Chapter 3
Direct Sums of Cyclic Groups

Abstract The study of important classes of abelian groups begins in this chapter. Not counting the finite and finitely generated groups, the class of direct sums of cyclic groups is perhaps the best understood class.

We give a fairly detailed account of free abelian groups, and discuss the presentation of groups *via* generators and defining relations. Several sections are devoted to direct sums of cyclic groups (called Σ-cyclic groups); these groups share most useful properties, and can easily be characterized by cardinal invariants. We present a few criteria for such groups, and establish several remarkable results, e.g. Kulikov's theorem that passage to subgroups preserves Σ-cyclicity. We draw attention to the method of smooth chains, which became the most important tool in the theory, and provides basic machinery for several results to come.

We shall cover some of the aspects of almost free groups, but shall not pursue their theory farther, due to the sophisticated set-theoretical arguments required.

In this chapter, in a number of proofs we have to use purity, so readers should be familiar with the fundamental results on pure subgroups (in Chapter 5) before studying the second part of this chapter.

1 Freeness and Projectivity

Free Abelian Groups By a **free (abelian) group** is meant a direct sum of infinite cyclic groups. If these cyclic groups are generated by the elements x_i $(i \in I)$, then the free group will be

$$F = \oplus_{i \in I} \langle x_i \rangle.$$

The set $\{x_i\}_{i \in I}$ is a **basis** of F. The elements of F are linear combinations

$$g = n_1 x_{i_1} + \cdots + n_k x_{i_k} \qquad (k \geq 0) \tag{3.1}$$

with different x_i and non-zero integers n_i. In view of the definition of direct sums, two such linear combinations represent the same element of F exactly if they differ at most in the order of the terms. Addition is performed in the obvious way by adding the coefficients of the same x_i.

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We can define F formally by starting with a set $X = \{x_i\}_{i\in I}$ of symbols, called **a free set of generators**, and declaring F as the set of all formal expressions (3.1) under the mentioned equality and addition. We say that F is **the free group on the set** X.

Example 1.1. An immediate example for a free group is the multiplicative group of positive rational numbers. The prime numbers form a free set of generators.

Needless to say, F is, up to isomorphism, uniquely determined by the cardinal number $\kappa = |I|$ of the index set I. Thus we are justified to write F_κ for the free group with κ free generators. κ is also called the **rank** of the free group F, in symbols, $\operatorname{rk} F = \kappa$ (for the discussion of rank, see Sect. 4).

Theorem 1.2. *The free groups F_κ and F_λ are isomorphic exactly if the cardinals κ and λ are equal.*

Proof. We need only verify the 'only if' part of the assertion. Observe that if F is a free group with free generators x_i $(i \in I)$, then an element (3.1) of F belongs to pF if and only if $p|n_1, \dots, p|n_k$. Hence, if p is a prime, then F/pF is a vector space over the prime field $\mathbb{Z}/p\mathbb{Z}$ of characteristic p with basis $\{x_i + pF\}_{i\in I}$, and so its cardinality is $p^{|I|}$ or $|I|$ according as I is finite or infinite. Thus $|F/pF|$ completely determines $|I|$. □

The Universal Property Free groups enjoy a universal property formulated in the next theorem which is frequently used for the definition of free groups.

Theorem 1.3 (Universal Property of Free Groups). *Let X be a free set of generators of the free group F. Any function $f : X \to A$ of X into any group A extends uniquely to a homomorphism $\phi : F \to A$. This property characterizes free sets of generators, and hence free groups.*

Proof. Write $X = \{x_i\}_{i\in I}$, and $f(x_i) = a_i \in A$. There is only one way f can be extended to a homomorphism $\phi: F \to A$, namely, by letting

$$\phi g = \phi(n_1 x_{i_1} + \cdots + n_k x_{i_k}) = n_1 a_{i_1} + \cdots + n_k a_{i_k}.$$

(The main point is that the uniqueness of (3.1) guarantees that ϕ is well defined.) It is immediate that ϕ preserves addition.

To verify the second part, assume that a subset X of a group F has the stated property. Let G be a free group with a free set $Y = \{y_i\}_{i\in I}$ of generators, where the index set is the same as for X. By hypothesis, the correspondence $f: \ x_i \mapsto y_i$ extends to a homomorphism $\phi : F \to G$; this cannot be anything else than the map $n_1 x_{i_1} + \cdots + n_k x_{i_k} \mapsto n_1 y_{i_1} + \cdots + n_k y_{i_k}$. ϕ is injective, because the linear combination of the y_i is 0 only in the trivial case. ϕ is obviously surjective, and so it is an isomorphism. □

Mapping X onto a generating system of a given group, we arrive at the following result which indicates that *the group $\mathbb{Z}$ is a generator of the category $\mathcal{A}b$* ('generator' in the sense used in category theory).

Corollary 1.4. *Every group with at most κ generators is an epimorphic image of a free group with κ generators.* □

Consequently, every group A can be embedded in a short exact sequence

$$0 \to H \to F \xrightarrow{\phi} A \to 0,$$

where F is free group, and $H = \operatorname{Ker} \phi$. (We will see shortly that H is likewise free.) This is called a **free resolution** of A. It is far from being unique, because both F and ϕ can be chosen in many ways.

If κ is an infinite cardinal, then F_κ has 2^κ subsets, and hence at most 2^κ subgroups and factor groups. We conclude that *there exist at most* 2^κ *pairwise non-isomorphic groups of cardinality* $\leq \kappa$. (We will learn in Corollary 3.8 in Chapter 11, that 2^κ is the precise number.)

The next two theorems are fundamental, they are quoted most frequently.

Theorem 1.5. *Suppose that B is a subgroup of a group A such that A/B is a free group. Then B is a summand of A, i.e., $A = B \oplus C$ for a subgroup $C \cong A/B$.*

Proof. That only free factor groups can share the stated property will follow from Theorem 1.7. In order to show that free groups do have this property, by Lemma 2.4 in Chapter 2, it suffices to verify the claim for $A/B \cong \mathbb{Z}$ only, say $A/B = \langle a + B \rangle$ with $a \in A$. The elements of A/B are the cosets $n(a + B) = na + B$ ($n \in \mathbb{Z}$) (all different). Hence $A = B \oplus \langle a \rangle$ is immediate. □

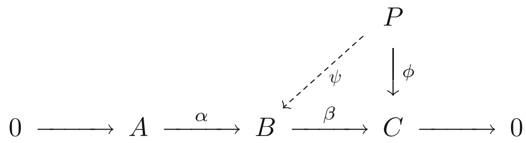
This theorem can also be phrased by saying that an exact sequence $0 \to B \to A \to F \to 0$ with a free group F is necessarily splitting.

Subgroups of Free Groups In the next theorem we study the subgroups of free abelian groups. Recall the famous result in group theory that subgroups of (non-commutative) free groups are again free. For abelian groups the situation is the same. To prove this, we use a well ordering of the index set.

Theorem 1.6. *Subgroups of free groups are free.*

Proof. Let F be a free group on the set X, which we now assume to be well ordered, say $X = \{x_\sigma\}_{\sigma<\tau}$ for some ordinal τ. Thus $F = \oplus_{\sigma<\tau} \langle x_\sigma \rangle$. For $\sigma < \tau$, define $F_\sigma = \oplus_{\rho<\sigma} \langle x_\rho \rangle$, and set $G_\sigma = G \cap F_\sigma$ for a subgroup $G < F$. Clearly, $G_\sigma = G_{\sigma+1} \cap F_\sigma$, so $G_{\sigma+1}/G_\sigma \cong (G_{\sigma+1} + F_\sigma)/F_\sigma$. The last factor group is a subgroup of $F_{\sigma+1}/F_\sigma \cong \langle x_\sigma \rangle$, thus either $G_{\sigma+1} = G_\sigma$ or $G_{\sigma+1}/G_\sigma$ is an infinite cyclic group. From Theorem 1.5 we conclude that $G_{\sigma+1} = G_\sigma \oplus \langle g_\sigma \rangle$ for some $g_\sigma \in G_{\sigma+1}$ (which is 0 if $G_{\sigma+1} = G_\sigma$). It follows that the elements g_σ generate the direct sum $\oplus_{\sigma<\tau} \langle g_\sigma \rangle$ in G. This must be all of G, since G is the union of the G_σ ($\sigma < \tau$). □

Projectivity Call a group P **projective** if every diagram



with exact row can be completed by a suitable homomorphism $\psi : P \to B$ to a commutative diagram, i.e. $\beta\psi = \phi$. We then say: ϕ is **lifted** to ψ.

Theorem 1.7. *A group is projective if and only if it is free.*

Proof. Let $\beta : B \to C$ be a surjective map, and F a free group with a homomorphism $\phi : F \to C$. For each x_i in a free set $X = \{x_i\}_{i \in I}$ of generators of F, we pick an element $b_i \in B$ such that $\beta b_i = \phi x_i$—this is possible, β being epic. Owing to Theorem 1.3, the correspondence $x_i \mapsto b_i$ $(i \in I)$ extends to a homomorphism $\psi : F \to B$. The maps $\beta\psi$ and ϕ are equal on the generators of F, so $\beta\psi = \phi$, and F is projective.

Next, let P be a projective group, and $\beta : F \to P$ an epimorphism, F a free group. By definition, the identity map $\mathbf{1}_P : P \to P$ can be lifted to a map $\psi : P \to F$, i.e. $\beta\psi = \mathbf{1}_P$. Thus ψP is a summand of F, so a free group by Theorem 1.6. The isomorphism $P \cong \psi P$ completes the proof. □

Thus 'free' and 'projective' have the same meaning for abelian groups. Therefore, free resolutions may also be called **projective resolutions**.

Projective Cover The **projective cover** of a group A is defined as a projective group P with a surjective map $\pi : P \to A$ such that $\operatorname{Ker} \pi$ is a superfluous subgroup of P. Projective covers are duals of injective hulls (to be discussed in Chapter 4), but in contrast to their dual counterparts, they rarely exist.

Example 1.8. (a) The cyclic group $\mathbb{Z}(p)$ has no projective cover. If it had one, $\mathbb{Z}$ would be a good candidate, but then the kernel would not be superfluous.

(b) However, $\mathbb{Z}(p)$ regarded as a $\mathbb{Z}_{(p)}$-module does have a projective cover, since $p\mathbb{Z}_{(p)}$ is superfluous in $\mathbb{Z}_{(p)}$.

Theorem 1.9. *A group has a projective cover if and only if it is free.*

Proof. We show that the zero-group is the only superfluous subgroup of a free group F. If $H \neq 0$ is a subgroup in F, then there is a prime p with $H \not\leq pF$ (since $\cap_p pF = 0$). Evidently, $(H + pF)/pF \neq 0$ is a summand of the $\mathbb{Z}/p\mathbb{Z}$-vector space F/pF, say, with complement G/pF for some $pF < G < F$. Then $G + H = F$ where G is a proper subgroup of F, so H cannot be superfluous. □

Defining Relations We shall discuss briefly the method of defining a group in terms of generators and relations. Though this is well known from general group theory, in the commutative case there are simplifications worthwhile to be pointed out.

Let $\{a_i\}_{i\in I}$ be a set of generators of a group A, and $\theta : F \to A$ an epimorphism from a free group $F = \oplus_{i\in I} \langle x_i \rangle$ such that $\theta x_i = a_i$ for each $i \in I$. Ker θ consists of those linear combinations $m_1 x_{i_1} + \cdots + m_k x_{i_k} \in F$ with integral coefficients m_i for which $m_1 a_{i_1} + \cdots + m_k a_{i_k} = 0$ holds in A. These equalities are called the **defining relations** relative to the generating system $\{a_i\}_{i\in I}$.

It follows that the group A is completely determined by giving a set $\{a_i\}_{i\in I}$ of generators along with the set of all defining relations:

$$A = \langle a_i \ (i \in I) |\ m_{j1} a_{i_1} + \cdots + m_{jk} a_{i_k} = 0 \ (j \in J) \rangle \tag{3.2}$$

(since we are dealing exclusively with abelian groups, the commutativity relations are not listed). Indeed, if (3.2) is given, then A is defined as the factor group F/H, where F is a free group on the free set $\{x_i\}_{i\in I}$ of generators, and H is the subgroup of F, generated by the elements $m_{j1} x_{i_1} + \cdots + m_{jk} x_{i_k}$ for all $j \in J$. The relations between the given generators of A are exactly those which are listed in (3.2), and their consequences. (The emphasis is on the non-existence of more relations.) Equation (3.2) is said to be a **presentation** of A.

Example 1.10. A presentation of a free group F with free generators $\{x_i\}_{i\in I}$ is given as $F = \langle x_i \ (i \in I) \mid \varnothing \rangle$ (there are no relations between the generators). Of course, there are numerous other presentations; e.g. $\mathbb{Z} = \langle x, y \mid 2x - 3y = 0 \rangle$.

Example 1.11. The group $C = \langle x \mid nx = 0 \rangle$ for $n \in \mathbb{N}$ is cyclic of order n.

★ **Notes.** The material on free groups is fundamental, and will be used in the future without explicit reference. Though in homological algebra, projectivity is predominant, in abelian group theory freeness seems to prevail. Fortunately, for abelian groups, freeness and projectivity are equivalent, while for modules, the projectives are exactly the direct summands of free modules. Projective modules are rarely free; they are free over principal ideal domains (but not even over Dedekind domains that are not PID), and over local rings (Kaplansky [2]).

Theorem 1.6 holds for modules over left principal ideal domains. Submodules of projectives are again projective if and only if the ring is left hereditary, i.e., all left ideals are projective. Theorem 1.2 holds over commutative rings or under the hypothesis that at least one of κ and λ is infinite. There exist, however, rings R such that all free R-modules $\neq 0$ with finite sets of generators are isomorphic. It is perhaps worthwhile pointing out that every R-module is free if and only if R is a field, and every R-module is projective exactly if R is a semi-simple artinian ring. The property that all R-modules have projective covers characterizes the perfect rings, introduced by H. Bass.

Hausen [6] defines a group P **κ-projective** for an infinite cardinal κ if it has the projective property with respect to all exact sequences $0 \to A \to B \to C \to 0$ with $|C| < \kappa$. She establishes various properties of κ-projective groups, e.g. P is κ-projective if and only if, for every subgroup G with $|P/G| < \kappa$, there is a summand H of P such that $G \leq H$ and G/H is a free group.

Exercises

(1) Let F be a free group on n free generators. If n elements $a_1, \ldots, a_n \in F$ generate F, then this set is a basis of F.

(2) Prove the following converse of Theorem 1.5: a group F is free if it has the property that whenever $B < A$ and $A/B \cong F$, then B is a summand of A.
(3) Give a presentation of $\mathbb{Z}(p^\infty)$, and one of $\mathbb{Q}$.
(4) Let A be presented by a set of generators and defining relations, and assume that the set of generators is the union of two disjoint subsets, $\{b_i\}_{i\in I}$ and $\{c_j\}_{j\in J}$, such that each of the defining relations contains only generators from the same subset. Then $A = B \oplus C$, where B is generated by the b_i, and C by the c_j.
(5) Let A be presented by a set of generators and defining relations, and B by a subset of these generators and defining relations. Show that letting the generators of B correspond to themselves *qua* generators of A induces a homomorphism $B \to A$.
(6) For every set of generators, there is a minimal set of defining relations relative to these generators (i.e., no relation can be omitted). [Hint: Theorem 1.6.]
(7) Let $0 \to A_1 \to A_2 \xrightarrow{\alpha} A_3 \to 0$ be an exact sequence, and $\phi_i \colon F_i \to A_i$ $(i = 1, 3)$ epimorphisms where F_i are free. If $\psi : F_3 \to A_2$ is such that $\alpha\psi = \phi_3$, then $\phi_1 \oplus \psi \colon F_1 \oplus F_3 \to A_2$ is epic, and its kernel is $\operatorname{Ker}\phi_1 \oplus \operatorname{Ker}\phi_3$.
(8) Let $0 \to F_1 \to F_2 \to \cdots \to F_n \to 0$ be an exact sequence of finitely generated free groups. Prove the equality $\sum_{k=1}^{n}(-1)^k \operatorname{rk} F_k = 0$.
(9) Assume $\{A_n \mid n \in \mathbb{Z}\}$ is a set of groups. Verify the existence of free groups F_n $(n \in \mathbb{Z})$ and a long sequence

$$\dots \xrightarrow{\alpha_{n-2}} F_{n-1} \xrightarrow{\alpha_{n-1}} F_n \xrightarrow{\alpha_n} F_{n+1} \xrightarrow{\alpha_{n+1}} \dots$$

such that $\alpha_{n-1}\alpha_n = 0$ and $\operatorname{Ker}\alpha_n / \operatorname{Im}\alpha_{n-1} \cong A_n$ for every $n \in \mathbb{Z}$.

2 Finite and Finitely Generated Groups

We turn our attention to groups with a finite number of generators. First, we discuss finite groups separately. Though this is a special case of the general theory of finitely generated groups (to be developed independently), a short, direct approach to the theory of finite groups is not without merit.

Finite Groups We start with a simple lemma.

Lemma 2.1. *Let A be a p-group that contains an element g of maximal order p^k for an integer $k > 0$. Then $\langle g\rangle$ is a direct summand of A.*

Proof. If A is infinite, then use Zorn's lemma to argue that there is a subgroup B of A maximal with respect to the property $B \cap \langle g\rangle = 0$. To show that $A^* = \langle g\rangle \oplus B$ equals A, by way of contradiction assume that some $a \in A$ does not belong to A^*. Replacing a by $p^i a$ if necessary, we may also suppose that $pa \in A^*$, i.e. $pa = mg + b$ for some $m \in \mathbb{Z}, b \in B$. By the maximality of the order of g, we have $p^{k-1}mg + p^{k-1}b = p^k a = 0$. Hence $p^{k-1}mg = 0$, so m must be divisible by p, say, $m = pm'$. Then $a' = a - m'g \notin A^*$ satisfies $pa' = b$. By the maximal choice of B, $\langle B, a'\rangle \cap \langle g\rangle \neq 0$, thus $0 \neq ra' + b' = sg$ for some $r, s \in \mathbb{Z}, b' \in B$. This can happen only if $(r, p) = 1$, since $pa' \in B$. But then $pa', ra' \in A^*$ implies $a' \in A^*$, a contradiction. $\square$

Fundamental Theorem on Finite Abelian Groups The first structure theorem in the history of group theory was the famous Basis Theorem on finite abelian groups.

Theorem 2.2 (Frobenius–Stickelberger [1]). *A finite group is the direct sum of a finite number of cyclic groups of prime power orders.*

Proof. Thanks to Theorem 1.2 in Chapter 2, the proof reduces at once to p-groups. In a finite p-group $A \neq 0$, we select an element g of maximal order. By the preceding lemma, $A = \langle g \rangle \oplus B$ for some subgroup B. Since B has a smaller order than A, a trivial induction completes the proof. $\square$

There is a uniqueness theorem attached to the preceding result. Again, it suffices to state it for p-groups.

Theorem 2.3. *Two direct decompositions of a finite p-group A into cyclic groups are isomorphic.*

Proof. In a direct decomposition of A collect the cyclic summands of equal orders into a single summand to obtain a courser decomposition $A = B_1 \oplus \cdots \oplus B_k$ where each B_i is 0 or a direct sum of cyclic groups of fixed order p^i. Evidently, $p^{k-1}A = p^{k-1}B_k$ is the socle of B_k, it is an elementary p-group, its dimension (as a $\mathbb{Z}/p\mathbb{Z}$-vector space) tells us the number of cyclic components in B_k. As this socle depends only on A, the number of cyclic summands of order p^k is independent of the choice of the direct sum representation of A. In general, $p^{i-1}A[p] = p^{i-1}B_i[p] \oplus \cdots \oplus p^{i-1}B_k[p]$ modulo $p^iA[p] = p^iB_{i+1}[p] \oplus \cdots \oplus p^iB_k[p]$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space $\cong p^{i-1}B_i[p]$ whose dimension is equal to the number of cyclic summands (of order p^i) in B_i. The same argument shows that this dimension is independent of the choice of the selected direct decomposition of A. $\square$

Finitely Generated Groups We proceed to the discussion of finitely generated groups. We start with a preliminary lemma.

Lemma 2.4 (Rado [1]). *Assume* $A = \langle a_1, \dots, a_k \rangle$, *and* $n_1, \dots, n_k$ *are integers such that* $\gcd\{n_1, \dots, n_k\} = 1$. *Then there exist elements* $b_1, \dots, b_k \in A$ *such that*

$$A = \langle b_1, \dots, b_k \rangle \qquad \textit{with } b_1 = n_1a_1 + \cdots + n_ka_k.$$

Proof. We induct on $n = |n_1| + \cdots + |n_k|$. If $n = 1$, then let $b_1 = \pm a_i$ for any i, and the claim is evident. Next let $n > 1$. Then at least two of the n_i are different from 0, say, $|n_1| \geq |n_2| > 0$. Since either $|n_1 + n_2| < |n_1|$ or $|n_1 - n_2| < |n_1|$, we have $|n_1 \pm n_2| + |n_2| + \cdots + |n_k| < n$ for one of the two signs. $\gcd\{n_1 \pm n_2, n_2, \dots, n_k\} = 1$ and the induction hypothesis imply that $A = \langle a_1, \dots, a_k \rangle = \langle a_1, a_2 \mp a_1, \dots, a_k \rangle = \langle b_1, \dots, b_k \rangle$ with $b_1 = (n_1 \pm n_2)a_1 + n_2(a_2 \mp a_1) + n_3a_3 + \cdots + n_ka_k = n_1a_1 + \cdots + n_ka_k$. $\square$

The main result on finitely generated groups is our next theorem which is regarded as the first major result in the abstract structure theory of infinite abelian groups. It plays an important role in several applications.

Theorem 2.5. *The following conditions on a group A are equivalent:*

(i) *A is finitely generated;*
(ii) *A is the direct sum of a finite number of cyclic groups;*
(iii) *the subgroups of A satisfy the maximum condition.*

Proof. (i) $\Rightarrow$ (ii) assume A is finitely generated, and a minimal generating set of A contains k elements. Pick such a set with k generators, say, $a_1, \dots, a_k$, with the additional property that a_1 has minimal order, i.e. no other set of k generators contains an element of smaller order. If $k = 1$, then $A = \langle a_1 \rangle$, and we are done. So let $k > 1$, and as a basis of induction, assume that $B = \langle a_2, \dots, a_k \rangle$ is a direct sum of cyclic groups. Thus it suffices to verify that $A = \langle a_1 \rangle \oplus B$, which will follow if we can show that $\langle a_1 \rangle \cap B = 0$.

By the choice of k, we have $o(a_1) > 1$. Working toward a contradiction, suppose that $\langle a_1 \rangle \cap B \neq 0$, i.e. $m_1 a_1 = m_2 a_2 + \dots + m_k a_k \neq 0$ with $0 < m_1 < o(a_1)$. Let $d = \gcd\{m_1, \dots, m_k\}$, and write $m_i = dn_i$. Then $\gcd\{n_1, \dots, n_k\} = 1$, and from Lemma 2.4 we conclude that $A = \langle a_1, \dots, a_k \rangle = \langle b_1, \dots, b_k \rangle$ with $b_1 = -n_1 a_1 + n_2 a_2 + \dots + n_k a_k$. Here $db_1 = 0$, thus $o(b_1) < o(a_1)$, contradicting the choice of a_1. Thus $\langle a_1 \rangle \cap B = 0$.

(ii) $\Rightarrow$ (iii) Let $A = \langle a_1 \rangle \oplus \dots \oplus \langle a_k \rangle$. If $k = 1$, then A is cyclic, and every non-zero subgroup is of finite index in A. Hence the subgroups satisfy the maximum condition. (iii) will follow by a trivial induction if we can show that $A = B \oplus C$ has the maximum condition on subgroups whenever both B and C share this property. If $A_1 \leq \dots \leq A_n \leq \dots$ is an ascending chain of subgroups in A, then $A_1 \cap B \leq \dots \leq A_n \cap B \leq \dots$ is one in B, so from some index m on, all $A_n \cap B$ are equal to $A_m \cap B$. For $n > m$ we have $A_n/(A_m \cap B) = A_n/(A_n \cap B) \cong (A_n + B)/B \leq A/B \cong C$, whence we conclude that from a certain index $t > m$ on all factor groups $A_t/(A_m \cap B)$, and hence all subgroups A_t, are equal.

(iii) $\Rightarrow$ (i) The set $\mathcal{S}$ of all finitely generated subgroups of A is not empty, so by hypothesis (iii) A contains a maximal finitely generated subgroup G. For any $a \in A$, $\langle G, a \rangle$ is still finitely generated. Hence $\langle G, a \rangle = G$, thus $A = G$, and A is finitely generated. ◻

Let us point out two immediate consequences of Theorem 2.5. First, *every finitely generated group is the direct sum of a finite group and a finitely generated free group* (follows from (ii)). Secondly, *subgroups of finitely generated groups are again finitely generated* (follows from (iii)).

The most essential part of the preceding theorem is the first implication. We give another quick proof, reducing it to Theorem 2.2. If we can show that A/T is free ($T = t(A)$), then $A \cong T \oplus A/T$ by Theorem 1.5, and we are done. Thus, it is enough to consider $A = \langle a_1, \dots, a_n \rangle$ torsion-free. To start the induction on n, there is nothing to prove if $n = 1$, since then $A \cong \mathbb{Z}$ trivially. Let $U/\langle a_n \rangle$ denote the torsion subgroup of $A/\langle a_n \rangle$. Then A/U is torsion-free and has a smaller number of generators, so it is free. Hence $A \cong U \oplus A/U$ (again by Theorem 1.5), where U is a finitely generated group isomorphic to a subgroup of $\mathbb{Q}$, so it is cyclic.

Stacked Basis Theorem A third proof of Theorem 2.5 is based on the following theorem which is of considerable interest in its own right (see the more general Theorem 6.5). We say $\{a_i\}_{i\in I}$ is a **basis** of A if $A = \oplus_{i\in I} \langle a_i\rangle$.

Theorem 2.6. *If H is a subgroup of the free group F of finite rank k, then F and H have 'stacked bases:'*

$$F = \langle a_1\rangle \oplus \cdots \oplus \langle a_k\rangle \quad \textit{and} \quad H = \langle b_1\rangle \oplus \cdots \oplus \langle b_k\rangle$$

such that there are non-negative integers $m_1, \ldots, m_k$ satisfying

$$b_i = m_i a_i \quad (i = 1, \ldots, k) \quad \textit{and} \quad m_{i-1}|m_i \quad (i = 2, \ldots, k).$$

Proof. We select a free basis $\{x_1, \ldots, x_k\}$ of F with the following extremal property: H contains an element $b_1 = n_1x_1 + \cdots + n_kx_k$ with a minimal positive coefficient n_1. In other words, for another basis of F, or for another permutation of the basis elements, or for other elements of H, the leading positive coefficient is never less than n_1.

The first observation is that $n_1|n_i$ $(i = 2, \ldots, k)$. For, if $n_i = q_in_1 + r_i$ $(q_i, r_i \in \mathbb{Z}, 0 \le r_i < n_1)$, then we can write $b_1 = n_1a_1 + r_2x_2 + \cdots + r_kx_k$ where $\{a_1 = x_1 + q_2x_2 + \cdots + q_kx_k, x_2, \ldots, x_k\}$ is a new basis of F. By the special choice of $\{x_1, \ldots, x_k\}$, we must have $r_2 = \cdots = r_k = 0$. The same argument shows that if $b = s_1x_1 + \cdots + s_kx_k$ $(s_i \in \mathbb{Z})$ is any element of H, then $s_1 = qn_1$ for some $q \in \mathbb{Z}$. Hence $b - qb_1 \in \langle x_2\rangle \oplus \cdots \oplus \langle x_k\rangle = F_1$. We conclude that F has a decomposition $F = \langle a_1\rangle \oplus F_1$ such that $H = \langle b_1\rangle \oplus H_1$, where $b_1 = n_1a_1$ and $H_1 \le F_1$. Using induction hypothesis for the pair H_1, F_1, we infer that F has a basis $\{a_1, \ldots, a_k\}$ and H has a basis $\{b_1, \ldots, b_k\}$ such that $b_i = m_ia_i$ for some non-negative integers m_i.

It remains to establish the divisibility relation $m_1|m_2$ (the others will follow by induction). Write $m_2 = tm_1 + r$ with $t, r \in \mathbb{Z}, 0 \le r < m_1$. Then $\{a = a_1 + ta_2, a_2, \ldots, a_k\}$ is a new basis of F, in terms of which we have $b_1 + b_2 = m_1a_1 + (tm_1 + r)a_2 = m_1a + ra_2 \in H$. The minimality of $m_1 = n_1$ implies $r = 0$. □

With the aid of Theorem 2.6, we can reprove the implication (i) $\Rightarrow$ (ii) in Theorem 2.5. If A is generated by k elements, then $A \cong F/H$, where F is a free group on a set of k elements. Choosing stacked bases for F and H, as described in Theorem 2.6, we obtain

$$A \cong \langle a_1\rangle/\langle m_1a_1\rangle \oplus \cdots \oplus \langle a_k\rangle/\langle m_ka_k\rangle.$$

Consequently, A is the direct sum of cyclic groups: the ith summand is cyclic of order m_i if $m_i > 0$, and infinite cyclic if $m_i = 0$. The numbers m_i are called **elementary divisors**.

Fundamental Theorem on Finitely Generated Groups Of course, the numbers m_i in Theorem 2.6 are not necessarily prime powers, but we can decompose the finite summands into direct sums of cyclic groups of prime power orders. Cyclic groups of prime power orders are indecomposable (and so are the infinite cyclic groups), so we can claim the fundamental theorem:

Theorem 2.7 (Fundamental Theorem on Finitely Generated Abelian Groups). *A finitely generated group is the direct sum of finitely many indecomposable cyclic groups, each of which is of prime power order or infinite cyclic.* □

Whenever one has a direct decomposition, then the standard question is: to what extent is the decomposition unique? This question is fully answered in the following theorem.

Theorem 2.8. *Any two direct decompositions of a finitely generated group into indecomposable cyclic groups are isomorphic.*

Proof. If A is finitely generated, then by Theorem 2.7 $A = tA \oplus F$ where $F \cong A/tA$ is finitely generated free. Both summands are uniquely determined by A up to isomorphism. Theorems 2.3 and 1.2 guarantee the uniqueness of the decompositions of the summands, whence the claim is evident. □

Invariants Thus in the decompositions of a finitely generated group A, the orders of the indecomposable cyclic summands (but not the summands themselves) are uniquely determined. These orders are referred to as the **invariants** of A. For instance, the invariants of $A \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}(p) \oplus \mathbb{Z}(p^2) \oplus \mathbb{Z}(q^3) \oplus \mathbb{Z}(q^3)$ (with primes p, q) are: $\infty, \infty, \infty, p, p^2, q^3, q^3$. We also say: A is **of type** $(\infty, \infty, \infty, p, p^2, q^3, q^3)$.

Consequently, with every finitely generated group A, a finite system of symbols ∞ and prime powers is associated. Not only is it uniquely determined by A, but it also determines A up to isomorphism, i.e. two finitely generated groups are isomorphic if and only if they have the same system of invariants (maybe in different orders)—this fact is expressed by saying that this is a **complete system of invariants**. Moreover, these invariants are **independent** in the sense that, for an arbitrary choice of a finite system of symbols ∞ and prime powers, there exists a finitely generated group exactly with this system of invariants (this is obvious).

Example 2.9. Let $C(m)$ denote the multiplicative group of those residue classes of integers modulo the integer $m = p_1^{r_1} \cdots p_k^{r_k}$ (canonical form) which are relatively prime to m. Its order is given by Euler's totient function $\varphi(m)$. Elementary number theory tells us that

(a) $C(m)$ is the direct product of the groups $C(p_i^{r_i})$ for $i = 1, \ldots, k$;
(b) for odd primes p, $C(p^r)$ is cyclic of order $\varphi(p^r) = p^r - p^{r-1}$;
(c) $C(4)$ is cyclic of order 2, while $C(2^r)$ $(r \geq 3)$ is of type $(2, 2^{r-2})$.

Kaplansky's Test Problems In his famous little red book [K], Kaplansky raises the question about criteria for satisfactory structure theorems. He lists two test problems that such theorems must pass in order to qualify 'satisfactory.' These are:

Test Problem I. If the group G is isomorphic to a direct summand of H, and H is isomorphic to a direct summand of G, are then G and H isomorphic?

Test Problem II. If $G \oplus G \cong H \oplus H$, are G and H isomorphic?

Evidently, the structure theorem on finitely generated groups passes the test with flying colors: both answers are easy 'yes.' However, some of the theorems that will be discussed later on will fail one or both test problems.

★ **Notes.** Whenever it seems instructive or interesting, we shall make historical remarks that are intended to give a sense of the way in which the subject has developed, but are in no way a comprehensive survey of the relevant contributions. As far as the fundamental theorem on finite abelian groups is concerned, it is not clear how far back in time one needs to go to trace its origin. It was F.C. Gauss who established a decomposition in number theory reminiscent to it. That time the concept of a group was unknown, it took a long time to formulate and to prove the fundamental theorem in the present form; see Frobenius–Stickelberger [1]. The theorem on finitely generated groups may be credited to H.J.S. Smith [Phil. Trans. **151**, 293–326 (1861)]. He reduced matrices with integral entries to canonical form that bears his name.

This is the first time we encounter a structure theorem, so a few comments are in order. Such a theorem (on any class of algebraic systems) is supposed to be in terms of easily recognizable invariants, like natural numbers, cardinal or ordinal numbers, but they can be matrices with integral entries, etc. '*Invariants*' mean by definition that they are exactly the same for isomorphic objects. A set of invariants is *complete* if we can reconstruct from it the object within the class by using a method typical for the class (for finitely generated groups, this method consists in forming the direct sum of cyclic groups with the given invariants as orders). Finally, *independence* means that the system of invariants can be chosen arbitrarily, i.e. without additional restriction (in this case, arbitrary prime powers and the sign ∞, each with arbitrary multiplicities). The system of invariants for finitely generated groups is most satisfactory, it has served as a prototype for structure theorems in algebra.

The Kaplansky test problems have been discussed for various classes, mostly with negative answers. de Groot modified Test Problem I by asking the isomorphy of G and H if G has a summand $G_1 \cong H$ and H has a summand $H_1 \cong G$ such that, in addition, $G/G_1 \cong H/H_1$ is also satisfied.

There are numerous generalizations of the theorems in this section. Kaplansky [J. Indian Math. Soc. **24**, 279–281 (1960)] proved that, for integral domains R, the torsion parts of finitely generated R-modules are summands exactly if R is a Prüfer domain. There is an extensive literature on commutative rings over which finitely generated torsion modules are Σ-cyclic. Unless the ring is left noetherian, finitely generated left modules are different from finitely presented ones which are somewhat better manageable. Finitely presented R-modules are Σ-cyclic if and only if R is an elementary divisor ring, i.e., every matrix over R can be brought to a diagonal form by left and right multiplications by unimodular matrices (Kaplansky [Trans. Amer. Math. Soc. **66**, 464–491 (1949)]). In this case, Theorem 2.6 still holds true.

Exercises

(1) A group is finite if and only if its subgroups satisfy both the maximum and the minimum conditions.

(2) A finite group A is cyclic exactly if $|A[p]| \leq p$ for every prime p.

(3) (a) If the integer m divides the order of the finite group A, then A has both a subgroup and a factor group of order m.
(b) (G. Frobenius) In a finite p-group, the number of subgroups of a fixed order (dividing the order of the group) is $\equiv 1 \bmod p$.
(4) A group is isomorphic to a subgroup of the finite group A if and only if it is isomorphic to a factor group of A.
(5) The number of non-isomorphic groups of order $m = p_1^{r_1} \cdots p_k^{r_k}$ (canonical form of m) is equal to $P(r_1) \cdots P(r_k)$, where $P(r)$ stands for the number of partitions of r into positive integers.
(6) If A, B are finite groups such that, for every integer m, they contain the same number of elements of order m, then $A \cong B$.
(7) A set $\{a_1, \dots, a_k\}$ of generators of a finite group is a basis if and only if the product $o(a_1) \cdots o(a_k)$ is minimal among the products of orders for all generating sets.
(8) In a finitely generated group, every generating set contains a finite set of generators.
(9) (a) The sum of all the elements of a finite group A is 0, unless A contains just one element a of order 2, in which case the sum is equal to this a.
(b) From (a) derive Wilson's congruence $(p-1)! \equiv -1 \bmod p$, p a prime.
(10) Let A, B be finitely generated groups. There is a group C such that both A and B have summands isomorphic to C, and every group that is isomorphic to summands of both A and B is isomorphic to a summand of C.
(11) Any set of pairwise non-isomorphic finite (finitely generated) groups has cardinality $\leq \aleph_0$.
(12) (Cohn, Honda, E. Walker) Finitely generated groups A have the cancellation property: $A \oplus B \cong A \oplus C$ implies $B \cong C$, or equivalently, if $G = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A \cong A_2$, then $B \cong C$. [Hint: enough for $A_1 = \langle a \rangle$ cyclic of order ∞ or prime power p^r.]
(13) If A and B are finitely generated groups, and if each is isomorphic to a subgroup of the other, then $A \cong B$.
(14) A surjective endomorphism of a finitely generated group is an automorphism.

3 Factorization of Finite Groups

In most cases, the fundamental theorem is instrumental in solving problems related to finite abelian groups. However, there are notable exceptions where it seems the fundamental theorem is totally irrelevant. One of these is Hajós' theorem on the 'factorization' of finite abelian groups.

The problem goes back to a famous conjecture by H. Minkowski in 1896 on tiling the n-dimensional Euclidean space by n-dimensional cubes. If the space is filled gapless such that no two cubes have common interior points, then it was conjectured that there exist cubes sharing $n-1$-dimensional faces. The conjecture was rephrased as an abelian group-theoretical problem, and solved in this form by G. Hajós. We discuss briefly this celebrated result. The proof involves group rings, and therefore at some point we have to switch to the multiplicative notation. It is reasonable to do this right away.

Thus in this section, all groups are finite, written multiplicatively. Accordingly, 1 will denote the identity element of groups.

Direct Products of Subsets If $S_1, \dots, S_k$ are non-empty subsets of a multiplicative group G, then we say that G is **a direct product** of these subsets, in notation,

$$G = S_1 \dot{\times} \dots \dot{\times} S_k, \tag{3.3}$$

if each element $g \in G$ can be written uniquely as $g = s_1 \cdot \ldots \cdot s_k$ with $s_i \in S_i$. This definition is in line with the definition of direct sum of subgroups. We will call the components S_i **factors** of G, and (3.3) **a factorization of** G. We obviously have

(A) *The cardinality of a factor is a divisor of the group order.*
(B) *Every subgroup H of G is a factor: $G = H\dot{\times}S$ if S is a complete set of representatives* mod H.
(C) *A factor S_i can be replaced by gS_i with any $g \in G$.* For this reason, there is no loss of generality in assuming that each factor contains $1 \in G$.

Periodic and Cyclic Subsets A subset P is called **periodic** and a non-unit $g \in G$ a **period** of P if $gP = P$. Subgroups are trivially periodic. If g is a period, then so are the elements $\neq 1$ of $\langle g \rangle$. In this case, P is the set union of certain cosets mod $\langle g \rangle$.

Lemma 3.1. *If $G = \langle a \rangle$ is cyclic of order p^n, and $G = S\dot{\times}T$, then either S or T is periodic.*

Proof. Set $S = \{a^{n_1}, \dots, a^{n_k}\}$, $T = \{a^{m_1}, \dots, a^{m_\ell}\}$ $(n_i, m_j \geq 0)$ and form the polynomials $S(z) = z^{n_1} + \cdots + z^{n_k}$, $T(z) = z^{m_1} + \cdots + z^{m_\ell}$ (with indeterminate z). Hypothesis implies

$$S(z)T(z) \equiv 1 + z + z^2 + \cdots + z^{p^n - 1} \mod z^{p^n} - 1.$$

It follows that $S(z)T(z)$ is divisible by the p^nth cyclotomic polynomial $\Phi_n(z) = 1 + z^{p^{n-1}} + \cdots + z^{(p-1)p^{n-1}}$. This polynomial is known to be irreducible over $\mathbb{Q}$, so one of the factors, say, $S(z)$ is divisible by $\Phi_n(z)$. Hence we conclude that $a^{p^{n-1}}$ is a period of S. □

Our main concern is with factors that are **cyclic subsets** in the sense that they are of the form

$$[a]_n = \{1, a, \dots, a^{n-1}\} \qquad (2 \leq n \leq o(a))$$

for some $a \in G$. We need two preliminary lemmas.

Lemma 3.2 (Hajós [1]). *A cyclic subset is periodic if and only if it is a group.*

Proof. Let $P = [a]_n$ be periodic with period $g \in G$, so $P = \langle g \rangle \dot{\times} S$ for some $S \subset G$ where $1 \in S$ may be assumed. Evidently, $g = a^t$ for some $t \in \mathbb{N}$, and P contains $1, a, \dots, a^{t-1}$, the powers of a^t, as well as their cosets mod $\langle a^t \rangle$. This means $\langle a \rangle \subseteq P$, so P is a (cyclic) group. □

Lemma 3.3 (Hajós [1]). *A cyclic subset C can be written as a direct product of cyclic subsets of prime orders such that C is a subgroup if and only if one of the factor cyclic subsets is a subgroup.*

Proof. Suppose $C = [c]_n$, and let $n = p_1 \cdots p_k$, a product of primes. It is an easy computation to show that

$$[c]_n = [c]_{p_1} \dot{\times} [c^{p_1}]_{p_2} \dot{\times} \dots \dot{\times} [c^{p_1 \cdots p_{k-1}}]_{p_k}.$$

If C is a subgroup, i.e. if $c^n = 1$, then the last factor is also a subgroup. For the converse, we show that if $C = \langle a \rangle \dot{\times} S$ for some $1 \neq a \in C,\ S \subset C$, then C has to be a subgroup. In fact, a is then a period of C, and the claim follows from Lemma 3.2. □

Hajós' Theorem We can now state the main theorem.

Theorem 3.4 (Hajós [1]). *If a finite group G is the direct product of cyclic subsets,*

$$G = [a_1]_{n_1} \dot{\times} \dots \dot{\times} [a_k]_{n_k},$$

then one of the factors is a subgroup.

Proof. In view of Lemma 3.3, for the proof we may assume that the orders n_i of the factors are primes p_i. Suppose $[a_k]_{p_k}$ is not a subgroup, i.e. $a_k^{p_k} \neq 1$. Then from $a_k G = G$ we derive that

$$[a_1]_{p_1} \dot{\times} \dots \dot{\times} [a_{k-1}]_{p_{k-1}} \cdot a_k^{p_k} = [a_1]_{p_1} \dot{\times} \dots \dot{\times} [a_{k-1}]_{p_{k-1}}, \tag{3.4}$$

that is, the product on the right is periodic with period $a_k^{p_k}$. Delete as many factors as possible until no more factor can be omitted without violating the periodicity of the product. Let $a \in G$ denote a period of a shortest periodic subset $P = [a_1]_{p_1} \dot{\times} \dots \dot{\times} [a_h]_{p_h}$.

Consider the subgroup $H = \langle a_1, \dots, a_h \rangle$ of G. As P is a factor of G, it is also a factor of H, thus $|P|$ divides $|H|$, i.e. $p_1 \cdots p_h \mid |H|$. If we can show that $|H|$ is the product of not more than h primes, then $P = H$ will follow. We will then have a similar direct product decomposition for H, a group of smaller order, so observing that the case $h = 1$ is trivial, an obvious induction will complete the proof.

It remains to substantiate the claim concerning the order of the subgroup H. We interrupt the proof to verify a lemma that will do the job.

The Crucial Lemma The crux of the problem is to find a proper statement, more general than actually needed for the proof, that will allow an induction to complete the proof. We need the group ring $\mathbb{Z}[G]$ to formulate such a lemma. $\mathbb{Z}[G]$ consists of elements of the form

$$\mathfrak{x} = m_1 g_1 + \cdots + m_\ell g_\ell \quad (g_i \in G, m_i \in \mathbb{Z}) \tag{3.5}$$

which we add and multiply according to the usual rules, respecting the multiplication rules in G.

In what follows we will assume that the expression of $\mathfrak{x}$ is canonical, i.e. all the g_i are different, and all $m_i \neq 0$. With this in mind, we go on to define $\langle \mathfrak{x} \rangle$ as the subgroup of G generated by the elements g_i in (3.5), and denote by $\pi(\mathfrak{x})$ the number of prime factors in the order of $\langle \mathfrak{x} \rangle$. Finally, the symbol $\mathfrak{a}$ will have double meaning: for an $a \in G$, it is either $1 + a + \cdots + a^{p-1}$ for a prime p, or $1 - a$. Thus $\mathfrak{a} \in \mathbb{Z}[G]$ and $\langle \mathfrak{a} \rangle = \langle a \rangle$.

Lemma 3.5 (Hajós [1]). *Assume that the equation*

$$\mathfrak{x}\, \mathfrak{a}_1 \cdots \mathfrak{a}_k = 0 \tag{3.6}$$

holds in the group ring $\mathbb{Z}[G]$, where $a_i \in G$, $\mathfrak{x} \in \mathbb{Z}[G]$. If no factor $\mathfrak{a}_i$ can be deleted without violating the validity of the equation, then

$$\pi(\mathfrak{x}, \mathfrak{a}_1, \cdots, \mathfrak{a}_k) - \pi(\mathfrak{x}) < k. \tag{3.7}$$

Proof. We begin with the verification in case $k = 1$. Thus we have $\pi(\mathfrak{x}\,\mathfrak{a}) = 0$ with non-zero factors, and what we wish to prove amounts to $a \in \langle \mathfrak{x} \rangle$. If $\mathfrak{a} = 1 - a$, then $\mathfrak{x} = a\mathfrak{x}$, which implies that there are $b_1, b_2 \in G$ in the normal form of $\mathfrak{x}$ such that $b_1 = ab_2$. Hence $a \in \langle \mathfrak{x} \rangle$ in this case. If $\mathfrak{a} = 1 + a + \cdots + a^{p-1}$ for some prime p, then by multiplication by $1 - a$ we get $\mathfrak{x}(1 - a^p) = 0$, whence $a^p \in \langle \mathfrak{x} \rangle$. On the other hand, from $\mathfrak{x}(a + \cdots + a^{p-1}) = -\mathfrak{x}$ we conclude that $b_1 = a^i b_2$ for some $b_1, b_2 \in \langle \mathfrak{x} \rangle$ and $1 \le i \le p - 1$. Thus also $a^i \in \langle \mathfrak{x} \rangle$, and therefore $a \in \langle \mathfrak{x} \rangle$.

We continue with induction on $n = \pi(\mathfrak{a}_1) + \cdots + \pi(\mathfrak{a}_k)$. If $n = 1$, then $k = 1$, and we are done. Assuming $k \ge 2$, we rewrite (3.6) in the form $(\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j)\mathfrak{a}_{j+1} \cdots \mathfrak{a}_k = 0$ for $j < k$, and apply the induction hypothesis to obtain

$$\pi((\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j), \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_k) - \pi(\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j) < k - j \qquad (1 \le j < k).$$

The index of the subgroup $\langle (\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j), \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_k \rangle$ in $\langle \mathfrak{x}, \mathfrak{a}_1 \ldots, \mathfrak{a}_j, \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_k \rangle$ evidently divides the index of $\langle \mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j \rangle$ in $\langle \mathfrak{x}, \mathfrak{a}_1, \ldots \mathfrak{a}_k \rangle$ (cf. Exercise 1). Hence, from the last inequality we get

$$\pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_k) - \pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_j) < k - j \qquad (1 \le j < k). \tag{3.8}$$

If $\pi(\mathfrak{a}_j) = 1$ for all $j \le k$, then clearly $\pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k-1}) - \pi(\mathfrak{x}) \le k - 1$, along with (3.8) for $j = k - 1$ yields (3.7). If, e.g., $\pi(\mathfrak{a}_k) \ge 2$, then by multiplication by $1 - a_k$ or by $1 + a_k + \cdots + a_k^{p-1}$ for some prime p, we can replace the factor $\mathfrak{a}_k$ by

$\mathfrak{a}_0 = 1 - a_0$ with $1 \leq \pi(\mathfrak{a}_0) = \pi(\mathfrak{a}_k) - 1$. After deleting superfluous factors $\mathfrak{a}_i$, and renumbering, we get

$$\mathfrak{x}\,\mathfrak{a}_1, \ldots \mathfrak{a}_\ell \mathfrak{a}_0 = 0 \qquad (0 \leq \ell \leq k-1)$$

where no factor can be omitted, not even the last one. By induction hypothesis, $\pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell, \mathfrak{a}_0) - \pi(\mathfrak{x}) \leq \ell$. In case $\ell = 0$, we have $\pi(\mathfrak{x}, \mathfrak{a}_0) - \pi(\mathfrak{x}) = 0$, and $a_0 \in \langle \mathfrak{x} \rangle$, thus $\pi(\mathfrak{x}, \mathfrak{a}_k) - \pi(\mathfrak{x}) \leq 1$. This, together with (3.8) for $j = k-1$, leads to (3.7). If $\ell \geq 1$, then manifestly $\pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell) - \pi(\mathfrak{x}) \leq \ell$, hence along with (3.8) for $j = \ell$ it yields the desired (3.7). □

Resuming the proof of Theorem 3.4, we rewrite (3.4) (after deleting superfluous factors) as an equation in $\mathbb{Z}[G]$:

$$\mathfrak{a}_1 \cdots \mathfrak{a}_h \cdot (1 - a) = 0$$

where $\mathfrak{a}_i = 1 + a_i + \cdots + a_i^{p_i - 1}$ $(i = 1, \ldots, h)$. Applying Lemma 3.5 to the case $\mathfrak{x} = 1$, we obtain $\pi(a_1, \ldots, a_h, a) = \pi(\mathfrak{a}_1, \ldots, \mathfrak{a}_h, a) \leq h$, and *a fortiori* $\pi(a_1, \ldots, a_h) \leq h$. As pointed out above, this completes the proof. □

Example 3.6 (Hajós). Theorem 3.4 may fail if the factors are not cyclic. This is shown by the following examples.

(a) Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ be a direct product of cyclic groups where a, b, c are generators of orders $4, 4, 2$, respectively. Then

$$G = \{1, a\} \dot{\times} \{1, b\} \dot{\times} \{1, a^2, ab^2, a^3b^2, c, a^2bc, a^2b^3c, b^2c\}.$$

(b) Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ where all the generators a, b, c are of order 4. Then

$$G = \{1, a\} \dot{\times} \{1, b\} \dot{\times} \{1, c\} \dot{\times} \{1, a^2b, b^2c, c^2a, a^2b^3, b^2c^3, c^2a^3, a^2b^2c^2\}.$$

★ **Notes.** The proof above is based on the original proof by Hajós [1] with essential simplifications due to L. Rédei and T. Szele. Various modified versions of the problem have been considered. One version requires the factors to be simulated subsets: a subset S of a group is *simulated* if it is obtainable from a subgroup by replacing an element by an arbitrary group element. There is an extensive literature on this difficult subject, most advanced papers are written recently by A.D. Sands and S. Szabó. There are remarkable connections to tessellations.

It is hard to understand why so far no evidence of a link has been found between the fundamental theorem on finite abelian groups and the Hajós theorem. Such a link would probably avoid group rings, but it seems doubtful we could have found our way through without making use of them.

A generalized, still unsolved version of Minkowski's conjecture was formulated by O.H. Keller. Its algebraized version says that if $G = S \dot{\times} [a_1]_{n_1} \dot{\times} \cdots \dot{\times} [a_k]_{n_k}$ with a subset $S \subset G$, then one of the elements $a_i^{n_i}$ equals $s_1 s_2^{-1}$ for some $s_1, s_2 \in S$.

Exercises

All groups are finite, written multiplicatively.

(1) If A, B, C are finite index subgroups of the group G, and $B \leq C$, then $[AC : AB]$ divides $[C : B]$.
(2) If in a group G, the subset $P = [a]_p \dot{\times} [b]_q$ $(a, b \in G)$ is periodic with different primes p, q, then one of the factors is a subgroup.
(3) (Sands) Let G be cyclic of order 8. Find $G = S\dot{\times}T$ such that none of S, T can be replaced by a subgroup. [Hint: $\langle a \rangle = \{1, a^2\}\dot{\times}\{1, a, a^4, a^5\}$.]
(4) (de Bruijn) A subset S of a cyclic group of order n is periodic if and only if there is a proper divisor d of n such that $S(z)$ (defined above in Lemma 3.1) is divisible by the polynomial $f(z) = (z^n - 1)(z^d - 1)^{-1}$.
(5) Assume G is a finite group of one of the types $(2, 2, 2)$, $(2, 2^2)$, $(2, 2, 3)$, $(2, 3, 3)$, $(3, 3^2)$, $(3, 3, 3)$. If $G = S\dot{\times}T$ for subsets S, T, then S or T is periodic. [Hint: S or T contains 2 or 3 elements.]
(6) (de Bruijn) Let G be an elementary 2-group with generators $a_1, a_2, a_3, b_1, b_2, b_3$. None of the factors is periodic in the factorization

$$G = \{1, a_1a_3b_1, a_2a_3, a_1a_2b_1, b_2, a_1a_2a_3b_2, a_1b_1b_2, a_2a_3b_1b_2\}\cdot$$
$$\cdot\{1, a_1, a_2, a_1a_2, b_3, a_3b_3, b_1b_3, a_3b_1b_3\}.$$

(7) (de Bruijn) Let $G = \langle a \rangle$ be cyclic of order 72. It factorizes into two non-periodic subsets: $\{1, a^8, a^{16}, a^{18}, a^{26}, a^{34}\}$ and $\{a^{18}, a^{54}, a^{24}, a^{60}, a^{48}, a^{12}, a^{17}, a^{41}, a^{65}, a^{45}, a^{69}, a^{21}\}$.

4 Linear Independence and Rank

Motivated by linear independence and dimension in vector spaces, we are in search for corresponding notions in groups.

Linear Independence Linear independence in groups can be defined in two inequivalent ways: one permits only elements of infinite order to be in the system, while the other makes no such restriction, and as a result, it is useful for torsion and mixed groups as well. With that said, we proceed to introduce the more useful version.

A set $\{a_1, \ldots, a_k\}$ of non-zero elements in a group is called **linearly independent**, or briefly, **independent** if

$$n_1a_1 + \cdots + n_ka_k = 0 \ \ (n_i \in \mathbb{Z}) \ \text{ implies } \ n_1a_1 = \cdots = n_ka_k = 0. \tag{3.9}$$

More explicitly, this means that $n_i = 0$ if $o(a_i) = \infty$ and $o(a_i)|n_i$ if $o(a_i)$ is finite. By definition, 0 is not allowed to be in an independent system. An infinite family L of group elements is **(linearly) independent** if every finite subset of L is independent. Thus independence is by definition a property of finite character.

Lemma 4.1. *A subset $L = \{a_i\}_{i\in I}$ $(0 \notin L)$ of a group is independent if and only if*

$$\langle L\rangle = \oplus_{i\in I} \langle a_i\rangle. \tag{3.10}$$

Proof. If L is independent, then the intersection of the cyclic group $\langle a_i\rangle$ with the subgroup generated by $L \setminus \{a_i\}$ is necessarily 0; hence, $\langle L\rangle$ is the direct sum of the $\langle a_i\rangle$ for $i \in I$. Conversely, if (3.10) holds, then a linear combination $n_1a_{i_1} + \cdots + n_ka_{i_k} = 0$ (with different $i_1, \ldots, i_k \in I$) can hold only in the trivial way: $n_1a_{i_1} = \cdots = n_ka_{i_k} = 0$. □

An element $g \in A$ is said to **depend on** a subset L of A if there is a **dependence relation**

$$0 \neq ng = n_1a_1 + \cdots + n_ka_k \qquad (n, n_i \in \mathbb{Z}) \tag{3.11}$$

for some elements $a_i \in L$. Thus g depends on L exactly if $\langle g\rangle \cap \langle L\rangle \neq 0$. A subset K depends on L if every element of K depends on L.

Every element a in an independent system can be replaced, without violating independence, by a non-zero multiple ma. Therefore, by replacing elements of finite order by multiples of prime power order, from every independent system we can get one in which each element is either of infinite or of prime power order.

An independent system M in A is **maximal** if there is no independent system in A that properly contains M. Every element $\neq 0$ of A depends on a maximal independent system. By Zorn's Lemma, *every independent system is contained in a maximal one.* Moreover, if the original system contained only elements of infinite or prime power orders, then a maximal one containing it can also be chosen to have this property.

Lemma 4.2. *An independent system is maximal if and only if it generates an essential subgroup.*

Proof. It suffices to observe that a non-zero element $a \in A$ depends on an independent system M if and only if $\langle a\rangle \cap \langle M\rangle \neq 0$. □

Rank of a Group By the **rank** $\mathrm{rk}(A)$ of a group A is meant the cardinal number of a maximal independent system containing only elements of infinite and prime power orders. If we consider only independent systems with elements of infinite order (of orders that are powers of a fixed prime p) which are maximal with respect to this property, then the cardinality of the system is called the **torsion-free rank** $\mathrm{rk}_0(A)$ (**p-rank** $\mathrm{rk}_p(A)$) of A. From the definitions it is evident that the equation

$$\mathrm{rk}(A) = \mathrm{rk}_0(A) + \sum_p \mathrm{rk}_p(A) \tag{3.12}$$

holds with p running over all primes. Obviously, $\mathrm{rk}(A) = 0$ means $A = 0$.

At this point the natural question is: how unique are these various ranks? In order to legitimize them, we need to show:

Theorem 4.3. *The ranks* $\mathrm{rk}(A), \mathrm{rk}_0(A), \mathrm{rk}_p(A)$ *of a group* A *are invariants of* A.

Proof. It suffices to prove that $\mathrm{rk}_0(A)$ and $\mathrm{rk}_p(A)$ are independent of the choice of the maximal independent system defining them.

It is routine to check that $\mathrm{rk}_0(A) = \mathrm{rk}(A/tA)$. As a consequence, in proving the invariance of $\mathrm{rk}_0(A)$, we may assume without loss of generality that A is a torsion-free group. Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_\ell\}$ be two maximal independent systems in A. Then there are integers m, m_i, n, n_j with $m, n \neq 0$ such that $ma_i = \sum_{j=1}^{\ell} m_{ij} b_j$ and $nb_j = \sum_{i=1}^{k} n_{ji} a_i$. Hence

$$mna_i = \sum_{h=1}^{k} \sum_{j=1}^{\ell} n_{ij} m_{jh} a_h$$

where the corresponding coefficients on both sides must be equal. This means that the product of matrices $\|n_{ij}\| \cdot \|m_{jh}\|$ is a scalar matrix mnE_k (E_k denotes the $k \times k$ identity matrix). This is impossible if $k < \ell$, thus $k \geq \ell$ must hold. For reasons of symmetry, $k = \ell$ follows, i.e. equivalent finite independent systems contain the same number of elements. This tells us that $\mathrm{rk}_0(A)$ is well defined whenever it is finite.

If $\mathrm{rk}_0(A)$ is infinite, then we show that $\mathrm{rk}_0(A) = |A|$ (A is still torsion-free). The inequality $\leq$ is obvious. To prove the converse, we choose a maximal independent system $L = \{a_i\}_{i \in I}$. For every $0 \neq g \in A$, there is $n \in \mathbb{N}$ such that $ng \in \langle L \rangle$, and if $ng = ng'$ ($g' \in A$), then $g = g'$. Hence we conclude that $|A| \leq |L|\aleph_0 = |L|$.

Turning to the ranks $\mathrm{rk}_p(A)$, it is clear that $\mathrm{rk}_p(A) = \mathrm{rk}(T_p)$ where T_p denotes the p-component of $T = tA$. Hence it is enough to verify the claim for p-groups A. Now if $\{a_i\}_{i \in I}$ is a maximal independent system, then so is $\{p^{m_i - 1} a_i\}_{i \in I}$ where $p^{m_i} = o(a_i)$. Therefore, $\mathrm{rk}_p(A)$ is the same as the rank of the socle $s(A)$. The socle is a $\mathbb{Z}/p\mathbb{Z}$-vector space, its dimension is obviously the same as its rank as a group. The uniqueness of the vector space dimension implies the uniqueness of $\mathrm{rk}(s(A)) = \mathrm{rk}_p(A)$. □

There is another important cardinal invariant associated with groups. This is the dimension of the $\mathbb{Z}/p\mathbb{Z}$-vector space $A/(tA + pA)$ which we shall call the p-**corank** of A, and will be denoted as

$$\mathrm{rk}^p(A) = \dim A/(tA + pA).$$

We will see later that this is the rank of the torsion-free part of p-basic subgroups of A.

★ **Notes.** The torsion-free rank of A is often defined as the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes A$ (then the uniqueness of $\mathrm{rk}_0(A)$ follows from that of the vector space dimension). The rank as we use here has been generalized to modules, called *Goldie dimension*.

Exercises

(1) Show that $\mathrm{rk}(\mathbb{Q}) = 1, \mathrm{rk}(\mathbb{Q}/\mathbb{Z}) = \aleph_0$, and $\mathrm{rk}(J_p) = 2^{\aleph_0}$ for each prime p.
(2) Prove that $\mathrm{rk}(A) = 1$ exactly if A is isomorphic to a subgroup of $\mathbb{Q}$ or to a subgroup of $\mathbb{Z}(p^\infty)$ for some prime p.
(3) Let B be a subgroup of A. Prove that: (i) $\mathrm{rk}(B) \leq \mathrm{rk}(A)$; (ii) $\mathrm{rk}(A) \leq \mathrm{rk}(B) + \mathrm{rk}(A/B)$; (iii) $\mathrm{rk}_0(A) = \mathrm{rk}_0(B) + \mathrm{rk}_0(A/B)$.
(4) The non-zero subgroups B_i $(i \in I)$ of A generate their direct sum in A if and only if every subset $L = \{b_i\}_{i \in I}$ with one $b_i \neq 0$ from each B_i is independent.
(5) A group of rank $\kappa \geq \aleph_0$ has 2^κ different subgroups.

5 Direct Sums of Cyclic Groups

The simplest kinds of infinitely generated groups are the direct sums of cyclic groups. These groups admit a satisfactory classification as we shall see below. We will feel fortunate if we are able to prove that certain groups under consideration are direct sums of cyclic groups.

For brevity, a direct sum of cyclic groups will be called a **Σ-cyclic group**.

Kulikov's Theorem A Σ-cyclic p-group contains no elements $\neq 0$ of infinite height. However, the absence of elements of infinite height does not ensure that a p-group is Σ-cyclic. We are looking for criteria under which a p-group is Σ-cyclic.

Theorem 5.1 (Kulikov [1]). *A p-group A is Σ-cyclic if and only if it is the union of a countable ascending chain of subgroups,*

$$A_0 \leq A_1 \leq \cdots \leq A_n \leq \ldots, \tag{3.13}$$

such that the heights of elements $\neq 0$ in A_n (computed in A) are bounded.

Proof. The stated condition is necessary: if A is a Σ-cyclic p-group, then in a decomposition, collect the cyclic summands of the same order p^n, for every n. If we denote their direct sum by B_n, then the subgroups $A_n = B_1 \oplus \cdots \oplus B_n$ $(n < \omega)$ (with bound $n-1$ on the heights) satisfy the stated condition.

For the proof of sufficiency, suppose that the chain (3.13) is as stated. Since we may adjoin the trivial subgroup to the beginning of the chain (3.13) and repeat subgroups without violating the hypothesis, it is clear that there is no loss of generality in assuming that $n-1$ is a bound for the heights in A_n, that is, $A_n \cap p^n A = 0$ for every $n < \omega$.

Accordingly, we consider the set of all chains $0 = C_0 \leq C_1 \leq \cdots \leq C_n \leq \ldots$ of subgroups of A such that

$$A_n \leq C_n \quad \text{and} \quad C_n \cap p^n A = 0 \quad \text{for every } n < \omega.$$

Define the chain of the C_n to be less than or equal to the chain of the B_n if and only if $C_n \leq B_n$ for all $n < \omega$. The set of all such chains in A is non-empty and is easily seen to be inductive, so Zorn's lemma applies to conclude that there exists a chain $0 = G_0 \leq G_1 \leq \cdots \leq G_n \leq \ldots$ that is maximal in the sense defined. Needless to say, $A = \bigcup_{n<\omega} G_n$.

The group G_n contains only elements of order $\leq p^n$, so $G_n \cap p^{n-1}A$ is in the socle of G_n. Select a $\mathbb{Z}/p\mathbb{Z}$-vector space basis L_n of $G_n \cap p^{n-1}A$, and set $L = \cup_{n<\omega} L_n$. For every $c_i \in L$ of $h(c_i) = n_i$ choose an $a_i \in A$ such that $p^{n_i} a_i = c_i$. The claim is that $A' = \langle \ldots, a_i, \ldots \rangle = \oplus_i \langle a_i \rangle$ is equal to A.

First we show that $\langle L \rangle = A[p]$. Since evidently $\langle L_n \rangle = G_n \cap p^{n-1}A$, all the elements $\neq 0$ in $\langle L_n \rangle$ are of height exactly $n-1$, so the $\langle L_n \rangle$ generate their direct sum, $\langle L \rangle = \oplus_{n<\omega} \langle L_n \rangle$. Assume, as a basis of induction on k, that $G_k[p] = \langle L_1 \rangle \oplus \cdots \oplus \langle L_k \rangle$. Let $a \in G_{k+1}[p] \setminus G_k$. By maximality, $\langle G_k, a \rangle \cap p^k A \neq 0$, thus $0 \neq g + ra = b \in p^k A$ with some $g \in G_k, r \in \mathbb{Z}$, where $r = 1$ may be assumed. Therefore, $g + a \in G_{k+1} \cap p^k A = \langle L_{k+1} \rangle$, thus a and hence $G_{k+1}[p]$ is contained in $\langle L_1 \rangle \oplus \cdots \oplus \langle L_{k+1} \rangle$. Consequently, $\langle L \rangle = A[p]$ follows.

Assume that, for some $n \in \mathbb{N}$, we have proved that every element of A of order $\leq p^n$ belongs to A'; for $n = 1$, this was done in the preceding paragraph. Pick an $a \in A$ of order p^{n+1} $(n \geq 1)$. Then $p^n a \in \langle L \rangle$, so we have $p^n a = m_1 c_1 + \cdots + m_\ell c_\ell$ with some $c_j \in L, m_j \in \mathbb{Z}$. Let $c_1, \ldots, c_r$ be of height $\geq n$, and $c_{r+1}, \ldots, c_\ell$ of height $< n$. Then in the equation

$$p^n a - m_1 c_1 - \cdots - m_r c_r = m_{r+1} c_{r+1} + \cdots + m_\ell c_\ell$$

the left-hand side is of height $\geq n$, while the right-hand side is contained in G_{n-1}; so both sides are 0. If we write $m_j c_j = p^n m_j' a_j$ $(j \leq r)$, then $a - m_1' a_1 - \cdots - m_r' a_r$ is of order $\leq p^n$, so it is contained in A' by induction hypothesis. Hence $a \in A'$ as well. $\square$

Prüfer's Theorems As corollaries we obtain the following two important, frequently quoted results.

Theorem 5.2 (Prüfer [1], Baer [1]). *A bounded group is* Σ*-cyclic.*

Proof. If A is bounded, then it can have but a finite number of non-zero p-components A_p. These components are also bounded, so we can apply Theorem 5.1 with all subgroups in (3.13) equal A_p, to conclude that each A_p is Σ-cyclic. □

Theorem 5.3 (Prüfer [1]). *A countable p-group is* Σ*-cyclic if and only if it contains no elements* $\neq 0$ *of infinite height.*

Proof. Only the 'if' part requires a verification. Suppose A is a countable p-group without elements of infinite height. If $\{a_0, \ldots, a_n, \ldots\}$ is a generating set of A, then A is the union of its finite subgroups $A_n = \langle a_0, \ldots, a_n \rangle$ $(n < \omega)$, where the heights of the elements are obviously bounded. The claim follows from Theorem 5.1. □

The following example shows that countability is an essential hypothesis in Theorem 5.3.

Example 5.4 (Kurosh). Let A be the torsion part of the direct product of the cyclic groups $\mathbb{Z}(p), \ldots, \mathbb{Z}(p^n), \ldots$. Then A is a p-group of the power of the continuum, without elements of infinite height (by the way, each $\mathbb{Z}(p^n)$ is a summand of A). Assume, by way of contradiction, that A is Σ-cyclic, say, $A = \oplus_{n<\omega} B_n$ where B_n is a direct sum of cyclic groups of fixed order p^n. Consider the socles $S_n = \oplus_{n \leq i < \omega} p^{i-1} B_i$; they form, with increasing n, an infinite properly descending chain such that S_n consists of those elements of $A[p]$ which are of heights $\geq n-1$. Clearly,

$$a = (c_1, \ldots, c_n, \ldots) \in A[p] \qquad (c_n \in \mathbb{Z}(p^n))$$

is of height $\geq n-1$ if and only if $c_1 = \cdots = c_{n-1} = 0$. This shows that each factor group S_n/S_{n+1} $(n = 1, 2, \ldots)$ is of order p. Hence $B_n[p] \cong S_n/S_{n+1}$ implies that the B_n are finite, and so A is countable, a contradiction. An $\aleph_1$-generated pure subgroup of A containing the direct sum $\oplus_n \mathbb{Z}(n)$ yields an example of smallest cardinality.

A quicker counterexample is available if we make use of the isomorphism of basic subgroups: no uncountable p-group with countable basic subgroup is Σ-cyclic.

Kulikov's criterion can be generalized to arbitrary cardinalities as follows (we make use of purity which will be discussed in Chapter 5).

Theorem 5.5 (Hill [13]). *A p-group A is* Σ*-cyclic if it is the union of an ascending chain* (3.13) *of* Σ*-cyclic pure subgroups* A_n $(n < \omega)$.

Proof. For countable A, sufficiency is easy: list the generators in a sequence: $a_1, a_2, \ldots, a_i, \ldots$. If $0 = B_0 < B_1 \leq \cdots \leq B_i$ is a chain of finite pure subgroups of A such that $a_1, \ldots, a_j \in B_j$ for all $j \leq i$, then choose a finite summand B_{i+1} of an A_n containing both B_i and a_{i+1}; such an n must exist. Then B_i as a bounded pure subgroup is a summand of A, and $A = \cup_{i<\omega} B_i$. Since $B_{i+1} = B_i \oplus C_i$ for some $C_i \leq A$, we get $A = \oplus_{i \in \mathbb{N}} C_i$.

The proof for the uncountable case is the exact analog of Theorem 7.5; we leave the details to the reader. □

Isomorphy of Decompositions Though a group may have several decompositions into a direct sum of cyclic groups, one can establish a strong uniqueness statement, just as in the finitely generated case. (Actually, one can prove more: the Krull-Schmidt property holds for Σ-cyclic groups.)

Theorem 5.6. *Any two direct decompositions of a group into direct sums of infinite cyclic groups and cyclic groups of prime power orders are isomorphic.*

Proof. First assume that A is a p-group. Collecting the cyclic summands of the same order, we get a decomposition $A = \oplus_{n<\omega} B_n$ where B_n is a direct sum of cyclic groups of the same order p^n. As in Example 5.4, we can argue that $B_n[p] \cong S_n/S_{n+1}$ where S_n is the set of elements of heights $\geq n-1$ in $A[p]$. The latter group is independent of the representation of A as direct sum of cyclic p-groups, and the dimension of S_n/S_{n+1} as a $\mathbb{Z}/p\mathbb{Z}$-vector space determines the number of cyclic summands of order p^n in any decomposition of A as a Σ-cyclic group.

In the general case, $A = B \oplus C$ where B is a Σ-cyclic torsion group and C is a free group. Then both B and C have unique decompositions (rk C being well defined), so the same holds for A. □

Subgroups of Σ-Cyclic Groups It is extremely important and most useful that the property of being Σ-cyclic is inherited by subgroups.

Theorem 5.7 (Kulikov [2]). *Subgroups of Σ-cyclic groups are again Σ-cyclic.*

Proof. First we dispose of the case when the group A is a p-group. By Theorem 5.1, A is the union of an ascending chain $A_0 \leq A_1 \leq \cdots \leq A_n \leq \ldots$ of subgroups, where the heights of elements of A_n are bounded, say, k_n is a bound in A_n. A subgroup B is the union of the chain

$$A_0 \cap B \leq A_1 \cap B \leq \cdots \leq A_n \cap B \leq \ldots$$

where the heights of elements of $A_n \cap B$, computed in B, do not exceed k_n. By virtue of Theorem 5.1, B is Σ-cyclic.

Turning to the general case, let A be an arbitrary Σ-cyclic group, and B a subgroup of A. Clearly, $tB = B \cap tA$, and so

$$B/tB = B/(B \cap tA) \cong (B + tA)/tA \leq A/tA,$$

where A/tA is a free group. By Theorem 1.6, B/tB is free, whence Theorem 1.5 implies that $B = tB \oplus C$ for some free subgroup C of B. By what has been shown in the preceding paragraph, tB is a direct sum of cyclic p-groups. Thus B is Σ-cyclic. □

Corollary 5.8 (Kulikov [2]). *Any two direct decompositions of a Σ-cyclic group have isomorphic refinements.*

Proof. In view of Theorem 5.7, each summand is Σ-cyclic. Replacing each summand by a direct sum of cyclic groups of orders ∞ or prime power, we arrive at refinements that are isomorphic, as is guaranteed by Theorem 5.6. □

The next lemma provides information about pure subgroups in free groups.

Lemma 5.9. (a) *A finite rank pure subgroup of a free group is a summand.*

(b) (Erdős [1]) *A pure subgroup H of a free group F contains a summand of F whose rank is the same as the rank of H.*

Proof. (a) A finite rank pure subgroup H is contained in a finitely generated summand of the free group F. Then F/H is finitely generated and torsion-free, so a free group. Therefore, H is a summand of F.

(b) If H is of finite rank, then it is a summand of F, and we are done. So assume that H is of infinite rank κ. Let $B = \{b_\alpha \mid \alpha < \kappa\}$ be a basis of F, and consider finite subsets B_i of B such that $\langle B_i \rangle \cap H \neq 0$. Select a maximal pairwise disjoint set S of such subsets B_i, and a non-zero h_i in each $\langle B_i \rangle \cap H$. Then the pure subgroup $\langle h_i \rangle_\star$ is a summand of $\langle B_i \rangle$, and hence $K = \oplus \langle h_i \rangle_\star$ is a summand of F, and so of H. Write $F = \langle S \rangle \oplus G$ where G is generated by the basis elements not in any member of S. Now $G \cap H \neq 0$ is impossible, because then the basis elements b_α occurring in a linear combination of a non-zero element in this intersection form a finite subset disjoint from every finite subset in S—this contradicts the maximality of S. Therefore, $G \cap H = 0$. Manifestly, the cardinality of the set of all basis elements b_α occurring in members of S is the same as the cardinality of S. Hence $G \cap H = 0$ implies that $\mathrm{rk}\, K = \mathrm{rk}\langle S \rangle = \mathrm{rk}\, F/G \geq \mathrm{rk}\, H = \kappa$. □

★ **Notes.** Various properties of Σ-cyclic groups have been investigated that are shared by larger classes of groups. The name of *Fuchs-5-group* is used in the literature for a group in which every infinite set is contained in a direct summand of the same cardinality. Trivial examples for such groups are direct sums of countable groups. Hill [8] proved that for every uncountable cardinal κ there exist p-groups with this property that need not be direct sums of countable groups. The existence of non-free $\aleph_1$-separable torsion-free groups shows that not all torsion-free Fuchs-5-groups are direct sums of countable subgroups.

Exercises

(1) For a group A, the following conditions are equivalent: (a) A is elementary; (b) every subgroup of A is a summand; (c) A is torsion with trivial Frattini subgroup; (d) A contains no proper essential subgroup.

(2) The direct product of $\kappa \geq \aleph_0$ copies of the cyclic group $\mathbb{Z}(p^k)$ is a direct sum of 2^κ copies of $\mathbb{Z}(p^k)$.

(3) Let A, B be Σ-cyclic groups.

  (a) $A \oplus A \cong B \oplus B$ implies $A \cong B$.

  (b) $A^{(\aleph_0)} \cong B^{(\aleph_0)}$ fails to imply $A \cong B$ even if A, B are finitely generated.

(4) Let A be a countable direct sum of cyclic groups of order p^2, and $B \cong A \oplus \mathbb{Z}(p)$. The isomorphy classes of subgroups (and factor groups) of A are equal to those of B, but $A \ncong B$.

(5) (Dlab)

(a) Let A be a bounded p-group, and $S = \{a_i\}_{i\in I}$ a subset of A such that the cosets $a_i + pA$ $(i \in I)$ generate A/pA. Then S generates A.
(b) Every generating set of a bounded p-group contains a minimal generating set (i.e. no generator can be omitted).

(6) (Szele) Improve on Example 5.4 by exhibiting an example of cardinality $\aleph_1$.
(7) Let $B = \oplus_{k\in\mathbb{N}} \mathbb{Z}(p^k)$.

(a) Every countable p-group is an epimorphic image of B.
(b) Each p-group of infinite cardinality κ is an epic image of $B^{(\kappa)}$.

(8) A is Σ-cyclic if it contains a Σ-cyclic subgroup G such that A/G is bounded.
(9) (Dieudonné [1]) Let G be a p-group that contains a subgroup A such that G/A is Σ-cyclic. Suppose that A is the union of a chain $A_0 \leq A_1 \leq \cdots \leq A_n \leq \ldots$ such that the heights of elements of A_n, computed in G, are bounded. Then G is Σ-cyclic.
(10) Let A, G be p-groups, and assume $C < A$ with Σ-cyclic A/C. If the homomorphism $\phi : C \to G$ does not decrease heights, then it extends to a homomorphism $A \to G$. [Hint: if $p^n a \in C$, there is $g \in G$ with $\phi(p^n a) = p^n g$.]
(11) An **equational class** or **variety** of groups is a class of groups that is closed under isomorphism, the formations of subgroups, epic images, and direct products. Prove that the following is a complete list of equational classes of abelian groups:

(a) the class of all abelian groups;
(b) for every positive integer n, the class of n-bounded abelian groups.

6 Equivalent Presentations

This section is concerned with special kind of presentations. First, Σ-cyclic groups will be considered.

Presentation with Stacked Basis We say that the group A has a **presentation with stacked bases** if there is a short exact sequence $0 \to H \to F \xrightarrow{\phi} A \to 0$ where $F = \oplus_{i\in I} \langle x_i\rangle$ is a free group and $H = \oplus_{i\in I} \langle n_i x_i\rangle$ is a free subgroup with $n_i \geq 0$ (see Theorem 2.6).

An obvious necessary condition for a group to be presented with stacked bases is that it be a Σ-cyclic group. Kaplansky raised the question whether or not every presentation of a Σ-cyclic group is with stacked bases. The affirmative answer was given by Cohen–Gluck [1]. In our treatment we follow closely their argument.

As a first step, we reduce the proof of the theorem to the torsion case. This is one of the rare situations when the discussion for torsion groups cannot be delegated to p-groups.

Lemma 6.1 (Cohen–Gluck [1]). *Let F be a free group and $A = B \oplus C$ any group with a free summand C. Given an epimorphism $\phi\colon F \to A$, F admits a decomposition $F = F_1 \oplus F_2$ such that $\phi(F_1) = B$ and $F_2 \cong C$.*

Proof. Let $\gamma: A \to C$ denote the projection along B. Then $F = F_1 \oplus F_2$ with $F_1 = \operatorname{Ker} \gamma\phi$ and $F_2 \cong \operatorname{Im} \gamma\phi = C$. The inclusion $B \leq \phi F_1$ cannot be proper. □

Next, we reduce the proof to the countable case; this is a main step, supported primarily by a straightforward back-and-forth argument.

Lemma 6.2 (Cohen–Gluck [1]). *Suppose $0 \to H \to F \xrightarrow{\phi} A \to 0$ is an exact sequence, and both F and A are direct sums of countable groups. Then there exist ('matching') direct decompositions*

$$F = \oplus_{\sigma<\tau} F_\sigma \qquad \textit{and} \qquad A = \oplus_{\sigma<\tau} A_\sigma \tag{3.14}$$

for some ordinal τ such that, for each $\sigma < \tau$,

(i) *F_σ is countable; and*
(ii) *$\phi F_\sigma = A_\sigma$.*

Proof. There is nothing to prove if A is countable, so suppose A is uncountable. Let $F = \oplus_{i\in I} G_i$ and $A = \oplus_{j\in J} B_j$ be decompositions with countable summands. For any $k \in I$, there is a countable subset Y_0 of J such that $\phi G_k \leq \oplus_{j\in Y_0} B_j$ and a countable subset X_0 of I such that $\oplus_{j\in Y_0} B_j \leq \phi(\oplus_{i\in X_0} G_i)$. Arguing the same way repeatedly, we obtain countable ascending chains of countable subsets $X_0 \subset \dots \subset X_n \subset \dots$ and $Y_0 \subset \dots \subset Y_n \subset \dots$ of I and J, respectively, satisfying

$$\oplus_{j\in Y_n} B_j \leq \phi(\oplus_{i\in X_n} G_i) \leq \oplus_{j\in Y_{n+1}} B_j \qquad (n < \omega).$$

If I_0 and J_0 denote the unions of the X_n and the Y_n, respectively, then let $F_0 = \oplus_{i\in I_0} G_i$ and $A_0 = \oplus_{j\in J_0} B_j$. They are clearly countably generated summands of F and A, respectively, such that $\phi F_0 = A_0$.

Assume that we have already found, for some ordinal σ, smooth chains of subsets $I_0 \subset \dots \subset I_\rho \subset \dots \subset I_\sigma$ and $J_0 \subset \dots \subset J_\rho \subset \dots \subset J_\sigma$ $(\rho \leq \sigma)$ of I and J, respectively, such that for all $\rho+1 \leq \sigma$, the sets $I_{\rho+1} \setminus I_\rho$ and $J_{\rho+1} \setminus J_\rho$ are countable, and the groups $F'_\rho = \oplus_{i\in I_{\rho+1}\setminus I_\rho} G_i$, $A_\rho = \oplus_{j\in J_{\rho+1}\setminus J_\rho} B_j$ satisfy $\phi(\oplus_{\rho<\sigma} F'_\rho) = \oplus_{\rho<\sigma} A_\rho$. Using a back-and-forth argument, we adjoin to I_σ and J_σ countable subsets U and V, respectively, such that putting $I_{\sigma+1} = I_\sigma \cup U$ and $J_{\sigma+1} = J_\sigma \cup V$, condition (ii) will be satisfied for $F'_{\sigma+1} = \oplus_{i\in U} G_i$, $A_{\sigma+1} = \oplus_{j\in V} B_j$. We repeat this argument transfinitely until the index sets I and J are exhausted, where—as usual—at limit ordinals we take unions of the previously selected subsets. Finally, we get decompositions satisfying $\phi(\oplus_{\rho<\sigma} F'_\rho) = \oplus_{\rho<\sigma} A_\rho$ for all $\sigma < \tau$.

These are not yet decompositions we are looking for, we still have to modify them to obtain ones satisfying (i)–(ii). Suppose that we have found F_ρ to satisfy $\phi F_\rho = A_\rho$ for all $\rho < \sigma$. Consider the diagram

$$\begin{array}{ccc} F'_\sigma & \xrightarrow{\phi} & \oplus_{\rho\le\sigma} A_\rho \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\delta} \\ \oplus_{\rho<\sigma} F_\rho & \xrightarrow{\phi} & \oplus_{\rho<\sigma} A_\rho \end{array}$$

where δ denotes the projection with kernel A_σ. Since the map in the bottom row is surjective and F'_σ is a free group, we can find a map ψ making the diagram commute. Clearly, $F_\sigma = \{x - \psi x \mid x \in F'_\sigma\}$ is isomorphic to F'_σ. Furthermore, $\phi(x - \psi x) = \phi x - \phi\psi x = \phi x - \delta\phi x \in A_\sigma$ shows that $\phi F_\sigma \le A_\sigma$. This inclusion cannot be proper, thus $\phi F_\sigma = A_\sigma$. As $\oplus_{\rho<\sigma} F_\rho \oplus F'_\sigma = \oplus_{\rho\le\sigma} F_\rho$, we may replace F'_σ by F_σ for each $\sigma < \tau$ inductively, to obtain $\phi F_\sigma = A_\sigma$ for all $\sigma < \tau$. □

The Torsion Case We are now prepared to tackle the torsion case. The starting point is a preliminary lemma (valid for arbitrary groups).

Lemma 6.3. *Let $F = F_1 \oplus F_2$ be a free group, and $\phi\colon F \to A = A_1 \oplus A_2$ an epimorphism such that $A_1 \le \phi F_1$. Then in the given direct decomposition, F_2 can be replaced by some $G \le F$ satisfying $\phi G \le A_2$.*

Moreover, if F' is a summand of F_2 with $\phi F' \le A_2$, then G can be chosen so as to contain F'.

Proof. Let $\pi\colon A \to A_1$ denote the projection with kernel A_2. The projectivity of F_2 guarantees the existence of ρ making the square

$$\begin{array}{ccc} F_2 & \overset{\rho}{\dashrightarrow} & F_1 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi\phi} \\ A_1 \oplus A_2 & \xrightarrow{\pi} & A_1 \end{array}$$

commutative. Setting $G = (1-\rho)F_2$, evidently $\pi\phi G = \pi(\phi - \phi\rho)F_2 = 0$. We conclude that $F_1 \oplus F_2 = F_1 \oplus G$, establishing the first claim. For the rest, it is enough to observe that the map ρ can be chosen so as to act trivially on the summand F'. □

The following lemma is a crucial ingredient in the proof of Theorem 6.5 to guarantee that no generator of F is left out in the successive decompositions. Theorem 6.2 permits us to confine ourselves to countable groups.

Lemma 6.4. *Let* $0 \to H \to F \xrightarrow{\phi} A \to 0$ *be a presentation of a* Σ*-cyclic torsion group* A*,* F *a countable free group. If* A_0 *is a finitely generated summand of* A*, then there are direct decompositions*

$$F = F_1 \oplus F_2 \quad \textit{and} \quad H = (H \cap F_1) \oplus (H \cap F_2)$$

such that

(a) F_1 *is finitely generated and* ϕF_1 *contains* A_0*; and*
(b) $A = \phi F_1 \oplus \phi F_2$.

Proof. Write $A = A_0 \oplus A_0'$ where A_0' is the complement of A_0 in a direct decomposition of A into cyclic groups of prime power orders. Apply Lemma 6.3 to this decomposition to get $F = F_0 \oplus G_2$ with $\phi G_2 \le A_0'$.

Choose a summand $A_2 \le \phi G_2$ of A_0', say $A_0' = A_1 \oplus A_2$ with finitely generated A_1. Again by Lemma 6.3, we argue that $\phi F_0 \le A_0 \oplus A_1$ may be assumed. In this way, we obtain a decomposition $A = A_0 \oplus A_1 \oplus A_2$, where $A_0 \le \phi F_0$ and $A_2 \le \phi G_2$. That $A_1 = (A_1 \cap \phi F_0) + (A_1 \cap \phi G_2)$ should be clear. Assuming that the cyclic summands in A are decomposed into their p-components, for any p, either ϕF_0 or ϕG_2 contains an element of A_1 of maximal p-power order; this generates a summand C of A_1. If C is contained in ϕF_0, then write $A_1 = C \oplus B_1$, and with the aid of Lemma 6.3 we can change G_2 to a summand G such that ϕG has trivial projection on $A_0 \oplus C = B_0$, and at the same time replace A_0 by B_0, and A_1 by B_1 to obtain $A = B_0 \oplus B_1 \oplus A_2$.

We continue in a similar fashion, next adjoining a summand of B_1 to A_2, etc. After a finite number of steps, we arrive at a decomposition $F = F_1 \oplus F_2$, satisfying (i) and (ii). □

The Stacked Basis Theorem Equipped with these lemmas, we are well prepared for the proof of the main result. We keep the same notation.

Theorem 6.5 (Cohen–Gluck [1]). *Every presentation of a* Σ*-cyclic group has stacked bases.*

Proof. In view of Lemma 6.1 and 6.2, the proof can be reduced to the case, in which A is a countable Σ-cyclic torsion group. Then F can also be assumed to be countable, say $F = \oplus_{i \in \mathbb{N}} \langle x_i \rangle$. We will be done if we reduce the problem to the finitely generated case, because then a simple reference to Theorem 2.6 will complete the proof.

By the preceding lemma, there is a decomposition $F = F_{11} \oplus F_{12}$ such that F_{11} is finitely generated, $x_0 \in F_{11}$, and $H = (H \cap F_{11}) \oplus (H \cap F_{12})$. Next, F admits a decomposition $F = F_{21} \oplus F_{22}$ where F_{21} is finitely generated, contains F_{11} and x_1, and H splits accordingly. Continuing in the same way, we obtain an ascending chain $F_{11} \le F_{21} \le \dots$ of summands of F, for which $H \cap F_{n1}$ is a summand of H. The union of the F_{n1} must be all of F. If we define $A_n (n < \omega)$ *via* $A_0 = 0$, $F_{n1} = F_{n-1,1} \oplus A_n$, and let $B_n = H \cap A_n$, then $F = \oplus_{n<\omega} A_n$ and $H = \oplus_{n<\omega} B_n$ are decompositions into finitely generated summands such that A_n and B_n are stacked. The reduction to the finitely generated case has been accomplished, and the proof is completed. □

Equivalent Presentations of Torsion-Free Groups The last theorem asserts that every presentation of a Σ-cyclic group is equivalent to one with stacked bases in the sense made precise by the following definition.

Let F, F' be free groups, H, H' subgroups such that $F/H \cong F'/H'$. We say that F/H and F'/H' are **equivalent presentations** of $A \cong F/H$ if there is an isomorphism $\xi : F \to F'$ carrying H onto H'.

In general, not much can be said about the situations when two presentations of a group have to be equivalent. However, the case of torsion-free groups provides an interesting, though not so easy positive example.

First of all, note that an obvious necessary condition for the equivalence of two presentations F/H and F'/H' of A is that $\operatorname{rk} F = \operatorname{rk} F'$ and $\operatorname{rk} H = \operatorname{rk} H'$. Our next purpose will be to show that, if A is torsion-free, then the sole condition $\operatorname{rk} H = \operatorname{rk} H'$ will be enough to ensure the equivalence of the presentations F/H and F'/H'.

We require an interesting preliminary lemma.

Lemma 6.6 (Erdős [1]). *Let F be a free group, and H a pure subgroup of F. F has a basis which is a complete set of representatives* mod H *if and only if* $|F/H| = \operatorname{rk} H$.

Proof. If F has such a basis $B = \{b_\alpha \mid \alpha < \kappa\}$ with (only) b_0 contained in H, then by the purity of H, B must be infinite, and obviously $|F/H| = |B| = \operatorname{rk} F$. For each $b_\alpha \in B$ there is a unique $b_\beta \in B$ such that $b_\alpha + b_\beta \in H$. If $b_\alpha = b_\beta$, then $2b_\alpha \in H$, so $b_\alpha = b_0$, by purity. The elements $b_\alpha + b_\beta$ $(b_\alpha \neq b_\beta)$ and b_0 form a basis of a summand of F contained in H. Hence $|B| \leq \operatorname{rk} H$, and necessity is established.

Turning to the proof of the sufficiency, suppose $|F/H| = \operatorname{rk} H$. From Lemma 5.9 we derive that H contains a summand G of F such that $\operatorname{rk} G = |F/H|$. Choose a basis $Y = \{y_j\}$ of G, and extend it to a basis $B = \{b_\alpha \mid \alpha < \kappa\}$ of F. Well-order B in such a way that the elements of Y precede the rest of the basis elements in B. Each element $h \in H$ can be written uniquely as a linear combination $h = t_1 b_{\alpha_1} + \cdots + t_s b_{\alpha_s}$ $(t_i \in \mathbb{Z})$ with non-zero terms such that $\alpha_1 < \cdots < \alpha_s$. To simplify our wording, we will say that the ordinal α_s is associated with h. If among the elements $h \in H$ associated with the same α_s there is one, say h', with $|t_s| = 1$, then in the basis B the element b_{α_s} can be replaced by h', without violating the basis character of the set. In doing so for all possible ordinals α_s inductively, Y remains unchanged, and the new basis (which we continue denoting by B) will have the additional property that if $h \in H$ is associated with α_s, and in the expression for h the coefficient of b_{α_s} is 1 in absolute value, then necessarily $b_{\alpha_s} = \pm h \in H$.

Split the basis B into two disjoint subsets, $B = B' \cup B''$, such that $Y \leq B' = B \cap H$. We keep B'', but change B' in order to obtain a basis $B^\star$ of F which is a complete set of representatives mod H, as desired.

First, observe that different elements b_α and b_β of B'' must belong to different cosets mod H. Indeed, otherwise $h = b_\alpha - b_\beta \in H$ is associated with either b_α or b_β, so either $b_\alpha \in H$ or $b_\beta \in H$, which is impossible, B'' being disjoint from H. Of course, there are cosets mod H which do not intersect B''. Since $B' \subset H$ implies that each coset mod H is represented by an element of the subgroup $\langle B'' \rangle$, for each coset mod H disjoint from B'' we can choose a representative $x_i \in \langle B'' \rangle$. Thus $B'' \cup X$ (with $X = \{x_i \mid i \in I\}$ for some index set I) is a complete set of representatives mod H.

Next we show that $|X| = |B'|$. On one hand, $\operatorname{rk} H = |F/H| = |Y| \leq |B'| \leq \operatorname{rk} F$ implies $|B'| = \operatorname{rk} F = |F/H|$. On the other hand, let b_γ be the first element of B'' in the chosen well-ordering of B. No two of the elements of the form $b_\alpha - b_\gamma$ ($b_\alpha \in B''$) belong to the same coset mod H, and none of these is congruent mod H to a $b_\beta \in B''$ (again, otherwise $b_\alpha - b_\gamma - b_\beta \in H$ would be associated with either b_α or b_β, etc.). Thus there are at least $|B''|$ many cosets of H which do not intersect B''; hence, $|B''| \leq |X|$ follows. This together with $|B''| + |X| = \operatorname{rk} F$ yields $|X| = \operatorname{rk} F$. Hence $|B'| = |X|$, so there is a bijection between the set of elements $\{b_i\}$ of B' and the set of cosets $\{x_i + H\}$ (where we have the corresponding elements carrying the same index i). If in the basis B, $b_i \in B'$ will be replaced by $b_i + x_i$, then we obtain a new basis $B^\star$ of F which is at the same time a complete set of representatives mod H. □

We are now able to verify the main result mentioned earlier.

Theorem 6.7 (Erdős [1]). *Two presentations, F/H and F'/H', of a torsion-free group are equivalent if and only if* $\operatorname{rk} H = \operatorname{rk} H'$.

Proof. To verify sufficiency, suppose $\operatorname{rk} H = \operatorname{rk} H'$; as noted above, this implies $\operatorname{rk} F = \operatorname{rk} F'$. We prove more than stated, viz. we show that every isomorphism $\psi : F/H \to F'/H'$ is induced by an isomorphism $\phi : F \to F'$ carrying H onto H'.

Since A is torsion-free, both H and H' are pure. Ignoring the trivial case, we may suppose that $\operatorname{rk} H$ is infinite. We distinguish three cases.

Case I: $\operatorname{rk} H = |A|$. Then the same is true for $\operatorname{rk} H'$. In view of Lemma 6.6, there exist a basis B of F and a basis B' of F' which are complete sets of representatives mod H and mod H', respectively. The correspondence $B \to B'$ which maps $b \in B$ upon $b' \in B'$ if and only if ψ maps the coset $b + H$ upon $b' + H'$ extends uniquely to an isomorphism $\phi : F \to F'$ under which H' is the image of H. Thus the two presentations are equivalent.

Case II: $\operatorname{rk} H > |A|$. Let G be a free group whose rank is $\operatorname{rk} H$. Replace F by $F \oplus G$ and F' by $F' \oplus G$, but keep H and H'. Application of Case I to $A \oplus G$ implies the existence of an isomorphism $\phi : F \oplus G \to F' \oplus G$ with $\phi H = H'$ inducing ψ. It is self-evident that $\phi F = F'$.

Case III: $\operatorname{rk} H < |A|$. There is a decomposition $F = F_1 \oplus F_2$ such that $H \leq F_1$ and $\operatorname{rk} H = \operatorname{rk} F_1 < \operatorname{rk} F_2 = |A|$. Thus $A = F_1/H \oplus F_2$, and ψ yields a similar decomposition $A' = F'_1/H' \oplus F'_2$. Case I guarantees the existence of an isomorphism $F_1 \to F'_1$ mapping H upon H'; this along with $F_2 \to F'_2$ (restriction of ψ) yields an isomorphism $\phi : F \to F'$. □

★ **Notes.** Hill–Megibben [4] furnished another proof of Theorem 6.5 as a corollary to a more general result which they proved on equivalent presentations of arbitrary abelian groups. F/H and F'/H' are equivalent presentations if and only if, for each prime p,

$$\dim(H + pF)/pF = \dim(H' + pF')/pF'.$$

For Dedekind domain, A.I. Generalov and M.V. Zheludev [St. Petersburg Math. J. **7**, 619–661 (1996)] characterized equivalent presentations. No such study is available for larger classes of domains, but several special cases have been investigated.

Various generalizations of the stacked basis theorem may be found in the literature. Let us mention Ould-Beddi–Strüngmann [1] where homogeneous completely decomposable groups are considered. Osofsky [1] studied a kind of dual to the stacked basis theorem. She proved that if H is a subgroup of a free group F such that F/H is p^n-bounded, then for every decomposition $F/H = \oplus C_i$ with cyclic groups C_i there is a decomposition $F = \oplus F_i$ such that $C_i = F_i/(H \cap F_i)$.

Cutler–Irwin–Pfaendtner–Snabb [1] have a nice generalization of Lemma 6.6. They show that a pure subgroup H in a Σ-cyclic group G contains a summand K of G such that $\mathrm{rk}_0(K) = \mathrm{rk}_0(H)$ and $\mathrm{rk}_p(K) = \mathrm{rk}_p(H)$ for each p. See Lemma 6.12 in Chapter 5, for the torsion case.

Exercises

(1) (Erdős) Let H be a subgroup of a group G such that G/H is torsion-free. There is a generating system of G which is a complete set of representatives mod H if and only if $|H| \leq |G/H|$. [Hint: Lemma 6.2 with a presentation of G.]

(2) (Hill–Megibben) If $A = F/H$ is a presentation of an infinite group such that F is free and $\mathrm{rk}\, F > |A|$, then there is a direct decomposition $F = F_1 \oplus F_2$ such that $\mathrm{rk}\, F_1 = |A|$ and $F_2 \leq H$.

(3) Let $H_0 < \cdots < H_n < \ldots$ be a countable ascending chain of summands of a free group F.

   (a) The union $H = \cup_{n<\omega} H_n$ need not be a summand of F.
   (b) H contains a summand of F whose rank is $\sum_{n<\omega} \mathrm{rk}(H_n)$. [Hint: H is pure in F, and apply Exercise 1.]

(4) (Erdős) Let $A = \oplus_{i \in I} A_i$ be a direct sum of torsion-free groups. If F is a free group and $\phi : F \to A$ is an epimorphism, then there is a decomposition $F = \oplus_{i \in I} F_i$ such that $\phi : F_i \to A_i$ for each $i \in I$. [Hint: represent $A_i = F_i'/H_i'$ such that $\sum_{i \in I} \mathrm{rk}(H_i') \leq \mathrm{rk}(\mathrm{Ker}\, \phi)$, and apply Lemma 6.6.]

7 Chains of Free Groups

We are looking for criteria for a group to be free, especially when the union of a chain of free subgroups is again free. In this section and in the next one, we have to use frequently purity to be discussed in Chapter 5.

Pontryagin's Criterion In a few cases useful criteria for freeness can be established. The one which is most often used works for countable torsion-free groups.

Theorem 7.1 (Pontryagin [1]). *A countable torsion-free group is free if and only if each of its finite rank subgroups is free. Equivalently, for every $n \in \mathbb{N}$, the subgroups of rank $\leq n$ satisfy the maximum condition.*

Proof. Because of Theorem 1.6, necessity is evident. For sufficiency, let $A = \langle a_0, \dots, a_n, \dots \rangle$ be a countable torsion-free group all of whose subgroups of finite rank are free. Define $A_0 = 0, A_n = \langle a_0, \dots, a_{n-1}\rangle_*$ $(n \in \mathbb{N})$ (the purification of $\langle a_0, \dots, a_{n-1}\rangle$ in A). Then $\operatorname{rk} A_n \leq n$ and $\operatorname{rk} A_{n+1} \leq \operatorname{rk} A_n + 1$. Therefore, either A is of finite rank—in which case there is nothing to prove—or there is a subsequence B_n of the A_n, such that $\operatorname{rk} B_n = n$, and A is the union of the strictly ascending chain $0 = B_0 < B_1 < \dots < B_n < \dots$. Now B_{n+1}/B_n is torsion-free of rank 1 and finitely generated, thus $B_{n+1}/B_n \cong \mathbb{Z}$. From Theorem 1.5 we obtain $B_{n+1} = B_n \oplus \langle b_n \rangle$ for some $b_n \in A$. This shows that the elements $b_0, b_1, \dots, b_n, \dots$ generate the direct sum $\oplus_{n<\omega}\langle b_n\rangle$, whence $A = \oplus_{n<\omega}\langle b_n\rangle$ is immediate.

By Theorem 1.6, the second formulation is equivalent to the first one. □

Corollary 7.2. *Suppose $0 = G_0 < G_1 < \dots < G_n < \dots$ is a chain of countable free groups such that each G_n is pure in the union G of the chain. Then G is free.*

Proof. A finite rank subgroup of G is contained in some G_n, so it is free. The claim is immediate from Theorem 7.1. □

If we have a chain like in Corollary 7.2 with the G_n as summands in a larger group F, the union G need not be a summand of F.

Example 7.3. Let G be a free group that is the union of a countable chain of infinite rank summands $G_0 < G_1 < \dots < G_i < \dots$. Our claim is that there exists a countable free group F containing G such that each G_i is, but G is not a summand of F.

Let $0 \to H \to F' \to \mathbb{Q} \to 0$ be a presentation of $\mathbb{Q}$ with countable free F'. Let H_n $(n < \omega)$ be a chain of finite rank summands of the free group H with union H. Then F'/H_n is free for all $n < \omega$. Next, pick free groups $F_0 \cong G_0$ and $F_i \cong G_i/G_{i-1}$ $(i \geq 1)$. It is evident that

$$G \cong H \oplus \oplus_{i<\omega} F_i \quad \text{and} \quad G_n \cong H_n \oplus \oplus_{n\leq i<\omega} F_i \quad (n < \omega).$$

Finally, we embed G in a free group $F \cong F' \oplus \oplus_{i<\omega} F_i$ imitating the embedding of H in F' and keeping the G_i fixed. This F is as desired.

The Eklof–Shelah Criterion The following lemmas provide us with versatile criteria for a group to be free.

Lemma 7.4. *Let, for some ordinal τ,*

$$0 = A_0 < A_1 < \dots < A_\sigma < \dots \quad (\sigma < \tau) \tag{3.15}$$

be a smooth chain of pure subgroups of a group A such that $A = \bigcup_{\sigma<\tau} A_\sigma$. If, for each $\sigma < \tau$, the factor group $A_{\sigma+1}/A_\sigma$ is free, then A is free.

Proof. In view of the stated condition, $A_{\sigma+1} = A_\sigma \oplus B_\sigma$ for each $\sigma < \tau$, for a suitable subgroup B_σ of $A_{\sigma+1}$ Theorem 1.5. If X_σ denotes a basis of B_σ, then the set union $X = \bigcup_{\sigma<\tau} X_\sigma$ is a basis for A. □

We can now verify the Eklof–Shelah criterion which provides a necessary and sufficient condition for freeness.

Theorem 7.5 (Eklof–Shelah). *Let κ be an uncountable regular cardinal, and assume $0 = A_0 < A_1 < \cdots < A_\sigma < \ldots (\sigma < \kappa)$ is a smooth chain of pure subgroups of a group A such that*

(i) *all the A_σ are free groups of cardinality $< \kappa$, and*
(ii) $A = \bigcup_{\sigma<\kappa} A_\sigma$.

Then A is free if and only if the set

$$E = \{\sigma < \kappa \mid \exists\, \rho > \sigma \;\; \text{such that } A_\rho / A_\sigma \text{ is not free}\}$$

is not stationary in κ.

Proof. Suppose A is free. Consider a filtration $\{B_\sigma\}_{\sigma<\kappa}$ of A with summands. The set C of indices σ of those subgroups A_σ which appear in the filtration $\{B_\sigma\}_{\sigma<\kappa}$ is a cub in κ, so $\{A_\sigma\}_{\sigma\in C}$ provides a filtration of A with summands. We see that A/A_σ is free for all $\sigma \in C$, and so C does not intersect the set E. This proves that E is not stationary.

Conversely, assume that E is not stationary. Then there is a cub $C \subset \kappa$ which does not intersect E. Evidently, $\{A_\sigma\}_{\sigma\in C}$ is still a filtration of A. Relabeling, we have a filtration $\{A_\sigma\}_{\sigma<\kappa}$ where all factor groups $A_{\sigma+1}/A_\sigma$ are free. By Lemma 7.4, A is free. ☐

Remark. For future applications we point out that both Lemma 7.4 and Theorem 7.5 hold for p-groups A_σ if 'free' is replaced throughout by 'Σ-cyclic.' The proofs are the same with obvious changes.

The next lemma teaches us how to create from a short chain of direct sums with large factor groups a long chain with small factor groups. (The main interest is in the torsion-free case, but no such restriction is needed.)

Lemma 7.6. *Assume*

$$0 = G_0 < G_1 < \cdots < G_n < \ldots$$

is a chain of groups that are pure in the union $G = \cup_{n<\omega} G_n$, where each G_n is a direct sum of countable groups. Then there is a smooth chain

$$0 = A_0 < A_1 < \cdots < A_\sigma < \ldots \quad (\sigma < \tau) \tag{3.16}$$

of pure subgroups A_σ of G such that

(i) *$A_\sigma \cap G_n$ is a summand of G_n, for every $n < \omega$ and $\sigma < \tau$; and*
(ii) *$A_{\sigma+1}/A_\sigma$ $(\sigma + 1 < \tau)$ is countable and the union of an ascending chain of pure subgroups, isomorphic to summands of G_n $(n < \omega)$.*

Proof. We start choosing a fixed direct decomposition of each G_n into countable summands, and define an $H(\aleph_0)$-family $\mathcal{H}_n$ of summands of G_n to consist of all direct sums of subsets of components in the chosen decomposition. We break the proof into three steps.

Step 1. The collection

$$\mathcal{G}_n = \{A \in \mathcal{H}_n \mid A + G_i \text{ is pure in } G_n \text{ for each } i < n\}$$

is a $G(\aleph_0)$-family of subgroups of G_n.
All that we have to check is that $\mathcal{G}_n$ satisfies the countability condition for a $G(\aleph_0)$-family, since the other conditions are obvious. Let $A \in \mathcal{G}_n$, and H_0 a countable subgroup of G_n. Suppose that we already have a chain $A = B_0 < B_1 < \cdots < B_m$ of subgroups in $\mathcal{H}_n$ such that

1. $A + H_0 \leq B_1$;
2. B_{j+1}/B_j is countable for all $j < m$; and in addition,
3. for each $j < m$ and for each $i < n$, $(B_{j+1} + G_i)/(A + G_i)$ contains a purification of $(B_j + G_i)/(A + G_i)$ in $G_n/(A + G_i)$.

To find a next member B_{m+1} of the chain, for each $i < n$, let $V_i \subset G_n$ be a countable set that—along with $(B_m + G_i)/(A + G_i)$—generates a pure subgroup in $G_n/(A + G_i)$. Thus $H_{m+1} = \bigcup_{i<n} V_i$ is likewise a countable set. Consequently, there is a $B_{m+1} \in \mathcal{H}_n$ such that $B_m + H_{m+1} \subset B_{m+1}$ and B_{m+1}/B_m is countable. Then, for each $i < n$, $(B_{m+1} + G_i)/(A + G_i)$ contains the purification of $(B_m + G_i)/(A + G_i)$ in $G_n/(A + G_i)$. The union B of the chain of the B_m for all $m < \omega$ is a member of $\mathcal{H}_n, B/A$ is evidently countable, and our construction guarantees that $(B + G_i)/(A + G_i)$ is pure in $G_n/(A + G_i)$. Thus $B + G_i$ is pure in G_n, i.e. $B \in \mathcal{G}_n$.

Step 2. The family

$$\mathcal{B} = \{A \leq G \mid A \cap G_n \in \mathcal{G}_n \text{ for each } n < \omega\}$$

is a $G(\aleph_0)$-family of subgroups in G.
Again, only the countability condition requires a proof. Since there are but countably many indices n to deal with, a similar back-and-forth argument (ω times) suffices to ensure that for each $A \in \mathcal{B}$, there exists an $A' \in \mathcal{B}$ such that A'/A is countable, as needed.

Step 3. At this point we know that $\mathcal{B}$ is a $G(\aleph_0)$-family satisfying (i), so we can extract a smooth chain (3.16) of pure subgroups with countable factor groups. Evidently, the group $A_{\sigma+1}/A_\sigma$ is the union of the ascending chain of groups $[A_\sigma + (G_n \cap A_{\sigma+1})]/A_\sigma \cong (G_n \cap A_{\sigma+1})/(G_n \cap A_\sigma)$ $(n < \omega)$, all summands of G_n in the chosen direct decomposition. □

Hill's Criterion The following result is a far-reaching generalization of Pontryagin's theorem.

Theorem 7.7 (Hill [9]). *The union G of a countable ascending chain*

$$0 = G_0 < G_1 < \cdots < G_n < \ldots$$

of pure subgroups, each of which is free, is a free group.

Proof. The given chain can be replaced by a chain of the A_σ as stated in Lemma 7.6. Apply Corollary 7.2 to the factor groups $A_{\sigma+1}/A_\sigma$ to conclude that they are free. A simple reference to Theorem 7.5 completes the proof. □

It should be pointed out that this theorem fails to hold for longer chains, as is shown by Theorem 8.6 below.

Before we go on, we would like to mention an important consequence of Hill's theorem. This is a special case of Shelah's compactness theorem 9.2 for limit ordinals cofinal with ω.

Corollary 7.8 (Hill [21]). *Suppose λ is an infinite cardinal whose cofinality is ω. A group of cardinality λ is free provided that all of its subgroups of cardinalities $< \lambda$ are free.*

Proof. Evidently, a group of cardinality λ is the union of a countable ascending chain of pure subgroups whose cardinalities are $< \lambda$. By hypothesis, each of these is free, so the claim follows right away from Theorem 7.7. □

Another criterion worthwhile recording is the following. (Observe the difference between Theorem 7.5 and Lemma 7.9.)

Lemma 7.9 (Eklof [5]). *Let κ be a regular cardinal, and*

$$0 = A_0 < A_1 < \cdots < A_\sigma < \cdots < A_\kappa = A \tag{3.17}$$

a smooth chain of free groups such that A_σ/A_ρ is free whenever ρ is a successor ordinal and $\rho < \sigma < \kappa$. If the set

$$E = \{\lambda < \kappa \mid \lambda \text{ limit ordinal},\ A_{\lambda+1}/A_\lambda \text{ not free}\}$$

is not stationary in κ, then A is a free group, and so is A/A_ρ for every successor ordinal ρ.

Proof. Suppose E is not stationary, i.e. there is a cub $C \subset \kappa$ that does not intersect E. Those A_σ whose indices belong to C form a chain like (3.17); we may assume that (3.17) is this subchain. In this chain, $A_{\sigma+1}/A_\sigma$ is free for all $\sigma < \kappa$, so Theorem 7.5 implies that A is free. The second claim follows by applying the result to A/A_ρ. □

When Torsion-Free has to Be Free A rather remarkable feature of free groups was discovered by Griffith [5]. A torsion-free group containing a free subgroup with bounded factor group is easily seen to be again free, and interestingly, the same conclusion can be reached under much weaker conditions on the factor group. The following theorem is a slightly modified version of Griffith's theorem.

Theorem 7.10. *Let A be a torsion-free group, and F a free subgroup of A. If A/F is a p-group that admits an $H(\aleph_0)$-family of subgroups such that all the factor groups in $H(\aleph_0)$ are reduced, then A is free* (*and* $\cong F$).

Proof. If A is of finite rank, then hypothesis implies A/F is a reduced p-group of finite rank, so it is finite (cp. Theorem 5.3). As a finitely generated torsion-free group, A is free.

Next assume F is of countable rank, and write $F = \oplus_{n<\omega} Z_n$ with $Z_n \cong \mathbb{Z}$. For $n < \omega$, set $F_n = \oplus_{i \le n} Z_i$ and $A_n = \langle F_n \rangle_*$. Manifestly, $A_n/F_n = A_n/(A_n \cap F) \cong (A_n + F)/F \le A/F$, which shows that A_n/F_n is reduced, and hence it must be a finite p-group. Therefore, A_n is free of finite rank by the preceding paragraph. Hence A is the union of a countable ascending chain $\{A_n\}_{n<\omega}$ of pure free subgroups, and by Corollary 7.2 we conclude that A itself is free.

Turning to the uncountable case, set $F = \oplus_{i \in I} Z_i$ with $Z_i \cong \mathbb{Z}$, and for the p-group $T = A/F$, select an $H(\aleph_0)$-family $\mathcal{H}$ as stated. As T is reduced, and A is torsion-free, every non-zero element of A is divisible but by a finite number of integers. We are going to define by transfinite induction a smooth chain $\varnothing = I_0 \subset I_1 \subset \cdots \subset I_\sigma \subset \cdots \subset I_\tau = I$ of subsets of I such that for all $\sigma < \tau$, we have

(a) $|I_{\sigma+1} \setminus I_\sigma| \le \aleph_0$, and
(b) $(A_\sigma + F)/F$ (which is $\cong A_\sigma/F_\sigma$) is a subgroup $T_\sigma \in \mathcal{H}$, where $F_\sigma = \oplus_{i \in I_\sigma} Z_i$ and $A_\sigma = \langle F_\sigma \rangle_*$.

Suppose σ is an ordinal such that, for all $\rho < \sigma$, the subsets I_ρ have been selected as required. If σ is a limit ordinal, then we set $I_\sigma = \cup_{\rho<\sigma} I_\rho$, as is forced by continuity. In this case, (b) will be satisfied, since $\mathcal{H}$ is closed under unions. If $\sigma = \rho + 1$ and $I_\rho \neq I$, pick any $i \in I \setminus I_\rho$. Note that $(A_\rho + F)/F = T_\rho$ is a subgroup of countable index in C_1/F where $C_1 = \langle A_\rho + Z_i \rangle_* + F$, so there is a subgroup $B_1 \le A$ such that $C_1/F \le B_1/F \in \mathcal{H}$ and $|B_1/C_1| \le \aleph_0$. There is a countable subset $J_1 \subseteq I \setminus I_\rho$ for which $\langle A_\rho \oplus \oplus_{j \in J_1} Z_j \rangle_* + F$ contains B_1. We keep repeating this process, to define an ascending chain of countable subsets J_n of $I \setminus I_\rho$, along with subgroups C_n and B_n of A (for $n < \omega$) such that $C_1 \le B_1 \le C_2 \le B_2 \le \dots$. If we set

$$I_\sigma = I_\rho \cup \cup_{n<\omega} J_n, \quad F_\sigma = \oplus_{j \in I_\sigma} Z_j, \quad \text{and } A_\sigma = \langle F_\sigma \rangle_*,$$

then $\cup_{n<\omega} C_n/F = \cup_{n<\omega} B_n/F$ will be a subgroup $T_\sigma \in \mathcal{H}$, and (a)-(b) will be satisfied for this σ. The factor group A_σ/A_ρ is torsion-free and countable; it contains $(A_\rho + F_\sigma)/A_\rho$ as a free subgroup such that the factor group is isomorphic to

$$A_\sigma/(A_\rho + F_\sigma) = A_\sigma/[A_\sigma \cap (A_\rho + F)] \cong (A_\sigma + F)/(A_\rho + F) \cong T_\sigma/T_\rho$$

which is a countable reduced p-group. Therefore, we can apply the countable case to derive that A_σ/A_ρ is a free group. Hence the chain of the A_σ ($\sigma < \tau$) has free factor groups, thus their union A is a free group. □

Example 7.11. Reduced totally projective p-groups (Sect. 6 in Chapter 11) admit an $H(\aleph_0)$-family as stated in the theorem. However, no uncountable p-group with countable basic subgroups has such a family.

The Summand Intersection Property We say that a group A has the **summand intersection property** if the intersection of two summands in A is likewise a summand of A. If the same holds for infinite intersections as well, then we refer to it as the **strong summand intersection property**. Needless to say, this property is shared by very special groups only. We are looking for free groups with this property.

Proposition 7.12 (Kaplansky [K], Wilson [1]). *All free groups have the summand intersection property. A free group has the strong summand intersection property if and only if it is countable.*

Proof. Let F be a free group, and $F = B_i \oplus C_i$ ($i = 1, 2$) direct decompositions. Then $F/(B_1 \cap B_2)$ is isomorphic to a subgroup of the free group $F/B_1 \oplus F/B_2$, so is itself free. Hence $B_1 \cap B_2$ is a summand of F.

If F is a countable free group, then the same argument with countable summands leads to a countable factor group contained in a product of countable free groups F/B_i. Theorem 8.2 below implies that this factor group is free, so the intersection of countably many summands is a summand.

Finally, suppose $|F| \geq \aleph_1$. Let A be a torsion-free, non-free group of cardinality $\aleph_1$, and $\phi: F \to A$ a homomorphism that is a bijection between a basis $\{b_\sigma \mid \sigma < \kappa\}$ of F and the elements of A. Select homomorphisms $\phi_i: F \to A$ with cyclic images whose kernels C_i are summands of F containing $K = \operatorname{Ker}\phi$. This can be done such that the intersection $\cap_{i \in I} C_i = K$. But K cannot be a summand, since F/K is not free. □

★ **Notes.** In this section, we have collected the most useful results on free groups. They have fascinating features, no wonder that their theory attracted so many researchers. For criteria on the existence of a basis, we refer to Kertész [1], Fuchs [1]. In view of the very useful chain criteria of freeness, basis criteria are hardly used.

The summand intersection property for free groups was observed by Kaplansky [K]. More on this property can be found in Wilson [1], Arnold–Hausen [1], Albrecht–Hausen [1]. Hausen [9] proved that $A^{(I)}$ has the summand intersection property if $\operatorname{End} A$ is a PID. This property was investigated by Kamalov [1] for non-free groups, and by Chekhlov [2] for torsion groups.

Exercises

(1) A countable group is Σ-cyclic if and only if every finite set of its elements is contained in a finitely generated direct summand.

(2) (a) A subset $\{a_i\}_{i\in I}$ of a torsion-free group A is a basis of A if and only if it is a minimal generating system such that, for every finite subset $\{a_1, \dots, a_n\}$, if $a \in A$ depends on $\{a_1, \dots, a_n\}$, then $a \in \langle a_1, \dots, a_n \rangle$.
(b) The same with "minimal generating system" replaced by "maximal independent subset."
(3) In any presentation of $\mathbb{Z}^{\aleph_0}$, there are continuously many generators and continuously many relations.
(4) (Danchev) In a p-group A, the $p^{\omega}A$-high subgroups are Σ-cyclic if and only if $A[p]$ is the union of an ascending chain T_n $(n < \omega)$ such that the finite heights in T_n are bounded.
(5) The summand intersection property is inherited by summands.
(6) (Wilson) A torsion group has the summand intersection property if and only if each of its p-components is either cocyclic or elementary.
(7) (Wilson, Hausen) A has the summand intersection property if and only if for every direct decomposition $A = B \oplus C$, the kernel of any map $B \to C$ is a summand of A.

8 Almost Free Groups

Almost free groups are those (necessarily) torsion-free groups in which all subgroups of smaller sizes are free. More precisely, for an infinite cardinal κ, we say that a group A is κ-**free** if every subgroup of A whose rank is $< \kappa$ is free [AG]. The problem of finding the cardinals κ for which there exist κ-free groups that fail to be κ^+-free was raised in [IAG]. As it turns out, it is an intricate problem, requiring sophisticated machinery from set theory. It has been studied extensively, and a significant amount of information has already been gained, but still much remains to be done. Here we aim simply at giving a taste of the subject. The objective is to understand how close almost free groups are to being free.

κ-Free Groups Since purification does not increase rank, it is clear that if A is κ-free, then every subgroup of rank $< \kappa$ is contained in a pure free subgroup of the same rank. Thus the collection $\mathfrak{C}$ of pure free subgroups of rank $< \kappa$ is witness for κ-freeness. In view of Theorem 7.7, $\mathfrak{C}$ is closed under taking unions of countable chains.

Example 8.1. In this new terminology, Pontryagin's theorem 7.1 can be rephrased by saying that a countable $\aleph_0$-free group is free.

(A) *If $\kappa < \lambda$ are infinite cardinals, then λ-free implies κ-free.* In particular, free groups are trivially κ-free for every cardinal κ.
(B) *Subgroups and direct sums of κ-free groups are κ-free.*
(C) *Extension of a κ-free group by a κ-free group is κ-free.* More generally, we have:

(D) *Let* $0 = A_0 < \cdots < A_\sigma < \cdots < A_\tau = A$ *be a smooth chain of groups such that all the factor groups* $A_{\sigma+1}/A_\sigma$ *are* κ*-free. Then* A *is also* κ*-free.* In fact, let X be a pure subgroup of A with $|X| < \kappa$. Then in the smooth chain $X \cap A_\sigma$ ($\sigma < \tau$) each factor group $(X \cap A_{\sigma+1})/(X \cap A_\sigma)$ is torsion-free of cardinality $< \kappa$, and therefore it is isomorphic to the free subgroup $(A_\sigma + (X \cap A_{\sigma+1}))/A_\sigma \leq A_{\sigma+1}/A_\sigma$. An appeal to Lemma 7.4 completes the proof.

The Baer–Specker Group The next theorem is concerned with a prototype for $\aleph_1$-freeness; actually, the group is of major interest.

Theorem 8.2 (Baer [6], Specker [1]). *The direct product of infinitely many infinite cyclic groups is* $\aleph_1$*-free, but not free.*

Proof. Write $A = \prod_{i \in I} \langle a_i \rangle$, where I is an infinite set, and $\langle a_i \rangle \cong \mathbb{Z}$ for each i. The first step in the proof is to show that every finite subset $\{x_1, \ldots, x_m\} \subset A$ is contained in a finitely generated direct summand of A whose complement is a direct product of infinite cyclic groups.

We induct on m. If $m = 1$ and $x_1 \neq 0$, then $x_1 = (\ldots, n_i a_i, \ldots)$ with $n_i \in \mathbb{Z}$. If there is an index $j \in I$ such that $|n_j| = 1$, then the jth component $\langle a_j \rangle$ in the direct product can be replaced by $\langle x_1 \rangle$, i.e. $A = \langle x_1 \rangle \oplus A_j$, where A_j is the set of elements with vanishing jth coordinate, so it is also a product of infinite cyclic groups. If the minimum n of the $|n_i|$ with $n_i \neq 0$ is greater than 1, then setting $n_i = q_i n + r_i$ with $q_i, r_i \in \mathbb{Z}, 0 \leq r_i < n$, define $y_1 = (\ldots, q_i a_i, \ldots), y_2 = (\ldots, r_i a_i, \ldots) \in A$ so that $x_1 = n y_1 + y_2$. There must be an index $j \in I$ with $|q_j| = 1$ and $r_j = 0$, thus $A = \langle y_1 \rangle \oplus A_j$, where $y_2 \in A_j$ with coefficients $0 \leq r_i < n$. By induction on n, A_j has a finitely generated summand B' containing y_2, and so $\langle y_1 \rangle \oplus B'$ is a finitely generated summand of A containing x_1 such that it has a direct product of infinite cyclic groups as a complement.

Assume that $m > 1$, and $A = B \oplus C$ where B is finitely generated containing $\{x_1, \ldots, x_{m-1}\}$, and C is a direct product of copies of $\mathbb{Z}$. Setting $x_m = b + c$ ($b \in B$, $c \in C$) and embedding c in a finitely generated summand C' of C, we obtain a finitely generated summand $B \oplus C'$ of A, containing $\{x_1, \ldots, x_m\}$, again with a complement that is a direct product of infinite cyclic groups.

The next step is to show that A is $\aleph_1$-free. Let G be a countable subgroup of A. A maximal independent set of a finite rank subgroup G' of G is contained in a finitely generated summand B of A, so by torsion-freeness, $G' \leq B$. Thus G' is free, and Theorem 7.1 implies that G is free.

It remains to prove that A itself is not free. We exhibit a non-free subgroup of A. Let p be any prime, and H the subgroup of $A' = \prod_{i<\omega} \langle a_i \rangle$ (a summand of A) that consists of all vectors $b = (n_0 a_0, n_1 a_1, \ldots, n_i a_i, \ldots)$ such that, for every integer $k > 0$, almost all coefficients n_i are divisible by p^k. Manifestly, H contains the direct sum $S = \oplus_{i<\omega} \langle a_i \rangle$, and has cardinality $2^{\aleph_0}$. Since each coset of H mod pH can be represented by some element of S, H/pH cannot be uncountable. If H were free, we would have $|H/pH| = |H|$, so H cannot be free. □

The countable direct product of infinite cyclic groups, i.e. the group $\mathbb{Z}^{\aleph_0}$, is often called the **Baer-Specker group**.

An immediate consequence is the following result that shows that in countable groups free summands can be collected into a single summand.

Corollary 8.3 (K. Stein). *A countable torsion-free group A can be decomposed as $A = F \oplus N$ where F is a free group, and N has no free factor group. N is uniquely determined by A.*

Proof. Define N as the intersection of the kernels of all homomorphisms $\eta: A \to \mathbb{Z}$. Then A/N is isomorphic to a countable subgroup of the direct product $\prod_\eta \mathbb{Z}$, so it is free in view of Theorem 8.2. N is then a summand of A, and we have $A = F \oplus N$ with F free. From the definition of N it is evident that N cannot have a non-trivial map into $\mathbb{Z}$. □

The next two examples show that there exist very large $\aleph_1$-free groups A such that $\mathrm{Hom}(A, \mathbb{Z}) = 0$, and it may also happen that an $\aleph_1$-free group is isomorphic to the countable direct sum and to the countable direct product of itself.

Example 8.4 (G. Reid [1]). Let $\kappa \geq \aleph_1$ be a non-measurable cardinal, and N the subgroup of $\mathbb{Z}^\kappa$ consisting of vectors with countable support. Then $A = \mathbb{Z}^\kappa/N$ is $\aleph_1$-free. To see this, let $b_n + N$ $(n < \omega)$ be a list of elements in a countable subgroup F of A. Clearly, each b_n has uncountable support, and each sum $b_i + b_j$ is equal to some b_k modulo a countable index set. Thus if we change the representatives b_n by dropping all the indices in these countably many index sets, then the new representatives form a subgroup $F' \cong F$. By Theorem 8.2, F' is free, so A is $\aleph_1$-free.

A homomorphism $\phi : A \to \mathbb{Z}$ may be viewed as a map $\phi^* : \mathbb{Z}^\kappa \to \mathbb{Z}$ such that $\phi^*(N) = 0$. By Theorem 2.8 in Chapter 13, $\phi^* = 0$, which means $\mathrm{Hom}(A, \mathbb{Z}) = 0$.

Example 8.5. There exists an $\aleph_1$-free group which is not free and isomorphic both to the direct sum of countably many copies of itself, and to the direct product of countable many copies of itself. See Proposition 4.9 in Chapter 13.

Strongly κ-Free The study of almost free groups brings a stronger version of κ-freeness into the picture. Let κ be a regular cardinal. A group A is said to be **strongly κ-free** if every subgroup of cardinality $< \kappa$ is contained in a free subgroup C of cardinality $< \kappa$ such that A/C is κ-free. Evidently, free groups are strongly κ-free for any cardinal κ.

It is not obvious that if $\kappa < \lambda$ are infinite cardinals, then strongly λ-free implies strongly κ-free, but it is true. In fact, if a subgroup B of cardinality $< \kappa$ is contained in a free subgroup C of cardinality $< \lambda$ with λ-free A/C, then B is contained in a summand C' of C of cardinality $< \kappa$. Because of (C), A/C' is κ-free, being an extension of the free group C/C' by the κ-free group A/C.

The fine nuance between strongly κ-free and just plainly κ-free groups can be better understood if we compare filtrations.

Lemma 8.6 (Eklof–Mekler [EM]). *Let A be a group of cardinality κ, where κ is an uncountable regular cardinal.*

(a) *A is κ-free exactly if it has a filtration $\{A_\sigma \mid \sigma < \kappa\}$ with free subgroups A_σ of cardinality $< \kappa$.*
(b) *A is strongly κ-free if and only if it admits a filtration $\{A_\sigma \mid \sigma < \kappa\}$ with free subgroups A_σ of cardinality $< \kappa$ such that, for all $\sigma < \tau < \kappa$, the factor groups $A_{\tau+1}/A_{\sigma+1}$ are free.*

Proof. (a) Since every subgroup of cardinality $< \kappa$ is contained in some member of a κ-filtration, the stated condition evidently implies the κ-freeness of A. Conversely, if A is κ-free, then the subgroups in any κ-filtration of A are free.

(b) For sufficiency, it is enough to observe that the stated condition is equivalent to that A is κ-free, and for every $A_{\sigma+1}$, the factor group $A/A_{\sigma+1}$ is κ-free. To prove the converse, assume A is strongly κ-free, and $\{a_\sigma \mid \sigma < \kappa\}$ is a well-ordered list of elements of A. We construct a filtration $\{A_\sigma \mid \sigma < \kappa\}$ of A as desired, with the additional property that $a_\rho \in A_\sigma$ for all $\rho < \sigma < \kappa$. Suppose that, for some $\sigma < \kappa$, we have a chain $\{A_\rho \mid \rho \leq \sigma\}$ satisfying the requisite properties. Choose for $A_{\sigma+1}$ a subgroup of cardinality $< \kappa$ that contains both A_σ and a_σ such that $A/A_{\sigma+1}$ is κ-free. Then the factor groups $A_{\tau+1}/A_{\sigma+1}$ are free for all $\tau > \sigma$, and by the κ-freeness of A, $A_{\sigma+1}$ is free. □

Observe that in (b) we have not said anything about the freeness of the factor groups A_τ/A_σ at limit ordinals σ.

Lemma 8.7. *The Baer-Specker group $P = \mathbb{Z}^{\aleph_0}$ is not strongly $\aleph_1$-free.*

Proof. We prove that the direct sum $S = \mathbb{Z}^{(\aleph_0)}$ is not contained in any countable subgroup G with $\aleph_1$-free P/G. Anticipating theorems that we will prove later on, the proof is quick. Corollary 1.12 in Chapter 6 asserts that P/S is algebraically compact, thus for every intermediate pure subgroup $S \leq G < P$, the factor group P/G is torsion-free and algebraically compact (see Lemma 8.1 in Chapter 9). Therefore, it contains a subgroup isomorphic to either $\mathbb{Q}$ or J_p for some prime p, and consequently, P/G can never be $\aleph_1$-free. □

Uncountable Chains If we wish to consider chains of free groups of cofinality exceeding ω, then we are confronted with a more complicated situation. In order to guarantee that the union of long chains of free groups will again be free, it is necessary to impose restrictions on the factors in the chain. A typical result is as follows.

Theorem 8.8 (Fuchs–Rangaswamy [4]). *Suppose κ is an uncountable regular cardinal, and $0 = F_0 < F_1 < \cdots < F_\sigma < \ldots$ $(\sigma < \kappa)$ is a smooth chain of groups such that, for every $\sigma < \kappa$,*

(a) *F_σ is free of cardinality $\leq \kappa$, and*
(b) *F_σ is a pure subgroup of $F_{\sigma+1}$.*

(i) *The union F of the chain is free provided the set*

$$S = \{\sigma < \kappa \mid \exists\, \rho > \sigma \text{ such that } F_\rho/F_\sigma \text{ is not } \kappa\text{-free}\}$$

is not stationary in κ.

(ii) *If all $|F_\sigma| < \kappa$, and S is stationary in κ, then F is κ-free, but not free.*

Proof. The proof is similar to the one in Lemma 7.6, but the construction of the $G(\kappa)$-families becomes complicated at limit ordinals. The details are too long to be reproduced here. □

Large Almost Free Non-free Groups We next prove that in the constructible universe, there exist large κ-free groups that are not free.

Theorem 8.9 (Gregory [1]). *Assume* V = L. *For every uncountable regular cardinal κ that is not weakly compact, there exists a κ-free group of cardinality κ which is not free.*

Proof. Let E be a stationary subset of κ that consists of limit ordinals cofinal with ω; see Lemma 4.5 in Chapter 1. Suppose for a moment that we have succeeded in constructing a group F as the union of a smooth chain of subgroups F_σ $(\sigma < \kappa)$ satisfying the following conditions for all $\sigma < \rho < \kappa$:

(i) F_σ is free of cardinality $|\sigma| \cdot \aleph_0$;
(ii) if $\sigma \in E$, then the quotient $F_{\sigma+1}/F_\sigma$ is not free;
(iii) if $\sigma \notin E$, then F_ρ/F_σ is free of cardinality $|\rho| \cdot \aleph_0$.

Then F is of cardinality κ and κ-free. Working toward contradiction, suppose F is free. Then there exists a cub $C \subset \kappa$ such that F_ρ/F_σ is free for each pair $\sigma < \rho$ in C. For such a pair of indices, the exact sequence $0 \to F_{\sigma+1}/F_\sigma \to F_\rho/F_\sigma \to F_\rho/F_{\sigma+1} \to 0$ must split because of (iii). This means that $F_{\sigma+1}/F_\sigma$ must be free for $\sigma \in E \cap C$, contrary to (ii).

It remains to construct a smooth chain of groups F_σ $(\sigma < \kappa)$ with the listed properties. Starting with F_0 free of rank $\aleph_0$, we proceed to define F_σ $(\sigma > 0)$ *via* transfinite induction as follows. Assume that σ is an ordinal $< \kappa$ such that the groups F_ρ $(\rho < \sigma)$ have already been constructed, and they satisfy conditions (i)–(iii) up to σ. To define F_σ we distinguish three cases.

Case 1. If σ is a limit ordinal, then we have no choice: $F_\sigma = \bigcup_{\rho<\sigma} F_\rho$. Since $E \cap \sigma$ is not stationary in σ Lemma 4.5 in Chapter 1, (iii) allows us to apply Theorem 7.5 to claim that F_σ is free. Hence conditions (i)–(iii) hold for all ordinals $\leq \sigma$.

Case 2. If $\sigma = \rho + 1$ and $\rho \notin E$, then we simply let $F_\sigma = F_\rho \oplus X$ where X is a countable free group.

Case 3. The critical case is when $\sigma = \rho + 1$ and $\rho \in E$. In view of the choice of E, we have cf $\rho = \omega$, so ρ is the supremum of an increasing sequence of non-limit ordinals $\rho_0 < \rho_1 < \cdots < \rho_n < \ldots$ $(n < \omega)$. Consider the chain $F_{\rho_0} < \cdots < F_{\rho_n} < \ldots$ of free groups whose union is the free group F_ρ. We are in the situation of Example 7.3, and so we can define $F_{\sigma+1}$ such that the F_{ρ_n} $(n < \omega)$ are, but F_ρ is not a summand of $F_{\sigma+1}$. With this choice, (i)–(iii) will be satisfied by all ordinals $\leq \sigma$. □

For cardinals $\aleph_n$ $(n \geq 1)$, the existence of a stationary E of property Lemma 4.5 in Chapter 1 can be established without the hypothesis V = L, therefore we can state:

Corollary 8.10 (Eklof [2], Griffith [7], Hill [13]). *For every integer $n > 1$, there is a non-free $\aleph_n$-free group of cardinality $\aleph_n$.* □

The Σ-Cyclic Case Several results proved above carry over to torsion and mixed groups provided we can interpret freeness in an appropriate way. This can be done by introducing κ**-cyclic** groups meaning that every subgroup of cardinality $< \kappa$ is Σ-cyclic.

A proof similar to Theorem 8.9 applies to verify:

Corollary 8.11 (Eklof [2]). *If κ is an uncountable regular cardinal that is not weakly compact, then there exist κ-cyclic torsion groups of cardinality κ that are not Σ-cyclic.*

For every $n > 0$, there are $\aleph_n$-cyclic torsion groups of cardinality $\aleph_n$ which are not Σ-cyclic.

Proof. Obvious modification to Theorem 8.9 is that 'free' should be replaced by 'Σ-cyclic,' 'κ-free' by 'κ-cyclic,' and purity should be assumed throughout. In place of Example 7.3, a modified example should be referred to where a pure-projective resolution of $\mathbb{Z}(p^\infty)$ is used. □

Mittag-Leffler Groups Most recently, a lot of attention has been devoted to Mittag-Leffler modules. In the group case, a satisfactory characterization is available. In the definition, we need tensor products: M is a **Mittag-Leffler group** if for every collection $\{A_i\}_{i\in I}$ of groups, the natural map

$$\phi : M \otimes \prod_{i\in I} A_i \to \prod_{i\in I} (M \otimes A_i)$$

given by $\phi(x \otimes (\ldots, a_i, \ldots)) \mapsto (\ldots, x \otimes a_i, \ldots)$ is monic ($x \in M, a_i \in A_i$).

Example 8.12. (a) Cyclic groups are Mittag-Leffler. This is trivial for $\mathbb{Z}$, and follows for $M = \mathbb{Z}(n)$ from the fact that both the domain and the image of ϕ are then isomorphic to $\prod_{i\in I} (A_i/nA_i)$.

(b) The Prüfer group $H_{\omega+1}$ (of length $\omega + 1$) is not Mittag-Leffler. The natural map $H_{\omega+1} \otimes \prod_{n<\omega} \mathbb{Z}(p^n) \to \prod_{n<\omega} H_{\omega+1} \otimes \mathbb{Z}(p^n)$ is not monic. (See the proof of Theorem 8.14.)

Lemma 8.13 (M. Raynaud, L. Gruson). *The class of Mittag-Leffler groups is closed under taking pure subgroups, pure extensions and arbitrary direct products.*

Proof. Starting with a pure-exact sequence $0 \to M' \to M \to M'' \to 0$, we form the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' \otimes \prod A_i & \longrightarrow & M \otimes \prod A_i & \longrightarrow & M'' \otimes \prod A_i & \longrightarrow & 0 \\
 & & \phi' \downarrow & & \phi \downarrow & & \downarrow \phi'' & & \\
0 & \longrightarrow & \prod (M' \otimes A_i) & \longrightarrow & \prod (M \otimes A_i) & \longrightarrow & \prod (M'' \otimes A_i) & \longrightarrow & 0
\end{array}$$

with pure-exact rows (see Corollary 3.7 in Chapter 5). Evidently, if ϕ is monic, then so is ϕ'. If both ϕ' and ϕ'' are monic, then Lemma 2.6 in Chapter 1 (or a simple diagram-chasing) shows that ϕ has to be monic as well. Finally, for direct products, the claim will be a simple consequence of Theorem 8.14 and Exercise 1. □

It is not difficult to characterize Mittag-Leffler groups.

Theorem 8.14 (Raynaud–Gruson). *A group is Mittag-Leffler if and only if it is* $\aleph_1$*-cyclic.*

Proof. We start the proof by showing that M is Mittag-Leffler if and only if each of its countable pure subgroups is Mittag-Leffler. One direction the claim follows from Lemma 8.13.

For the converse, assume ϕ maps $\sum_{j=1}^{n}(x_j \otimes b_j)$ $(x_j \in M,\ b_j \in \prod A_i)$ to 0. Then by Lemma 1.12 in Chapter 8 and by the remark after it, the same sum vanishes in $\phi(M' \otimes \prod A_i)$ for a countable pure subgroup $M' \leq M$ containing the x_j's. Hence M cannot be Mittag-Leffler if its countable pure subgroups are not, but it is if its countable pure subgroups are Mittag-Leffler.

It remains to prove that countable Mittag-Leffler groups are Σ-cyclic. Suppose N is a countable p-group which has elements $\neq 0$ of infinite p-heights. Clearly, all non-zero elements of $\prod_{n\in\mathbb{N}}(N \otimes \mathbb{Z}(p^n))$ have finite p-heights. However, $\prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$ has summands $\cong J_p$, so $N \otimes \prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$ has summands $\cong N \otimes J_p \cong N$ with elements of infinite heights. Thus N is not Mittag-Leffler. A countable Mittag-Leffler group must therefore have separable p-components, so its torsion subgroup is Σ-cyclic.

Next, let M be of finite torsion-free rank $n > 0$ such that M/tM is not finitely generated. Then M contains a subgroup N such that $N/tM \cong \mathbb{Z}^n$ and M/N is an infinite torsion group. First assume M/N is reduced. Then it is Σ-cyclic of the form $\oplus_{i<\omega} \mathbb{Z}(p_i^{k_i})$ with $k_i \in \mathbb{N}$, where the p_i are not necessarily different primes, but each prime may occur at most n times. We tensor the exact sequence

$$0 \to N \cong tM \oplus \mathbb{Z}^n \to M \to \oplus_{i<\omega} \mathbb{Z}(p_i^{k_i}) \to 0$$

with $A = \prod_j \mathbb{Z}(p_j)$ where p_j varies over the (infinitely many) different primes in the set of the p_i. In the long exact sequence connecting Tor and $\otimes$, the map $\mathrm{Tor}(N, A) \to \mathrm{Tor}(M, A)$ is an isomorphism as N, M share the same torsion subgroup, thus the induced sequence

$$0 \to \mathrm{Tor}(\oplus_{i<\omega} \mathbb{Z}(p_i^{k_i}), A) \xrightarrow{\delta} (tM \oplus \mathbb{Z}^n) \otimes A \to M \otimes A \to \dots$$

is exact. We calculate: $\mathrm{Tor}(\oplus_{i<\omega} \mathbb{Z}(p_i^{k_i}), A) \cong \oplus_i \mathbb{Z}(p_i)$, and note that this Tor is sent by the connecting map δ into $\mathbb{Z}^n \otimes A$. Therefore, $M \otimes A$ must contain an image of the divisible group $A/\oplus_j \mathbb{Z}(p_j)$. But $\prod_j(M \otimes \mathbb{Z}(p_j))$ is reduced, thus $M \otimes \prod_j \mathbb{Z}(p_j) \to \prod_j(M \otimes \mathbb{Z}(p_j))$ is not monic. Such an M cannot be Mittag-Leffler.

A similar proof applies to show that M cannot be Mittag-Leffler if M/tM contains a rank 1 pure subgroup that is p-divisible for some prime p (in this case, we tensor with $A = \prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$). The conclusion is that if M is Mittag-Leffler, then the finite rank pure subgroups of M/tM are free, i.e. M/tM is free if it is countable by Theorem 7.1. Therefore, a countable Mittag-Leffler group is Σ-cyclic. □

★ **Notes.** The Baer–Specker group has been investigated from various points of view, it is an excellent source of ideas. We point out that, among others, Blass–Irwin have several interesting papers on this group and its subgroups. In their paper [2], several interesting subgroups are dealt with. In the other paper [1], a core class for $\aleph_1$-freeness is discussed: a well-defined class of non-free $\aleph_1$-free groups of cardinality $\aleph_1$ such that every non-free $\aleph_1$-free group of cardinality $\aleph_1$ contains a subgroup from the class. Another interesting result is the existence of indecomposable $\aleph_1$-free groups by Palyutin [1] (under CH) which was generalized to rigid $\aleph_1$-free groups of cardinality $\aleph_1$ by Göbel–Shelah [2].

Eda [4] shows that a group is $\aleph_1$-free if and only if it is contained in $\mathbb{Z}^{(\mathbf{B})}$ for some Boolean lattice **B**. To illustrate the importance of $\aleph_1$-freeness, we also mention several topological connections. L. Pontryagin proved that a connected compact abelian group G is locally connected exactly if its character group Char G is $\aleph_1$-free, and J. Dixmier showed that it is arcwise connected if and only if $\mathrm{Ext}(\mathrm{Char}\, G, \mathbb{Z}) = 0$ (which is stronger than $\aleph_1$-freeness). We also point out that for a compact connected group G, the nth homotopy group $\pi_n(G) = 0$ for all $n > 1$, while $\pi_1(G) = \mathrm{Hom}(\mathrm{Char}\, G, \mathbb{Z})$ is always $\aleph_1$-free.

That $\aleph_n$-free groups need not be $\aleph_{n+1}$-free was proved by Hill, Griffith, and then by Eklof. Mekler–Shelah [2] study regular cardinals κ for which κ-free implies strongly κ-free or κ^+-free. Gregory [1] proved in L the most interesting Theorem 8.9. Assuming V = L, Rychkov [3] proves that for each uncountable regular, not weakly compact cardinal κ, there exist p-groups A of final rank κ such that every subgroup C of cardinality $< \kappa$ is contained in a Σ-cyclic direct summand of cardinality $|C|\aleph_0$, but A itself is not Σ-cyclic, not even the direct sum of two subgroups of final ranks κ.

Mittag-Leffler modules were introduced by M. Raynaud and L. Gruson [Invent. Math. **13**, 1–89 (1971)].

Exercises

(1) (a) A direct product of $\aleph_1$-free groups is $\aleph_1$-free.
  (b) The same may fail for larger cardinals.
  (c) Derive from Theorem 8.14 that a direct product of Mittag-Leffler groups is Mittag-Leffler.

(2) In a free group F, a subgroup G of cardinality $< \kappa$ for which F/G is κ-free is a summand.

(3) An extension of a free group by a strongly κ-free group is strongly κ-free.

(4) Let A be a direct product of infinite cyclic groups, and B the subgroup of A whose elements are the vectors with countable support. B is $\aleph_1$-free, but not free.

(5) In the Baer–Specker group $A = \prod_{k\in\mathbb{N}} \langle e_k\rangle$, let D denote the $\mathbb{Z}$-adic closure of $S = \oplus_k \langle e_k\rangle$. Prove that D consists of all vectors $x = \sum m_k e_k$ such that, for every $n \in \mathbb{N}$, n divides almost all m_k.

(6) Is it possible to define Mittag-Leffler groups by using only countable index sets I?

(7) If M is Mittag-Leffler, then so is M/N for every finitely generated subgroup N of M.

9 Shelah's Singular Compactness Theorem

The question as to when κ-free implies κ^+-free turns out to be extremely complicated for regular cardinals κ (see Magidor–Shelah [1]). As far as singular cardinals are concerned, the same question can be fully answered; this is shown by the next theorem, a most powerful result.

The following lemma will be required in the proof of Theorem 9.2.

Lemma 9.1 (Eklof–Mekler [EM]). *If κ is a regular cardinal, then a κ^+-free group is strongly κ-free.*

Proof. By way of contradiction, assume that A is κ^+-free, but not strongly κ-free. This means that A contains a subgroup B of cardinality $< \kappa$ which is not contained in any subgroup of A of cardinality $< \kappa$ with κ-free factor group. Set $C_0 = B$, and let C_1 be a pure subgroup of A of cardinality $< \kappa$ that contains C_0 such that C_1/C_0 is not free. Repeat this with C_1 in the role of C_0 to obtain C_2, and continue this process transfinitely up to κ steps, taking unions at limit ordinals. We get a chain $C_0 < C_1 < \cdots < C_\sigma < \ldots$ $(\sigma < \kappa)$ where none of the factor groups $C_{\sigma+1}/C_\sigma$ is free. The union $C = \bigcup_{\sigma<\kappa} C_\sigma$ has cardinality κ, and is not free because of Theorem 7.5. This contradicts the κ^+-freeness of A. $\square$

Theorem 9.2 (Shelah [1]). *For a singular cardinal λ, a λ-free group of cardinality λ is free.*

Proof. Suppose A is λ-free of cardinality λ. Let $\{\kappa_\nu \mid \nu < \mathrm{cf}(\lambda)\}$ be a smooth increasing sequence of cardinals $> \mathrm{cf}(\lambda)$ with λ as supremum, and $\{A_\nu \mid \nu < \mathrm{cf}(\lambda)\}$ a smooth chain of pure subgroups of A with union A such that $|A_\nu| = \kappa_\nu$. Set

$$\mathcal{P}_\nu = \{B < A \mid |B| \leq \kappa_\nu \text{ and } A/B \text{ is } \kappa_\nu^+\text{–free}\}.$$

Since A is λ-free for all $\kappa < \lambda$, by Lemma 9.1 it is strongly λ-free for all $\kappa < \lambda$ (including limit ordinals $< \lambda$); thus, every subgroup of A of cardinality $\leq \kappa_\nu$ is contained in a member of $\mathcal{P}_\nu$. For all $\nu < \mathrm{cf}(\lambda)$, define subgroups $B_{\nu k}$ $(k < \omega)$ and subsets $X_{\nu k}$ $(k < \omega)$ such that

(i) $B_{\nu k} \in \mathcal{P}_\nu$ $(\nu < \mathrm{cf}(\lambda),\ k < \omega)$;
(ii) $X_{\nu k}$ is a basis of $B_{\nu k}$ $(\nu < \mathrm{cf}(\lambda),\ k < \omega)$;
(iii) $A_\nu < B_{\nu 0} < B_{\nu 1} < \cdots < B_{\nu k} < \ldots$ and $X_{\nu 0} \subset X_{\nu 1} \subset \cdots \subset X_{\nu k} \subset \ldots$ for each $\nu < \mathrm{cf}(\lambda)$;
(iv) $B_{\nu,k-1} \leq \langle B_{\nu k} \cap X_{\nu+1,k-1}\rangle$ for each $\nu < \mathrm{cf}(\lambda)$, $0 < k < \omega$;
(v) for a limit ordinal $\mu < \mathrm{cf}(\lambda)$, $X_{\mu k}$ is the union of a chain of subsets $Y_{\mu k}(\nu)$ where $|Y_{\mu k}(\nu)| = \kappa_\nu$ $(\nu < \mu)$, and $Y_{\mu k}(\nu) \subset B_{\nu k+1}$ for all $\nu < \mu$.

The construction is by induction on κ. In the first step, we define the subgroups $B_{\nu 0}$ $(\nu < \mathrm{cf}(\lambda))$ recursively on ν. Let B_{00} be any member of $\mathcal{P}_0$ that contains A_0. If, for some $\mu < \mathrm{cf}(\lambda)$, the $B_{\nu 0}$ have been defined for all $\nu < \mu$, then pick $B_{\mu 0} \in \mathcal{P}_\mu$ such that it contains $A_\nu + \sum_{\nu<\mu} B_{\nu 0}$; this can be done in view of the cardinality hypotheses. We are led to a well-ordered ascending chain $B_{00} < B_{10} < \cdots < B_{\nu 0} < \ldots$ (that need not be smooth) where $B_{\nu 0}$ has cardinality κ_ν. Choose any basis $X_{\nu 0}$ for $B_{\nu 0}$. For limit ordinals ν, represent $X_{\nu 0}$ as the union of a chain of subsets $Y_{\nu 0}(\sigma)$ where $Y_{\nu 0}(\sigma)$ has cardinality κ_σ $(\sigma < \nu)$.

The next step is to define $B_{\mu k}$ along with $X_{\mu k}$ after all $B_{\nu j}, X_{\nu j}$ (and $Y_{\nu j}(\sigma)$ only for limit ordinals ν) have been defined for all $j < k$ and for all $\nu < \mathrm{cf}(\lambda)$, and $B_{\nu k}, X_{\nu k}$, and $Y_{\nu k}(\sigma)$ for all $\nu < \mu$. Choose $B_{\mu k} \in \mathcal{P}_\mu$ so as to satisfy (iv), and to contain all of the following: (a) $B_{\mu,k-1}$; (b) $B_{\nu k}$ for all $\nu < \mu$; and (c) the sets $Y_{\nu,k-1}(\mu)$ for limit ordinals $\nu > \mu$. As $B_{\mu,k-1}$ is a summand of $B_{\mu k}$, we can select a basis $X_{\mu k}$ of $B_{\mu k}$ that contains $X_{\mu,k-1}$. If μ happens to be a limit ordinal, we choose the $Y_{\mu k}(\nu)$ $(\nu < \mu)$ so as to satisfy (v). An easy cardinality argument convinces us that this can be done in view of the hypothesis that $\kappa_\mu > \mathrm{cf}(\lambda)$. It is obvious that conditions (i)–(v) are satisfied.

We claim that the subgroups $B_\nu = \bigcup_{k<\omega} B_{\nu k}$ $(\nu < \mathrm{cf}(\lambda))$ form a smooth chain $B_0 < B_1 < \cdots < B_\nu < \ldots$ $(\nu < \mathrm{cf}(\lambda))$ with free factor groups $B_{\nu+1}/B_\nu$. Observe that if μ is a limit ordinal, then in view of

$$B_\mu = \bigcup_{k<\omega} B_{\mu k} = \bigcup_{k<\omega} \langle X_{\mu k}\rangle = \bigcup_{k<\omega}\bigcup_{\nu<\mu} \langle Y_{\mu k}(\nu)\rangle \leq \bigcup_{k<\omega}\bigcup_{\nu<\mu} B_{\mu k+1} = \bigcup_{\nu<\mu} B_\nu,$$

the chain of the B_ν is continuous. Since (iv) implies that B_ν is generated by $B_\nu \cap X_{\nu+1}$ (where $X_\nu = \bigcup_{k<\omega} X_{\nu k}$), $B_{\nu+1}/B_\nu$ is indeed free. By Theorem 7.5, the group $A = \bigcup_\nu B_\nu$ is free. □

In Chapter 14, a more general form of the Singular Compactness Theorem will be needed (for Butler groups); we state it here for groups without proof. This axiomatic form is due to Eklof–Mekler [EM], generalizing W. Hodges' version [Algebra Universalis **12**, 205–220 (1981)].

Assume $\mathcal{F}$ is a class of groups such that $0 \in \mathcal{F}$, and for each $G \in \mathcal{F}$, there is given a family $\mathcal{B}(G)$ of sets of subgroups of G. We say that G is **'free'** if $G \in \mathcal{F}$ and $\mathfrak{B}$ is a **'basis'** of G if $\mathfrak{B} \in \mathcal{B}(G)$. The subgroups $B \in \mathfrak{B}$ are called 'free' factors of G.

For a fixed infinite cardinal μ, the following properties (i)-(v) are required for every 'free' group G, and for every 'basis' $\mathfrak{B}$ of G.

- (i) $\mathfrak{B}$ is closed under unions of chains.
- (ii) If $B \in \mathfrak{B}$ and $g \in G$, then there is a $C \in \mathfrak{B}$ that contains both B and g, and is such that $|C| \leq |B| + \mu$.
- (iii) Every $B \in \mathfrak{B}$ is 'free' (i.e., 'free' factors are 'free'); and moreover, the set $\{C \in \mathfrak{B} \mid C \leq B\} = \mathfrak{B} \restriction B$ is a 'basis' for B.
- (iv) If B is a 'free' factor of G, then for every 'basis' $\mathfrak{B}'$ of B, there exists a 'basis' $\mathfrak{B}$ of G such that $\mathfrak{B}' = \mathfrak{B} \restriction B$.
- (v) Suppose B_σ $(\sigma < \kappa)$ is a smooth chain of 'free' subgroups of G with 'bases' $\mathfrak{B}_\sigma$ satisfying $\mathfrak{B}_\rho \restriction B_\sigma = \mathfrak{B}_\sigma$ for all $\sigma < \rho < \kappa$ (in particular, $B_\sigma \in \mathfrak{B}_\rho$). Then the union $B = \bigcup_{\sigma<\kappa} B_\sigma$ is a 'free' subgroup of G such that $\bigcup_{\sigma<\kappa} \mathfrak{B}_\sigma$ is a 'basis' of B.

Theorem 9.3. *Suppose that the class $\mathcal{F}$ of groups satisfies conditions* (i)-(v) *for cardinal* μ, *and the cardinality* λ *of the group* $G \in \mathcal{F}$ *is a singular cardinal* $> \mu$. *G is 'free' if, for every cardinal* $\kappa < \lambda$, *there is a family* $\mathcal{C}_\kappa$ *of subgroups of G of cardinality* κ *satisfying the following conditions:*

(a) $\mathcal{C}_\kappa$ *is a subclass of* $\mathcal{F}$;
(b) $\mathcal{C}_\kappa$ *is closed under unions of chains of lengths* $\leq \kappa$;
(c) *every subset of G of cardinality* $\leq \kappa$ *is contained in a subgroup that belongs to* $\mathcal{C}_\kappa$. □

★ **Notes.** Hill [13] showed that $\aleph_\omega$-free groups of cardinality $\aleph_\omega$ are free, defeating the conjecture that κ-free never implies κ^+-free. In a subsequent paper, he proved the same for $\aleph_{\omega_1}$-free groups. Based on these results, Shelah conjectured and proved the general theorem on singular cardinals. (The term 'compact' is designated in the sense used in logic, not as in topology: properties of small substructures imply the same for the entire structure.)

Various generalizations of the compactness theorem are available in the literature which we do not wish to review here. Let us point out that Hodges [loc.cit.] published an interesting proof of the singular compactness theorem, based on Shelah's ideas. The κ-Shelah game on a group A (for a regular cardinal κ) is introduced; it is played by two players. The players take turns to choose subgroups of A of cardinalities $< \kappa$ to build an increasing chain $\{B_n\}_{n<\omega}$ of subgroups. The players know what subgroups have been chosen at previous steps. B_n is chosen by player I if n is even and by player II if n is odd. Player II wins if for every odd integer n, B_n is a free summand of B_{n+2}, otherwise player I is the winner. The κ-Shelah game is determinate in the sense that one of the players has a winning strategy. It is then shown that player I has no winning strategy, so player II wins. Being a 'free summand' is used in a more general sense in order to obtain a singular compactness result more general than our Theorem 9.3.

Exercises

(1) Let A be a p-group of singular cardinality λ. If all subgroups of A of cardinalities $< \lambda$ are Σ-cyclic, then A too is Σ-cyclic. [Hint: the λ-free vector space $A[p]$ is free.]
(2) Let A be a group of singular cardinality λ. If A is λ-cyclic, then it is Σ-cyclic.

10 Groups with Discrete Norm

Normed vector spaces play a most important role in functional analysis. In abelian group theory, the idea of an integer-valued (more generally, a discrete) norm leads to an interesting characterization of free groups—a result that has several important applications.

Discrete Norm A **norm** on a group A is a function $\| \ldots \|: A \to \mathbb{R}$ such that

(i) $\| a \| \geq 0$ for all $a \in A$; and $\| a \| = 0$ exactly if $a = 0$;
(ii) $\| a + b \| \leq \| a \| + \| b \|$ for all $a, b \in A$;
(iii) $\| ma \| = |m|.\| a \|$ for each $m \in \mathbb{Z}$ and $a \in A$.
A norm $\| \ldots \|$ is called **discrete** if it also satisfies:
(iv) there is a real number $\epsilon > 0$ such that $\| a \| \geq \epsilon$ for all $0 \neq a \in A$.
(Requirement (iv) is *a priori* less than demanding that the norms be always integers.) We record the following elementary facts.

(A) *A group with a norm has to be torsion-free.* This follows at once from properties (i) and (iii).
(B) *If* $\| \ldots \|$ *is a (discrete) norm, then so is* $r\| \ldots \|$ *for every positive* $r \in \mathbb{R}$.
(C) *Subgroups inherit the norm function. Discreteness is inherited as well.*
(D) *A norm* $\| \ldots \|$ *on a torsion-free group extends uniquely to a norm on its divisible hull* (for divisible hull, see Sect. 2 in Chapter 4). Needless to say, an extended norm is never discrete.

Example 10.1. A free abelian group F admits a discrete norm. In fact, if $\{e_i\}_{i \in I}$ is a free basis of F, then

$$\| \sum n_i e_i \| = \sum |n_i| \qquad (n_i \in \mathbb{Z})$$

defines a discrete norm on F. Another way of furnishing F with a discrete norm is by setting

$$\| \sum n_i e_i \| = \max |n_i|.$$

It would be futile to look for other groups as examples, because—as is shown by the theorem below—only the free groups admit discrete norms.

The discussion starts with the finite rank case.

Lemma 10.2 (Lawrence [1], Zorzitto [1]). *A finite rank torsion-free group with discrete norm is free.*

Proof. Let A be torsion-free of finite rank with a discrete norm $\| \ldots \|$. By induction on the rank, we prove that A is free.

Without loss of generality, we may assume that $\| a \| \geq 1$ for all non-zero $a \in A$, and that there is an $x_0 \in A$ whose norm is $< 3/2$. Under this hypothesis on the norm, x_0 is evidently not divisible in A by any integer > 1, hence the cyclic subgroup $\langle x_0 \rangle$ must be pure in A. Therefore, if A is of rank 1, then $\langle x_0 \rangle$ is all of A.

Let A be of rank $n + 1$, and assume the claim holds for groups of rank $n \geq 1$. Starting with x_0, pick a maximal independent set $\{x_0, x_1, \ldots, x_n\}$ in A. The factor group $A^\star = A/\langle x_0 \rangle$ is torsion-free of rank n. It is straightforward to check that one can define a norm μ in $A^\star$ by setting

$$\mu(r_1 x_1^\star + \cdots + r_n x_n^\star) = |r_1| \cdot \| x_1 \| + \cdots + |r_n| \cdot \| x_n \|$$

where the coefficients r_i are rational numbers, and stars indicate cosets mod $\langle x_0 \rangle$. Supposing $A^\star$ is not free, induction hypothesis implies that $A^\star$ cannot have a discrete norm, so some coset $y^\star = s_1 x_1^\star + \cdots + s_n x_n^\star$ ($s_i \in \mathbb{Q}$) has a norm $< 1/4$. There is an $a \in A$ such that $a = s_0 x_0 + s_1 x_1 + \cdots + s_n x_n$ for some $s_0 \in \mathbb{Q}$. By adding to a an integral multiple of x_0 if necessary, we can assume that $|s_0| \leq 1/2$. But then

$$\| a \| \leq |s_0| \cdot \| x_0 \| + |s_1| \cdot \| x_1 \| + \cdots + |s_n| \cdot \| x_n \| < 1/2 \cdot 3/2 + 1/4 = 1,$$

a contradiction. Thus A^*, and hence A, is free. □

Free Groups and Discrete Norm We can now verify the main result.

Theorem 10.3 (Steprāns [1]). *A group admits a discrete norm if and only if it is free.*

Proof. In view of our example above, it is enough to show that a group A with a discrete norm $\| \dots \|$ is free. We induct on the rank κ of A. The preceding lemma settles the case if κ is finite, so assume that κ is an infinite cardinal, and that the claim holds for groups of rank $< \kappa$. If $\kappa = \aleph_0$, then finite rank subgroups are free, so Pontryagin's theorem 7.1 implies that A is free.

Next, let κ be an uncountable regular cardinal, and $0 = A_0 < A_1 < \cdots < A_\sigma < \dots$ ($\sigma < \kappa$) a smooth chain of pure subgroups of the group A such that the A_σ are of cardinality $< \kappa$, and $A = \bigcup_{\sigma<\kappa} A_\sigma$. By induction hypothesis, the subgroups A_σ are free. Consider the set

$$E = \{\sigma < \kappa \mid \exists\, \rho > \sigma \text{ such that } A_\rho / A_\sigma \text{ is not free}\},$$

and suppose E is a stationary set in κ. Without loss of generality, we may assume that $\rho = \sigma + 1$ in the definition of E by thinning out the chain. For each $\sigma \in E$, pick elements $x_{\sigma\tau}$ (where τ runs over a suitable index set) such that $\{x_{\sigma\tau} + A_\sigma\}_\tau$ is a maximal independent set of $A_{\sigma+1}/A_\sigma$. As above, define a norm μ_σ in $A_{\sigma+1}/A_\sigma$ by setting

$$\mu_\sigma\Big(\sum_\tau r_{\sigma\tau}(x_{\sigma\tau} + A_\sigma)\Big) = \sum_\tau |r_{\sigma\tau}| \cdot \| x_{\sigma\tau} \|$$

where the coefficients $r_{\sigma\tau}$ are rational numbers, and of course, all sums are finite. Since $A_{\sigma+1}/A_\sigma$ has cardinality $< \kappa$ and is not free, the norm μ_σ cannot be discrete. Thus there is a coset $y_\sigma + A_\sigma$ with norm $< \frac{1}{2}$, say, $y_\sigma = \sum_\tau s_{\sigma\tau} x_{\sigma\tau} + z_\sigma$ for some z_σ in A_σ.

For convenience, we assume that the underlying set of A consists of all ordinals $< \kappa$, and A_σ ($\sigma \in E$) is just the set of ordinals $< \sigma$. Then the correspondence $\psi : \sigma \mapsto z_\sigma$ is a regressive function from E into κ. Fodor's theorem (Jech [J]) implies that there exist a $z \in A$ and a stationary subset E' of E such that $\psi(\sigma) = z$ for all $\sigma \in E'$.

Choose different $\sigma, \rho \in E'$ such that $y_\sigma \neq y_\rho$ whose cosets have norm $< \frac{1}{2}$. We then have

$$\|y_\sigma - y_\rho\| = \|(\sum_\tau s_{\sigma\tau} x_{\sigma\tau} + z) - (\sum_\nu s_{\rho\nu} x_{\rho\nu} + z)\| \leq$$

$$\leq \sum_\tau |s_{\sigma\tau}| \cdot \| x_{\sigma\tau} \| + \sum_\nu |s_{\rho\nu}| \cdot \| x_{\rho\nu} \| = \mu_\sigma(y_\sigma + A_\sigma) + \mu_\rho(y_\rho + A_\rho) < 1,$$

a contradiction. We conclude that E is not stationary, and hence Theorem 7.5 implies that A is free.

To complete the proof for singular cardinals κ, it suffices to refer to Shelah's singular compactness theorem 9.2. □

Corollaries To underscore the significance of this result, we record a few applications of this theorem.

Let A be an arbitrary group, and X an index set. The set of all functions $f : X \to A$ such that f assumes but a finite number of distinct values in A is a subgroup $B(X, A)$ of the cartesian power A^X. In case $A = \mathbb{Z}$, this subgroup consists of the bounded integer-valued functions on X.

Theorem 10.4 (Specker [1], Nöbeling [1]). *The group $B(X, \mathbb{Z})$ of bounded functions on any set X into $\mathbb{Z}$ is a free abelian group.*

Proof. For the application of Theorem 10.3 all that we have to note is that the group $B(X, \mathbb{Z})$ carries a discrete norm. In fact, the norm of a function $f \in B(X, \mathbb{Z})$ is defined as the maximum of the absolute values of integers in the range of f. □

An immediate corollary is a far-reaching generalization.

Corollary 10.5 (Kaup–Kleane [1]). *The group of all finite-valued functions on a set X into any group A is a direct sum of copies of A.*

Proof. In view of the last theorem, it suffices to verify the isomorphism $B(X, A) \cong A \otimes B(X, \mathbb{Z})$. Let h_Y denote the characteristic function of the subset Y of X, i.e. $h_Y(x) = 1$ or 0 according as $x \in Y$ or not. Every $f \in B(X, A)$ can be written as

$$f = a_1 h_{Y_1} + \cdots + a_k h_{Y_k} \qquad (a_i \in A)$$

for some k and disjoint subsets $Y_1, \ldots, Y_k$ of X. If the characteristic functions h_Y are viewed as elements of $B(X, \mathbb{Z})$, then f can be identified with the element $a_1 \otimes h_{Y_1} + \cdots + a_k \otimes h_{Y_k}$ of $A \otimes B(X, \mathbb{Z})$. □

An interesting corollary is concerned with continuous functions on a compact space. J. de Groot considered the group $C(X, \mathbb{Z})$ of all continuous functions from a topological space X into the discrete group of the integers $\mathbb{Z}$. Of special interest is the case in which X is a compact space. In this case, a continuous function from X to $\mathbb{Z}$ is finite-valued, i.e. $C(X, \mathbb{Z})$ is a subgroup of $B(X, \mathbb{Z})$. As such it is free:

Corollary 10.6 (de Groot). *The group of all continuous functions from a compact space into the discrete group of the integers is free.* □

Yamabe [1] considered, for groups A, bilinear, positive definite functions $f : A \times A \to \mathbb{Z}$. Note that such a function f defines a discrete norm as usual *via* $\| a \| = \sqrt{f(a,a)}$ for $a \in A$. This leads us to

Corollary 10.7. *If A is a group such that there is a bilinear, positive definite function f from $A \times A$ into the integers $\mathbb{Z}$, then A has to be a free group.* □

★ **Notes.** This section is a typical example how a difficult question can sometimes be rephrased to an easier one by making it more general. Specker [1] could prove only under the CH that the group of bounded sequences of the integers is free. Nöbeling [1] succeeded in solving the more general problem on bounded functions of integers by induction on what he called Specker groups. Bergman [1] provided another proof by establishing an even more general theorem on commutative torsion-free rings generated by idempotents. Finally, the powerful theorem on groups with discrete norm was proved. It is due to Steprāns [1] who proved it after Lawrence [1], Zorzitto [1] settled the countable case. As shown above, this result has important applications.

Hill [14] found an interesting generalization of Bergman's version by dropping the condition of torsion-freeness: the additive group of a commutative ring generated by idempotents is Σ-cyclic.

Exercises

(1) Find a discrete norm on a free group of rank $\aleph_0$ that is not a multiple of any of examples in Example 10.1.

(2) Let $P = \mathbb{Z}^X$ and $B = B(X, \mathbb{Z})$ for an infinite set X. Show that P/B is divisible. [Hint: for $a \in P, n \in \mathbb{N}$ find $c \in P$ with $a = nc + b$ with $b \in B$.]

(3) (Nöbeling) Recalling that every element $f \in B(X, A)$ can be written as $f = a_1 h_{Y_1} + \cdots + a_k h_{Y_k} (a_i \in A)$ for some k and disjoint subsets $Y_1, \ldots, Y_k$ of X, call a subgroup S of $B(X, A)$ a **Specker group** if $f \in S$ implies that $Ah_{Y_1}, \ldots, Ah_{Y_k}$ are contained in S. Prove that the following conditions are equivalent for a subgroup S of $B(X, \mathbb{Z})$:

  (a) S is a Specker group;
  (b) $f \in S$ implies h_Y where Y denotes the support of f;
  (c) S is a pure subgroup and a subring in $\mathbb{Z}^X$. [Hint: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).]

(4) The intersection of Specker subgroups in $B(X, \mathbb{Z})$ is again a Specker group.

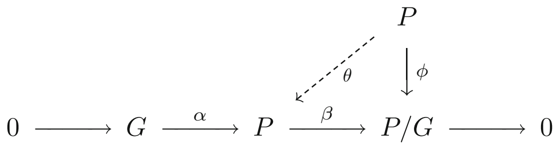
(5) If S is a Specker subgroup in $B(X, \mathbb{Z})$ and $Y \subseteq X$, then Sh_Y is also a Specker group.

(6) (Bergman) Let R be a commutative ring with identity whose additive group R^+ is torsion-free. If R is generated as a ring by a set E of idempotents, then R^+ is a free group. It can be freely generated by idempotents that are products of elements of E. [Hint: assume E is a multiplicative semigroup, well-order its elements and show that the elements of E which are not linear combinations of preceding elements of E in the ordering form a basis of R^+.]

11 Quasi-Projectivity

Another fundamental concept in this circle of ideas is that of quasi-projectivity. It is a natural generalization of projectivity where the projective property is required only with respect to the group itself (so ‘self-projective’ would probably be a better name).

Quasi-Projective Groups Thus, a group P is called **quasi-projective** if for every exact sequence with P in the middle and for every homomorphism $\phi : P \to P/G$

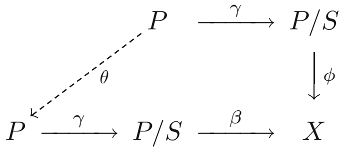


there exists an endomorphism θ of P making the triangle commute: $\beta\theta = \phi$. Free groups are quasi-projective, but not only these.

Example 11.1. (a) All cyclic groups are quasi-projective.
(b) Elementary groups are quasi-projective.

A few properties that are worth noting are as follows.

(A) *Summands of quasi-projective groups are quasi-projective.* If $P = X \oplus Y$ and $G \leq X$, then for any homomorphism $\phi : X \to X/G$, the map $\phi \oplus \mathbf{1}_Y : P \to X/G \oplus Y = P/G$ lifts to a $\theta : P \to P$, and $\theta \restriction X$ is a desired endomorphism of X.
(B) *A torsion group $P = \oplus_p P_p$ is quasi-projective if and only if its p-components P_p are.* Necessity follows from (A), and sufficiency is straightforward.
(C) *Factor groups modulo fully invariant subgroups inherit quasi-projectivity.* To see this, let S be fully invariant in the quasi-projective group P, and $\beta : P/S \to X$ an epimorphism. If $\phi : P/S \to X$ is any map and $\gamma : P \to P/S$ is the canonical homomorphism, then by the quasi-projectivity of P, there is a map $\theta : P \to P$ such that $\beta\gamma\theta = \phi\gamma$.



Since S is fully invariant in P, θ induces a map $\theta' : P/S \to P/S$ such that $\gamma\theta = \theta'\gamma$. γ can be canceled in $\beta\theta'\gamma = \phi\gamma$, thus $\beta\theta' = \phi$.
(D) *Let G be a subgroup of a quasi-projective P such that P/G is isomorphic to a summand A of P. Then G is a summand of P.* Let $\alpha : A \to P$ and $\rho : P \to A$ be

the canonical injection and projection maps, respectively. If $\beta : A \to P/G$ is an isomorphism, and $\gamma : P \to P/G$ is the canonical map, then by quasi-projectivity there is a $\theta : P \to P$ rendering the diagram

$$\begin{array}{ccc} P & \xrightarrow{\rho} & A \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\beta} \\ P & \xrightarrow{\gamma} & P/G \end{array}$$

commutative. Define the homomorphism $\delta : P/G \to P$ as $\delta = \theta\alpha\beta^{-1}$; then $\gamma\delta = \gamma\theta\alpha\beta^{-1} = \beta\rho\alpha\beta^{-1} = \mathbf{1}_{P/G}$. This means that the exact sequence $0 \to G \to P \xrightarrow{\gamma} P/G \to 0$ is splitting.

(E) *Let G be a subgroup of the quasi-projective group P such that there is an epimorphism $\rho : G \to P$. Then $K = \operatorname{Ker}\rho$ is a summand of G.* Let $\alpha : G/K \to P$ be the isomorphism induced by ρ. We have an injection $\beta : P \to P/K$ with $\operatorname{Im}\beta = G/K$ and $\beta\alpha = \mathbf{1}_{G/K}$. By quasi-projectivity, there is a $\theta : P \to P$ such that $\gamma\theta = \beta$ where $\gamma : P \to P/K$ denotes the canonical map. We argue that $\theta(P) \leq \gamma^{-1}(G/K) = G$. Now $\delta = \theta\alpha : G/K \to G$ satisfies $\gamma\delta = \mathbf{1}_{G/K}$, thus $0 \to K \to G \xrightarrow{\gamma} G/K \cong P \to 0$ is a splitting exact sequence.

Structure of Quasi-Projective Groups A complete classification of quasi-projective groups can be given in terms of cardinal invariants, based on the following theorem.

Theorem 11.2 (Fuchs–Rangaswamy [2]). *A group is quasi-projective if and only if either*

(i) *it is a free group; or*

(ii) *it is a torsion group such that each of its p-components is a direct sum of cyclic groups of fixed order p^{k_p}.*

Proof. Free groups F are obviously quasi-projective, and (C) implies that the groups F/nF are also quasi-projective for every $n \in \mathbb{N}$. By (B), the same holds for the direct sum $\oplus(F/p^{k_p}F)$ with different primes p. As $F/p^{k_p}F$ is a direct sum of cyclic groups of fixed order p^{k_p}, the sufficiency follows.

Conversely, assume P is quasi-projective. If P is torsion, then it cannot have a summand $\mathbb{Z}(p^\infty)$, because by (A) this summand would be quasi-projective, so by (D) it would contain every $\mathbb{Z}(p^n)$ as a summand—this is impossible. Thus P is reduced. It cannot have a summand of the form $C = \mathbb{Z}(p^n) \oplus \mathbb{Z}(p^m)$ with $n > m$, since there is an epimorphism $\mathbb{Z}(p^n) \to \mathbb{Z}(p^m)$ whose kernel is not a summand of C (cp. (D)). Therefore, the p-components of P are bounded by some p^k with no cyclic summands of different orders. Hence (ii) holds for P if torsion.

If P is torsion-free, then let F be a free subgroup of P generated by a maximal independent set, so that P/F is a torsion group. Let $\gamma : P \to P/F$ denote the natural map. We distinguish two cases according as P is of finite or infinite rank. If $\operatorname{rk} P$ is finite, then for every map $\phi : P \to P/F$ there is a $\theta : P \to P$ with $\phi = \theta\gamma$, and for different ϕ we have different θ. If P/F were infinite, then it had continuously many automorphisms (see Sect. 2 in Chapter 17, Exercise 3), so there would be this many choices for ϕ. But a finite rank P has only countably many endomorphisms. Thus P/F must be finite, P is finitely generated, so P is finitely generated free. If $\mathrm{rk}\, P$ is infinite, then we can find a surjective map $F \to P$, and (E) shows that P is isomorphic to a summand of F, so it is free.

Finally, suppose P is mixed. (C) implies that P/tP is quasi-projective, so free: $P = tP \oplus F$ with F a free group. If none of the summands is 0, then there is an epimorphism $F \to C \leq tP$, C cyclic, whose kernel is not a summand of F, contradicting (D). Thus P cannot be mixed. □

★ **Notes.** The listed properties of quasi-projectivity were borrowed from the pioneering paper L. Wu–J.P. Jans [Ill. J. Math. **11**, 439–448 (1967)], and from Fuchs–Rangaswamy [2]. The majority of the results (e.g., Theorem 11.2 is an exception) are valid for modules as well.

Exercises

(1) Describe the complete set of cardinal invariants attached to a quasi-projective group.
(2) $P^{(\kappa)}$ is quasi-projective for every cardinal κ whenever P is quasi-projective.
(3) (a) Suppose $P = \oplus_{n<\omega} P_n$ where the P_n are fully invariant in P. P is quasi-projective if and only if every P_n is quasi-projective.
  (b) Claim (a) may fail if the summands are not fully invariant.
(4) Fully invariant subgroups inherit quasi-projectivity.
(5) Only quasi-projective groups admit quasi-projective covers.
(6) Let $G < P$, P a quasi-projective group. Then $|\,\mathrm{End}\, P/G| \leq |\,\mathrm{End}\, P|$.

Problems to Chapter 3

PROBLEM 3.1. Characterize almost free groups in which the intersection of two direct summands is again a summand.

PROBLEM 3.2. For which ordinals σ do there exist (strongly) $\aleph_\sigma$-free groups that are not (strongly) $\aleph_{\sigma+1}$-free?

PROBLEM 3.3 (IRWIN). Is there a core class of $\aleph_1$-free groups? That is, a small collection of $\aleph_1$-free groups, each of cardinality $\aleph_1$, such that every $\aleph_1$-free group contains a member of this class.

Cf. Blass–Irwin [1].

PROBLEM 3.4. Let A be the free lattice-ordered group generated by the partially ordered group G. Relate A to G as groups.