \pi_i A_i = A$, we have $\eta = 0$.

Any $\chi \in H$ is of the form $\chi = (\dots, \chi_{ij}, \dots) \in \prod \operatorname{Hom}(A_i, C_j)$ where the coordinates χ_{ij} satisfy the requisite postulates. Define $\eta : A \to C$ as follows: if $a = \pi_i a_i$, then for this i set $\eta a = (\dots, \chi_{ij} a_i, \dots) \in \prod C_j$. It is straightforward to verify the independence of ηa of the choice of i as well as the homomorphism property of η. Considering that $\xi_{ij}\chi = \chi_{ij}$ and $\xi_{ij}\xi\eta = \rho_j\eta\pi_i = \chi_{ij}$, we must have $\xi\eta = \chi$, showing that ξ is epic. Thus ξ is an isomorphism. □

Hom and Exact Sequences Next we prove a most frequently used application of the Hom functor.

Theorem 1.12. *If* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is an exact sequence, then so are the induced sequences*

$$0 \to \operatorname{Hom}(G, A) \xrightarrow{\alpha_*} \operatorname{Hom}(G, B) \xrightarrow{\beta_*} \operatorname{Hom}(G, C) \tag{7.3}$$

and

$$0 \to \operatorname{Hom}(C, G) \xrightarrow{\beta^*} \operatorname{Hom}(B, G) \xrightarrow{\alpha^*} \operatorname{Hom}(A, G) \tag{7.4}$$

for every group G. Equation (7.3) *can always be completed to an exact sequence with* $\to 0$ *if G is a free group, and* (7.4) *if G is a divisible group.*

Proof. Let $\eta : G \to A$. If $\alpha\eta = 0$, then α monic implies $\eta = 0$, so α_* is also monic. Furthermore, $\beta\alpha\eta = 0$ shows that $\beta_*\alpha_* = 0$ as well. If $\xi \colon G \to B$ is such that $\beta\xi = 0$, then $\operatorname{Im} \xi \le \operatorname{Ker} \beta = \operatorname{Im} \alpha$, so there is a $\phi : G \to A$ with $\xi = \alpha\phi$. Thus (7.3) is exact. If we continue with $\to 0$, then exactness at $\operatorname{Hom}(G, C)$ would mean that for every $\zeta \colon G \to C$ there is a $\xi \colon G \to B$ such that $\beta\xi = \zeta$, this holds for free groups G, due to their projective property.

Next let $\eta \colon C \to G$. If $\eta\beta = 0$, then $\eta = 0$, since β is epic; thus β^* is monic. From $\eta\beta\alpha = 0$ we obtain $\alpha^*\beta^* = 0$. Assume $\xi : B \to G$ satisfies $\xi\alpha = 0$. This means that $\operatorname{Ker}\beta = \operatorname{Im}\alpha \le \operatorname{Ker}\xi$, so there is a $\phi\colon C \to G$ with $\xi = \phi\beta$. Thus (7.4) is exact. If G is injective, then for every $\zeta : A \to G$ there is a $\xi : B \to G$ such that $\zeta = \xi\alpha$, so $\to 0$ can be added to (7.4). □

More can be said if we start with a pure-exact sequence.

Proposition 1.13 (Fuchs [11]). *If the sequence given in the preceding theorem is pure-exact (p-pure-exact), then so are* (7.3) *and* (7.4).

Proof. First, let $\eta : G \to B$, $\xi : G \to A$ satisfy $n\eta = \alpha\xi$ ($n \in \mathbb{N}$). Thus $n\eta$ maps G into αA, and so $\operatorname{Im}\eta \le n^{-1}\alpha A$. By Theorem 2.10 in Chapter 5, $n^{-1}\alpha A = \alpha A \oplus X$ where $nX = 0$. If π denotes the projection onto the first summand, then $\phi = \alpha^{-1}\pi\eta\colon G \to A$ satisfies $n\phi = \xi$, establishing the first claim.

Next, assume $n\eta = \xi\beta$ holds for $\eta : B \to G$, $\xi : C \to G$ and $n \in \mathbb{N}$. Then $\eta(n\alpha) = \xi\beta\alpha = 0$ shows that $n\alpha A \le \operatorname{Ker}\eta$. Owing to Theorem 2.10 in Chapter 5, there is a direct decomposition $B/(n\alpha A) = \alpha A/(n\alpha A) \oplus B'/(n\alpha A)$ for some $B' \le B$. Define ϕ as the composite map $B \to B/(n\alpha A) \to B'/(n\alpha A) \to G$, where the second map is the canonical projection, while the third is induced by η. Clearly, $n\phi = n\eta$ and $\alpha A \le \operatorname{Ker}\phi$. Because of this inclusion, there is a homomorphism $\theta : C \to G$ such that $\phi = \theta\beta$. Hence $n(\theta\beta) = n\phi = n\eta = \xi\beta$, completing the proof. □

Small Groups A group G is said to be **small** if there is a natural isomorphism

$$\operatorname{Hom}(G, \oplus_{i\in I} C_i) \cong \oplus_{i\in I} \operatorname{Hom}(G, C_i)$$

for every set of groups C_i. Equivalently, the image of every homomorphism of G into an infinite direct sum is already contained in the direct sum of a finite number of summands. If the groups C_i in the definition are restricted to a class $\mathcal{C}$ of groups (e.g., to torsion-free groups), then G is called **$\mathcal{C}$-small**. In particular, if $\mathcal{C}$ is the direct sum of copies of G itself, then G is **self-small**.

Example 1.14. (a) Finitely generated groups are small, while finite rank torsion-free groups are $\mathcal{F}$-small, where $\mathcal{F}$ denotes the class of torsion-free groups.
(b) The quasi-cyclic group $\mathbb{Z}(p^\infty)$ is not small: it has a homomorphic image in $\oplus_{\aleph_0}\mathbb{Z}(p^\infty)$ that has non-zero projection in every summand (elements of order p^n have non-zero coordinates in the first n summands).

(a) Epic images of small groups are small.
(b) A finite direct sum of groups is small if and only if each component is small.
(c) *A group is small if and only if it is finitely generated.* If G is small, then its copy in an injective group must be contained in the direct sum of finitely many summands, whence we infer that G is of finite rank. The torsion subgroup of G cannot have infinitely many non-zero p-components, nor a quasi-cyclic summand (see Example 1.14b), so it must be finite. G/tG is a small torsion-free group, hence all of its torsion homomorphic images have to be small, so finite. Hence G/tG is a finite extension of a finitely generated free subgroup, so itself finitely generated.
(d) *A torsion-free group is small in the category of torsion-free groups if and only if it is of finite rank.*

★ **Notes.** The group structure of Hom has been the main topic of numerous investigations. It is impossible to survey them without the extensive knowledge of the material in later chapters. Perhaps the most important results are due to Pierce [1] that give a very precise description of Hom in case of p-groups, making use of Theorem 2.1. No comparable study is expected for torsion-free groups.

One question which we would like to point out here is concerned with the problem as to what extent the functor $\mathrm{Hom}(A, *)$ determines the group A. That A is by no means determined by this functor was proved by Hill [12] for p-groups and by Sebel'din [2] for torsion-free groups. The counterexamples are: 1) $A = \oplus_{2^{\aleph_0}} B$ and $A' = \oplus_{2^{\aleph_0}} \overline{B}$ where $B = \oplus_{\aleph_0} \oplus_{n<\omega} \mathbb{Z}(p^n)$ ($\overline{B}$ is the torsion-completion of B, see Sect. 3 in Chapter 10); and 2) $A = \oplus_{\aleph_0} \mathbb{Z} \oplus \mathbb{Q}$ and $A' = A \oplus \mathbb{Q}$. Then $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(A', G)$ holds for all G. Albrecht [7] deals with this question for p-groups and cotorsion groups.

An important generalization of homomorphism groups is concerned with groups with distinguished subgroups. The objects of the category $\mathcal{A}b_n$ are $\mathbf{A} = \{A; A_i (i < n)\}$ where $A \in \mathcal{A}b$, and $A_0, \dots, A_{n-1}$ are fixed subgroups of A. If $\mathbf{C} = \{C; C_i (i < n)\}$ is another object in this category, then $\phi : \mathbf{A} \to \mathbf{C}$ is a morphism if $\phi \in \mathrm{Hom}(A, C)$ such that $\phi(A_i) \leq C_i$ for all $i < n$. Results on homomorphism groups in such categories are instrumental in several questions concerning ordinary homomorphism groups.

Exercises

(1) Show that $\mathrm{Hom}(A, C)$ is isomorphic to a subgroup of C^A.
(2) We have $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(C, A)$ and $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong A$ if both A and C are finite groups.
(3) If A is torsion-free and C is divisible, then $\mathrm{Hom}(A, C)$ is divisible.
(4) Prove that $\mathrm{Hom}(A, \mathbb{Z}(m)) \cong \mathrm{Hom}(A/mA, \mathbb{Z}(m))$ for all $m \in \mathbb{N}$.
(5) If C is torsion-free, then $\mathrm{Hom}(\mathbb{Q}, C)$ is isomorphic to the maximal divisible subgroup of C.
(6) If the sequence $0 \to A \to B \to C \to 0$ is pure-exact, then (7.3) can be completed with $\to 0$ if G is Σ-cyclic, and (7.4) can so be completed if G is pure-injective.
(7) (a) If A is a torsion group, then the set union $\cup \,\mathrm{Im}\, \alpha$, taken for all $\alpha \in \mathrm{Hom}(A, C)$, is a subgroup of C.
    (b) The same is not necessarily true if A is torsion-free. [Hint: A of rank 2 with $\mathrm{End}\, A \cong \mathbb{Z}$, and $C = A \oplus A$.]
(8) Prove $\mathrm{End}\, J_p \cong J_p$ *via* the isomorphism $\mathrm{End}\, J_p \cong \varprojlim_n \mathrm{Hom}(J_p, \mathbb{Z}(p^n))$.
(9) Describe the structures of $\mathrm{End}(\oplus_\kappa \mathbb{Q})$ and $\mathrm{End}(\oplus_\kappa J_p)$ for a cardinal κ.
(10) If either A or C is a p-group, then $\mathrm{Hom}(A, C)$ is a J_p-module.
(11) If $\alpha \in \mathrm{Aut}\, A$, $\gamma \in \mathrm{Aut}\, C$, then $\mathrm{Hom}(\alpha, \gamma)$ is an automorphism of $\mathrm{Hom}(A, C)$.
(12) (Gerdt) G is small if and only if $G \leq \oplus_{i \in I} C_i$ implies $G \leq \oplus_{i \in J} C_i$ for some finite subset $J \subset I$. (Thus it suffices to consider monomorphisms.)
(13) (Gerdt) If $\mathcal{D}$ is the class of divisible groups, then $\mathcal{D}$-small groups are small.

2 Algebraically Compact Homomorphism Groups

Having considered elementary properties of Hom as well as the exact sequences involving Homs, we turn our attention to special situations when $\mathrm{Hom}(A, C)$ is of great interest. We concentrate on cases in which Hom is algebraically compact.

Hom for Torsion Groups We start with the remarkable fact that if A is a torsion group, then $\mathrm{Hom}(A, C)$ has to be algebraically compact, and hence it can be characterized by invariants describable in terms of the invariants of A and C.

Theorem 2.1 (Harrison [2], Fuchs [11]). *If A is a torsion group, then* $\mathrm{Hom}(A, C)$ *is a reduced algebraically compact group, for any C.*

Proof. It suffices to prove that if A is a p-group, then $H = \mathrm{Hom}(A, C)$ is complete in its p-adic topology. To show that H is Hausdorff, suppose $\eta \in H$ is divisible by every power of p. If $a \in A$ is of order p^k, and if $\chi \in H$ satisfies $p^k\chi = \eta$, then $\eta a = p^k\chi a = \chi p^k a = 0$ shows that $\eta = 0$. Next, let $\eta_1, \dots, \eta_n, \dots$ be a Cauchy sequence in the p-adic topology of H; dropping to a subsequence if necessary, we may assume it is neat: $\eta_{n+1} - \eta_n \in p^n H$ for each n, i.e. $\eta_{n+1} - \eta_n = p^n\chi_n$ for some $\chi_n \in H$. Let

$$\eta = \eta_1 + (\eta_2 - \eta_1) + \dots + (\eta_{n+1} - \eta_n) + \dots .$$

This is a well-defined map $A \to C$, since for $a \in A$ of order p^k, we have $(\eta_{n+1} - \eta_n)a = 0$ for all $n \geq k$, so that the image $\eta a = \eta_1 a + (\eta_2 - \eta_1)a + \dots + (\eta_k - \eta_{k-1})a$ is well defined. Furthermore,

$$\eta - \eta_n = (\eta_{n+1} - \eta_n) + (\eta_{n+2} - \eta_{n+1}) + \dots = p^n(\chi_n + p\chi_{n+1} + \dots),$$

where the infinite sum in the parentheses belongs to H. Thus $\eta - \eta_n \in p^n H$, and η is the limit of the given Cauchy sequence. Consequently, H is complete. □

We give a second, shorter proof based on Theorem 1.11. As a torsion group, A is the direct limit of its finite subgroups A_i. By Theorem 1.11, $\mathrm{Hom}(A, C)$ is then the inverse limit of the groups $\mathrm{Hom}(A_i, C)$ which are bounded in view of Sect. 1(F). Hence $\mathrm{Hom}(A, C)$ is the inverse limit of complete groups, and the assertion follows from Sect. 2, Exercise 7 in Chapter 6.

The invariants of $\mathrm{Hom}(A, C)$ for p-groups A can be computed, but the computation is very technical and lengthy, so we just refer the interested reader to Pierce [1]; see also Fuchs [IAG].

Hom for Compact Groups A most interesting case is when the Hom is a compact group. Next, we take a look at this situation.

We want to give an algebraic characterization of those groups that can carry a compact group topology. In Sect. 5 in Chapter 6 we have introduced the group $\mathrm{Char}\, G = \hat{G}$ as the group of all continuous homomorphisms of the topological group G into the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and observed that $\hat{G}$ is compact if and only if G is discrete. Actually, this is all that we need from the Pontryagin duality to describe the structure of compact groups:

Proposition 2.2. *A group A can carry a compact group topology if and only if it is of the form*

$$A = \mathrm{Hom}(G, \mathbb{T})$$

for some group G. □

Here Hom can be viewed in the algebraic or in the topological sense.

Character Groups Accordingly, our problem became purely algebraic: to classify the groups of the form $\mathrm{Hom}(G, \mathbb{T})$. Algebraically, $\mathbb{T}$ is nothing else than the direct product of quasi-cyclic groups, one for each prime p. Hence

$$\mathrm{Char}\ \ G \cong \prod_p \mathrm{Hom}(G, \mathbb{Z}(p^\infty)).$$

Consequently, it suffices to deal with $\mathrm{Hom}(G, \mathbb{Z}(p^\infty))$ only.

In describing the structure of this Hom, crucial role is played by the p-basic subgroups of G. So let us fix a p-basic subgroup B of G, and write

$$B = \oplus_{n=0}^{\infty} B_n \quad \text{where} \quad B_0 = \oplus_{\kappa_0} \mathbb{Z},\ B_n = \oplus_{\kappa_n} \mathbb{Z}(p^n) \ \text{ for } \ n \geq 1.$$

Here κ_n ($n \geq 0$) are cardinal numbers, uniquely determined by G. The p-component of G/B is of the form $\oplus_\kappa \mathbb{Z}(p^\infty)$; the fact that the cardinal κ depends on the choice of B is not relevant (as we shall see below), but it can be made unique by choosing e.g. a lower basic subgroup in the p-component of G. Finally, we let $\lambda = \mathrm{rk}_0(G/B)$. A full characterization of $\mathrm{Hom}(G, \mathbb{Z}(p^\infty))$ may be given with the aid of these cardinal numbers.

Theorem 2.3 (Fuchs [10]). *Using the above notation, for any group G we have*

$$\mathrm{Hom}(G, \mathbb{Z}(p^\infty)) \cong \prod_{\kappa_0} \mathbb{Z}(p^\infty) \oplus \prod_{n=1}^{\infty} \prod_{\kappa_n} \mathbb{Z}(p^n) \oplus \prod_{\kappa} J_p \oplus \prod_{\lambda \aleph_0} \mathbb{Q}. \tag{7.5}$$

Proof. The p-pure exact sequence $0 \to B \to G \to G/B \to 0$ induces the p-pure-exact sequence

$$0 \to \mathrm{Hom}(G/B, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(G, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(B, \mathbb{Z}(p^\infty)) \to 0.$$

Now Theorem 1.7 shows that

$$\mathrm{Hom}(B, \mathbb{Z}(p^\infty)) = \prod_{n=0}^{\infty} \mathrm{Hom}(B_n, \mathbb{Z}(p^\infty)) \cong \prod_{\kappa_0} \mathbb{Z}(p^\infty) \oplus \prod_{n=1}^{\infty} \prod_{\kappa_n} \mathbb{Z}(p^n).$$

If we write $G/B = \oplus_\kappa \mathbb{Z}(p^\infty) \oplus H$ with zero p-component for H, then because of $\mathrm{Hom}(\oplus_\kappa \mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong \prod_\kappa \mathrm{End}(\mathbb{Z}(p^\infty)) \cong \prod_\kappa J_p$, it remains to evaluate $\mathrm{Hom}(H, \mathbb{Z}(p^\infty))$. The p-pure subgroup L in H generated by a maximal independent set of elements of infinite order is torsion-free and p-divisible, and H/L is a torsion group with zero p-component, so $L = \oplus_\lambda \mathbb{Q}^{(p)}$. (This group L is not unique, not even its cardinality is well defined, but this does not influence the outcome.) The exactness of $0 \to L \to H \to H/L \to 0$ implies that of $0 = \mathrm{Hom}(H/L, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(H, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(L, \mathbb{Z}(p^\infty)) \to 0$, thus we obtain

$$\mathrm{Hom}(H, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(L, \mathbb{Z}(p^\infty)) \cong \prod_\lambda \mathrm{Hom}(\mathbb{Q}^{(p)}, \mathbb{Z}(p^\infty)) = \prod_\lambda (\prod_{\aleph_0} \mathbb{Q}),$$

where we have used Example 1.5. We observe that $\mathrm{Hom}(G/B, \mathbb{Z}(p^\infty))$ is algebraically compact, so its purity in $\mathrm{Hom}(G, \mathbb{Z}(p^\infty))$ implies that it is a summand. This completes the proof. □

If we determine the cardinal numbers $\kappa_0, \kappa_n, \kappa, \lambda$ for all primes, then Char G will be the direct product of groups (7.5) with p ranging over all primes. The group on the right side of (7.5) does not depend on the choice of κ, since the second summand always has a summand that is the product of κ' copies of J_p where $\kappa' = \mathrm{fin\ rk}\, tB$, so (7.5) has always the product of fin rk tG copies of J_p. A similar comment applies to the choice of λ (see also Theorem 2.6 below).

Observe that the first and the fourth summands in (7.5) come from elements of infinite order, while the two middle summands from the torsion subgroup of G. Hence:

Corollary 2.4. Char *G is reduced if and only if G is a torsion group, and is divisible if and only if G is torsion-free.* □

Since groups G can be found with arbitrarily chosen cardinals κ_n and κ, for every prime p, we can conclude:

Corollary 2.5 (Hulanicki [1], Harrison [1]). *A reduced group is the character group of some* (*torsion*) *group exactly if it is the direct product of finite cyclic groups and groups J_p for* (*distinct or equal*) *primes p.* □

For divisible groups, a simple inequality must be satisfied.

Theorem 2.6 (Hulanicki [1], Harrison [1]). *A divisible group $\neq 0$ is the character group of some* (*torsion-free*) *group if and only if it is of the form*

$$\prod_p \prod_{\mu_p} \mathbb{Z}(p^\infty) \oplus \prod_\mu \mathbb{Q} \qquad \text{where} \ \ \mu \geq \aleph_0.$$

Proof. If G is a torsion-free group, then its rank is, in the above notation, $\kappa_0(p) + \lambda(p)$ (the dependence on p must be indicated, but the sum is the same for every prime p). This shows that Char G will have the stated form with $\mu_p = \kappa_0(p)$, unless $\kappa_0(p) = 0$ for every prime p. In this case, the direct sum with $\prod_{\aleph_0} \mathbb{Q}$ does not change the isomorphy class of the first direct product.

Conversely, given a divisible group of the stated form, it is an easy exercise to check that μ may be replaced by $\mu + \sum_p \mu_p$. This says, in short, that $\mu \geq \mu_p$ may be assumed. Define G as a direct sum of rational groups G_i such that, for every prime p, μ_p of them satisfy $pG_i \neq G_i$ and μ of them satisfy $pG_i = G_i$. Then Char G will be as desired. □

Example 2.7. (a) For discrete groups $\mathbb{Z}(p^\infty), \mathbb{Q}$ we have Char $\mathbb{Z}(p^\infty) \cong J_p$ and Char $\mathbb{Q} \cong \mathbb{R}$.
(b) For discrete J_p, Char $J_p \cong \mathbb{Z}(p^\infty) \oplus \oplus_{2^{\aleph_0}} \mathbb{Q}$.

Corollary 2.8 (Kakutani). *The character group of a group of infinite cardinality κ is of the power 2^κ.*

Proof. The cardinality of an infinite group G is the sum of the cardinalities used in Theorem 2.3, taken for all primes p. (7.5) implies that then the group Char G must have cardinality 2^κ. □

The theorems above are convincing evidence that the algebraic structure of compact groups is extremely special. The cardinality of the set of all non-isomorphic groups of infinite cardinality $\leq 2^\kappa$ is 2^{2^κ}, but the number of those that can carry a compact group structure is minuscule. For instance, if $\kappa = \aleph_\alpha$ with $|\alpha| \leq \aleph_0$, then there are only countably many, pairwise (algebraically) non-isomorphic groups of cardinality κ that can be compact topological groups, provided we assume GCH. Indeed, then the cardinal invariants in Theorems 2.3 and 2.6 can be chosen not more than countably many ways, and they are unique due to GCH.

Example 2.9. This is an example of a group that can carry one and only one compact group topology: J_p^κ for any cardinal κ (it is compact in the finite index topology). In fact, Theorem 2.3 shows that the only discrete group whose character group is $\cong J_p^\kappa$ is the group $\oplus_\kappa \mathbb{Z}(p^\infty)$ where κ is unique if GCH holds (note that $|J_p^\kappa| = 2^\kappa$).

In contrast, some groups may be furnished with as many distinct compact topologies as possible, as is shown by the following theorem:

Theorem 2.10 (Fuchs [10]). *For any infinite cardinal κ, there exist 2^κ non-isomorphic compact topological groups of power 2^κ that are algebraically all isomorphic.*

Proof. In the proof we refer to Corollary 3.8 in Chapter 11 that asserts the existence of 2^κ non-isomorphic p-groups of cardinality κ: they can be chosen with isomorphic basic subgroups $\oplus_{n=1}^\infty \oplus_\kappa \mathbb{Z}(p^n)$, and they have the same final rank κ. By virtue of Theorem 2.3, their character groups are algebraically isomorphic to $\prod_{n=1}^\infty \prod_\kappa \mathbb{Z}(p^n) \oplus \prod_\kappa J_p$; however, by the Pontryagin duality theory, they are not isomorphic as topological groups. □

While we are still on the subject of compactness, it is worthwhile pointing out that Hom preserves (algebraic) compactness in the second argument.

Theorem 2.11. *If A is (algebraically) compact, then* $\operatorname{Hom}(G,A)$ *is (algebraically) compact for every group G.*

Proof. $\operatorname{Hom}(G,A)$ is isomorphic to a subgroup of the group A^G of all functions from G to A. If A is compact, then A^G is a compact group in which Hom is a closed subset. Hence $\operatorname{Hom}(G,A)$ is a compact group. (For the proof of algebraic compactness, use the summand property.) □

Linearly Compact Groups The structure of linearly compact groups (see Sect. 3 in Chapter 6) is similar to the compact case, though there are some notable differences. First and foremost is that the Kaplansky duality replaces the Pontryagin duality.

In Kaplansky's theory, the duality is established between the category of linearly compact and the category of discrete p-adic modules, for a fixed prime p. Thus only those abelian groups are participating in the duality that are also J_p-modules. The characters are continuous homomorphisms into the discrete p-adic module $\mathbb{Z}(p^\infty)$. If M is a discrete p-adic module, then its character module $\operatorname{Hom}_{J_p}(M,\mathbb{Z}(p^\infty))$ is a linearly compact J_p-module, furnished with the compact-open topology. On the other hand, if M is a linearly compact J_p-module, then the continuous homomorphisms of M into $\mathbb{Z}(p^\infty)$ yield a discrete J_p-module.

Example 2.12. The Kaplansky dual of the discrete group $\mathbb{Z}(p^\infty)$ is the linearly compact group J_p, and *vice versa*.

Consequently, the linearly compact p-adic modules are, from the pure algebraic point of view, nothing else than the groups $\operatorname{Hom}_{J_p}(M,\mathbb{Z}(p^\infty))$ where M ranges over the class of discrete p-adic modules. Hence, from the proof above on the character groups one can derive:

Theorem 2.13 (Fuchs [15]). *A group admits a linearly compact topology if and only if it is the direct product of groups of the following types:*

(a) *Cocyclic groups:* $\mathbb{Z}(p^n), \mathbb{Z}(p^\infty)$ *for any prime p and $n \in \mathbb{N}$;*
(b) *The additive group J_p of the p-adic integers and the additive group of the field $\mathbb{Q}_p^*$ of the p-adic numbers, for each prime p.* □

★ **Notes.** The character group of a discrete left module (over any ring) is a right module that is compact in the compact-open topology. Theorem 2.11 also extends to modules. It was S. Lefschetz who introduced linearly compact vector spaces, and later the theory was extended to modules. Linearly compact modules have an extensive theory.

Exercises

(1) Prove that $\operatorname{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}^{\aleph_0}$.
(2) (a) If A is algebraically compact, and if H is a pure subgroup of a group G, then $\operatorname{Hom}(G,A) \cong \operatorname{Hom}(H,A) \oplus \operatorname{Hom}(G/H,A)$.
   (b) Show that then Char $G \cong$ Char $H\oplus$ Char G/H (isomorphism in the algebraic sense only).

(3) The additive group $\mathbb{R}$ of the reals can be furnished with infinitely many distinct topologies, each yielding non-isomorphic compact groups. [Hint: Char $(\oplus_n \mathbb{Q})$.]
(4) The group $A = \mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ is the character group of $\oplus_{\aleph_0}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$. [Hint: Exercise 9 in Sect. 3 in Chapter 6.]
(5) If C is a complete group, then $\mathrm{Hom}(A, C)$ is the inverse limit of bounded groups.
(6) (a) Assume A is equipped with a non-discrete compact topology. Then it has a subgroup (algebraically) isomorphic either to 1) J_p for some p, or to 2) an infinite direct product of cyclic groups of prime orders. [Hint: Theorem 2.3.]
(b) A group admits a non-discrete locally compact topology if and only if it has a subgroup of kind 1) or 2).
(7) (Faltings) Let A be a p-group. Then $t(\mathrm{Hom}(A, \mathbb{Z}(p^\infty)) \cong A$ if and only if A is torsion-complete with finite UK-invariants.
(8) Calculate the invariants of the algebraically compact group $\mathrm{Hom}(A, C)$ in case A is Σ-cyclic and $C = \oplus_\kappa \mathbb{Z}(p^\infty)$.
(9) Find the invariants of $\mathrm{Hom}(A, C)$ if A is torsion-free and $C = \oplus_\kappa \mathbb{Z}(p^\infty)$. [Hint: take p-basic in A.]
(10) Give a detailed proof of Theorem 2.13 for linearly compact groups.

3 Small Homomorphisms

This section should be read after getting familiar with the basic material from Chapter 10; in particular, with large subgroups to be discussed in Sect. 2 there.

Small Homomorphisms Let A, C be p-groups. Following Pierce [1], we call a homomorphism $\phi : A \to C$ **small** if $\mathrm{Ker}\,\phi$ contains a large subgroup of A. In other words, the Pierce condition (Sect. 2 in Chapter 10) must be satisfied: given $k > 0$, there exists $n > 0$ such that

$$p^n A[p^k] \leq \mathrm{Ker}\,\phi.$$

Example 3.1. The map η in the proof of Szele's theorem 6.10 in Chapter 6 is a small endomorphism of the p-group A; its image is a basic subgroup.

(A) *Elements of infinite height belong to the kernel of every small homomorphism,* since the first Ulm subgroup is contained in each large subgroup (see Sect. 2(D) in Chapter 10).
(B) *The small homomorphisms $\phi : A \to C$ form a subgroup of the group* $\mathrm{Hom}(A, C)$. Observe that if $p^{n_1} A[p^k] \leq \mathrm{Ker}\,\phi_1$ and $p^{n_2} A[p^k] \leq \mathrm{Ker}\,\phi_2$, then $p^n A[p^k] \leq \mathrm{Ker}(\phi_1 + \phi_2)$ holds with $n = \max\{n_1, n_2\}$. The group of small homomorphisms will be denoted by $\mathrm{Hom}_s(A, C)$.
(C) $\mathrm{Hom}_s(A, C) = \mathrm{Hom}(A, C)$ *whenever either A or C is bounded.* The latter holds as $p^m A$ is a large subgroup of A for each $m > 0$.

(D) *The factor group* $\mathrm{Hom}(A, C)/\mathrm{Hom}_s(A, C)$ *is torsion-free.* This follows from the fact that if $p^m\phi$ is a small homomorphism for some $\phi: A \to C$ and $m > 0$, then ϕ must be small, as well.

(E) *If* $\phi: A \to C$ *is a small homomorphism, then* $B + \mathrm{Ker}\,\phi = A$ *for any basic subgroup* B *of* A. This is a consequence of the definition of large subgroups.

Lemma 3.2 (Pierce [1]). *Let B be a basic subgroup of the p-group A. There is a natural isomorphism*

$$\mathrm{Hom}_s(A, C) \to \mathrm{Hom}_s(B, C)$$

given by the restriction map: $\phi \mapsto \phi \restriction B$ *where* $\phi: A \to C$.

Proof. If $\phi: A \to C$ is a small homomorphism, then (E) shows that, for every $a \in A$, the image ϕa is the same as ϕb if $a \equiv b \bmod \mathrm{Ker}\,\phi$ ($b \in B$). Hence it is clear that $\phi \restriction B$ is different for distinct ϕ's.

Conversely, we show that if we are given a small homomorphism $\psi: B \to C$, then we can extend it to a small $\phi: A \to C$. By definition, there is a large subgroup $B(\underline{u})$ of B contained in $\mathrm{Ker}\,\psi$. Then $L = A(\underline{u})$ is a large subgroup of A such that $L \cap B = B(\underline{u})$ by the purity of B in A. Since $A/L = (L + B)/L \cong B/(L \cap B)$ and there is a natural homomorphism $B/(L \cap B) \to B/\mathrm{Ker}\,\psi$, we have the composite map $\phi: A \to A/L \to B/\mathrm{Ker}\,\psi$, which is evidently small and coincides with ψ on B. □

Hom_s As a Summand Perhaps more interesting is that $\mathrm{Hom}_s(A, C)$ is a summand of $\mathrm{Hom}(A, C)$. This is demonstrated by the next theorem.

Theorem 3.3 (Pierce [1]). *For p-groups A, C,* $\mathrm{Hom}_s(A, C)$ *is a direct summand of* $\mathrm{Hom}(A, C)$*, complete in its p-adic topology. We have*

$$\mathrm{Hom}(A, C) \cong \tilde{F} \oplus \mathrm{Hom}_s(A, C)$$

where $\tilde{F}$ is the p-adic completion of a free group F.

Proof. To show that $\mathrm{Hom}_s(A, C)$ is complete, let $\phi_1, \dots, \phi_i, \dots$ be a neat Cauchy sequence in $\mathrm{Hom}_s(A, C)$. It is Cauchy also in $\mathrm{Hom}(A, C)$, thus, by the completeness of this Hom, it has a limit in $\mathrm{Hom}(A, C)$, which must be $\psi = \phi_1 + (\phi_2 - \phi_1) + \cdots + (\phi_{i+1} - \phi_i) + \dots$. It remains to show that ψ is small. By the Cauchy property, $\phi_{i+1} - \phi_i = p^i\psi_i$ for some $\psi_i \in \mathrm{Hom}_s(A, C)$. Pick a $k \in \mathbb{N}$ and let $p^{n_i}A[p^k] \leq \mathrm{Ker}\,\psi_i$ for suitable $n_i \in \mathbb{N}$, for each i. Since $A[p^k] \leq \mathrm{Ker}\,p^k\psi_i \leq \mathrm{Ker}\,p^i\psi_i$ whenever $i \geq k$, if we choose $n = \max\{n_0, n_1, \dots, n_k, k\}$, then $p^nA[p^k] \leq \mathrm{Ker}(\phi_{i+1} - \phi_i)$ for all i, showing that ψ is a small homomorphism. This proves that $\mathrm{Hom}_s(A, C)$ is a complete group.

Since by (D) $\mathrm{Hom}(A, C)/\mathrm{Hom}_s(A, C)$ is torsion-free, $\mathrm{Hom}_s(A, C)$ is by algebraic compactness a summand of $\mathrm{Hom}(A, C)$. A complementary summand is p-adically complete (as a summand of $\mathrm{Hom}(A, C)$) and torsion-free, so it must be the p-adic completion of a free group. (The rank of F can be computed as λ^κ where κ (λ) denotes the final rank of the basic subgroup of A (resp. C) if these are infinite.) □

The special case $A = C$ leads to the subgroup $\mathrm{End}_s(A)$ of $\mathrm{End}\,A$ consisting of the small endomorphisms of A. It is a two-sided ideal: that $\eta\phi$ is small for all $\eta \in \mathrm{End}\,A$ whenever $\phi \in \mathrm{End}_s A$ is pretty obvious. That the same holds for $\phi\eta$ too follows easily, see, e.g., Exercise 2. Thus $\mathrm{End}(A)/\mathrm{End}_s(A)$ is a torsion-free ring on an algebraically compact group.

Exact Sequence for Hom_s We now prove an analogue of Theorem 1.12 for Hom_s.

Proposition 3.4 (Pierce [1]). *Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be an exact sequence of p-groups. Then the induced sequence*

$$0 \to \mathrm{Hom}_s(G, A) \xrightarrow{\alpha_*} \mathrm{Hom}_s(G, B) \xrightarrow{\beta_*} \mathrm{Hom}_s(G, C)$$

is likewise exact for every group G. If the first sequence is pure-exact, then so is the induced sequence even if we append $\to 0$ *to the end.*

Proof. For the first part, the only non-obvious claim is $\mathrm{Im}\,\alpha_* \geq \mathrm{Ker}\,\beta_*$. If $\eta \in \mathrm{Hom}_s(G, B)$ satisfies $\beta\eta = 0$, then $\beta(\eta G) = 0$, so $\eta G \leq \mathrm{Im}\,\alpha$. Hence there is a $\xi : G \to A$ such that $\alpha\xi = \eta$. Since evidently $\mathrm{Ker}\,\xi = \mathrm{Ker}\,\eta$ (α being monic), we have $\xi \in \mathrm{Hom}_s(G, A)$, as desired.

Assume the given sequence is pure-exact, and $\theta \in \mathrm{Hom}_s(G, C)$. If H denotes a basic subgroup of G, then by Theorem 4.3 in Chapter 5 there is a homomorphism $\gamma : H \to B$ such that $\beta\gamma = \theta \restriction H$. It is readily checked that the map $\phi : G = H + \mathrm{Ker}\,\theta \to B$ is well defined if we apply γ to H and send $\mathrm{Ker}\,\theta$ to 0. Furthermore, ϕ is small, and $\beta_*(\phi) = \theta$, so β_* is surjective. To verify purity, let $\phi \in \mathrm{Hom}_s(G, B)$ satisfy $p^k\phi \in \alpha(\mathrm{Hom}_s(G, A))$. Then by Proposition 1.13 there is a $\psi : G \to A$ satisfying $\alpha(p^k\psi) = p^k\phi$. From the equality of the kernels we derive that $p^k\psi$ is a small homomorphism. Hence ψ is small as well, and the proof is complete. □

Note that we do not claim that the sequence for $\mathrm{Hom}_s(*, G)$ is exact. As Pierce [1] points out, in contrast to Proposition 3.4, this is not true in general.

★ **Notes.** Megibben [2] shows that an unbounded torsion-complete p-group has a non-small homomorphism into a separable p-group C if and only if C has an unbounded torsion-complete subgroup. A result by Monk [2] states that the finite direct decompositions of $\mathrm{End}\,A/\mathrm{End}_s\,A$ are induced by those of $\mathrm{End}\,A$, so that they correspond to certain direct decompositions of A.

The concept of small homomorphism has been extended to the torsion-free and mixed cases by Corner–Göbel [1]. The general version, called **inessential homomorphism**, is based on the ideal $\mathrm{Ines}\,A$ of $\mathrm{End}\,A$; this is the set of all kinds of endomorphisms that are always present in groups (like those with finite rank images). Interested readers are advised to consult this interesting paper.

Exercises

(1) (Pierce) Every small homomorphism $\phi: p^k A \to p^k C$ can be extended to a small homomorphism $A \to C$.

(2) (Pierce) (a) $\phi: A \to C$ is small if and only if for every $k > 0$ there is an $n > 0$ such that $o(a) \leq p^k$ and $h(a) \geq n$ imply $\phi(a) = 0$.
(b) ϕ is small if and only if, for every $k > 0$, there is an $n > 0$ such that $p^n a \neq 0$ $(a \in A)$ implies $o(\phi a) \leq o(p^k a)$.

(3) (Pierce) Prove that $\mathrm{Hom}_s(A_1 \oplus A_2, C) \cong \mathrm{Hom}_s(A_1, C) \oplus \mathrm{Hom}_s(A_2, C)$ and $\mathrm{Hom}_s(A, C_1 \oplus C_2) \cong \mathrm{Hom}_s(A, C_1) \oplus \mathrm{Hom}_s(A, C_2)$.

(4) (Pierce) If G is a pure subgroup of the p-group A, then every small homomorphism $\phi: G \to C$ extends to a small homomorphism $A \to C$.

(5) (Pierce) Let A, C be arbitrary p-groups. $\mathrm{Hom}_s(A, C) = \mathrm{Hom}(A, C)$ if either (a) A has bounded basic subgroups and C is reduced, or (b) C is bounded.

(6) The composite of two small homomorphisms is small.

(7) Suppose A, C are p-groups. $\phi : A \to C$ is small exactly if $\phi(A^1) = 0$ and the induced map $A/A^1 \to C$ is small.

(8) Let A be a p-group with unbounded basic subgroup B. Prove Szele's theorem 6.10 in Chapter 5 by first mapping B onto itself by a small endomorphism, and then applying Lemma 3.2.

(9) Let A be a p-group and B a basic subgroup of A. Then, for every p-group C, the torsion subgroups of $\mathrm{Hom}(A, C)$ and $\mathrm{Hom}(B, C)$ are isomorphic. [Hint: the elements in these Homs are small homomorphisms.]

Problems to Chapter 7

PROBLEM 7.1. Can the groups $\mathrm{Hom}(M, \mathbb{Z})$ be characterized for monotone subgroups M of $\mathbb{Z}^{\aleph_0}$?

See Sect. 2 in Chapter 13 for monotone subgroups.

PROBLEM 7.2. Call $\phi \in \mathrm{Hom}(A, B)$ a *right universal homomorphism* for $A \to B$ if every $\psi \in \mathrm{Hom}(A, B)$ factors uniquely as $\psi = \eta\phi$ with $\eta \in \mathrm{End}\, B$. It is *left universal* if $\psi = \phi\chi$ with unique $\chi \in \mathrm{End}\, A$. Study the cases when uniqueness is not required, so Hom is singly generated (on the right or on the left) over End.

Right universal homomorphisms, called *localizations,* were discussed by Dugas [3] for torsion-free groups. Left universal homomorphisms were completely described by Chachólski–Farjoun–Göbel–Segev [1] for divisible groups B under the name of *cellular cover*, and for arbitrary abelian groups by Fuchs–Göbel [2]. See also Dugas [4].