Chapter 2
Direct Sums and Direct Products

Abstract The concept of direct sum is of utmost importance for the theory. This is mostly due to two facts: first, if we succeed in decomposing a group into a direct sum, then it can be studied by investigating the summands separately, which are, in numerous cases, simpler to deal with. We shall see that almost all structure theorems in abelian group theory involve, explicitly or implicitly, some direct decomposition. Secondly, new groups can be constructed as direct sums of known or previously constructed groups.

Accordingly, there are two ways of approaching direct sums: an internal and an external way. Both will be discussed here along with their basic features. The external construction leads to the unrestricted direct sum, called direct (or cartesian) product, which will also play a prominent role in our future discussions. We present interesting results reflecting the fundamental differences in the behavior of direct sums and products in the infinite case. Pull-back and push-out diagrams will also be dealt with.

Important concepts are the direct and inverse limits that we shall use on several occasions. The final section of this chapter discusses completions in linear topologies.

A reader who is well versed in group theory can skip much of this chapter.

1 Direct Sums and Direct Products

Internal Direct Sum Let B, C be subgroups of the group A, and assume they satisfy

(i) $B + C = A$; and
(ii) $B \cap C = 0$.

Condition (i) tells us that every element $a \in A$ can be written as $a = b + c$ with $b \in B, c \in C$, while (ii) implies that such b, c are unique. For, if $a = b' + c'$ with $b' \in B, c' \in C$, then $b - b' = c' - c \in B \cap C = 0$. We will refer to b, c as the **coordinates** of a (in the given direct sum decomposition of A). In this case we write $A = B \oplus C$, and call A the **(internal) direct sum** of its subgroups B and C. (Recall that if (ii) is satisfied, we say that B and C are **disjoint**.)

Let B_i $(i \in I)$ be a set of subgroups in A subject to the following conditions:

(i) $\sum_{i \in I} B_i = A$, i.e. the subgroups B_i combined generate A; and
(ii) for every $i \in I$, $B_i \cap \sum_{j \neq i} B_j = 0$.

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Again, (i) means that every element $a \in A$ can be written as a finite sum $a = b_{i_1} + \cdots + b_{i_n}$ with b_{i_j} belonging to different **components** B_{i_j}, while (ii) states that such an expression is unique. We then write

$$A = B_1 \oplus \cdots \oplus B_n \qquad \text{or} \qquad A = \oplus_{i \in I} B_i$$

according as the index set is finite or infinite. We call these **direct decompositions** of the group A, and the B_i **(direct) summands** of A. If $A = B \oplus C$, C is a **complementary summand** or a **complement** to B. A is called **(directly) indecomposable** if $A = B \oplus C$ implies that either $B = 0$ or $C = 0$.

Let $a \in A = B \oplus C$, and write $a = b + c$ with $b \in B, c \in C$. The maps

$$\pi : A \to B,\ \rho : A \to C \quad \text{given by } \pi : a \mapsto b,\ \rho : a \mapsto c$$

are surjective maps; they can also be regarded as endomorphisms of A. They satisfy $\pi b = b$, $\pi c = 0$, $\rho c = c$, $\rho b = 0$ as well as $\pi a + \rho a = a$, thus

$$\pi^2 = \pi, \quad \rho^2 = \rho, \quad \rho\pi = 0 = \pi\rho, \quad \pi + \rho = \rho + \pi = \mathbf{1}_A. \tag{2.1}$$

If we mean by a **projection** an idempotent endomorphism, and by **orthogonal** endomorphisms those with 0 products (in both orders), then (2.1) may be expressed by saying that *a direct decomposition* $A = B \oplus C$ *defines a pair of orthogonal projections with sum* $\mathbf{1}_A$. Conversely, any pair π, ρ of endomorphisms satisfying (2.1) yields a direct decomposition $A = \pi A \oplus \rho A$. In fact, idempotency and orthogonality imply that any element common to πA and ρA must be both reproduced and annihilated by π and ρ, so $\pi A \cap \rho A = 0$, while $\pi + \rho = \mathbf{1}_A$ guarantees that $\pi A + \rho A = A$.

If A is the direct sum of several subgroups, $A = \oplus_{i \in I} B_i$, the decomposition can also be described in terms of pairwise orthogonal projections. The ith projection $\pi_i : A \to B_i$ assigns to the element $a = b_{i_1} + \cdots + b_{i_n}$ the term $b_i \in B_i$ (which can very well be 0). Then we have:

(a) $\pi_i \pi_j = 0$ or π_i according as $i \neq j$ or $i = j$;
(b) for every $a \in A$, almost all of $\pi_i a$ are 0, and $\sum_{i \in I} \pi_i a = a$.

Conversely, if $\{\pi_i \mid i \in I\}$ is a set of endomorphisms of A satisfying (a) and (b), then A is the direct sum of the subgroups $\pi_i A$.

Some of the most useful properties of direct sums are listed as follows:

(A) If $A = B \oplus C$, then $C \cong A/B$. Thus the complement of B in A is unique up to isomorphism.
(B) If $A = B \oplus C$, and if G is a subgroup of A containing B, then we have $G = B \oplus (G \cap C)$.
(C) If $a \in A = B \oplus C$, and if $a = b + c$ ($b \in B, c \in C$), then $o(a) = \operatorname{lcm}\{o(b), o(c)\}$ provided both orders are finite. Otherwise, $o(a) = \infty$.
(D) If $A = \oplus_{i \in I} B_i$ and if $C_i \leq B_i$ for each i, then $\sum_i C_i = \oplus_i C_i$.

(E) If $A = \oplus_i B_i$, where each B_i is a direct sum $B_i = \oplus_j C_{ij}$, then $A = \oplus_i \oplus_j C_{ij}$. This is a **refinement** of the given decomposition of A. Conversely, if $A = \oplus_i \oplus_j C_{ij}$, then $A = \oplus_i B_i$ where $B_i = \oplus_j C_{ij}$.

(F) If in the exact sequence $0 \to B \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$, $\operatorname{Im}\alpha$ is a summand of A, then $A \cong B \oplus C$. In this case, we say that the exact sequence is **splitting**. Any map $\gamma : C \to A$ satisfying $\beta\gamma = \mathbf{1}_C$ is called a **splitting map**; then $A = \operatorname{Ker}\beta \oplus \operatorname{Im}\gamma$. Of course, there is another map: $\delta : A \to B$ with $\delta\alpha = \mathbf{1}_B$ indicating splitting: $A = \operatorname{Im}\alpha \oplus \operatorname{Ker}\delta$.

Two direct decompositions of A, $A = \oplus_i B_i$ and $A = \oplus_j C_j$ are called **isomorphic** if there is a bijection between the two sets of components, B_i and C_j, such that corresponding components are isomorphic.

We now prove a fundamental result.

Lemma 1.1. *Let $C = \langle c \rangle$ be a finite cyclic group where $o(c) = m = p_1^{r_1} \cdots p_k^{r_k}$ with different primes p_i. Then C has a decomposition into a direct sum*

$$C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_k \rangle \qquad (o(c_i) = p_i^{r_i})$$

with uniquely determined summands.

Proof. Define $m_i = mp_i^{-r_i}$ and $c_i = m_i c$ $(i = 1, \dots, k)$. Then the m_i are relatively prime, so there are $s_i \in \mathbb{Z}$ such that $s_1 m_1 + \cdots + s_k m_k = 1$. Then $c = s_1 m_1 c + \cdots + s_k m_k c = s_1 c_1 + \cdots + s_k c_k$ shows that the c_i generate C. Clearly, $\langle c_i \rangle$ is of order $p_i^{r_i}$, so disjoint from $\langle c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_k \rangle$ which has order m_i. Hence we conclude that $C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_k \rangle$.

The uniqueness of the summands $\langle c_i \rangle$ (but not of the generators c_i) follows from the fact that $\langle c_i \rangle$ is the only subgroup of C that contains all the elements whose orders are powers of p_i. □

Decomposition of Torsion Groups One of the most important applications of direct sums is the following theorem that plays a fundamental role in abelian group theory.

Theorem 1.2. *A torsion group A is the direct sum of p-groups A_p belonging to different primes p:*

$$A = \oplus_p A_p.$$

The A_p are uniquely determined by A.

Proof. Given A, let A_p consist of all $a \in A$ whose orders are powers of the prime p. Since $0 \in A_p$, A_p is not empty. If $a, b \in A$, i.e. $p^m a = 0 = p^n b$ for integers $m, n \geq 0$, then $p^{n+m}(a - b) = 0$, so $a - b \in A_p$, and A_p is a subgroup of A. If $p_1, \dots, p_k$ are primes $\neq p$, then $A_p \cap (A_{p_1} + \cdots + A_{p_k}) = 0$, since every element of $A_{p_1} + \cdots + A_{p_k}$ is annihilated by a product of powers of $p_1, \dots, p_k$. Thus the A_p generate their direct sum in A; it must be all of A, as it is obvious in view of Lemma 1.1.

If $A = \oplus_p B_p$ is another decomposition of A into p-groups B_p with different primes p, then by the definition of the A_p we have $B_p \leq A_p$ for each p. If we had $B_p < A_p$ for some p, then $\oplus_p B_p$ could not equal A. $\square$

The subgroups A_p are called the **primary components** or the ***p*-components** of A. They are, as is seen from the definition, fully invariant in A. If A is not torsion, then the p-components T_p of its torsion part $T = tA$ may be referred to as the p-components of A. (In this case, however, T_p need not be a summand of A.) Theorem 1.2 is of utmost importance as it makes it possible to reduce the structure theory of torsion groups to primary groups.

Example 1.3. The group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of all complex numbers that are nth roots of unity for some integer $n > 0$. It is a torsion group whose p-component is $\mathbb{Z}(p^\infty)$ (this corresponds to the subgroup of all p^kth roots of unity ($k = 0, 1, 2, \ldots$)). Hence

$$\mathbb{Q}/\mathbb{Z} \cong \oplus_p \mathbb{Z}(p^\infty).$$

Another crucial direct sum decomposition is a trivial consequence of a vector space theorem.

Theorem 1.4. *An elementary group is a direct sum of cyclic groups of prime orders.*

Proof. By Theorem 1.2 only p-groups need to be considered. An elementary p-group is a $\mathbb{Z}/p\mathbb{Z}$-vector space, and as such it is the direct sum of one-dimensional spaces, i.e. of groups of order p. $\square$

External Direct Sum While the internal direct sum serves to break a group into smaller pieces, in case of external direct sums we glue together groups to create a new larger group.

We start with two unrelated groups, B and C, and construct a new group A that is the direct sum of two subgroups B' and C', such that $B' \cong B$, $C' \cong C$. The set of all pairs (b, c) with $b \in B, c \in C$ forms a group A under the rules:

(a) $(b_1, c_1) = (b_2, c_2)$ if and only if $b_1 = b_2, c_1 = c_2$,
(b) $(b_1, c_1) + (b_2, c_2) = (b_1 + b_2, c_1 + c_2)$.

Then $B' = \{(b, 0) \mid b \in B\} \cong B$, $C' = \{(0, c) \mid c \in C\} \cong C$ under the correspondences $b \mapsto (b, 0), c \mapsto (0, c)$; they are subgroups of A such that $A = B' \oplus C'$ (internal direct sum). If we think of B, C being identified with B', C' under the indicated mappings, then we may also write $A = B \oplus C$, and call A the **external direct sum** of B and C. (We write $A \cong B \oplus C$ to say that A is a direct sum of two subgroups isomorphic to B and C.)

Direct Products A **vector** $(\ldots, b_i, \ldots)$ over the set $\{B_i\}_{i \in I}$ of groups has exactly one **coordinate** b_i from B_i, *viz.* in the ith position, for each $i \in I$. Such a vector can also be interpreted as a function f defined over I such that $f(i) = b_i \in B_i$ for every $i \in I$. Equality and addition of vectors are defined coordinate-wise (for functions, we would say point-wise). In this way, the set of all vectors becomes a group C, called the **direct product** or the **cartesian product** of the groups B_i; in notation:

$$C = \prod_{i \in I} B_i.$$

The correspondence $\rho_i : b_i \mapsto (\dots, 0, b_i, 0, \dots, 0, \dots)$ where b_i is the ith coordinate and 0's are everywhere else, is an isomorphism of B_i with a subgroup B_i' of C. The groups B_i' $(i \in I)$ generate their direct sum B in C which consists of all vectors with finite support, where **support** means supp $c = \{i \in I \mid c_i \neq 0\}$ if $c = (\dots, c_i, \dots) \in C$. B is the external direct sum of the B_i, $B = \oplus_{i \in I} B_i$. Clearly, $B = C$ whenever I is finite.

For a group A, and for a set I, $A^{(I)} = \oplus_{i \in I} A$ will denote the direct sum of $|I|$ copies of A, and the symbol $A^I = \prod_{i \in I} A$ will stand for the direct product of $|I|$ copies of A. The corresponding notations $A^{(\kappa)}$ and A^κ for a cardinal κ should be clear.

The external direct sums and direct products can also be described in terms of systems of maps. The functions

$$\rho_B \colon b \mapsto (b, 0), \quad \rho_C \colon c \mapsto (0, c), \quad \pi_B \colon (b, c) \mapsto b, \quad \pi_C \colon (b, c) \mapsto c$$

are called the **(coordinate) injection** and **projection** maps, respectively. They satisfy

$$\pi_B \rho_B = \mathbf{1}_B, \quad \pi_C \rho_C = \mathbf{1}_C, \quad \pi_B \rho_C = 0 = \pi_C \rho_B, \quad \rho_B \pi_B + \rho_C \pi_C = \mathbf{1}_{B \oplus C}.$$

For an arbitrary number of components B_i $(i \in I)$, we have injections ρ_i and projections π_i satisfying

$$B_i \xrightarrow{\rho_i} C = \prod_{i \in I} B_i \xrightarrow{\pi_i} B_i$$

where $\rho_i b_i = (\dots, 0, b_i, 0, \dots)$, $\pi_i(\dots, b_j, \dots, b_i, \dots) = b_i$ satisfy the conditions: (i) $\pi_j \rho_i = \mathbf{1}_{B_i}$ or 0 according as $i = j$ or $i \neq j$; and (ii) $\sum_{i \in I} \rho_i \pi_i = \mathbf{1}_C$ (formally). Similarly for an infinite direct sum $\oplus_{i \in I} B_i$, in which case any given element is annihilated by almost all π_i.

The following 'universal' properties are crucial.

Theorem 1.5. *Let $\beta_i \colon B_i \to A$ $(i \in I)$ denote arbitrary homomorphisms, and $\rho_i \colon B_i \to \oplus_{i \in I} B_i$ the injection maps. There is a unique homomorphism $\phi \colon \oplus_{i \in I} B_i \to A$ such that $\beta_i = \phi \rho_i$ for every i.*

Proof. Write $b \in \oplus_{i \in I} B_i$ in the form $b = \rho_1 \pi_1 b + \dots + \rho_n \pi_n b$ where the π_i are the projection maps of the direct sum. It is immediately checked that $\phi b = \beta_1 \pi_1 b + \dots + \beta_n \pi_n b \in A$ defines a homomorphism $\phi \colon \oplus_{i \in I} B_i \to A$ with $\beta_i = \phi \rho_i$.

If ϕ' is another such map, then $(\phi - \phi')\rho_i = 0$ for each i, so $(\phi - \phi')b$ vanishes for all $b \in \oplus_{i\in I} B_i$, i.e. $\phi = \phi'$. $\square$

Theorem 1.6. *Let $\alpha_i\colon A \to B_i$ $(i \in I)$ denote homomorphisms and $\pi_i\colon \prod_{i\in I} B_i \to B_i$ the projection maps. There exists a unique map $\psi : A \to \prod_{i\in I} B_i$ such that $\alpha_i = \pi_i\psi$ for each $i \in I$.*

Proof. Define $\psi(a) = (\dots, \alpha_i a, \dots) \in \prod_{i\in I} B_i$. This is obviously a homomorphism satisfying $\alpha_i = \pi_i\psi$. If also ψ' has the same property, then $\pi_i(\psi - \psi')a = 0$ for all $a \in A$, thus $(\psi - \psi')a = 0$. This means $\psi = \psi'$. $\square$

A notational agreement: if $\alpha_i\colon A_i \to B_i$ $(i \in I)$ are homomorphisms, then $\oplus_{i\in I}\alpha_i$ will denote the map $\oplus_{i\in I} A_i \to \oplus_{i\in I} B_i$ that carries the ith coordinates to the ith coordinates as given by α_i. The map $\prod_{i\in I}\alpha_i : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ has similar meaning.

For a group G, the **diagonal map** $\Delta_G : G \to \prod G$ (arbitrary number of components) acts as $\Delta_G : g \mapsto (\dots, g, \dots, g, \dots)$, and the **codiagonal map** $\nabla_G\colon \oplus G \to G$ as $\nabla_G\colon (\dots, g_i, \dots) \mapsto \sum_i g_i$.

Subdirect Products Among the subgroups of the direct product, the subdirect products are most important. A group G is a **subdirect product** of the groups B_i $(i \in I)$ if it is a subgroup of the direct product $A = \prod_{i\in I} B_i$ such that $\pi_i G = B_i$ for all projections $\pi_i : A \to B_i$. This means that for every $b_i \in B_i$, G contains at least one vector whose ith coordinate is exactly b_i. If $K_i = \mathrm{Ker}(\pi_i \restriction G)$, then $\cap_{i\in I} K_i = 0$. Conversely, if K_i are subgroups of a group G such that $\cap_{i\in I} K_i = 0$, then G is a subdirect product of the factor groups G/K_i, *via*

$$g \mapsto (\dots, g + K_i, \dots) \in \prod_{i\in I}(G/K_i) \qquad \text{where } g \in G.$$

If the index set I is finite, then we also say that we have a subdirect sum.

Lemma 1.7 (Łoś). *Every group is a subdirect product of cocyclic groups.*

Proof. For every non-zero a in group A, let K_a be a subgroup of A maximal without a (argue with Zorn). Thus every subgroup of A that properly contains K_a also contains a, i.e. the coset $a + K_a$ is a cogenerator in A/K_a, so this factor group is cocyclic. Since $\cap_{0\neq a\in A} K_a = 0$, it follows that A is a subdirect product of the cocyclic groups A/K_a. $\square$

There are numerous subdirect products contained in a direct product of groups, but there is no complete survey of them. The only exception is the case of subdirect sums of two groups.

Let G be a subdirect sum of B and C. The elements $b \in B$ with $(b, 0) \in G$ form a subgroup $B_0 \leq B$ and the elements $c \in C$ with $(0, c) \in G$ form a subgroup $C_0 \leq C$. It is straightforward to check that the correspondence $b + B_0 \mapsto c + C_0$ whenever $(b, c) \in G$ is an isomorphism of B/B_0 with C/C_0. Thus G consists of those $(b, c) \in B \oplus C$ for which the canonical epimorphisms $B \to B/B_0$ and $C \to C/C_0$ map b

and c upon corresponding cosets. The groups B_0 and C_0 are called the **kernels** of the subdirect sum. Conversely, if we are given the groups B, C along with epimorphisms $\beta : B \to F$, $\gamma : C \to F$ for some group F, then the elements $(b, c) \in B \oplus C$ with $\beta b = \gamma c$ form a group G that is a subdirect sum of B and C. It is easy to verify the isomorphisms

$$G/B_0 \cong C, \quad G/C_0 \cong B, \quad B/B_0 \cong G/(B_0 \oplus C_0) \cong C/C_0.$$

We mention that the subdirect sum G in the preceding paragraph may also be obtained as a pull-back of the maps β, γ where $B \xrightarrow{\beta} B/B_0 \cong C/C_0 \xleftarrow{\gamma} C$. See Exercise 3 in Sect. 3.)

Let K be an ideal in the Boolean lattice of all subsets of I; then the **K-product** $\prod_{\mathsf{K}} A_i$ is the set of all vectors in $\prod_{i \in I} A_i$ whose supports belong to K. The **κ-product** $\prod_{i \in I}^{<\kappa} A_i$ consists of vectors with support $< \kappa$.

Ultraproducts The following construction is based on the notion of filters. Let I be an infinite index set and $\mathcal{F}$ a filter on the subsets of I. The **filtered direct product** of groups A_i $(i \in I)$ is a subgroup of the direct product $A = \prod_{i \in I} A_i$ consisting of all vectors $a = (\dots, a_i, \dots) \in A$ for which the null-set $n(a) = \{i \in I \mid a_i = 0\} \in \mathcal{F}$. It is routine to check that this is in fact a (pure) subgroup of A, which we shall denote as $\prod_{i \in I}^{\mathcal{F}} A_i$. The factor group

$$\prod_{i \in I} A_i/\mathcal{F} = \prod_{i \in I} A_i/\prod_{i \in I}^{\mathcal{F}} A_i$$

is called the **reduced product** with respect to $\mathcal{F}$. Thus $a, b \in \prod_{i \in I} A_i$ are equal in $\prod_{i \in I} A_i/\mathcal{F}$ exactly if supp $(a - b) \in \mathcal{F}$.

The most important special case is when $\mathcal{F}$ is an ultrafilter $\mathcal{U}$. Then $\prod_{i \in I} A_i/\mathcal{U}$ is called the **ultraproduct** of the A_i. If $\mathcal{U}$ is a principal ultrafilter, i.e. it consists of those subsets of I that contain a fixed $j \in I$, then $\prod_{i \in I}^{\mathcal{U}} A_i = \prod_{i \in J} A_i$ where $J = I \setminus \{j\}$. In this case, the ultraproduct is just A_j. Therefore, only ultraproducts with respect to non-principal ultrafilters are of real interest.

Example 1.8. The filter $\mathcal{F}$ that consists of all subsets of I with finite complements is non-principal, and $\prod_{i \in I}^{\mathcal{F}} A_i$ contains the direct sum $\oplus_{i \in I} A_i$.

★ **Notes.** A noteworthy generalization of direct powers, studied by Balzerzyk [3], Eda [1], relies on a complete Boolean lattice $\mathbf{B}$ with $\mathbf{0}$ as smallest and $\mathbf{1}$ as largest element. By the **Boolean power** $A^{(\mathbf{B})}$ of the group A is meant the set of functions $f : A \to \mathbf{B}$ such that (i) $f(a) \wedge f(b) = \mathbf{0}$ if $a \neq b$ in A, and (ii) $\bigvee_{a \in A} f(a) = \mathbf{1}$. The sum $f + g$ of two functions is defined *via*

$$(f + g)(a) = \vee_{a = x + y}(f(x) \wedge g(y))$$

for all possible $x, y \in A$ satisfying $x + y = a$. In case $\mathbf{B}$ is the power-set of a set I, then the elements $f \in A^{(\mathbf{B})}$ are in a bijective correspondence with the elements $\bar{f} \in A^I$ such that $f(a) = \{i \in I \mid \bar{f}(i) = a\} \in \mathbf{B}$ where $a \in A$.

The primary decomposition Theorem 1.2 is of central importance in abelian group theory. Its roots are in elementary number theory; this kind of decomposition was used by C.F. Gauss. In its complete, final form is due to Frobenius–Stickelberger [1]. The result generalizes straightforwardly to torsion modules over Dedekind domains.

In contrast to Theorem 1.2, Theorem 1.4 easily generalizes to arbitrary modules: if a module is the union of simple submodules, then it is a direct sum of simple modules (it is then called semi-simple). Semi-simple modules may be characterized by the property that every submodule is a direct summand.

The result on the subdirect sum of two groups is due to R. Remak; he dealt with finite, not necessarily commutative groups. Ultraproducts have profound implications in various areas, especially in model theory. See Eklof [1] for their structure.

Exercises

(1) Let B, C be subgroups of A, and $B \oplus C$ their external direct sum. There is an exact sequence $0 \to B \cap C \to B \oplus C \to B + C \to 0$.

(2) Determine when the direct product of infinitely many torsion groups is again a torsion group.

(3) If $0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$ are exact sequences for $i \in I$, then so are

$$0 \to \oplus A_i \xrightarrow{\oplus\alpha_i} \oplus B_i \xrightarrow{\oplus\beta_i} \oplus C_i \to 0 \quad \text{and} \quad 0 \to \prod A_i \xrightarrow{\prod\alpha_i} \prod B_i \xrightarrow{\prod\beta_i} \prod C_i \to 0.$$

(4) If G is a subdirect sum of B and C, then $B + G = B \oplus C = G + C$.

(5) Let B, C be subgroups of A such that $B \cap C = 0$. If $(B + C)/C$ is a summand of A/C, then B is a summand of A.

(6) (a) The subdirect sum of $\mathbb{Z}(p^m)$ and $\mathbb{Z}(p^n)$ $(0 < m \le n)$ with kernels $\mathbb{Z}(p^{m-k})$ and $\mathbb{Z}(p^{n-k})$ is isomorphic to $\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^{m-k})$.

(b) The subdirect sum of $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}(p^\infty)$ with kernels $\mathbb{Z}(p^m)$ and $\mathbb{Z}(p^n)$ $(0 < m \le n)$ is isomorphic to $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^m)$.

(7) A group A is called **subdirectly irreducible** if in any representation of A as a subdirect product of groups A_i, one of the coordinate projections $\pi_i : A \to A_i$ is an isomorphism. Prove that A is subdirectly irreducible if and only if it is cocyclic.

2 Direct Summands

Direct Summands In this section, we collect a few criteria for a subgroup to be a summand. We start with the most frequently used criterion.

Lemma 2.1. *A subgroup B of A is a summand of A if and only if A has an idempotent endomorphism π satisfying $\pi A = B$; equivalently, the injection $B \to A$ followed by $\pi : A \to B$ is the identity $\mathbf{1}_B$ of B.*

Proof. If $A = B \oplus C$, then the projection π on the first summand, viewed as an element of $\operatorname{End} A$, is as desired. Conversely, if π is an idempotent endomorphism, then $A = \pi A \oplus (1 - \pi)A$. □

Putting it in a different way, lemma states that B is a summand of A exactly if the identity map of B extends to an endomorphism $A \to B$.

If B is a summand of A, then the complementary summand is unique up to isomorphisms (recall: it is $\cong A/B$), but it is far from being unique as a subgroup. The following result explains how to obtain from one complement all the other complements.

Lemma 2.2. *Let $A = B \oplus C$ be a direct decomposition with projections β, γ. If also $A = B \oplus C_0$ with projections β_0, γ_0, then, for some endomorphism θ of A, we have*

$$\beta_0 = \beta + \beta\theta\gamma, \qquad \gamma_0 = \gamma - \beta\theta\gamma. \tag{2.2}$$

Conversely, if the maps β_0, γ_0 are of the form (2.2), then $A = B \oplus \gamma_0 A$.

Proof. If we are given the two direct decompositions, then let $\theta = \gamma - \gamma_0$. Then $B \leq \operatorname{Ker}\theta$, so $\theta = \theta\beta + \theta\gamma = \theta\gamma$. If $a = b + c = b_0 + c_0$ with $b, b_0 \in B, c \in C, c_0 \in C_0$, then $\theta a = c - c_0 = b_0 - b \in B$, thus $\beta\theta = \theta$. Hence $\gamma_0 = \gamma - \theta = \gamma - \beta\theta\gamma$ and $\beta_0 = \mathbf{1}_A - \gamma_0 = \beta + \gamma - \gamma_0 = \beta + \beta\theta\gamma$.

Conversely, if β_0, γ_0 are obtained from β, γ as given in (2.2) with any $\theta \in \operatorname{End} A$, then $\beta_0 + \gamma_0 = \mathbf{1}_A, \beta_0^2 = \beta_0, \gamma_0^2 = \gamma_0$, and $\beta_0\gamma_0 = \gamma_0\beta_0 = 0$. Thus $A = \beta_0 A \oplus \gamma_0 A$ where $\beta_0 A = \beta A = B$. □

If β is a central idempotent (commutes with all endomorphisms), then $\beta_0 = \beta + \theta\beta\gamma = \beta$ and $\gamma_0 = \gamma$. Thus the complements cannot be changed (they are fully invariant in A).

In general, a subgroup of a direct sum does not decompose along the summands. However, there is an important exceptional case.

Lemma 2.3. *If $A = B \oplus C$, and if G is a fully invariant subgroup of A, then*

$$G = (G \cap B) \oplus (G \cap C).$$

Proof. Let β, γ be the projections attached to the given direct sum. By full invariance, both βG and γG are subgroups of G. Evidently, βG and γG generate a direct sum in A, and since $\beta + \gamma = \mathbf{1}_A$, we have $G = \beta G \oplus \gamma G$. Since $\beta G \leq G \cap B$ and $\gamma G \leq G \cap C$, and proper inclusion is out of question, we have $\beta G = G \cap B$ and $\gamma G = G \cap C$. □

The following is a useful lemma.

Lemma 2.4 (Kaplansky [K]). *If the factor group A/B is a direct sum: $A/B = \oplus_{i\in I}(A_i/B)$, and if B is a direct summand in every A_i, say, $A_i = B \oplus C_i$, then B is a summand of A. More precisely,*

$$A = B \oplus (\oplus_{i\in I} C_i).$$

Proof. It is clear that the groups B and the C_i generate A. Assume that $b + c_1 + \cdots + c_n = 0$ for some $b \in B$ and $c_j \in C_j$ ($j = 1, \dots, n$). Passing mod B, we obtain $(c_1 + B) + \cdots + (c_n + B) = B$, whence the given direct sum forces $c_j \in B$ for every j. Thus $c_j \in C_j \cap B = 0$, and hence also $b = 0$. Consequently, B and the C_i generate their direct sum in A. □

Summands of Large Direct Sums The following theorem has several applications in the study of properties inherited by summands.

Theorem 2.5 (Kaplansky [2]). *Summands of a direct sum of countable groups are also direct sums of countable groups.*

Proof. Let $A = \oplus_{i\in I} A_i = B \oplus C$ where each summand A_i is countable. Pick any summand A_1, a generating system $\{a_j\}_{j\in J}$ of A_1, and write $a_j = b_j + c_j$ ($b_j \in B$, $c_j \in C$). Note that each b_j and each c_j has but a finite number of non-zero coordinates in the direct sum $A = \oplus_i A_i$. Collecting all the A_i that contain at least one non-zero coordinate of some b_j or c_j, and then forming their direct sum, we obtain a countable direct summand X_1 of A. Next, we repeat the same process with X_1 replacing A_1: select a generating system for X_1 and collect all the A_i which have non-zero coordinates of the B- and C-coordinates of the generators, to obtain a larger countable summand X_2 of A. Continuing the same way, we get a chain $X_1 \leq X_2 \leq \cdots \leq X_n \leq \dots$ of countable summands of A whose union is a countable summand $\bar{A}_1$ such that $\bar{A}_1 = (B \cap \bar{A}_1) \oplus (C \cap \bar{A}_1)$.

A smooth chain of summands S_σ of A is defined as follows. Each S_σ is a direct sum of some A_i. Set $S_0 = 0$. If S_σ is defined for an ordinal σ and $S_\sigma < A$, then pick an A_i not in S_σ and let $S_{\sigma+1} = S_\sigma + \bar{A}_i$ (where $\bar{A}_i$ is obtained by repeating the above process for A_i using components not in S_σ). For limit ordinals σ we set $S_\sigma = \cup_{\rho<\sigma} S_\rho$. It is evident that for some ordinal $\tau \leq |A|$ we will reach $S_\tau = A$. It is also clear that $S_{\sigma+1}/S_\sigma$ is countable, and every S_σ is a direct sum of a subset of the A_i such that $S_\sigma = (B \cap S_\sigma) \oplus (C \cap S_\sigma)$ for all $\sigma \leq \tau$. Setting $B \cap S_{\sigma+1} = (B \cap S_\sigma) \oplus B_\sigma$, it is clear that the B_σ are countable and generate their direct sum in B. Since the B_σ together generate B, we have $B = \oplus_{\sigma<\tau} B_\sigma$, as claimed. □

Example 2.6. Let G be any countable group, and $A = \oplus_{\sigma<\omega_1} G_\sigma$ where $G_\sigma \cong G$ for each σ. If $A = B \oplus C$, then both B and C are direct sums of countable groups (not necessarily isomorphic to G).

★ **Notes.** Kaplansky's Theorem 2.5 holds for countably generated modules over arbitrary rings. It has been extended to κ-generated modules by C. Walker [2] for any infinite cardinal κ.

Exercises

(1) (Grätzer) Let B be a subgroup of A, and C a B-high subgroup in A. Then $A = B \oplus C$ if and only if $pa = b + c$ $(a \in A, b \in B, c \in C)$ for a prime p implies $b \in pB$.

(2) Suppose $C < B < A$. Prove that

(a) if B is a summand of A, then B/C is a summand of A/C;
(b) if C is a summand of A and B/C is a summand of A/C, then B is a summand of A.

(3) Let B be a summand of A, and let $\{\pi_i \mid i \in I\}$ be the set of all projections of A onto B. These projections form a semigroup such that $\pi_i\pi_j = \pi_j$.

(4) A group A has no summand isomorphic to itself if and only if one-sided units in its endomorphism ring $\mathrm{End}\,A$ are twosided.

(5) Let η denote an endomorphism of A.

(a) If, for some n, $\mathrm{Im}\,\eta^{n+1} = \mathrm{Im}\,\eta^n$, then $\mathrm{Ker}\,\eta^n + \mathrm{Im}\,\eta^n = A$.
(b) If, for some n, $\mathrm{Ker}\,\eta^{n+1} = \mathrm{Ker}\,\eta^n$, then $\mathrm{Ker}\,\eta^n \cap \mathrm{Im}\,\eta^n = 0$.

(6) Assume $A = B \oplus C = B' \oplus C'$, and let $\beta : A \to B, \beta' : A \to B'$ denote the projections in the given decompositions. Then $B \cong B'$ if and only if there are $\phi, \psi \in \mathrm{End}\,A$ such that $\phi\psi = \beta$ and $\psi\phi = \beta'$.

(7) (a) (Grätzer–Schmidt) Let B be a direct summand of A. The intersections of all the complements of B in A is the maximal fully invariant subgroup of A that is disjoint from B. [Hint: Lemma 2.2.]
(b) A complement to a direct summand of A is unique if and only if it is fully invariant in A.

(8) Call a subgroup G of A **projection-invariant** if $\pi G \leq G$ for every projection π of A onto a summand. Prove that: (a) G is projection-invariant in A if and only if $\pi G = G \cap \pi A$ for all projections π; (b) intersections of projection-invariant subgroups are projection-invariant, and so are subgroups generated by projection-invariant subgroups; (c) Lemma 2.2 holds for projection-invariant G; (d) a projection-invariant summand is a fully invariant subgroup.

(9) (Kulikov) A direct decomposition $A = \oplus_{i \in I} A_i$ has a common refinement with every direct decomposition of A if and only if every A_i is projection-invariant.

(10) (Fuchs) $B < A$ is an **absolute direct summand** of A if $A = B \oplus C$ for every B-high subgroup C. (a) Prove that B is an absolute direct summand if and only if it is either injective (see Chapter 4) or A/B is a torsion group whose p-component is annihilated by p^k whenever $B \setminus pB$ contains an element of order p^k. (b) Find all absolute direct summands of a bounded group.

(11) (Irwin–Walker) Let $A = \oplus_{i \in I} A_i$ and $B_i \leq A_i$ for each i. If C_i is B_i-high in A_i, then $\oplus_i C_i$ is $\oplus_i B_i$-high in A.

(12) (Enochs) Let A be a p-group and $A = B \oplus C = B' \oplus C'$ direct decompositions of A such that $B[p] = B'[p]$. Then $A = B \oplus C' = B' \oplus C$. [Hint: use induction of the order of $a \in A$ to show $a \in B \oplus C'$.]

(13) (C. Walker) Generalize Theorem 2.5 to larger cardinalities κ: summands of direct sums of κ-generated groups are of the same kind.

(14) A **supplement subgroup** S to some $C < A$ is defined to be minimal with respect to the property $A = C + S$. S has this property if and only if $S \cap C$ is superfluous in A. [Hint: use the modular law in both directions.]

3 Pull-Back and Push-Out Diagrams

Pull-Backs With the aid of direct sums, we can describe two important methods in constructing certain commutative diagrams.

Theorem 3.1. *Given the homomorphisms $\alpha : A \to C$ and $\beta : B \to C$, there exists a group G, unique up to isomorphism, along with homomorphisms $\gamma : G \to A$, $\delta : G \to B$, such that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & A \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

is commutative, and if

$$\begin{array}{ccc} G' & \xrightarrow{\gamma'} & A \\ \big\downarrow{\scriptstyle\delta'} & & \big\downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

is any commutative diagram, then there exists a unique homomorphism $\phi : G' \to G$ such that $\gamma\phi = \gamma'$ and $\delta\phi = \delta'$.

Proof. Given α, β, define G as the subgroup of the direct sum $A \oplus B$ consisting of all pairs (a, b) ($a \in A$, $b \in B$) such that $\alpha a = \beta b$, and let $\gamma : (a, b) \mapsto a$, $\delta : (a, b) \mapsto b$. This makes the first diagram commutative.

If the second diagram is commutative, then define $\phi : G' \to G$ as $\phi g' = (\gamma' g', \delta' g')$ for $g' \in G'$; here $(\gamma' g', \delta' g') \in G$, since $\alpha\gamma' = \beta\delta'$. Evidently, $\gamma\phi g' = \gamma' g'$ and $\delta\phi g' = \delta' g'$ for every $g' \in G'$. It is easy to see that $\operatorname{Ker}\gamma = (0, \operatorname{Ker}\beta)$ and $\operatorname{Ker}\delta = (\operatorname{Ker}\alpha, 0)$. Therefore, if $\phi' : G' \to G$ also satisfies $\gamma\phi' = \gamma'$, $\delta\phi' = \delta'$, then $\gamma(\phi - \phi') = 0 = \delta(\phi - \phi')$, and so $\operatorname{Im}(\phi - \phi') \leq \operatorname{Ker}\gamma \cap \operatorname{Ker}\delta = 0$. Hence $\phi - \phi' = 0$, thus ϕ is unique.

The uniqueness of G can be verified by considering a G_0 with the same properties. Then by what has already been shown, there are unique maps $\phi : G \to G_0$, $\phi_0 : G_0 \to G$ with the indicated properties. Then $\phi_0\phi : G \to G$ is a unique map (if applied to the case $G' = G$), so it must be the identity; the same holds for $\phi\phi' : G_0 \to G_0$, whence the uniqueness of G is manifest. □

Push-Outs The group G of the preceding theorem is called the **pull-back** of the maps α and β. Our next task is to prove the dual, where the group H will be called the **push-out** of α and β.

Theorem 3.2. *Assume that $\alpha : C \to A$, $\beta : C \to B$ are homomorphisms. There exist a group H, unique up to isomorphism, and homomorphisms $\gamma : A \to H$, $\delta : B \to H$, such that the diagram*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ \beta\downarrow & & \downarrow\gamma \\ B & \xrightarrow{\delta} & H \end{array}$$

is commutative, and for every commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ \beta\downarrow & & \downarrow\gamma' \\ B & \xrightarrow{\delta'} & H' \end{array}$$

there is a unique homomorphism $\psi : H \to H'$ satisfying $\psi\gamma = \gamma'$ and $\psi\delta = \delta'$.

Proof. Starting with α, β, define H as the factor group of $A \oplus B$ modulo the subgroup $X = \{(\alpha c, -\beta c) \mid c \in C\}$. Let $\gamma : a \mapsto (a, 0) + X$, $\delta : b \mapsto (0, b) + X$ $(a \in A, b \in B)$ be the maps induced by the injections. Then $\gamma\alpha c = \delta\beta c$ for every $c \in C$ assures the commutativity of the first diagram.

If the second diagram is commutative, then we let $\psi : (a, b) + X \mapsto \gamma' a + \delta' b \in H'$. One can readily check that this definition is independent of the chosen representative (a, b) of the coset, and moreover, it satisfies $\psi\gamma = \gamma'$ and $\psi\delta = \delta'$. The uniqueness follows from the simple fact that $\operatorname{Im}\gamma$ and $\operatorname{Im}\delta$ generate H, and therefore, if $\psi'\gamma = \gamma', \psi'\delta = \delta'$ for some $\psi' : H \to H'$, then $(\psi - \psi')\gamma = 0 = (\psi - \psi')\delta$ implies that $\psi - \psi'$ maps the whole of H upon 0. An argument similar to the one at the end of the proof of the preceding theorem establishes the uniqueness of H. □

The following observations are of importance.

(a) *If in the pull-back diagram, α is monic, then so is δ; if α is epic, so is δ.* In view of the uniqueness of the pull-back diagram, it is enough to prove the claim for the group G as constructed in the proof above. That $\operatorname{Ker}\alpha = 0$ implies $\operatorname{Ker}\delta = 0$ is immediately seen from the proof. Furthermore, if α is epic, then to every $b \in B$ there is an $a \in A$ such that $\alpha a = \beta b$, and so δ is also epic.

(b) *If in the push-out diagram, α is monic, then so is δ; if α is epic, so is δ.* Again, we need only show this for H as defined above. Now clearly $\operatorname{Ker}\delta = 0$ whenever $\operatorname{Ker}\alpha = 0$. If α is epic, then to every $a \in A$ there is a $c \in C$ with $\alpha c = a$, and so δ maps $b + \beta c$ upon $(0, b + \beta c) + X = (a, b) + X$. Hence δ is epic as well.

Exercises

(1) If $B = 0$ in the pull-back diagram above, then $G \cong \operatorname{Ker}\alpha$.
(2) (a) If $C = 0$ in the pull-back diagram, then $G \cong A \oplus B$.
(b) If $C = 0$ in the push-out diagram, then $H \cong A \oplus B$.
(3) If both α, β are surjective in the pull-back diagram, then G is a subdirect sum of A and B with kernels $\operatorname{Ker}\alpha, \operatorname{Ker}\beta$.
(4) The pull-back diagram above is a push-out diagram (for γ, δ) exactly if the map $\nabla(\alpha \oplus \beta) : A \oplus B \to C$ is surjective.
(5) If in the diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[d] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \end{tikzcd}$$

each of the two squares is a pull-back, then the outer rectangle is also a pull-back.
(6) Formulate and prove the dual of the preceding exercise for push-outs.
(7) Using the notations of the above pull-back and push-out diagrams, the sequences $0 \to G \to A \oplus B \to C \to 0$ and $0 \to C \to A \oplus B \to H \to 0$ (with the obvious maps) are exact.

4 Direct Limits

Direct Systems Let $\{A_i \ (i \in I)\}$ be a system of groups where the index set I is partially ordered and **directed (upwards)** in the sense that to each pair $i, j \in I$, there is a $k \in I$ such that both $i \leq k$ and $j \leq k$. Suppose that for every pair $i, j \in I$ with $i \leq j$, there is a homomorphism $\pi_i^j : A_i \to A_j$ (called **connecting map**) subject to the conditions:

(i) π_i^i is the identity map of A_i for all $i \in I$; and
(ii) if $i \leq j \leq k$ in I, then $\pi_j^k \pi_i^j = \pi_i^k$.

In this case, $\mathfrak{A} = \{A_i \ (i \in I);\ \pi_i^j\}$ is called a **direct system**. (If the index set is ω, then it suffices to specify only π_n^{n+1} for all $n < \omega$, because the other π_n^m are then determined by rule (ii).) By the **direct** or **injective limit**, or **colimit** of $\mathfrak{A}$ is meant a group A_* such that

(a) there are maps $\pi_i \colon A_i \to A_*$ such that $\pi_i = \pi_j \pi_i^j$ holds for all $i \leq j$;
(b) if G is any group, and $\rho_i \colon A_i \to G$ $(i \in I)$ are maps satisfying $\rho_i = \rho_j \pi_i^j$ for all $i \leq j$, then there is a *unique* map $\alpha \colon A_* \to G$ such that $\rho_i = \alpha \pi_i$ for all $i \in I$.

We write: $A_* = \varinjlim_{i \in I} A_i$, and call the maps $\pi_i \colon A_i \to A_*$ **canonical**.

Theorem 4.1. *A direct system $\mathfrak{A} = \{A_i \ (i \in I);\ \pi_i^j\}$ of groups has a limit, unique up to isomorphism.*

Proof. We form the direct sum $A = \oplus_i A_i$, and consider the subgroup $B \leq A$ generated by the elements $a_i - \pi_i^j a_i$ for all $a_i \in A_i$ and for all $i \leq j$ in I. Our claim is that $A/B = A_*$ is the direct limit of $\mathfrak{A}$.

The elements of A/B are cosets of the form $a_{i_1} + \cdots + a_{i_n} + B$ with $a_{i_j} \in A_{i_j}$. If $i \in I$ is such that $i_1, \ldots, i_n$ are all $\leq i$, then this coset is $\pi_{i_1}^i a_{i_1} + \cdots + \pi_{i_n}^i a_{i_n} + B$, since $a_{i_1} + \cdots + a_{i_n} - \pi_{i_1}^i a_{i_1} - \cdots - \pi_{i_n}^i a_{i_n} = (a_{i_1} - \pi_{i_1}^i a_{i_1}) + \cdots + (a_{i_n} - \pi_{i_n}^i a_{i_n}) \in B$. Thus every element in A/B can be written as $a_i + B$ for some $a_i \in A_i$. In particular, B consists of all finite sums of the form $b = a_{i_1} + \cdots + a_{i_n}$ with $a_{i_j} \in A_{i_n}$ for which there is an $i \in I$ such that $i_1, \ldots, i_n \leq i$ and $\pi_{i_1}^i a_{i_1} + \cdots + \pi_{i_n}^i a_{i_n} = 0$.

Consider the maps $\pi_i \colon A_i \to A/B$ acting as $a_i \mapsto a_i + B$. They obviously satisfy $\pi_i = \pi_j \pi_i^j$ $(i \leq j)$. If G is any group as stated in (b), then define $\alpha : A/B \to G$ by $\alpha(a_i + B) = \rho_i a_i$. Owing to $\rho_i = \rho_j \pi_i^j$, this definition is independent of the choice of the coset representative, and since α is evidently additive, α is a genuine homomorphism. It satisfies $\rho_i = \alpha \pi_i$ for all $i \in I$, as required. If $\alpha' \colon A/B \to G$ also satisfies $\rho_i = \alpha' \pi_i$ for all $i \in I$, then $(\alpha - \alpha')\pi_i = 0$ for each $i \in I$, thus $\alpha - \alpha'$ sends every $a_i + B = \pi_i a_i$ to 0, i.e. $\alpha = \alpha'$. It follows that A/B is a limit of the given direct system, so we can write $A_* = A/B$.

To show that A_* is unique up to isomorphism, suppose that also A_0 shares properties (a)–(b). Then there exist unique maps $\alpha \colon A_* \to A_0$ and $\alpha_0 \colon A_0 \to A_*$ as required by (b). Also, $\alpha_0 \alpha \colon A \to A$ and $\alpha \alpha_0 \colon A_0 \to A_0$ are unique, so they are the identity maps. Consequently, $A_0 \cong A_*$. $\square$

We now list some of the most useful properties of direct limits.

(A) *A_* is the set union of the subgroups $\pi_i A_i$ $(i \in I)$.*
(B) *If $\pi_i a_i = 0$ for some $a_i \in A_i$, then there is a $j \geq i$ such that $\pi_i^j a_i = 0$.* Indeed, if $\pi_i a_i = 0$, then $a_i \in B$, and the proof above establishes this claim.
(C) *If every π_i^j is a monic map, then all the π_i are monomorphisms.* This follows from (B).
(D) *If J is a cofinal subset of I, then the system restricted to J has the same direct limit: $\varinjlim_J A_i \cong \varinjlim_I A_i$.* In fact, if the first group is A'/B', then $a_j + B' \mapsto a_j + B$ is an isomorphism of A'/B' with A/B.

Example 4.2. Let $\{A_i\ (i \in I)\}$ be the collection of all subgroups of a group A where the index set I is partially ordered by the rule: $i \le j$ if and only if $A_i \le A_j$. Let $\pi_i^j : A_i \to A_j$ denote the injection map for $i \le j$. Then $\mathfrak{A} = \{A_i\ (i \in I); \pi_i^j\}$ is a direct system with limit A.

Example 4.3. If we admit only finitely generated subgroups of A in the direct system $\mathfrak{A} = \{A_i\ (i \in I); \pi_i^j\}$ with the injection maps π_i^j, the direct limit is still A. In the special case where A is arbitrary torsion-free, we get A as the direct limit of finitely generated free groups.

Example 4.4. Let $A = \bigoplus_{j \in J} C_j$ be a direct sum. Let i range over the set I of finite subsets of J, so that $i \le k$ in I means that i is a subset of k. If we define $A_i = \bigoplus_{j \in i} C_j$ for all $i \in I$, and $\pi_i^k : A_i \to A_k$ to be the obvious inclusion map, then we get a direct system whose limit is A.

Maps Between Direct Systems We consider homomorphisms between direct limits that are induced by homomorphisms between direct systems. If $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ and $\mathfrak{B} = \{B_i\ (i \in I);\ \rho_i^j\}$ are direct systems with the same index set I, then by a **homomorphism** $\Phi : \mathfrak{A} \to \mathfrak{B}$ we mean a set of homomorphisms $\Phi = \{\phi_i : A_i \to B_i \mid i \in I\}$ such that the diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{\pi_i^j} & A_j \\ {\scriptstyle\phi_i}\big\downarrow & & \big\downarrow{\scriptstyle\phi_j} \\ B_i & \xrightarrow{\rho_i^j} & B_j \end{array}$$

commute for all $i \le j$ in I.

Lemma 4.5. *If Φ is a homomorphism between the direct systems $\mathfrak{A}$ and $\mathfrak{B}$, then there exists a unique morphism $\Phi_* : A_* = \varinjlim A_i \to B_* = \varinjlim B_i$ making all the diagrams*

$$\begin{array}{ccc} A_i & \xrightarrow{\pi_i} & A_* \\ {\scriptstyle\phi_i}\big\downarrow & & \big\downarrow{\scriptstyle\Phi_*} \\ B_i & \xrightarrow{\rho_i} & B_* \end{array}$$

commute (π_i, ρ_i denote the canonical morphisms). Φ_* is an epimorphism (monomorphism) if all the ϕ_i are epimorphisms (monomorphisms).

Proof. Since the maps $\rho_i\phi_i : A_i \to B_*$ satisfy the condition $\rho_j\phi_j\pi_i^j = \rho_j\rho_i^j\phi_i = \rho_i\phi_i$ for every pair $i \le j$, the existence of a unique homomorphism $\Phi_* : A_* \to B_*$ such that $\rho_i\phi_i = \Phi_*\pi_i$ for each $i \in I$ is guaranteed. This proves the first assertion.

If all the ϕ_i are epic, then the subgroups $\rho_i B_i = \rho_i\phi_i A_i$ cover B_*, so Φ_* must be epic. If all the ϕ_i are monic, then pick an $a \in \operatorname{Ker}\Phi_*$. There is $j \in I$ such that $a = \pi_j a_j$ for some $a_j \in A_j$. Hence $\rho_j\phi_j a_j = \Phi_*\pi_j a_j = \Phi_* a = 0$, and so by (B) we have a $k \ge j$ with $\rho_j^k\phi_j a_j = 0$. But $\rho_j^k\phi_j = \phi_k\pi_j^k$ and ϕ_k is monic, so $\pi_j^k a_j = 0$, whence $\pi_j a_j = 0$ and $a = 0$. □

We now move to three direct systems: $\mathfrak{A}$, $\mathfrak{B}$ as above, and a third one, $\mathfrak{C} = \{C_i\ (i \in I);\ \sigma_i^j\}$, all with the same directed index set I. If $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ and $\Psi\colon \mathfrak{B} \to \mathfrak{C}$ are homomorphisms between them such that the sequence $0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$ is exact for each $i \in I$, then we say that the sequence

$$0 \to \mathfrak{A} \xrightarrow{\Phi} \mathfrak{B} \xrightarrow{\Psi} \mathfrak{C} \to 0 \tag{2.3}$$

is exact. It is an important fact that direct limits of exact sequences is exact. More precisely,

Theorem 4.6. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be direct systems over the same index set I, and $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ and $\Psi : \mathfrak{B} \to \mathfrak{C}$ homomorphisms between them. If the sequence (2.3) is exact, then the sequence*

$$0 \to A_* = \varinjlim_i A_i \xrightarrow{\Phi_*} B_* = \varinjlim_i B_i \xrightarrow{\Psi_*} C_* = \varinjlim_i C_i \to 0$$

of direct limits is likewise exact.

Proof. Exactness at A_* and C_* is guaranteed by Lemma 4.5, so we prove exactness only at B_*. By Lemma 4.5, the diagram

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\pi_i"'] & B_i \arrow[r, "\psi_i"] \arrow[d, "\rho_i"'] & C_i \arrow[r] \arrow[d, "\sigma_i"] & 0 \\
0 \arrow[r] & A_* \arrow[r, "\Phi_*"] & B_* \arrow[r, "\Psi_*"] & C_* \arrow[r] & 0
\end{tikzcd}$$

is commutative for all $i \in I$. If $a \in A_*$, then $\pi_i a_i = a$ for some $a_i \in A_i$, so $\Psi_*\Phi_* a = \Psi_*\Phi_*\pi_i a_i = \Psi_*\rho_i\phi_i a_i = \sigma_i\psi_i\phi_i a_i = 0$. Next let $b \in \operatorname{Ker}\Psi_*$. For some $b_i \in B_i$, we have $\rho_i b_i = b$, whence $\sigma_i\psi_i b_i = \Psi_*\rho_i b_i = \Psi_* b = 0$. There exists $j \in I$ with $\sigma_i^j\psi_i b_i = 0$, thus $\psi_j b_j = \psi_j\rho_i^j b_i = 0$. Since the top row in the diagram is exact, there is an $a_j \in A_j$ with $\phi_j a_j = b_j$. Setting $a = \pi_j a_j$, we arrive at $\Phi_* a = \Phi_*\pi_j a_j = \rho_j\phi_j a_j = \rho_j\rho_i^j b_i = \rho_i b_i = b$, i.e. $b \in \operatorname{Im}\Phi_*$, and the bottom row is exact at B_*. $\square$

Exercises

(1) Show that $\varinjlim_n \mathbb{Z}(p^n) = \mathbb{Z}(p^\infty)$, using inclusion maps.
(2) Let $A_n \cong \mathbb{Z}$ $(n < \omega)$ with $\pi_n^{n+1} : A_n \to A_{n+1}$ multiplication by n. Prove that $\varinjlim_n A_n \cong \mathbb{Q}$.
(3) If every π_i^j is an onto map, then all the π_i are epimorphisms.
(4) A group is locally cyclic if and only if it is a direct limit of cyclic groups.

(5) (a) Let A_* be the limit of the direct system $\mathfrak{A} = \{A_i\ (i \in I); \pi_i^j\}$, and $a \in A_*$. There exist a $j \in I$ and an $a_j \in A_j$ such that $\pi_j a_j = a$ and $o(a_j) = o(a)$.
(b) Direct limit of torsion (torsion-free) groups is again torsion (torsion-free).
(6) If G is finitely generated, and $\alpha : G \to A_*$ (notations as above), then there exist a $j \in I$ and an $\alpha_j : G \to A_j$ such that $\alpha = \pi_j \alpha_j$.
(7) If $\mathfrak{A} = \{A_i\ (i \in I); \pi_i^j\}$ and $\mathfrak{B} = \{B_i\ (i \in I); \rho_i^j\}$ are direct systems of groups, then $\mathfrak{A} \oplus \mathfrak{B} = \{A_i \oplus B_i\ (i \in I); \pi_i^j \oplus \rho_i^j\}$ is likewise a direct system whose direct limit is the direct sum of the direct limits of $\mathfrak{A}$ and $\mathfrak{B}$.
(8) What is wrong with the following argument? Because of Theorem 4.6, the sequence $0 \to \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty) \to 0$ must be exact, since it can be obtained as the direct limit of the exact sequences $0 \to \mathbb{Z}(p^m) \to \mathbb{Z}(p^{2m}) \to \mathbb{Z}(p^m) \to 0$ $(m \in \mathbb{N})$.

5 Inverse Limits

Inverse Systems Inverse limits are dual to direct limits: we just reverse the arrows.

Assume $\{A_i \mid i \in I\}$ is a collection of groups, indexed by a poset I, and for each pair $i, j \in I$ of indices with $i \leq j$ there is given a **connecting** homomorphism $\pi_i^j : A_j \to A_i$ such that

(i) π_i^i is the identity map of A_i for all $i \in I$; and
(ii) if $i \leq j \leq k$ in I, then $\pi_i^j \pi_j^k = \pi_i^k$.

In this case, $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ is called an **inverse system**. By the **inverse** or **projective limit**, or simply **limit**, of this inverse system is meant a group A^* such that

(a) there are maps $\pi_i : A^* \to A_i$ such that $\pi_i = \pi_i^j \pi_j$ for all $i \leq j$; and
(b) if G is any group with maps $\rho_i : G \to A_i$ $(i \in I)$ subject to $\rho_i = \pi_i^j \rho_j$ for $i \leq j$, then there is a *unique* map $\phi : G \to A^*$ satisfying $\rho_i = \pi_i \phi$ for all $i \in I$.

We write: $A^* = \varprojlim_{i \in I} A_i$, and call the maps $\pi_i : A^* \to A_i$ **canonical**.

Theorem 5.1. *An inverse system $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ of groups has a limit, unique up to isomorphism.*

Proof. Consider the subgroup A^* in the direct product $A = \prod_{i \in I} A_i$ that consists of all vectors $a = (\dots, a_i, \dots)$ whose coordinates satisfy $\pi_i^j a_j = a_i$ for all $i \leq j$. This is in fact a subgroup as is seen immediately. The projection maps $\pi_i : a \mapsto a_i$ satisfy $\pi_i = \pi_i^j \pi_j$, so (a) holds for A^*.

To verify (b), let G be a group as stated in (b), and for $g \in G$ define $\phi : g \mapsto (\dots, \rho_i g, \dots) \in \prod_i A_i$. Owing to the condition $\rho_i = \pi_i^j \rho_j$, we have $\phi g \in A^*$. Clearly, $\phi : G \to A^*$ satisfies $\rho_i = \pi_i \phi$ for all $i \in I$. If $\rho_i = \pi_i \phi'$ holds also for $\phi' : G \to A^*$, then $\pi_i(\phi - \phi') = 0$ for all i, so every coordinate projection of $(\phi - \phi')G$ is 0, hence $\phi = \phi'$.

In order to establish the uniqueness of A^*, we can mimic the argument at the end of the last paragraph almost word-by-word. $\square$

It is worthwhile noting the following properties of inverse limits.

(A) *If I is directed, and if in the inverse system $\mathfrak{A} = \{A_i \ (i \in I);\ \pi_i^j\}$ all connecting maps π_i^j are monomorphisms, then so are all the π_i.* In fact, assume $a \in A^*$ is such that $\pi_i a = 0$. Given $j \in I$, there is a $k \in I$ with $i, j \le k$. Then $\pi_i^k \pi_k a = \pi_i a = 0$, whence π_i^k monic implies $\pi_k a = 0$. Therefore, $\pi_j a = \pi_j^k \pi_k a = 0$ for all $j \in I$, and so $a = 0$. (Exercise 5 will show that, in general, the same fails for epimorphisms.)

(B) *If I is directed, and if J is a cofinal directed subset in I, then we have $\varprojlim_{i \in I} A_i = \varprojlim_{j \in J} A_j$.*

(C) *A^* is the intersection of kernels of certain endomorphisms of $\prod_i A_i$.* For, every pair $i \le j$ in I defines an endomorphism

$$\theta_{ij} : (\dots, a_i, \dots, a_j, \dots) \mapsto (\dots, a_i - \pi_i^j a_j, \dots, a_j, \dots).$$

Comparing this with the proof of Theorem 5.1, it becomes evident that $A^* = \cap_{i \le j} \operatorname{Ker} \theta_{ij}$.

(D) *If all the groups in the inverse system $\mathfrak{A} = \{A_i \ (i \in I);\ \pi_i^j\}$ are Hausdorff topological groups and the connecting maps π_i^j are continuous homomorphisms, then the inverse limit A^* is a closed subgroup of $\prod_i A_i$ (which is equipped with the product topology), and the canonical maps $\pi_i : A^* \to A_i$ are continuous.* Indeed, then the endomorphisms θ_{ij} in (C) are continuous, so their kernels as well as the intersection of the kernels are closed subgroups. A^* carries the topology inherited from $\prod_i A_i$, so the continuity of the π_i is obvious.

Example 5.2. Let $A = \prod_{\alpha \in J} B_\alpha$ be the direct product of the groups B_α. Let I denote the set of all finite subsets of J, partially ordered by inclusion. For $i \in I$, set $A_i = \oplus_{\alpha \in i} B_\alpha$, and for $i \le j$ in I let π_i^j be the projection map $A_j \to A_i$. This gives rise to an inverse system $\mathfrak{A} = \{A_i \ (i \in I);\ \pi_i^j\}$. We now claim: $A^* = \varprojlim_{i \in I} A_i \cong A$. To prove this, let $\pi_i : A^* \to A_i$ be the ith canonical map, and $\rho_i : A \to A_i$ the ith projection map. By definition, there is a unique map $\phi : A \to A^*$ such that $\pi_i \phi = \rho_i$. If $\phi a = 0$ for some $a \in A$, then $\rho_i a = \pi_i \phi a = 0$ for all $i \in I$, so ϕ is monic. If $a^* = (\dots, a_i, \dots, a_j, \dots) \in A^*$, then write $a_i = b_{\alpha_1} + \dots + b_{\alpha_k}$ with $b_{\alpha_\ell} \in B_{\alpha_\ell}$ if $i = \{\alpha_1, \dots, \alpha_k\}$. If $i \le j$, then by the choice of π_i^j, the B_α-coordinates of a_i are identical with the corresponding coordinates of a_j, so a^* defines a unique $(\dots, b_\alpha, \dots) \in A$. A glance at the definition of ϕ in the proof of Theorem 5.1 shows that $\phi(\dots, b_\alpha, \dots) = a^*$, so ϕ is epic as well.

Example 5.3. Let $C_n = \langle c_n \rangle$ be cyclic groups of order p^n $(n \in \mathbb{N})$, and define maps $\pi_n^{n+1} : C_{n+1} \to C_n$ induced by $c_{n+1} \mapsto c_n$. Now $\mathfrak{C} = \{C_n \ (n \in \mathbb{N});\ \pi_n^m\}$ is an inverse system, and our claim is that $C^* = \varprojlim_{n \in \mathbb{N}} C_n \cong J_p$. If $\pi_n : C^* \to C_n$ is the canonical map, and if we define $\rho_n : J_p \to C_n$ via $\rho_n(1) = c_n$, then by definition there is a unique map $\phi : J_p \to C^*$ such that $\pi_n \rho = \rho_n$ for all $n \in \mathbb{N}$. Since only $0 \in J_p$ can belong to all $\operatorname{Ker} \rho_n$, $\operatorname{Ker} \phi = 0$ is clear. Now let $c = (b_1, \dots, b_n, \dots) \in C^*$ with $b_n = k_n c_n$ $(k_n \in \mathbb{Z})$; by the choice of π_n^{n+1} we have $k_{n+1} \equiv k_n \bmod p^n$, so there is a p-adic integer σ such that $\sigma \equiv k_n \bmod p^n$ for all n. We conclude that $\rho_n \sigma = b_n$, and ϕ must be epic.

Example 5.4 (The Intersection of Subgroups is an Inverse Limit). Let $\{A_i \mid i \in I\}$ denote a set of subgroups of a group A closed under finite intersections. We partially order I by reverse inclusion. The groups A_i, along with the injection maps $\pi_i^j : A_j \to A_i$ $(i \leq j)$, form an inverse system. Its inverse limit will be $\cap_{i \in I} A_i$, because only the constant vectors in $\prod_{i \in I} A_i$ can belong to the inverse limit A^*.

Maps Between Inverse Systems Assume $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ and $\mathfrak{B} = \{B_i\ (i \in I);\ \rho_i^j\}$ are inverse systems, indexed by the same poset I. A **homomorphism** $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a set $\{\phi_i : A_i \to B_i\ (i \in I)\}$ of homomorphisms subject to the requirement that all diagrams of the form

$$\begin{array}{ccc} A_j & \xrightarrow{\pi_i^j} & A_i \\ \phi_j \downarrow & & \downarrow \phi_i \\ B_j & \xrightarrow{\rho_i^j} & B_i \end{array}$$

be commutative for all $i \leq j$.

Lemma 5.5. *If $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism between inverse systems, then there exists a unique map $\Phi^* : A^* = \varprojlim_{i \in I} A_i \to B^* = \varprojlim_{i \in I} B_i$ such that, for every $i \in I$, the diagram*

$$\begin{array}{ccc} A^* & \xrightarrow{\pi_i} & A_i \\ \Phi^* \downarrow & & \downarrow \phi_i \\ B^* & \xrightarrow{\rho_i} & B_i \end{array}$$

commutes (with canonical maps π_i, ρ_i). Φ^* is monic, if so are the ϕ_i.

Proof. The homomorphisms ϕ_i $(i \in I)$ induce a homomorphism $\bar{\phi} = \prod_i \phi_i : \prod_i A_i \to \prod_i B_i$. The commutativity of the diagram before the lemma shows that if $a = (\dots, a_i, \dots) \in A^*$, then $\bar{\phi} a \in B^*$, hence we can define $\Phi^* : A^* \to B^*$ as the restriction of $\bar{\phi}$. With this Φ^* we have $\phi_i \pi_i a = \phi_i a_i = \rho_i \Phi^* a$, establishing the commutativity of the diagram. If also $\Phi_0 : A^* \to B^*$ makes the diagram commute for every i, then $\rho_i(\Phi^* - \Phi_0) = 0$ for every i, thus $\Phi^* = \Phi_0$.

Finally, if all the ϕ_i are monic, and if $\Phi^* a = 0$ for some $a \in A^*$, then $\phi_i \pi_i a = \rho_i \Phi^* a = 0$ implies $\pi_i a = 0$ for every i, whence $a = 0$. $\square$

For the inverse limits of exact sequences, we have a somewhat weaker result than for direct limits.

Theorem 5.6. *Assume $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}, \mathfrak{B} = \{B_i\ (i \in I);\ \rho_i^j\}$, and $\mathfrak{C} = \{C_i\ (i \in I);\ \sigma_i^j\}$ are inverse systems over the same index set I. Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ and $\Psi : \mathfrak{B} \to \mathfrak{C}$ be homomorphisms. If the sequence $0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$ is exact* *for every $i \in I$, then the sequence*

$$0 \to A^* = \varprojlim_i A_i \xrightarrow{\Phi^*} B^* = \varprojlim_i B_i \xrightarrow{\Psi^*} C^* = \varprojlim_i C_i \tag{2.4}$$

of inverse limits is likewise exact.

Proof. Exactness at A^* follows from Lemma 5.5. From the definition of Φ^*, Ψ^* it is evident that $\Psi^*\Phi^* = 0$. If π_i, ρ_i, σ_i denote the canonical maps, then by Lemma 5.5 the diagram

$$\begin{tikzcd}
0 \arrow[r] & A^* \arrow[r, "\Phi^*"] \arrow[d, "\pi_i"'] & B^* \arrow[r, "\Psi^*"] \arrow[d, "\rho_i"'] & C^* \arrow[d, "\sigma_i"] & \\
0 \arrow[r] & A_i \arrow[r, "\phi_i"] & B_i \arrow[r, "\psi_i"] & C_i \arrow[r] & 0
\end{tikzcd}$$

is commutative for each $i \in I$. In order to show the exactness of the top row at B^*, let $b \in \operatorname{Ker}\Psi^*$. In view of $\psi_i\rho_i b = \sigma_i\Psi^* b = 0$ and the exactness of the bottom row, for every $i \in I$ there is an $a_i \in A_i$ satisfying $\phi_i a_i = \rho_i b$. For $j > i$, $\phi_i\pi_i^j a_j = \rho_i^j\phi_j a_j = \rho_i^j\rho_j b = \rho_i b = \phi_i a_i$, whence $\pi_i^j a_j = a_i$ as ϕ_i is monic. We infer that $a = (\dots, a_i, \dots, a_j, \dots) \in A^*$. For this a we have $\rho_i\Phi^* a = \phi_i\pi_i a = \phi_i a_i = \rho_i b$ for every i, so $\Phi^* a = b$. $\square$

Exercise 5 will show that, in general, Theorem 5.6 cannot be improved by putting $\to 0$ at the end of the exact sequence (2.4). A noteworthy special case when the exact sequence of inverse limits is exact is as follows.

Proposition 5.7. *If in Theorem 5.6 we specialize:* $\mathfrak{A} = \{A_n\ (n < \omega);\ \pi_n^{n+1}\}$, $\mathfrak{B} = \{B_n\ (n < \omega);\ \rho_n^{n+1}\}$, $\mathfrak{C} = \{C_n\ (n < \omega);\ \sigma_n^{n+1}\}$, *and assume that all the maps* π_n^{n+1} *are epic, then the sequence of inverse limits is exact:*

$$0 \to A^* = \varprojlim_n A_n \xrightarrow{\Phi^*} B^* = \varprojlim_n B_n \xrightarrow{\Psi^*} C^* = \varprojlim_n C_n \to 0.$$

Proof. Let $c^* = (c_0, \dots, c_n, \dots)$ represent an element of C^*. We now construct by induction an element $b^* = (b_0, \dots, b_n, \dots) \in B^*$ such that $\Psi^* b^* = c^*$. As ψ_0 is surjective, there is $b_0 \in B_0$ with $\psi_0 b_0 = c_0$. Suppose that, for some $n < \omega$, we have found $b_i \in B_i$ for all $i \leq n$ such that $\psi_i b_i = c_i$ and $\rho_{i-1}^i b_i = b_{i-1}$. Choose any $b'_{n+1} \in B_{n+1}$ mapped upon c_{n+1} by ψ_{n+1}. Then $b_n - \rho_n^{n+1} b'_{n+1} = \phi_n a_n$ for some $a_n \in A_n$. If $a_{n+1} \in A_{n+1}$ is such that $\pi_n^{n+1} a_{n+1} = a_n$ (which exists by hypothesis), then $b'_{n+1} + \phi_{n+1} a_{n+1} \in B_{n+1}$ is our choice for the next coordinate b_{n+1} in b^*. It is clear that then $b^* \in B^*$ is as desired. $\square$

Derived Functor of $\mathcal{I}nv$ The inverse systems of abelian groups (with a fixed index set I) and the morphisms between them form a category $\mathcal{I}nv(I)$. The functor

$\mathcal{I}nv(I) \mapsto \mathcal{A}b$ assigning to an inverse system its inverse limit is left-exact. Since its right-exactness fails in general, the inverse limit functor has a derived functor, denoted $\lim^1$. This is especially interesting in case the index set is ω when the inverse system looks like

$$C_0 \overset{\gamma_1}{\longleftarrow} C_1 \overset{\gamma_2}{\longleftarrow} \dots \overset{\gamma_n}{\longleftarrow} C_n \overset{\gamma_{n+1}}{\longleftarrow} C_{n+1} \overset{\gamma_{n+2}}{\longleftarrow} \dots .$$

Then $\varprojlim_n^1 C_n \cong \operatorname{Coker} \psi$ where

$$\psi : (\dots, c_n, \dots) \mapsto (\dots, c_n - \gamma_{n+1} c_{n+1}, \dots).$$

denotes the **Eilenberg map** $\psi : \prod_{n<\omega} C_n \to \prod_{n<\omega} C_n$; see Jensen [Je], as well as Schochet [1]. The functor $\varprojlim^1$ will be discussed later in Proposition 6.9 in Chapter 9. We just point out here that, for an exact sequence of inverse systems in Proposition 5.7, there is an exact sequence

$$0 \to \varprojlim_n A_n \to \varprojlim_n B_n \to \varprojlim_n C_n \to \varprojlim_n{}^1 A_n \to \varprojlim_n{}^1 B_n \to \varprojlim_n{}^1 C_n \to 0.$$

Example 5.8. Consider the inverse system $\{\mathbb{Z}; n!\}$: $\mathbb{Z} \overset{1!}{\longleftarrow} \mathbb{Z} \overset{2!}{\longleftarrow} \dots \overset{n!}{\longleftarrow} \mathbb{Z} \overset{(n+1)!}{\longleftarrow} \mathbb{Z} \longleftarrow \dots$ The Eilenberg map ψ carries the vector $(k_0, \dots, k_n, \dots)$ ($k_n \in \mathbb{Z}$) to the vector $(k_0 - k_1, k_1 - 2!k_2, \dots, k_n - (n+1)!k_{n+1}, \dots)$. This vector is divisible modulo $\oplus\, \mathbb{Z}$ (which group is in $\operatorname{Ker} \psi$), so $\operatorname{Coker} \psi$ is divisible. An easy argument shows that the cokernel is torsion-free and its cardinality is the continuum, hence $\varprojlim_n^1 \{\mathbb{Z}; n!\} \cong \mathbb{Q}^{\aleph_0}$.

Example 5.9. We now consider three inverse systems: $\{\mathbb{Z}, p\}$: $\mathbb{Z} \overset{\dot{p}}{\longleftarrow} \mathbb{Z} \overset{\dot{p}}{\longleftarrow} \mathbb{Z} \overset{\dot{p}}{\longleftarrow} \dots$, $\{\mathbb{Z}, 1\}$: $\mathbb{Z} \overset{\mathbf{1}}{\longleftarrow} \mathbb{Z} \overset{\mathbf{1}}{\longleftarrow} \mathbb{Z} \overset{\mathbf{1}}{\longleftarrow} \dots$, and $\{\mathbb{Z}/p^n\mathbb{Z}, \pi\}$: $0 \longleftarrow \mathbb{Z}/p\mathbb{Z} \overset{\pi}{\longleftarrow} \mathbb{Z}/p^2\mathbb{Z} \overset{\pi}{\longleftarrow} \dots$ (with canonical maps π). They fit into the exact sequence

$$0 \to \{\mathbb{Z}, p\} \to \{\mathbb{Z}, 1\} \to \{\mathbb{Z}/p^n\mathbb{Z}, \pi\} \to 0$$

of inverse systems. The $\varprojlim$-$\varprojlim^1$ exact sequence (see above) yields the exact sequence $0 \to \mathbb{Z} \to \varprojlim \mathbb{Z}/p^n\mathbb{Z} \to \varprojlim^1 \{\mathbb{Z}, p\} \to 0$, whence

$$\varprojlim{}^1 \{\mathbb{Z}, p\} \cong J_p/\mathbb{Z} \cong \mathbb{Q}^{\aleph_0}.$$

★ **Notes.** The so-called Mittag-Leffler condition (not stated) is a most useful sufficient criterion to guarantee that $\to 0$ can be put at the end of (2.4). See Jensen [Je].

Exercises

(1) If $\mathfrak{A} = \{A_i \ (i \in I);\ \pi_i^j\}$ and $\mathfrak{B} = \{B_i \ (i \in I);\ \rho_i^j\}$ are inverse systems, then $\mathfrak{A} \oplus \mathfrak{B} = \{A_i \oplus B_i \ (i \in I);\ \pi_i^j \oplus \rho_i^j\}$ is again an inverse system. Its limit is the direct sum of the limits of $\mathfrak{A}$ and $\mathfrak{B}$.

(2) Let $C_n = \langle c_n \rangle$ be cyclic of order n, and for $n|m$ let $\pi_n^m : C_m \to C_n$ be the homomorphism induced by $c_m \mapsto c_n$. Then $\mathfrak{C} = \{C_n \ (n \in \mathbb{N}); \pi_n^m\}$ is an inverse system where $\mathbb{N}$ is partially ordered by the divisibility relation. Show that $\varprojlim_n \mathfrak{C} \cong \prod_p J_p$.
(3) Let $A_n \cong \mathbb{Z}(p^\infty)$, and let $\pi_n^{n+1} : \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$ be the multiplication by p. Then the inverse limit of the inverse system $\mathfrak{A} = \{A_n \ (n \in \mathbb{N}); \pi_n^{n+1}\}$ is isomorphic to the group of all p-adic numbers.
(4) The inverse limit of torsion-free groups is torsion-free, but the inverse limit of torsion groups need not be torsion.
(5) Let $B_n = \langle b_n \rangle \cong \mathbb{Z}$ and $\pi_n^m : b_m \mapsto b_n$ for all $n \leq m$ in $\mathbb{N}$. Let $C_n = \langle c_n \rangle \cong \mathbb{Z}(p^n)$ and $\rho_n^m : c_m \mapsto c_n$ for $n \leq m$. Show that

   (a) $\mathfrak{B} = \{B_n \ (n \in \mathbb{N}); \pi_n^m\}$ and $\mathfrak{C} = \{C_n \ (n \in \mathbb{N}); \rho_n^m\}$ are inverse systems, and the epimorphisms $\phi_n : b_n \to c_n (n \in \mathbb{N})$ define a map $\Phi : \mathfrak{B} \to \mathfrak{C}$.
   (b) The induced homomorphism $\Phi^* : B^* \to C^*$ between the inverse limits is not epic. [Hint: $\mathbb{Z} \to J_p$.]

(6) The inverse limit of splitting exact sequences need not be exact.
(7) Let $\mathfrak{A} = \{A_n \ (n < \omega); \pi_n^{n+1}\}$ be an inverse system where the maps π_n^{n+1} are epimorphisms, but not isomorphisms. Then the inverse limit A^* has cardinality at least the continuum.

6 Direct Products vs. Direct Sums

One aspect of direct products that deserves special attention is related to their homomorphisms. There is a remarkable contrast between homomorphisms from a direct sum and from a direct product: those from direct sums are completely determined by their restrictions to the components, but not much can be said about homomorphisms from a direct product, except when either the components or the target groups satisfy restrictive conditions. A most fascinating result is concerned with homomorphisms of direct products into direct sums—this is the case that we wish to explore here. What is a surprising, if not recondite, phenomenon about it is that it works only up to the first measurable cardinal.

Before entering into the discussion, a simple remark might be helpful on infinite sums in direct products $A = \prod_{i \in I} A_i$. Infinite sums $\sum_{j \in J} x_j$ do make sense when the terms are vectors $x_j = (\dots, a_{ji}, \dots)$ $(a_{ji} \in A_i)$ such that, for each $i \in I$, only a finite number of ith coordinates $a_{ji} \neq 0$. (Actually, $\sum_{j \in J} x_j$ is then a convergent sum in the product topology.)

Example 6.1. Let $A = \prod_{n<\omega} A_n$ be a countable product. Then $x = \sum_{n<\omega} x_n$ is a well-defined element of A if $x_n = (0, \dots, 0, a_{nn}, a_{n,n+1}, \dots)$ $(a_{ni} \in A_i)$ (n zeros).

Maps from Direct Product into Direct Sum We start with a special case which has independent interest. ('Reduced' means no divisible subgroup $\neq 0$, and $C^1 = \cap_{n \in \mathbb{N}} nC$ denotes the first Ulm subgroup of C.)

Theorem 6.2 (Chase [1], Ivanov [5]). *Let* $A = \prod_{i<\omega} A_i$ *denote a countable product of groups, and* $\phi : A \to C = \oplus_{j\in J} C_j$ *a homomorphism into the direct sum of reduced groups* C_j*. Then there exist integers* $m > 0$, k*, as well as a finite subset* $J_0 \subseteq J$ *such that*

$$\phi(mB_k) \le (\oplus_{j\in J_0} C_j) + (\oplus_{j\in J} C_j^1),$$

where $B_k = \prod_{k\le i<\omega} A_i$ *(summand of* A*).*

Proof. Let $\phi_j : A \to C_j$ denote the map ϕ followed by the jth coordinate projection. Assume the claim is false. Then we can find inductively an increasing sequence $1 = m_0 < m_1 < \cdots < m_k < \ldots$ of integers, a sequence of elements $b_k \in m_k B_k$, and indices $j_k \in J$ such that

$$m_k | m_{k+1}, \quad \phi_{j_k}(b_\ell) = 0 \text{ for } \ell < k \quad \text{and} \quad \phi_{j_k}(b_k) \notin m_{k+1} C_{j_k}$$

for all $k < \omega$. Indeed, if, for some $k < \omega$, we have b_ℓ, m_ℓ and j_ℓ for all $\ell \le k$ at hand as required, then j_{k+1} will be selected as an index not in $\cup_{\ell\le k}(\operatorname{supp} \phi(b_\ell))$ such that $\phi_{j_{k+1}}(m_{k+1}B_{k+1}) \not\le \cap_{n\in\mathbb{N}} nC_{j_{k+1}}$ for some proper multiple m_{k+1} of m_k; this can be done, since otherwise the claim would be true. Only a finite number of b_k have non-zero coordinates in any A_i, therefore, the infinite sum $a = \sum_{k<\omega} b_k$ is a well-defined element in A. Consider

$$\phi_{j_k} a = \phi_{j_k}(\sum_{\ell<\omega} b_\ell) = \phi_{j_k}(\sum_{\ell<k} b_\ell) + \phi_{j_k}(b_k) + \phi_{j_k}(\sum_{k<\ell<\omega} b_\ell),$$

and observe that in the last sum the first term is 0, and the third term is contained in $m_{k+1}C_{j_k}$, but the second term is not. Since ϕa has a finite support in C, this equation can hold only for a finite number of indices k—an obvious contradiction. $\square$

We state the following theorem for p-groups that involves transfinite heights; its proof runs parallel to the preceding one.

Theorem 6.3 (Zimmermann-Huisgen [1]). *Let* $A = \prod_{i<\omega} A_i$ *be a countable product of* p*-groups, and* $\phi : A \to C = \oplus_{j\in J} C_j$ *a homomorphism into the direct sum of reduced* p*-groups* C_j*. Given a limit ordinal* τ*, there exist an integer* $k < \omega$*, an ordinal* $\sigma < \tau$*, and a finite subset* $J_0 \subseteq J$ *such that*

$$\phi(p^\sigma \prod_{k\le i<\omega} A_i) \le (\oplus_{j\in J_0} C_j) + (\oplus_{j\in J} p^\tau C_j). \qquad \square$$

The Measurable Cardinal Phenomenon If we wish to extend the preceding results to uncountable direct products, then we are confronted with an unusual phenomenon. There is a natural boundary to the extension: the first measurable cardinal. The reader who wishes to avoid the following delicate set-theoretical arguments can safely assume that there are no such cardinals in our model of ZFC, and jump to Theorem 6.5.

Recall that a cardinal κ is **measurable** if a set X of cardinality κ admits a countably additive measure μ such that μ assumes only two values: 0 and 1, and satisfies $\mu(X) = 1$, while $\mu(\{x\}) = 0$ for all $x \in X$. Here 'countably additive' means that if X_i $(i < \omega)$ are pairwise disjoint subsets of X, then $\mu(\cup_{i<\omega} X_i) = \sum_{i<\omega} \mu(X_i)$.

Let f be a function $\mathbf{B} \to G$ where $\mathbf{B} = \mathbf{2}^X$ is the Boolean lattice of all subsets of a set X, and $G \neq 0$ is a group. We will say f is a **G-valued measure on X** if it satisfies the following conditions:

(i) $f(\{x\}) = 0$ for every singleton $\{x\} \in \mathbf{B}$;
(ii) if $V \subset U$ are subsets of X, then $f(U) = 0$ implies $f(V) = 0$;
(iii) if U, V are disjoint subsets of X, then $f(U \cup V) = f(U) + f(V)$;
(iv) if U_i $(i < \omega)$ are pairwise disjoint subsets of X, then there is $n \in \mathbb{N}$ such that $f(\cup_{i<\omega} U_i) = f(U_0) + \cdots + f(U_n)$ and $f(U_i) = 0$ for all $i > n$.

We call f *non-trivial* if $f(X) \neq 0$. The following striking argument is due to J. Łoś.

Lemma 6.4. *If a non-trivial group-valued measure exists on the subsets of the set X, then $|X|$ is a measurable cardinal.*

Proof. Assume $f : \mathbf{2}^X \to G$ is a non-trivial G-valued measure on X, $G \neq 0$ any group. We show that then there exists a non-trivial countably additive $\{0, 1\}$-valued measure on X.

Consider all subsets $U \subset X$ such that $f(U) = 0$. From (i)–(iv) we conclude that these U form a countably additive ideal $\mathbf{I}$ in the Boolean lattice $\mathbf{B}$ of all subsets of X. It is readily checked that f induces a countably additive G-valued measure $\bar{f}$ on the Boolean quotient $\mathbf{B}/\mathbf{I}$. Let $\bar{U}_0, \dots, \bar{U}_i, \dots$ be pairwise disjoint elements in $\mathbf{B}/\mathbf{I}$. We can find representatives $U_i \subseteq X$ of the $\bar{U}_i$ which are still pairwise disjoint. By condition (iv), $f(U_i) \neq 0$ can hold only for a finite set of indices i; in other words, $\mathbf{B}/\mathbf{I}$ is a finite Boolean lattice. Thus $\mathbf{B}/\mathbf{I}$ has but a finite number of atoms, and on them $\bar{f}$ is not 0. Hence we derive a $\{0, 1\}$-valued measure μ' on $\mathbf{B}/\mathbf{I}$ by selecting an atom in $\mathbf{B}/\mathbf{I}$ and define $\mu'(\bar{U})$ to be 1 or 0 according as $\bar{U}$ does or does not contain the selected atom. In the obvious manner, μ' gives rise to a $\{0, 1\}$-valued measure μ on $\mathbf{B}$, showing that the set X is measurable. □

It is remarkable that Theorems 6.2 and 6.3 generalize to larger products provided that the cardinality of the set of components is not measurable.

Theorem 6.5 (Dugas–Zimmermann-Huisgen [1]). *Let $A = \prod_{i \in I} A_i$ be a direct product, and $\phi : A \to C = \oplus_{j \in J} C_j$ a homomorphism where the C_j are reduced groups. If $|I|$ is not a measurable cardinal, then there are an integer $m \neq 0$ and finite subsets $I_0 \subseteq I, J_0 \subseteq J$ such that*

$$\phi(m \prod_{i \in I \setminus I_0} A_i) \leq (\oplus_{j \in J_0} C_j) + (\oplus_{j \in J} C_j^1).$$

Proof. Consider the set $\mathcal{S}$ of all subsets S of I such that for the product $A_S = \prod_{i\in S} A_i$ the statement of the theorem holds (i.e., if I is replaced by S). Evidently, if $S \in \mathcal{S}$, then all subsets of S also belong to $\mathcal{S}$. Furthermore, $\mathcal{S}$ is not only closed under finite unions (which is evident), but also under countable unions. In fact, if $S_k \in \mathcal{S}$ $(k < \omega)$ are pairwise disjoint subsets, then, for some $n < \omega$ we have $\cup_{n<k<\omega} S_k \in \mathcal{S}$—this follows by applying Theorem 6.2 to the countable product $\phi : \prod_{k<\omega}(\prod_{i\in S_k} A_i) \to C$.

Once this has been established, in order to complete the proof it suffices to repeat the arguments in Łoś' theorem to conclude that if the claim fails, then I must be measurable. □

Example 6.6. To show that the last theorem may indeed fail for a measurable index set I, let each A_i denote a copy of the Σ-cyclic p-group $B = \oplus_{k<\omega} \mathbb{Z}(p^k)$, and let $C = B$. To define $\phi : A \to B$, pick an $a = (\dots, a_i, \dots) \in A = \prod_{i\in I} A_i$. a has only countably many different coordinates (as elements of B), so the supports of the equal ones give rise to a countable partition of I into disjoint subsets, exactly one of which has measure 1, and the rest have measure 0. If $b \in B$ is the element for which the support is of measure 1, then we set $\phi(a) = b$. It is easy to see that this gives rise to a well-defined homomorphism. It violates the conclusion of Theorem 6.5: $m \operatorname{Im} \phi$ is not contained in any finite direct sum of cyclic groups in C, for any integer $m > 0$.

The proof of Theorem 6.5 also applies to verify:

Theorem 6.7 (Zimmermann-Huisgen [1]). *Let $A = \prod_{i\in I} A_i$ be a product of p-groups, and $\phi : A \to C = \oplus_{j\in J} C_j$ a homomorphism where the C_j are reduced p-groups. Given a limit ordinal ρ, if $|I| = \kappa$ is not measurable, then there exist an ordinal $\sigma < \rho$, as well as finite subsets $I_0 \subseteq I, J_0 \subseteq J$, such that*

$$\phi(p^\sigma \prod_{i\in I\setminus I_0} A_i) \leq (\oplus_{j\in J_0} C_j) + (\oplus_{j\in J}\, p^\rho C_j). \qquad \square$$

★ **Notes.** The peculiar behavior of homomorphisms from a countable direct product into an infinite direct sum was noticed by Chase [1]. The same phenomenon of larger direct products was observed by Dugas–Zimmermann-Huisgen [1] up to the first measurable cardinal (just as in the case of slender groups). By using $\aleph_1$-complete ultrafilters, Eda [1] gave a generalization to all cardinals; see Lemma 2.13 in Chapter 13.

Ivanov [1] proves various theorems on so-called **Fuchs-44 groups** with respect to a class $\mathfrak{A}$, which is closed under extensions, submodules, and direct products. G is such a group if for every $\phi : G \to \oplus_{i\in I} A_i$ with $A_i \in \mathfrak{A}$, there are $m \in \mathbb{N}$ and a finite subset $J \subset I$ such that $\phi(mG) \leq \oplus_{i\in J} A_i$.

Exercises

(1) The group $A = \prod_{k\in\mathbb{N}} \mathbb{Z}(p^k)$ has no unbounded Σ-cyclic p-group as an epimorphic image.

(2) Let $A = \mathbb{Z}^\kappa$ where κ is not measurable, and F a free group. Show that the image of every homomorphism $A \to F$ is finitely generated.
(3) (Keef) Let A_i $(i \in I)$ be an infinite set of unbounded separable p-groups. There is no epimorphism $\prod_{i \in I} A_i \to \oplus_{i \in I} A_i$.

7 Completeness in Linear Topologies

Groups that are complete in some linear topology are very special. Therefore, we examine completeness and the completion processes.

Linear Topologies Assume that a linear topology is defined on the group A in terms of a filter $\mathbf{u}$ in the lattice $\mathbf{L}(A)$ of subgroups of A. The subgroups $U \in \mathbf{u}$ form a base of open neighborhoods about 0; we label them by a directed index set I, so that $i \leq j$ for $i, j \in I$ means that $U_i \geq U_j$. Thus I as a (directed) poset is dual-isomorphic to a subset of $\mathbf{u}$ (which has the natural order relation by inclusion).

By a **net** in A we mean a set $\{a_i\}_{i \in I}$ of elements in A, indexed always by I. A net is said to **converge** to a **limit** $a \in A$ if to every $i \in I$ there is a $j \in I$ such that

$$a_k - a \in U_i \qquad \text{for all } k \geq j.$$

If A is Hausdorff in the topology, then limits are unique; if, however, A fails to be Hausdorff, then limits are determined only up to mod $\cap_i U_i$. The classical proof applies to show that a subgroup B of A is closed in the topology if and only if it contains the limits of convergent nets whose elements belong to B.

A net $\{a_i\}_{i \in I}$ is a **Cauchy net** if to any given $i \in I$, there is a $j \in I$ such that

$$a_k - a_\ell \in U_i \qquad \text{whenever } \ k, \ell \geq j.$$

Since the U_i are subgroups, $a_k - a_j, a_\ell - a_j \in U_i$ implies $a_k - a_\ell \in U_i$, for the Cauchy character of a net it suffices to require that $a_k - a_j \in U_i$ for all $k \geq j$. Clearly, cofinal subnets of a Cauchy net are again Cauchy nets, and such a subnet converges if and only if the larger net also converges; moreover, the limits are then the same. To facilitate discussion and to simplify notation, we shall concentrate without loss of generality on Cauchy nets $\{b_i\}_{i \in I}$ which are **neat** in the sense that, for every $i \in I$, $b_k - b_i \in U_i$ holds for $k \geq i$ (i.e., $j = i$ can be chosen). If a neat Cauchy net $\{b_i\}_{i \in I}$ converges to a limit $b \in A$, then it converges **neatly**: $b_k - b \in U_i$ for all $k \geq i$. In a group whose topology satisfies the first axiom of countability, Cauchy sequences $\{a_n \mid n < \omega\}$ satisfying $a_{n+1} - a_n \in U_n$ for all $n \in \mathbb{N}$ are neat.

Topological Completeness A group A is said to be **complete** in a topology if it is Hausdorff, and every (neat) Cauchy net in A has a limit in A. Observe that we mean by complete groups only Hausdorff groups.

Lemma 7.1. *A subgroup of a complete group is closed if and only if it is complete in the induced topology.*

Proof. Let G be a subgroup in the complete group C. First assume G is closed in C, and $\{g_i\}_{i\in I}$ is a Cauchy net in G (in the inherited topology). The net is Cauchy in C too, so it converges to a limit $c \in C$ which must be in G, since G is closed. Thus G is complete. Conversely, suppose G is complete in the induced topology, and $c \in C$ is the limit of a Cauchy net $\{g_i\}_{i\in I}$ with $g_i \in G$. It is a Cauchy net in G as well, so has a limit in G, which cannot be anything else than c. Thus G is closed in C. $\square$

In the next result the countability hypothesis is essential.

Lemma 7.2. *Let B be a closed subgroup of a complete group A that satisfies the first axiom of countability. Then the factor group A/B is complete in the induced topology.*

Proof. Since B is closed, A/B is Hausdorff. Consider a base of neighborhoods about 0 in A such that $U_1 \geq \dots \geq U_m \geq \dots$ with $\cap_{m\in\mathbb{N}} U_m = 0$. Let $\{a_m + B \mid m \in \mathbb{N}\}$ be a Cauchy sequence in A/B; without loss of generality, we assume that it is neat, i.e. $a_{m+1} - a_m + B \subseteq U_m + B$. We want to lift this Cauchy sequence to a Cauchy sequence $\{c_m \mid m \in \mathbb{N}\}$ in A. Let $c_1 = a_1$, and assume that $c_1, \dots, c_m \in A$ have already been chosen such that $c_i \in a_i + B$ and $c_i - c_{i-1} \in U_{i-1}$ for $i = 2, \dots, m$. Then $a_{m+1} - c_m = u_m + b_m$ for some $u_m \in U_m$, $b_m \in B$, and set $c_{m+1} = a_{m+1} - b_m \in a_{m+1} + B$ to have $c_{m+1} - c_m \in U_m$. If $\lim\{c_m\} = a \in A$, then $a + B$ is the limit of the sequence $\{a_m + B \mid m \in \mathbb{N}\}$ in A/B. $\square$

Recall that if $\{A_j \mid j \in J\}$ is a family of groups, each equipped with a linear topology, say, defined by the filter $\mathbf{u}_j$ in $\mathbf{L}(A_j)$, then the direct product $A^* = \prod_{j\in J} A_j$ is given the **product (Tychonoff) topology**: a subbase of neighborhoods of 0 consists of the subgroups $\pi_j^{-1} U_{ji}$ where $\pi_j : A^* \to A_j$ is the jth coordinate projection, and $U_{ji} \in \mathbf{u}_j$. The product topology is again linear, and the π_j are continuous, open homomorphisms. The direct sum $A = \oplus_{j\in J} A_j$ is a dense subgroup of A^*.

We should also mention the box topology on the direct product; this topology is used, e.g., when the components are viewed in the $\mathbb{Z}$-adic topology, and we want to have this topology on their direct product. We now assume that the same poset I serves to index a base of neighborhoods about 0 in each A_j. If $\{U_{ji} \leq A_j \mid i \in I\}$ is a base in the topology of A_j (where $U_{ji} \leq U_{jk}$ whenever $k \leq i$ in I), then the **box topology** on $A^* = \prod_{j\in J} A_j$ is defined to have the subgroups

$$U_i = \prod_{j\in J} U_{ji} \qquad (i \in I)$$

as a base of neighborhoods about 0. The box topology on A^* satisfies the first countability hypothesis if all the A_j do. The inclusion $U_i \leq \pi_j^{-1} U_{ji}$ for all j shows that the box topology is finer than the product topology. Hence the projections π_j are continuous in the box topology as well.

Example 7.3. Actually, there are several methods of furnishing a direct product with a linear topology. E.g., let $G = \prod_{j\in I} A_j$ be a product, and $\mathcal{F}$ a filter on the index set I. For each $X \in \mathcal{F}$ we form the subgroup

$$V_X = \{g = (\dots, a_j, \dots) \in \prod_{j\in I} A_j \mid n(g) \in X\}$$

(where $n(g) = \{j \in I \mid a_j = 0\}$ denotes the null-set of g), and declare the subgroups V_X $(X \in \mathcal{F})$ as a base of neighborhoods about 0. This linear topology is Hausdorff if and only if $\mathcal{F}$ is a **free filter**, i.e. $\cap_{X\in\mathcal{F}} X = \varnothing$.

Example 7.4. Choose the filter of subsets of I with finite complements. Then the topology defined in Example 7.3 is the product topology. If A_i $(i \in I)$ are non-trivial groups in the discrete topology, then $\prod_i A_i$ is complete in the product topology. The direct sum $\oplus_i A_i$ is dense in the direct product.

Completions The rest of this section is devoted to the completion of groups in linear topologies. There are two important completion processes: one is *via* Cauchy nets, and another is by using inverse limits. We will employ the second method which fits better to linear topologies.

Let A be a group with linear topology (not necessarily Hausdorff), and $\{U_i \mid i \in I\}$ a base of neighborhoods of 0, with I a directed index set: $i \leq j$ in I if and only if $U_i \geq U_j$. Define the groups $C_i = A/U_i$, and for $j \geq i$ in I, the homomorphisms $\pi_i^j : C_j \to C_i$ via $\pi_i^j : a + U_j \mapsto a + U_i$. The limit of the arising inverse system $\mathfrak{C} = \{C_i\ (i \in I); \pi_i^j\}$ will be denoted by $\breve{A}$: it is furnished with the topology inherited from the product topology of $\prod C_i$. Thus, if π_i denotes the ith projection $\prod C_i \to C_i$, then a subbase of neighborhoods of 0 in $\breve{A}$ is given by the subgroups $\breve{U}_i = \breve{A} \cap \pi_i^{-1}0$. Evidently, $\theta_A : a \mapsto (\dots, a + U_i, \dots) \in \breve{A}$ is a homomorphism $A \to \breve{A}$ which is continuous and open, and $\theta_A U_i = \theta_A A \cap \breve{U}_i$ holds for each $i \in I$. It is clear that $\operatorname{Ker}\theta_A$ is the intersection of all U_i.

Lemma 7.5. *For every group A with a linear topology, the group $\breve{A}$ is complete in the induced topology, and the image of the map $\theta : A \to \breve{A}$ is a dense subgroup of $\breve{A}$.*

Proof. Let $\breve{a} = (\dots, a_i + U_i, \dots) \in \breve{A}$, and let $\breve{U}_i \subset \breve{A}$ be an open set. As $\theta_A a_i$ lies in the $\breve{U}_i$-neighborhood of $\breve{a}$, $\theta_A A$ is dense in $\breve{A}$. Therefore, to prove completeness, we need only verify the convergence of Cauchy nets in $\theta_A A$ to elements of $\breve{A}$. A neat Cauchy net in $\theta_A A$ is the image of a neat Cauchy net $\{b_i\}_{i\in I}$ in A. We claim that $\breve{b} = (\dots, b_i + U_i, \dots)$ is the limit of $\{\theta_A b_i\}_{i\in I}$. First, $\breve{b} \in \breve{A}$, since $\pi_i^j(b_j + U_j) = b_j + U_i = b_i + U_i$ for $j \geq i$. Secondly, the ith coordinate of $\theta_A b_i - \breve{b}$ is 0, so it belongs to the open set $\breve{U}_i$. $\square$

Observe that the completion is always Hausdorff, and $\theta_A : A \to \breve{A}$ is monic if and only if A had a Hausdorff topology to start with.

Lemma 7.6. *If ϕ is a continuous homomorphism of the group A into a complete group C, then there is a unique continuous homomorphism $\breve{\phi} : \breve{A} \to C$ such that $\breve{\phi}\theta_A = \phi$.*

Proof. Let $\{a_i \mid i \in I\}$ be a Cauchy net in A converging to the element $\breve{a} \in \breve{A}$. Continuity implies that $\{\phi a_i \mid i \in I\}$ is a Cauchy net in C. If $c \in C$ is its limit, then the only possible way of defining a continuous $\breve{\phi}$ is to let $\breve{\phi} : \breve{a} \mapsto c$. The rest of the claim is straightforward. □

From this lemma it also follows that the completion $\breve{A}$ of A is unique up to topological isomorphism. Also, $\theta_A : A \to \breve{A}$ is a natural map, for if $\phi : A \to C$ is a continuous homomorphism, then the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\theta_A"'] & C \arrow[d, "\theta_C"] \\ \breve{A} \arrow[r, "\breve{\phi}"] & \breve{C} \end{tikzcd}$$

commutes where $\breve{\phi}$ is the map whose existence was established in Lemma 7.6.

Our main interest lies in the $\mathbb{Z}$-adic topology, and in completions in that topology. Therefore, if we say that '*a group is complete*,' then we always mean completeness in the $\mathbb{Z}$-adic or p-adic topology (whichever is obvious), unless stated otherwise. Furthermore, we shall use the special notation $\tilde{A}$ for the completion of A in the $\mathbb{Z}$-adic topology.

In the next theorem we refer to linear compactness; see Sect. 3 in Chapter 6.

Theorem 7.7. *Let A be any group.*

(i) *Its completion in the $\mathbb{Z}$-adic (p-adic) topology carries the $\mathbb{Z}$-adic (p-adic) topology.*
(ii) *Its completion in the finite index topology has a compact topology.*
(iii) *Its completion in the Prüfer topology carries a linearly compact topology.*

Proof. (i) Let $\tilde{A} = \varprojlim_{n\in\mathbb{N}} A/nA$, or, equivalently, $\tilde{A} = \varprojlim_{n\in\mathbb{N}} A/n!A$ whenever we consider the collection of subgroups $U_n = n!A$ ($n \in \mathbb{N}$) as a decreasing sequence of neighborhoods about 0. The elements in the induced $\tilde{U}_n$ have nth coordinates 0, and it is easy to see that the conditions on the coordinates of elements on $\tilde{A}$ imply that all the ith coordinates in $\tilde{U}_n$ are 0 for $i < n$, while all those for $i > n$ are divisible by $n!$. This means that $\tilde{U}_n = n!\tilde{A}$.

(ii) In the finite index topology, the groups A/U_i are finite, so they are compact. Thus the product $\prod_i A/U_i$ is compact, and the inverse limit $\breve{A} = \varprojlim_{i\in I} A/U_i$ as a closed subgroup is also compact.

(iii) The proof is similar to the one in (ii), using the linear compactness of A/U_i in the Prüfer case.

Example 7.8. Let $A = \oplus_{n<\omega} A_n$ be furnished with the topology where the subgroups $U_k = \oplus_{k\le n<\omega} A_n$ form a base of neighborhoods about 0. The completion of A in this topology is the direct product $\prod_{n<\omega} A_n$.

If the topology fails to satisfy the first axiom of countability, then completeness may occur in an unexpected situation. This is demonstrated by the following example where, for a limit ordinal λ, the p^λ-topology of a p-group A is defined by declaring the subgroups $p^\sigma A$ $(\sigma < \lambda)$ as a base of neighborhoods of 0.

Example 7.9. Suppose λ is a limit ordinal not cofinal with ω, and let A_σ $(\sigma < \lambda)$ be p-groups such that A_σ has length σ. Then the A_σ are discrete (and hence complete) in the p^λ-topology. Consequently, $A^* = \prod_{\sigma<\lambda} A_\sigma$ is complete in the p^λ-topology which is now the box topology on A^* (cf. Exercise 2).

Strangely enough, $A = \oplus_{\sigma<\lambda} A_\sigma$ is complete in the p^λ-topology. To prove this, we show that A is closed in A^*. Assume the contrary, i.e. there is $x = (\dots, a_\sigma, \dots) \in A^* \setminus A$ in the closure of A. We can find a sequence $\sigma_1 < \dots < \sigma_n < \dots$ of ordinals with $a_{\sigma_n} \neq 0$. Let $\sup \sigma_n = \sigma' < \lambda$ and $y \in A$ such that $x - y \in p^{\sigma'} \prod_{\sigma<\lambda} A_\sigma$. Then x and y have equal coordinates in every A_ρ with $\rho < \sigma'$ which contradicts the fact that x has infinitely many and y only finitely many non-zero coordinates for $\rho < \sigma'$.

$\mathbb{Z}$-adic Completeness Direct products of complete groups are complete in the product topology. We wish to point out the following result on the $\mathbb{Z}$-adic topology.

Lemma 7.10. *A direct product is complete in the $\mathbb{Z}$-adic topology if and only if every component is complete in its $\mathbb{Z}$-adic topology.*

Proof. Summands inherit $\mathbb{Z}$-adic topology and completeness. Conversely, assume every A_j in $G = \prod_{j\in J} A_j$ is $\mathbb{Z}$-adically complete and $\{g_i \mid i \in I\}$ is a neat Cauchy net in G. Then $\{\pi_j g_i \mid i \in I\}$ is a neat Cauchy net in A_j, and if $a_j \in A_j$ is the limit of this net, then $g \in G$ with $\pi_j g = a_j$ is the limit of $\{g_i \mid i \in I\}$. $\square$

★ **Notes.** While the completion in the Prüfer topology may be viewed as a 'linear compactification,' completion in the finite index topology is not at all compactification. The latter process kills the first Ulm subgroup of the group, so it is an embedding only for groups that are Hausdorff in the finite index topology. A genuine 'compactification' can be accomplished by the so-called Stone compactification. This is the process of embedding A in the group $\mathrm{Hom}(\mathrm{Hom}(A, \mathbb{T}), \mathbb{T})$, where $\mathbb{T}$ denotes the circle group $\mathbb{R}/\mathbb{Z}$ (the inner Hom is furnished with the discrete topology, and the outer with the compact-open topology).

Exercises

(1) (a) The completions of the groups A and $A/\cap U_i$ are the same.
 (b) A and A/A^1 have the same $\mathbb{Z}$-adic completion.
(2) A direct product is complete in the box topology if and only if every component is complete.
(3) Every compact (linearly compact) group is complete in its topology.
(4) The direct product of discrete groups is Hausdorff and complete in every **u**-topology where **u** is a free filter.
(5) The inverse limit of complete groups is complete. (Careful with the topology.)
(6) Compare the completions of a group in the finite index and in the Prüfer topologies.

Problems to Chapter 2

PROBLEM 2.1 (J. Dauns). Suppose A has the property that every summand B of A has a decomposition $B = B_1 \oplus B_2$ with $B_1 \cong B_2$. Is then $A \cong A \oplus A$?

PROBLEM 2.2. Study the Boolean powers $A^{(\mathbf{B})}$ of a group A.

Cf. Balcerzyk [3], and especially, Eda [2].

PROBLEM 2.3. Represent a p-group A as a direct limit $A \cong \varinjlim_n A[p^n]$. How does the structure of A change if the connecting monomorphisms $A[p^n] \to A[p^{n+1}]$ are modified?

PROBLEM 2.4. Suppose $\phi : A = \prod A_i \to C = \prod^{<\aleph_1} C_j$ is a homomorphism of a product into an $\aleph_1$-product. Can we say something about where the image must be contained (like Theorem 6.5)?