Chapter 4
Divisibility and Injectivity

Abstract Most perfect objects in the category of abelian groups are those groups in which we can also 'divide:' for every element a and for every positive integer n, the equation $nx = a$ has a (not necessarily unique) solution for x in the group. These objects are the divisible groups which are universal in the sense that every group can be embedded as a subgroup in a suitable divisible group.

The divisible groups form one of the most important classes of abelian groups. In our presentation, we focus on their most prominent properties, many of them may serve as their characterization. Their outstanding feature is that they coincide with the injective groups, and as such they are direct summands in every group containing them as subgroups. Moreover, they constitute a class in which the groups admit a satisfactory characterization in terms of cardinal invariants.

The concluding topic for this chapter is concerned with a remarkable duality between maximum and minimum conditions on subgroups.

1 Divisibility

Since multiplication of group elements by integers makes sense, it is natural to consider divisibility of group elements by integers. Divisibility offers a great deal of information on how an element fits in the group.

Divisibility of Elements We shall say that the element a of the group A is **divisible** by $n \in \mathbb{N}$, in symbols: $n|a$, if the equation

$$nx = a \quad (a \in A) \tag{4.1}$$

is solvable for x in A, i.e., there exists a $b \in A$ such that $nb = a$. Evidently, (4.1) is solvable if and only if $a \in nA$.

We list some elementary consequences of the definition.

(a) If $x = b$ is a solution to (4.1), then the coset $b + A[n]$ is the set of all solutions of (4.1).
(b) If A is torsion-free, then (4.1) has at most one solution.
(c) If $\gcd\{n, o(a)\} = 1$, then (4.1) is solvable. For if $r, s \in \mathbb{Z}$ are such that $nr + o(a)s = 1$, then $x = ra$ satisfies $nx = nra = nra + o(a)sa = a$.

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(d) $m|a$ and $n|a$ imply $\text{lcm}\{m,n\}|a$. Indeed, if r, s satisfy $mr + ns = d = \gcd\{m,n\}$, and if $b, c \in A$ are such that $mb = a = nc$, then $(\text{lcm}\{m,n\})(rc + sb) = mnd^{-1}(rc + sb) = md^{-1}ra + nd^{-1}sa = a$.
(e) $n|a$ and $n|b$ with $a, b \in A$ imply $n|(a \pm b)$.
(f) If $A = B \oplus C$ is a direct sum, then $n|a = b + c$ ($b \in B, c \in C$) if and only if both $n|b$ in B and $n|c$ in C. The same holds for infinite direct sums and direct products.
(g) If $\alpha: A \to B$ is a homomorphism, then $n|a$ in A implies $n|\alpha b$ in B.
(h) If p is a prime, then $p^k|a$ is equivalent to $k \leq h_p(a)$.

Divisibility of Groups A group D is called **divisible** if

$$n|d \quad \text{for all } d \in D \text{ and all } 0 \neq n \in \mathbb{Z}.$$

Thus D is divisible exactly if $nD = D$ for every integer $n \neq 0$.

Example 1.1. The groups 0, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}(p^\infty)$, $\mathbb{R}$ are divisible groups, but no non-zero cyclic group is divisible.

A most useful criterion of divisibility is our next lemma.

Lemma 1.2. *A group D is divisible if and only if every homomorphism $\xi: \mathbb{Z} \to D$ can be extended to a homomorphism $\eta: \mathbb{Q} \to D$.*

Proof. Let ξ be a homomorphism of $\mathbb{Z}$ into the divisible group D. We think of $\mathbb{Q}$ as the union of the chain $\langle 1 \rangle < \langle (2!)^{-1} \rangle < \cdots < \langle (n!)^{-1} \rangle < \ldots$ of infinite cyclic groups. The element $\xi(1) = d_1 \in D$ is divisible by 2, so there is a $d_2 \in D$ with $2d_2 = d_1$, and we can extend ξ to $\xi_2 : \langle (2!)^{-1} \rangle \to D$ by letting $\xi_2((2!)^{-1}) = d_2$. If we have an extension $\xi_n : \langle (n!)^{-1} \rangle \to D$ with $\xi_n((n!)^{-1}) = d_n$, then we select a $d_{n+1} \in D$ satisfying $(n+1)d_{n+1} = d_n$, and define $\xi_{n+1}(((n+1)!)^{-1}) = d_{n+1}$ to extend ξ_n. At the end of this stepwise process we arrive at a desired homomorphism $\eta: \mathbb{Q} \to D$ (whose restriction to $\langle (n!)^{-1} \rangle$ equals ξ_n).

Conversely, assume that the group D has the indicated property, and pick any $d \in D$. There is a homomorphism $\xi : \mathbb{Z} \to D$ with $\xi(1) = d$. By hypothesis, ξ can be extended to a map $\eta : \mathbb{Q} \to D$. Then the element $\eta(n^{-1})$ satisfies $n\eta(n^{-1}) = d$, establishing the divisibility of d. □

A group D is said to be p**-divisible** (p a prime) if $p^k D = D$ for every positive integer k. Since $p^k D = p \cdots pD$, it is obvious that p-divisibility is implied by $pD = D$, i.e. every element of D is divisible by p. Then every element is of infinite p-height in D.

(A) *A group is divisible if and only if it is p-divisible for every prime p.* Indeed, if $pD = D$ for every prime p and $n = p_1 p_2 \cdots p_k$ with primes p_i, then $nD = p_1 p_2 \cdots p_k D = p_1 \cdots p_{k-1} D = \cdots = p_1 D = D$.
(B) *A p-group is divisible if and only if it is p-divisible.* In view of (c), for a p-group A we always have $qA = A$ whenever the primes p, q are different.
(C) *A p-group D is divisible exactly if every element of order p is of infinite height.* Only sufficiency requires a proof. So assume every element of order p is of

infinite height, and let $a \in D$ be of order p^k. We induct on k to prove that $p|a$. For $k = 1$, the claim is included in the hypothesis, so assume $k > 1$. By hypothesis, $p^{k-1}a$ has infinite height, thus $p^{k-1}a = p^k b$ for some $b \in D$. Since $o(a - pb) \leq p^{k-1}$, we have $p|a - pb$ by induction hypothesis, whence $p|a$, indeed.

(D) *Epimorphic images of a divisible group are divisible.* This is an immediate consequence of (g) above.

(E) *A direct sum (direct product) of groups is divisible if and only if each component is divisible.* This follows at once from (f).

(F) *If D_i ($i \in I$) are divisible subgroups of A, then so is their sum $\sum_{i \in I} D_i$.* This is evident in view of (e).

An immediate consequence is that the sum of all divisible subgroups of a group is again divisible, so we have the following result:

Lemma 1.3. *Every group A contains a maximal divisible subgroup D. D contains all divisible subgroups of A.* □

Groups that contain no divisible subgroups other than 0 are called **reduced**.

Embedding in Divisible Groups Recall that free groups are universal in the sense that every group is an epic image of a suitable free group. The next result shows that divisible groups have the dual universal property.

Theorem 1.4. *Every group can be embedded as a subgroup in a divisible group.*

Proof. The infinite cyclic group $\mathbb{Z}$ can be embedded in a divisible group, namely, in $\mathbb{Q}$. Hence every free group can be embedded in a direct sum of copies of $\mathbb{Q}$, which is a divisible group. Now if A is an arbitrary group, then $A \cong F/H$ for some free group F and a subgroup H of F. If we embed F in a divisible group D, then A will be isomorphic to the subgroup F/H of the divisible group D/H. □

It follows that, for every group A, there is an exact sequence

$$0 \to A \to D \to E \to 0$$

with D and (hence) E divisible.

★ **Notes.** There is no need to emphasize the relevance of divisibility in the theory of abelian groups: the reader will soon observe that a very large number of theorems rely on this concept. (In early literature, divisible groups were called *complete* groups.)

Exercises

(1) The additive group of any field of characteristic 0 is divisible.
(2) The factor group $J_p/\mathbb{Z}$ is divisible.

(3) A group is divisible exactly if it satisfies one of the following conditions:

  (a) it has no finite epimorphic image $\neq 0$;
  (b) it has no maximal subgroups (it coincides with its own Frattini subgroup).

(4) If $\{a_i\}_{i \in I}$ is a generating set (or a maximal independent set) in a group D, and if $n|a_i$ in D for every $i \in I$ and every $n \in \mathbb{N}$, then D is divisible.
(5) A direct sum (direct product) of groups is reduced if and only if every component is reduced.
(6) Let $0 \to A \to B \to C \to 0$ be an exact sequence. If both A and C are divisible (p-divisible), then so is B; if both are reduced, then B is also reduced.
(7) The maximal divisible subgroup of a torsion-free group coincides with the first Ulm subgroup of the group.
(8) Let A be the direct product, and B the direct sum of the groups B_n $(n < \omega)$. A/B is a divisible group if and only if, for every prime p, $pB_n = B_n$ holds for almost all n.
(9) Direct limits of divisible groups are divisible.
(10) (Szélpál) Assume A is a group such that all non-zero factor groups of A are isomorphic to A. Show that $A \cong \mathbb{Z}(p)$ or $A \cong \mathbb{Z}(p^\infty)$ for some p.
(11) Using Hom and Ext, show that (a) D is divisible if and only if $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, D) = 0$; (b) A is reduced if and only if $\mathrm{Hom}(\mathbb{Q}, A) = 0$.

2 Injective Groups

Injective groups are dual to projective groups; they are defined by dualizing the definition of projectivity.

Injectivity A group D is said to be **injective** if, for every diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\alpha} & A & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi} & \swarrow{\scriptstyle \eta} & & & & & \\ & & D & & & & & & \end{array}$$

with exact rows and a homomorphism $\xi : B \to D$, there is a homomorphism $\eta : A \to D$ making the triangle commute: $\eta\alpha = \xi$. If B is identified with its image in A, then the injectivity of D can be interpreted as the extensibility of any homomorphism $\xi : B \to D$ to a homomorphism of any group A containing B into D.

Our next purpose is to show that the injective groups are precisely the divisible groups.

Theorem 2.1 (Baer [8]). *A group is injective if and only if it is divisible.*

Proof. That an injective group is divisible follows at once from Lemma 1.2. In order to verify the converse, let D be a divisible group, and $\xi : B \to D$ a homomorphism from a subgroup B of the group A. Consider all groups G between B and A, such that ξ has an extension $\theta : G \to D$. The set S of all pairs (G, θ) is partially ordered by setting

$$(G, \theta) \leq (G', \theta') \quad \text{if and only if} \quad G \leq G' \text{ and } \theta = \theta' \restriction G.$$

S is not empty, since $(B, \xi) \in S$, and is inductive, since every chain (G_i, θ_i) $(i \in I)$ has an upper bound in S, *viz.* (G, θ) where $G = \bigcup_{i \in I} G_i$ and $\theta = \bigcup_{i \in I} \theta_i$. By Zorn's lemma, there exists a maximal pair (G_0, θ_0) in S. We claim: $G_0 = A$.

By way of contradiction, suppose $G_0 < A$. If $a \in A \setminus G_0$ is such that $na = g \in G_0$ for some $n \in \mathbb{N}$, then choose a minimal such n. By the divisibility of D, some $d \in D$ satisfies $nd = \theta_0 g$. It is straightforward to check that

$$x + ka \mapsto \theta_0 x + kd \qquad (x \in G_0,\ 0 \leq k < n)$$

is a genuine homomorphism of $\langle G_0, a \rangle$ into D. If $na \notin G_0$ for all $n \in \mathbb{N}$, then the correspondence $x + ka \mapsto \theta_0 x + kd$ $(x \in G_0)$ is a homomorphism for any choice of $d \in D$ (no restriction on $k \in \mathbb{Z}$). In either case, $G_0 < A$ contradicts the maximality of (G_0, θ_0). Hence $\theta_0 : A \to D$ is a desired extension of ξ. □

Baer's Criterion From the proof we derive the famous **Baer criterion for injectivity** (whose real importance lies in the fact that its analogue holds for modules over any ring):

Corollary 2.2. *A group D is injective if and only if, for each $n \in \mathbb{N}$, every homomorphism $\mathbb{Z}n \to D$ extends to a homomorphism $\mathbb{Z} \to D$.*

Proof. A careful analysis of the preceding proof shows that the only place where the injectivity of D was needed was to assure the existence of a $d \in D$. The same conclusion can be reached if we extend the map $\mathbb{Z}n \to D$ given by $n \mapsto \theta_0 g$ to $\mathbb{Z} \to D$, and pick d as the image of 1. □

Another simple result is a trivial consequence of Theorem 2.1 and Corollary 2.2, but it is important enough to record it as a corollary.

Corollary 2.3. *Epic images of injective groups are injective.* □

The Summand Property We are now able to show that injective (divisible) subgroups are always summands.

Corollary 2.4 (Baer [8]). *A divisible subgroup D of a group A is a summand, $A = D \oplus C$ for some subgroup C of A. Here C can be chosen so as to contain any preassigned subgroup B of A with $D \cap B = 0$. (Thus D-high subgroups are always summands.)*

Proof. By Theorem 2.1, the identity map $\mathbf{1}_D : D \to D$ extends to a homomorphism $\eta : A \to D$. Therefore, $A = D \oplus \operatorname{Ker} \eta$. If $D \cap B = 0$, then the same argument implies that the map $D \oplus B \to D$ which is the identity on D and trivial on B extends to an $\eta : A \to D$. Then evidently, $B \leq \operatorname{Ker} \eta$. □

Given a group A, consider the subgroup generated by all divisible subgroups of A. From Sect. 1(F) we know that D is divisible, it is the **maximal divisible subgroup** of A. By Corollary 2.4, $A = D \oplus C$, where evidently, the summand C has to be reduced. We thus have the first part of

Theorem 2.5. *Every group A is the direct sum of a divisible group D and a reduced group C,*

$$A = D \oplus C.$$

D is a uniquely determined subgroup of A, C is unique up to isomorphism.

Proof. To verify the second claim, it is clear that if $A = D \oplus C$ with D divisible and C reduced, then D ought to be the unique maximal divisible subgroup of A. Hence D is unique, and a complement is as always unique up to isomorphism. □

A consequence of the last theorem is that a problem on abelian groups can be often reduced to those on divisible and reduced groups.

We now summarize as a main result:

Theorem 2.6. *For a group, the following conditions are equivalent:*

(i) *it is divisible;*
(ii) *it is injective;*
(iii) *it is a direct summand in every group containing it.*

Proof. We had proved earlier the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), so only (iii) $\Rightarrow$ (i) is needed to complete the proof. Let D satisfy (iii). Theorem 1.4 shows that $D \leq E$ for a divisible group E. Then (iii) implies D is a summand of E, so (i) follows. □

Injective Hulls Theorems 1.4 and 2.1 guarantee that every group can be embedded in an injective group. This important fact can be considerably improved by establishing the existence of a *minimal* embedding.

A minimal divisible (injective) group containing the group A is called a **divisible hull** or **injective hull** of A, it will be denoted as $E(A)$. The main result on injective hulls is as follows.

Theorem 2.7. *Any injective group containing A contains an injective hull of A. The injective hull of A is unique up to isomorphism over A.*

Proof. Let E be an injective group containing A. There exists a subgroup C of E maximal with respect to the property $A \cap C = 0$. For each $c_0 \in C$ and prime p, there is $x \in E$ such that $px = c_0 \in C$. If $x \notin C$, then $c + kx = a \neq 0$ $(c \in C, a \in A)$ for some $k \in \mathbb{Z}$ prime to p. Then $pc + kc_0 = pa$ must be 0, which shows that kc_0, and hence c_0 is divisible by p in C. That is, C is divisible. By Corollary 2.4, we can write $E = C \oplus D$ with $A \leq D$. Clearly, D is divisible, and by the maximality of C, D cannot have any proper summand still containing A. Therefore, D is minimal divisible containing A, and by the choice of C, A is essential in D.

If D_1 and D_2 are two minimal injective groups containing A, then because of Theorem 2.1, the identity map $\mathbf{1}_A$ of A extends to a homomorphism $\phi : D_1 \to D_2$. Since ϕD_1 is divisible containing A, we have $A \leq \phi D_1 \leq D_2$. By the minimality of D_2, the last $\leq$ must be equality. Now $\phi|_A = \mathbf{1}_A$ implies $\operatorname{Ker}\phi \cap A = 0$ which means that ϕ is monic. Hence ϕ is an isomorphism between D_1 and D_2 leaving A element-wise fixed. □

An important consequence of the preceding proof is

Corollary 2.8. *An injective group E containing A is an injective hull of A if and only if A is an essential subgroup in E.* □

Example 2.9.

(a) The injective hull of a torsion-free group A is its divisible hull; it can also be obtained as $\mathbb{Q} \otimes A$.
(b) In order to get the injective hull of a p-group A with basic subgroup $B = \oplus_{i \in I} \langle b_i \rangle$, embed each $\langle b_i \rangle$ in a quasi-cyclic group C_i, and form $A + \sum_{i \in I} C_i$. (Check that this group is divisible, and A is essential in it.)

Quasi-Injectivity We now embark upon a noteworthy generalization of injectivity. A group A is called **quasi-injective** if every homomorphism of every subgroup into A extends to an endomorphism of A.

(A) Summands of quasi-injective groups are again quasi-injective.
(B) Powers of a quasi-injective group are quasi-injective.
(C) A torsion group A is quasi-injective if and only if all of its p-components A_p are quasi-injective.
(D) Neither direct sums nor direct products of quasi-injective groups are necessarily quasi-injective (this is trivial from Theorem 2.11).

Theorem 2.10 (Johnson–Wong). *A group is quasi-injective if and only if it is a fully invariant subgroup of its injective hull.*

Proof. First assume A is fully invariant in an injective group E, and let $\phi\colon B \to A$ where $B \leq A$. By injectivity, ϕ extends to an endomorphism $\psi\colon E \to E$ which—by full invariance—must map A into itself.

Conversely, let A be quasi-injective, and η an endomorphism of the injective hull E of A. The subgroup $B = \{a \in A \mid \eta a \in A\}$ is mapped by η into A, so η extends to a map $\psi : A \to A$. If $(\eta - \psi)a \in A$ for an $a \in A$, then $\eta a \in A$, so $a \in B$. Thus $(\eta - \psi)A \cap A = 0$, whence $(\eta - \psi)A = 0$ by the essential character of A. This means that $\eta \restriction A = \psi$, and so $\eta(A) \leq A$, indeed. □

From full invariance it follows that if $E = E_1 \oplus E_2$ is a direct decomposition of the injective hull of a quasi-injective group A, then A has a corresponding decomposition: $A = (A \cap E_1) \oplus (A \cap E_2)$.

A consequence of the last lemma is that every group A admits a **quasi-injective hull**: a smallest quasi-injective group containing A as a subgroup. This is simply the fully invariant subgroup generated by A in the injective hull $E(A)$ of A. This characterization of quasi-injective groups enables us to prove the following structural result on them.

Theorem 2.11 (Kil'p [1]). *A group is quasi-injective exactly if it either injective or is a torsion group whose p-components are direct sums of isomorphic cocyclic groups.*

Proof. If A is a group as stated, then it is immediately seen that it is fully invariant in its injective hull.

Conversely, let A be a quasi-injective group, and E its injective hull. If A contains an element a of infinite order, then for every $b \in E$ there is a map $\phi : \langle a \rangle \to E$ with $\phi a = b$, so the only fully invariant subgroup of E containing a is E itself. Thus $A = E$ in this case. If A is a torsion group, then its p-components A_p are likewise quasi-injective, so fully invariant in their injective hulls E_p. The latter group is a direct sum of copies of $\mathbb{Z}(p^\infty)$ (see Theorem 3.1), and its non-zero fully invariant subgroups are the direct sums of copies of a fixed subgroup $\mathbb{Z}(p^k)$ of $\mathbb{Z}(p^\infty)$, where $k \in \mathbb{N}$ or $k = \infty$. □

More on Quasi-Injectivity Quasi-injective groups have several remarkable properties which led to various generalizations of quasi-injectivity in module categories (for abelian groups some of them coincide with quasi-injectivity). We mention a few interesting facts for illustration.

Proposition 2.12. *A quasi-injective group has the following properties:*

(i) *it is a* CS-*group: high subgroups are summands;*
(ii) *it is an extending group: every subgroup is contained as an essential subgroup in a direct summand;*
(iii) *a subgroup that is isomorphic to a summand is itself a summand;*
(iv) *if B, C are summands and $B \cap C = 0$, then $B \oplus C$ is also a summand.*

Proof. (i) $\Leftrightarrow$ (ii) is routine.

(ii) Let G be a subgroup of the quasi-injective group A, and let E_1 denote the injective hull of G in the injective hull E of A. Then $E = E_1 \oplus E_2$ holds for some $E_2 \leq E$, and $A = (A \cap E_1) \oplus (A \cap E_2)$. Clearly, G is essential in the first summand of A.
(iii) Let H be a subgroup, and G a summand of A with inverse isomorphisms $\gamma : G \to H$, $\beta : H \to G$. Now β followed by the injection map $G \to A$ extends to an endomorphism $\alpha : A \to A$. If this is followed by the projection $A \to G$ and then by γ, then the composite is $\mathbf{1}_H$. As this extends to $A \to H$, H is a summand.
(iv) Let $A = B \oplus B'$ and $\pi : A \to B'$ the projection. Then $B \oplus C = B \oplus \pi C$ where $\pi \restriction C$ is an isomorphism. By (iii), πC is a summand of A and hence of B'. It follows $B \oplus C$ is a summand of A. □

★ **Notes.** The true significance of injectivity of groups lies not only in its extremely important role in the theory of abelian groups, but also in the fact that it admits generalizations to modules over any ring such that most of its relevant features carry over to the general case. The injective property was discovered by Baer [8]. He also proved that for the injectivity of a left R-module M, it is necessary and sufficient that every homomorphism from every left ideal L of R into M extends to an R-homomorphism $\mathsf{R} \to M$. This extensibility property, with L restricted to principal left ideals generated by non-zero divisors, is perhaps the most convenient way to define divisible R-modules. It is then immediately clear that an injective module is necessarily divisible. For modules over integral domains, the coincidence of injectivity and divisibility characterizes the Dedekind domains (see Cartan–Eilenberg [CE]). For torsion-free modules over Ore domains divisibility always implies injectivity.

Mishina [3] calls a group A **weakly injective** if every endomorphism of every subgroup extends to an endomorphism of A. Besides quasi-injective groups, only the groups of the form $A = D \oplus R$ have this property where D is torsion divisible and R is a rational group.

Epimorphic images of injective left R-modules are again injective if and only if R is left hereditary (left ideals are projective); an integral domain is hereditary exactly if it is Dedekind. Note that the semi-simple artinian rings are characterized by the property that all modules over them are injective.

It is an easy exercise to show that over left noetherian rings every left module contains a maximal injective submodule. This is not necessarily a uniquely determined submodule, unless the ring is, in addition, left hereditary. E. Matlis [Pac. J. Math. **8**, 511–528 (1958)] and Z. Papp [Publ. Math. Debrecen **6**, 311–327 (1959)] proved that every injective module over a left noetherian ring R is a direct sum of directly indecomposable ones. If, in addition, R is commutative, then the indecomposable injective R-modules are in a one-to-one correspondence with the prime ideals P of R, namely, they are the injective hulls of R/P (for $\mathsf{R} = \mathbb{Z}$ take $\mathsf{P} = (0)$ or (p)); cf. Matlis [loc. cit.]. It is remarkable that direct sums (and direct limits) of injective left modules are again injective if and only if the ring is left noetherian.

Before the term 'essential extension' was generally accepted, some authors were using 'algebraic extension' instead. Szele [3] developed a theory of 'algebraic' and 'transcendental' extensions of groups, modeled after field theory, where 'algebraic' meant 'essential,' while 'transcendental' was used for 'non-essential' extensions. He established the analogue of algebraic closure (injective hull) a few years before the Eckman–Schopf paper on the existence of injective hulls was published.

Quasi-injectivity was introduced by R.E. Johnson and E.T. Wong [J. London Math. Soc. **36**, 260–268 (1961)]. As far as generalizations of quasi-injectivity are concerned, interested readers are referred to the monographs S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Notes 17 (1990), and N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, Pitman Research Notes 313 (1994).

Exercises

(1) (Kertész) A group is divisible if and only if it is the endomorphic image of every group containing it.

(2) If $A = \oplus_{i \in I} B_i$, then $E(A) = \oplus_{i \in I} E(B_i)$ where $E(*)$ stands for the injective hull.

(3) Every automorphism of a subgroup A of an injective group D is induced by an automorphism of D.

(4) A direct sum of copies of $\mathbb{Z}(p^\infty)$ is injective as a J_p-module as well.

(5) Let A be a torsion-free group, and E the set of all pairs $(a, m) \in A \times \mathbb{Z}$ with $m \neq 0$ subject to

   (a) $(a, m) = (b, n)$ if and only if $mb = na$,
   (b) $(a, m) + (b, n) = (na + mb, mn)$.

   Show that E is a divisible hull of the image of A under the map $a \mapsto (a, 1)$.
(6) If C is a subgroup of the group B such that B/C is isomorphic to a subgroup H of G, then there exists a group A containing B such that $A/C \cong G$.
(7) Given A and integer $n > 0$, there exists an essential extension C of A such that $A = nC$. Is C unique up to isomorphism (over A)?
(8) (a) (Charles, Khabbaz) A subgroup A of a divisible group D is the intersection of divisible subgroups of D if and only if, for every prime p, $A[p] = D[p]$ implies $pA = A$.
   (b) (Bergman) Every group is the intersection of divisible subgroups in a suitable divisible group. [Hint: push-out of two different injective hulls.]
(9) (Szele) Let B be a subgroup of A. Call an $a \in A$ of infinite or prime power order **algebraic over** B if $a = 0$ or $\langle a \rangle \cap B \neq 0$. A is **algebraic over** B if every $a \in A$ is algebraic over B.

   (a) A is algebraic over B if and only if B is an essential subgroup of A.
   (b) A is a maximal algebraic extension of B exactly if $A = E(B)$.
   (c) Derive Theorem 2.1 from the existence of maximal algebraic extensions in E.

(10) The group $A \cong \mathbb{Z}(p^2) \oplus \mathbb{Z}(p)$ is a CS-group, but not quasi-injective.

3 Structure Theorem on Divisible Groups

Structure of Divisible Groups The groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ were among our first examples for divisible groups. The main theorem of this section will show that there are no divisible groups other than the direct sums of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ with various primes p.

Theorem 3.1. *A divisible group D is the direct sum of groups each of which is isomorphic either to the additive group $\mathbb{Q}$ of rational numbers and or to a quasi-cyclic group $\mathbb{Z}(p^\infty)$:*

$$D \cong (\oplus_\kappa \mathbb{Q}) \oplus \oplus_p (\oplus_{\kappa_p} \mathbb{Z}(p^\infty)).$$

The cardinal numbers κ, κ_p (for every prime p) form a complete and independent system of invariants for D.

Proof. The torsion part $T = tD$ of D is divisible, so by Corollary 2.4 it is a summand: $D = T \oplus E$, where E is torsion-free and divisible. The p-components

T_p of $T = \oplus_p T_p$ are divisible, so it suffices to show that T_p is a direct sum of copies of $\mathbb{Z}(p^\infty)$, and E is a direct sum of copies of $\mathbb{Q}$.

Owing to divisibility, for each $a \in T_p$ we can find a sequence $a = a_0, a_1, a_2, \ldots, a_n, \ldots$ in T_p such that $pa_{n+1} = a_n$ for $n = 0, 1, \ldots$. Thus every element embeds in a subgroup $\cong \mathbb{Z}(p^\infty)$. Consider the set S of subgroups B_i in T_p that are direct sums of subgroups $\cong \mathbb{Z}(p^\infty)$, and partially order S by declaring $B_i \leq B_j$ if B_i is a summand of B_j. Use Zorn's lemma to argue that T_p contains a maximal $B \in S$. Such a B is injective, so $T_p = B \oplus C$. If $C \neq 0$, then it must contain a subgroup $\cong \mathbb{Z}(p^\infty)$, contradicting the maximal choice of B. Hence $T_p = B$, and T_p is a direct sum of copies of $\mathbb{Z}(p^\infty)$. The proof for E is similar, making use of the embeddability of every element in a subgroup $\cong \mathbb{Q}$.

To show that the cardinal numbers of the summands $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ do not depend on the special choice of the decompositions, it is enough to note that these cardinals are exactly the ranks $\mathrm{rk}_0(D)$ and $\mathrm{rk}_p(D)$, which are uniquely determined by D. They do form a complete system of invariants for D, since if given $\mathrm{rk}_0(D) = \kappa$ and $\mathrm{rk}_p(D) = \kappa_p$, we can uniquely reconstruct D as a direct sum of κ copies of $\mathbb{Q}$ and κ_p copies of $\mathbb{Z}(p^\infty)$ for each prime p. Moreover, that these cardinals can be chosen arbitrarily is obvious. □

Note that Corollary 2.8 implies that

$$\mathrm{rk}_0(E(A)) = \mathrm{rk}_0(A) \quad \text{and} \quad \mathrm{rk}_p(E(A)) = \mathrm{rk}_p(A) \text{ for every prime } p.$$

Consequently, the structure of the divisible hull $E(A)$ of A is completely determined by the ranks of A.

Example 3.2. The additive group $\mathbb{R}$ of the real numbers is a torsion-free divisible group of the power of the continuum $2^{\aleph_0}$. Hence $\mathbb{R} \cong \oplus_\kappa \mathbb{Q}$, where $\kappa = 2^{\aleph_0}$.

Example 3.3. The multiplicative group of the positive real numbers is a torsion-free divisible group of the power of the continuum $2^{\aleph_0}$. It is isomorphic to $\mathbb{R}$ under the correspondence $r \mapsto \log r$.

Example 3.4. The multiplicative group $\mathbb{R}^\times$ of the non-zero real numbers is the direct product of the group in Example 3.3 and the multiplicative cyclic group $\langle -1 \rangle \cong \mathbb{Z}(2)$.

Example 3.5. The multiplicative group $\mathbb{T}$ of complex numbers of absolute value 1, the **circle group**, is isomorphic to $\mathbb{R}/\mathbb{Z}$, the (additive) group of reals mod 1. The torsion subgroup is $\cong \mathbb{Q}/\mathbb{Z}$, so the p-components are quasi-cyclic p-groups. Therefore,

$$\mathbb{R}/\mathbb{Z} \cong \oplus_p \mathbb{Z}(p^\infty) \oplus (\oplus_\kappa \mathbb{Q})$$

where again $\kappa = 2^{\aleph_0}$.

Example 3.6. The multiplicative group $\mathbb{C}^\times$ of all complex numbers $\neq 0$ is the direct product of the circle group and the multiplicative group of the positive real numbers. It is isomorphic to its subgroup: the circle group.

Cogenerators A group C is called a **cogenerator** of the category $\mathcal{A}b$ of abelian groups if every abelian group is contained in a suitable direct product of copies of C, or, equivalently, every non-zero abelian group has a non-trivial homomorphism into C.

Theorem 3.7. *A group is a cogenerator of $\mathcal{A}b$ if and only if it has a summand isomorphic to $\mathbb{Q}/\mathbb{Z}$. Thus $\mathbb{Q}/\mathbb{Z}$ is the minimal cogenerator of the category $\mathcal{A}b$.*

Proof. Every non-trivial homomorphic image of a quasi-cyclic group is isomorphic to the group itself, so any cogenerator C must contain $\mathbb{Z}(p^\infty)$ for each prime p. These $\mathbb{Z}(p^\infty)$ generate their direct sum in any group. Such a direct sum is $\cong \mathbb{Q}/\mathbb{Z}$, an injective group, so a summand of C.

Conversely, any group $A \neq 0$ has a non-trivial homomorphism into $\mathbb{Q}/\mathbb{Z}$. For, $0 \neq a \in A$ implies $\langle pa \rangle < \langle a \rangle$ for some prime p, and then the cyclic group $\langle a \rangle$ can be mapped upon the cyclic subgroup $\langle c \rangle$ of order p in $\mathbb{Z}(p^\infty)$ *via* $a \mapsto c$. This map extends to a homomorphism of $A \to \mathbb{Q}/\mathbb{Z}$. □

It is an important fact that the endomorphism rings of injective groups are very special. We do not discuss them here, because later we will learn more about these rings. We refer to Theorem 4.3 in Chapter 16.

Exercises

(1) Find the cardinal invariants for the following groups: (a) the direct product of κ copies of $\mathbb{Z}(p^\infty)$; (b) the direct product of κ copies of $\mathbb{Q}/\mathbb{Z}$; (c) the direct product of κ copies of $\mathbb{R}/\mathbb{Z}$; here κ denotes an infinite cardinal.
(2) Any two direct decompositions of a divisible group have isomorphic refinements.
(3) If A, B are divisible groups, each containing a subgroup isomorphic to the other, then $A \cong B$.
(4) If A is divisible, and B is a group such that $A \oplus A \cong B \oplus B$, then $B \cong A$.
(5) Find minimal cogenerators for the following categories: (a) torsion-free groups; (b) torsion groups; (c) p-groups.
(6) (Szele) A group contains no two distinct isomorphic subgroups if and only if it is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$.
(7) (Kertész) Let A be a p-group in which the heights of elements of finite heights are bounded by an integer $m > 0$. Then A is the direct sum of cocyclic groups. [Hint: $p^m A$ is divisible.]
(8) (a) (E. Walker) Any torsion-free group of infinite rank is a subdirect sum of copies of the group $\mathbb{Q}$.
(b) An unbounded p-group is a subdirect sum of quasi-cyclic groups.
(9) (a) For every infinite cardinal κ, there is a group U_κ of cardinality κ which contains an isomorphic copy of every group of cardinality $\leq \kappa$.
(b) In the set of groups U_κ with the indicated property there is one that is isomorphic to a summand of every other one.
(10) (W.R. Scott) An infinite group A is a **Jónsson group** if every proper subgroup has cardinality $< |A|$. Prove that A is a Jónsson group if and only if $A \cong \mathbb{Z}(p^\infty)$ for some prime p. [Hint: it is indecomposable, divisible, torsion.]

(11) A group is **hopfian** if its surjective endomorphisms are automorphisms; it is **cohopfian** if its injective endomorphisms are automorphisms. Show that the only hopfian-cohopfian torsion-free groups are the finite direct sums of $\mathbb{Q}$. (It is difficult to construct an infinite hopfian-cohopfian p-group.)

4 Systems of Equations

By the definition of divisible groups D, all 'linear' equations of the form $nx = d \in D$ with positive integers n are solvable for x in D. It is natural to raise the question of solvability of *systems* of linear equations in D. We are going to show that all consistent systems of linear equations are solvable in any divisible group.

Systems of Linear Equations By a **system of equations** over a group A is meant a set of equations

$$\sum_{j \in J} n_{ij} x_j = a_i \qquad (a_i \in A,\ i \in I) \tag{4.2}$$

where the coefficients n_{ij} are integers such that, for any fixed $i \in I$, almost all $n_{ij} = 0$; here, $\{x_j\}_{j \in J}$ is a set of unknowns, while I, J are index sets of arbitrary cardinalities. Equation (4.2) is a **homogeneous system** if $a_i = 0$ for all $i \in I$. We say that $x_j = g_j \in A$ $(j \in J)$ is a **solution** to (4.2) if Eq. (4.2) are satisfied in A whenever the x_j are replaced by the g_j. Sometimes, it is convenient to view a solution $x_j = g_j \in A$ $(j \in J)$ as an element $(\dots, g_j, \dots)$ in the direct product A^J.

For the solvability of the system (4.2), a trivial necessary condition is that it be **consistent** in the sense that, if a linear combination of the left sides of some equations vanishes (i.e., the coefficients of all the unknowns are 0), then it equals $0 \in A$ when the corresponding right-hand sides are substituted. Following Kertész, we give another, more versatile interpretation of the consistency and solvability of systems of equations.

The left members of the equations in (4.2) may be thought of as elements of the free group F on the set $\{x_j\}_{j \in J}$ of unknowns. Let H denote the subgroup of F generated by the left hand sides of the equations in (4.2). It is readily checked that the correspondence

$$\sum_{j \in J} n_{ij} x_j \mapsto a_i \qquad (i \in I) \tag{4.3}$$

induces a homomorphism $\eta : H \to A$ if and only if every representation of 0 as a linear combination of the left-hand sides is mapped by η upon 0, i.e. if the system is consistent in the sense above. Accordingly, we call (4.2) a **consistent system** if (4.3) extends to a homomorphism $\eta : H \to A$.

Clearly, two consistent systems define the same pair (H, η) exactly if the equations of either system are linear combinations of the equations of the other system, i.e. if the two systems of equations are **equivalent.** Thus a consistent system (or any of its equivalent systems) may be viewed as a pair (H, η), where H is a subgroup of the free group F on the set of unknowns, and η is a homomorphism $H \to A$.

Solvability of Systems of Equations Manifestly, $x_j = g_j \in A$ $(j \in J)$ is a solution of (4.2) if and only if the correspondence

$$x_j \mapsto g_j \qquad (j \in J) \tag{4.4}$$

extends to a homomorphism $\chi : F \to A$ whose restriction to H is η. Moreover, the extensions $\chi : F \to A$ of $\eta : H \to A$ are in a bijective correspondence with the solutions of (4.2), so we may use the notation (F, χ) for a solution of (4.2).

Theorem 4.1 (Gacsályi [1]). *Every consistent system of equations over A is solvable in A if and only if A is an injective group.*

Proof. The necessity is evident, since a single equation $nx = a \in A$ with $n \neq 0$ is a consistent system. Turning to the proof of sufficiency, let (H, η) be a consistent system of equations over a divisible group A. By Theorem 2.1, η extends to a homomorphism $\chi : F \to A$, that is, a solution exists. □

Consistency being a property of finite character, we conclude at once:

Corollary 4.2 (Gacsályi [1]). *A system of equations over an injective group D is solvable in D if and only if every finite subsystem has a solution in D.* □

It is worthwhile mentioning the following characterization of summands in terms of solvability of equations.

Proposition 4.3 (Gacsályi [1]). *A subgroup B of a group A is a direct summand exactly if every system of equations over B that is solvable in A can also be solved in B.*

Proof. If B is a summand, say, $A = B \oplus C$, then the B-coordinates of a solution in A provide a solution in B.

Conversely, assume that any system over B is solvable in B whenever it has a solution in A. For each coset u of A mod B, select a representative $a(u) \in A$, and consider the system

$$x_u + x_v - x_{u+v} = a(u) + a(v) - a(u+v) \in B \qquad \text{for all } u, v \in A/B.$$

By hypothesis, it has a solution $x_u = b(u) \in B$. Then the representatives $a(u) - b(u)$ of the cosets u form a subgroup C of A, and $A = B \oplus C$. □

★ **Notes.** The idea of considering systems of equations over a group was suggested by Szele whose student Gacsályi developed the theory in two papers. Generalizations to modules are due to Kertész who published several papers on the subject, starting with [Publ. Math. Debrecen **4**, 79–86 (1955)].

Exercises

(1) A system of equations over a group A is consistent if and only if it is solvable in some group containing A.
(2) A system of equations over an injective group contains maximal solvable subsystems.
(3) Prove that, for any prime p, the equation system

$$x_1 - px_2 = 1,\ x_2 - p^2 x_3 = 1,\ \ldots,\ x_n - p^n x_{n+1} = 1, \ldots$$

over $\mathbb{Z}$ is not solvable in $\mathbb{Z}$, though each of its finite subsystems is solvable.
(4) A homogeneous system $(H, 0)$ over an arbitrary group A admits a non-trivial solution in A if and only if there exists a non-zero homomorphism $\phi\colon F/H \to A$ (notation as above). The maps ϕ are in a bijective correspondence with the non-trivial solutions of the system.
(5) A homogeneous system of n equations with $n + 1$ unknowns over any group $A \neq 0$ always has a non-trivial solution in A.

5 Finitely Cogenerated Groups

We turn our attention to a concept dual to finite generation.

Finite Cogeneration A set C of non-zero elements in a group A is called a set of **cogenerators** if, every non-zero subgroup of A contains an element of C. Equivalently, for any group G, and for any homomorphism $\phi : A \to G$, $C \cap \operatorname{Ker} \phi = \varnothing$ implies that ϕ is monic.

Example 5.1. For a set C of cogenerators, the subgroup $\langle C \rangle$ is an essential subgroup in A, and the set of elements in an essential subgroup with 0 omitted is always a set of cogenerators.

Example 5.2. In a cocyclic group $\mathbb{Z}(p^k)$ $(k \in \mathbb{N} \cup \infty)$, a generator of its minimal subgroup $\mathbb{Z}(p)$ is a singleton cogenerator of this group.

A group is **finitely cogenerated** if it has a finite set of cogenerators. The following theorem is an analogue of Theorem 2.5 in Chapter 3, and points out a beautiful duality between maximum and minimum conditions.

Theorem 5.3 (Prüfer [1], Kurosh [1], Yahya [1]). *For a group A, the following conditions are equivalent:*

(i) *A is finitely cogenerated;*
(ii) *A is an essential extension of a finite group;*
(iii) *A is torsion of finite rank;*
(iv) *A is a direct sum of a finite number of cocyclic groups;*
(v) *the subgroups of A satisfy the minimum condition.*

Proof. (i) $\Rightarrow$ (ii) By hypothesis, A has a finite set C of cogenerators. A cannot have elements a of infinite order, for otherwise we could select a cyclic subgroup in $\langle a\rangle$ disjoint from C. Thus the elements in C are of finite order, whence $\langle C\rangle$ is finite. A must be an essential extension of $\langle C\rangle$, so (ii) follows.

(ii) $\Leftrightarrow$ (iii) is straightforward.

(ii) $\Rightarrow$ (iv) Let A be an essential extension of a finite subgroup B. It follows that A is a torsion group with a finite number of non-zero p-components, and in order to prove (iv), we may without loss of generality assume that A is a p-group. Write $A = D \oplus F$ where D is divisible and F reduced. As $A[p] = B[p]$ is finite, there is a bound p^m for the heights of elements in $F[p]$, whence $p^{m+1}F = 0$ follows, i.e. F is finite. Both D and F are direct sums of cocyclic groups, their number ought to be finite, due to the finiteness of the socle.

(iv) $\Rightarrow$ (v) Observe that if A is quasi-cyclic, then it enjoys the minimum condition on subgroups. To complete the proof, we show that if $A = U \oplus V$ and both of U, V have the minimum condition of subgroups, then the same holds for A. If $B_1 \geq B_2 \geq \cdots \geq B_n \geq \ldots$ is a descending chain of subgroups in A, then in the chain $B_1 \cap U \geq B_2 \cap U \geq \cdots \geq B_n \cap U \geq \ldots$ there is a minimal member, say $B_m \cap U$. Then the chain $B_n/(B_m \cap U) = B_n/(B_n \cap U) \cong (B_n + U)/U \leq A/U \cong V$ for $n \geq m$ also contains a minimal member, say $B_t/(B_m \cap U)$. Then B_t is minimal in the given chain.

Finally, (v) $\Rightarrow$ (i). The minimum condition is inherited by subgroups, so A cannot contain elements of infinite order, neither can the socle of A be infinite. Thus the socle contains a finite set of cogenerators. □

From the equivalence of (i) and (v) we infer that factor groups of finitely cogenerated groups are again finitely cogenerated. Observe that (ii) is equivalent to the finiteness of the socle in a torsion group.

Countable Number of Subgroups The groups in Theorem 5.3 have but countably many subgroups. There are only few other groups with this special property.

Proposition 5.4 (Rychkov–Fomin [1]). *A group A has fewer than continuously many subgroups if and only if it is an extension of a finitely generated group by a finite rank divisible subgroup of* $\mathbb{Q}/\mathbb{Z}$*. Then the set of subgroups is countable.*

Proof. It is an easy exercise to show that finitely generated groups and finite rank subgroups of $\mathbb{Q}/\mathbb{Z}$ have but a countable number of subgroups. A proof like (iv) $\Rightarrow$ (v) above shows that this remains true for their extensions.

For the converse, observe that a group of infinite rank has at least $2^{\aleph_0}$ subgroups, since different subsets of a maximal independent set generate different subgroups. Thus, if A has less than $2^{\aleph_0}$ subgroups, then $\operatorname{rk} A$ is finite. A maximal independent set in such an A generates a finitely generated subgroup F. The factor group A/F is torsion, it must also be of finite rank, so it satisfies the minimum condition on subgroups (Theorem 5.3); if its finite part is included in F, then A/F is divisible of finite rank. The group $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ has $2^{\aleph_0}$ subgroups, so $A/F < \mathbb{Q}/\mathbb{Z}$. □

The Prüfer Topology As another application of Theorem 5.3, we show:

Proposition 5.5. *Let S be a finite subset in the group A, and B a subgroup of A maximal disjoint from S. Then A/B satisfies the minimum condition for subgroups. If $|S| = 1$, then A/B is cocyclic.*

Proof. Every subgroup of A that contains B properly must intersect S, i.e., every non-zero subgroup of A/B contains one of the cosets $s + B$ with $s \in S$. Hence A/B is finitely cogenerated, and a reference to Theorem 5.3 completes the proof. □

Recall that the **Prüfer topology** of a group A is defined by declaring those subgroups U of A as a base of open neighborhoods of 0 for which A/U satisfies the minimum condition on subgroups. The preceding proposition is nothing else than asserting that the Prüfer topology is always Hausdorff.

★ **Notes.** While groups with maximum condition on subgroups are quite familiar to many mathematicians, because they have numerous applications, groups with minimum condition are often ignored due to their limited occurrence.

Exercises

(1) Let $0 \to A \to B \to C \to 0$ be an exact sequence of groups. B satisfies the minimum condition on subgroups if and only if so do A and C.
(2) An endomorphism η of a group with minimum condition is an automorphism if and only if $\operatorname{Ker} \eta = 0$.
(3) If A has minimum condition on subgroups, and if $A \oplus B \cong A \oplus C$, then $B \cong C$.
(4) If A has the minimum condition on subgroups and the group B satisfies $A \oplus A \cong B \oplus B$, then $B \cong A$.
(5) If A is finitely cogenerated, then a minimal set of cogenerators is contained in the socle of A.
(6) (Kulikov) Suppose A is a direct sum of cocyclic groups.
    (a) Every summand of A is likewise a direct sum of cocyclic groups.
    (b) Any two direct decompositions of A have isomorphic refinements.
(7) (de Groot) Let A and B be direct sums of cocyclic groups. If each is isomorphic to a pure subgroup of the other, then $A \cong B$.

(8) A group satisfies the minimum condition on fully invariant subgroups exactly if it is a direct sum of groups $\mathbb{Q}$, $\mathbb{Z}(p^{\infty})$ for finitely many different primes p, and a bounded group.

Problems to Chapter 4

PROBLEM 4.1. Call A *endo-divisible* if every E-homomorphism $\mathsf{L} \to A$ from a principal left ideal L of $\mathsf{E} = \operatorname{End} A$ extends to $\mathsf{E} \to A$. Which groups are endo-divisible?

PROBLEM 4.2. Characterize the quasi-injective hull of a p-group over its endomorphism ring.