Chapter 16
Endomorphism Rings

Abstract With an abelian group A one associates the ring End A of its endomorphisms. This is an associative ring with 1 which frequently reflects several relevant features of the group. Information about direct decompositions is certainly stored in this ring. It is quite challenging to unveil hidden relations between a group and its endomorphism ring.

Quite a lot of information is available for the endomorphism rings of p-groups. A celebrated theorem by Baer and Kaplansky shows that torsion groups with isomorphic endomorphism rings ought to be isomorphic. Moreover, the endomorphism rings of separable p-groups can be characterized ring-theoretically. However, in the torsion-free case, we can offer nothing anywhere near as informative or complete as for torsion groups. As a matter of fact, on one hand, there exists a large variety of non-isomorphic torsion-free groups (even of finite rank) with isomorphic endomorphism rings, and on the other hand, endomorphism rings of torsion-free groups seem to be quite general: every countable rank reduced torsion-free ring with identity appears as an endomorphism ring of a torsion-free group. The situation is not much better even if we involve the finite topology. The mixed case is of course more difficult to survey.

Though the problem concerning the relations between group and ring properties has attracted much attention, our current knowledge is still far from being satisfactory. The main obstacle to developing a feasible in-depth theory is probably the lack of correspondence between relevant group and relevant ring properties. However, there is a great variety of examples of groups with interesting endomorphism rings, and we will list a few which we think are more interesting. In some cases we have to be satisfied with just stating the results in order to avoid tiresome ring-theoretical arguments. In some proofs, however, we had no choice but to refer to results on rings which can be found in most textbooks on graduate algebra. There are very good surveys on endomorphism rings by Russian algebraists, e.g. Krylov–Tuganbaev [1], and especially, the book by Krylov–Mikhalev–Tuganbaev [KMT].

In this chapter, we require some, but no more than a reasonable acquaintance with standard facts on associative rings.

1 Endomorphism Rings

Rings of Endomorphisms It is a familiar fact that the endomorphisms $\alpha, \beta, \dots$ of an abelian group A form a ring under the addition and multiplication of homomorphisms:

$$(\alpha + \beta)a = \alpha a + \beta a \quad \text{and} \quad (\alpha\beta)a = \alpha(\beta a) \quad (\forall a \in A).$$

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The **endomorphism ring** End A of A is an associative ring with identity. By abuse of notation, we will use the same symbol for the endomorphism ring and for its additive group: the **endomorphism group**.

Example 1.1.

(a) If $A = \langle e \rangle \cong \mathbb{Z}$, then every $\alpha \in \operatorname{End} A$ is completely determined by αe. Since e is a free generator, every correspondence $e \mapsto ne$ for any $n \in \mathbb{Z}$ extends to an endomorphism. The operations with endomorphisms are like with integers, so $\operatorname{End} A \cong \mathbb{Z}$ (ring isomorphism).
(b) A similar argument shows that if A is a cyclic group of order m, then we have $\operatorname{End} A \cong \mathbb{Z}/m\mathbb{Z}$ (as rings).

Example 1.2.

(a) From Example 1.4 in Chapter 7 we obtain the isomorphism $\operatorname{End}(\mathbb{Z}(p^\infty)) \cong J_p$.
(b) $\operatorname{End}(\mathbb{Q}/\mathbb{Z}) \cong \prod_p J_p = \tilde{\mathbb{Z}}$. This follows from (a).

Example 1.3. Example 1.6 in Chapter 7 shows that $\operatorname{End} J_p \cong J_p$ for every prime p.

Example 1.4. Let R denote a rational group, $1 \in R$. Here again, an endomorphism α is fully determined by $\alpha 1 = r \in R$, so α is simply a multiplication by the rational number $r \in R$. As endomorphisms respect divisibility, $\dot{r}$ can be an endomorphism of R only if every prime factor p of the denominator of r (in its reduced form) satisfies $pR = R$. That the converse is also true is seen immediately. Thus $\operatorname{End} R$ is a subring of $\mathbb{Q}$ whose type is the largest idempotent type $\leq \mathbf{t}(R)$, i.e. $\mathbf{t}(R) : \mathbf{t}(R)$.

We continue with a few elementary observations.

(A) *A group isomorphism $\phi : A \to C$ induces a ring isomorphism $\phi^* :$ End $A \to$ End C via $\phi^* : \alpha \mapsto \phi\alpha\phi^{-1}$.*
(B) *Suppose $A = B \oplus C$ with $\epsilon : A \to B$ the projection map. Then the identification* $\operatorname{End} B = \epsilon(\operatorname{End} A)\epsilon$ *can be made.* Indeed, for $\alpha \in \operatorname{End} A$, $\epsilon\alpha\epsilon \in \operatorname{End} B$, while if $\beta \in \operatorname{End} B$, then $\beta = \epsilon\beta\epsilon$ may be regarded as an element in $\operatorname{End} A$.
(C) *If ϵ is a central idempotent in* End A*, then ϵA is a fully invariant summand of A.*
(D) *If $A = A_1 \oplus \cdots \oplus A_n$ is a direct decomposition with fully invariant summands, then*

$$\operatorname{End} A \cong \operatorname{End} A_1 \oplus \cdots \oplus \operatorname{End} A_n.$$

If the summands are not fully invariant, then we get only a matrix representation, see Proposition 1.14.

An idempotent $\epsilon \neq 0$ is said to be **primitive** if it cannot be written as a sum of two non-zero orthogonal idempotents. The following claim is rather obvious in view of (D).

(E) *For an idempotent $\epsilon \neq 0$ of* End A*, the summand ϵA is indecomposable if and only if ϵ is a primitive idempotent.*

We have already made frequent use of the fact that direct decompositions correspond to idempotent endomorphisms. This interplay between direct decompositions and endomorphisms is constantly used. At this point, we insert the following lemma that will be indispensable throughout.

Lemma 1.5. *There is a bijection between the finite direct decompositions* $A = A_1 \oplus \cdots \oplus A_n$ *of a group* A*, and decompositions of* End A *into finite direct sums of left ideals,*

$$\mathrm{End}\, A = \mathsf{L}_1 \oplus \cdots \oplus \mathsf{L}_n. \tag{16.1}$$

If $A_i = \epsilon_i A$ *with pairwise orthogonal idempotents* ϵ_i*, then* $\mathsf{L}_i = (\mathrm{End}\, A)\epsilon_i$ *for* $i = 1, \ldots, n$.

Proof. Suppose $A = \epsilon_1 A \oplus \cdots \oplus \epsilon_n A$ with mutually orthogonal idempotents ϵ_i. The well-known Peirce decomposition of End A yields $\mathrm{End}\, A = (\mathrm{End}\, A)\epsilon_1 \oplus \cdots \oplus (\mathrm{End}\, A)\epsilon_n$. Conversely, if (16.1) holds with left ideals L_i of End A, then—as is readily checked—we have $\mathsf{L}_i = (\mathrm{End}\, A)\epsilon_i$ where ϵ_i is the ith coordinate of the identity of End A. These ϵ_i are orthogonal idempotents with sum 1, hence $A = \epsilon_1 A \oplus \cdots \oplus \epsilon_n A$ follows at once. It is pretty clear that the indicated correspondence is a bijection. □

As far as isomorphic summands are concerned, the basic information is recorded in the next lemma.

Lemma 1.6. *Suppose that* B_1, B_2 *are summands of* A*, corresponding to idempotents* $\epsilon_1, \epsilon_2 \in \mathsf{E} = \mathrm{End}\, A$*. The following are equivalent:*

(i) $B_1 \cong B_2$;
(ii) *there exist* $\alpha, \beta \in \mathsf{E}$ *such that* $\alpha = \epsilon_1\alpha\epsilon_2, \beta = \epsilon_2\beta\epsilon_1$*, and* $\alpha\beta = \epsilon_1$*,* $\beta\alpha = \epsilon_2$;
(iii) $\mathsf{E}\epsilon_1 \cong \mathsf{E}\epsilon_2$ *as left* E*-modules;*
(iv) $\epsilon_1\mathsf{E} \cong \epsilon_2\mathsf{E}$ *as right* E*-modules.*

Proof.

(i) ⇒ (iii) Let $\gamma : B_1 \to B_2$ and $\delta : B_2 \to B_1$ be inverse isomorphisms. Thus $\gamma\delta = \epsilon_2$ and $\delta\epsilon_2 = \delta$ imply that $\mathsf{E}\delta = \mathsf{E}\epsilon_2$, because each of δ and ϵ_2 is contained in the left E-ideal generated by the other. Multiplication by δ and γ yield $\mathsf{E}\epsilon_1 \to \mathsf{E}\epsilon_1\delta \to \mathsf{E}\delta = \mathsf{E}\epsilon_2 \to \mathsf{E}\delta\gamma = \mathsf{E}\epsilon_1$. The composite is multiplication by ϵ_1 which is the identity on $\mathsf{E}\epsilon_1$, so multiplication by γ is an isomorphism.

(iii) ⇔ (ii) Choose an isomorphism $\phi : \mathsf{E}\epsilon_1 \to \mathsf{E}\epsilon_2$, and let $\phi(\epsilon_1) = \alpha$, $\phi^{-1}(\epsilon_2) = \beta$ with $\alpha, \beta \in \mathsf{E}$. Then $\alpha = \epsilon_1\alpha\epsilon_2, \beta = \epsilon_2\beta\epsilon_1$, and also $\phi^{-1}\phi : \epsilon_1 \mapsto \alpha = \alpha\epsilon_2 \mapsto \alpha\beta$, thus $\alpha\beta = \epsilon_1$; similarly, $\beta\alpha = \epsilon_2$. Conversely, if $\alpha, \beta \in \mathsf{E}$ are as stated, then $\mathsf{E}\alpha = \mathsf{E}\epsilon_2$. Thus $\epsilon_1 \mapsto \alpha$ induces an epimorphism $\phi : \mathsf{E}\epsilon_1 \to \mathsf{E}\epsilon_2$. ϕ is an isomorphism, because if $\phi(\rho\epsilon_1) = \rho\alpha = 0$ for some $\rho \in \mathsf{E}$, then also $\rho\epsilon_1 = \rho\alpha\beta = 0$.

(ii) ⇔ (iv) Condition (ii) is left-right symmetric, so by the last proof (ii) is also equivalent to $\epsilon_1\mathsf{E} \cong \epsilon_2\mathsf{E}$.

(iv) ⇒ (i) Let $\phi : \epsilon_1\mathsf{E} \to \epsilon_2\mathsf{E}$ be an isomorphism. Define a map $B_1 \to B_2$ as $\epsilon_1(a) \to \phi(\epsilon_1)(a)$ $(a \in A)$. It has the obvious inverse, so it has to be an isomorphism. □

In the next result we are referring to the completion $\tilde{A}$ and the cotorsion hull $A^\bullet$ of a group A.

Proposition 1.7.

(i) *Let A be a group with $A^1 = 0$ (i.e., Hausdorff in the $\mathbb{Z}$-adic topology). For every $\eta \in \operatorname{End} A$, there is a unique $\tilde{\eta} \in \operatorname{End} \tilde{A}$ such that $\tilde{\eta} \restriction A = \eta$.*
(ii) *For a reduced torsion group T, there is a natural isomorphism $\operatorname{End} T \cong \operatorname{End} T^\bullet$ between the endomorphism rings.*

Proof.

(i) The hypothesis $A^1 = 0$ allows us to regard A as a pure subgroup of $\tilde{A}$. By the pure-injectivity of $\tilde{A}$, η can be extended to an $\tilde{\eta} : \tilde{A} \to \tilde{A}$ which must be unique in view of the density of A in $\tilde{A}$.
(ii) The exact sequence $0 \to T \to T^\bullet \to D \to 0$ with torsion-free divisible D induces the exact sequence $0 = \operatorname{Hom}(D, T^\bullet) \to \operatorname{Hom}(T^\bullet, T^\bullet) \to \operatorname{Hom}(T, T^\bullet) \to \operatorname{Ext}(D, T^\bullet) = 0$. Hence and from the obvious $\operatorname{Hom}(T, T^\bullet) \cong \operatorname{Hom}(T, T)$ the claim is evident. □

Inessential Endomorphisms When studying the endomorphisms of reduced p-groups, we always have to deal with endomorphisms to and from cyclic summands. These as well as the small endomorphisms are 'inessential' endomorphisms: they do not reveal much about the group structure. The relevant information about the group encoded in the endomorphism ring is actually in the other endomorphisms. Also, certain torsion-free groups (like separable groups) admit endomorphisms that provide hardly any information about group. The idea of formalizing this phenomenon is due to Corner–Göbel [1]. We define this concept in the local case.

Let A be a reduced p-local group with $A^1 = 0$, and B a basic subgroup of A. Evidently, $B \leq A \leq \tilde{B}$, and every $\eta \in \operatorname{End} A$ extends uniquely to an $\tilde{\eta} \in \operatorname{End} \tilde{B}$. Now $\eta \in \operatorname{End} A$ is called **inessential** if $\tilde{\eta}(\tilde{B}) \leq A$. The inessential endomorphisms form an ideal of $\operatorname{End} A$, denoted Ines A.

Example 1.8. Let $A \cong (J_p)^{(\mathbb{N})}$. Thus $B \cong (\mathbb{Z}_{(p)})^{(\mathbb{N})}$ and $\tilde{B} < (J_p)^{\mathbb{N}}$. In this case, Ines A consists of those $\eta \in \operatorname{End} A$ for which $\operatorname{Im} \eta$ is contained in a finite direct sum of the J_p.

Example 1.9. Let A be a separable p-group, and B its basic subgroup. Now the ideal Ines A coincides with set of the small endomorphisms (Sect. 3 in Chapter 7).

If $\eta \in \operatorname{End}_s A$, then for every $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $\eta(p^n A[p^k]) = 0$. Then also $\tilde{\eta}(p^n \tilde{B}[p^k]) = 0$, thus $\tilde{\eta}$ is also small. By Sect. 3(E) in Chapter 7, $B + \operatorname{Ker} \tilde{\eta} = \tilde{B}$, so $\tilde{\eta}(\tilde{B}) = \tilde{\eta}(B) = \eta(B) \leq A$. Thus $\operatorname{End}_s A \leq$ Ines A.

Suppose η is not small, i.e. for some $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$, $\eta(p^n A[p^k]) \neq 0$. We can select independent elements $a_i \in A$ of orders $\leq p^k$ and of increasing heights n_i such that $\eta a_i \neq 0$. We may, in addition, assume that the elements $\eta(a_i)$ are also independent. Then the sum $c = \sum_{i<\omega} a_i$ converges in $\tilde{B}$, but $\tilde{\eta}(c) \neq 0$. Consequently, Ines $A \leq \operatorname{End}_s A$, and so Ines $A = \operatorname{End}_s A$.

Annihilator Ideals in End A There is a fundamental correspondence between certain subgroups of a group, and certain left ideals of its endomorphism ring.

For a subgroup G of the group A, we set

$$G^{\perp} = \{\eta \in \operatorname{End} A \mid \eta(g) = 0 \ \forall g \in G\},$$

and for a left ideal L of End A, we define

$$\mathsf{L}^{\perp} = \{a \in A \mid \eta(a) = 0 \ \forall \eta \in \mathsf{L}\}.$$

Evidently, $G^{\perp}$ is a left ideal in End A, called **annihilator left ideal**, and $\mathsf{L}^{\perp}$ is a subgroup of A, called **kernel subgroup**.

(a) For every subgroup $G \leq A$, we have $G^{\perp\perp\perp} = G^{\perp}$, and $\mathsf{L}^{\perp\perp\perp} = \mathsf{L}^{\perp}$ for every left ideal L in End A.
(b) A subgroup $G \leq A$ is a kernel subgroup exactly if $G^{\perp\perp} = G$, and a left ideal L is an annihilator left ideal if and only if $\mathsf{L}^{\perp\perp} = \mathsf{L}$.
(c) The correspondences $G \mapsto G^{\perp}$ and $\mathsf{L} \mapsto \mathsf{L}^{\perp}$ are inclusion reversing inverse maps between the set of kernel subgroups of A and the set of annihilator left ideals of End A.

Example 1.10.

(a) Consider the group $\mathbb{Z}(p^{\infty})$ and its endomorphism ring J_p. The annihilator ideals are $p^n J_p$, and $\mathbb{Z}(p^n)(\leq \mathbb{Z}(p^{\infty}))$ are the kernel subgroups.
(b) Let $A \cong \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$, an n-dimensional $\mathbb{Q}$-vector space. Then $\mathsf{E} = \operatorname{End} A$ is isomorphic to the $n \times n$-matrix ring over $\mathbb{Q}$. Every subspace is a kernel subgroup and every left ideal an annihilator.

Finite Topology Matrix representation of linear transformations is an important issue in linear algebra. For groups it is possible to establish a similar, though much less informative, representation of endomorphisms of direct sums. Since we do not wish to restrict our study to finite direct decompositions, we need to introduce a natural topology in endomorphism rings.

Actually, endomorphism rings admit various topologies defined in terms of the underlying groups. They play an increasingly important role in the study of groups (also modules, rings, etc.). We will discuss the most significant topology, the so-called **finite topology** that was introduced into the theory of abelian groups by Szele [8].

The terminology comes from the fact that the open neighborhoods are defined in terms of finite subsets of the group: every finite subset X in A defines an open set about $\alpha \in \operatorname{End} A$, viz.

$$U_X(\alpha) = \{\eta \in \operatorname{End} A \mid \eta x = \alpha x \ \ \forall x \in X\}.$$

It is clear that $U_X(\alpha) = \cap_{x \in X} U_x(\alpha)$; also, $U_X(\alpha) = \alpha + U_X(0)$ for each $\alpha \in \operatorname{End} A$. Thus, the finite topology can more conveniently be defined with the aid of a subbase of neighborhoods of 0, by the open sets $U_X = \{\eta \in \operatorname{End} A \mid \eta X = 0\}$, taken for all finite subsets $X \leq A$. The finite topology is evidently Hausdorff, and moreover, it is linear: the open sets U_X are left ideals in End A. As a consequence, the continuity of the addition in End A is immediate. Moreover, we can state:

Theorem 1.11. *The endomorphism ring* End$A$ *of a group* A *is a complete topological ring in the finite topology.*

Proof. To prove that ring multiplication is continuous, let $\alpha, \beta \in \operatorname{End} A$, and let $\alpha\beta + U_X$ be an open neighborhood of $\alpha\beta$. Since U_X is a left ideal and since $U_{\beta X}\beta \leq U_X$, the desired continuity follows from

$$(\alpha + U_{\beta X})(\beta + U_X) \subseteq \alpha\beta + U_{\beta X}\beta + U_X \subseteq \alpha\beta + U_X.$$

Therefore, End$A$ is a topological ring.

To verify completeness, let $\{\alpha_X\}_{X\in I}$ denote a Cauchy net where the index set I (the set of finite subsets of A) is partially ordered by inclusion. Reminder: the Cauchy net satisfies: given $X \in I$, $\alpha_Y - \alpha_Z \in U_X$ holds for all $Y, Z \in I$ containing some $X_0 \in I$; here $X \subseteq X_0$ may be assumed, so that $\alpha_Y x = \alpha_Z x$ $(\forall x \in X)$ for large Y, Z. Therefore, the common value of the $\alpha_Y x$ for large Y defines an element $x' \in A$, and it is readily seen that the assignment $\alpha : x \mapsto x'$ is an endomorphism of A. This is the limit of our Cauchy net, as is evident from $\alpha - \alpha_Y \in U_X$ for all $Y \supset X_0$. □

Observe that the factor group $(\operatorname{End} A)/U_X$ is isomorphic to a subgroup of a direct sum of copies of A, because U_X is the kernel of the homomorphism $\operatorname{End} A \to A^X$ defined as $\theta \mapsto (\theta x_1, \dots, \theta x_n)$, where $X = \{x_1, \dots, x_n\}$.

The finite topology raises several questions. An immediate one is whether or not the finite topology can be discrete for an infinite group.

Theorem 1.12 (Arnold–Murley [1]). *The endomorphism ring of a group* A *is discrete in the finite topology only if* A *is self-small. The converse holds for countable* End$A$.

Proof. Suppose A is not self-small, so there exists a map $\phi : A \to \oplus_{i<\omega} A_i$ where $A_i \cong A$ and $\pi_i\phi \neq 0$ for all i; here π_i denotes the ith projection. Define $G_n = \{a \in A \mid \pi_i\phi(a) = 0 \ \forall i \geq n\}$, an increasing chain of proper subgroups with union A. Then $\{\eta \in \operatorname{End} A \mid \eta(G_n) = 0\} \neq 0$ for all $n \leq \omega$. Any finite subset X of A is contained in some G_n, so $U_X \neq 0$. Thus no open set in the finite topology is 0, and End$A$ is not discrete in the finite topology.

Conversely, suppose that End$A$ is not discrete in the finite topology. Let $U_0 > \dots > U_n > \dots$ be a strictly descending chain of neighborhoods in End$A$; if End$A$ is countable, then there is no harm in assuming that $\cap_{n<\omega} U_n = 0$ as well. Choose $\eta_n \in U_n$ such that $\eta_n \notin U_{n+1}$. Define a homomorphism $\phi : A \to \oplus_{i<\omega} A_i$ with $A_i \cong A$ by setting $\phi = \sum_{n<\omega} \eta_n$. This means that A is not self-small. □

Example 1.13.

(a) The finite topology on the endomorphism ring of A is discrete if A is finite, or torsion-free of finite rank, or a rigid group, but not discrete if A is an infinite p-group.

(b) J_p is self-small, but its endomorphism ring is not discrete in the finite topology. The same holds for torsion-complete p-groups with finite UK-invariants.

When the Finite Topology is Compact Compactness being always of particular interest, let us turn our attention to the question as to when End$A$ is compact in the finite topology.

Before stating the relevant result, consider the fully invariant subgroup $O_x = \{\theta x \mid \theta \in \operatorname{End} A\}$ of A which we will call the **orbit of** $x \in A$. The evaluation map $\theta \mapsto \theta x$ is a group homomorphism of End G onto O_x whose kernel is U_x. This leads to the group isomorphism $(\operatorname{End} A)/U_x \cong O_x$ (which is moreover an End A-module map).

Proposition 1.14. *The endomorphism ring* End A *of a group* A *is compact in the finite topology if and only if* A *is a torsion group whose* p*-components are finitely cogenerated.*

Proof. Knowing that End A is complete in the finite topology, for the compactness of End A it is necessary and sufficient that all neighborhoods U_x be of finite indices. This is the case exactly if the orbits are finite.

First, suppose End A is compact. As $\langle x \rangle$ is a subgroup of O_x, A ought to be a torsion group. By Corollary 2.3 in Chapter 5, every non-zero p-component A_p of A contains a cocyclic summand; let C_p be one of minimal order in A_p. The orbit of its cogenerator is the socle of A_p, so it must be finite. Thus A_p is finitely cogenerated (hence a finite direct sum of cocyclic groups).

Conversely, if $A = \oplus_p A_p$ with finitely cogenerated A_p, then for every $n \in \mathbb{N}$, $A[n]$ is finite. But $x \in A[n]$ implies $O_x \leq A[n]$, thus all U_x are of finite indices in End A (and the same holds for all neighborhoods U_X). Consequently, End A is compact in the finite topology. □

Though the endomorphism ring of a p-group is rarely algebraically compact as a ring, we can still claim that its endomorphism group is algebraically compact. In fact, from Theorem 2.1 in Chapter 7 we know that $\operatorname{Hom}(A, *)$ is algebraically compact whenever A is a p-group. The invariants can be computed by making use of theorems by Pierce [1].

Matrix Representations of Endomorphisms Once we have a ring topology in End A, it makes sense to form convergent infinite sums. An infinite sum $\sum_{i \in I} \alpha_i$ with $\alpha_i \in \operatorname{End} A$ is **convergent** if, for each $x \in A$, almost all $\alpha_i x = 0$, in which case $\alpha \in \operatorname{End} A$ is its limit where $\alpha x = \sum_i \alpha_i x$ for $x \in A$. A matrix $\|\alpha_{ji}\|$ with entries in End A is said to be **column-convergent** if for each column i, the sum $\sum_j \alpha_{ji}$ is convergent.

Suppose $A = \oplus_{i \in I} A_i$ is a (finite or infinite) direct sum, and ϵ_i are the associated projections, viewed as mutually orthogonal idempotents in End A. Every $a \in A$ can be written as $a = \sum_i \epsilon_i a$ where almost all terms vanish. For $\alpha \in \operatorname{End} A$, we then have $\alpha a = \sum_i \alpha \epsilon_i a = \sum_{i,j} (\epsilon_j \alpha \epsilon_i) a$. In this way, with every $\alpha \in \operatorname{End} A$ we associate an $I \times I$-matrix $\|\alpha_{ji}\|$ where $\alpha_{ji} = \epsilon_j \alpha \epsilon_i$. If $\|\beta_{ji}\|$ with $\beta_{ji} = \epsilon_j \beta \epsilon_i$ is the matrix associated with $\beta \in \operatorname{End} A$, then the matrices associated with $\alpha - \beta$ and $\alpha\beta$ are precisely the difference matrix $\|\alpha_{ji} - \beta_{ji}\|$ and the product matrix $\| \sum_k \alpha_{jk} \beta_{ki} \|$, respectively. We conclude that $\phi \colon \alpha \to \|\alpha_{ji}\|$ is a ring homomorphism whose kernel is evidently 0.

For a fixed index i, $\alpha \epsilon_i a = \sum_j \alpha_{ji} a$ converges for every $a \in A$, indicating that the matrix $\|\alpha_{ji}\|$ is column-convergent. Conversely, if a matrix with entries $\alpha_{ji} \in \epsilon_j (\operatorname{End} A) \epsilon_i$ is column-convergent, then it must come from an $\alpha \in \operatorname{End} A$, namely, from $\alpha = \sum_{i,j} \alpha_{ji}$. This is a convergent sum, because for each $a \in A$, the sum $\sum_{i,j} \alpha_{ji} a$ is finite. If $\epsilon_j(\mathrm{End}\, A)\epsilon_i$ is identified with $\mathrm{Hom}(A_i, A_j)$, then we can state our findings as follows.

Proposition 1.15. *Let* $A = \oplus_{i \in I} A_i$ *be a direct sum decomposition of* A. *Then* $\mathrm{End}\, A$ *is isomorphic to the ring of all column-convergent* $I \times I$*-matrices* $\|\alpha_{ji}\|$ *where* $\alpha_{ji} \in \mathrm{Hom}(A_i, A_j)$. □

Needless to say, if the A_i are small objects, then every column contains but a finite number of non-zero entries, and if the group is a finite direct sum, then its endomorphism ring will be the full matrix ring with the indicated entries.

Example 1.16. Let $A = \oplus_{i \in I} \langle a_i \rangle$ be a free group. In the matrix representation of $\mathrm{End}\, A$, the entries are integers, and in each column, almost all entries are 0.

Example 1.17. If $D = \oplus_{i \in I} D_i$ is a torsion-free divisible group with $D_i \cong \mathbb{Q}$, then we are in the same situation as in the preceding example, the only difference is that the entries of the matrix can be arbitrary rational numbers.

Example 1.18. Let $A = A_0 \oplus (\oplus_p A_p)$, where A_0 is torsion-free, while the A_p are p-groups belonging to different primes. Then the matrices representing the endomorphisms are $\omega \times \omega$-matrices of the form

$$\begin{pmatrix} \alpha_{00} & 0 & 0 & \dots & 0 & \dots \\ \alpha_{20} & \alpha_{22} & 0 & \dots & 0 & \dots \\ \alpha_{30} & 0 & \alpha_{33} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{p0} & 0 & 0 & \dots & \alpha_{pp} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where $\alpha_{00} \in \mathrm{End}\, A_0$, $\alpha_{p0} \in \mathrm{Hom}(A_0, A_p)$, and $\alpha_{pp} \in \mathrm{End}\, A_p$.

★ **Notes.** There is an extensive literature on endomorphism rings; this is one of the most investigated areas in the theory. Readers interested in the subject are referred to Krylov–Mikhalev–Tuganbaev [KMT] where a large amount of material on the endomorphism rings of abelian groups is presented in a systematic manner with full proofs. Important results are also available on endomorphism rings of groups (or modules) with distinguished submodules; these are instrumental in deriving results on endomorphism rings.

The finite topology provides additional information about the relation between the group and its endomorphism ring. It is a rare possibility of defining the finite topology intrinsically (i.e., solely in the endomorphism ring, without referring to the group), but it seems it is a most relevant feature in cases when the endomorphism ring determines the group, like for torsion groups, certain completely decomposable groups, etc. Recently, May [7] discusses the use of finite topology.

There are a few ideals in the endomorphism ring that are of interest from the group theoretical point of view. The most widely studied ideal is the Jacobson radical $\mathsf{J}(A)$. In general, there is no satisfactory characterization for $\mathsf{J}(A)$ in terms of the group A, only in special cases. Another notable ideal is Ines A, the set of inessential endomorphisms. For a study of right ideals, see Faticoni [2]. No detailed discussion will be given here to the ideals of $\mathrm{End}\, A$.

Let us point out some results by Mishina on endomorphism rings. In her paper [3], she characterizes the groups A for which every endomorphism of each subgroup extends to an endomorphism of A: these groups are either divisible, or torsion with homogeneous p-components. In another paper [4], she shows that all the endomorphisms of all factor groups A/C lift to A if and only if A is either free, or torsion with homogeneous p-components, or else the direct sum of a divisible torsion group and a finitely generated free group. (Similar results are proved for automorphisms, see Sect. 1, Exercise 8 in Chapter 17.)

Recently, several publications deal with the so-called *algebraic entropy* which was recently introduced in abelian groups. The paper Dikranjan–Goldsmith–Salce–Zanardo [1] contains lots of interesting results on the entropy of endomorphisms.

Exercises

(1) If $|A| = p^n$, then $|\operatorname{End} A| \leq p^{n^2}$.
(2) (a) If $G \leq A$, then $\operatorname{Hom}(A, G)$ is a right ideal in $\operatorname{End} A$.
(b) If G is fully invariant, then $\operatorname{Hom}(A, G)$ is a two-sided ideal.
(3) (Lawver) All endomorphic images of A are fully invariant exactly if, for every $a \in A$ and for all $\eta, \xi \in \operatorname{End} A$, there is a $b \in A$ such that $(\eta\xi)a = \xi b$.
(4) Show that the finite topology of $\operatorname{End} A$ for a separable torsion-free group A can be defined intrinsically (i.e. without reference to A).
(5) Describe the finite topology of J_p as an endomorphism ring of $\mathbb{Z}(p^\infty)$ and as that of J_p.
(6) (a) The direct sum and the direct product of elementary p-groups T_p for different primes p have isomorphic endomorphism rings.
(b) However, these endomorphism rings are not isomorphic as topological rings (equipped with the finite topology).
(7) For an infinite group A, $\operatorname{End} A$ is always infinite. Give examples where $|A| < |\operatorname{End} A|$, and where $|A| > |\operatorname{End} A|$.
(8) (a) Let $\{G_i\}_{i \in I}$ be a system of subgroups of A which is directed upwards under inclusion such that $\cup_{i \in I} G_i = A$. Define a topology in $\operatorname{End} A$ by declaring the set of left ideals $\mathsf{L}_i = \{\theta \in \operatorname{End} A \mid \theta G_i = 0\}$ as a base of neighborhoods about 0. Show that $\operatorname{End} A$ is a complete group in this topology, and if the G_i are fully invariant in A, then $\operatorname{End} A$ is a topological ring.
(b) (Pierce) Let A be a p-group and $G_i = A[p^i]$ $(i < \omega)$. Then $\operatorname{End} A$ is a complete topological ring in this topology.
(9) A self-small group is not the direct sum of infinitely many non-zero groups, but it may be decomposed into the direct sum of any finite number of non-zero summands. [Hint: torsion-complete with standard basic.]
(10) A group is self-small if all of its endomorphisms $\neq 0$ are monic.
(11) (Dlab) Let D be a divisible p-group of countable rank. Representing endomorphisms by matrices over J_p, show that in each column almost all entries are divisible by p^k for every $k > 0$.
(12) Suppose $A = \oplus_{i \in I} A_i$ with countable summands A_i. In the matrix representations of endomorphisms, every column contains at most countably many non-zero entries.
(13) The set $\operatorname{End}_{\mathsf{E}} A$ of E-endomorphisms of A is the center of the ring $\mathsf{E} = \operatorname{End} A$.
(14) For any group A, the center of the ring $\operatorname{End}(A \oplus \mathbb{Z})$ is $\cong \mathbb{Z}$.

2 Endomorphism Rings of *p*-Groups

It is a rather intriguing question to find general properties shared by the endomorphism rings of p-groups. Fortunately, substantial information is available, and we wish to discuss some relevant results.

Role of Basic Subgroups We start with the observation that underscores the relevance of basic subgroups also from the point of view of endomorphisms: *any endomorphism of a reduced p-group is completely determined by its action on a basic subgroup.* Actually, a stronger statement holds: any homomorphism of a p-group A into a reduced group C is determined by its restriction to a basic subgroup B of A. The exact sequence $0 \to B \to A \to A/B \to 0$ induces the exact sequence $0 = \text{Hom}(A/B, C) \to \text{Hom}(A, C) \to \text{Hom}(B, C)$ which justifies our claim.

The Finite Topology It should be pointed out that in case of reduced p-groups, the finite topology of the endomorphism ring can be defined intrinsically, without reference to the underlying group. If $x \in A$, then there are a projection $\epsilon : A \to \langle c \rangle$ onto a cyclic summand and a $\theta \in \mathsf{E}$ such that $x = \theta c$. Manifestly, the neighborhood U_x (annihilating x) is nothing else than the left annihilator ideal $(\theta\epsilon)^{\perp}$.

Proposition 2.1. *Let A be a separable p-group.*

(i) *The finite topology of its endomorphism ring* E *can be defined by taking the left annihilators of the primitive idempotents.*
(ii) *In the finite topology of* E*, the left ideal* E_0 *of* E *generated by the primitive idempotents is dense, and its completion is* E.

Proof. As $U_x \leq U_{px}$ and every element is contained in a finite summand, the U_ϵ for primitive idempotents form a subbase.

For (ii) we show that for every $\theta \in \mathsf{E}$ and for every neighborhood U_x, the coset $\theta + U_x$ intersects E_0. Now $U_x = (p^k\epsilon)^{\perp}$ (if $\langle x \rangle = p^k\epsilon A$) is a left ideal, so $1 - \epsilon \in U_x$ implies $-\theta(1 - \epsilon) \in U_\epsilon$ whence $\theta\epsilon \in \theta + U_\epsilon$ follows. □

Structure of End for *p*-Groups We are in the fortunate situation that a lot is known about the endomorphism rings of p-groups. As a matter of fact, Liebert [3] gave a complete characterization in the separable case. A much less informative, but perhaps more attractive information is recorded in the next theorem.

Before stating the theorem, we recall a definition: a ring E is a **split extension** of a subring R by an ideal L of E if there exists a ring homomorphism $\rho : \mathsf{E} \to \mathsf{R}$ that is the identity on R, and $\mathsf{L} = \text{Ker}\rho$. We write $\mathsf{E} = \mathsf{R} \oplus \mathsf{L}$ (direct sum in the group sense).

Theorem 2.2 (Pierce [1]). *For a p-group A,* End *A is a split extension*

$$\text{End}\, A \cong \mathsf{R} \oplus \text{End}_s A,$$

where R *is a ring whose additive group is the completion of a free p-adic module, and* $\text{End}_s A$ *is the ideal of small endomorphisms of A.*

Proof. This is an immediate consequence of Theorem 3.3 in Chapter 7. □

Baer–Kaplansky Theorem The endomorphism ring of an abelian group may contain more or less information about the group itself. There are arbitrarily large torsion-free groups whose endomorphism rings are just $\mathbb{Z}$, or a subring of $\mathbb{Q}$, so their endomorphism rings reveal no more about their structure than indecomposability. But, on the other hand, if A is cocyclic, then $\operatorname{End} A$ (now $\cong \mathbb{Z}/p^k\mathbb{Z}$ or $\cong J_p$) completely characterizes A among the torsion groups. Indeed, in this case $\operatorname{End} A$ has only two idempotents: 0 and 1, so any torsion group C with $\operatorname{End} C \cong \operatorname{End} A$ must be indecomposable, and hence cocyclic. It is a trivial exercise to check that only $C \cong \mathbb{Z}(p^k)$ ($k \in \mathbb{N}$ or ∞) is a possibility.

Example 2.3. It can very well happen that a torsion group and a torsion-free group have isomorphic endomorphism rings. Example: J_p is the endomorphism ring of both $\mathbb{Z}(p^\infty)$ and J_p. Exercise 6 in Sect. 1 provides an example of the same situation for a torsion and a mixed group.

It is natural to wonder in which cases the group can entirely be recaptured from its endomorphism ring, or, more accurately, when the isomorphy of endomorphism rings implies that the groups themselves are isomorphic. We will show that this is always the case for torsion groups. The proof relies on a distinguished feature of torsion groups: they have lots of summands, even indecomposable summands, so an adequate supply of idempotent endomorphisms is at our disposal.

We start with a preparatory lemma.

Lemma 2.4 (Richman–Walker [1]). *Let* $\mathsf{E} = \operatorname{End} A$ *where* A *is a* p*-group.*

(i) *If* A *is a bounded or a divisible group, then* $A \cong \mathsf{E}\epsilon$ *for an idempotent* $\epsilon \in \mathsf{E}$.
(ii) *If* A *has an unbounded basic subgroup, then there are idempotent endomorphisms* π_i ($i \in \mathbb{N}$) *such that*

$$A \cong \varinjlim \mathsf{E}\pi_i$$

with connecting maps $\gamma_i \colon \eta\pi_i \mapsto \eta\pi_i\pi_{i+1}$ ($\eta \in \mathsf{E}, i \in \mathbb{N}$).

Proof. Let A be a bounded p-group, and $\epsilon \in \mathsf{E}$ a primitive idempotent of maximal order p^k. Then $\epsilon A = \langle c \rangle$ is a summand of A. We claim that $\alpha : \mathsf{E}\epsilon \to A$ is a group isomorphism where $\alpha \colon \eta\epsilon \mapsto \eta\epsilon(c) = \eta(c)$ ($\eta \in \mathsf{E}$). It is clearly a homomorphism, and it is surjective, since c can be mapped onto every element of A by a suitable η. Moreover, it is monic, because if we have $\eta(c) = 0$ for some $\eta \in \mathsf{E}$, then also $\eta\epsilon = 0$. Similar argument applies if A is divisible.

Next, let A have an unbounded basic subgroup. Then there exist primitive idempotents $\epsilon_i \in \mathsf{E}$ of orders p^{n_i} with $n_1 < \cdots < n_i < \dots$ such that $A = C_1 \oplus \cdots \oplus C_i \oplus A_i$ with $C_j < A_i$ for all $j > i$, where $C_i = \epsilon_i A$ is cyclic of order p^{n_i}. Choose endomorphisms $\phi_i \colon A \to C_i$ satisfying $\phi_i \restriction C_i = \mathbf{1}_{C_i}$, $\phi_i(C_{i+1}) = C_i$, while $\phi_i(C_j) = 0$ for $j \neq i, i+1$ and $\phi_i(A_{i+1}) = 0$. Then for every $i \in \mathbb{N}$, $\pi_i = \epsilon_i + \phi_i\epsilon_{i+1}$ is an idempotent endomorphism with $\operatorname{Im} \pi_i = C_i$. Evidently, we can select generators c_i of C_i inductively so as to satisfy $\phi_i(c_{i+1}) = c_i$ for all i.

Consider the E-modules $\mathsf{E}\pi_i$ ($i \in \mathbb{N}$). The E-map $\gamma_i : \mathsf{E}\pi_i \to \mathsf{E}\pi_{i+1}$ defined by $\gamma_i(\pi_i) = \pi_i\pi_{i+1}$ is monic; indeed, if $\eta\pi_i\pi_{i+1} = 0$ ($\eta \in \mathsf{E}$), then $\eta\pi_i(c_i) = \eta\pi_i\pi_{i+1}(c_{i+1}) = 0$ implies $\eta\pi_i = 0$ (since π_i is 0 everywhere else). We thus have a direct system $\mathsf{E}\pi_i$ ($i \in \mathbb{N}$) of left E-modules with connecting maps γ_i. As above, we can show that the map $\alpha_i : \eta\pi_i \mapsto \eta(c_i)$ of $\mathsf{E}\pi_i$ to $A[p^{n_i}]$ is an E-isomorphism. Moreover, the diagram

$$\begin{array}{ccc} \mathsf{E}\pi_i & \xrightarrow{\alpha_i} & A[p^{n_i}] \\ \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \beta_i} \\ \mathsf{E}\pi_{i+1} & \xrightarrow{\alpha_{i+1}} & A[p^{n_{i+1}}] \end{array}$$

(where β_i is the inclusion map) commutes for each i, because $\beta_i\alpha_i(\eta\pi_i) = \eta(c_i)$ and $\alpha_{i+1}\gamma_i(\eta\pi_i) = \alpha_{i+1}(\eta\pi_i\pi_{i+1}) = \eta\pi_i(c_{i+1}) = \eta(\epsilon_i + \phi_i\epsilon_{i+1})(c_{i+1}) = \eta\phi_i(c_{i+1}) = \eta(c_i)$. Since $A = \varinjlim A[p^{n_i}]$, we obtain $A \cong \varinjlim \mathsf{E}\pi_i$. □

Theorem 2.5 (Baer [9], Kaplansky [1]). *If A and C are torsion groups whose endomorphism rings are isomorphic, then $A \cong C$.*

Moreover, every ring isomorphism ψ : $\operatorname{End} A \to \operatorname{End} C$ is induced by a group isomorphism $\phi : A \to C$; i.e. $\psi : \eta \mapsto \phi\eta\phi^{-1}$.

Proof. The proof can at once be reduced to p-groups. So suppose A, C are p-groups, and $\psi : \operatorname{End} A \to \operatorname{End} C$ is a ring isomorphism. To simplify notation, write $\mathsf{E} = \operatorname{End} A$.

If A is bounded or divisible, then by Lemma 2.4(i) we have $A \cong \mathsf{E}\epsilon$ for an idempotent $\epsilon \in \mathsf{E}$. Similarly, $C \cong \psi(\mathsf{E})\psi(\epsilon)$ whence the existence of an isomorphism $\phi : A \to C$ is immediate.

If $A = B \oplus D$ where B is bounded and D is divisible, then $\operatorname{End} A$ has a torsion-free part $\cong \operatorname{End} D$ that is an ideal, whose two-sided annihilator is $\cong \operatorname{End} B$. These ideals are carried by ψ to the corresponding ideals of $\operatorname{End} C$. The settled cases imply that A and C have isomorphic bounded and divisible subgroups.

If A has unbounded basic subgroup, then in view of Lemma 2.4(ii) there are idempotents $\pi_i \in \mathsf{E}$ such that $A \cong \varinjlim \mathsf{E}\pi_i$, and clearly $C \cong \varinjlim \psi(\mathsf{E})\psi(\pi_i)$ with corresponding connecting maps. Thus again $A \cong C$ follows.

We proceed to the second claim. For simplicity we identify A with $\mathsf{E}\epsilon$ or with $\varinjlim \mathsf{E}\pi_i$, as the case may be, and similarly for C. Let $\phi : A \to C$ be the isomorphism induced by ψ from $\mathsf{E}\epsilon$ to $\psi(\mathsf{E})\psi(\epsilon)$, or by ψ between the direct limits. Then the endomorphism $\psi(\eta)$ ($\eta \in \mathsf{E}$) acting on C can also be obtained by using ϕ^{-1}, applying η and followed by ϕ, i.e. $\psi(\eta) = \phi\eta\phi^{-1}$, as claimed. □

Since all the primitive idempotents are contained in the ideal $\operatorname{End}_s A$ of small endomorphisms, from the foregoing proof we conclude:

Corollary 2.6 (Pierce [1]). *The torsion groups A and C are isomorphic exactly if the rings $\operatorname{End}_s A$ and $\operatorname{End}_s C$ are isomorphic.* □

An immediate corollary to Theorem 2.5 is the following remarkable fact:

Corollary 2.7 (Baer [9]). *Every automorphism of the endomorphism ring of a torsion group is inner.*

Proof. Let α be an automorphism of $\operatorname{End} A$, where A is a torsion group. Theorem 2.5 asserts that it must act as $\alpha : \eta \mapsto \phi\eta\phi^{-1}$ for some automorphism ϕ of A ($\eta \in \operatorname{End} A$). Here ϕ is viewed as a unit in $\operatorname{End} A$. □

Liebert's Theorem The following characterization of the endomorphism rings of separable p-groups is an important document, though it seems difficult to use it. We state it without proof.

Theorem 2.8 (Liebert [3]). *For a ring* E*, there exists a separable p-group A such that* $\operatorname{End} A \cong \mathsf{E}$ *if and only if the following conditions are satisfied:*

(i) *the sum* E_0 *of all minimal non-nil left ideals is a ring whose additive group is a p-group, and whose left annihilator in* E *is* 0*;*
(ii) *if* π, ρ *are primitive idempotents in* E*, then the additive group of* $\pi\mathsf{E}\rho$ *is cyclic;*
(iii) *if* π, ρ *are primitive idempotents in* E *such that* $o(\pi) \leq o(\rho)$*, then the left annihilator of* $\mathsf{E}\rho$ *is contained in the left annihilator of* $\mathsf{E}\pi$*, and* $\mathsf{E}\pi\mathsf{E}\rho = \mathsf{E}\rho[o(\pi)]$*;*
(iv) *a proper right ideal of* E_0 *whose left annihilator in* E_0 *is nilpotent is not a summand in* E_0*;*
(v) E *is complete in its finite topology.* □

Center of the Endomorphism Ring We proceed to identify the center of the endomorphism ring of a torsion group. The general case immediately reduces to primary groups, and it is more elegant to formulate the result for p-groups.

Theorem 2.9 (Charles [1], Kaplansky [K]). *The center of the endomorphism ring of a p-group consists of multiplications by p-adic integers or by elements of the residue class ring* $\mathbb{Z}/p^k\mathbb{Z}$ *according as the group is unbounded or bounded by* p^k (*with minimal k*).

Proof. Multiplication by a rational or a p-adic integer is an endomorphism of any p-group A; it evidently commutes with every $\eta \in \operatorname{End} A$.

Conversely, γ in the center of $\operatorname{End} A$ means that the map $\gamma : A \to A$ is an $\operatorname{End} A$-module homomorphism. We now appeal to Lemma 6.1(i) below to infer that γ must act as multiplication by a $\rho \in J_p$. □

The Jacobson Radical The Jacobson radical $\mathsf{J}(A)$ of the endomorphism ring E of a p-group A has been extensively studied, but so far there is no satisfactory characterization. Pierce [1] compared it to the set

$$\mathsf{H}(A) = \{\eta \in \mathsf{E} \mid \eta(p^n A[p]) \leq p^{n+1} A[p]\},$$

which is an ideal of E located between $p\mathsf{E}$ and E (also called **Pierce radical**); it consists of all endomorphisms that strictly increase the heights of elements of finite heights in the socle. From the point of view of endomorphism rings of p-groups,

$\mathsf{H}(A)$ seems to be more tractable than the Jacobson radical. We now prove results that involve $\mathsf{H}(A)$.

Lemma 2.10 (Pierce [1]).

(i) *For every reduced p-group A,* $\mathsf{H}(A)$ *contains the Jacobson radical* $\mathsf{J}(A)$ *of* End A.
(ii) *For a separable p-group A,* $\mathsf{J}(A) = \mathsf{H}(A)$ *if and only if, for all* $a \in A[p]$ *and all* $\eta \in \mathsf{H}(A)$*, the infinite sum* $\sum_{n<\omega} \eta^n(a)$ *converges* (*in the p-adic topology of A*).
(iii) *If A is torsion-complete, then* $\mathsf{J}(A) = \mathsf{H}(A)$.

Proof.

(i) Assume $\eta \in \mathsf{J}(A)$, but $\eta \notin \mathsf{H}(A)$. Thus for some $n < \omega$, there is an $a \in A[p]$ of finite height n such that also $h_p(\eta a) = n$. Then a is in the socle of a summand $\langle b \rangle$, and ηa is in the socle of a summand $\langle c \rangle$, both of order p^{n+1}. Evidently, there is a $\xi \in \operatorname{End} A$ mapping $\langle c \rangle$ onto $\langle b \rangle$ such that $\xi(\eta a) = a$. Now $\eta \in \mathsf{J}(A)$ implies $\xi\eta \in \mathsf{J}(A)$, so $1 - \xi\eta$ is an automorphism of A. However, $(1 - \xi\eta)a = 0$, a contradiction.
(ii) If $\eta \in \mathsf{H}(A)$ and $\mathsf{H}(A) = \mathsf{J}(A)$, then $1 - \eta$ is an automorphism of A. Thus, given $a \in A[p]$, there is a $b \in A[p]$ such that $(1 - \eta)b = a$. It follows that the partial sums $b_n = a + \eta(a) + \cdots + \eta^n(a) = (1 - \eta^{n+1})b$ satisfy $b - b_n = \eta^{n+1}(b)$. Since $\eta \in \mathsf{H}(A)$ guarantees that $h_p(\eta^{n+1}(b)) > n$, the sequence b_n converges to b.

Conversely, if we know that all the infinite sums of the stated kind converge, then we can show that $1 - \eta$ is an automorphism of A for all $\eta \in \mathsf{H}(A)$. For each $a \in A[p]$, we have $h_p((1 - \eta)a) = h_p(a)$, and therefore $\operatorname{Ker}(1 - \eta) = 0$. To see that $\operatorname{Im}(1 - \eta) = A$, the proof goes by induction on the order of $a \in A$, to verify the existence of a $c \in A$ such that $(1 - \eta)c = a$. If $o(a) = p$, then for c we choose $\sum_{n<\omega} \eta^n(a)$. If $o(a) = p^{k+1}$, then $b' = \sum_{n<\omega} \eta^n(p^k a)$ must belong to $p^k A$, because all the partial sums belong to this (closed) subgroup. Hence $b' = p^k u$ for some $u \in A$, and we have $(1 - \eta)p^k u = p^k a$. Now $a - (1 - \eta)u$ is of order $\leq p^k$, so by the induction hypothesis there is a $v \in A$ satisfying $(1 - \eta)v = a - (1 - \eta)u$. Then $c = u + v$ is mapped by $1 - \eta$ upon a.
(iii) If A is a torsion-complete p-group, then the sums in (ii) always converge, and therefore, $\mathsf{J}(A) = \mathsf{H}(A)$. □

Example 2.11. Let $A = \oplus_{n<\omega} \langle a_n \rangle$ where $o(a_n) = p^n$. Then $\mathsf{J}(\operatorname{End} A) \neq \mathsf{H}(\operatorname{End} A)$. Indeed, the correspondence $\eta \colon a_n \mapsto p a_{n+1}$ defines an endomorphism in $\mathsf{H}(A)$, and the sequence $(1 + \eta + \cdots + \eta^k)a_1$ $(k < \omega)$ does not converge.

Proposition 2.12 (Pierce [1]).

(i) *For a reduced p-group A, there is a ring embedding*

$$\psi \colon (\operatorname{End} A)/\mathsf{H}(A) \to \prod_{n<\omega} \mathsf{M}_{f_n(A)} \qquad (16.2)$$

where $\mathsf{M}_{f_n(A)}$ *denotes a matrix ring over the prime field* F_p *whose dimension is the nth UK-invariant* $f_n(A)$ *of* A. Im ψ *contains the direct sum of the matrix rings.*

(ii) *For a separable p-group A,* ψ *is an isomorphism if and only if A is torsion-complete.*

Proof.

(i) Every $\eta \in \operatorname{End} A$ induces a linear transformation η_n of the $f_n(A)$-dimensional vector space $p^nA[p]/p^{n+1}A[p]$. Thus $\eta \mapsto (\eta_n)_{n<\omega}$ induces a ring homomorphism from End A to the right-hand side of (16.2), whose kernel is exactly $\mathsf{H}(A)$.

If we write the basic subgroup of A as $B = \oplus_{n<\omega} B_n$ where B_n is the direct sum of cyclic groups of order p^{n+1}, then $p^nA[p]/p^{n+1}A[p] \cong B_n[p]$. It is clear that any linear transformation η_n on $B_n[p]$ extends to an endomorphism of B_n, and then to an endomorphism η of A. This is always true for a simultaneous extension of a finite number of η_ns, and also for infinitely many provided A is torsion-complete.

(ii) That ψ is an isomorphism only if the separable p-group A is torsion-complete can be seen from the representation of elements in separable p-groups as $\sum_{n<\omega} b_n$ with $b_n \in B_n$. If an arbitrarily chosen collection $\eta_n \in \operatorname{End} B_n$ $(n < \omega)$ extends to $\eta \in \operatorname{End} A$, then all $\sum_{n<\omega} \eta(b_n)$ must be contained in A, thus $\overline{B} \leq A$. □

The proof shows that Im ψ is a subdirect product of the matrix rings.

★ **Notes.** The study of the endomorphism rings of p-groups A was initiated by Pierce [1]; in this important paper, he proved several relevant results. In a sequel [3] to [1], he characterizes End A within a class of rings when A is separable with a prescribed basic subgroup. A more satisfactory realization theorem is due to Liebert [1, 3] who characterized the endomorphism rings as rings, first for bounded, and later for separable p-groups. (Though it is an important contribution, it still falls short of the true significance, since the conditions are not illuminating.) See also Liebert [4]. Goldsmith [2] examines endomorphism rings of non-separable p-groups.

Generalizing Corner [5], Dugas–Göbel [3] show that for every reduced ring R over J_p, whose additive group is torsion-free and algebraically compact, there exists a separable p-group A such that End A is a split extension of R and $\operatorname{End}_s A$. They also prove, for every cardinal κ, the existence of κ separable p-groups, all with such a fixed R, so that all homomorphisms between them are small.

Ideals in endomorphism rings have been discussed by Liebert [2] and Monk [1]. According to Hausen [4] and Ivanov [1], the sum of nilpotent ideals is the collection of all $\eta \in \operatorname{End} A$, for which there is a finite chain $0 = A_0 < A_1 < \cdots < A_n = A$ of fully invariant subgroups such that $\eta A_{i+1} \leq A_i$. Hausen [5] generalizes the ideal $\mathsf{H}(A)$ by defining I_A as the collection of all $\eta \in \operatorname{End} A$ for which there is a finite sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = \tau$ of ordinals such that $\eta(p^{\sigma_i}A[p]) \leq p^{\sigma_{i+1}}A[p]$ for $i = 0, 1, \ldots, n-1$, where τ denotes the length of A. Her theorems are concerned with totally projective groups.

Several authors discussed a modified version of the Baer–Kaplansky theorem: when the isomorphism of the Jacobson radicals of the endomorphism rings implies the isomorphism of the groups. See, e.g., Flagg [1], Hausen–Johnson [1], Hausen–Praeger–Schultz [1], Schultz [2]. Puusemp [1] showed that the conclusion of Theorem 2.5 remains true if only the isomorphy of the multiplicative endomorphism semigroups is assumed. There are theorems similar to Theorem 2.5 on certain mixed groups. For the endomorphism semigroup, see also Sebel'din [3]. In several papers, May–Toubassi (see, e.g., [1]) showed that two mixed local groups of torsion-free rank 1 with totally projective torsion subgroups are isomorphic if their endomorphism rings are isomorphic; for a survey, see May [4]. Files–Wickless [1] extended the Baer–Kaplansky theorem to a class of global mixed groups. Files [1] proved the isomorphy of reduced local Warfield groups with isomorphic endomorphism rings.

Nunke [6] has a nice generalization of Theorem 2.8: If A, C are unbounded p-groups, then End A and End C embed in End(Tor(A, C)) in such a way that they are centralizers of each other, and their intersection is precisely the center of End(Tor(A, C)). The special case $C \cong \mathbb{Z}(p^\infty)$ yields Theorem 2.9.

A fairly large literature deals with the problem as to when End A equals the subring generated by its units (i.e., by Aut A). Castagna [1] gives an example where this subring is a proper subring. Hill [6] shows that if $p > 2$, then every endomorphism of a totally projective p-group is the sum of two automorphisms. Hill–Megibben–Ullery [1] prove the same for local Warfield groups. See also Goldsmith–Meehan–Wallutis [1] where the unit sum numbers (number of units needed to be added to get the endomorphisms) are investigated.

Bunina–Mikhalëv [1] investigate when the endomorphism rings of two p-groups are elementarily equivalent.

Exercises

(1) Give more examples to show that a torsion-free group and a torsion group may have isomorphic endomorphism rings.

(2) (Levi) Find the endomorphisms which map every subgroup into itself. In particular, for p-groups.

(3) If A is a torsion group, then the $\mathbb{Z}$-adic topology of End A is finer than its finite topology. [Hint: $n(\text{End}\,A) \leq U_x$ if $nx = 0$.]

(4) Let A be a separable p-group with basic subgroup B. Then End A is a closed subring in End $\overline{B}$ (in the finite topology).

(5) (Szele–Szendrei) A torsion group A has commutative endomorphism ring exactly if $A \leq \mathbb{Q}/\mathbb{Z}$.

(6) Find two non-isomorphic p-groups with isomorphic endomorphism groups.

(7) Verify the analogue of the Baer–Kaplansky theorem for adjusted cotorsion groups.

(8) For a divisible p-group D, the Jacobson radical of End D is equal to p End D.

(9) (a) Assume A is a torsion group. The endomorphisms of A with finitely cogenerated images form an ideal $\mathsf{V}(A)$ in End A. It is the ideal generated by the primitive idempotents.
 (b) Follow the proof of the Baer–Kaplansky theorem to conclude that a ring isomorphism $\mathsf{V}(A) \cong \mathsf{V}(C)$ implies $A \cong C$ provided that C, too, is torsion.

(10) Let A be a torsion-complete p-group, and $\eta \in$ End A with Ker $\eta = 0$. If η maps a basic subgroup into a basic subgroup, then $\eta \in$ Aut A.

(11) (D'Este [1]) A group C is said to be an *E-dual* of A if End A and End C are anti-isomorphic rings. A reduced p-group has an E-dual if and only if it is torsion-complete with finite UK-invariants. [Hint: summands B_n in a basic subgroup $B = \oplus_n B_n$ are finite; A is separable and $\overline{B} \leq A$.]

3 Endomorphism Rings of Torsion-Free Groups

In contrast to torsion groups, non-isomorphic torsion-free groups may very well have isomorphic endomorphism rings. Another major difference in the behavior of endomorphism rings between torsion and torsion-free groups lies in the fact that only minor restrictive conditions hold for the torsion-free case. We will see in Theorem 7.1 that only slight restriction on the ring (to be cotorsion-free) is enough to guarantee that it is an endomorphism ring of a torsion-free group.

Example 3.1 (Corner [2]). A ring whose additive group is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$ cannot be the endomorphism ring of an abelian group. For, such a group must be torsion-free divisible, and it cannot be of rank 1, neither of rank ≥ 2, because then the rank of its endomorphism ring is 1, and ≥ 4, respectively. (Similar argument holds for $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$.)

Example 3.2 (Şąsiada [4], Corner [2]). There exist torsion-free groups of finite rank whose endomorphism groups are isomorphic, but their endomorphism rings are not. Indeed, by Theorem 3.3 below, there exist such groups with endomorphism rings isomorphic to the ring $\mathbb{Z} \oplus \mathbb{Z}$ and to the ring of the Gaussian integers $\mathbb{Z} + \mathbb{Z}i$.

Corner's Theorem We enter our study of torsion-free endomorphism rings with the following striking theorem. Though it is a corollary to Theorem 7.1 that is by far more general, we offer a full proof of this historically important result that opened new prospects in the theory; this proof is needed, because we will give no detailed proof for Theorem 7.1. The idea of localization in the proof is due to Orsatti [2], so Corner's method will be needed merely in the more tractable local case.

Theorem 3.3 (Corner [2]). *Every countable reduced torsion-free ring is isomorphic to the endomorphism ring of some countable reduced torsion-free group.*

Proof. Let R be a countable, p-local reduced torsion-free ring. It is a $\mathbb{Z}_{(p)}$-algebra, Hausdorff in its p-adic topology, so it is a pure subring in its p-adic completion $\tilde{\mathsf{R}}$, which is a torsion-free J_p-algebra with the same identity. Choose a maximal set $\{z_n\ (n < \omega)\}$ in R that is linearly independent over J_p. Thus, for every $a \in \mathsf{R}$, there is a dependence relation

$$p^k a = \pi_1 z_1 + \cdots + \pi_m z_m \qquad (\pi_i \in J_p),$$

for some k and m, where the coefficients π_i are uniquely determined up to factors p^j. We denote by S the pure subring of J_p generated by $\mathbb{Z}_{(p)}$ and the π_i for all $a \in \mathsf{R}$. Clearly, S is still countable. We proceed with the construction which requires several steps.

(a) Suppose $\pi_1, \dots, \pi_n \in J_p$ are linearly independent over S. We claim: $\pi_1 r_1 + \cdots + \pi_n r_n = 0$ ($r_i \in \mathsf{R}$) implies that all $r_i = 0$. In fact, for sufficiently large $\ell \in \mathbb{N}$, there are relations $p^\ell r_i = \rho_{i1} z_1 + \cdots + \rho_{im} z_m$ with $\rho_{ij} \in \mathsf{S}$, thus $\sum_i \sum_j \pi_i \rho_{ij} z_j = 0$. By the independence of the z_j, we have $\sum_i \pi_i \rho_{ij} = 0$ ($j = 1, \dots, m$) whence, by hypothesis on the π_i, all $\rho_{ij} = 0$, and $r_i = 0$.

(b) For each non-zero $a \in \mathsf{R}$ choose a pair ρ_a, σ_a of p-adic integers such that the set $\{\rho_a, \sigma_a \mid 0 \neq a \in \mathsf{R}\}$ is algebraically independent over S. This is possible, for S is countable, and the transcendence degree of J_p over S is the continuum. Set

$$e_a = \rho_a 1 + \sigma_a a \in \tilde{\mathsf{R}}, \tag{16.3}$$

and define the group A as the pure subgroup

$$A = \langle \mathsf{R}, \mathsf{R}e_a \ \forall a \in \mathsf{R}\rangle_* \leq \tilde{\mathsf{R}}. \tag{16.4}$$

Obviously, A is countable, reduced, and torsion-free.

(c) It is evident from the definition that A is a left R-module. It is faithful, as different elements of R act differently on $1 \in A$. Consequently, R is isomorphic to a subring of $\operatorname{End} A$.

(d) In order to show that it is not a proper subring, select an $\eta \in \operatorname{End} A$. Since R is pure and dense in A (which is pure and dense in $\tilde{\mathsf{R}}$), it follows that $\tilde{A} = \tilde{\mathsf{R}}$. By Proposition 2.10 in Chapter 6, η extends uniquely to a J_p-endomorphism $\tilde{\eta}$ of $\tilde{\mathsf{R}}$. Then

$$\eta e_a = \tilde{\eta}(\rho_a 1 + \sigma_a a) = \rho_a(\tilde{\eta}1) + \sigma_a(\tilde{\eta}a) = \rho_a(\eta 1) + \sigma_a(\eta a)$$

for any $a \in A$. More explicitly, write

$$p^k(\eta e_a) = b_0 + \sum_{i=1}^{n} b_i e_{a_i},\ p^k(\eta 1) = c_0 + \sum_{i=1}^{n} c_i e_{a_i},\ p^k(\eta a) = d_0 + \sum_{i=1}^{n} d_i e_{a_i}$$

for $a_i, b_i, c_i, d_i \in \mathsf{R}$, and for some $k, n \in \mathbb{N}$. Substitution yields

$$\begin{aligned} b_0 + &\sum_{i=1}^{n} b_i(\rho_{a_i} 1 + \sigma_{a_i} a_i) \\ &= \rho_a[c_0 + \sum_{i=1}^{n} c_i(\rho_{a_i} 1 + \sigma_{a_i} a_i)] + \sigma_a\left[d_0 + \sum_{i=1}^{n} d_i(\rho_{a_i} 1 + \sigma_{a_i} a_i)\right] \end{aligned}$$

where we may assume that $a_1 = a$. Comparing the corresponding coefficients on both sides, we use algebraic independence to argue that $b_1 = c_0$, $b_1 a = d_0$, while all other b_i, c_i, d_i vanish. This means $p^k(\eta 1) = c_0$, $p^k(\eta a) = c_0 a$ which thus holds for all $a \in \mathsf{R}$. Therefore, with the notation $\eta 1 = c \in \mathsf{R}$, we have $\eta a = ca$ for all $a \in \mathsf{R}$, showing that η acts on R as left multiplication by $c \in \mathsf{R}$. The same holds for $\tilde{\eta}$ and for $\eta = \tilde{\eta} \restriction A$. This completes the proof of the local case.

(e) Moving to the global case, suppose R is as stated in the theorem. We get $\tilde{\mathsf{R}} = \prod_p \tilde{\mathsf{R}}_{(p)}$ where $\tilde{\mathsf{R}}_{(p)}$ is the p-adic completion of the reduced part of the localization $\mathsf{R}_{(p)} = \mathbb{Z}_{(p)} \otimes \mathsf{R}$. For $a \in \mathsf{R}$ we write $a = (\dots, a_p, \dots)$ with $a_p \in \mathsf{R}_{(p)}$.

Just as in (b), for each $a \in \mathsf{R}$ choose ρ_a, σ_a now as $\rho_a = (\dots, \rho_{pa}, \dots), \sigma_a = (\dots, \sigma_{pa}, \dots)$ with $\rho_{pa}, \sigma_{pa} \in J_p$ algebraically independent over $\mathsf{S}_{(p)}$ for each p; note that if $\mathsf{R}_{(p)} = 0$, then $\rho_{pa} = \sigma_{pa} = 0$ can be chosen. Defining e_a as in (16.3) and A as in (16.4), A becomes a countable subgroup of $\tilde{\mathsf{R}}$. As in (c), we argue that R is isomorphic to a subring of $\operatorname{End} A$. Every $\eta \in \operatorname{End} A$ extends uniquely to $\tilde{\eta} \in \operatorname{End}(\tilde{\mathsf{R}}^+)$ which must act coordinate-wise in each $\tilde{\mathsf{R}}_{(p)}$, because these are fully invariant subrings in $\tilde{\mathsf{R}}$. By the local case, $\tilde{\eta}$ is left multiplication by the $\mathsf{R}_{(p)}$-component of $\eta 1 = c \in \mathsf{R}$, thus $\tilde{\eta}$ must agree with the left multiplication by c on all of $\tilde{\mathsf{R}}$, in particular on A. This establishes the claim that $\operatorname{End} A \cong \mathsf{R}$.

□

Since there is a set of cardinality $2^{\aleph_0}$ of pairwise disjoint finite subsets of algebraically independent elements of J_p, and since these define non-isomorphic torsion-free groups in the above construction, it is clear that there are $2^{\aleph_0}$ non-isomorphic solutions in Theorem 3.3.

The Topological Version Another point of interest emerges if the endomorphism rings are equipped with the finite topology. Then all endomorphism rings of countable reduced torsion-free groups can be characterized, even if they are uncountable. Note that the necessity of the condition stated in the next theorem is immediate: each left ideal L in Theorem 3.4 is defined to consist of all $\eta \in \operatorname{End} A$ that annihilate a fixed $a \in A$. However, the proof of sufficiency involves more ring theory than we care to get into, and therefore we state the theorem without proof.

Theorem 3.4 (Corner [4]). *A topological ring* R *is isomorphic to* $\operatorname{End} A$ *for a countable reduced torsion-free group* A *if and only if it is complete in the topology with a base of neighborhoods of* 0 *consisting of left ideals* L_n $(n < \omega)$ *such that the factor groups* R/L_n *are countable, reduced and torsion-free.* □

Quasi-Endomorphism Ring This is a most useful tool in dealing with finite rank torsion-free groups. The set of quasi-endomorphisms of a torsion-free A (see Sect. 9 in Chapter 12) is a $\mathbb{Q}$-algebra

$$\mathbb{Q}\operatorname{End} A = \mathbb{Q} \otimes \operatorname{End} A.$$

The fully invariant pure subgroups of A form a lattice $\mathfrak{F}$ where $G \cap H$ and $(G+H)_*$ are the lattice operations $(G, H \in \mathfrak{F})$.

Lemma 3.5 (Reid [3]). *Let* A *be a torsion-free group. The correspondences*

$$G \mapsto \mathbb{Q} \otimes G \qquad \textit{and} \qquad M \mapsto M \cap A$$

are inverse to each other between the lattice $\mathfrak{F}$ of fully invariant pure subgroups G and the $\mathbb{Q}\,\mathrm{End}\,A$*-submodules M of* $\mathbb{Q} \otimes A$.

Proof. Straightforward. □

Isomorphic Endomorphism Rings It seems that it does not make much sense to pose the question as to when the isomorphy of endomorphism rings of torsion-free groups implies the isomorphy of the groups themselves. Surprisingly it has an answer, albeit in a very special case, by the following theorem.

Theorem 3.6 (Sebel'din [1]). *Suppose that* $\mathrm{End}\,A \cong \mathrm{End}\,C$ *where A and C are direct sums of rational groups each of which is p-divisible for almost all primes p. Then* $A \cong C$.

Proof. If A has κ summands of type $\mathbb{Z}_{(p)}$ for a prime p, then $\mathrm{End}\,A$ has κ orthogonal primitive idempotents of this type. Hence from $\mathrm{End}\,A \cong \mathrm{End}\,C$ it follows that A and C must have the same numbers of summands of types $\mathbb{Z}_{(p)}$. Factor out the ideals of elements of types $\mathbb{Z}_{(p)}$ for all p, and repeat the same argument for types $\mathbb{Z}_{(p,q)}$ for different primes p, q (i.e., for rational groups r-divisible for all primes $r \neq p, q$). We can then conclude in the same way the equality of the numbers of summands of these types. If we keep doing this, including more and more primes, then the claim follows. □

Sebel'din points out that this is a sharp result: the hypothesis on A in the preceding theorem cannot be weakened: if A is of rank 1 and if its type $\mathbf{t}$ is finite at infinitely many primes, then there are non-isomorphic C of rank one with isomorphic endomorphism ring.

In an another special case, more can be stated:

Theorem 3.7 (Wolfson [2]). *Let* A, C *be homogeneous separable torsion-free groups of type* $\mathbf{t}(\mathbb{Z})$. *If* $\psi : \mathrm{End}\,A \to \mathrm{End}\,C$ *is a ring isomorphism, then there exists an isomorphism* $\phi : A \to C$ *such that* $\psi(\eta) = \phi\eta\phi^{-1}$ *for all* $\eta \in \mathrm{End}\,A$.

Proof. If $\epsilon \in \mathrm{End}\,A$ is a primitive idempotent, then $A \cong (\mathrm{End}\,A)\epsilon$, and the same holds for C. Hence the existence of ϕ is immediate. The rest follows as in the proof of Theorem 2.5. □

★ **Notes.** Theorem 3.3 is one of the most significant theorems on endomorphism rings. It has been generalized, see Theorem 7.2. Corner [2] also proves that for finite rank groups, Theorem 3.3 has a noteworthy improvement: a reduced torsion-free ring of rank n is isomorphic to the endomorphism ring of a torsion-free group of rank $\leq 2n$. This is the sharpest result in general. Zassenhaus [1] found conditions for a ring of rank n to be the endomorphism ring of a group of the same rank. For a generalization of this result, see Dugas–Göbel [6].

Theorem 3.3 has been generalized by several authors, see Sect. 7. It was Göbel [1] who observed that Corner's theorem should be valid for uncountable groups under suitable assumptions. The first generalization to arbitrary cardinalities was given by Dugas–Göbel [2] for cotorsion-free rings under the hypothesis V = L.

There are numerous results on the endomorphism rings of a few selected classes of torsion-free groups. For example, Dugas–Thomé [2] discuss the Butler version, while the endomorphism rings of separable torsion-free groups were characterized by Metelli–Salce [1] in the homogeneous case, by Webb [1] for homogeneously decomposable groups, and by Bazzoni–Metelli [1] in the general case. The conditions are similar to those given by Liebert [3] for p-groups. Blagoveshchenskaya [1] investigates the case of countable torsion-free groups.

Faticoni [Fat] observes and demonstrates that for torsion-free finite rank groups, the endomorphism ring modulo the nilradical is more tractable than the ring itself. Chekhlov [3] considers groups whose idempotent endomorphisms are central, including the separable and cotorsion cases. For a reduced torsion-free A, Krylov [5] defines the ideal $\mathsf{H}(A)$ of $\mathsf{E} = \operatorname{End} A$ as the set of all $\eta \in \mathsf{E}$ for which $h_p(\eta x) > h_p(x)$ for all $x \in A$ if the latter height is finite (this being the torsion-free analogue of the ideal denoted by the same symbol in the torsion case). If A is of finite rank, then the Jacobson radical J of E contains $\mathsf{H}(A)$ and is nilpotent mod it. For more on End, see also Krylov [6].

Exercises

(1) If A is torsion-free, then $p \operatorname{End} A = \operatorname{End} A$ exactly if $pA = A$.
(2) Every reduced torsion-free ring of rank one is the endomorphism ring of a torsion-free group of rank $n \in \mathbb{N}$.
(3) (Corner) If $\operatorname{End} A$ is countable, reduced and torsion-free, then A must be reduced and torsion-free.
(4) (Corner) A ring whose additive group is $\cong J_p \oplus J_p$ cannot be the endomorphism ring of any group.
(5) For an arbitrarily large cardinal κ, there exist torsion-free groups of cardinality κ whose endomorphism rings have cardinality 2^κ.
(6) For a finite rank torsion-free A, $\operatorname{End} A$ has no divisors of zero if and only if all endomorphisms are monic if and only if $\mathbb{Q} \operatorname{End} A$ is a division ring.
(7) The center of the endomorphism ring of a homogeneous separable torsion-free group is a subring of $\mathbb{Q}$.
(8) (Hauptfleisch) Suppose that $\operatorname{End} A \cong \operatorname{End} C$ where A and C are homogeneous separable torsion-free groups of types $\mathbf{t}$ and $\mathbf{s}$, respectively.
    (a) If $\mathbf{t} = \mathbf{s}$, then every isomorphism between the endomorphism rings is induced by an isomorphism between the groups.
    (b) If $\mathbf{t} \neq \mathbf{s}$, then $A \otimes S \cong C \otimes T$ where T, S are rational groups of type $\mathbf{t}$ and $\mathbf{s}$, respectively.
(9) Is Theorem 3.7 true for any type $\mathbf{t}$?
(10) (J. Reid, Orsatti) Call a ring R **subcommutative** if for all $r, s \in \mathsf{R}$ there is $t \in \mathsf{R}$ such that $rt = sr$.
    (a) The ring of the integral quaternions $q = a+bi+cj+dk$ with $a, b, c, d \in \mathbb{Z}$ is subcommutative.
    (b) If $\operatorname{End} A$ is subcommutative, then endomorphic images of A are fully invariant subgroups.
    (c) Conclude that the full invariance of endomorphic images does not imply the commutativity of the endomorphism ring.

4 Endomorphism Rings of Special Groups

Our next concern is with the endomorphism rings of some important types of groups, like projective, injective, etc. groups.

Endomorphism Rings of Free Groups We start with free groups. It might be helpful to note in advance that if F is a free group, then the image of an endomorphism η is also a free subgroup, and therefore $\operatorname{Ker}\eta$ is always a summand of F.

At this point, we need a definition: a ring R is called **Baer ring** if the left (or, equivalently, the right) annihilator of a non-empty subset of R is generated by an idempotent.

Theorem 4.1 (Wolfson [1], Rangaswamy [2], Tsukerman [1]). *The endomorphism ring of a free group is a Baer ring if and only if the group is countable.*

Proof. We show that, for a free group F, $\operatorname{End} F$ is a Baer ring if and only if F enjoys the strong summand intersection property, i.e. intersections of any number of summands are summands. Then the claim will follow from Proposition 7.12 in Chapter 3.

Assume F has the strong summand intersection property. Consider the right annihilator $\mathsf{N} \le \operatorname{End} F = \mathsf{E}$ of a subset $X \subset \mathsf{E}$, and let $K = \cap_{\xi \in X} \operatorname{Ker} \xi$. Then K is a summand of F by hypothesis, thus $K = \epsilon F$ for an idempotent $\epsilon \in \mathsf{E}$. Hence $\xi\epsilon = 0$ for all $\xi \in X$, moreover, $\xi\alpha = 0$ for an $\alpha \in \mathsf{E}$ if and only if $\alpha F \le \epsilon F$—which holds if and only if $\epsilon\alpha = \alpha$, i.e. $\alpha \in \epsilon\mathsf{E}$. Thus $\mathsf{N} = \epsilon\mathsf{E}$, and $\operatorname{End} F$ is a Baer ring.

Conversely, suppose $\operatorname{End} F = \mathsf{E}$ is a Baer ring, and $K = \cap_{\xi \in X} \operatorname{Ker} \xi$ for some set $X \subset \mathsf{E}$, where the summands are written in the form $\operatorname{Ker} \xi$. Define N as the right annihilator ideal of X, thus $\mathsf{N} = \epsilon\mathsf{E}$ with an idempotent ϵ. Then $K = \epsilon F$ is a summand of F. □

Endomorphism Rings of Algebraically Compact Groups Next, we consider algebraically compact groups. We know from Theorem 2.11 in Chapter 7 that their endomorphism groups are also algebraically compact, but we can now prove much more.

Theorem 4.2.

(a) *The endomorphism ring of an algebraically compact group is a right algebraically compact ring.*
(b) *The same holds if the group is torsion-complete.*

Proof.

(a) We show: if A is algebraically compact, then $\mathsf{E} = \operatorname{Hom}(A, A)$ is an algebraically compact right E-module (the first A being viewed as a left E-module). Let N be a pure submodule in a right E-module M. It suffices to prove that the group homomorphism

$$\operatorname{Hom}_{\mathsf{E}}(M, \operatorname{Hom}(A, A)) \to \operatorname{Hom}_{\mathsf{E}}(N, \operatorname{Hom}(A, A))$$

induced by the inclusion map $N \to M$ is surjective. In view of a well-known natural isomorphism, this can be rewritten as

$$\mathrm{Hom}(M \otimes_{\mathsf{E}} A, A) \to \mathrm{Hom}(N \otimes_{\mathsf{E}} A, A).$$

Since $N \to M$ is a pure inclusion as right E-modules, $N \otimes_{\mathsf{E}} A \to M \otimes_{\mathsf{E}} A$ is a pure inclusion group-theoretically. Therefore, by the pure-injectivity of A, the map in question is in fact surjective.

(b) A reduced torsion group A has the same endomorphism ring as its cotorsion hull $A^{\bullet}$. For a torsion-complete group, the cotorsion and pure-injective hulls are identical. □

Endomorphism Rings of Divisible Groups We now turn to injective groups; their endomorphism rings are very special, indeed.

Theorem 4.3. *The endomorphism ring* E *of a divisible group* D *is a right algebraically compact ring whose Jacobson radical* J *consists of those* $\eta \in \mathsf{E}$ *for which* $\mathrm{Ker}\,\eta$ *is an essential subgroup in* D.

Idempotents mod J *lift, and* E/J *is a von Neumann regular ring.*

Proof. By Theorem 4.2, E is an algebraically compact ring. Let $\chi \in \mathsf{E}$ be such that $\mathrm{Ker}\,\chi$ is essential in D. Then $\mathrm{Ker}\,\eta\chi$ is also essential in D for every $\eta \in \mathsf{E}$. Since $\mathrm{Ker}\,\eta\chi \cap \mathrm{Ker}(1-\eta\chi) = 0$, we obtain that $\mathrm{Ker}(1-\eta\chi) = 0$, so $1-\eta\chi$ is a monomorphism. Evidently, $\mathrm{Ker}\,\eta\chi \leq \mathrm{Im}(1-\eta\chi)$, therefore $\mathrm{Im}(1-\eta\chi)$ is an essential divisible subgroup, whence it follows that, for all η, $1-\eta\chi$ is an automorphism of D, i.e. an invertible element in E. Thus $\chi \in \mathsf{J}$. Conversely, if $\chi \in \mathsf{J}$, then let K be a subgroup in D with $K \cap \mathrm{Ker}\,\chi = 0$. By the injectivity of D, there is a $\xi : D \to D$ such that $\xi\chi x = x$ for all $x \in K$. This means that $K \leq \mathrm{Ker}(1-\xi\chi)$ where $1-\xi\chi \in \mathrm{Aut}\,D$ (as $\chi \in \mathsf{J}$). Thus $K = 0$, and $\mathrm{Ker}\chi$ is essential in D.

If $\chi \in \mathsf{E}$ satisfies $\chi^2 - \chi \in \mathsf{J}$, then $N = \mathrm{Ker}(\chi^2-\chi)$ is essential in D. Now $\chi N \cap \mathrm{Ker}\,\chi = 0$, for if χa ($a \in N$) is annihilated by χ, then $0 = \chi(\chi a) = \chi a$. The subgroup $\chi N \oplus \mathrm{Ker}\,\chi$ is essential in D, since it contains N: every $a \in N$ can be written in the form $a = \chi a + (1-\chi)a \in \chi N \oplus \mathrm{Ker}\,\chi$. Thus $D = B \oplus C$ where $\chi N \leq B, \mathrm{Ker}\,\chi \leq C$. If $\epsilon : D \to B$ denotes the projection with $\epsilon C = 0$, then $\chi - \epsilon$ annihilates every $a = \chi a + (1-\chi)a \in N$. Therefore, $\chi - \epsilon \in \mathsf{J}$, showing that χ lifts to ϵ mod J.

Assume now that $\chi \in \mathsf{E}$, and $K \leq D$ is maximal with respect to the property $K \cap \mathrm{Ker}\,\chi = 0$. Then $K + \mathrm{Ker}\,\chi \leq \mathrm{Ker}(\chi - \chi\xi\chi)$ where $\xi \in \mathsf{E}$ is as above. Hence $\mathrm{Ker}(\chi - \chi\xi\chi)$ is essential in D, and so $\chi - \chi\xi\chi \in \mathsf{J}$. We conclude that E/J is a von Neumann regular ring. □

Although we won't do it here, with some extra effort we can even prove that the factor ring E/J is self-injective; see Notes.

Endomorphism Rings of Cotorsion Groups Finally, we prove something concerning endomorphism groups of cotorsion groups that should not be surprising.

Theorem 4.4. *The endomorphism group of a cotorsion group is cotorsion.*

Proof. If C is an adjusted cotorsion group, then $\operatorname{End} C \cong \operatorname{End} tC$ by Proposition 1.7(ii). As tC is torsion, $\operatorname{End} tC$ is algebraically compact.

In general, let $G = C \oplus A \oplus D$ where C is adjusted cotorsion, A is reduced torsion-free algebraically compact, and D is divisible. Since homomorphism groups into algebraically compact groups are algebraically compact, $\operatorname{Hom}(G, A \oplus D)$ is algebraically compact (Theorem 2.11 in Chapter 7). To find the structure of $\operatorname{Hom}(G, C)$, observe that $\operatorname{Hom}(D, C) = 0$, thus it remains to check $\operatorname{Hom}(A, C)$. If B is a basic subgroup of the complete group A, then we have an exact sequence $0 \to B \to A \to \oplus \mathbb{Q} \to 0$. This implies the exactness of $0 = \operatorname{Hom}(\oplus \mathbb{Q}, C) \to \operatorname{Hom}(A, C) \to \operatorname{Hom}(B, C) \to \operatorname{Ext}(\oplus \mathbb{Q}, C) = 0$, thus the middle terms are isomorphic. As B is the direct sum of copies of $\mathbb{Z}_{(p)}$ for various primes p, $\operatorname{Hom}(B, C)$ will be the direct product of copies of $C_{(p)}$ for various primes p, so cotorsion. □

★ **Notes.** In general, endomorphism rings of cotorsion groups need not be cotorsion rings, but they are in the torsion-free case (Theorem 4.2). For the endomorphism ring of an algebraically compact group, a result similar to the divisible case can be established: idempotents mod its Jacobson radical J lift, and E/J is a von Neumann regular self-injective ring. This follows from a ring-theoretical result by Zimmermann–Zimmermann-Huisgen [Math. Z. **161**, 81–93 (1978)] stating that the radical of a pure-injective ring has this property.

Exercises

(1) (Szélpál) (a) $\operatorname{End} A$ is a torsion ring if and only if A is a bounded group.

(b) $\operatorname{End} A$ is torsion-free if and only if the reduced part of A is torsion-free, and $pA \neq A$ implies $A[p] = 0$.

(2) The endomorphism ring of a cotorsion group need not be algebraically compact as a group.

(3) The endomorphism ring of the additive group of any injective module is algebraically compact.

(4) (Rangaswamy) For a torsion group A, $\operatorname{End} A$ is a Baer ring if and only if, for every prime p, A_p is either divisible or elementary.

5 Special Endomorphism Rings

So far our investigations relating a group and its endomorphism ring have mainly been concerned with the question as to how the group structure is reflected in the endomorphism ring. In this section we approach this problem from the opposite direction, and the dominant theme will be the kind of influence properties of endomorphism rings have on the underlying group. The results in this section would seem to suggest that specific endomorphism rings are indeed special. Needless to say, we are primarily interested in conventional ring properties.

It seems sensible at this point to stress that one should not have high expectations of the interrelations between interesting group and important ring properties. We keep emphasizing that significant group and ring properties seldom match: groups whose endomorphism rings are of special interest in ring theory are few and far between.

We are going to survey several cases of interest.

Elementary Properties We start assembling a few simple observations that will simplify subsequent arguments. We will abbreviate $\mathsf{E} = \mathrm{End}\, A$.

(a) *Suppose* $\alpha \in \mathsf{E}$. *If* $n|\alpha$, *then* $\alpha A \leq nA$, *and if* $n\alpha = 0$, *then* $\alpha A \leq A[n]$ *for* $n \in \mathbb{N}$. Indeed, if $\beta \in \mathsf{E}$ satisfies $n\beta = \alpha$, then $\alpha A = n\beta A \leq nA$, and if $n\alpha = 0$, then $n\alpha A = 0$, so $\alpha A \leq A[n]$.

(b) *If* $\mathrm{End}\, A = \mathsf{R}_1 \oplus \mathsf{R}_2$ *is a ring direct sum, then* $A = B_1 \oplus B_2$ *where* $\mathsf{R}_i \cong \mathrm{End}\, B_i$ *and* $\mathrm{Hom}(B_i, B_j) = 0$ *for* $i \neq j$. This is a simple consequence of Sect. 1(D).

(c) *If A has a sequence of direct decompositions*

$$A = A_0 \oplus \cdots \oplus A_n \oplus C_n \qquad \textit{with} \quad C_n = A_{n+1} \oplus C_{n+1} \quad (n < \omega)$$

where $A_n \neq 0$ $(n < \omega)$, *then neither the left nor the right* (*principal*) *ideals of* E *satisfy the minimum condition.* To verify this, let $\pi_n : A \to C_n$ denote the obvious projection. Then $\pi_n \pi_{n+1} = \pi_{n+1}$, but no $\alpha \in \mathsf{E}$ exists satisfying $\pi_{n+1}\alpha = \pi_n$, because $\mathrm{Im}\, \pi_{n+1}\alpha \leq \mathrm{Im}\, \pi_{n+1} < \mathrm{Im}\, \pi_n$. This establishes the proper inclusion $\pi_{n+1}\mathsf{E} < \pi_n \mathsf{E}$ for every n. Also, $\pi_{n+1}\pi_n = \pi_{n+1}$, and there is no $\beta \in \mathsf{E}$ with $\beta\pi_{n+1} = \pi_n$, since clearly $\mathrm{Ker}\, \pi_n < \mathrm{Ker}\, \pi_{n+1} \leq \mathrm{Ker}\, \beta\pi_{n+1}$. This shows that $\mathsf{E}\pi_{n+1} < \mathsf{E}\pi_n$.

(d) *If* γ *is in the center of* E, *then* $\mathrm{Im}\, \gamma$ *and* $\mathrm{Ker}\, \gamma$ *are fully invariant subgroups of* A. For each $\eta \in \mathsf{E}$, we have $\eta\gamma(A) = \gamma\eta(A) \leq \mathrm{Im}\, \gamma$, and if $a \in \mathrm{Ker}\, \gamma$, then $\gamma\eta(a) = \eta(\gamma(a)) = 0$ means $\eta(a) \in \mathrm{Ker}\, \gamma$.

(e) *In a torsion ring* (*with or without identity*), *the divisible subgroup is contained in the annihilator of the ring.* Thus $1 \in \mathsf{R}$ implies R is reduced.

Recall that an element $\alpha \in \mathsf{R}$ is **regular** (in the sense of von Neumann) if $\alpha\beta\alpha = \alpha$ for some $\beta \in \mathsf{R}$, and π**-regular** if α^m is regular for some $m \in \mathbb{N}$.

Lemma 5.1 (Rangaswamy [1]). *An endomorphism* α *of* A *is a* (*von Neumann*) *regular element in* $\mathrm{End}\, A$ *if and only if both* $\mathrm{Im}\, \alpha$ *and* $\mathrm{Ker}\, \alpha$ *are summands of* A.

Proof. Assume $\alpha \in \mathsf{E} = \mathrm{End}\, A$ satisfies $\alpha\beta\alpha = \alpha$ for some $\beta \in \mathsf{E}$. Since $\alpha\beta$ and $\beta\alpha$ are idempotents, they are projections of A, so their images and kernels are summands of A. The inclusions $\mathrm{Im}\, \alpha\beta\alpha \leq \mathrm{Im}\, \alpha\beta \leq \mathrm{Im}\, \alpha$ and $\mathrm{Ker}\, \alpha \leq \mathrm{Ker}\, \beta\alpha \leq \mathrm{Ker}\, \alpha\beta\alpha$ imply $\mathrm{Im}\, \alpha = \mathrm{Im}\, \alpha\beta$ and $\mathrm{Ker}\, \alpha = \mathrm{Ker}\, \beta\alpha$, so $\mathrm{Im}\, \alpha, \mathrm{Ker}\, \alpha$ are summands of A.

Conversely, suppose $\mathrm{Im}\, \alpha = B$ and $\mathrm{Ker}\, \alpha = K$ are summands of A, say, $A = B \oplus C = K \oplus H$ for some $C, H \leq A$. Because of $K \cap H = 0$, α maps H isomorphically into, and evidently onto, B. This means, there is a $\beta \in \operatorname{End} A$ that annihilates C and is inverse to $\alpha \restriction H$ on B. Writing $a = g + h \in K \oplus H$, we get $(\alpha\beta\alpha)a = (\alpha\beta)\alpha h = \alpha h = \alpha a$ for every $a \in A$, whence $\alpha\beta\alpha = \alpha$ follows. □

Simple Endomorphism Rings Our survey of special endomorphism rings starts with division rings.

Proposition 5.2 (Szele [2]). *The endomorphism ring of A is a division ring if and only if $A \cong \mathbb{Q}$ or $A \cong \mathbb{Z}(p)$ for some prime p.*

A division ring is the endomorphism ring of a group if and only if it is the additive group of a prime field.

Proof. If $\operatorname{End} A$ is a division ring, then every non-zero $\alpha \in \operatorname{End} A$ is an automorphism, which entails, for every prime p, either $pA = A$ or $pA = 0$. It also follows that $0, 1$ are the only idempotents in $\operatorname{End} A$. After ruling out $\mathbb{Z}(p^\infty)$, it is clear that the only possibility is either $A \cong \mathbb{Z}(p)$ for some p, or else $A \cong \mathbb{Q}$. □

By a **simple ring** is meant a ring R whose only ideals are 0 and R.

Theorem 5.3. $\operatorname{End} A$ *is a simple ring if and only if, for some integer n, either $A \cong \oplus_n \mathbb{Q}$ or $A \cong \oplus_n \mathbb{Z}(p)$ for a prime p.*

A simple ring is the endomorphism ring of a group if and only if it is a complete matrix ring of finite order over a prime field.

Proof. Sufficiency is obvious from the matrix representation of direct sums. Conversely, if $\mathsf{E} = \operatorname{End} A$ is a simple ring, then for every prime p, either $p\mathsf{E} = 0$ or $p\mathsf{E} = \mathsf{E}$. In the first alternative, (a) above implies that $A \leq A[p]$, i.e. A is an elementary p-group. If $p\mathsf{E} = \mathsf{E}$ for every p, then (a) applied to $\alpha = 1$ shows that A is divisible. In this case, the socle of E, if not zero, would be a non-trivial ideal, therefore E must be torsion-free, and division by every p is an automorphism of A. Consequently, A is likewise torsion-free divisible. Our conclusion is that either $A \cong \oplus \mathbb{Z}(p)$ or $A \cong \oplus \mathbb{Q}$. The endomorphisms mapping A onto a finite rank subgroup form an ideal in E, by simplicity this must be all of E. Hence A is a finite direct sum, and we are done, since the second assertion is pretty clear from our argument. □

Artinian Endomorphism Rings The preceding theorem shows that simple rings can be endomorphism rings only if they are left and right artinian. It is not difficult to identify all artinian endomorphism rings.

Theorem 5.4.

(i) $\operatorname{End} A$ *is left* (*right*) *artinian if and only if*

$$A = B \oplus D$$

where B is finite and D is torsion-free divisible of finite rank.

(ii) (Szász [1]) *The same holds for left* (*right*) *perfect rings.*

Proof.

(i) If $\mathsf{E} = \operatorname{End} A$ is left (right) artinian, then in the set of ideals $\{n\mathsf{E} \mid n \in \mathbb{N}\}$ there is a minimal one, say, $m\mathsf{E}$; clearly, it must be divisible. (a) above shows that $nm|m\mathbf{1}_A$ implies $mA \le nmA$ for every $n \in \mathbb{N}$, i.e. mA is divisible. Consequently, $A = B \oplus D$ where $mB = 0$ and D is divisible. From (c) we derive that both B and D are of finite rank; in particular, B is a finite group. The presence of $\mathbb{Z}(p^\infty)$ is ruled out, since a ring with $\mathbf{1}$ cannot have a non-zero annihilator.

For the converse, assume A has the stated decomposition. Then the two summands are fully invariant, so $\operatorname{End} A = \operatorname{End} B \oplus \operatorname{End} D$. Hence $\operatorname{End} B$ is finite and $\operatorname{End} D$ is a complete matrix ring of finite order over $\mathbb{Q}$, so $\operatorname{End} A$ is artinian.

(ii) The same proof goes through under the milder hypothesis that the principal left (right) ideals satisfy the minimum condition, i.e. $\operatorname{End} A$ is a left (right) perfect ring. □

PID Endomorphism Rings We have seen that the endomorphism rings of Procházka-Murley groups were PID. Clearly, a group with PID endomorphism ring is indecomposable. More specific results can be stated about the behavior of summands with PID endomorphism rings. A typical result is recorded in the next proposition. We recall that quasi-isomorphic groups whose endomorphism rings are PID have isomorphic endomorphism rings (cf. Lemma 9.8 in Chapter 12).

Proposition 5.5. *Let $A = A_1 \oplus \cdots \oplus A_n$ where the summands are quasi-isomorphic finite rank torsion-free groups and have endomorphism rings isomorphic to the same principal ideal domain* D.

(i) *For every endomorphism η of A,* $\operatorname{Ker} \eta$ *is a summand of A. It is the direct sum of* D*-modules quasi-isomorphic to the A_i.*

(ii) *A pure subgroup of A that is quasi-isomorphic to one* (*and hence to each*) *of A_i, is a summand of A.*

Proof.

(i) An endomorphism η of A can be represented as an $n \times n$ matrix $\mathbb{M} = \|\eta_{ij}\|$ with entries in the field Q of quotients of D. For the proof, we multiply $\mathbb{M}$ by a suitable $0 \ne \gamma \in \mathsf{D}$ so as to have all the entries contained in D. By hypothesis, D is a PID, so there exist invertible matrices $\mathbb{A} = \|\alpha_{ki}\|, \mathbb{B} = \|\beta_{j\ell}\| \in \mathsf{M}_n(\mathsf{D})$ (the full matrix ring of order n over D) such that $\mathbb{AMB} = \|\delta_{k\ell}\|$ is a diagonal matrix: $\delta_{k\ell} = 0$ for $k \ne \ell$ (and $\delta_{kk}|\delta_{\ell\ell}$ if $k < \ell$). An easy calculation shows that its diagonal entries are

$$\delta_{kk} = \sum_{i,j} \alpha_{ki}(\gamma\eta_{ij})\beta_{jk} \in \gamma\mathsf{D} \quad (k \le n),$$

where the indicated inclusion $\delta_{kk} \in \gamma\mathsf{D}$ is justified by the fact (as we shall see in the next paragraph) that the diagonal elements represent endomorphisms of quasi-isomorphic modules, so by Lemma 9.8 in Chapter 12 they have to be elements of $\operatorname{End} A_i = \mathsf{D}$. (One can also argue that $\gamma\mathbb{AMB}$ represents a map

$A \to \rho A$ which must evidently be γ times of a map $A \to A$.) Consequently, the matrix of η admits a diagonal form with entries in Q.

This fact translated to maps asserts that there are automorphisms ρ, σ of A (corresponding to $\mathbb{A}, \mathbb{B}$) such that $\chi = \rho\eta\sigma$ is represented by a diagonal matrix. The zero columns in this matrix correspond to the kernel of χ; consequently, the kernel is of the form $\operatorname{Ker}\chi = B_1 \oplus \cdots \oplus B_m$ with $m \leq n$. Each component B_j is obtained as a sum of isomorphic copies of some of the components A_i, so it is evidently quasi-isomorphic to the A_i. Hence for $\operatorname{Ker}\eta = \operatorname{Ker}\rho\eta = \operatorname{Ker}\chi\sigma^{-1} = \sigma(\operatorname{Ker}\chi)$ we have

$$\operatorname{Ker}\eta = \sigma(B_1 \oplus \cdots \oplus B_m) = \sigma B_1 \oplus \cdots \oplus \sigma B_m.$$

(ii) Let $\pi_i : A \to A_i$ denote the projections in the given direct decomposition, and let us fix quasi-isomorphisms $\alpha_i : A_i \to C$ ($i = 1, \dots, n$) where C is a pure subgroup in A. Evidently, $\alpha_i\pi_i \restriction C$ is an endomorphism of C, say, it is equal to some $\epsilon_i \in \mathsf{D}$ (Lemma 9.8 in Chapter 12). Let $\epsilon = \gcd\{\epsilon_1, \dots, \epsilon_n\}$ be calculated in D; thus, we have $\epsilon = \tau_1\epsilon_1 + \cdots + \tau_n\epsilon_n$ for suitable $\tau_i \in \mathsf{D}$. Then $\eta = \sum_i \tau_i\alpha_i\pi_i : A \to C$ is an endomorphism of A whose restriction to C acts like ϵ.

We use induction on n to complete the proof. If $n = 1$, then $C \leq A_1$, and by purity equality holds. Next suppose $n > 1$. Evidently, $\xi = \eta - \epsilon$ is a non-zero endomorphism of A that annihilates C. Owing to (i), $\operatorname{Ker}\xi$ is a summand of A, and also a direct sum of groups quasi-isomorphic to the A_i. Since $\xi \neq 0$, this summand has less than n components, so induction hypothesis applies. The claim that C is a summand follows at once. □

Noetherian Endomorphism Rings It is a trivial observation that if $\operatorname{End} A$ is noetherian, then A decomposes into the direct sum of a finite number of indecomposable groups. It seems difficult to say much more in general about groups with noetherian endomorphism rings: just consider arbitrarily large torsion-free groups whose endomorphism rings are $\cong \mathbb{Z}$. The only hope is to put aside torsion-free groups, and concentrate on torsion groups. Luckily, we have a complete description in this case.

Theorem 5.6. *Suppose A is a torsion group.* $\operatorname{End} A$ *is left* (*or right*) *noetherian exactly if A is finitely cogenerated.*

Proof. If $\operatorname{End} A$ is noetherian, then A is a finite direct sum of indecomposable (i.e. cocylic) groups, so it is finitely cogenerated. Conversely, if A is finitely cogenerated, then by Theorem 5.3 in Chapter 4, $A = B \oplus D$ with finite B and finite rank divisible D. In the additive decomposition $\operatorname{End} A = \operatorname{Hom}(B, D) \oplus \operatorname{End} D$, the first summand is finite, while the second summand is a finite direct sum of complete matrix rings over the p-adic integers, for various primes. Thus $\operatorname{End} A$ is a finite extension of a two-sided noetherian ring. □

Notice a kind of duality: a torsion group A has the maximum (minimum) condition of subgroups if and only if $\operatorname{End} A$ has the minimum (maximum) condition on left (or right) ideals.

Regular Endomorphism Rings Our next project in this section is concerned with (von Neumann) regular rings.

Example 5.7.

(a) The endomorphism ring of an elementary group is regular. This is obvious from the fact that every subgroup is a summand; see Lemma 5.1.
(b) A direct product $\prod_p A_p$ of elementary p-groups A_p, for different primes p, has also regular endomorphism ring. This follows from the isomorphism $\mathrm{End}(\prod_p A_p) \cong \mathrm{End}(\oplus_p A_p)$.

Example 5.8. Torsion-free divisible groups have regular endomorphism rings.

Theorem 5.9. *If* $\mathrm{End}\, A$ *is a von Neumann regular ring, then* $A = D \oplus G$ *where*

(i) D *is torsion-free divisible; it is* 0 *if* G *is not torsion;*
(ii) G *is a pure subgroup between* $\oplus_{p\in P} T_p$ *and* $\prod_{p\in P} T_p$, *where* T_p *is an elementary* p-*group, and* P *is a set of primes.*

Proof. If $\mathrm{End}\, A$ is a regular ring, then to every $\alpha \in \mathrm{End}\, A$ there is a $\beta \in \mathrm{End}\, A$ such that $\alpha\beta\alpha = \alpha$. Specifically, if α is multiplication by p, then $p\beta p = p$, which implies that A contains no elements of order p^2, so A_p, the p-component of A, is an elementary group. Since also $p = p\beta p\beta p = p\beta p\beta p\beta p = \dots$, it is clear that, for every $a \in A$, $h_p(pa) = \infty$. Therefore, pG is p-divisible. Write $A = D \oplus G$ with D divisible and G reduced; at this point we already know that D is torsion-free, and $T = tG$ is an elementary group. We have $G = T_p \oplus G(p)$ with p-divisible $G(p)$, and it follows that $G(p) = pG$. As G/T_p is p-divisible, G/T is divisible.

The intersection $\cap_p pG$ is clearly a torsion-free subgroup of G. For every $a \in G$, pa belongs to every G_q (for every prime q), and is divisible in G_q by every p^k ($k \geq 2$) uniquely. This means $\cap_p pG$ is divisible, and hence 0. Thus the intersection of the kernels of the projections $\pi_p : G \to T_p$ is 0, and consequently, G is isomorphic to a subgroup of the direct product $\prod_{p\in P} T_p$, containing $T = \oplus_{p\in P} T_p$ with divisible G/T.

If G is not torsion and $D \neq 0$, then A has an endomorphism mapping $G \to D$ non-trivially whose kernel contains T, but is not a summand (G has no proper summand containing T). This is in violation to Lemma 5.1. □

Semi-Local Endomorphism Rings Finally, we consider local and semi-local endomorphism rings. Recall that a ring R is said to be **semi-local** if R/J is a semi-simple artinian ring; as usual, here J denotes the Jacobson radical of R. R is **local** if R/J is a division ring.

Lemma 5.10 (Orsatti [1]). *A torsion group has local endomorphism ring if and only if it is cocyclic.*

Proof. A local ring has only two idempotents: 0 and 1, therefore a group with local endomorphism ring is indecomposable. A torsion group is indecomposable if and only if it is cocyclic. Since $\mathrm{End}\, \mathbb{Z}(p^n) \cong \mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{End}\, \mathbb{Z}(p^\infty) \cong J_p$ are local rings, the claim is evident. □

For the semi-local case we prove:

Theorem 5.11 (Călugăreanu [1]). *A group A has semi-local endomorphism ring if and only if $A = T \oplus G$ where T is a finitely cogenerated group, and G is torsion-free with semi-local* End G.

In case G has finite rank, End G *is semi-local exactly if $pG = G$ for almost all primes p.*

Proof. Assume End A is semi-local. A semi-local ring has but a finite number of orthogonal idempotents, hence A is a finite direct sum of indecomposable groups. This must be true for $tA = T$ as well, whence $A = T \oplus G$ where T and G are as stated (because summands inherit semi-local endomorphism rings). If $pG \neq G$, then also $p \operatorname{End} G \neq \operatorname{End} G$, and there is at least one maximal left ideal L_p between $p \operatorname{End} G$ and End G. Since $\mathsf{L}_p \neq \mathsf{L}_q$ if $p \neq q$ are primes, and since in a semi-local ring the Jacobson radical is the intersection of finitely many maximal left ideals, $pG < G$ can hold only for a finite number of primes.

Conversely, if A is as stated, then both T and G have semi-local endomorphism rings, and it is easy to check that the same is true for A (the radical of End A is then the direct sum of the radicals of End T and End G). If G is of finite rank, then so is End A, and because $\operatorname{End} A / p \operatorname{End} A$ is not 0 but for finitely many primes p, and then it is finite, the same must hold for $(\operatorname{End} A)/\mathsf{J}$. Hence J is a finite intersection of maximal left ideals, and $(\operatorname{End} A)/\mathsf{J}$ is a finite direct sum of simple artinian rings. □

★ **Notes.** It was Szele who initiated a systematic study of groups whose endomorphism rings belong to a class of rings of interest, and since then this has been a recurrent theme in the literature. While the theorems above convince us that some familiar ring properties of the endomorphism ring might impose severe restrictions on the groups, it is unlikely that classes of groups whose endomorphism rings share some other prominent ring properties, like commutativity, hereditariness etc. will admit a satisfactory description in the near future, since these do not impose much restriction on the groups. There exists an extensive literature on special endomorphism rings of restricted classes of groups, many of these results are technical or involve more advanced ring theory.

For more results on cases when End A is regular, or when its principal right ideals are projective, see Glaz–Wickless [1]. Karpenko–Misyakov [1] describe the groups whose endomorphism rings have regular center. Mader [5] investigates the maximal regular ideal in endomorphism rings. Groups with π-regular endomorphism rings were studied in Fuchs–Rangaswamy [1]; the results are similar to Example 5.7. See F. Kasch–A. Mader, *Regularity and substructures of Hom* (2009) where regular homomorphisms were studied.

Misyakova [1] examines the case when the endomorphism ring is semi-prime (i.e., the intersection of prime ideals is 0). Salce–Menegazzo [1] investigate groups with linearly compact endomorphism rings. Several papers are devoted to self-injective endomorphism rings, see Albrecht [2], Ivanov [6], Rangaswamy [3].

Exercises

(1) The endomorphism ring of a separable torsion-free group A is left (or right) noetherian if and only if A is completely decomposable of finite rank.

(2) Find groups in which every endomorphism is either an automorphism or nilpotent.
(3) (Szele–Szendrei)

(a) If $\operatorname{End} A$ is commutative, then the p-components A_p are cocyclic, and $A/t(A)$ is p-divisible for every p with $A_p \neq 0$.
(b) A splitting mixed group A has commutative endomorphism ring exactly if both $\operatorname{End} t(A)$ and $\operatorname{End}(A/t(A))$ are commutative, and A satisfies the condition in (a).

(4) The endomorphism ring of $A = \oplus_p \mathbb{Z}(p) \oplus \prod_p \mathbb{Z}(p)$ is not regular.
(5) (Rangaswamy) Kernels and images of all the endomorphisms of A are pure subgroups in A if and only if $t(A)$ is elementary and $A/t(A)$ is divisible.
(6) (Rangaswamy) If the kernel of an $\eta \in \operatorname{End} A$ is a summand of A, then the right annihilator of η is a projective right ideal of $\operatorname{End} A$. [Hint: if $\epsilon : A \to \operatorname{Ker} \eta$ is projection, then $\operatorname{Ann} \eta = \epsilon \operatorname{End} A$.]
(7) (Călugăreanu) A divisible group has semi-local endomorphism ring exactly if it is of finite rank.
(8) If A is a p-group and $\operatorname{End} A$ is Dedekind-finite (i.e. $\alpha\beta = \mathbf{1}$ ($\alpha, \beta \in \operatorname{End} A$) implies $\beta\alpha = \mathbf{1}$), then the UK-invariants of A are finite.

6 Groups as Modules Over Their Endomorphism Rings

Since every group A is a left module over its own endomorphism ring $\mathsf{E} = \operatorname{End} A$, it is tempting to strive for a better understanding as to how A behaves as an E-module. A large number of special cases have already been investigated (mostly restricted to subcategories of $\mathcal{A}b$), but so far no systematic study is available. Here we collect a few special cases of interest in order to give a flavor of this interesting topic.

Preliminaries A few trivial facts to keep in mind:

(A) A group A is a faithful E-module (i.e. no non-zero element of A is annihilated by E).
(B) The E-submodules are exactly the fully invariant subgroups.
(C) If $\phi : A \to N$ is an E-homomorphism into an E-module N, then $\operatorname{Ker} \phi$ is a fully invariant subgroup of A.

Our starting point is a brief analysis of the E-homomorphisms into and from A. The basic facts for the primary case are summarized in the following lemma.

Lemma 6.1. *Let A be a p-group with a fully invariant subgroup H.*

(i) *An endomorphism is an E-map if and only if it is multiplication by a p-adic integer.*
(ii) *If A is either bounded or has an unbounded basic subgroup, then a homomorphism $\alpha : A \to A/H$ is an E-map if and only if it is the canonical homomorphism followed by multiplication by a p-adic integer.*

Proof. The 'if' parts of both claims are obvious, so we proceed to prove necessity. Let $A = C \oplus U$ be a decomposition of A where C is a cocyclic group. If C is cyclic of order p^n, then let $c \in C$ denote a generator, and if $C \cong \mathbb{Z}(p^\infty)$, then pick any non-zero element $c \in C$, say, of order p^n. Choose an $\eta \in \mathsf{E}$ so as to satisfy:

$$\eta\colon\ c \mapsto c + x, \quad u \mapsto tu \quad (\forall\, u \in U) \tag{16.5}$$

where t is any integer; $x \in A[p^n]$ is chosen arbitrarily if C is cyclic, while if C is quasi-cyclic, then x can be any element of order $\le p^n$ in the divisible subgroup of A.

(i) We need a brief calculation. Suppose $\theta \in \mathsf{E}$ is an E-map, and write $\theta(c) = kc + v \in A$ with $k \in \mathbb{Z}, v \in U$. Then

$$\eta\theta(c) = kc + kx + tv \quad \text{and} \quad \theta\eta(c) = kc + v + \theta(x).$$

The equality $\eta\theta = \theta\eta$ holds for all possible choices of t and x; this yields $v = 0$ and $\theta(x) = kx$. Hence we conclude that if C is cyclic, then θ acts on $A[p^n]$ as multiplication by k. If C is quasi-cyclic, then the same can be said only about the action of θ in the divisible subgroup of H. The integer k depends on the order of c, but for different elements the numbers must match, so it follows by usual arguments that there is a $\rho \in J_p$ such that θ acts as multiplication by ρ. In case A is bounded by p^n, then only multiplications mod p^n need to be considered.

(ii) Assume A is as stated above in (ii), so it has a cyclic summand $C = \langle c \rangle$ of order p^n with maximal n or with arbitrarily large n. If $A = C \oplus U$, then clearly, $A/H = C/(H \cap C) \oplus U/(H \cap U)$. For an E-map $\alpha : A \to A/H$, we set $\alpha c = k\bar{c} + \bar{v}$ with $k \in \mathbb{Z}, v \in U$ where bars indicate cosets mod H. If η is as above in (16.5), then $\eta\bar{c} = \bar{c} + \bar{x}$ and $\eta\bar{u} = t\bar{u}$. We then have

$$\eta\alpha(c) = \eta(k\bar{c} + \bar{v}) = k\bar{c} + k\bar{x} + t\bar{v},\ \alpha\eta(c) = \alpha(c + x) = k\bar{c} + \bar{v} + \alpha x.$$

These have to be equal for all permissible choices of x and t, whence $\bar{v} = 0$, $\alpha c = k\bar{c}$, and $\alpha x = k\bar{x}$. Hence α acts as multiplication by the integer k on $A[p^n]$ mod H, in particular, on $\langle \bar{c} \rangle \cong \langle c \rangle/(H \cap \langle c \rangle)$. As in (i), we argue that these numbers k define a $\rho \in J_p$ such that $\alpha x = \rho\bar{x}$ for all $x \in A$. Consequently, $\alpha = \phi\rho$ where $\phi : A \to A/H$ is the canonical map.

(From Theorem 6.11 below it will follow that (ii) fails for p-groups of the form $B \oplus D$ with bounded B and divisible $D \neq 0$.) □

From the preceding lemma it is obvious that the ring of E-endomorphisms of a p-group is isomorphic to the center of E.

To formulate the precise statements in the following theorems, we will need new definitions. To simplify, following [KMT], we adopt an easy terminology: a group will be called **endo-P** if it has property P as a left E-module.

Endo-Finitely-Generated Groups First, we exhibit a few examples for endo-finitely generated groups. Some are even endo-cyclic.

Example 6.2. An immediate example for an endo-finitely-generated group is any bounded group B. If $b \in B$ is an element of maximal order, then B is generated by b over $\mathsf{E} = \operatorname{End} B$. Thus $B = \mathsf{E}b$ is endo-cyclic.

Example 6.3. A torsion-free divisible group is also endo-cyclic. The direct sum of a torsion-free divisible group and a bounded group is endo-finitely generated.

Example 6.4. An E-group A (see Sect. 6 in Chapter 18 for definition) is endo-cyclic: if $a \in A$ corresponds to $1 \in \mathsf{E}$, then $A = \mathsf{E}a$.

Endo-finitely generated torsion groups admit a full characterization.

Proposition 6.5. *A torsion group is endo-finitely-generated if and only if it is bounded.*

Proof. As pointed out in Example 6.2, a bounded group is endo-cyclic. On the other hand, if A is unbounded torsion, then a finite set can generate over $\operatorname{End} A$ only a bounded subgroup of A. □

Endo-Artinian and Endo-Noetherian Groups It is not difficult to characterize the groups that are artinian over their endomorphism rings; however, only little has been established for the noetherian case.

Theorem 6.6 (Krylov–Mikhalev–Tuganbaev[KMT]).

(i) *A group A is endo-artinian if and only if $A = B \oplus D$ where B is a bounded group, and D is a divisible group with a finite number of non-zero p-components.*
(ii) *A is endo-noetherian exactly if $A = B \oplus C$ where B is a bounded group, and C is torsion-free endo-noetherian.*

Proof.

(i) Assume A is endo-artinian. Then there is a minimal subgroup among the subgroups of the form nA ($n \in \mathbb{N}$), say, mA is minimal. This mA is divisible, so $A = B \oplus mA$ with $mB = 0$. Clearly, mA can have but a finite number of p-components $\neq 0$, because these are E-submodules.

Conversely, if A is of the form stated in the theorem, then B has only a finite number of fully invariant subgroups. The torsion-free part of D has no proper fully invariant subgroup $\neq 0$, while the fully invariant proper subgroups of a p-component D_p of D are of the form $D_p[p^k]$, so they satisfy the minimum condition.
(ii) Supposing A is endo-noetherian, there is a maximal subgroup in the set of subgroups of the form $A[n]$ ($n \in \mathbb{N}$). This must coincide with the torsion subgroup tA of A, thus $A = A[m] \oplus C$ for some $m \in \mathbb{N}$ and torsion-free C. Such a C must be endo-noetherian, because the fully invariant subgroups of A containing $A[m]$ correspond to those of C.

For the converse, note that all fully invariant subgroups of $A = B \oplus C$ are among the direct sums of fully invariant subgroups of B and C. □

Endo-noetherian torsion-free groups are abundant, and it seems impossible to characterize them reasonably.

Example 6.7. The strongly irreducible torsion-free groups of finite rank are endo-noetherian, since their non-zero fully invariant subgroups have finite indices.

Endo-Projective Groups As far as endo-projective groups are concerned, only the torsion groups have a satisfactory characterization, the general case seems to be out of reach at this time. It is an easy exercise to see that a torsion group is endo-projective exactly if its p-components are, thus it suffices to deal with p-groups.

Theorem 6.8 (Richman–Walker [1]). *A p-group is endo-projective if and only if it is bounded.*

Proof. Suppose the p-group A is E-projective. If A is bounded, then by Lemma 2.4(i), $A \cong \mathsf{E}\epsilon$ with an idempotent ϵ, thus A is a summand of the free module E.

If A is unbounded, then we concentrate on the group homomorphism

$$\chi\colon \operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) \otimes_{\mathsf{E}} A \to \operatorname{Hom}_{\mathsf{E}}(A, A)$$

acting as $(\theta \otimes a)(a') \mapsto \theta(a')(a)$, where $\theta \in \operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E})$ and $a, a' \in A$. Since A is a p-group, $\operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) \otimes_{\mathsf{E}} A$ is also a p-group, while $\operatorname{Hom}_{\mathsf{E}}(A, A)$ is the center of E that is isomorphic to J_p (cf. Theorem 2.9). Thus $\chi = 0$. It is clear that $\theta(a')(a) = 0$ for all $a, a' \in A$ only if $\operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) = 0$. But if A is E-projective, then A is a summand of a direct sum of copies of E, so $\operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) \neq 0$. Therefore, no unbounded A can be endo-projective. □

Example 6.9. Examples of endo-projective torsion-free groups are $\mathbb{Z}$ and $\mathbb{Q}$. Also, $\mathbb{Q} \oplus \mathbb{Q}$ is endo-projective as it is isomorphic to $\mathsf{E}\epsilon$ for an idempotent ϵ.

Endo-Projective Dimension It is natural to inquire about the endo-projective dimension of groups. We will denote by p.d.A the E-projective dimension of A.

Theorem 6.10 (Douglas–Farahat [1]).

(i) *A torsion group is either endo-projective or has endo-projective dimension* 1.
(ii) *The same holds for a divisible group.*

Proof.

(i) It suffices to deal with p-groups A. Let ϵ_n denote the projection of A onto a summand $C_n \cong \mathbb{Z}(p^n)$. Then $A[p^n]$ is E-projective, being isomorphic to $\operatorname{Hom}(C_n, A) \cong \mathsf{E}\epsilon_n$. Thus p.d.$A = 0$ if A is bounded.

Next suppose A has an unbounded basic subgroup. Then A is the union of a countable ascending chain of E-projective subgroups $A[p^n]$ for integers n for which A has cyclic summands of order p^n. A well-known lemma of Auslander states that if the groups in such a chain have projective dimensions $\leq m$, then the projective dimension of the union is $\leq m + 1$; in our case, $m = 0$. Thus p.d.$A \leq 1$ in this case.

Finally, let $A = B \oplus D$ where B is p^n-bounded, and $D \neq 0$ is divisible; we may assume $B \neq 0$, since the divisible case will be settled in (ii). Consider the submodules $A_k = A[p^{n+k}]$ for $k < \omega$ whose union is A. For a cyclic summand C

of order p^n, we have $A_0 \cong \mathrm{Hom}(C, A)$ as E-modules, so p.d.$A_0 = 0$. If we prove that p.d.$(A_k/A_{k-1}) = 1$, then another form of Auslander's lemma on projective dimensions will assure that p.d.A cannot exceed 1. It is immediately seen that the factor group A_k/A_{k-1} is generated over E by any element $d \in D$ of order p^{n+k}, so it is isomorphic to E/L where $\mathsf{L} = \{\eta \in \mathsf{E} \mid \eta d \in A_{k-1}\}$ is a left ideal of E. Write $A = D' \oplus A'$ where $d \in D' \cong \mathbb{Z}(p^\infty)$, and define $\chi \in \mathsf{E}$ as multiplication by p on D' and the identity on A'. It follows that $\mathsf{L} = \mathsf{E}\chi \cong \mathsf{E}$, so L is E-projective. Hence p.d.$(A_k/A_{k-1}) =$ p.d.$(\mathsf{E}/\mathsf{L}) = 1$, and p.d.$A \leq 1$.

(ii) If D is a divisible group that is not torsion, then $D \cong \mathsf{E}\pi$ where π denotes the projection onto a summand $\cong \mathbb{Q}$, since $D = \mathsf{E}d$ for any $d \in D$ of infinite order. In this case, D is evidently endo-projective.

If D is a divisible p-group, then form a projective resolution of $\mathbb{Z}(p^\infty)$ as a J_p-module: $0 \to H \to F \to \mathbb{Z}(p^\infty) \to 0$ where F, H are free J_p-modules (submodules of free are free as J_p is a PID). Now $\mathsf{E} = \mathrm{End}\, D$ is a (torsion-free, hence) flat J_p-module, so the tensored sequence

$$0 \to \mathsf{E} \otimes_{J_p} H \to \mathsf{E} \otimes_{J_p} F \to \mathsf{E} \otimes_{J_p} \mathbb{Z}(p^\infty) = D \to 0$$

is exact. Moreover, it is exact even as an E-sequence. The first two tensor products are free E-modules, whence p.d.$D \leq 1$ is the consequence of a Kaplansky inequality for projective dimensions in an exact sequence. □

In contrast, the endo-projective dimension of a torsion-free group can be any integer or ∞; see the Notes.

Endo-Quasi-Projective Groups Turning our attention to the quasi-projective case, we can prove the following characterization for p-groups.

Theorem 6.11 (Fuchs [19]). *A p-group is endo-quasi-projective if and only if it is bounded or has an unbounded basic subgroup.*

Proof. Evidently, a group A is quasi-projective over E if and only if, for each fully invariant subgroup H of A, every E-homomorphism $\alpha : A \to A/H$ factors as $\alpha = \phi\theta$ where $\phi : A \to A/H$ is the canonical map, and $\theta : A \to A$ is a suitable E-map, i.e. multiplication by some $\rho \in J_p$.

To prove necessity, we have to rule out the case when $A = B \oplus D$ where $p^m B = 0$ for some $m \in \mathbb{N}$ and $D \neq 0$ is divisible. Let $\phi : A \to A/H$ be the canonical map with $H = A[p^m]$. Choose $\alpha : A \to A/H$ so as to satisfy $\alpha B = 0$ and $\alpha \restriction D : D \to D/(D[p^m])$ an isomorphism; clearly, α is an E-map. There is no $\rho \in J_p$ such that $\alpha = \phi\rho$, since $\alpha D[p^m] \neq 0$, but $\phi\rho D[p^m] = 0$ for all $\rho \in J_p$. Consequently, A must be as stated.

For sufficiency, note that a bounded group is by Theorem 6.8 endo-projective, while for groups with unbounded basic subgroups, an appeal to Lemma 6.1(ii) completes the proof. □

Endo-Flat Groups In order to describe the endo-flat torsion groups, we may again restrict our consideration to p-groups.

Theorem 6.12 (Richman–Walker [1]). *A p-group is endo-flat if and only if it is bounded or has an unbounded basic subgroup.*

Proof. If A is bounded, then it is E-projective, so a fortiori E-flat. It is shown in Theorem 2.4 that if A is a p-group with an unbounded basic subgroup, then $A[p^n]$ is E-projective (for any $n \in \mathbb{N}$ if there is a primitive idempotent of order p^n). Hence A is E-flat as the direct limit of E-projective groups.

It remains to show that $A = B \oplus D$ is not endo-flat if B is bounded and $D \neq 0$ is divisible. If $p^n B = 0$ (with smallest n), then $\mathsf{E} = \operatorname{End} A$ is additively the direct sum of a p^n-bounded group and a torsion-free group, and the same must hold for E-projective modules. Direct limits of groups that are direct sums of p^n-bounded and torsion-free groups are again of the same kind. Therefore, if $D \neq 0$, then A cannot be flat. □

We find it rather surprising that a torsion group is endo-flat if and only if it is endo-quasi-projective.

Example 6.13. Let A be a torsion-free group whose endomorphism ring is a PID (e.g. any rigid group). Then A is endo-flat, because it is torsion-free as an E-module. For more examples see Faticoni–Goeters [1].

Endo-Injective Groups We do not wish to discuss endo-injectivity in general, we only wish to prove that finite groups share this property. We will then state the general result without proof (in order to avoid the search for additional structural information).

Proposition 6.14. *Finite groups are endo-injective.*

Proof. We prove that a finite p-group F is endo-injective. For convenience, let us abbreviate $\operatorname{Hom}(F, \mathbb{Z}(p^\infty)) = F^\circ$. If F is viewed as a left E-module, then module theory tells us that F° becomes a right E-module, and $F^{\circ\circ}$ a left E-module. For finite F, we have the canonical E-isomorphism $F \to F^{\circ\circ}$. Since the action of E on F on the left is the same as the action of E on F° on the right, from Theorem 6.8 we infer that F° is a projective right E-module. We appeal to the natural isomorphism

$$\operatorname{Ext}^1_{\mathsf{E}}(C, \operatorname{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}^{\mathsf{E}}_1(B, C), \mathbb{Q}/\mathbb{Z})$$

that holds if C is a left and B is a right E-module (see Cartan–Eilenberg [CE]). Choose $B = F^\circ$, then its projectivity implies that the right-hand side is 0, so the same holds for the Ext on the left for each C. Hence $\operatorname{Hom}_{\mathbb{Z}}(F^\circ, \mathbb{Q}/\mathbb{Z})) = F^{\circ\circ} \cong F$ is an injective left E-module. □

The main result on endo-injectivity is the following theorem.

Proposition 6.15 (Richman–Walker [3]). *A group A is endo-injective if and only if it is of the form*

$$A = \prod_p A_p \oplus D$$

where each A_p is a finite p-group, D is a divisible group of finite rank, and either

(i) $D = 0$*; or*
(ii) *D is not a mixed group, and almost all* $A_p = 0$*.* □

★ **Notes.** There is an extensive literature on groups (and more generally, on modules) that are of special kind over their own endomorphism rings. Significant advances have been made, and the proofs are getting more and more involved in ring theory. This area is certainly a gold mine for research problems. More on this subject may be found in the monograph Krylov–Mikhalev–Tuganbaev [KMT].

Some sample results from the rich selection:

(i) Reid [5, 6] has an in-depth study of the endo-finitely generated case.
(ii) Niedzwecki–Reid [1] described cyclic endo-projective groups A. Their result states that $A = \mathsf{R}^+ \oplus M$ where R is an E-ring and M is an E-module, i.e. an R-module with $\mathrm{Hom}_{\mathsf{R}}(\mathsf{R}, M) = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}, M)$.
(iii) See Arnold–Pierce–Reid–Vinsonhaler–Wickless [1] for more on endo-projectivity. They show that a torsion-free group is endo-projective whenever it is endo-quasi-projective, and an endo-projective group is 2-generated.
(iv) For arbitrary torsion-free groups, endo-quasi-projectivity was investigated by Vinsonhaler–Wickless [1], Vinsonhaler [1], Bowman–Rangaswamy [1].
(v) A is endo-flat exactly if for every finitely generated left ideal L of $\mathrm{End}\, A$, the canonical map $\mathsf{L} \otimes A \to \mathsf{L}A$ is an isomorphism (Goeters–Reid [1]). Cf. also Albrecht–Faticoni [1], Albrecht–Goeters–Wickless [1].
(vi) A completely decomposable group is endo-flat exactly if in the set of types of summands, two types with upper bound have also a lower bound (Richman–Walker [6]). This result is generalized by Rangaswamy [5] to separable groups.
(vii) Detachable p-groups are endo-quasi-injective (Richman [4]). Here 'detachable' means that every element a in the socle is contained in a summand C such that $p^\sigma C[p] = \langle a \rangle$ for $\sigma = h_p(a)$. (Separable and totally projective p-groups are detachable.) For more endo-quasi-injective groups, see Poole–Reid [1].
(viii) Endo-torsion and other module properties were discussed by Faticoni [3].
(ix) A torsion-free A is endo-noetherian if and only if it is quasi-isomorphic to a finite direct sum of strongly irreducible groups (Paras [1]).
(x) Endo-uniserial groups (the fully invariant subgroups form a chain) have been investigated by Hausen [7], [8], and Dugas–Hausen [1].
(xi) Endomorphism rings with chain conditions were studied by Albrecht [1].
(xii) The question when a group is endo-slender was considered by Huber [2], Mader [4], Eda [5]. Eda proved that a necessary and sufficient condition for a reduced vector group $A = \prod_{i \in I} R_i$ to be endo-slender is that for each infinite subset J of I, there exists an index $i \in I$ such that the set $\{j \in J \mid \mathbf{t}(R_j) \leq \mathbf{t}(R_i)\}$ is infinite.
(xiii) Self-injective endomorphism rings are subjects of several of papers, see e.g. Rangaswamy [3], Ivanov [6], Albrecht [2].
(xiv) Goldsmith–Vámos [1] deal with the so-called clean endomorphism rings.

After it has been observed that injective modules over any ring have algebraically compact additive groups, Richman–Walker [3] characterized the algebraically compact groups that are injective over the endomorphism ring. Vinsonhaler–Wickless [2] consider the endo-injective hull of separable p-groups: they are reduced algebraically compact groups, whose p-basic subgroups can be calculated.

Projective dimensions over the endomorphism ring were studied by Douglas–Farahat [1]. Inter alia they prove that, for any group A and for its powers A^I, the endo-projective dimensions are equal. That the endo-projective dimension of a finite rank torsion-free group can be any integer or ∞ was shown by Bobylev [1] for countable rank torsion-free groups. In the finite rank case, the same was proved for finite dimensions by Angad-Gaur [1], employing Corner type constructions.

Vinsonhaler–Wickless [3] discussed the flat dimension of completely decomposable torsion-free groups, and proved that it can be any integer.

Wickless [2] studied the torsion subgroup as an endo-submodule. Properties of $\mathrm{Hom}(C,A), \mathrm{Ext}(C,A), A \otimes C, \mathrm{Tor}(A,C)$, viewed as modules over $\mathrm{End}\,A$ or $\mathrm{End}\,C$, are popular subjects in the Russian literature. This is an interesting area of research which already produced several noteworthy results.

Exercises

(1) If A is endo-finite, and H is a fully invariant subgroup of A, then A/H is also endo-finite. [Hint: cosets of an E-generating set for A.]
(2) Subgroups and finite direct sums of endo-artinian groups are likewise endo-artinian.
(3) Finite direct sums and summands of endo-noetherian groups are again endo-noetherian.
(4) Find the groups that are both endo-artinian and endo-noetherian.
(5) For any group C, the group $A = \mathbb{Z} \oplus C$ is endo-cyclic and endo-projective.
(6) (Niedzwecki–Reid) If A is generated by n elements over $\mathrm{End}\,A$, then A^n is cyclic over $\mathrm{End}(A^n)$.
(7) There are torsion-free groups A of arbitrarily large cardinality κ such that—as $\mathrm{End}\,A$-modules—they cannot be generated by less than κ elements.
(8) (Wu–Jans) If A is endo-quasi-projective, then A/H is also endo-quasi-projective for any fully invariant subgroup H.
(9) (Poole–Reid) Let A be a direct sum of fully invariant subgroups A_i. Then A is endo-quasi-injective if and only if each A_i is endo-quasi-injective.
(10) (a) (Brameret, Feigelstock) A p-group is endo-uniserial if and only if it is divisible or a direct sum of copies of $\mathbb{Z}(p^n)$ and $\mathbb{Z}(p^{n+1})$ for some $n \in \mathbb{N}$.
(b) (Hausen) A torsion-free endo-uniserial group of finite rank is p-local for some p, and every non-zero fully invariant subgroup is of finite index.
(11) Find groups, other than finite groups, that are both endo-projective and endo-injective.

7 Groups with Prescribed Endomorphism Rings

It is natural to ask the question: if we are given a ring, under what conditions is it an endomorphism ring? And if it is an endomorphism ring, how can we find groups to fit? These questions have been answered for separable p-groups partially by Pierce [1], and completely by Liebert [3]; the conditions are quite complicated (see Theorem 2.8), and so are the constructions.

We now consider the questions in general. For torsion-free groups, we will have a mild sufficient condition on the ring to satisfy to be an endomorphism ring, but the real challenge is to construct groups once the target ring has been selected. Torsion-free groups of totally different, even arbitrarily large cardinalities may have isomorphic endomorphism rings. We will return to torsion groups, but prefer to think of the endomorphism rings in the form of Theorem 2.2, and our aim will be to find groups once the ring R is prescribed. Unfortunately, for mixed groups, only very little can be said.

The Torsion-Free Case Corner's Theorem 3.3 provides a satisfactory condition for countable rings. Of course, one prefers conditions that are applicable in arbitrary cardinalities. It turns out that the study of this problem in full generality requires sophisticated machinery that would go beyond the scope of this volume. Therefore, we just state the theorems without elaborating the proofs. Before doing so, let us recall that the endomorphism ring a group of cardinality κ has cardinality at most 2^κ.

For cotorsion-free rings the result is most satisfactory.

Theorem 7.1 (Dugas–Göbel [2], Shelah [4]). *Let* R *denote a cotorsion-free ring, and* κ *a cardinal satisfying* $\kappa^{\aleph_0} = \kappa$ *and* $\kappa^+ \geq |\mathsf{R}|$. *There exists a group* A *of cardinality* κ^+ *such that*

$$\operatorname{End} A \cong \mathsf{R}.$$

Moreover, there is a rigid system of 2^κ *groups with this property.* □

There are various proofs of this theorem, all rely on some set-theoretical principle, like the Black Box or the Shelah elevator. The group A is constructed as a sandwich group between the direct sum B of copies of R^+ and its $\mathbb{Z}$-adic completion $\tilde{B}$. In this way, we could get too many endomorphisms, and the essence of the construction is to find out how to get rid of unwanted maps. This is a difficult and tiring technical process—and additional principles are needed to accomplish this task. For details, we refer to the excellent expositions Eklof–Mekler [EM], Göbel–Trlifaj [GT], or the original papers.

In the publications, slightly different conditions are stated for the cardinals involved. The main issue in the preceding theorem is that, under fairly general conditions, there exist arbitrarily large groups having the given ring as their endomorphism ring. Interestingly, for small cardinals a different approach seems to be necessary.

We should mention a topological version that is actually stronger than Theorem 7.1 inasmuch as it gives an explicit necessary and sufficient condition (recall that all endomorphism rings are complete in the finite topology). It extends Theorem 3.4 to arbitrary cardinalities.

Theorem 7.2 (Dugas–Göbel [5]). *A topological ring* R *is isomorphic to the finitely topologized endomorphism ring of a cotorsion-free group if and only if* R *is complete and Hausdorff in a topology that admits a basis of neighborhoods of* 0 *consisting of right ideals* N *with cotorsion-free factor groups* R/N. □

The Torsion Case The problem for torsion groups is a different ball game. The endomorphism rings are extremely restricted: they are built on algebraically compact groups, and admit unavoidable small endomorphisms. The big problem is that these small endomorphisms form an ideal that already totally characterizes the group itself. Thus the only hope is to have some control on the endomorphism rings modulo small endomorphisms. We know from Theorem 2.2 that $\mathrm{End}\, A$ is a split extension of the ideal $\mathrm{End}_s A$ by an algebraically compact ring that faithfully reflects direct decompositions with unbounded summands. We have something to say about this ring.

Theorem 7.3 (Corner [5], Dugas–Göbel [3]). *Let* R *denote a ring whose additive group is the p-adic completion of a free group. There exists a separable p-group A such that*

$$\mathrm{End}\, A \cong \mathsf{R} \oplus \mathrm{End}_s A. \qquad \square$$

Moreover, if the ring R is given as stated, then for every infinite cardinal κ, there exists a family $\{A_\sigma \mid \sigma < \kappa\}$ of separable p-groups such that for each σ, $\mathrm{End}\, A_\sigma \cong \mathsf{R} \oplus \mathrm{End}_s A_\sigma$, and $\mathrm{Hom}(A_\rho, A_\sigma) = \mathrm{Hom}_s(A_\rho, A_\sigma)$ whenever $\rho \neq \sigma$.

Reduced Mixed Groups Corner–Göbel [1] gave a unified treatment of constructing groups from prescribed endomorphism rings that included mixed groups as well. On a different vein, Franzen–Goldsmith constructed a functor from the category of reduced torsion-free groups to the category of reduced mixed group to transfer results from one category to the other. They proved:

Theorem 7.4 (Franzen–Goldsmith [1]). *For every reduced countable torsion-free ring* R *there exists a reduced countable mixed group A such that A/tA is divisible, and*

$$\mathrm{End}\, A \cong \mathsf{R} \oplus \mathrm{Hom}(A, tA). \qquad \square$$

★ **Notes.** Starting with Corner's epoch making Theorem 3.3, a significant amount of work has been done on constructing torsion-free groups, even for arbitrarily large cardinalities, with prescribed endomorphism rings. Using a consequence of V = L, Dugas–Göbel [2] proved that every cotorsion-free ring is an endomorphism ring. For the general version, see Shelah [4]. The condition of cotorsion-freeness appears in almost all of the publications.

The consequences of Theorem 7.2 are legion, especially in constructing fascinating examples. It seems easier to construct a ring with required properties than large groups satisfying prescribed conditions. We have already taken advantage of this procedure.

May [5] considers cases when the rings are not necessarily cotorsion-free. Arnold–Vinsonhaler [3] proved that every reduced ring R whose additive group is a finite rank Butler group is the endomorphism ring of a finite rank Butler group provided that $p\mathsf{R} \neq \mathsf{R}$ holds for at least five primes p. Another kind of realization theorem is due to Dugas–Irwin–Khabbaz [1] who considered subrings of the Baer-Specker ring $\mathsf{P} = \mathbb{Z}^{\aleph_0}$ as split extensions of the ideal of finite rank endomorphisms as endomorphism rings of pure subgroups of P.

Exercises

(1) Prove that there exists no group whose endomorphism ring is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$ or to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ or to $J_p \oplus J_p$ for any prime p.

(2) Prove the existence of a mixed group with divisible torsion-free part such that its endomorphism ring has a homomorphic image of a ring with countable free additive group. [Hint: Theorem 7.4.]

Problems to Chapter 16

PROBLEM 16.1. Is the anti-isomorphic ring of an endomorphism ring also an endomorphism ring?

PROBLEM 16.2. Find the groups such that every endomorphism is the sum of (two) monomorphisms.

There are several publications on the unit sum number, see Notes to Sect. 2.

PROBLEM 16.3 (Megibben). Investigate the endomorphism rings of $\aleph_1$-free $\aleph_1$-separable torsion-free groups.

PROBLEM 16.4. Which groups have endo-projective cover?

PROBLEM 16.5. Relate the endo-flat cover of a group to the group.

PROBLEM 16.6. Discuss pure-projective and pure-injective dimensions over endomorphism rings.

PROBLEM 16.7. Find the stable range of endomorphism rings.

PROBLEM 16.8. Let S be a ring containing $\operatorname{End} A$ as a subring, sharing the same identity. When does a group G exist that contains A and satisfies $\operatorname{End} G \cong \mathsf{S}$?

PROBLEM 16.9. The endo-Goldie-dimension of a torsion-free group can be arbitrarily large. Given the endomorphism ring, can we say something about the minimum?

PROBLEM 16.10. Which groups A have the property that every automorphism of $\operatorname{Aut} A$ extends to an automorphism of $\operatorname{End} A$?

PROBLEM 16.11. For a subgroup C of A define $\dot{C} = \cap_{\eta \in S} \operatorname{Ker} \eta$ where $S = \{\eta \in \operatorname{End} A \mid \eta C = 0\}$. Which subgroups satisfy $\dot{C} = C$?

PROBLEM 16.12. Any polynomial identity (besides commutativity) in endomorphism rings?