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László Fuchs

Abelian Groups

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Dedicated with love
to my wife Shula,
our daughter Terry, and son David

Preface

The theory of abelian groups is a branch of (abstract) algebra, which deals with commutative groups, named after the Norwegian mathematician Niels H. Abel. Curiously enough, it is rather independent of general group theory: its methods bear only a slight resemblance to the non-commutative case. (However, there is a close relationship to the theory of modules, especially over integral domains.)

This is my third book on abelian groups. My first book was published in 1958; it was a reasonably comprehensive systematic summary of the theory at that time. With the advent of homological methods and the explosion of the theory of torsion groups, an extended version was needed that was published in 1970 and 1973. Even two volumes could not claim any more a comprehensive status, but they seem to have presented accurately the main streams in the theory. Today, after four decades of developments and thousands of publications, it is hardly imaginable to have a comparably complete volume. As a consequence, I had to scale down my goals. By no means could I be content with just a broad introductory volume, so I had in mind a monograph which goes much beyond being a mere introduction and concentrates on most of the advanced ideas and methods of today's research. I do hope that this volume will provide a thorough and accurate picture of the current main trends in abelian group theory.

The audience I have in mind with this book is anybody with a reasonable mathematical maturity interested in algebra. My aim has been to make the material accessible to a mathematician who would like to study abelian groups or who is looking for particular results on abelian groups needed in his/her field of specialization. I have made an effort to keep the exposition as self-contained as possible, but I had to be satisfied with stating a few very important theorems without proofs whenever the proofs would not fit in this volume. The specific prerequisites are sound knowledge of the rudiments of abstract algebra, in particular, basic group and ring theory. Also required is some exposure to category theory, topology, and set theory. A few results where more advanced set theory is indispensable are included for those who are willing to indulge in a bit more sophisticated set theory. In order to fill the need of a reference source for experts, much additional information about the topics discussed is included also in the "Notes" at the end of sections with reference to relevant publications. Many results discussed are collected from scattered articles in the literature; some theorems appear here for the first time in a book.

The writing of such a book requires, inevitably, making tough choices on what to include and what to leave out. I have tried to be selective so that the central ideas stand out. My guiding principle has been to give preference to important methods and typical results as well as to topics, which contain innovative ideas or which I felt were particularly instructive. It is no secret that several flourishing areas of module theory have their origin in abelian groups where the situation is often more transparent—I have paid special attention to these results as well. As I have tried to make the book more accessible for starters, generality is sometimes sacrificed in favor of an easier proof. I have shied away from theorems with too technical proofs unless the result, I believe, is theoretically or methodically extremely important. In the selection of additional topics, the guide was my personal interest (which I view as an indisputable privilege of authors).

In order to give a fair idea of the current main streams in abelian group theory, it is impossible to ignore the numerous undecidable problems. Abelian group theory is not only distinguished by its rich collection of satisfactory classification theorems, but also for having an ample supply of interesting undecidable problems. Shelah's epoch-making proof of the undecidability of Whitehead's problem in ZFC marked the beginning of a new era in the theory of abelian groups with set-theoretical methods assuming the leading role. The infusion of ideas from set theory created as radical a change in the subject as Homological Algebra did a quarter of a century before. Therefore, I had to take a more penetrating approach to set-theoretical methods and to discuss undecidable problems as well, but I treat these fascinating problems as interesting special cases of theoretical importance, rather than as a main object of a systematic study. Though a short survey of set-theoretical background is given, the reader is well advised to consult other sources.

I am a bit leisurely, primarily in the first half of the book, with the method of presentation and the proofs in order to assist students who want to learn the subject thoroughly. I have included a series of exercises at the end of each section. As is customary, some of them are simple to test the reader's comprehension, but the majority give noteworthy extensions of the theory or sidelines to enrich the topics studied in the same section. A student should conscientiously solve several exercises to check how he/she commands the material. Serious attempts have been made to provide ample examples to illustrate the theory. Good examples not only serve to motivate the results, but also provide an explicit source of ideas for further research.

Past experience suggests that listing open problems is a good way of encouraging young researchers to start thinking seriously on the subject. In view of this, I am listing open problems at the end of each chapter, with a brief commentary if needed. I have not given any serious thought to their solutions; so a lucky reader may find quick answers to some of them.

This volume follows basically the same line of development as my previous books on abelian groups. With the kind permission of the publishing company Elsevier, I could include portions of my two volume book "Infinite Abelian Groups." The book was out of print by 1990, and at that time I was playing with the idea of revising it, but decided to join forces with my friend Luigi Salce to write a monograph on modules over non-noetherian integral domains. When I felt ready to get involved in working on a book on abelian groups, Hurricane Katrina interfered: it forced us to leave our home for three-and-a-half years. It took me some time to brace myself for such a big task as writing a book. When I realized that a mere revision would not be satisfactory, I had no choice but to start working on a new book.

The first two chapters of this volume are introductory, and the reader who is knowledgeable in group theory will want to skip most of the sections very quickly. For those who are not so familiar with more advanced set-theoretical methods, it might help to read Sect. 4 carefully before proceeding further. The text starts in earnest in Chapter 3, which provides a thorough discussion of direct sums of cyclic groups. Chapter 4 is devoted to divisibility and injectivity, while Chapter 5 explores pure subgroups along with basic subgroups. In Chapter 6, we introduce algebraically compact groups motivated by purity and topological compactness.

In the next three chapters, we embark on the fundamental homological machinery needed for abelian groups. The insight developed here is essential for the rest of the book. We conclude Chapter 9 with the theory of cotorsion groups.

Armed with the powerful tools of homological algebra, we proceed to the next six chapters, which form the backbone of the theory, exploring the structure of various classes of abelian groups. The structure theory is a vast field of central importance that is not easy to organize into a coherent scheme. Starting with torsion groups, we first deal with p-groups that contain no elements of infinite height. Then we go on to explore the Ulm-Zippin theory of countable p-groups leading to the highlights of the impressive theory of totally projective p-groups. Chapters 12 and 13 provide a large amount of material on torsion-free groups—an area that has shown great advancement in the past quarter of century. We explore indecomposable groups, slender groups, and vector groups. The discussion culminates in the proof of the undecidability of the famous Whitehead problem. Chapter 14 serves as a broad introduction to the fascinating theory of Butler groups. In Chapter 15, the main results on mixed groups are presented.

The final three chapters deal with endomorphism rings, automorphism groups, and the additive groups of rings. Some ideas are introduced that interact between abelian groups and rings.

I stress that the reader should by no means take this book as a complete survey of the present state of affairs in abelian group theory. A number of significant and more advanced results in several areas of the theory, as well as a wealth of important topics (like group algebras) are left out, not to mention topics like the more advanced theory of finite and infinite rank torsion-free groups, as well as p-groups more general than totally projective groups. In spite of all these, I sincerely hope that the material discussed provides a significant amount of information that will open up new and promising vistas in our subject.

As far as the bibliography is concerned, references are not provided beyond the list of works quoted in the text and in the "Notes." No reference is given to the exercises. This self-imposed restriction was necessary in view of the vast literature on abelian groups. The reader who is interested in going beyond the contents of this book should use the list of references as a guide to other sources. References outside the theory of abelian groups are usually given in an abbreviated form embedded in the "Notes." The system of cross references is simple: lemmas, theorems etc. are numbered in each chapter separately. Books are cited by letters in square brackets. I have attempted to give credit to the results, and I apologize if I have made mistakes or omissions.

A remark is in order about notational conventions. We are using the functional notation for maps, thus $\phi(x)$, or simply ϕx, is the image of x under the map ϕ. Accordingly, the product $\phi\psi$ of two maps is defined as $(\phi\psi)(x) = \phi(\psi(x))$. This is not a universally adopted convention in abelian group theory, but is predominant. The "Table of Notations" should be consulted if symbols are not clear. I have introduced new notation or terminology only in a few places where I found the frequently used terms clumsy or confusing, or if a name was missing.

I am grateful to my friends and colleagues Lutz Strüngmann, Luigi Salce, Kulumani M. Rangaswamy, Claudia Metelli, Brendan Goldsmith, and Ulrich Albrecht for their comments on portions of an earlier version of the manuscript. I have greatly benefitted from the comments I received from them. I apologize for the errors which remain in the text.

Metairie, Louisiana, USA László Fuchs
March 27, 2015

Contents

Table of Notations

Set Theory

$\subseteq, \subset$: subset, proper subset
$\cup, \cap$: set union, intersection
$\times$: cartesian product
$X \setminus Y = \{x \in X \mid x \notin Y\}$
$|S|$: cardinality of the set S
$\varnothing$: empty set
$\{a \in A \mid *\}$: set of elements of A satisfying $*$
$\{a_i\}_{i \in I}$: the set of all a_i with $i \in I$
$\kappa, \lambda, \mu, \ldots$: cardinals
$\alpha, \beta, \ldots, \rho, \sigma, \tau$: ordinals
$\aleph_{-1}$: finite, $\aleph_0$: countable cardinal
$2^{\aleph_0}$: continuum
$\aleph_\sigma$: σth infinite cardinal, σth aleph
ω: first infinite ordinal or the set $\{0, 1, \ldots, n, \ldots\}$
ω_σ: σth initial ordinal ($|\omega_\sigma| = \aleph_\sigma$)
cf σ: cofinality of the ordinal σ
$\Rightarrow$: implication
$\Leftrightarrow$: equivalence, if and only if
$\forall$: for all
$\exists$: there exists
$\neg$: negation
ZFC : Zermelo-Fraenkel axioms of set theory + Axiom of Choice
CH: Continuum Hypothesis
GCH: Generalized Continuum Hypothesis
V: model of set theory
L: Gödel's Axiom of Constructibility
V = L: Axiom L is assumed
$\diamondsuit$: Jensen's Diamond Principle

MA: Martin's Axiom
$\mathcal{A}b$: category of abelian groups
$\mathcal{C}, \mathcal{D}, \ldots$: categories
$\mathcal{V}, \mathcal{V}_p$: category of valuated groups or vector spaces
$\mathcal{G}, \mathcal{H}, \ldots$: $G(\kappa), H(\kappa)$-families of subgroups

Maps

$\rightarrow$: mapping, homomorphism
$A \xrightarrow{\alpha} B$: map α from A to B
$\mapsto$: corresponds to
$\dot{\rightarrow}$: quasi-homomorphism
$\mathbf{1}_A$: identity map on A
$\dot{n}$: multiplication by integer n
$\phi \restriction A$: restriction of map ϕ to A
Im ϕ: image of map ϕ
Ker ϕ: kernel of map ϕ
Coker ϕ: cokernel of map ϕ
Δ, ∇: diagonal, codiagonal map
$\oplus \phi_i$: direct sum of maps ϕ_i
$\prod \phi_i$: direct product of maps ϕ_i
$\cup_i \phi_i$: union of a chain of maps
$\mathfrak{e} : 0 \to A \to B \to C \to 0$: exact sequence

Groups, rings

0: number 0, element 0, or subgroup $\{0\}$
$\mathbb{N}$: set of positive integers
$\mathbb{Z}$: group of integers
$\mathbb{Q}$: group of rational numbers
$\mathbb{R}$: group of real numbers
$\mathbb{C}$: group of complex numbers
$\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$: group of complex numbers of absolute value 1
$\mathbb{Z}(n)$: cyclic group of order n
$\mathbb{Z}(p^\infty)$: quasi-cyclic p-group
$\mathbb{Z}_{(p)}$: group or ring of rational numbers with denominators coprime to p
$\mathbb{Q}^{(p)}$: group or ring of rational numbers whose denominators are powers of p
J_p: group or ring of p-adic integers
$\mathbb{Q}_p^*$: group or field of p-adic numbers
H_σ: generalized Prüfer group of length σ

$N_\sigma(p)$: Nunke group of length σ for prime p
P_β: Walker group of length β
$A, B, C, G, H, T, \ldots$: groups
A_p: p-component of group A
$A_{(p)}$: localization of A at prime p
$E(A)$: injective hull of A
$\tilde{A}$: $\mathbb{Z}$-adic or p-adic completion of A
$\hat{A}$: Pontryagin dual of A
$\overline{B}$: torsion-complete group with basic subgroup B
$A^\bullet$: cotorsion hull of A
$A(\mathbf{t})$: subgroup of elements of types $\geq \mathbf{t}$
$A^*(\mathbf{t})$: subgroup generated by types $> \mathbf{t}$
$A[\mathbf{t}], A^*[\mathbf{t}]$: subgroups defined in terms of $\mathbf{t}$
$\mathfrak{G}, \mathfrak{H}$: non-commutative groups
$\mathfrak{z}(S)$: centralizer of set S in a group
$\mathsf{R}, \mathsf{S}, \ldots$: rings
R^+: additive group of ring R
$U(\mathsf{R})$: group of units of ring R
$\mathsf{F}^\times$: multiplicative group of field F
J: Jacobson radical of the ring
${}_{\mathsf{R}}M$: left module M over R

Special Notation

gcd: greatest common divisor
lcm: least common multiple
rk A: rank of A
$\mathrm{rk}_0\, A$, p-rk A: torsion-free rank, p-rank of A
$\mathrm{rk}^p A$: p-corank of A
fin rk A: final rank of p-group A
supp: support of a vector
bpd(A): balanced projective dimension of A
Type(A): type set of torsion-free finite rank group A
IT(A), OT(A): inner, outer type of torsion-free group A

Symbols

$\cong$: isomorphism
$\approx$: near-isomorphism
$\sim$: quasi-isomorphism
$\cong_{\mathcal{C}}$: isomorphism in category $\mathcal{C}$

$o(a)$: order of element a
$n|a$: integer n divides group element a
$h_p(a)$: (transfinite) height of element a at prime p
$\chi(a)$: characteristic of element a in torsion-free A
$A(\chi) = \{a \in A \mid \chi(a) \geq \chi\}$
$\mathbf{t}(a)$: type of element a
$B \leq A, B < A$: B is a (proper) subgroup of A
$\lhd$: normal subgroup
$B + C, \sum B_i$: subgroup generated by subgroups B, C, by subgroups B_i
$\langle S \rangle$: subgroup generated by S
$\langle S \rangle_*$: pure subgroup generated by S in a torsion-free group
$|A : B|$: index of subgroup B in A
$t(A) = tA$: torsion subgroup of A
$nA = \{na \mid a \in A\}$ where $n \in \mathbb{N}$
$A[n] = \{a \in A \mid na = 0\}$ where $n \in \mathbb{N}$
$n^{-1}B = \{a \in A \mid na \in B\}$ for $B < A$
C^-: $\mathbb{Z}$-adic closure of subgroup C in the given group
$p^\sigma A$: set of elements in A of p-height $\geq \sigma$
$A^1 = \cap_{n \in \mathbb{N}} nA$: first Ulm subgroup of A
A^σ: σth Ulm subgroup of A
$A_\sigma = A^\sigma / A^{\sigma+1}$: σth Ulm factor of A
$\underline{u} = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$: indicator, increasing sequence of ordinals and ∞
$A(\underline{u})$: fully invariant subgroup of A defined by $\underline{u}$
$A \| B$: A is compatible with B
$\mathbf{L}(A)$: lattice of subgroups in A
$\mathfrak{T}$: lattice of all types
$\mathbf{t}, \mathbf{s}, \dots$: types in torsion-free groups
$\mathbf{t}(A)$: type of the torsion-free group A
$\ell(A)$: length of p-group A
$f_\sigma(A) = p^\sigma A[p] / p^{\sigma+1} A[p]$: σth UK-invariant of p-group A
$f_\sigma(A, G)$: σth Hill invariant of A relative to subgroup G
$\mathrm{rk} A_{\mathbf{t}} = \mathrm{rk}(A(\mathbf{t}) / A^*(\mathbf{t}))$: Baer invariant for $\mathbf{t}$
$\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$: direct (inverse) system
$\mathbb{M}$, $\|a_{ik}\|$: matrix
$\mathbb{H}(a) = \|\sigma_{ik}\|$: height matrix of a in mixed group
$A(\mathbb{H}) = \{a \in A \mid \mathbb{H}(a) \geq \mathbb{H}\}$

Operations

$A \oplus C$: direct sum of A and C
$\oplus_{i \in I} A_i, \prod_{i \in I} A_i$: direct sum, direct product of the A_i with i changing over I
$\oplus_\kappa A, A^{(\kappa)}$: direct sum of κ copies of A

$\prod_\kappa A, , A^\kappa$: direct product of κ copies of A
$\prod_{i\in I}^{<\kappa} A_i$: κ-product of A_i
$\prod_{\mathsf{K}} A_i$: K-product (K=ideal in power-set of I)
$A^{(\mathbf{B})}$: Boolean power of A
$\mathrm{Hom}(A, C)$: group of homomorphisms from A to C
$\mathrm{Hom}_s(A, C)$: group of small homomorphisms
$\mathbb{Q}\,\mathrm{Hom}(A, C)$: group of quasi-homomorphisms
$\mathrm{Hom}_{\mathsf{R}}(A, C)$: group of R-homomorphisms between R-modules ${}_{\mathsf{R}}A, {}_{\mathsf{R}}C$
$\mathrm{End}\,A$: endomorphism ring (group) of A
$\mathrm{End}_s\,A$: ring of small endomorphisms of A
$\mathbb{Q}\,\mathrm{End}\,A$: quasi-endomorphism ring of torsion-free A
$\mathrm{Aut}\,A$: automorphism group of A
$A \otimes C$: tensor product of A and C
$A \otimes_{\mathsf{R}} C$: tensor product of A_{R} and ${}_{\mathsf{R}}C$ over R
$\mathrm{Tor}(C, A)$: torsion product of C and A
$\mathrm{Ext}(C, A)$: group of extensions of A by C
$\mathrm{Pext}(C, A)$: group of pure extensions of A by C
$\mathrm{Bext}(C, A), \mathrm{PBext}(C, A)$: group of (pre)balanced extensions of A by C
$\mathrm{Char}\,A$: character group of A in Pontryagin duality
$\mathrm{Mult}\,A$: group of ring multiplications on A
$\mathrm{Tr}_A(G)$: trace of A in G
$\varinjlim, \varprojlim$: direct, inverse limit
$\varprojlim^1$: first derived functor of inverse limit

Chapter 1
Fundamentals

Abstract The aim of this introductory chapter is twofold. First, to refresh the reader's memory about the basics before entering into the study of more serious topics. A fairly large portion of the material can be found in standard textbooks on algebra. So, the reader may skip some or all of the sections and turn to the appropriate places when needed. Secondly, we have to fix the fundamental terminology to be used throughout this volume. In the 1950s, as in every rapidly developing field, the terminology in abelian groups often varied from author to author. Slowly standardization took place, and we will use here the widely accepted, the most familiar or the more appropriate names for the concepts, with a couple of exceptions where new terminology is warranted.

The topics covered in this chapter include maps, diagrams, sets, categories. Most of the proofs will be omitted as they are standard and available in textbooks.

For the comfort of the reader, especially, of those who look for references, we avoid the use of abbreviations in the text as far as reasonable.

1 Basic Definitions

Throughout this book, 'group' will mean an additively written abelian group. That is, a **group** is a set A satisfying the following conditions:

1. with every pair $a, b \in A$, an element $a + b \in A$ is associated, called the **sum** of a and b;
2. **associative law**: $a + (b + c) = (a + b) + c$ for all $a, b, c \in A$;
3. **commutative law**: $a + b = b + a$ for all $a, b \in A$;
4. there is an element 0, called **zero**, such that $a + 0 = a$ for all $a \in A$;
5. to each $a \in A$ there exists an $x \in A$ satisfying $a + x = 0$; this is $x = -a$, the **inverse** to a.

Note that a group is never empty. The associative law enables us to write a sum of more than two summands without parentheses, and due to the commutative law, the terms of a sum may be permuted. We write $a - b$ to mean $a + (-b)$, and $-a - b$ for the inverse of $a + b$. The sum $a + \cdots + a$ (n summands) is abbreviated as na (called a **multiple** of a), and $-a - \cdots - a$ (n summands) as $-na$ with $n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers). A sum without terms is 0, thus $0a = 0$ for all $a \in A$. Notice that we do not make distinction in the notation between the integer 0 and the group element 0.

The same symbol is used for a group and for the set of its elements. The **order** of a group A is the cardinal number $|A|$ of the set of its elements. According as $|A|$

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is finite, countable, or uncountable, we call A a **finite, countable** or **uncountable** group.

Subgroups A subset B of a group A is a **subgroup** if the elements of B form a group under the same operation. A non-empty subset B of a group A is a subgroup if and only if $a, b \in B$ implies $a + b \in B$ and $b \in B$ implies $-b \in B$, or more simply, if and only if $a, b \in B$ implies $a - b \in B$. If A is a finite group and B is a subgroup, then by Lagrange's theorem $|B|$ is a divisor of $|A|$. Every subgroup of A contains the element 0, and the subgroup $\{0\}$ consisting of the element 0 alone is called the **trivial** subgroup of A (there being no danger of confusion, the trivial subgroup will be denoted by 0 instead of $\{0\}$). A subgroup of A, different from A, is said to be a **proper** subgroup. We shall use the following convention: $B \leq A$ will denote that B is a subgroup of A, while $B < A$ will mean that B is a proper subgroup. In contrast, whenever we wish to claim that X is merely a subset (resp. a proper subset) of A, then we shall write $X \subseteq A$ (resp. $X \subset A$).

The set-theoretic intersection $B \cap C$ of two subgroups B, C of A is again a subgroup. More generally, if $\{B_i\}_{i \in I}$ is a family of subgroups of A, for any index set I, then their intersection $B = \cap_{i \in I} B_i$ is likewise a subgroup of A. If I is the empty set, then we agree that $B = A$. If the subgroups B and C satisfy $B \cap C = 0$, we will say that B and C are **disjoint**. This terminology is not consistent with the set-theoretical meaning of the word, but it is an accepted agreement that the group-theoretical disjointness should mean that the subgroups have nothing else in common other than what they ought to have.

If S is a subset of A, the symbol $\langle S \rangle$ will denote the subgroup of A **generated by** S, i.e. the intersection of all subgroups of A that contain S. If $S = \{\dots, a_i, \dots\}$, then we write $\langle S \rangle = \langle \dots, a_i, \dots \rangle$, or simply $\langle S \rangle = \langle a_i \rangle_{i \in I}$. This $\langle S \rangle$ consists of all **linear combinations** (i.e., sums of the form) $n_1 a_1 + \dots + n_k a_k$ $(a_i \in S, n_i \in \mathbb{Z})$ where k denotes a non-negative integer. We set $\langle \varnothing \rangle = 0$. In case $\langle S \rangle = A$, we say that S is a **generating system** of A, and its elements are **generators**. If B, C are subgroups of A, then the subgroup $\langle B, C \rangle$ they generate consists of the elements $b + c$ $(b \in B, c \in C)$. We may write, therefore, $\langle B, C \rangle = B + C$. For a possibly infinite collection of subgroups $\{B_i\}_{i \in I}$ of A, the subgroup B they generate consists of all finite sums $b_{i_1} + \dots + b_{i_k}$ with $b_{i_j} \in B_{i_j}$; we shall then write $B = \sum_{i \in I} B_i$.

A group $\langle a \rangle$, generated by a single element a, is called **cyclic**. The order of this cyclic group is called the **order** of the element a; notation: $o(a)$. The order of an element can be a positive integer or the symbol ∞. If a is of finite order n, then $na = 0$, so $\langle a \rangle = \{0, a, \dots, (n-1)a\}$, while if $o(a) = \infty$, then $\langle a \rangle = \{0, \pm a, \dots, \pm na, \dots\}$ (all different elements).

By a **superfluous subgroup** of a group A is meant a subgroup $G \leq A$ such that $X + G = A$ holds for a subgroup $X \leq A$ only if $X = A$, i.e. the elements of G are superfluous in any generating system. Evidently, subgroups of a superfluous subgroup are superfluous, and the sum of two superfluous subgroups is again superfluous.

Example 1.1.

(a) The only superfluous subgroup of the group $\mathbb{Z}$ of integers is the 0 subgroup. In fact, if $n\mathbb{Z}$ is any non-zero subgroup and $1 < m \in \mathbb{N}$ satisfies $\gcd(n, m) = 1$, then $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$, but $m\mathbb{Z} \neq \mathbb{Z}$.
(b) In a cyclic group of prime power order, all proper subgroups are superfluous.

The set of all subgroups of a group A is partially ordered under the inclusion relation. It is a lattice, where $B \cap C$ and $B + C$ are the lattice-operations 'inf' and 'sup,' respectively, for subgroups B, C of A. This lattice $\mathbf{L}(A)$ has a minimum and a maximum element (0 and A), and satisfies the important **modular law**: for subgroups B, C, D of A,

$$B + (C \cap D) = (B + C) \cap D \qquad \text{provided } B \leq D.$$

In fact, the inclusion relation $\leq$ being obvious, we need only verify that every $d \in (B + C) \cap D$ belongs to the subgroup on the left. We can write $d = b + c$ with $b \in B, c \in C$. Thus $d - b = c$ belongs to both D and C. Hence $c \in C \cap D$, and $d = b + c \in B + (C \cap D)$.

Factor Groups If $B \leq A$ and $a \in A$, then $a+B = \{a+b \mid b \in B\}$ is called a **coset** of A **modulo** B. Any element in a coset, also called a **representative** of the coset, identifies the coset. The cosets of A mod B are pairwise disjoint, and A is their set-theoretical union. Elements $x, y \in A$ are **congruent** (written $x \equiv y$) mod B, if they belong to the same coset mod B. A set consisting of exactly one representative from each coset of A mod B is a **complete set of representatives** mod B. Its cardinality is the **index** of B in A, denoted as $|A : B|$. If this cardinal number is finite, we call B **of finite index** in A. If A is a finite group, then $|A : B| = |A|/|B|$.

The cosets of A mod B form a group $\bar{A} = A/B$ known as the **factor group** (or **quotient group**) of A mod B. In this group, the sum of two elements C_1, C_2 (which are cosets in A) is defined as the coset consisting of the elements $\{c_1 + c_2 \mid c_1 \in C_1, c_2 \in C_2\}$; it is uniquely determined as the coset represented by any of its elements. The zero of $\bar{A}$ is the coset B, and the inverse of a coset C is the coset $-C = \{-c \mid c \in C\}$.

We shall frequently refer to the natural bijection between the subgroups of the factor group A/B and the subgroups of A containing B. The elements of A contained in elements of a subgroup $\bar{C}$ of $\bar{A} = A/B$ form a subgroup C of A containing B. On the other hand, if $B \leq C \leq A$, then the cosets of A mod B containing only elements from C form a subgroup $\bar{C} \leq A/B$. This correspondence $C \leftrightarrow \bar{C}$ is a bijection, and we may write $\bar{C} = C/B$. Note that $|\bar{C}| = |C : B|$ and $|\bar{A} : \bar{C}| = |A : C|$.

The Torsion Subgroup If every element of a group A is of finite order, then A is called a **torsion** (or **periodic**) group, while A is **torsion-free** if all its elements, with the exception of 0 (which has order 1), are of infinite order. **Mixed** groups contain both non-zero elements of finite order and elements of infinite order. By a **primary** group or p**-group** is meant a group the orders of whose elements are powers of a fixed prime p. An abelian group A has a unique p-Sylow subgroup A_p for each prime p: A_p consists of all elements of A whose orders are powers of p.

Theorem 1.2. *In a group A, the set T of elements of finite order is a subgroup. T is a torsion group and A/T is torsion-free.*

Proof. Since $0 \in T$, the set T is not empty. If $a, b \in T$, i.e. $ma = 0, nb = 0$ for some $m, n \in \mathbb{N}$, then $mn(a - b) = 0$, and so $a - b \in T$. Thus T is a torsion subgroup. To show that A/T is torsion-free, let $a + T$ be a coset of finite order, i.e. $m(a + T) \subseteq T$ for some $m \in \mathbb{N}$. This means that $ma \in T$, so $n(ma) = 0$ for some $n \in \mathbb{N}$. Thus a is of finite order, i.e. $a \in T$, and $a + T = T$ is the zero of A/T. Consequently, A/T is torsion-free. □

We shall call T the **(maximal) torsion subgroup** or the **torsion part** of A, and we shall denote it by $t(A)$ or tA. (If we refer to *the* torsion subgroup of A, then we always mean the maximal torsion subgroup.)

The following notations are typical for abelian groups; they will be used all the time without explanation. Given a group A and an integer $n \in \mathbb{N}$, define the subgroups:

$$nA = \{na \mid a \in A\} \quad \text{and} \quad A[n] = \{a \in A \mid na = 0\}.$$

Thus $b \in nA$ if and only if the equation $nx = b$ has a solution for x in A, and $c \in A[n]$ exactly if $o(c)|n$. A fundamental concept is the **pure subgroup**. Purity will be discussed in Chapter 5, here we just state the definition: a subgroup G of A is a pure subgroup if $nG = G \cap nA$ for every $n \in \mathbb{N}$, i.e. if whenever the equation $nx = g \in G$ admits a solution in A for x, then it is also solvable in G.

Ulm Subgroups The first **Ulm subgroup** of A is defined as

$$A^1 = \cap_{n \in \mathbb{N}} nA.$$

The second Ulm subgroup is $A^2 = (A^1)^1$, etc. We shall also need the σth Ulm subgroups for ordinals σ; these are defined transfinitely by the rules: $A^{\sigma+1} = (A^\sigma)^1$, and $A^\rho = \cap_{\sigma<\rho} A^\sigma$ if ρ is a limit ordinal (see Sect. 4). (We view $A = A^0$.) The **Ulm length** of A is the smallest cardinal τ such that $A^{\tau+1} = A^\tau$ (which exists by cardinality reason). The σth **Ulm factor** of A is the factor group

$$A_\sigma = A^\sigma / A^{\sigma+1}.$$

Given $a \in A$, the largest non-negative integer n for which the equation $p^n x = a$ is solvable for $x \in A$ is said to be the p**-height** $h_p(a)$ of a. If this equation is solvable for every integer $n > 0$, then a is **of infinite** p**-height**, $h_p(a) = \infty$. The element 0 is of infinite height at every prime. If it is completely clear from the context which prime p is meant, then we may simply talk of the **height** of a and write $h(a)$. (In Chapter 11, we shall discuss transfinite heights.)

The **socle** $s(A)$ of a group A is the set (actually the subgroup) of all the elements $a \in A$ such that $o(a)$ is a square-free integer. If $s(A) = A$, A is called an **elementary** group. If A is a p-group, then $s(A) = A[p]$.

A subgroup E of A is **essential** if its intersection with any non-zero subgroup of A is non-zero. It is easily checked that (i) E is essential in A if and only if it contains the socle of A and A/E is torsion; (ii) the property of being an essential subgroup is transitive; and (iii) the intersection of two essential subgroups is again an essential subgroup.

★ **Notes.** Theorem 1.2 is an elementary, but fundamental result. It means that a typical group can be thought of as being a composite of a torsion and a torsion-free group. This, however, does not reduce the theory of mixed groups to those of these constituents, since a major issue that remains is to find out how they are glued together to form the mixed groups. It is hard to trace the history of Theorem 1.2.

Generalizations of 'torsion' exist for modules, albeit not over all rings. If we mean by a 'torsion' element one whose annihilator in the ring is $\neq 0$, then the left Ore domains are exactly those rings R for which in every left R-module M the torsion elements form a submodule T and M/T has no torsion $\neq 0$. There is an extensive literature on torsion theories in module categories, even in additive categories; see, e.g., J. Golan's book *Torsion Theories* (1986).

Exercises

(1) The associativity and commutativity laws can be combined into a single law: $(a+b)+c = a+(c+b)$ for all $a, b, c \in A$.

(2) (a) Let $B_1, \dots, B_k$ be subgroups of the group A, and let $B = B_1 \cap \dots \cap B_k$. The index $|A : B|$ is not larger than the product of the indices $|A : B_i|$.
(b) The intersection of a finite number of subgroups of finite index is again a subgroup of finite index.

(3) Let B, C be subgroups of A.

(a) For every $a \in A$, the cosets $a + B$ and $a + (B + C)$ have non-zero intersections with the same cosets mod C.
(b) A coset mod B contains exactly $|B : (B \cap C)|$ pairwise incongruent elements mod C.

(4) (O. Ore) The group A has a common system of representatives mod two of its subgroups, B and C, if and only if $|B : (B \cap C)| = |C : (B \cap C)|$. [Hint: for necessity use Exercise 3; for sufficiency, divide the cosets mod B into blocks mod $(B + C)$, and define a bijective correspondence within the blocks.]

(5) (N.H. McCoy) (a) Let B, C, G be subgroups of A such that G is contained in the set union $B \cup C$. Then either $G \leq B$ or $G \leq C$. [Hint: if $b \in (B \cap G) \setminus C$, then $c \in C \cap G$ implies $b + c \in B \cap G, c \in B \cap G$.]
(b) The same fails for the set union of three subgroups.

(6) If $n = p_1^{i_1} \cdots p_k^{i_k}$ is the canonical representation of the integer $n > 0$, then $nA = p_1^{i_1}A \cap \dots \cap p_k^{i_k}A$ and $A[n] = A[p_1^{i_1}] \oplus \dots \oplus A[p_k^{i_k}]$.

(7) (Honda) If $B \leq A$ and $m \in \mathbb{N}$, set $m^{-1}B = \{a \in A \mid ma \in B\}$. Prove that (a) $m^{-1}B$ is a subgroup of A containing B; (b) $m^{-1}0 = A[m]$; (c) $m^{-1}mB = B + A[m]$; (d) $m(m^{-1}B) = B \cap mA$; (e) $m^{-1}n^{-1}B = (mn)^{-1}B$ where $n \in \mathbb{N}$.

(8) A superfluous subgroup is contained in every maximal subgroup.
(9) (B.H. Neumann) Let $a_i + B_i$ ($i = 1, \dots, n$) be cosets of subgroups B_i of A such that their set union is all of A. Then one of B_i has finite index in A. [Hint: induct on n.]
(10) Let B be a subgroup of A. (a) If B is torsion, then $B \leq tA$. (b) If A/B is torsion-free, then $tA \leq B$.
(11) For every $n \in \mathbb{N}$, we have $n(tA) = t(nA)$.
(12) Prove the **triangle inequality** for the heights:

$$h_p(a + b) \geq \min\{h_p(a), h_p(b)\}$$

for the elements a, b of any group A. Equality holds if $h_p(a) \neq h_p(b)$.
(13) (a) Prove the inclusion relations $(B \cap C) + (B \cap D) \leq B \cap (C + D)$ and $B + (C \cap D) \leq (B + C) \cap (B + D)$ for subgroups B, C, D of a group A.
(b) Find examples where proper inclusions hold.

2 Maps and Diagrams

Homomorphisms Let A and B be arbitrary groups. A **homomorphism** $\alpha : A \to B$ (often denoted as $A \xrightarrow{\alpha} B$) is a function that associates with every element $a \in A$ a unique element $b \in B$, written as $\alpha(a) = b$ (or simply as $\alpha a = b$), such that it preserves addition:

$$\alpha(a_1 + a_2) = \alpha(a_1) + \alpha(a_2) \qquad \text{for all } a_1, a_2 \in A.$$

A is the **domain** and B is the **codomain** or **range** of α. If there is no need to name the homomorphism, then we write simply $A \to B$.

A homomorphism $\alpha : A \to B$ gives rise to two subgroups: $\operatorname{Ker}\alpha \leq A$ and $\operatorname{Im}\alpha \leq B$. $\operatorname{Ker}\alpha$ is the **kernel** of α: the set of all $a \in A$ with $\alpha a = 0$, while $\operatorname{Im}\alpha$, the **image** of α, consists of all $b \in B$ such that there is an $a \in A$ with $\alpha a = b$. The factor group $B/\operatorname{Im}\alpha$ is called the **cokernel** of α; notation: $\operatorname{Coker}\alpha$. If $\operatorname{Im}\alpha = B$, then α is **surjective** or **epic**; we also say that it is an **epimorphism**. If $\operatorname{Ker}\alpha = 0$, α is said to be **injective** or **monic**; also, α is a **monomorphism.** If both $\operatorname{Im}\alpha = B$ and $\operatorname{Ker}\alpha = 0$, then α is a **bijection**; in this case, it is called an **isomorphism**. The groups A and B are **isomorphic** (denoted as $A \cong B$) if there is an isomorphism $\alpha : A \to B$. Then the inverse map $\alpha^{-1} : B \to A$ exists, and is again an isomorphism. Abstractly, no distinction is made between isomorphic groups, unless they are distinct subgroups of the same larger group under consideration. If G is a subgroup of both A and B, and if $\alpha : A \to B$ fixes the elements of G, then α is a **homomorphism over** G.

A homomorphism α with $\operatorname{Im}\alpha = 0$ is referred to as a **zero homomorphism**; it will be denoted by 0. If $A \leq B$, then the map that assigns to every $a \in A$ itself may be regarded as a homomorphism of A into B; it is called the **injection** or **inclusion map**. The injection $0 \to A$ is the unique homomorphism of the group 0 into A. If $\alpha : A \to B$ and $C \leq A$, then the **restriction** $\gamma = \alpha \restriction C$ has the domain C and codomain B. In this case, α is viewed as an **extension** of γ, written $\gamma \leq \alpha$.

If $\alpha_0 < \cdots < \alpha_n < \ldots$ is a chain of extensions of maps (i.e., each map is an extension of its predecessor), then $\cup_{n<\omega}\alpha_n$ is their 'union.' This is a map from the union of the domains of the α_n to the common codomain.

Let $\alpha : A \to B$ and $\beta : B \to C$ be homomorphisms between groups; here, the codomain of α is the same as the domain of β. The composite map $A \to B \to C$, called the **product** of α and β and denoted by $\beta \circ \alpha$ or simply by $\beta\alpha$, is a homomorphism $A \to C$ (notice the order of factors!). Keep in mind that $\beta\alpha$ acts according to the rule

$$(\beta\alpha)a = \beta(\alpha a) \qquad \text{for all } a \in A.$$

The associative law $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ holds whenever the products $\beta\alpha$ and $\gamma\beta$ are defined. α is **right-cancellable** (i.e. $\beta\alpha = \gamma\alpha$ always implies $\beta = \gamma$) exactly if α is an epimorphism, and **left-cancellable** ($\alpha\beta = \alpha\gamma$ always implies $\beta = \gamma$) if and only if it is a monomorphism. The product of two epimorphisms (monomorphisms) is again one.

If $\alpha, \beta : A \to B$ are homomorphisms, then their sum $\alpha + \beta$ is again a homomorphism $A \to B$ defined as

$$(\alpha + \beta)a = \alpha a + \beta a \qquad \text{for all } a \in A.$$

Under this operation, the homomorphisms from A to B form an abelian group, denoted as $\operatorname{Hom}(A, B)$ (to be studied in Chapter 7).

A homomorphism of A into itself is called an **endomorphism**, and an isomorphism with itself an **automorphism**. The endomorphisms of A form a ring $\operatorname{End} A$, called the **endomorphism ring** of A, and the automorphisms of A form a group $\operatorname{Aut} A$, called the **automorphism group** of A (which is rarely commutative). The identity automorphism $\mathbf{1}_A$ of A is the identity both in $\operatorname{End} A$ and in $\operatorname{Aut} A$. A subgroup of A that is carried into itself by every endomorphism (automorphism) of A is said to be a **fully invariant** (**characteristic**) subgroup of A.

Both the sum and the intersection of fully invariant subgroups of A are fully invariant in A. Thus the fully invariant subgroups of A form a sublattice in the lattice $\mathbf{L}(A)$ of subgroups of A. If S is a subset of the group A, then we can talk about the fully invariant subgroup generated by S: this is the intersection of all fully invariant subgroups containing S, and coincides with the set of all sums of the images of elements in $\langle S \rangle$ under endomorphisms of A.

Let $\alpha : A \to B$ be a homomorphism, and set $K = \operatorname{Ker}\alpha$. The map $a + K \mapsto \alpha a$ from the factor group A/K to $\operatorname{Im}\alpha$ is an isomorphism. Thus $\operatorname{Im}\alpha \cong A/K$. The map

$A \to A/K$ acting as $a \mapsto a + K$ is called the **natural** or **canonical** homomorphism. There is an important isomorphism

$$B/(B \cap C) \cong (B + C)/C \qquad \text{if } B, C \le A,$$

known as the **first isomorphism theorem** by Emmy Noether. The natural isomorphism is $b + (B \cap C) \mapsto b + C$ for $b \in B$.

If $C \le B \le A$, then there is an epimorphism $A/C \to A/B$ acting as $a + C \mapsto a + B$, whose kernel is B/C. This leads to the **second isomorphism theorem**:

$$A/B \cong (A/C)/(B/C) \qquad \text{where } C \le B \le A.$$

Exact Sequences A sequence of groups A_i and homomorphisms α_i,

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_k} A_k \qquad (k \ge 2),$$

is called an **exact sequence** if $\operatorname{Im} \alpha_i = \operatorname{Ker} \alpha_{i+1}$ for $i = 1, \dots, k-1$. In particular, the sequence $0 \to A \xrightarrow{\alpha} B$ is exact if and only if α is monic, and $B \xrightarrow{\beta} C \to 0$ is exact if and only if β is epic. The exactness of $0 \to A \xrightarrow{\alpha} B \to 0$ is equivalent to α being an isomorphism. The most frequently used exact sequences are of the form

$$\mathfrak{e}: \; 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

and are called **short exact sequences**; here, α is an injection of A in B such that β is a surjective map with $\operatorname{Im} \alpha$ as kernel. Whenever convenient, we may identify A with the subgroup $\operatorname{Im} \alpha$ of B, and C with the quotient group B/A; in other words, A can be treated as a subgroup, and C as a factor group of B.

Another important exact sequence arises from an arbitrary group homomorphism $\eta\colon A \to B$. This is the sequence

$$0 \to \operatorname{Ker} \eta \xrightarrow{\alpha} A \xrightarrow{\eta} B \xrightarrow{\beta} \operatorname{Coker} \eta \to 0$$

where α denotes the embedding, and β the canonical map mod $\operatorname{Im} \eta$.

Diagrams The balance of this section is devoted to diagrams. Diagrams not only assist us in understanding statements, but they also clarify proofs, and most importantly, help intuition.

Roughly speaking, a **diagram** consists of capital letters, representing groups, and arrows between certain pairs of capital letters, representing homomorphisms between the corresponding groups. A diagram is **commutative** if we get the same composite homomorphism no matter how we move along directed arrows on different paths from one group to another group in the diagram. For instance, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C' \end{array}$$

is commutative exactly if the homomorphisms $\beta\mu$ and $\mu'\alpha$ of A into B' are equal, and the same holds for the homomorphisms $\gamma\nu$ and $\nu'\beta$ of B into C'. Then the equality of the homomorphisms $\gamma\nu\mu, \nu'\beta\mu, \nu'\mu'\alpha$ is a simple consequence. In diagrams, the identity map is often denoted by the sign of equality, as, e.g., in

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \| & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\gamma} & C \end{array}$$

This diagram is commutative exactly if $\gamma = \beta\alpha$.

The following two lemmas are dual to each other. In the proofs, the technique with maps is instructive.

Lemma 2.1. *A diagram*

$$\begin{array}{ccccccc} & & & & G & & \\ & & & & \downarrow{\scriptstyle\eta} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \end{array}$$

with exact row can be embedded in a commutative diagram

$$\begin{array}{ccccccc} & & & & G & & \\ & & & \swarrow{\scriptstyle\phi} & \downarrow{\scriptstyle\eta} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \end{array}$$

exactly if $\beta\eta = 0$. *In this case,* $\phi : G \to A$ *is uniquely determined.*

Proof. If such a ϕ exists, then $\eta = \alpha\phi$ implies $\beta\eta = \beta\alpha\phi = 0\phi = 0$, thus the stated condition is necessary. Conversely, if $\beta\eta = 0$ holds, then $\operatorname{Im}\eta \le \operatorname{Ker}\beta$. By the exactness of the row, α is monic and $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, so $\operatorname{Im}\eta \le \operatorname{Im}\alpha$. Hence the map $\phi = \alpha^{-1}\eta : G \to A$ is well defined and obviously satisfies $\alpha\phi = \eta$. If $\phi' : G \to A$ also satisfies $\alpha\phi' = \eta$, then $\alpha\phi' = \alpha\phi$ implies $\phi' = \phi$, since α is a monomorphism. ☐

In order to save space, the two diagrams may be combined to a single one:

$$
\begin{tikzcd}
 & & G \arrow[dl, dashed, "\phi"'] \arrow[d, "\eta"] & \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C
\end{tikzcd}
$$

where the solid arrows represent given maps, while the broken arrow designates a homomorphism to be "filled in."

Lemma 2.2. *A diagram*

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] \arrow[d, "\eta"'] & C \arrow[r] \arrow[dl, dashed, "\psi"] & 0 \\
 & G & &
\end{tikzcd}
$$

with exact row can be completed by a map $\psi : C \to G$ so as to get a commutative triangle exactly if $\eta\alpha = 0$. Such a ψ is unique.

Proof. If such a ψ exists, then from $\eta = \psi\beta$ we obtain $\eta\alpha = \psi\beta\alpha = \psi 0 = 0$, and the necessity is clear. Conversely, assume $\eta\alpha = 0$, and define $\psi : C \to G$ as follows. For $c \in C$ set $\psi c = \eta b$ with some $b \in B$ satisfying $\beta b = c$. This is a good definition, for if also $\psi b' = c$ for $b' \in B$, then $b' - b \in \operatorname{Ker}\beta = \operatorname{Im}\alpha \leq \operatorname{Ker}\eta$, so $\eta b' = \eta b$. Uniqueness follows from the surjectivity of β. $\square$

Next, we prove a lemma that has several applications. In its proof, the preceding lemmas are used without explicit reference.

Lemma 2.3. *Suppose*

$$
\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[d, "\phi"] & B' \arrow[r, "\beta'"] \arrow[d, "\psi"] \arrow[dl, dashed, "\sigma"] & C' \arrow[r] \arrow[d, "\eta"] \arrow[dl, dashed, "\rho"] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}
$$

is a commutative diagram with exact rows. There exists a map $\sigma : B' \to A$ making the upper triangle commute (i.e., $\sigma\alpha' = \phi$) if and only if there is a map $\rho\colon C' \to B$ making the lower triangle commute (i.e., $\beta\rho = \eta$).

Proof. If $\sigma : B' \to A$ satisfies $\sigma\alpha' = \phi$, then $(\psi - \alpha\sigma)\alpha' = \alpha\phi - \alpha\phi = 0$ implies the existence of a map ρ: $C' \to B$ such that $\rho\beta' = \psi - \alpha\sigma$. Hence from $\beta\rho\beta' = \beta\psi - \beta\alpha\sigma = \eta\beta'$ we obtain $\beta\rho = \eta$. Conversely, if there is a map ρ: $C' \to B$ such that $\beta\rho = \eta$, then $\tau = \psi - \rho\beta'$ satisfies $\beta\tau = \beta\psi - \beta\rho\beta' = \eta\beta' - \eta\beta' = 0$. Therefore, there is a map $\sigma : B' \to A$ with $\alpha\sigma = \tau$. It satisfies $\alpha\sigma\alpha' = \psi\alpha' - \rho\beta'\alpha' = \alpha\phi$, that is, $\sigma\alpha' = \phi$, as desired. □

The next two lemmas give us an opportunity to get acquainted with a simple, but extremely useful technique, called **diagram chasing**. The first lemma is a prelude to the second one.

Lemma 2.4. *Suppose we are given a commutative square as the center square in the diagram*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\alpha \arrow[r] \arrow[d, dashed, "\mu'"'] & A \arrow[r, "\alpha"] \arrow[d, "\mu"'] & A' \arrow[r] \arrow[d, "\nu"] & \operatorname{Coker}\alpha \arrow[r] \arrow[d, dashed, "\nu'"] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] & B \arrow[r, "\beta"] & B' \arrow[r] & \operatorname{Coker}\beta \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Then μ induces a map $\mu' :$ Ker $\alpha \to$ Ker β, and ν induces a map $\nu' :$ Coker $\alpha \to$ Coker β making the left and the right squares commute. μ' is monic if so is μ, and ν' is epic if so is ν.

Proof. The map $\mu' = \mu \restriction \operatorname{Ker}\alpha$ carries evidently $\operatorname{Ker}\alpha$ into B. We have to check that it lands in $\operatorname{Ker}\beta$. So we pick an $a \in \operatorname{Ker}\alpha$, and want to show that $\mu a \in \operatorname{Ker}\beta$, i.e. $\beta\mu a = 0$. This is indeed true, since $\beta\mu = \nu\alpha$ by the commutativity of the central square, and $\alpha a = 0$ by the choice of a. It is also clear that μ' is monic if μ is monic.

A kind of dual argument applies to the other half of the claim. Let $x \in \operatorname{Coker}\alpha$, i.e. x is a coset $a' + \operatorname{Im}\alpha$ $(a' \in A')$. Then $\nu a' + \operatorname{Im}\beta$ is independent of the selection of the representative a' of the coset, because if $a'' = a' + \alpha a$ for an $a \in A$, then $\nu a'' = \nu a' + \nu\alpha a = \nu a' + \beta\mu a \in \nu a' + \operatorname{Im}\beta$. It is immediate that the correspondence $a' + \operatorname{Im}\alpha \mapsto \nu a' + \operatorname{Im}\beta$ is a homomorphism $\nu' : \operatorname{Coker}\alpha \to \operatorname{Coker}\beta$. If ν is epic, then for every coset $b' + \operatorname{Im}\beta$ $(b' \in B')$ there exists an $a' \in A'$ such that $\nu a' = b' + \operatorname{Im}\beta$, whence the surjectivity of ν' is evident. □

The Snake Lemma The next, forbiddingly looking diagram is not as formidable as it appears at the first sight. The lemma provides us with a long exact sequence that will have significant applications in later chapters.

Lemma 2.5 (Snake Lemma *or* Kernel-Cokernel Sequence). *Let*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\alpha \arrow[r, "\mu_1"] \arrow[d] & \operatorname{Ker}\beta \arrow[r, "\nu_1"] \arrow[d] & \operatorname{Ker}\gamma \arrow[r, "\delta"] \arrow[d] & {} \\
0 \arrow[r] & A \arrow[r, "\mu"] \arrow[d, "\alpha"] & B \arrow[r, "\nu"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu'"] \arrow[d] & B' \arrow[r, "\nu'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
{} \arrow[r, "\delta"] & \operatorname{Coker}\alpha \arrow[r, "\mu_2"] \arrow[d] & \operatorname{Coker}\beta \arrow[r, "\nu_2"] \arrow[d] & \operatorname{Coker}\gamma \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commutative diagram, where the two middle rows and all the columns are exact. Then there exist maps making the top and bottom rows exact, as well as a map $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$*, called* **connecting homomorphism***, making the long sequence*

$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma \to 0$$

exact.

Proof. To start with, note that the vertical maps from the kernels are viewed as inclusions, while those into the cokernels as canonical epimorphisms.

First we define the connecting homomorphism δ. Pick an element $c \in \operatorname{Ker}\gamma$. There is a $b \in B$ such that $\nu b = c$. As $\gamma c = 0$, we have $\nu'\beta b = 0$, which implies that some $a' \in A'$ satisfies $\mu' a' = \beta b$. Define $\delta c = a' + \operatorname{Im}\alpha \in \operatorname{Coker}\alpha$. It is urgent to check that this is independent of the choice of $b \in B$. This is easily done: another choice $b^* = b + \mu x$ ($x \in A$) leads to $a^* = a' + \alpha x$ which gives the same coset in $\operatorname{Coker}\alpha$ as a'.

Since the maps μ_1, ν_1 act like μ, ν, it is easily seen that the sequence of kernels is exact. A similar comment settles the exactness of the sequence of the cokernels. It remains to check exactness at $\operatorname{Ker}\gamma$ and $\operatorname{Coker}\alpha$.

It is clear that the composite maps at these places are 0. Clearly, $\delta c = 0$ if and only if $a' = \alpha a$ for some $a \in A$. Then $\beta b = \mu' a' = \beta\mu a$, thus $b - \mu a \in \operatorname{Ker}\beta$. As $c = \nu b = \nu(b - \mu a)$, the exactness at $\operatorname{Ker}\gamma$ follows at once. Similarly, $a' + \operatorname{Im}\alpha \in \operatorname{Ker}\mu_2$ ($a' \in A'$) means that $\mu' a' \in \operatorname{Im}\beta$, so there is a $b \in B$ with $\mu' a' = \beta b$. Then $\nu b = c \in \operatorname{Ker}\gamma$ is mapped by δ upon $a' + \operatorname{Im}\alpha$. $\square$

The 5- and the 3 × 3-Lemma Finally, we state two more lemmas that are often useful in applications. The direct proofs are rather technical and omitted. They can be found, e.g., in Mac Lane [M], but they can also be derived from the Snake Lemma.

Lemma 2.6 (The 5-Lemma). *Suppose*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} & & \downarrow{\scriptstyle\gamma_4} & & \downarrow{\scriptstyle\gamma_5} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

is a commutative diagram with exact rows. Then

(a) *if γ_1 is epic and γ_2, γ_4 are monic, then γ_3 is monic;*
(b) *if γ_5 is monic and γ_2, γ_4 are epic, then γ_3 is epic;*
(c) *if γ_1 is epic, γ_5 is monic, and γ_2, γ_4 are isomorphisms, then γ_3 is an isomorphism.* □

Lemma 2.7 (The 3×3-Lemma). *Assume that the diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 & \longrightarrow & 0 \\ & & {\scriptstyle\lambda_1}\downarrow & & {\scriptstyle\mu_1}\downarrow & & \downarrow{\scriptstyle\nu_1} & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{\alpha_2} & B_2 & \xrightarrow{\beta_2} & C_2 & \longrightarrow & 0 \\ & & {\scriptstyle\lambda_2}\downarrow & & {\scriptstyle\mu_2}\downarrow & & \downarrow{\scriptstyle\nu_2} & & \\ 0 & \longrightarrow & A_3 & \xrightarrow{\alpha_3} & B_3 & \xrightarrow{\beta_3} & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

is commutative and has exact columns. If the first two rows or the last two rows are exact, then all three rows are exact. □

★ **Notes.** Only with the advance of category theory became the maps (homomorphisms) between groups, and more generally, between algebraic systems, major players in algebra. Their overall importance is manifest.

We will often draw diagrams to support proofs, though frequently, diagrams that can easily be supplied by the reader will be skipped in order to save space. We will see that even simple minded diagrams prove very effective in understanding claims, and can be extremely helpful in following proofs. Readers are encouraged to draw diagrams whenever possible, and it is hoped that serious students will fall into this useful habit.

Exercises

(1) Assume $\alpha : A \to B$ and $\beta : B \to C$. Prove that

  (a) $\operatorname{Ker} \beta\alpha \geq \operatorname{Ker} \alpha$, and equality holds if β is a monomorphism;
  (b) $\operatorname{Im} \beta\alpha \leq \operatorname{Im} \beta$, and equality holds if α is an epimorphism.

(2) Let again $\alpha : A \to B$ and $\beta : B \to C$.

  (a) If $\beta\alpha$ is a monomorphism, then α is monic, but β need not be monic.
  (b) If $\beta\alpha$ is an epimorphism, then β is, but α is not necessarily epic.

(3) For every group A and for every positive integer m there are exact sequences $0 \to A[m] \to A \to mA \to 0$ and $0 \to mA \to A \to A/mA \to 0$.

(4) Suppose $\alpha : A \to B$ and $B' \leq B$.

  (a) If we write $\alpha^{-1}B' = \{a \in A \mid \alpha a \in B'\}$, then we have $\alpha(\alpha^{-1}B') \leq B'$.
  (b) For $A' \leq A$, the following is true: $A' \leq \alpha^{-1}(\alpha A')$.

(5) Let $\eta \in \operatorname{End} A$ and $m \in \mathbb{N}$. Then $\operatorname{Ker} m\eta = m^{-1} \operatorname{Ker} \eta$.

(6) Let B and C be subgroups of A such that $A = B + C$, and $\beta : B \to G$, $\gamma : C \to G$ homomorphisms into the same group G. There is an $\alpha : A \to G$ with $\alpha \restriction B = \beta$ and $\alpha \restriction C = \gamma$ if and only if β and γ are equal on $B \cap C$.

(7) For every $m \in \mathbb{N}$, and for every fully invariant subgroup H of A, the subgroups mH and $H[m]$ are likewise fully invariant in A.

(8) Full invariance is a transitive property: a fully invariant subgroup of a fully invariant subgroup is fully invariant.

(9) (a) If H is a fully invariant subgroup of A, and η is an endomorphism of A, then the correspondence $a + H \mapsto \eta a + H$ is an endomorphism of A/H.
  (b) If H is fully invariant in A, and G/H is fully invariant in A/H, then G is fully invariant in A.

(10) If H_i $(i \in I)$ are fully invariant (characteristic) subgroups of A, then so are $\cap_{i \in I} H_i$ and $\sum_{i \in I} H_i$.

(11) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence, and $B' \leq B$. Then there exist $A' \leq A$ and $C' \leq C$ such that the sequence $0 \to A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C' \to 0$ is exact where $\alpha' = \alpha \restriction A'$ and $\beta' = \beta \restriction B'$.

(12) (a) In a diagram

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] \arrow[d, dashed, "\phi"] & B \arrow[d, "\beta"] \\
0 \arrow[r] & C \arrow[r, "\gamma"] & D
\end{tikzcd}$$

  with exact row, an arrow $\phi : A \to C$ can be filled in to make the diagram commute if and only if $\operatorname{Im} \beta\alpha \leq \operatorname{Im} \gamma$. Such a ϕ is unique.
  (b) Formulate and prove the dual of (a).

3 Fundamental Examples

Our next order of business is to get acquainted with the groups that we will meet all the time in the sequel.

Cyclic Groups They have been defined above as groups that can be generated by a single element: $C = \langle c \rangle$. A cyclic group can be finite or infinite.

The elements of an infinite cyclic group C generated by c are nc (all distinct) with n running over the additive group $\mathbb{Z}$ of integers. C is isomorphic to $\mathbb{Z}$, an isomorphism is given by the correspondence $nc \mapsto n \in \mathbb{Z}$. Thus all infinite cyclic groups are isomorphic. Along with c, also $-c$ can be a generator of C, but no other element alone can generate C.

The list of elements of a cyclic group $C = \langle c \rangle$ of order m is: $0, c, 2c, \dots,$ $(m-1)c$. Because of $mc = 0$, we compute in C just as with the integers mod m. Consequently, C is isomorphic to the additive group of residue classes of the integers mod m; this group is $\mathbb{Z}/m\mathbb{Z}$. Thus all finite cyclic groups of the same order are isomorphic; we shall use the notation $\mathbb{Z}(m)$. (In the literature, often $\mathbb{Z}_m$ is used, though $\mathbb{Z}_p$ is in conflict with the notation used for the localization of $\mathbb{Z}$ at p.) Along with c, every kc with $\gcd(k, m) = 1$ can serve as a single generator of C, but only these.

The simple abelian groups (0 is the only proper subgroup) are of prime order; they are cyclic, isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. They can be generated by any element $\neq 0$.

Theorem 3.1. *Subgroups of cyclic groups are cyclic.*

Proof. Let $C = \langle c \rangle$ be any cyclic group, and B a subgroup in C. If $B = 0$, then B is evidently cyclic, so for the rest of the proof we may assume that $B \neq 0$. Therefore, there exists $kc \in B$ with $kc \neq 0$. Then also $-kc \in B$, so B must contain multiples of c with positive coefficients. Among such coefficients there is a minimal one, say, n. We are going to show that $B = \langle nc \rangle$. As $nc \in B$, the inclusion $\langle nc \rangle \leq B$ is obvious. To prove the reverse inclusion, let sc be an arbitrary element of B where $s \in \mathbb{Z}$. Euclidean division yields $s = qn + r$ with $q, r \in \mathbb{Z}$ such that $0 \leq r < n$. Now $rc = (s - qn)c = sc - q(nc) \in B$, whence the choice of n implies $r = 0$. Hence $sc = q(nc) \in B$, proving that $B \leq \langle nc \rangle$. $\square$

An infinite cyclic group $C = \langle c \rangle$ has infinitely many subgroups; these are: $\langle mc \rangle$ for $m = 0, 1, 2, \dots$. The subgroups of a finite cyclic group $C = \langle c \rangle$ of order n are $\langle mc \rangle$ for positive divisors m of n.

Cocyclic Groups A cyclic group can be characterized as a group C containing an element c such that every homomorphism $\phi : A \to C$ (for any group A) with $c \in \operatorname{Im}\phi$ is surjective. Dualizing this concept, we arrive at the definition of cocyclic groups: C is **cocyclic** if there is an element $c \in C$ such that any homomorphism $\phi : C \to A$ with $c \notin \operatorname{Ker}\phi$ is monic. In this situation, c is called a **cogenerator** of C. Since every subgroup is the kernel of a suitable homomorphism, a cogenerator c must belong to all non-zero subgroups of C. Hence the intersection of all non-zero subgroups of a cocyclic group C is not 0; this is the unique smallest non-zero subgroup in C, it is a simple group generated by c. Conversely, if a group has a unique smallest subgroup $\neq 0$, then the group is cocyclic, and any non-zero element in the smallest subgroup is a cogenerator.

Example 3.2. A cyclic group $C = \langle c \rangle$ of prime power order p^k is cocyclic, where any element of order p is a cogenerator. This follows from the simple fact that the only subgroups of such a C are those in the chain

$$0 < \langle p^{k-1}c \rangle < \cdots < \langle pc \rangle < \langle c \rangle.$$

Let p denote a prime number. The p^nth complex roots of unity, with n running over all integers > 0, form an infinite multiplicative group; in accordance with our convention, we will switch to the additive notation. This group, called a **quasi-cyclic** group or a group **of type** p^∞ (notation: $\mathbb{Z}(p^\infty)$), can be defined as follows: it is generated by elements $c_1, c_2, \ldots, c_n, \ldots$ such that

$$pc_1 = 0,\ pc_2 = c_1,\ \ldots,\ pc_{n+1} = c_n,\ \ldots \qquad (n \in \mathbb{N}).$$

Here $o(c_n) = p^n$, and $\mathbb{Z}(p^\infty)$ is the union of the ascending chain of cyclic subgroups $\langle c_n \rangle$; these are the only non-zero proper subgroups of $\mathbb{Z}(p^\infty)$.

Theorem 3.3. *A group $C \neq 0$ is cocyclic if and only if it is isomorphic to $\mathbb{Z}(p^k)$ for some prime p and for some $k \in \mathbb{N} \cup \{\infty\}$.*

Proof. Let c be a cogenerator of C. Then $\langle c \rangle$ is the smallest subgroup $\neq 0$ of C, and therefore c must be of prime order p. Since c lies in every non-zero subgroup of C, C cannot contain elements of infinite order, neither elements whose order contains a prime factor $\neq p$, i.e. C is a p-group. As a basis of induction, assume that for an integer n, C contains at most one subgroup C_n of order p^n, and this is cyclic, say, $C_n = \langle c_n \rangle$, containing all elements of orders $\leq p^n$. This is evidently true for $n = 1$. If A, B are subgroups of order p^{n+1} in C, then there are elements $a \in A \setminus C_n$ and $b \in B \setminus C_n$, and their orders must be p^{n+1}. We may pick a, b such that $pa = c_n = pb$. Hence $p(a - b) = 0$, so $a - b = tc_n$ for some $t \in \mathbb{Z}$. We conclude that $a = b + tpb$ and $b = a - tpa$, i.e. $a \in \langle b \rangle$ and $b \in \langle a \rangle$. Consequently, $A = \langle a \rangle = \langle b \rangle = B$ is cyclic, and is the only subgroup of order p^{n+1} in C. Moreover, it must contain all the elements of order $\leq p^{n+1}$. Thus C is the union of an ascending chain of cyclic groups of orders p^n, so it must be of the form $C = \mathbb{Z}(p^k)$ for some $k \leq \infty$. $\square$

The proof also shows that all quasi-cyclic groups belonging to the same prime p are isomorphic. Since the proper subgroups of $\mathbb{Z}(p^\infty)$ are cyclic of type $\mathbb{Z}(p^k)$, all of its quotient groups $\neq 0$ are isomorphic to $\mathbb{Z}(p^\infty)$.

Elementary *p*-Groups An **elementary** group is defined as a group the orders of whose elements are square-free integers. We will see later on that such a group decomposes into the direct sum of elementary p-groups, for different primes p. Therefore, here we will focus our attention on elementary p-groups.

In an elementary p-group A all the elements $\neq 0$ have order p, i.e. $pa = 0$ for all $a \in A$. This means that the integers that are congruent mod p operate the same way on the elements of A, so that A can be viewed as a vector space over the prime field $\mathbb{Z}/p\mathbb{Z}$ of characteristic p. Hence A admits a basis $\{a_i\}_{i\in I}$ and every element $a \in A$ can be expressed uniquely as

$$a = r_1 a_{i_1} + \cdots + r_n a_{i_n} \quad (r_i \in \mathbb{Z}/p\mathbb{Z},\ i_j \in I).$$

The cardinality of a basis is uniquely determined by the group: it is its **dimension** as a $\mathbb{Z}/p\mathbb{Z}$-vector space. It is also determined by the order of the group.

Rational Groups Under addition, the rational numbers form a torsion-free group, called the **full rational group**, denoted by $\mathbb{Q}$. Like $\mathbb{Z}(p^\infty)$, $\mathbb{Q}$ can also be obtained as the union of an infinite ascending chain of cyclic groups, this time the cyclic groups are infinite:

$$\mathbb{Z} = \langle 1\rangle < \langle 2!^{-1}\rangle < \cdots < \langle n!^{-1}\rangle < \ldots .$$

Thus $\mathbb{Q}$ has a generating system $\{c_1 = 1, c_2, \ldots, c_n, \ldots\}$ such that

$$2c_2 = c_1,\ 3c_3 = c_2,\ \ldots,\ (n+1)c_{n+1} = c_n,\ \ldots .$$

It is easily seen that $\mathbb{Q}$ is **locally cyclic** in the sense that its finitely generated subgroups are cyclic. In fact, every finite set of its elements is contained in some $\langle n!^{-1}\rangle$; therefore, the subgroup they generate is a subgroup of a cyclic group, so itself cyclic. $\mathbb{Q}$ contains numerous proper subgroups that are not finitely generated, as the group $\mathbb{Z}_{(p)}$ of all rational numbers whose denominators are prime to the prime p, or the group $\mathbb{Q}^{(p)}$ of rational numbers whose denominators are powers of p. The subgroups of $\mathbb{Q}$, called **rational groups**, are the building bricks of torsion-free groups, and as such they are of fundamental importance in the theory of torsion-free groups; see Chapters 12–14.

Every proper factor group $\mathbb{Q}/A$ of $\mathbb{Q}$ (i.e., $A \neq 0$) is a torsion group, since every non-zero rational number has a non-zero multiple in A. In particular, $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the group of all complex roots of unity, an isomorphism being given by the map $r + \mathbb{Z} \mapsto e^{2ir\pi}$ (where $r \in \mathbb{Q}$, $i = \sqrt{-1}$, and e is the base of natural logarithm). We have $\mathbb{Q}/\langle r\rangle \cong \mathbb{Q}/\mathbb{Z}$ for every rational number $r \neq 0$, while $\mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Z}(p^\infty)$. ($\mathbb{Z}(p^\infty)$ is the p-Sylow subgroup of $\mathbb{Q}/\mathbb{Z}$.)

***p*-adic Integers** The p-adic integers appear naturally on the scene in a variety of ways; they play a substantial role in several branches of abelian group theory.

Let p be a prime, and $\mathbb{Z}_{(p)}$ the ring of rational numbers whose denominators are prime to p (this is the **localization of $\mathbb{Z}$ at** p, it is a discrete valuation ring). The non-zero ideals in $\mathbb{Z}_{(p)}$ are principal ideals generated by p^k with $k = 0, 1, \ldots$. If the set of these ideals $p^k\mathbb{Z}_{(p)}$ is declared to be a fundamental system of neighborhoods of 0, then $\mathbb{Z}_{(p)}$ becomes a (Hausdorff) topological ring, and we may form its completion J_p in this topology (this completion process is described in more detail for groups in Sect. 7 in Chapter 2). J_p is again a ring, called the **ring of** p**-adic integers**, whose non-zero ideals are $p^k J_p$ with $k = 0, 1, 2, \ldots$, and which is complete (i.e., every Cauchy sequence is convergent) in the topology defined by its ideals.

For all practical reasons, the elements of J_p may be represented as power series in p. Note that $\{0, 1, \ldots, p-1\}$ is a complete set of representatives of $\mathbb{Z}_{(p)}$ mod $p\mathbb{Z}_{(p)}$, and more generally, $\{0, p^k, 2p^k, \ldots, (p-1)p^k\}$ is a complete set of representatives of $p^k\mathbb{Z}_{(p)}$ mod $p^{k+1}\mathbb{Z}_{(p)}$. Let $\sigma \in J_p$, and $a_0, \ldots, a_n, \ldots$ a sequence in $\mathbb{Z}_{(p)}$ converging to σ (dropping to a subsequence, we may accelerate convergence, and assume $a_n - \sigma \in p^n J_p$ for all $n \in \mathbb{N}$). Owing to the definition of convergence, almost all a_n belong to the same coset mod $p\mathbb{Z}_{(p)}$, say, to the one represented by some $s_0 \in \{0, 1, \ldots, p-1\}$. Almost all differences $a_n - s_0$ belong to the same coset of $p\mathbb{Z}_{(p)}$ mod $p^2\mathbb{Z}_{(p)}$, say, to the one represented by $s_1 p$. So proceeding, σ defines a sequence $s_0, s_1 p, \ldots, s_n p^n, \ldots$, which is the same for every sequence converging to σ. Accordingly, we assign to σ the (formal) infinite series

$$\sigma = s_0 + s_1 p + s_2 p^2 + \cdots + s_n p^n + \ldots \qquad (s_n \in \{0, 1, \ldots, p-1\}).$$

Its partial sums $b_n = s_0 + s_1 p + \cdots + s_n p^n$ $(n = 0, 1, 2, \ldots)$ form a Cauchy sequence in $\mathbb{Z}_{(p)}$ that converges to σ in J_p because $\sigma - b_n \in p^k J_p$ for $n \geq k$. From the uniqueness of limits it follows that in this way different elements of J_p are associated with different series, and since every infinite series $s_0 + s_1 p + \cdots + s_n p^n + \ldots$ defines an element $\sigma \in J_p$, we may identify the elements of J_p with the corresponding series.

Let us see how to compute in J_p. If $\rho = r_0 + r_1 p + \cdots + r_n p^n + \ldots$ with $r_n \in \{0, 1, \ldots, p-1\}$ is another p-adic integer, then the sum $\rho + \sigma = t_0 + t_1 p + \cdots + t_n p^n + \ldots$ and the product $\rho\sigma = v_0 + v_1 p + \cdots + v_n p^n + \ldots$ are computed as follows: $t_0 = r_0 + s_0 - \ell_0 p$, $t_n = r_n + s_n + \ell_{n-1} - \ell_n p$ and $v_0 = r_0 s_0 - m_0 p$, $v_n = r_0 s_n + r_1 s_{n-1} + \cdots + r_n s_0 + m_{n-1} - m_n p$ for $n = 1, 2, \ldots$ where the integers ℓ_n, m_n are uniquely determined by the rule that all of t_n, v_n are integers in the set $\{0, \ldots, p-1\}$. As to subtraction and division, note that if, e.g., $s_0 \neq 0$, then the negative of σ is $-\sigma = (p - s_0)p + (p - s_1 - 1)p + (p - s_2 - 1)p^2 + \ldots$, and the inverse σ^{-1} of σ exists if and only if $s_0 \neq 0$ in which case it may be computed by using the inverse rule of multiplication.

We shall denote by J_p both the ring and the group of the p-adic integers. $\mathbb{Q}_p^*$ will denote the field of quotients of J_p (and its additive group); its elements are of the form $p^{-k}\sigma$ with $\sigma \in J_p$, $k \in \mathbb{Z}$.

★ **Notes.** It will perhaps be instructive to mention a few important applications of abelian groups, in particular, to illustrate the groups in this section by pointing out a few applications outside group theory.

The most widespread applications of abelian groups outside algebra are in algebraic topology and algebraic geometry. A main step was the reinterpretation of Betti numbers as invariants of finite groups. In algebra, the additive groups of rings and the unit groups of commutative rings play a leading role, see Chapter 18. Elementary p-groups are the additive groups of fields of prime characteristics. The case $p = 2$ plays a prominent role in computer theory as well as in coding theory. Class groups of integral domains are abelian groups. As a matter of fact, L. Claborn [Pac. J. Math. **18**, 219–222 (1966)] proved that every abelian group occurs in this way.

Exercises

(1) Neither $\mathbb{Z}(p^\infty)$ nor $\mathbb{Q}$ is finitely generated, and any finite subset of a generating set of these groups can be dropped without spoiling the generating property.

(2) (a) Subgroups and quotient groups of locally cyclic groups are locally cyclic.
(b) The (multiplicative) group of complex roots of unity is locally cyclic.

(3) The additive group $\mathbb{R}$ of the real numbers is isomorphic to the multiplicative group of the positive reals. [Hint: $x \mapsto e^x$.]

(4) If B, C are subgroups of a cyclic group A, then $A/B \cong A/C$ implies $B = C$.

(5) In a cocyclic group all subgroups are fully invariant.

(6) An elementary group of order p^r contains exactly

$$(p^r - 1)(p^{r-1} - 1) \cdots (p^{r-t+1} - 1)/(p - 1)(p^2 - 1) \cdots (p^t - 1)$$

different subgroups of order p^t $(t \leq r)$.

(7) Let A be a finite group. Suppose A has a set of subgroups, say $B_1, \ldots, B_n$ $(n > 1)$, such that every non-zero element of A belongs to exactly one of the B_i. Show that A is an elementary p-group.

(8) A subgroup M of A is called **maximal** if $M < A$ and $M \leq B < A$ implies $B = M$.

(a) A subgroup is maximal if and only if its index is a prime number.
(b) The cyclic groups $\mathbb{Z}(p^k)$ $(k = 1, 2, \ldots)$ as well as J_p have exactly one maximal subgroup. $\mathbb{Z}$ has infinitely many maximal subgroups, while neither $\mathbb{Z}(p^\infty)$ nor Q has any maximal subgroup.

(9) (a) The intersection of all maximal subgroups of A of the same prime index p is equal to pA.
(b) The **Frattini subgroup** of A is defined as the intersection of all maximal subgroups of A. Show that it is the intersection of the subgroups pA with p running over all primes.
(c) Find the Frattini subgroups of $\mathbb{Z}(n)$, $\mathbb{Z}(p^\infty)$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{(p)}$, and J_p.

(10) Prove the isomorphisms $\mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Z}(p^\infty)$, $\mathbb{Z}_{(p)}/p^n\mathbb{Z}_{(p)} \cong \mathbb{Z}(p^n)$, $J_p/p^nJ_p \cong \mathbb{Z}(p^n)$.

(11) (a) Let A be an infinite group all of whose proper subgroups are finite. Then $A \cong \mathbb{Z}(p^\infty)$ for some prime p.
(b) An infinite group B satisfying $B/C \cong B$ for every proper subgroup C is isomorphic to $\mathbb{Z}(p^\infty)$ for some prime p.

(12) (a) Find the sum, product, and quotient of the 3-adic integers $\pi = 2 + 1 \cdot 3 + 2 \cdot 3^2 + 1 \cdot 3^3 + 2 \cdot 3^4 + \ldots$ and $\rho = 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + \ldots$.
(b) Verify that $-1 = (p - 1) + (p - 1)p + \cdots + (p - 1)p^n + \ldots$ in J_p for every prime p (thus $\rho = -1$ in (a)).

(13) Prove that a p-adic integer $\sigma = s_0 + s_1p + \cdots + s_np^n + \ldots$ (where $s_n \in \{0, \ldots, p - 1\}$) is a unit in J_p if and only if $s_0 \neq 0$.

4 Sets

Since the publications of Baer [6] and Kulikov [1], set theory has played a significant role in abelian group theory. Its importance catapulted with the epoch making paper Shelah [1], and since that it has been impossible to study abelian groups thoroughly without a working knowledge in the theory of sets.

First a word about **classes** and **sets**. We are going to deal with both of them (though only rarely with classes), but we avoid trying to explain what they are, as they are most fundamental concepts. We will assume that the reader has acquired a working knowledge about classes and sets. It is of vital importance to distinguish between them, the rule of thumb being that a set does have cardinality, while a proper class (i.e., a class that is not a set) is too big to be measured by any cardinal; e.g., all the abelian groups form a proper class.

ZFC Throughout we will take for granted the ZFC axioms of Set Theory, i.e. the axioms of Zermelo-Fraenkel along with the Axiom of Choice. They formalize the basic set-theoretical properties that mathematicians use every day without mentioning them explicitly. We do not list these axioms, as we do not plan to refer to any of them explicitly, and their exact forms will never be required. So, we just accept them with the usual understanding that, if nothing is said, then we work in a model of ZFC. However, occasionally, we will adjoin another set-theoretical hypothesis consistent with ZFC as an additional axiom, like the Continuum Hypothesis (CH), the Generalized Continuum Hypothesis (GCH), or Gödel's Axiom of Constructibility (L). We recall that it was a major accomplishment by P.J. Cohen to show that both CH and GCH are independent of ZFC, while K. Gödel proved that GCH (and hence CH) holds if L is assumed.

We will frequently refer to Zorn's lemma (known to be equivalent to the Axiom of Choice in ZFC) as a main tool in existence proofs. In order to formulate it, recall the definition of a **partially ordered set** or, in short, a **poset**. This is a set P equipped with a binary relation $\leq$ such that $a \leq a$ (reflexivity); $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry); $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity) for all $a, b, c \in P$. A subset C of P is a **chain** if $a, b \in C$ implies $a \leq b$ or $b \leq a$. An element $u \in P$ is an **upper bound** for C if $c \leq u$ for all $c \in C$, and P is **inductive** if every chain in P has an upper bound in P. An element $a \in P$ is **maximal** in P if $a \leq x \in P$ implies $a = x$. (In contrast, a **maximum** element $b \in P$ is one that satisfies the stronger relation: $x \leq b$ for all $x \in P$.)

Zorn's Lemma. *A partially ordered set that is inductive contains maximal elements.*

Thus, in the applications of this lemma, we need three checks: it is a set, it is partially ordered, and finally, it is inductive.

If S is a subset (or a subgroup) of the group A, then a subgroup $B \leq A$ that is maximal with respect to the property $B \cap S = \varnothing$ ($B \cap S = 0$) is called an **S-high** subgroup. Its existence is guaranteed by Zorn's lemma.

Ordinals and Cardinals We assume the reader is familiar with the theory of cardinals and ordinals. The cardinality of a set X will be denoted by $|X|$. If κ is a cardinal, κ^+ will stand for the first cardinal strictly larger than κ, and 2^κ for the cardinality of the set of all subsets of a set of cardinality κ. We adhere to the standard notation, and will use the symbol $\aleph_\sigma$ (Hebrew letter aleph) for the σth infinite cardinal. In particular, $\aleph_0$ stands for countable (strangely enough, $\aleph_{-1}$ may indicate finiteness), and $2^{\aleph_0}$ for the continuum. CH claims that $2^{\aleph_0} = \aleph_1$, and GCH asserts that $2^\kappa = \kappa^+$ for each infinite cardinal κ. Addition and multiplication of cardinals are trivialities: $\kappa + \lambda = \kappa\lambda = \max\{\kappa, \lambda\}$ if at least one of κ, λ is infinite.

If convenient, we will view an ordinal μ as the set of ordinals less than μ, and a cardinal number as an initial ordinal, i.e. an ordinal which is not equinumerous with any smaller ordinal. The σth initial ordinal is denoted by ω_σ; in particular, ω_0 (or just ω) is the smallest infinite ordinal. If for an ordinal σ and a cardinal κ, we write $\sigma < \kappa$ or $\sigma \in \kappa$, we mean that the cardinality of σ is less than κ; in particular, $n < \omega$ or $n \in \omega$ is another way of saying that n is a non-negative integer. The notation X_σ $(\sigma < \kappa)$ means that the sets X_σ are indexed by ordinals σ less than κ. Addition and multiplication of ordinals are associative, but not commutative; we will not need them except when decomposing an ordinal $\sigma = \omega\alpha + n$ into segments of length ω; here, α is an ordinal, $n \geq 0$ an integer, both uniquely determined by σ. Note that $2\omega = 2 + 2 + \cdots + 2 + \cdots = \omega \neq \omega 2 = \omega + \omega$, the latter being the second smallest limit ordinal.

An ordinal σ is a **successor ordinal** if it is of the form $\sigma = \rho + 1$ for some ordinal ρ (called the **immediate predecessor of** σ). If $\sigma > 0$ has no immediate predecessor, it is called a **limit ordinal.** Every infinite cardinal κ represents a limit ordinal. The **cofinality** of σ, cf σ, is the smallest cardinal λ (= initial ordinal) such that there is a subset C in σ whose cardinality is λ and whose supremum is σ, $\sup C = \sigma$. A cardinal κ is called **regular** if $\mathrm{cf}\,\kappa = \kappa$. Otherwise it is **singular**. For every cardinal κ, cf κ is regular. For instance, the cardinals $\aleph_n$ for integers $n \geq 0$ are regular, while $\aleph_\omega$ is a singular cardinal with $\mathrm{cf}\,\aleph_\omega = \omega$.

A consequence of Gödel's theorem is that it is consistent with ZFC that no regular limit cardinal exists. On the other hand, it is not possible to prove that the hypothesis of the existence of regular limit cardinals is consistent with ZFC.

Cubs and Stationary Sets Let κ be an uncountable regular cardinal. A subset C of κ (i.e., a set of ordinals $< \kappa$) is said to be **unbounded** if $\sup C = \kappa$, and **closed** if $X \subset C$ and $\sup X < \kappa$ imply $\sup X \in C$. A **cub** is a closed unbounded subset of κ. A cub C in κ is order-isomorphic to κ, and so it can be reindexed by using all the ordinals $< \kappa$, i.e. it can be written as $C = \{f(\alpha)\}_{\alpha<\kappa}$ for an order-preserving bijection $f : \kappa \to C$. It is easily seen by using a routine back-and-forth argument:

Lemma 4.1. *The intersection of two cubs in κ is again a cub in κ.*

Proof. The proof is the same as for Lemma 4.3 below, so we may skip it. □

A subset E of a regular cardinal κ is said to be **stationary** if it intersects every cub in κ. Actually, the intersection of a stationary set with a cub is again a stationary set—this follows at once from Lemma 4.1. It is easily seen that a stationary subset in κ must have cardinality κ.

It might be helpful to think of sets containing a cub as analogs of sets of measure 1, and stationary sets as analogs of sets of measure > 0.

Example 4.2.

(a) Cubs are obvious examples for stationary subsets.
(b) For a regular cardinal κ, the set E of those limit ordinals in $\lambda > \kappa$ that are cofinal with κ is a stationary subset of λ. To show that E intersects every cub C in λ, let μ be the supremum of a subset of cardinality κ in C. Since κ is regular, cf $\mu = \kappa$, and since κ^+ is regular, $\mu < \kappa^+ \leq \lambda$. Hence $\mu \in C \cap E$, and E is stationary in λ.

Filtrations By a **filtration** of a set X of cardinality κ (almost always uncountable regular) we mean a family $\{X_\alpha\}_{\alpha<\kappa}$ of subsets of X such that the following holds:

(i) $\alpha \leq \beta$ implies $X_\alpha \subseteq X_\beta$ (i.e., it is a well-ordered ascending chain);
(ii) $|X_\alpha| < \kappa$ for all $\alpha < \kappa$;
(iii) $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$ whenever β is a limit ordinal $< \kappa$ (we will refer to this by saying that the chain of the X_α is **continuous** or **smooth**);
(iv) $X = \bigcup_{\alpha<\kappa} X_\alpha$.

It is a routine exercise in elementary set theory to show that such filtrations always exist. Though a set has numerous filtrations, it ought to be emphasized that, for uncountable regular cardinals, it really does not matter which filtration is chosen, since we have:

Lemma 4.3. *If κ is an uncountable regular cardinal, and if $\{X_\alpha\}_{\alpha<\kappa}$ and $\{Y_\alpha\}_{\alpha<\kappa}$ are two filtrations of the same set X of cardinality κ, then*

$$E = \{\alpha < \kappa \mid X_\alpha = Y_\alpha\}$$

is a cub in κ.

Proof. It is evident that the set E is closed in κ. To prove that it is also unbounded, note that for every member X_α of the first filtration there is a member Y_β of the second filtration such that $\alpha \leq \beta$ and $X_\alpha \subset Y_\beta$. To this Y_β we can find an X_{α_1} such that $\beta \leq \alpha_1$ and $Y_\beta \subset X_{\alpha_1}$. There is a Y_{β_1} with $\alpha_1 \leq \beta_1$ and $X_{\alpha_1} \subset Y_{\beta_1}$. In this way, we obtain an increasing sequence of ordinals $\alpha \leq \beta \leq \alpha_1 \leq \beta_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \ldots$ (all $< \kappa$) along with an increasing sequence of subsets, $X_\alpha \subset Y_\beta \subset X_{\alpha_1} \subset Y_{\beta_1} \subset \cdots \subset X_{\alpha_n} \subset Y_{\beta_n} \subset \ldots$, such that the ordinal $\sup_{n<\omega}\alpha_n = \sup_{n<\omega}\beta_n = \gamma$ satisfies $X_\gamma = \bigcup_{n<\omega} X_{\alpha_n} = \bigcup_{n<\omega} Y_{\beta_n} = Y_\gamma$. This argument (which is called a **back-and-forth argument**) establishes the existence of an index γ such that $\alpha < \gamma < \kappa$ and $\gamma \in E$. Thus E is unbounded in κ. $\square$

Filters The set of subsets of a set X is called the **power set of** X, denoted $\mathcal{P}(X)$. As mentioned above, its cardinality is $2^{|X|}$. A **filter** $\mathcal{D}$ on a set X is a set of subsets of X such that

(i) $\emptyset \notin \mathcal{D}$, $X \in \mathcal{D}$;
(ii) if $Y \in \mathcal{D}$ and $Y \subset Z \subset X$, then $Z \in \mathcal{D}$; and
(iii) $U, V \in \mathcal{D}$ implies $U \cap V \in \mathcal{D}$.

The principal filter generated by an element $x \in X$ consists of all subsets of X that contain x. If κ is an infinite cardinal, we say that $\mathcal{D}$ is **κ-complete** if the intersection of any $< \kappa$ members of $\mathcal{D}$ also belongs to $\mathcal{D}$; if this is not the case, we say the filter is **κ-incomplete**. An **ultrafilter** on X is a filter $\mathcal{U}$ such that for every subset Y of X, either Y or its complement $X \setminus Y$ belongs to $\mathcal{U}$. An ultrafilter $\mathcal{U}$ on X may be interpreted as a $\{0, 1\}$-valued, finitely additive measure μ on the power set $\mathcal{P}(X)$ where $\mu(Y) = 1$ $(Y \subseteq X)$ means $Y \in \mathcal{U}$. Ultrafilters are the maximal filters on X, and every filter is contained in an ultrafilter.

Inaccessible and Measurable Cardinals A cardinal λ is said to be **(strongly) inaccessible** if it is a regular uncountable cardinal such that $\kappa < \lambda$ implies $2^\kappa < \lambda$. (It is thus a limit cardinal as well; in the presence of GCH, it is nothing else than a regular limit cardinal.)

A **weakly compact** cardinal κ is inaccessible, and has the property that, for every κ-complete Boolean lattice B of subsets of a set of cardinality κ, every κ-complete filter on B extends to a κ-complete ultrafilter on B. (It seems there is no simple way to define weak compactness.) A characteristic property of weak compactness is displayed in Lemma 4.6.

A cardinal λ is called **measurable** if there is a non-principal $\aleph_1$-complete ultrafilter on λ. Equivalently, a set X of cardinality λ admits a countably additive measure μ which assumes only the values 0 and 1, and which satisfies $\mu(X) = 1$ and $\mu(x) = 0$ for all singletons $\{x\} \subset X$. It is straightforward to see that if a cardinal is measurable, then so are all larger cardinals, and if there exists a measurable cardinal at all, then there is a smallest one. Assuming V = L (see next), no measurable cardinal exists.

Constructible Universe The **constructible universe** is a model of set theory obtained from ZFC by adjoining Gödel's Axiom of Constructibility. This axiom does not allow us to form all subsets of an already existing set, but only those which can be defined in terms of a selection principle. This model is denoted by L, and if we assume the Axiom of Constructibility, then we can indicate this briefly by writing V = L (V being the standard notation for the model we work in). Gödel proved that V = L is consistent with ZFC.

An important consequence of Gödel's axiom is the Diamond Principle; see R. Jensen [Ann. Math. Logic **4**, 229–308 (1972)].

Diamond Principle $\diamondsuit_\kappa$. *Let κ be an uncountable regular cardinal, and E a stationary subset of κ. Given a filtration $\{X_\alpha\}_{\alpha<\kappa}$ of a set X of cardinality κ, there is a family $\{S_\alpha\}_{\alpha\in E}$ of sets such that $S_\alpha \subset X_\alpha$, and for any subset Y of X, the set*

$$E_Y = \{\alpha \in E \mid Y \cap X_\alpha = S_\alpha\}$$

is a stationary subset of κ.

This is an amazing prediction principle: it says that no matter how we choose a subset Y, Y will meet stationarily many times the preassigned subsets X_α exactly in the predicted subsets S_α.

From the stated form of this principle we can derive another version of $\diamondsuit$; the following proof is standard. ($\times$ between sets denotes cartesian product.)

Lemma 4.4. ($\diamondsuit_\kappa$) *Let E be a stationary subset of an uncountable regular cardinal κ, and $\{X_\alpha\}_{\alpha<\kappa}$ a filtration of a set X of cardinality κ. Let Y be any countable set. There is a family $\{g_\alpha\}_{\alpha\in E}$ of functions*

$$g_\alpha : X_\alpha \to Y \times X_\alpha$$

such that, for any function $g : X \to Y \times X$, the set

$$E_g = \{\alpha \in E \mid g \restriction X_\alpha = g_\alpha\}$$

is stationary in κ.

Proof. Define $X' = X \times Y \times X$ and $X'_\alpha = X_\alpha \times Y \times X_\alpha$ ($\alpha < \kappa$), and apply the Diamond Principle to this filtration of X' to conclude the existence of subsets S_α of $X_\alpha \times Y \times X_\alpha$ ($\alpha \in E$) with the property stated above. If for an α, S_α is the graph of a function $X_\alpha \to Y \times X_\alpha$, then define g_α to be this function. Otherwise, define g_α to be any function $X_\alpha \to Y \times X_\alpha$ whatsoever.

Now pick any function $g : X \to Y \times X$, and let S denote its graph viewed as a subset of $X \times Y \times X$. From the Diamond Principle we obtain that the set E_g of α's with $g \restriction X_\alpha = g_\alpha$ is stationary in κ, as desired. □

Another result of Jensen's which will be needed is as follows.

Lemma 4.5 (V = L). *Let κ be a regular cardinal which is not weakly compact. There exists a stationary subset E of κ which consists of limit ordinals cofinal with ω such that, for every limit ordinal $\lambda < \kappa$, the set $\lambda \cap E$ is not stationary in λ.* □

In contrast, for weakly compact cardinals the following lemma holds.

Lemma 4.6. *Let κ be a weakly compact cardinal, and E a stationary subset of κ. There is a regular cardinal $\lambda < \kappa$ such that the set $\lambda \cap E$ is stationary in λ.* □

We now consider another set-theoretic hypothesis which is not a consequence of, but is consistent with, ZFC (but inconsistent with V = L). In order to formulate it, we require a couple of definitions.

Let P be a partially ordered set under a binary relation $\leq$. We say the elements $p, q \in P$ have an **upper bound** if some $r \in P$ satisfies both $p \leq r$ and $q \leq r$. A subset D of P is **directed** (upwards) if every pair of elements of D has an upper bound in D. A subset C of P is **cofinal** in P if for every $p \in P$ there is a $c \in C$ such that $p \leq c$.

Martin's Axiom (MA). *Let P be a partially ordered set such that every subset of P no two elements of which have an upper bound in P is countable* (*this is called* *the* **countable antichain condition**). *For every family $\{C_i\}_{i\in I}$ of cofinal subsets of P (where $|I| < 2^{\aleph_0}$) there is a directed subset D of P that intersects every C_i.*

R. Solovay and S. Tenenbaum [Ann. Math. **94**, 201–245 (1971)] proved that MA is consistent with ZFC and the negation of the Continuum Hypothesis ($\neg$ CH), i.e. ZFC + MA + ($\aleph_1 < 2^{\aleph_0}$) is consistent (provided so is ZFC).

★ **Notes.** The above incomplete notes on set theory are intended to recap needed facts only, not to serve as a systematic introduction. For more details, see, e.g., Eklof [5]. Let us point out for information that in some cases Martin's Axiom is not a strong enough hypothesis to prove a result; a stronger version is Shelah's Proper Forcing Axiom (not to be used in this volume). A powerful prediction principle, called Black Box, was developed by Shelah; for an algebraic version, see Göbel–Wallutis [1].

For convenience, we will often assume V = L even if the Diamond Principle or something weaker would suffice.

Exercises

(1) Let A be a group of cardinality $\kappa \geq \aleph_0$. The set of finitely generated subgroups of A has cardinality κ.
(2) Let B, X be subgroups of A. There exists a subgroup C of A such that (i) $B \leq C$; (ii) $C \cap X = B \cap X$; (iii) if $C \leq C' \leq A$ and $C' \cap X = B \cap X$, then $C = C'$.
(3) Let H be a fully invariant subgroup of A, and S a subset of A such that $H \cap S = \varnothing$. There exists a fully invariant subgroup G of A that is maximal with respect to the properties: (i) $H \leq G$; (ii) $G \cap S = \varnothing$.
(4) Assume $\diamondsuit_\kappa$ for an uncountable regular κ, and let E be a stationary subset of κ. For sets X, Y of cardinality κ with filtrations $\{X_\alpha\}_{\alpha<\kappa}$ and $\{Y_\alpha\}_{\alpha<\kappa}$, there exists a set $\{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha<\kappa}$ of functions such that for any function $f : X \to Y$, the set $\{\alpha \in E \mid f \restriction X = f_\alpha\}$ is stationary in κ.

5 Families of Subgroups

In several occasions we will need a collection of subgroups to characterize a group property or to prove a theorem. The simplest example for such a collection is an ascending chain of subgroups indexed by the natural numbers, like

$$0 = A_0 < A_1 < \cdots < A_n < \ldots$$

whose union is the group A, i.e. $A = \cup_{n<\omega} A_n$.

Transfinite Chains A more general version is a transfinite sequence or chain

$$0 = A_0 < A_1 < \cdots < A_\sigma < \ldots \qquad (\sigma < \tau)$$

of subgroups, indexed by the ordinals less than an ordinal τ, which is called the **length** of the chain. Here we usually assume that the chain is **continuous** or **smooth** to mean that $A_\sigma = \cup_{\rho<\sigma} A_\rho$ holds for limit ordinals $\sigma < \tau$ (so that there is no hole in the chain). In this case we talk about a **smooth** or a **continuous well-ordered ascending chain**.

In the preceding section, we have introduced set filtrations. More often, we shall deal with group filtrations. The definition is the same: for an infinite cardinal κ, by a **κ-filtration** of a group A is meant a smooth chain $\{A_\sigma\}_{\sigma<\kappa}$ of subgroups with union A, subject to the condition that $|A_\sigma| < \kappa$ for all $\sigma < \kappa$. The κ-filtrations are especially useful for uncountable regular cardinals κ. In this case, Lemma 4.3 is applicable, i.e. *the intersection of two κ-filtrations is again one.*

Sometimes we shall need a **pure κ-filtration**, where the subgroups are assumed to be pure subgroups.

***G*- and *H*-Families** We will encounter collections of subgroups that are no longer linearly ordered. Their significance is greatly enhanced by the fact that they are not only useful in some proofs, but also instrumental in characterizing groups with certain properties. Following P. Hill, we define various families of subgroups.

Let $\kappa = \aleph_\nu$ ($\nu \geq -1$). (Recall $\aleph_{-1}$ is to be interpreted as 'finite.') By an **$H(\kappa)$-family** of subgroups of A is meant a collection $\mathcal{H}$ of subgroups of A satisfying the following conditions:

H_1. $0, A \in \mathcal{H}$;
H_2. $\mathcal{H}$ is closed under unions, i.e. $A_i \in \mathcal{H}$ $(i \in I)$ implies $\sum_{i\in I} A_i \in \mathcal{H}$ for any index set I;
H_3. if $C \in \mathcal{H}$, and X is any subset of A of cardinality $\leq \kappa$, then there is a subgroup $B \in \mathcal{H}$ that contains both C and X, and is such that $|B/C| \leq \kappa$.

It is easily checked that in the presence of H_2, it suffices to assume H_3 only for $C = 0$.

A **$G(\kappa)$-family** $\mathcal{G}$ is defined similarly with H_2 replaced by the following weaker condition:

G_2. $\mathcal{G}$ is closed under unions of chains.

Obviously, every $H(\kappa)$-family is a $G(\kappa)$-family, but the converse fails in general, see Example 5.2 below. Sometimes we will need the rank versions of these families (for the definition of rank, see Sect. 4 in Chapter 3). $H^*(\kappa)$- and $G^*(\kappa)$-*families* are defined similarly for torsion-free groups A: in these cases the subgroups in the families are required to be pure subgroups (Sect. 1 in Chapter 5) and in condition H_3, 'rank' is to be used in place of 'cardinality.'

Example 5.1. Let X be a generating system of A. If we let Y run over all subsets of X, then the subgroups $\langle Y\rangle$ generated by the Y form an $H(\aleph_0)$-family of subgroups.

Example 5.2. This example relies on the concept of pure subgroup. We claim that every torsion-free group A has a $G(\aleph_0)$-family of pure subgroups. In fact, select a maximal independent set X in A, and for a subset Y of X, let A_Y denote the smallest pure subgroup of A that contains Y. Then the set of all A_Y is a $G(\aleph_0)$-family. However, this is in general not an $H(\aleph_0)$-family, since the sum of pure subgroups need not be pure.

We shall see that the existence of families of subgroups of various kinds has a strong influence on the group structure. Actually, it works in both directions: a global property of groups may turn out to be equivalent to having a family of a certain kind of subgroups.

A notable special case is when the group A (of cardinality κ) happens to be a direct sum of subgroups. If $A = \oplus_{i \in I} A_i$ ($|A_i| < \kappa$) is a direct sum decomposition (see Sect. 1 in Chapter 2), then the standard way of defining an $H(\kappa)$-family of summands in A is to consider the set of all partial summands in this decomposition: $B_J = \oplus_{i \in J} A_i$ with J ranging over all subsets of I.

In the next proof, we use again a back-and-forth argument.

Lemma 5.3. *For an uncountable regular κ, the intersection of at most κ families of $H(\kappa)$- or $G(\kappa)$-families of subgroups is again a family of the same kind.*

Proof. We give a detailed proof for the intersection of two $H(\kappa)$-families, the general case follows the same pattern. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two $H(\kappa)$-families of subgroups of A, and let $\mathcal{H}$ be their intersection. Conditions H_1 and H_2 are evidently satisfied for $\mathcal{H}$, so we proceed to the verification of H_3. Given any $B \in \mathcal{H}$ and a set X with $|X| \le \kappa$, we can find a $B_1 \in \mathcal{H}_1$ such that $B \cup X \le B_1$ and $|B_1/B| \le \kappa$, and then a $C_1 \in \mathcal{H}_2$ such that $B_1 \le C_1$ and $|C_1/B| \le \kappa$. Continuing, there is a $B_2 \in \mathcal{H}_1$ satisfying $C_1 \le B_2$ and $|B_2/B| \le \kappa$, and then again a $C_2 \in \mathcal{H}_2$ with $B_2 \le C_2$ and $|C_2/B| \le \kappa$, etc. We obtain an increasing sequence of subgroups $B \le B_1 \le C_1 \le \cdots \le B_n \le C_n \le \dots$ where $B_n \in \mathcal{H}_1$ and $C_n \in \mathcal{H}_2$. If we set $C = \bigcup_{n<\omega} B_n = \bigcup_{n<\omega} C_n$, then it is clear that $C \in \mathcal{H}$, and we also have the required $|C/B| \le \kappa$. □

Summands of Families The following lemma deals with families in direct summands.

Lemma 5.4. *Assume $A = B \oplus C$ has an $H(\kappa)$-family $\mathcal{A}$ of subgroups ($\kappa \ge \aleph_0$). Then the B-components B' of those $A' \in \mathcal{A}$ which decompose as $A' = B' \oplus C'$ with $C' \le C$ form an $H(\kappa)$-family $\mathcal{B}$ of subgroups in B.*

Proof. We prove that the set $\mathcal{B}$ of the B-components of the groups A' satisfies H_3. Let $B_0 \in \mathcal{B}$ and X a subset of B of cardinality $\le \kappa$. There are $A_0 \in \mathcal{A}$ such that $A_0 = B_0 \oplus C_0$ with $C_0 \le C$, and $A_1 \in \mathcal{A}$ such that $A_0 \cup X \subseteq A_1$ and $|A_1/A_0| \le \kappa$. There are subgroups $B_1 \le B, C_1 \le C$ such that $B_0 \le B_1, C_0 \le C_1, A_1 \le B_1 \oplus C_1$ with $|B_1/B_0| \le \kappa, |C_1/C_0| \le \kappa$. We can now find $A_2 \in \mathcal{A}$ with $B_1 \oplus C_1 \le A_2$ and $|A_2/A_1| \le \kappa$. Proceeding in a similar fashion, we obtain sequences $A_n \in \mathcal{A}, B_n \le B$, and $C_n \le C$ ($n < \omega$) with the following properties:

$$A_n \le B_n \oplus C_n \le A_{n+1}, \qquad |A_{n+1}/A_n| \le \kappa.$$

Then $A' = \cup_n A_n \in \mathcal{A}$ will satisfy $A' = (A' \cap B) \oplus (A' \cap C)$, whence $A' \cap B \in \mathcal{B}$ is immediate. To complete the proof, we note that $B_0 \cup X \subseteq A' \cap B$ and $|(A' \cap B)/B_0| \le |A'/A_0| \le \kappa$. □

If κ is an infinite cardinal, and if we have a $G(\kappa)$-family $\mathcal{G}$ of subgroups, then it is easy to extract from it a κ-filtration. The procedure is obvious: we start with $A_0 = 0$, and if we have $A_\sigma \in \mathcal{G}$, then for $A_{\sigma+1}$ we pick any $B \in \mathcal{G}$ with $|B/A_\sigma| \leq \kappa$, and take unions at limit ordinals. We have to reach the whole group for obvious cardinality reason.

***H*-Family from a Chain** The following ingenious lemma also holds when the group G is torsion-free and 'cardinality' is replaced by 'rank.'

Theorem 5.5 (Hill [16]). *Suppose the group G is the union of a smooth chain*

$$0 = G_0 < G_1 < \cdots < G_\alpha < \ldots \qquad (\alpha < \tau) \tag{1.1}$$

of subgroups such that for each $\alpha + 1 < \tau$,

$$G_{\alpha+1} = G_\alpha + A_\alpha$$

holds for some subgroup A_α of cardinality $\leq \kappa = \aleph_\nu$ where $\nu \geq -1$. Then G admits an $H(\kappa)$-family $\mathcal{C}$ of subgroups such that every $C \in \mathcal{C}$ has a smooth chain of subgroups C_β with union C whose factors are isomorphic to factors in the chain (1.1), and satisfy $C_{\beta+1} = C_\beta + A_\alpha$ for some $\alpha < \tau$.

If the groups G_α are pure in G, then the members of $\mathcal{C}$ can be chosen to be pure.

Proof. To start with, observe that hypotheses imply $G_\beta = \sum_{\alpha<\beta} A_\alpha$ for all $\beta < \tau$. Thus each $g \in G$ is contained in the sum of a finite number of A_α. A subset S of τ will be called **blocked** if every $\beta \in S$ satisfies

$$G_\beta \cap A_\beta \leq \sum_{\alpha \in S, \alpha < \beta} A_\alpha.$$

For a blocked subset S of τ, we will set $G(S) = \sum_{\alpha \in S} A_\alpha$, and claim that

$$\mathcal{C} = \{G(S) \mid S \text{ a blocked subset in } \tau\}$$

is a desired $H(\kappa)$-family in G. The proof is completed in several steps.

1° *Unions of blocked subsets of τ are blocked.* Let S_i $(i \in I)$ be blocked subsets of τ, and $\beta \in \bigcup_{i \in I} S_i$. Then $\beta \in S_j$ for some $j \in I$, and $G_\beta \cap A_\beta$ is evidently contained in the sum of the A_α for $\alpha < \beta$ with all $\alpha \in S_j$, and *a fortiori* with all $\alpha \in \bigcup_{i \in I} S_i$.

2° *A subset of τ of cardinality $\leq \kappa$ is contained in a blocked subset of τ of cardinality $\leq \kappa$.* By 1°, it suffices to verify this for a single $\beta < \tau$. We induct on β. If $\beta = 0$, then the claim is true, as $\{0\}$ is evidently a blocked subset of τ. The subgroup $G_\beta \cap A_\beta$ is of cardinality $\leq \kappa$, so it contains a generating set $\{a_i \mid i \in I\}$ with $|I| \leq \kappa$. The a_i's are in G_β, so each is contained in a finite sum of the A_α's with $\alpha < \beta$. By induction, there is a blocked subset $S' \subset \tau$ such that $|S'| \leq \kappa$, and all the a_i are contained in $G(S')$. There is no loss of generality in assuming that $S' \subset \beta$, since otherwise S' can be replaced by the blocked subset $S' \cap \beta$. To show that $S = S' \cup \{\beta\}$ is a blocked subset of τ, it suffices to check the definition for β. $G(S')$ contains all the a_i, hence it contains $G_\beta \cap A_\beta$ as well.

3° *$\mathcal{C}$ is an $H(\kappa)$-family of subgroups in G.* Obviously, both $\emptyset$ and τ are blocked subsets of τ. Since $\sum_{i\in I} G(S_i) = G\left(\bigcup_{i\in I} S_i\right)$ holds for blocked subsets $S_i \subset \tau$, $\mathcal{C}$ is closed under arbitrary unions. If X is a subset of G of cardinality $\leq \kappa$, then there is a blocked subset $S \subset \tau$ such that $|S| \leq \kappa$, $X \subset G(S)$, where $G(S)$ has cardinality $\leq \kappa$.

4° *$G(S)$ has chains as desired.* Let S be a blocked subset of τ, and form the chain $C_\beta = \sum_{\alpha\in S,\alpha<\beta} A_\alpha$ for all $\beta \in S$. This is a smooth chain of subgroups of $C = G(S)$ with union C. As $G_\beta \cap A_\beta \leq \sum_{\alpha\in S,\alpha<\beta} A_\alpha \leq G_\beta$ implies $\left(\sum_{\alpha\in S,\alpha<\beta} A_\alpha\right) \cap A_\beta = G_\beta \cap A_\beta$, we clearly have

$$C_{\beta+1}/C_\beta = \left(\sum_{\alpha\in S,\alpha\leq\beta} A_\alpha\right) \Big/ \left(\sum_{\alpha\in S,\alpha<\beta} A_\alpha\right) \cong A_\beta/(G_\beta \cap A_\beta) \cong G_{\beta+1}/G_\beta.$$

The second part of the claim is clear.

5° *The $G(S)$ are pure in G if the G_α are pure in G.* Let $g \in G(S)$, thus $g = a_{\alpha_1} + \cdots + a_{\alpha_k}$ with $a_{\alpha_j} \in A_{\alpha_j}$, where $\alpha_1 < \cdots < \alpha_k$ in S. If $nx = g$ for some $n \in \mathbb{Z}$ and $x \in G$, then by purity $x \in G_{\alpha_k+1}$ can be assumed, so let $x = y + b_{\alpha_k}$ with $y \in G_{\alpha_k}, b_{\alpha_k} \in A_{\alpha_k}$. Hence $ny = nx - nb_{\alpha_k} = a_{\alpha_1} + \cdots + a_{\alpha_k} - nb_{\alpha_k}$, so $a_{\alpha_k} - nb_{\alpha_k} \in G_{\alpha_k} \cap A_{\alpha_k} \leq \sum_{\alpha\in S,\alpha<\alpha_k} A_\alpha$. We obtain $ny \in \sum_{\alpha\in S,\alpha<\alpha_k} A_\alpha$, whence an induction on the largest index α_k implies that we may pick $y \in G(S)$; then $x \in G(S)$ as well. □

Almost Disjoint Subsets The following two lemmas will be needed in later proofs. They deal with the existence of almost disjoint sets. Two infinite sets are called **almost disjoint** if their intersection is finite. In the following lemmas, the symbol Ω stands for the initial ordinal of the power $2^{\aleph_0}$.

Lemma 5.6. *There is a set $\{S_\sigma \mid \sigma \in \Omega\}$ of countable sets S_σ that are pairwise almost disjoint.*

Proof. For an irrational real number r let S_r be a sequence of rational numbers with limit α. It is clear that $S_r \cap S_s$ is finite whenever $r \neq s$. □

The following generalization of Lemma 5.6 will be needed in Sect. 6 in Chapter 13. We state it without proof.

Lemma 5.7 (W. Sierpinski). *Let κ be an infinite cardinal. There exists a set $\mathcal{C}$ of subsets of κ such that*

(i) $|X| = \kappa$ *for each* $X \in \mathcal{C}$;
(ii) *if* $X \neq Y$ *in* $\mathcal{C}$, *then* $|X \cap Y| < \kappa$; *and*
(iii) $|\mathcal{C}| > \kappa$. □

For the proof of Proposition 8.7 in Chapter 14, almost disjoint sets seems insufficient, a collection of disjoint sets is needed. We establish the existence of two collections Σ and Σ' (both of the power of the continuum) of subsets of a countable set S, such that for each fixed $\sigma < \Omega$, $\{S_\sigma \cap S'_\rho \mid \rho < \Omega\}$ is a collection of almost disjoint sets, where $S_\sigma \in \Sigma$, $S'_\rho \in \Sigma'$.

Lemma 5.8 (Dugas–Thomé [1]). *Given a countably infinite set S, there exist two families,*

$$\Sigma = \{S_\sigma \mid \sigma < \Omega\} \quad \textit{and} \quad \Sigma' = \{S'_\sigma \mid \sigma < \Omega\},$$

of almost disjoint subsets of S such that

(i) $\bigcup_{\sigma<\Omega} S_\sigma = S$ *and* $\bigcup_{\sigma<\Omega} S'_\sigma = S$;
(ii) *for all $\sigma, \rho < \Omega$, the intersection $S_\sigma \cap S'_\rho$ is infinite.*

Proof. For $n < \omega$, let F_n denote the set of all functions f from the set $\bar{n} = \{0, 1, \dots, n-1\}$ to the set $\{0, 1\}$. Evidently, the set $S = \bigcup_{n<\omega}(\bar{n} \times F_n)$ is countable. For a map $\phi : \omega \to \{0, 1\}$, we define

$$S_\phi = \{(\bar{n}, \phi \restriction \bar{n}) \mid n < \omega\}.$$

Then the set $\Sigma = \{S_\phi \mid \phi : \omega \to \{0, 1\}\}$ is an almost disjoint family of countable subsets of S; its size is $2^{\aleph_0}$.

Next we select an almost disjoint family $\{T_\sigma \mid \sigma < \Omega\}$ of subsets of ω (see Lemma 5.6), and define the countable sets

$$S'_\sigma = \{(\bar{n}, f) \mid n \in T_\sigma, f \in F_n\}.$$

Then for $\sigma \neq \rho \in \Omega$, the intersection

$$S'_\sigma \cap S'_\rho = \{(\bar{n}, f) \mid n \in T_\sigma, f \in F_n\} \cap \{(\bar{n}, f) \mid n \in T_\rho, f \in F_n\} =$$
$$= \{(\bar{n}, f) \mid n \in T_\sigma \cap T_\rho, f \in F_n\}$$

has to be finite, because $T_\sigma \cap T_\rho$ is a finite set. Hence $\Sigma' = \{S'_\sigma \mid \sigma < \Omega\}$ is an almost disjoint family of countable subsets of S. On the other hand, for any $\phi : \omega \to \{0, 1\}$ and any $\sigma < \Omega$, the intersection

$$S_\phi \cap S'_\sigma = \{(\bar{n}, \phi \restriction \bar{n}) \mid n < \omega\} \cap \{(\bar{n}, f) \mid n \in T_\sigma, f \in F_n\} =$$
$$= \{(\bar{n}, \phi \restriction \bar{n}) \mid n \in T_\sigma\}$$

is infinite. $\square$

★ **Notes.** A word of warning is in order. We follow the customary definitions for the $H(\kappa)$- and $G(\kappa)$-families which do not confirm with the usual practice involving a cardinal κ (that would be $< \kappa$ rather than our $\leq \kappa$).

Hill prefers to use the term **Axiom-3 family** for a $G(\aleph_0)$-family; the name is chosen to remind that having such a family is a condition that is weaker than, but similar to, the first and the second axioms of countability hypotheses widely used in topology.

Exercises

(1) Give a detailed proof of Lemma 5.3 for the intersection of κ families.
(2) A group of cardinality κ^+ has a κ-filtration.
(3) If $\{A_\sigma\}_{\sigma<\kappa}$ is a κ-filtration of A, and C is a cub in κ (an infinite cardinal), then $\{A_\sigma\}_{\sigma\in C}$ is also a κ-filtration of A.
(4) Let B be a subgroup in A. If B has a $G(\kappa)$-family $\mathcal{B}$ of subgroups, then there is a $G(\kappa)$-family $\mathcal{A}$ in A such that $\mathcal{B} = \{B \cap X \mid X \in \mathcal{A}\}$.
(5) Suppose $\alpha : A \to B$ is an epimorphism.
   (a) $\alpha\mathcal{A}$ is an $H(\kappa)$-family in B if $\mathcal{A}$ is an $H(\kappa)$-family of subgroups in A.
   (b) If B has a $G(\kappa)$- (or $H(\kappa)$-)family $\mathcal{B}$ of subgroups, then there is a $G(\kappa)$- ($H(\kappa)$-)family $\mathcal{A}$ in A such that $\alpha\mathcal{A} = \mathcal{B}$.
(6) Prove Lemma 5.6 as follows: each S_r (r is a positive real number) is a set of lattice points in the first quarter of the plane. For r, we let S_r consist of the lattice points whose distances from the line $y = rx$ are less than 1.

6 Categories of Abelian Groups

In the theory of abelian groups it is often convenient and more suggestive to express situations in the categorical language. In fact, categories and functors appear to be the right unifying concepts, and the categorical point of view frequently provides a better understanding. We survey some basic facts on categories and exhibit some important concepts connected to them, confining ourselves to those that will be needed in this volume. We are not borrowing sophisticated results from category theory, we mostly use it as an appropriate language.

Categories A category $\mathcal{C}$ is a class of **objects** $A, B, C, \ldots$ and **morphisms** (sometimes called **arrows**) $\alpha, \beta, \gamma, \ldots$ satisfying the following axioms:

C1. With each ordered pair of objects, $A, B \in \mathcal{C}$, there is associated a *set* $\mathrm{Morph}_{\mathcal{C}}(A, B)$ (or simply $\mathrm{Morph}(A, B)$) of morphisms in $\mathcal{C}$ such that every morphism in $\mathcal{C}$ belongs to exactly one $\mathrm{Morph}(A, B)$. In case $\alpha \in \mathrm{Morph}(A, B)$, we write $\alpha : A \to B$ and call α a **map** from A to B. A is the **domain** and B the **range** or **codomain** of α.
C2. *Composition.* With $\alpha \in \mathrm{Morph}(A, B)$ and $\beta \in \mathrm{Morph}(B', C)$ there is associated a unique element in $\mathrm{Morph}(A, C)$, called their **product** $\beta\alpha$ if and only if $B = B'$.

C3. *Identity.* For every object A, there is a morphism $\mathbf{1}_A \in \text{Morph}(A, A)$, the **identity morphism** of A, such that $\mathbf{1}_A\alpha = \alpha$ and $\beta\,\mathbf{1}_A = \beta$ provided the products are defined.
C4. *Associativity.* Whenever the products are defined, the composition is associative: $\gamma(\beta\alpha) = (\gamma\beta)\alpha$.

One verifies at once that the identity $\mathbf{1}_A$ is uniquely determined by the object A. Hence there is a bijection between the objects A and the identities $\mathbf{1}_A$. In view of this, categories can be (and often are) defined in terms of morphisms only.

Two objects, $A, C \in \mathcal{C}$, are said to be **isomorphic** if there exist morphisms, $\alpha : A \to C$ and $\gamma : C \to A$, such that $\gamma\alpha = \mathbf{1}_A$ and $\alpha\gamma = \mathbf{1}_C$. Manifestly, isomorphism is an equivalence relation in $\mathcal{C}$.

In the category $\mathcal{A}b$ of abelian groups, the objects are the abelian groups, and the morphisms are the homomorphisms between them. In particular, $\text{Morph}(A, B) = \text{Hom}(A, B)$ (see Chapter 7), and the identities are the identity automorphisms of the groups.

The category $\mathcal{D}$ is a **subcategory of** $\mathcal{C}$ if the objects of $\mathcal{D}$ are objects in $\mathcal{C}$, and the morphisms of $\mathcal{D}$ are morphisms in $\mathcal{C}$. $\mathcal{D}$ is a **full subcategory** of $\mathcal{C}$ if it is a subcategory with $\text{Morph}_{\mathcal{D}}(A, B) = \text{Morph}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{D}$. The **cartesian product** $\mathcal{C} \times \mathcal{D}$ of two categories consists of the objects (C, D) and morphisms (γ, δ) with $C, \gamma \in \mathcal{C}$, and $D, \delta \in \mathcal{D}$, where the morphisms act coordinate-wise.

Functors If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$, and to each morphism $\alpha : A \to B$ a morphism $F(\alpha) : F(A) \to F(B)$ such that

(i) if a product $\beta\alpha$ is defined in $\mathcal{C}$, then $F(\beta)F(\alpha)$ is defined in $\mathcal{D}$ and $F(\beta\alpha) = F(\beta)F(\alpha)$,
(ii) F carries the identity $\mathbf{1}_A$ to the identity $\mathbf{1}_{F(A)}$.

Thus a covariant functor preserves domains, codomains, products, and identities. The **identity functor** $\mathbf{1}_{\mathcal{C}}$, defined by $\mathbf{1}_{\mathcal{C}}(A) = A, \mathbf{1}_{\mathcal{C}}(\alpha) = \alpha$ for all $A, \alpha \in \mathcal{C}$, is a covariant functor of $\mathcal{C}$ into itself. A **contravariant functor** $G : \mathcal{C} \to \mathcal{D}$ is defined similarly by reversing arrows: it assigns an object $G(A) \in \mathcal{D}$ to every object $A \in \mathcal{C}$, and a morphism $G(\alpha) : G(B) \to G(A)$ to every morphism $\alpha : A \to B$. It is subject to the conditions: $G(\mathbf{1}_A) = \mathbf{1}_{G(A)}, G(\beta\alpha) = G(\alpha)G(\beta)$ for $\alpha, \beta \in \mathcal{C}$ whenever $\beta\alpha$ is defined. The unqualified term 'functor' will mean 'covariant functor.'

Example 6.1. Let $t : \mathcal{A}b \to \mathcal{T}$ be the functor on the category of abelian groups to the subcategory $\mathcal{T}$ of torsion abelian groups such that for $A \in \mathcal{A}b$, $t(A)$ is the torsion subgroup of A, and for $\alpha : A \to B$ in $\mathcal{A}b$, $t(\alpha) = \alpha \restriction t(A) : t(A) \to t(B)$.

Example 6.2. For a positive integer n, let the functor $M_n : \mathcal{A}b \to \mathcal{A}b$ assign to $A \in \mathcal{A}b$ the subgroup nA, and to $\alpha : A \to B$ the induced homomorphism $\alpha \restriction nA : nA \to nB$.

Example 6.3.

(a) Let $\mathcal{B}_n$ be the full subcategory of $\mathcal{A}b$ consisting of the n-**bounded** groups for a fixed $n \in \mathbb{N}$. $F : \mathcal{A}b \to \mathcal{B}_n$ is a functor assigning to $A \in \mathcal{A}b$ the subgroup $A[n] \in \mathcal{B}_n$, and to $\alpha : A \to B$ the restriction $\alpha \restriction A[n]$.

(b) We get another functor $G\colon \mathcal{A}b \to \mathcal{B}_n$ by letting $G(A) = A/nA$ and $G(\alpha)\colon a + nA \mapsto \alpha a + nB$ for $\alpha : A \to B$.

Example 6.4. In this example, $\mathcal{F}$ is the full subcategory of $\mathcal{A}b$ that consists of the torsion-free groups. The functor $F\colon \mathcal{A}b \to \mathcal{F}$ assigns to a group A the factor group $A/t(A)$, and to $\alpha\colon A \to B$ the induced map $F(\alpha)\colon a + t(A) \mapsto \alpha a + t(B)$.

Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ are functors. The composite GF is a functor from $\mathcal{C}$ to $\mathcal{E}$ where $GF(A) = G(F(A))$ and $GF(\alpha) = G(F(\alpha))$ for all $A, \alpha \in \mathcal{C}$. Clearly, GF is covariant if both F and G are covariant or both are contravariant, and is contravariant if one of F, G is covariant and the other is contravariant.

We shall have occasions to consider functors in several variables, covariant in some of their variables, and contravariant in others. For instance, if $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, then a **bifunctor** $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, covariant in $\mathcal{C}$ and contravariant in $\mathcal{D}$, assigns to each pair $(C, D) \in \mathcal{C} \times \mathcal{D}$ of objects an object $F(C, D) \in \mathcal{E}$, and to each pair $\alpha : A \to C, \beta : B \to D$ of morphisms $\alpha \in \mathcal{C}, \beta \in \mathcal{D}$ a morphism $F(\alpha, \beta)\colon F(A, D) \to F(C, B)$ in $\mathcal{E}$ such that

$$F(\gamma\alpha, \delta\beta) = F(\gamma, \beta)F(\alpha, \delta) \quad \text{and} \quad F(\mathbf{1}_C, \mathbf{1}_D) = \mathbf{1}_{F(C,D)} \tag{1.2}$$

whenever $\gamma\alpha, \delta\beta$ are defined. The quintessence of these relations is made clear in the commutativity of the diagram

$$\begin{array}{ccc}
F(A,D) & \xrightarrow{F(\alpha,\mathbf{1}_D)} & F(C,D) \\
{\scriptstyle F(\mathbf{1}_A,\beta)}\downarrow & & \downarrow{\scriptstyle F(\mathbf{1}_C,\beta)} \\
F(A,B) & \xrightarrow{F(\alpha,\mathbf{1}_B)} & F(C,B)
\end{array}$$

In the theory of abelian groups, one encounters almost exclusively **additive functors**, i.e. functors F satisfying $F(\alpha + \beta) = F(\alpha) + F(\beta)$ for all morphisms α, β whenever $\alpha + \beta$ is defined. For an additive functor F, one always has $F(0) = 0$ where 0 stands for the zero group or for the zero homomorphism. Also $F(n\alpha) = nF(\alpha)$ holds for every $n \in \mathbb{Z}$.

Exact Sequences One of the fundamental questions concerning functors in abelian groups is to find out how they behave for subgroups and quotient groups. This is most efficiently studied in terms of exact sequences. If F is a covariant functor from a subcategory $\mathcal{C}$ of $\mathcal{A}b$ into the category $\mathcal{D}$, and if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence in $\mathcal{C}$, then F is called **left** or **right exact** according as

$$0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \qquad \text{or} \qquad F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \to 0$$

is exact in $\mathcal{D}$; F is **exact** if it is both left and right exact. For a contravariant F, the displayed sequences are replaced by

$$0 \to F(C) \xrightarrow{F(\beta)} F(B) \xrightarrow{F(\alpha)} F(A) \qquad \text{and} \qquad F(C) \xrightarrow{F(\beta)} F(B) \xrightarrow{F(\alpha)} F(A) \to 0,$$

respectively. Subfunctors of the identity (assigning a subgroup to every group) are always left exact, while quotient functors (assigning some factor group) are right exact.

Assume F and G are covariant functors $\mathcal{C} \to \mathcal{D}$. By a **natural transformation** $\Phi : F \to G$ is meant a function assigning to each object $A \in \mathcal{C}$ a morphism $\Phi_A : F(A) \to G(A)$ in $\mathcal{D}$ in such a way that for all morphisms $\alpha : A \to B$ in $\mathcal{C}$, the following diagram in $\mathcal{D}$ commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\alpha)} & F(B) \\ {\scriptstyle \Phi_A}\downarrow & & \downarrow{\scriptstyle \Phi_B} \\ G(A) & \xrightarrow{G(\alpha)} & G(B) \end{array}$$

In this case, we also say that Φ_A is a **natural morphism** from $F(A)$ to $G(A)$. If Φ_A is a bijection (isomorphism) for every $A \in \mathcal{C}$, then Φ is called a **natural equivalence (natural isomorphism)**. (It is important to understand that natural isomorphism is far more than just an isomorphism between two groups: it is an individual case of a functorial isomorphism.)

In general, two categories, $\mathcal{C}$ and $\mathcal{D}$, are called **equivalent** if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that FG is naturally equivalent to $\mathbf{1}_{\mathcal{D}}$, and GF is naturally equivalent to $\mathbf{1}_{\mathcal{C}}$.

Example 6.5. The **skeleton** of a category $\mathcal{C}$ is the category $\mathcal{S}$ such that (i) $\mathcal{S}$ is a full subcategory of $\mathcal{C}$; (ii) every object $A \in \mathcal{C}$ is isomorphic to a unique object $S \in \mathcal{S}$. The embedding functor $F : S \mapsto S$ from $\mathcal{S}$ into $\mathcal{C}$, and the assignment functor $G : A \mapsto S$ define an equivalence between $\mathcal{C}$ and $\mathcal{S}$.

In particular, if we select an abelian group from each isomorphy class in $\mathcal{A}b$, then the full subcategory whose objects are the selected groups is a skeleton of $\mathcal{A}b$.

Example 6.6. For a fixed group G, the assignments $F_G : G \oplus A \mapsto A$ and $(\gamma, \alpha) \mapsto \alpha$ for all A, α in $\mathcal{A}b$ and $\gamma \in \operatorname{End} G$ define a functor. Furthermore, each homomorphism $\phi : G \to H$ between groups gives rise to a natural transformation $\Phi : F_G \to F_H$.

Anticipating the functor Hom that will be discussed in Chapter 7, we can explain what we mean by adjoint functors.

Let $\mathcal{A}$ and $\mathcal{C}$ be two categories, and $F : \mathcal{A} \to \mathcal{C},\ G : \mathcal{C} \to \mathcal{A}$ functors between them. The functors are called **adjoint** (more precisely, F is the left adjoint of G, and G is the right adjoint to F) if we have

$$\operatorname{Hom}_{\mathcal{A}}(A, G(C)) \cong \operatorname{Hom}_{\mathcal{C}}(F(A), C) \qquad \forall A \in \mathcal{A},\ C \in \mathcal{C}, \tag{1.3}$$

where the isomorphism is natural both in A and in C.

Example 6.7. Consider the categories $\mathcal{A}b$ and $\mathcal{B}_n$ of abelian groups and n-bounded abelian groups, respectively. Define $F : \mathcal{A}b \mapsto \mathcal{B}_n$ via $A \mapsto A/nA$ (canonical map) and $G : \mathcal{B}_n \mapsto \mathcal{A}b$ via $G: C[n] \mapsto C$ (injection). Then F is the left adjoint of G as is demonstrated by $\mathrm{Hom}(A, C[n]) \cong \mathrm{Hom}(A/nA, C)$.

★ **Notes.** Category theory is the product of the twentieth century, initiated by S. Eilenberg and S. Mac Lane. The revolutionary new idea was to shift the emphasis from the objects (like groups, rings, etc.) to the maps between them, and instead of focusing on one particular object, the entire class of similar objects became the subject of study. This point of view has proved very fruitful, it has penetrated into many branches of mathematics. The category of abelian groups has been under strict scrutiny of category theorists.

Functorial subgroups were studied by B. Charles. Let $F: \mathcal{A}b \to \mathcal{A}b$ be a functor such that (i) $F(A) \leq A$, and (ii) if $\phi: A \to B$ is a morphism in $\mathcal{A}b$, then $F(\phi) = \phi \restriction F(A)$, for all groups A. Then $F(A)$ is a **functorial subgroup** of A. Nunke calls such a functor **subfunctor of the identity**; he develops a more extensive theory. The torsion subgroup and its p-components are prototypes of functorial subgroups.

Exercises

(1) Both the torsion groups and the torsion-free groups form a subcategory in $\mathcal{A}b$. The same holds for p-groups.
(2) A category with a single object is essentially a monoid (i.e., a semigroup with identity) of morphisms.
(3) Give a detailed proof of our claim that the cartesian product of two categories is again a category.
(4) Prove that the following is a category: the objects are commutative squares of the form

$$\begin{array}{ccc} A_1 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_3 & \longrightarrow & A_4 \end{array}$$

where A_i are groups. The morphisms are quadruples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of group maps $(\alpha_i : A_i \to B_i)$ making all the arising squares between the objects commutative.
(5) The composite of two natural transformations is again one. Does this composition obey the associative law?
(6) Prove that, for an integer $n > 0$, $F: A \to A/nA$ with $F(\alpha): a + nA \mapsto \alpha a + nB$ for $\alpha: A \to B$ is a functor from $\mathcal{A}b$ to $\mathcal{B}_n$.
(7) We get a functor $U: \mathcal{A}b \to \mathcal{A}b$ by assigning the Ulm subgroup A^1 to a group A, and the restrictions to the Ulm subgroups of the homomorphisms.
(8) The equivalence of categories is a reflexive, symmetric, and transitive relation.

7 Linear Topologies

In abelian groups, topology can be introduced in various ways which are natural in one sense or another. The importance of certain topologies will be evident from subsequent developments, especially when completeness will be discussed.

Linear Topologies The most important topologies on groups to be considered here are the **linear topologies**; this means that there is a base (fundamental system) of neighborhoods about 0 which consists of subgroups such that all the cosets of these subgroups form a base of open sets for the topology. A more formal definition for linear topologies can be given as follows.

Let $\mathbf{u}$ be a filter in the lattice $\mathbf{L}(A)$ of all subgroups of A. $\mathbf{u}$ defines a topology on A, if we declare the set of subgroups $U \in \mathbf{u}$ to be a **base of open neighborhoods** about 0, and for every $a \in A$, the cosets $a + U$ $(U \in \mathbf{u})$ as a base of open neighborhoods of a. Since the intersection of two cosets, $a + U$ and $b + V$, is either vacuous or a coset mod $U \cap V$ (this subgroup also belongs to $\mathbf{u}$ whenever $U, V \in \mathbf{u}$), all open sets will be unions of cosets $a + U$ with $a \in A$, $U \in \mathbf{u}$. The continuity of the group operations is obvious from the simple observation that $x - y \in a + U$ implies $(x + U) - (y + U) \subseteq a + U$. Thus A is always a topological group under the arising topology, which may be called the **u-topology** of A; $(A, \mathbf{u})$ will denote A as a topological group equipped with the $\mathbf{u}$-topology.

Note the following simple facts on $\mathbf{u}$-topologies.

(A) The $\mathbf{u}$-topology on a group A is **discrete** exactly if $\{0\} \in \mathbf{u}$, **indiscrete** if $\mathbf{u} = \{A\}$, and **Hausdorff** if and only if $\bigcap_{U \in \mathbf{u}} U = 0$.
(B) *Open subgroups are closed.* Indeed, the complement of an open subgroup B is open as the union of the open cosets $a + B$ $(a \notin B)$.
(C) If the $\mathbf{u}$-topology of A is Hausdorff, then it makes A into a 0-dimensional topological group. In fact, the subgroups $U \in \mathbf{u}$ are by (B) both open and closed.
(D) *The closure of a subgroup B of A in the $\mathbf{u}$-topology of A is given by the formula* $B^- = \bigcap_{U \in \boldsymbol{u}} (B + U)$.

Some topologies satisfy the **first axiom of countability**, i.e. there is a countable base of neighborhoods about 0. If $\{U_n \mid n < \omega\}$ is such a system of neighborhoods, then $\{U_0 \cap \cdots \cap U_n \mid n < \omega\}$ is also one; thus, in this case we may assume without loss of generality that the sequence of the U_n is decreasing. These topologies—if Hausdorff—are exactly the **metrizable** linear topologies; in fact, a metric can be introduced by setting $\|a\| = e^{-n}$ (e denotes the base of the natural logarithm) whenever $a \in A$ belongs to $U_n \setminus U_{n+1}$.

The $\mathbf{u}$-topology is considered **finer** than the $\mathbf{v}$-topology, and the $\mathbf{v}$-topology **coarser** than the $\mathbf{u}$-topology, if $\mathbf{v} \subseteq \mathbf{u}$. Thus the discrete topology is the finest, and the indiscrete topology is the coarsest of all topologies on any group.

A **compact** group is a topological group whose topology is compact.

The following special topologies are significant, especially the $\mathbb{Z}$-adic topology that will be used constantly. In Examples 7.1–7.4, every group homomorphism is continuous.

Example 7.1. The $\mathbb{Z}$-**adic topology** on a group A is defined by letting $\{nA \mid n \in \mathbb{N}\}$ be a base of neighborhoods about 0. This is a $\mathbf{u}$-topology, where $\mathbf{u}$ consists of all $U \leq A$ such that A/U is a bounded group. This topology is Hausdorff if and only if the first Ulm subgroup A^1 of A vanishes. A subgroup G of A is closed exactly if the first Ulm subgroup of A/G is 0.

Example 7.2. In the p-**adic topology** (for a prime p) the subgroups $p^k A$ $(k < \omega)$ are declared to form a base of neighborhoods about 0. This is likewise a $\mathbf{u}$-topology, with $\mathbf{u}$ consisting of all $U \leq A$ such that A/U is a bounded p-group.

Example 7.3. In order to define the **Prüfer topology**, we choose the filter $\mathbf{u}$ to consist of all $U \leq A$ such that A/U satisfies the minimum condition on subgroups (see Sect. 5 in Chapter 4). This is always a Hausdorff topology in which all subgroups are closed.

Example 7.4. In the **finite index topology**, the subgroups of finite indices constitute a base of neighborhoods of 0; equivalently, $\mathbf{u}$ consists of the subgroups of finite indices in A. This is coarser than both the $\mathbb{Z}$-adic and the Prüfer topologies.

Example 7.5. The following groups are usually viewed as being equipped with the interval topology (which is not a linear topology): the multiplicative group $\mathbb{T}$ of complex numbers of absolute value 1, and the additive group $\mathbb{R}$ of reals.

Let $(A, \mathbf{u})$ be a topological group, and B a subgroup of A. The **induced topology** on B has the subgroups $B \cap U$ with $U \in \mathbf{u}$ as a base of neighborhoods of 0. There is an induced topology on the factor group A/B as well, where the subgroups $(B + U)/B$ with $U \in \mathbf{u}$ are the neighborhoods of B/B. This topology is Hausdorff if and only if B is closed, and discrete when B is open.

We will need later the following simple lemma.

Lemma 7.6. *Assume $(A, \mathbf{u})$ is a Hausdorff topological group, and B is a subgroup of A. If $mB = 0$ for some $0 \neq m \in \mathbb{Z}$, then also $mB^- = 0$ for the closure B^- of B in the $\mathbf{u}$-topology of A.*

Proof. For an $a \in A$, $a \in B^-$ means that $a \in B + U$ for all $U \in \mathbf{u}$. Then $ma \in mU \leq U$, thus $ma \in \cap_{U \in \mathbf{u}} U = 0$. $\square$

If $\{(A_i, \mathbf{u}_i) \mid i \in I\}$ is a family of topological groups, then their cartesian product $\prod A_i$ is usually equipped with the **product (Tychonoff) topology**; this is a linear topology (provided all $\mathbf{u}_i$ are linear) in which a base $\mathbf{u}$ of the neighborhoods of 0 consists of all subgroups of the form $\prod X_i$ where $X_i \in \mathbf{u}_i$ such that $X_i = A_i$ for almost all $i \in I$.

A homomorphism $\phi : A \to C$ between two topological groups is a group homomorphism that is at the same time continuous in the topological sense, i.e. for every open set V about $0 \in C$ there is an open neighborhood U of $0 \in A$ such that $\phi(U) \subseteq V$. A topological isomorphism ϕ is a group isomorphism such that both ϕ and ϕ^{-1} are continuous.

Groups and Hausdorff Topology We now prove a theorem showing that all infinite abelian groups can be equipped with a non-discrete Hausdorff topology. The result is of theoretical importance, no use will be made of it in the sequel.

Theorem 7.7 (Kertész–Szele [1]). *Every infinite abelian group can be made into a non-discrete Hausdorff topological group.*

Proof. (We need some simple facts that will be proved only later on.) The Prüfer topology makes an infinite abelian group A into a Hausdorff topological group. This is a discrete topology if and only if A itself satisfies the minimum condition on subgroups. Then by Theorem 5.3 in Chapter 4, such an A, if infinite, contains a subgroup $\mathbb{Z}(p^\infty)$ for some p. The embedding of this subgroup in the multiplicative group of complex numbers of absolute value 1 induces a non-discrete (non-linear) topology on $\mathbb{Z}(p^\infty)$, and by translations one obtains a non-discrete topology on A. □

Functorial Topologies Following Charles [3], we introduce functorial topologies. A **functorial topology** is defined in terms of a functor T on the category $\mathcal{A}b$ of abelian groups to the category $\mathcal{L}$ of (linearly) topologized abelian groups such that $T(A)$ is the group A furnished with a linear topology and $T(\phi) = \phi$ for every homomorphism $\phi : A \to B$ in $\mathcal{A}$. (The main point is that a functorial topology assigns a topology to every group A such that all group homomorphisms are continuous.) In this sense, all of our examples above (with the exception of the last one) are functorial topologies.

A general method of obtaining a functorial linear topology is to choose an arbitrary class $\mathcal{X}$ of groups and to declare, for each group A, the subgroups $\operatorname{Ker}\phi$ taken for all $\phi : A \to X \in \mathcal{X}$ as a subbase of neighborhoods of $0 \in A$. The discrete groups in the arising topology $T(\mathcal{X})$ are exactly the subgroups of finite direct sums of groups in $\mathcal{X}$. Thus if we assume, to start with, that $\mathcal{X}$ is closed under taking finite direct sums and subgroups, then different choices of $\mathcal{X}$ define different functorial topologies.

It should be pointed out that there are functorial linear topologies which are not obtainable in the indicate way by using an appropriate class $\mathcal{X}$. The large subgroup topology of a p-group (Sects. 2–3 in Chapter 10) is an example for such a functorial topology.

★ **Notes.** It is an interesting question as to which groups can carry certain special kind of topology. Minimal and maximal functorial topologies, e.g., were studied by Boyer–Mader [1] and by Fay–Walls [1].

Exercises

(1) The $\mathbf{u}$-topology is finer than the $\mathbf{v}$-topology on a group A if and only if the identity map $(A, \mathbf{u}) \to (A, \mathbf{v})$ is continuous.
(2) In a linear topology, a subgroup is closed if and only if it is the intersection of open subgroups. [Hint: use (D).]
(3) The sum of two closed subgroups is again closed.
(4) Prove that (a) the Prüfer topology is always Hausdorff; (b) every subgroup is closed.
(5) Let G be a closed subgroup in a linear topology of A. If A/G satisfies the minimum condition on subgroups, then G is open.

(6) The group J_p is compact both in the finite index and in the p-adic topologies.
(7) Let B be a closed subgroup of the topological group A. Prove that

  (a) the canonical map $A \to A/B$ is an open, continuous homomorphism (as usual, A/B is furnished with the induced topology);
  (b) A/B is discrete if and only if B is open in A;
  (c) if $\phi : A \to C$ is an open, continuous epimorphism between topological groups, then the topological isomorphism $A/\operatorname{Ker}\phi \cong C$ holds.

(8) The topology $T(\mathcal{X})$ on a group A (see above) is Hausdorff if and only if homomorphisms from A into groups in $\mathcal{X}$ separate points (i.e., for each $0 \neq a \in A$ there are $X \in \mathcal{X}$ and $\phi : A \to X$ with $\phi a \neq 0$).
(9) Let $\mathbf{u}$ be a linear topology of the group A whose first Ulm subgroup $A^1 = 0$. All subgroups of A are closed in $\mathbf{u}$ if and only if $\mathbf{u}$ is finer than the finite index topology.

8 Modules

Numerous theorems on abelian groups can be generalized, *mutatis mutandis*, to unital modules over principal ideal domains, even over Dedekind domains. Some results admit generalizations to modules over all integral domains, or possibly to arbitrary associative rings. It is a delicate question to find the natural boundaries of the validity of a particular theorem, i.e. to describe the largest class of rings over which the theorem in question is still valid—discussion of problems like this is beyond our present aim. But we cannot avoid modules completely: those over the ring J_p of p-adic integers naturally appear in the scene in a variety of ways. *Nolens volens*, we have to deal with them.

Modules Let R be an associative ring with identity 1, and M an abelian group such that

(i) with $r \in \mathsf{R}$ and $a \in M$ there is associated an element $ra \in M$;
(ii) $r(a + b) = ra + rb$ for all $r \in \mathsf{R}, a, b \in M$;
(iii) $(r + s)a = ra + sa$ for all $r, s \in \mathsf{R}, a \in M$:
(iv) $(rs)a = r(sa)$ for all $r, s \in \mathsf{R}$ and $a \in M$;
(v) $1a = a$ for all $a \in M$.

In this case, M is said to be a **left R-module** or a **left module over R**. A right module is defined similarly, with the elements of R acting on the right (the crucial difference is in postulate (iv) which tells us which factor in a product is to be applied first). If R is commutative, no distinction is necessary between left and right R-modules.

A submodule N of a left R-module M is a subset that is an R-module under the operations of M, i.e. a subgroup of M such that $rN \leq N$ for all $r \in \mathsf{R}$. In this case, the factor group M/N becomes an R-module, where $r(a + N) = ra + N$ for cosets $a + N$ and for $r \in \mathsf{R}$.

For two R-modules, M and N, an **R-homomorphism** is a group homomorphism $\phi : M \to N$ that respects multiplication by elements $r \in \mathsf{R}$, i.e. $\phi(ra) = r\phi(a)$ for all $r \in \mathsf{R}$ and $a \in M$. The meaning of R-isomorphism, etc. should be obvious.

Example 8.1.

(a) If R is a field, then an R-module is just an R-vector space. In abelian group theory, vector spaces over the prime fields ($\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$) are ubiquitous.
(b) If R is the ring $\mathbb{Z}$ of integers, then every abelian group A can be viewed as a $\mathbb{Z}$-module under the natural definition of multiplication of $a \in A$ by $n \in \mathbb{Z}$: na is the nth multiple of a.

Occasionally, we will deal with p-local groups. These are exactly those groups in which the elements are uniquely divisible by every prime $\neq p$. They are genuine $\mathbb{Z}_{(p)}$-modules, i.e., modules over the localization of $\mathbb{Z}$ at the prime p. The torsion subgroup $t(A)$ of a p-local group A is a p-group and $A/t(A)$ is q-divisible torsion-free for all primes $q \neq p$.

Example 8.2.

(a) The group J_p is p-local, and so is the additive group of $\mathbb{Z}_{(p)}$.
(b) For any group A, the tensor product $A \otimes \mathbb{Z}_{(p)}$ is a p-local group.

***p*-adic Modules** Let us say a few words about modules over the rings $\mathbb{Z}_{(p)}$ and J_p. The following observation will be used frequently.

Lemma 8.3. *Every p-group is, in a natural way, a module over the ring J_p of the p-adic integers.*

Proof. If $\pi = s_0 + s_1 p + s_2 p^2 + \cdots + s_n p^n + \cdots \in J_p$, and if $a \in A$ is of order p^n, then $\pi a = (s_0 + s_1 p + s_2 p^2 + \cdots + s_{n-1} p^{n-1})a$ is the natural definition. The element on the right does not change if we use a longer partial sum for π. The module properties are pretty clear. $\square$

Modules over J_p are called ***p*-adic modules**. For any group A, $J_p \otimes A$ is a p-adic module, and $a \mapsto 1 \otimes a$ $(a \in A)$ is the canonical map $\phi : A \to J_p \otimes A$. ϕ is universal for A in the sense that if M is any p-adic module and $\alpha : A \to M$ any homomorphism, then there is a unique J_p-map $\psi : J_p \otimes A \to M$ such that $\alpha = \psi\phi$.

★ **Notes.** Though several theorems in abelian group theory can be phrased more naturally as statements on modules over integral domains, or just over $\mathbb{Z}_{(p)}$ or J_p, we hesitate to enter unexplored territory, and will phrase the results to abelian groups only. In this way, inevitably some flavor is lost, but strict limitations had to be honored. The only exceptions will be cases when p-adic modules will emerge naturally.

Exercises

(1) For any ring R, the cyclic left R-module Ra is R-isomorphic **left annihilator** $\operatorname{Ann} a$ of a.
(2) For a submodule N of an R-module M, $\operatorname{Ann} a \leq \operatorname{Ann}(a + N)$ for all $a \in M$.
(3) If R is an integral domain, then the elements a of an R-module M with $\operatorname{Ann} a \neq 0$ form a submodule in M (called the torsion submodule).
(4) Every torsion group is a module over the completion $\tilde{\mathbb{Z}}$ of $\mathbb{Z}$ in the $\mathbb{Z}$-adic topology. This ring is the cartesian product of the rings J_p for all primes p.

(5) Let $\mathsf{S} \to \mathsf{R}$ be a ring homomorphism preserving identities. Then every R-module M becomes an S-module *via* $sa = \phi(s)a$ for $a \in M$.

Problems to Chapter 1

PROBLEM 1.1. Characterize the lattices of fully invariant subgroups of torsion-free groups.

PROBLEM 1.2. What are the singular submodules of A over its endomorphism ring $\operatorname{End} A$?

Chapter 2
Direct Sums and Direct Products

Abstract The concept of direct sum is of utmost importance for the theory. This is mostly due to two facts: first, if we succeed in decomposing a group into a direct sum, then it can be studied by investigating the summands separately, which are, in numerous cases, simpler to deal with. We shall see that almost all structure theorems in abelian group theory involve, explicitly or implicitly, some direct decomposition. Secondly, new groups can be constructed as direct sums of known or previously constructed groups.

Accordingly, there are two ways of approaching direct sums: an internal and an external way. Both will be discussed here along with their basic features. The external construction leads to the unrestricted direct sum, called direct (or cartesian) product, which will also play a prominent role in our future discussions. We present interesting results reflecting the fundamental differences in the behavior of direct sums and products in the infinite case. Pull-back and push-out diagrams will also be dealt with.

Important concepts are the direct and inverse limits that we shall use on several occasions. The final section of this chapter discusses completions in linear topologies.

A reader who is well versed in group theory can skip much of this chapter.

1 Direct Sums and Direct Products

Internal Direct Sum Let B, C be subgroups of the group A, and assume they satisfy

(i) $B + C = A$; and
(ii) $B \cap C = 0$.

Condition (i) tells us that every element $a \in A$ can be written as $a = b + c$ with $b \in B, c \in C$, while (ii) implies that such b, c are unique. For, if $a = b' + c'$ with $b' \in B, c' \in C$, then $b - b' = c' - c \in B \cap C = 0$. We will refer to b, c as the **coordinates** of a (in the given direct sum decomposition of A). In this case we write $A = B \oplus C$, and call A the **(internal) direct sum** of its subgroups B and C. (Recall that if (ii) is satisfied, we say that B and C are **disjoint**.)

Let B_i $(i \in I)$ be a set of subgroups in A subject to the following conditions:

(i) $\sum_{i \in I} B_i = A$, i.e. the subgroups B_i combined generate A; and
(ii) for every $i \in I$, $B_i \cap \sum_{j \neq i} B_j = 0$.

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-319-19422-6_2

Again, (i) means that every element $a \in A$ can be written as a finite sum $a = b_{i_1} + \cdots + b_{i_n}$ with b_{i_j} belonging to different **components** B_{i_j}, while (ii) states that such an expression is unique. We then write

$$A = B_1 \oplus \cdots \oplus B_n \qquad \text{or} \qquad A = \oplus_{i \in I} B_i$$

according as the index set is finite or infinite. We call these **direct decompositions** of the group A, and the B_i **(direct) summands** of A. If $A = B \oplus C$, C is a **complementary summand** or a **complement** to B. A is called **(directly) indecomposable** if $A = B \oplus C$ implies that either $B = 0$ or $C = 0$.

Let $a \in A = B \oplus C$, and write $a = b + c$ with $b \in B, c \in C$. The maps

$$\pi : A \to B,\ \rho : A \to C \quad \text{given by } \pi : a \mapsto b,\ \rho : a \mapsto c$$

are surjective maps; they can also be regarded as endomorphisms of A. They satisfy $\pi b = b$, $\pi c = 0$, $\rho c = c$, $\rho b = 0$ as well as $\pi a + \rho a = a$, thus

$$\pi^2 = \pi, \quad \rho^2 = \rho, \quad \rho\pi = 0 = \pi\rho, \quad \pi + \rho = \rho + \pi = \mathbf{1}_A. \tag{2.1}$$

If we mean by a **projection** an idempotent endomorphism, and by **orthogonal** endomorphisms those with 0 products (in both orders), then (2.1) may be expressed by saying that *a direct decomposition $A = B \oplus C$ defines a pair of orthogonal projections with sum $\mathbf{1}_A$*. Conversely, any pair π, ρ of endomorphisms satisfying (2.1) yields a direct decomposition $A = \pi A \oplus \rho A$. In fact, idempotency and orthogonality imply that any element common to πA and ρA must be both reproduced and annihilated by π and ρ, so $\pi A \cap \rho A = 0$, while $\pi + \rho = \mathbf{1}_A$ guarantees that $\pi A + \rho A = A$.

If A is the direct sum of several subgroups, $A = \oplus_{i \in I} B_i$, the decomposition can also be described in terms of pairwise orthogonal projections. The ith projection $\pi_i : A \to B_i$ assigns to the element $a = b_{i_1} + \cdots + b_{i_n}$ the term $b_i \in B_i$ (which can very well be 0). Then we have:

(a) $\pi_i \pi_j = 0$ or π_i according as $i \neq j$ or $i = j$;
(b) for every $a \in A$, almost all of $\pi_i a$ are 0, and $\sum_{i \in I} \pi_i a = a$.

Conversely, if $\{\pi_i \mid i \in I\}$ is a set of endomorphisms of A satisfying (a) and (b), then A is the direct sum of the subgroups $\pi_i A$.

Some of the most useful properties of direct sums are listed as follows:

(A) If $A = B \oplus C$, then $C \cong A/B$. Thus the complement of B in A is unique up to isomorphism.
(B) If $A = B \oplus C$, and if G is a subgroup of A containing B, then we have $G = B \oplus (G \cap C)$.
(C) If $a \in A = B \oplus C$, and if $a = b + c$ ($b \in B, c \in C$), then $o(a) = \operatorname{lcm}\{o(b), o(c)\}$ provided both orders are finite. Otherwise, $o(a) = \infty$.
(D) If $A = \oplus_{i \in I} B_i$ and if $C_i \leq B_i$ for each i, then $\sum_i C_i = \oplus_i C_i$.

(E) If $A = \oplus_i B_i$, where each B_i is a direct sum $B_i = \oplus_j C_{ij}$, then $A = \oplus_i \oplus_j C_{ij}$. This is a **refinement** of the given decomposition of A. Conversely, if $A = \oplus_i \oplus_j C_{ij}$, then $A = \oplus_i B_i$ where $B_i = \oplus_j C_{ij}$.

(F) If in the exact sequence $0 \to B \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$, $\operatorname{Im}\alpha$ is a summand of A, then $A \cong B \oplus C$. In this case, we say that the exact sequence is **splitting**. Any map $\gamma : C \to A$ satisfying $\beta\gamma = \mathbf{1}_C$ is called a **splitting map**; then $A = \operatorname{Ker}\beta \oplus \operatorname{Im}\gamma$. Of course, there is another map: $\delta : A \to B$ with $\delta\alpha = \mathbf{1}_B$ indicating splitting: $A = \operatorname{Im}\alpha \oplus \operatorname{Ker}\delta$.

Two direct decompositions of A, $A = \oplus_i B_i$ and $A = \oplus_j C_j$ are called **isomorphic** if there is a bijection between the two sets of components, B_i and C_j, such that corresponding components are isomorphic.

We now prove a fundamental result.

Lemma 1.1. *Let $C = \langle c\rangle$ be a finite cyclic group where $o(c) = m = p_1^{r_1} \cdots p_k^{r_k}$ with different primes p_i. Then C has a decomposition into a direct sum*

$$C = \langle c_1\rangle \oplus \cdots \oplus \langle c_k\rangle \qquad (o(c_i) = p_i^{r_i})$$

with uniquely determined summands.

Proof. Define $m_i = mp_i^{-r_i}$ and $c_i = m_i c$ $(i = 1, \dots, k)$. Then the m_i are relatively prime, so there are $s_i \in \mathbb{Z}$ such that $s_1m_1 + \cdots + s_km_k = 1$. Then $c = s_1m_1c + \cdots + s_km_kc = s_1c_1 + \cdots + s_kc_k$ shows that the c_i generate C. Clearly, $\langle c_i\rangle$ is of order $p_i^{r_i}$, so disjoint from $\langle c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_k\rangle$ which has order m_i. Hence we conclude that $C = \langle c_1\rangle \oplus \cdots \oplus \langle c_k\rangle$.

The uniqueness of the summands $\langle c_i\rangle$ (but not of the generators c_i) follows from the fact that $\langle c_i\rangle$ is the only subgroup of C that contains all the elements whose orders are powers of p_i. □

Decomposition of Torsion Groups One of the most important applications of direct sums is the following theorem that plays a fundamental role in abelian group theory.

Theorem 1.2. *A torsion group A is the direct sum of p-groups A_p belonging to different primes p:*

$$A = \oplus_p A_p.$$

The A_p are uniquely determined by A.

Proof. Given A, let A_p consist of all $a \in A$ whose orders are powers of the prime p. Since $0 \in A_p$, A_p is not empty. If $a, b \in A$, i.e. $p^m a = 0 = p^n b$ for integers $m, n \geq 0$, then $p^{n+m}(a - b) = 0$, so $a - b \in A_p$, and A_p is a subgroup of A. If $p_1, \dots, p_k$ are primes $\neq p$, then $A_p \cap (A_{p_1} + \cdots + A_{p_k}) = 0$, since every element of $A_{p_1} + \cdots + A_{p_k}$ is annihilated by a product of powers of $p_1, \dots, p_k$. Thus the A_p generate their direct sum in A; it must be all of A, as it is obvious in view of Lemma 1.1.

If $A = \oplus_p B_p$ is another decomposition of A into p-groups B_p with different primes p, then by the definition of the A_p we have $B_p \leq A_p$ for each p. If we had $B_p < A_p$ for some p, then $\oplus_p B_p$ could not equal A. □

The subgroups A_p are called the **primary components** or the ***p*-components** of A. They are, as is seen from the definition, fully invariant in A. If A is not torsion, then the p-components T_p of its torsion part $T = tA$ may be referred to as the p-components of A. (In this case, however, T_p need not be a summand of A.) Theorem 1.2 is of utmost importance as it makes it possible to reduce the structure theory of torsion groups to primary groups.

Example 1.3. The group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of all complex numbers that are nth roots of unity for some integer $n > 0$. It is a torsion group whose p-component is $\mathbb{Z}(p^\infty)$ (this corresponds to the subgroup of all p^kth roots of unity ($k = 0, 1, 2, \dots$)). Hence

$$\mathbb{Q}/\mathbb{Z} \cong \oplus_p \mathbb{Z}(p^\infty).$$

Another crucial direct sum decomposition is a trivial consequence of a vector space theorem.

Theorem 1.4. *An elementary group is a direct sum of cyclic groups of prime orders.*

Proof. By Theorem 1.2 only p-groups need to be considered. An elementary p-group is a $\mathbb{Z}/p\mathbb{Z}$-vector space, and as such it is the direct sum of one-dimensional spaces, i.e. of groups of order p. □

External Direct Sum While the internal direct sum serves to break a group into smaller pieces, in case of external direct sums we glue together groups to create a new larger group.

We start with two unrelated groups, B and C, and construct a new group A that is the direct sum of two subgroups B' and C', such that $B' \cong B$, $C' \cong C$. The set of all pairs (b, c) with $b \in B, c \in C$ forms a group A under the rules:

(a) $(b_1, c_1) = (b_2, c_2)$ if and only if $b_1 = b_2, c_1 = c_2$,
(b) $(b_1, c_1) + (b_2, c_2) = (b_1 + b_2, c_1 + c_2)$.

Then $B' = \{(b, 0) \mid b \in B\} \cong B$, $C' = \{(0, c) \mid c \in C\} \cong C$ under the correspondences $b \mapsto (b, 0), c \mapsto (0, c)$; they are subgroups of A such that $A = B' \oplus C'$ (internal direct sum). If we think of B, C being identified with B', C' under the indicated mappings, then we may also write $A = B \oplus C$, and call A the **external direct sum** of B and C. (We write $A \cong B \oplus C$ to say that A is a direct sum of two subgroups isomorphic to B and C.)

Direct Products A **vector** $(\dots, b_i, \dots)$ over the set $\{B_i\}_{i \in I}$ of groups has exactly one **coordinate** b_i from B_i, *viz.* in the ith position, for each $i \in I$. Such a vector can also be interpreted as a function f defined over I such that $f(i) = b_i \in B_i$ for every $i \in I$. Equality and addition of vectors are defined coordinate-wise (for functions, we would say point-wise). In this way, the set of all vectors becomes a group C, called the **direct product** or the **cartesian product** of the groups B_i; in notation:

$$C = \prod_{i \in I} B_i.$$

The correspondence $\rho_i : b_i \mapsto (\dots, 0, b_i, 0, \dots, 0, \dots)$ where b_i is the ith coordinate and 0's are everywhere else, is an isomorphism of B_i with a subgroup B_i' of C. The groups B_i' $(i \in I)$ generate their direct sum B in C which consists of all vectors with finite support, where **support** means $\operatorname{supp} c = \{i \in I \mid c_i \neq 0\}$ if $c = (\dots, c_i, \dots) \in C$. B is the external direct sum of the B_i, $B = \oplus_{i \in I} B_i$. Clearly, $B = C$ whenever I is finite.

For a group A, and for a set I, $A^{(I)} = \oplus_{i \in I} A$ will denote the direct sum of $|I|$ copies of A, and the symbol $A^I = \prod_{i \in I} A$ will stand for the direct product of $|I|$ copies of A. The corresponding notations $A^{(\kappa)}$ and A^{κ} for a cardinal κ should be clear.

The external direct sums and direct products can also be described in terms of systems of maps. The functions

$$\rho_B \colon b \mapsto (b, 0), \quad \rho_C \colon c \mapsto (0, c), \quad \pi_B \colon (b, c) \mapsto b, \quad \pi_C \colon (b, c) \mapsto c$$

are called the **(coordinate) injection** and **projection** maps, respectively. They satisfy

$$\pi_B \rho_B = \mathbf{1}_B, \quad \pi_C \rho_C = \mathbf{1}_C, \quad \pi_B \rho_C = 0 = \pi_C \rho_B, \quad \rho_B \pi_B + \rho_C \pi_C = \mathbf{1}_{B \oplus C}.$$

For an arbitrary number of components B_i $(i \in I)$, we have injections ρ_i and projections π_i satisfying

$$B_i \xrightarrow{\rho_i} C = \prod_{i \in I} B_i \xrightarrow{\pi_i} B_i$$

where $\rho_i b_i = (\dots, 0, b_i, 0, \dots)$, $\pi_i(\dots, b_j, \dots, b_i, \dots) = b_i$ satisfy the conditions: (i) $\pi_j \rho_i = \mathbf{1}_{B_i}$ or 0 according as $i = j$ or $i \neq j$; and (ii) $\sum_{i \in I} \rho_i \pi_i = \mathbf{1}_C$ (formally). Similarly for an infinite direct sum $\oplus_{i \in I} B_i$, in which case any given element is annihilated by almost all π_i.

The following 'universal' properties are crucial.

Theorem 1.5. *Let $\beta_i : B_i \to A$ $(i \in I)$ denote arbitrary homomorphisms, and $\rho_i : B_i \to \oplus_{i \in I} B_i$ the injection maps. There is a unique homomorphism $\phi \colon \oplus_{i \in I} B_i \to A$ such that $\beta_i = \phi \rho_i$ for every i.*

Proof. Write $b \in \oplus_{i \in I} B_i$ in the form $b = \rho_1 \pi_1 b + \dots + \rho_n \pi_n b$ where the π_i are the projection maps of the direct sum. It is immediately checked that $\phi b = \beta_1 \pi_1 b + \dots + \beta_n \pi_n b \in A$ defines a homomorphism $\phi \colon \oplus_{i \in I} B_i \to A$ with $\beta_i = \phi \rho_i$.

If ϕ' is another such map, then $(\phi - \phi')\rho_i = 0$ for each i, so $(\phi - \phi')b$ vanishes for all $b \in \oplus_{i\in I} B_i$, i.e. $\phi = \phi'$. □

Theorem 1.6. *Let $\alpha_i : A \to B_i$ $(i \in I)$ denote homomorphisms and $\pi_i : \prod_{i\in I} B_i \to B_i$ the projection maps. There exists a unique map $\psi : A \to \prod_{i\in I} B_i$ such that $\alpha_i = \pi_i\psi$ for each $i \in I$.*

Proof. Define $\psi(a) = (\dots, \alpha_i a, \dots) \in \prod_{i\in I} B_i$. This is obviously a homomorphism satisfying $\alpha_i = \pi_i\psi$. If also ψ' has the same property, then $\pi_i(\psi - \psi')a = 0$ for all $a \in A$, thus $(\psi - \psi')a = 0$. This means $\psi = \psi'$. □

A notational agreement: if $\alpha_i : A_i \to B_i$ $(i \in I)$ are homomorphisms, then $\oplus_{i\in I} \alpha_i$ will denote the map $\oplus_{i\in I} A_i \to \oplus_{i\in I} B_i$ that carries the ith coordinates to the ith coordinates as given by α_i. The map $\prod_{i\in I} \alpha_i : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ has similar meaning.

For a group G, the **diagonal map** $\Delta_G : G \to \prod G$ (arbitrary number of components) acts as $\Delta_G : g \mapsto (\dots, g, \dots, g, \dots)$, and the **codiagonal map** $\nabla_G : \oplus G \to G$ as $\nabla_G : (\dots, g_i, \dots) \mapsto \sum_i g_i$.

Subdirect Products Among the subgroups of the direct product, the subdirect products are most important. A group G is a **subdirect product** of the groups B_i $(i \in I)$ if it is a subgroup of the direct product $A = \prod_{i\in I} B_i$ such that $\pi_i G = B_i$ for all projections $\pi_i : A \to B_i$. This means that for every $b_i \in B_i$, G contains at least one vector whose ith coordinate is exactly b_i. If $K_i = \mathrm{Ker}(\pi_i \restriction G)$, then $\cap_{i\in I} K_i = 0$. Conversely, if K_i are subgroups of a group G such that $\cap_{i\in I} K_i = 0$, then G is a subdirect product of the factor groups G/K_i, *via*

$$g \mapsto (\dots, g + K_i, \dots) \in \prod_{i\in I} (G/K_i) \qquad \text{where } g \in G.$$

If the index set I is finite, then we also say that we have a subdirect sum.

Lemma 1.7 (Łoś). *Every group is a subdirect product of cocyclic groups.*

Proof. For every non-zero a in group A, let K_a be a subgroup of A maximal without a (argue with Zorn). Thus every subgroup of A that properly contains K_a also contains a, i.e. the coset $a + K_a$ is a cogenerator in A/K_a, so this factor group is cocyclic. Since $\cap_{0\neq a\in A} K_a = 0$, it follows that A is a subdirect product of the cocyclic groups A/K_a. □

There are numerous subdirect products contained in a direct product of groups, but there is no complete survey of them. The only exception is the case of subdirect sums of two groups.

Let G be a subdirect sum of B and C. The elements $b \in B$ with $(b, 0) \in G$ form a subgroup $B_0 \leq B$ and the elements $c \in C$ with $(0, c) \in G$ form a subgroup $C_0 \leq C$. It is straightforward to check that the correspondence $b + B_0 \mapsto c + C_0$ whenever $(b, c) \in G$ is an isomorphism of B/B_0 with C/C_0. Thus G consists of those $(b, c) \in B \oplus C$ for which the canonical epimorphisms $B \to B/B_0$ and $C \to C/C_0$ map b

and c upon corresponding cosets. The groups B_0 and C_0 are called the **kernels** of the subdirect sum. Conversely, if we are given the groups B, C along with epimorphisms $\beta : B \to F$, $\gamma : C \to F$ for some group F, then the elements $(b, c) \in B \oplus C$ with $\beta b = \gamma c$ form a group G that is a subdirect sum of B and C. It is easy to verify the isomorphisms

$$G/B_0 \cong C, \quad G/C_0 \cong B, \quad B/B_0 \cong G/(B_0 \oplus C_0) \cong C/C_0.$$

We mention that the subdirect sum G in the preceding paragraph may also be obtained as a pull-back of the maps β, γ where $B \xrightarrow{\beta} B/B_0 \cong C/C_0 \xleftarrow{\gamma} C$. See Exercise 3 in Sect. 3.)

Let K be an ideal in the Boolean lattice of all subsets of I; then the K**-product** $\prod_{\mathsf{K}} A_i$ is the set of all vectors in $\prod_{i\in I} A_i$ whose supports belong to K. The κ**-product** $\prod_{i\in I}^{<\kappa} A_i$ consists of vectors with support $< \kappa$.

Ultraproducts The following construction is based on the notion of filters. Let I be an infinite index set and $\mathcal{F}$ a filter on the subsets of I. The **filtered direct product** of groups A_i $(i \in I)$ is a subgroup of the direct product $A = \prod_{i\in I} A_i$ consisting of all vectors $a = (\dots, a_i, \dots) \in A$ for which the null-set $n(a) = \{i \in I \mid a_i = 0\} \in \mathcal{F}$. It is routine to check that this is in fact a (pure) subgroup of A, which we shall denote as $\prod_{i\in I}^{\mathcal{F}} A_i$. The factor group

$$\prod_{i\in I} A_i/\mathcal{F} = \prod_{i\in I} A_i/\prod\nolimits_{i\in I}^{\mathcal{F}} A_i$$

is called the **reduced product** with respect to $\mathcal{F}$. Thus $a, b \in \prod_{i\in I} A_i$ are equal in $\prod_{i\in I} A_i/\mathcal{F}$ exactly if supp $(a - b) \in \mathcal{F}$.

The most important special case is when $\mathcal{F}$ is an ultrafilter $\mathcal{U}$. Then $\prod_{i\in I} A_i/\mathcal{U}$ is called the **ultraproduct** of the A_i. If $\mathcal{U}$ is a principal ultrafilter, i.e. it consists of those subsets of I that contain a fixed $j \in I$, then $\prod_{i\in I}^{\mathcal{U}} A_i = \prod_{i\in J} A_i$ where $J = I\setminus\{j\}$. In this case, the ultraproduct is just A_j. Therefore, only ultraproducts with respect to non-principal ultrafilters are of real interest.

Example 1.8. The filter $\mathcal{F}$ that consists of all subsets of I with finite complements is non-principal, and $\prod_{i\in I}^{\mathcal{F}} A_i$ contains the direct sum $\oplus_{i\in I} A_i$.

★ **Notes.** A noteworthy generalization of direct powers, studied by Balzerzyk [3], Eda [1], relies on a complete Boolean lattice $\mathbf{B}$ with $\mathbf{0}$ as smallest and $\mathbf{1}$ as largest element. By the **Boolean power** $A^{(\mathbf{B})}$ of the group A is meant the set of functions $f : A \to \mathbf{B}$ such that (i) $f(a) \wedge f(b) = \mathbf{0}$ if $a \neq b$ in A, and (ii) $\bigvee_{a\in A} f(a) = \mathbf{1}$. The sum $f + g$ of two functions is defined *via*

$$(f + g)(a) = \vee_{a=x+y}(f(x) \wedge g(y))$$

for all possible $x, y \in A$ satisfying $x + y = a$. In case $\mathbf{B}$ is the power-set of a set I, then the elements $f \in A^{(\mathbf{B})}$ are in a bijective correspondence with the elements $\bar{f} \in A^I$ such that $f(a) = \{i \in I \mid \bar{f}(i) = a\} \in \mathbf{B}$ where $a \in A$.

The primary decomposition Theorem 1.2 is of central importance in abelian group theory. Its roots are in elementary number theory; this kind of decomposition was used by C.F. Gauss. In its complete, final form is due to Frobenius–Stickelberger [1]. The result generalizes straightforwardly to torsion modules over Dedekind domains.

In contrast to Theorem 1.2, Theorem 1.4 easily generalizes to arbitrary modules: if a module is the union of simple submodules, then it is a direct sum of simple modules (it is then called semi-simple). Semi-simple modules may be characterized by the property that every submodule is a direct summand.

The result on the subdirect sum of two groups is due to R. Remak; he dealt with finite, not necessarily commutative groups. Ultraproducts have profound implications in various areas, especially in model theory. See Eklof [1] for their structure.

Exercises

(1) Let B, C be subgroups of A, and $B \oplus C$ their external direct sum. There is an exact sequence $0 \to B \cap C \to B \oplus C \to B + C \to 0$.

(2) Determine when the direct product of infinitely many torsion groups is again a torsion group.

(3) If $0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$ are exact sequences for $i \in I$, then so are

$$0 \to \oplus A_i \xrightarrow{\oplus\alpha_i} \oplus B_i \xrightarrow{\oplus\beta_i} \oplus C_i \to 0 \quad \text{and} \quad 0 \to \prod A_i \xrightarrow{\prod\alpha_i} \prod B_i \xrightarrow{\prod\beta_i} \prod C_i \to 0.$$

(4) If G is a subdirect sum of B and C, then $B + G = B \oplus C = G + C$.

(5) Let B, C be subgroups of A such that $B \cap C = 0$. If $(B + C)/C$ is a summand of A/C, then B is a summand of A.

(6) (a) The subdirect sum of $\mathbb{Z}(p^m)$ and $\mathbb{Z}(p^n)$ $(0 < m \leq n)$ with kernels $\mathbb{Z}(p^{m-k})$ and $\mathbb{Z}(p^{n-k})$ is isomorphic to $\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^{m-k})$.
   (b) The subdirect sum of $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}(p^\infty)$ with kernels $\mathbb{Z}(p^m)$ and $\mathbb{Z}(p^n)$ $(0 < m \leq n)$ is isomorphic to $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^m)$.

(7) A group A is called **subdirectly irreducible** if in any representation of A as a subdirect product of groups A_i, one of the coordinate projections $\pi_i : A \to A_i$ is an isomorphism. Prove that A is subdirectly irreducible if and only if it is cocyclic.

2 Direct Summands

Direct Summands In this section, we collect a few criteria for a subgroup to be a summand. We start with the most frequently used criterion.

Lemma 2.1. *A subgroup B of A is a summand of A if and only if A has an idempotent endomorphism π satisfying $\pi A = B$; equivalently, the injection $B \to A$ followed by $\pi : A \to B$ is the identity $\mathbf{1}_B$ of B.*

Proof. If $A = B \oplus C$, then the projection π on the first summand, viewed as an element of $\operatorname{End} A$, is as desired. Conversely, if π is an idempotent endomorphism, then $A = \pi A \oplus (1 - \pi)A$. $\square$

Putting it in a different way, lemma states that B is a summand of A exactly if the identity map of B extends to an endomorphism $A \to B$.

If B is a summand of A, then the complementary summand is unique up to isomorphisms (recall: it is $\cong A/B$), but it is far from being unique as a subgroup. The following result explains how to obtain from one complement all the other complements.

Lemma 2.2. *Let $A = B \oplus C$ be a direct decomposition with projections β, γ. If also $A = B \oplus C_0$ with projections β_0, γ_0, then, for some endomorphism θ of A, we have*

$$\beta_0 = \beta + \beta\theta\gamma, \qquad \gamma_0 = \gamma - \beta\theta\gamma. \tag{2.2}$$

Conversely, if the maps β_0, γ_0 are of the form (2.2), then $A = B \oplus \gamma_0 A$.

Proof. If we are given the two direct decompositions, then let $\theta = \gamma - \gamma_0$. Then $B \le \operatorname{Ker}\theta$, so $\theta = \theta\beta + \theta\gamma = \theta\gamma$. If $a = b + c = b_0 + c_0$ with $b, b_0 \in B, c \in C, c_0 \in C_0$, then $\theta a = c - c_0 = b_0 - b \in B$, thus $\beta\theta = \theta$. Hence $\gamma_0 = \gamma - \theta = \gamma - \beta\theta\gamma$ and $\beta_0 = \mathbf{1}_A - \gamma_0 = \beta + \gamma - \gamma_0 = \beta + \beta\theta\gamma$.

Conversely, if β_0, γ_0 are obtained from β, γ as given in (2.2) with any $\theta \in \operatorname{End} A$, then $\beta_0 + \gamma_0 = \mathbf{1}_A, \beta_0^2 = \beta_0, \gamma_0^2 = \gamma_0$, and $\beta_0\gamma_0 = \gamma_0\beta_0 = 0$. Thus $A = \beta_0 A \oplus \gamma_0 A$ where $\beta_0 A = \beta A = B$. $\square$

If β is a central idempotent (commutes with all endomorphisms), then $\beta_0 = \beta + \theta\beta\gamma = \beta$ and $\gamma_0 = \gamma$. Thus the complements cannot be changed (they are fully invariant in A).

In general, a subgroup of a direct sum does not decompose along the summands. However, there is an important exceptional case.

Lemma 2.3. *If $A = B \oplus C$, and if G is a fully invariant subgroup of A, then*

$$G = (G \cap B) \oplus (G \cap C).$$

Proof. Let β, γ be the projections attached to the given direct sum. By full invariance, both βG and γG are subgroups of G. Evidently, βG and γG generate a direct sum in A, and since $\beta + \gamma = \mathbf{1}_A$, we have $G = \beta G \oplus \gamma G$. Since $\beta G \le G \cap B$ and $\gamma G \le G \cap C$, and proper inclusion is out of question, we have $\beta G = G \cap B$ and $\gamma G = G \cap C$. $\square$

The following is a useful lemma.

Lemma 2.4 (Kaplansky [K]). *If the factor group A/B is a direct sum: $A/B = \oplus_{i \in I} (A_i/B)$, and if B is a direct summand in every A_i, say, $A_i = B \oplus C_i$, then B is a summand of A. More precisely,*

$$A = B \oplus (\oplus_{i \in I} C_i).$$

Proof. It is clear that the groups B and the C_i generate A. Assume that $b + c_1 + \cdots + c_n = 0$ for some $b \in B$ and $c_j \in C_j$ $(j = 1, \ldots, n)$. Passing mod B, we obtain $(c_1 + B) + \cdots + (c_n + B) = B$, whence the given direct sum forces $c_j \in B$ for every j. Thus $c_j \in C_j \cap B = 0$, and hence also $b = 0$. Consequently, B and the C_i generate their direct sum in A. □

Summands of Large Direct Sums The following theorem has several applications in the study of properties inherited by summands.

Theorem 2.5 (Kaplansky [2]). *Summands of a direct sum of countable groups are also direct sums of countable groups.*

Proof. Let $A = \oplus_{i \in I} A_i = B \oplus C$ where each summand A_i is countable. Pick any summand A_1, a generating system $\{a_j\}_{j \in J}$ of A_1, and write $a_j = b_j + c_j$ $(b_j \in B, c_j \in C)$. Note that each b_j and each c_j has but a finite number of non-zero coordinates in the direct sum $A = \oplus_i A_i$. Collecting all the A_i that contain at least one non-zero coordinate of some b_j or c_j, and then forming their direct sum, we obtain a countable direct summand X_1 of A. Next, we repeat the same process with X_1 replacing A_1: select a generating system for X_1 and collect all the A_i which have non-zero coordinates of the B- and C-coordinates of the generators, to obtain a larger countable summand X_2 of A. Continuing the same way, we get a chain $X_1 \leq X_2 \leq \cdots \leq X_n \leq \ldots$ of countable summands of A whose union is a countable summand $\bar{A}_1$ such that $\bar{A}_1 = (B \cap \bar{A}_1) \oplus (C \cap \bar{A}_1)$.

A smooth chain of summands S_σ of A is defined as follows. Each S_σ is a direct sum of some A_i. Set $S_0 = 0$. If S_σ is defined for an ordinal σ and $S_\sigma < A$, then pick an A_i not in S_σ and let $S_{\sigma+1} = S_\sigma + \bar{A}_i$ (where $\bar{A}_i$ is obtained by repeating the above process for A_i using components not in S_σ). For limit ordinals σ we set $S_\sigma = \cup_{\rho < \sigma} S_\rho$. It is evident that for some ordinal $\tau \leq |A|$ we will reach $S_\tau = A$. It is also clear that $S_{\sigma+1}/S_\sigma$ is countable, and every S_σ is a direct sum of a subset of the A_i such that $S_\sigma = (B \cap S_\sigma) \oplus (C \cap S_\sigma)$ for all $\sigma \leq \tau$. Setting $B \cap S_{\sigma+1} = (B \cap S_\sigma) \oplus B_\sigma$, it is clear that the B_σ are countable and generate their direct sum in B. Since the B_σ together generate B, we have $B = \oplus_{\sigma < \tau} B_\sigma$, as claimed. □

Example 2.6. Let G be any countable group, and $A = \oplus_{\sigma < \omega_1} G_\sigma$ where $G_\sigma \cong G$ for each σ. If $A = B \oplus C$, then both B and C are direct sums of countable groups (not necessarily isomorphic to G).

★ **Notes.** Kaplansky's Theorem 2.5 holds for countably generated modules over arbitrary rings. It has been extended to κ-generated modules by C. Walker [2] for any infinite cardinal κ.

Exercises

(1) (Grätzer) Let B be a subgroup of A, and C a B-high subgroup in A. Then $A = B \oplus C$ if and only if $pa = b + c$ $(a \in A, b \in B, c \in C)$ for a prime p implies $b \in pB$.

(2) Suppose $C < B < A$. Prove that

  (a) if B is a summand of A, then B/C is a summand of A/C;
  (b) if C is a summand of A and B/C is a summand of A/C, then B is a summand of A.

(3) Let B be a summand of A, and let $\{\pi_i \mid i \in I\}$ be the set of all projections of A onto B. These projections form a semigroup such that $\pi_i \pi_j = \pi_j$.

(4) A group A has no summand isomorphic to itself if and only if one-sided units in its endomorphism ring $\mathrm{End}\, A$ are twosided.

(5) Let η denote an endomorphism of A.

  (a) If, for some n, $\mathrm{Im}\, \eta^{n+1} = \mathrm{Im}\, \eta^n$, then $\mathrm{Ker}\, \eta^n + \mathrm{Im}\, \eta^n = A$.
  (b) If, for some n, $\mathrm{Ker}\, \eta^{n+1} = \mathrm{Ker}\, \eta^n$, then $\mathrm{Ker}\, \eta^n \cap \mathrm{Im}\, \eta^n = 0$.

(6) Assume $A = B \oplus C = B' \oplus C'$, and let $\beta : A \to B, \beta' : A \to B'$ denote the projections in the given decompositions. Then $B \cong B'$ if and only if there are $\phi, \psi \in \mathrm{End}\, A$ such that $\phi\psi = \beta$ and $\psi\phi = \beta'$.

(7) (a) (Grätzer–Schmidt) Let B be a direct summand of A. The intersections of all the complements of B in A is the maximal fully invariant subgroup of A that is disjoint from B. [Hint: Lemma 2.2.]
  (b) A complement to a direct summand of A is unique if and only if it is fully invariant in A.

(8) Call a subgroup G of A **projection-invariant** if $\pi G \leq G$ for every projection π of A onto a summand. Prove that: (a) G is projection-invariant in A if and only if $\pi G = G \cap \pi A$ for all projections π; (b) intersections of projection-invariant subgroups are projection-invariant, and so are subgroups generated by projection-invariant subgroups; (c) Lemma 2.2 holds for projection-invariant G; (d) a projection-invariant summand is a fully invariant subgroup.

(9) (Kulikov) A direct decomposition $A = \oplus_{i \in I} A_i$ has a common refinement with every direct decomposition of A if and only if every A_i is projection-invariant.

(10) (Fuchs) $B < A$ is an **absolute direct summand** of A if $A = B \oplus C$ for every B-high subgroup C. (a) Prove that B is an absolute direct summand if and only if it is either injective (see Chapter 4) or A/B is a torsion group whose p-component is annihilated by p^k whenever $B \setminus pB$ contains an element of order p^k. (b) Find all absolute direct summands of a bounded group.

(11) (Irwin–Walker) Let $A = \oplus_{i \in I} A_i$ and $B_i \leq A_i$ for each i. If C_i is B_i-high in A_i, then $\oplus_i C_i$ is $\oplus_i B_i$-high in A.

(12) (Enochs) Let A be a p-group and $A = B \oplus C = B' \oplus C'$ direct decompositions of A such that $B[p] = B'[p]$. Then $A = B \oplus C' = B' \oplus C$. [Hint: use induction of the order of $a \in A$ to show $a \in B \oplus C'$.]

(13) (C. Walker) Generalize Theorem 2.5 to larger cardinalities κ: summands of direct sums of κ-generated groups are of the same kind.

(14) A **supplement subgroup** S to some $C < A$ is defined to be minimal with respect to the property $A = C + S$. S has this property if and only if $S \cap C$ is superfluous in A. [Hint: use the modular law in both directions.]

3 Pull-Back and Push-Out Diagrams

Pull-Backs With the aid of direct sums, we can describe two important methods in constructing certain commutative diagrams.

Theorem 3.1. *Given the homomorphisms $\alpha : A \to C$ and $\beta : B \to C$, there exists a group G, unique up to isomorphism, along with homomorphisms $\gamma : G \to A$, $\delta : G \to B$, such that the diagram*

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C \end{tikzcd}$$

is commutative, and if

$$\begin{tikzcd} G' \arrow[r, "\gamma'"] \arrow[d, "\delta'"] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C \end{tikzcd}$$

is any commutative diagram, then there exists a unique homomorphism $\phi : G' \to G$ such that $\gamma\phi = \gamma'$ and $\delta\phi = \delta'$.

Proof. Given α, β, define G as the subgroup of the direct sum $A \oplus B$ consisting of all pairs (a, b) $(a \in A,\ b \in B)$ such that $\alpha a = \beta b$, and let $\gamma : (a, b) \mapsto a$, $\delta : (a, b) \mapsto b$. This makes the first diagram commutative.

If the second diagram is commutative, then define $\phi : G' \to G$ as $\phi g' = (\gamma' g', \delta' g')$ for $g' \in G'$; here $(\gamma' g', \delta' g') \in G$, since $\alpha\gamma' = \beta\delta'$. Evidently, $\gamma\phi g' = \gamma' g'$ and $\delta\phi g' = \delta' g'$ for every $g' \in G'$. It is easy to see that $\operatorname{Ker}\gamma = (0, \operatorname{Ker}\beta)$ and $\operatorname{Ker}\delta = (\operatorname{Ker}\alpha, 0)$. Therefore, if $\phi' : G' \to G$ also satisfies $\gamma\phi' = \gamma'$, $\delta\phi' = \delta'$, then $\gamma(\phi - \phi') = 0 = \delta(\phi - \phi')$, and so $\operatorname{Im}(\phi - \phi') \leq \operatorname{Ker}\gamma \cap \operatorname{Ker}\delta = 0$. Hence $\phi - \phi' = 0$, thus ϕ is unique.

The uniqueness of G can be verified by considering a G_0 with the same properties. Then by what has already been shown, there are unique maps $\phi : G \to G_0$, $\phi_0 : G_0 \to G$ with the indicated properties. Then $\phi_0\phi : G \to G$ is a unique map (if applied to the case $G' = G$), so it must be the identity; the same holds for $\phi\phi' : G_0 \to G_0$, whence the uniqueness of G is manifest. □

Push-Outs The group G of the preceding theorem is called the **pull-back** of the maps α and β. Our next task is to prove the dual, where the group H will be called the **push-out** of α and β.

Theorem 3.2. *Assume that $\alpha : C \to A$, $\beta : C \to B$ are homomorphisms. There exist a group H, unique up to isomorphism, and homomorphisms $\gamma : A \to H$, $\delta : B \to H$, such that the diagram*

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\gamma"] \\ B \arrow[r, "\delta"] & H \end{tikzcd}$$

is commutative, and for every commutative diagram

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\gamma'"] \\ B \arrow[r, "\delta'"] & H' \end{tikzcd}$$

there is a unique homomorphism $\psi : H \to H'$ satisfying $\psi\gamma = \gamma'$ and $\psi\delta = \delta'$.

Proof. Starting with α, β, define H as the factor group of $A \oplus B$ modulo the subgroup $X = \{(\alpha c, -\beta c) \mid c \in C\}$. Let $\gamma : a \mapsto (a, 0) + X$, $\delta : b \mapsto (0, b) + X$ $(a \in A, b \in B)$ be the maps induced by the injections. Then $\gamma\alpha c = \delta\beta c$ for every $c \in C$ assures the commutativity of the first diagram.

If the second diagram is commutative, then we let $\psi : (a, b) + X \mapsto \gamma' a + \delta' b \in H'$. One can readily check that this definition is independent of the chosen representative (a, b) of the coset, and moreover, it satisfies $\psi\gamma = \gamma'$ and $\psi\delta = \delta'$. The uniqueness follows from the simple fact that $\operatorname{Im}\gamma$ and $\operatorname{Im}\delta$ generate H, and therefore, if $\psi'\gamma = \gamma', \psi'\delta = \delta'$ for some $\psi' : H \to H'$, then $(\psi - \psi')\gamma = 0 = (\psi - \psi')\delta$ implies that $\psi - \psi'$ maps the whole of H upon 0. An argument similar to the one at the end of the proof of the preceding theorem establishes the uniqueness of H. □

The following observations are of importance.

(a) *If in the pull-back diagram, α is monic, then so is δ; if α is epic, so is δ.* In view of the uniqueness of the pull-back diagram, it is enough to prove the claim for the group G as constructed in the proof above. That $\operatorname{Ker}\alpha = 0$ implies $\operatorname{Ker}\delta = 0$ is immediately seen from the proof. Furthermore, if α is epic, then to every $b \in B$ there is an $a \in A$ such that $\alpha a = \beta b$, and so δ is also epic.

(b) *If in the push-out diagram, α is monic, then so is δ; if α is epic, so is δ.* Again, we need only show this for H as defined above. Now clearly $\operatorname{Ker}\delta = 0$ whenever $\operatorname{Ker}\alpha = 0$. If α is epic, then to every $a \in A$ there is a $c \in C$ with $\alpha c = a$, and so δ maps $b + \beta c$ upon $(0, b + \beta c) + X = (a, b) + X$. Hence δ is epic as well.

Exercises

(1) If $B = 0$ in the pull-back diagram above, then $G \cong \operatorname{Ker}\alpha$.
(2) (a) If $C = 0$ in the pull-back diagram, then $G \cong A \oplus B$.
(b) If $C = 0$ in the push-out diagram, then $H \cong A \oplus B$.
(3) If both α, β are surjective in the pull-back diagram, then G is a subdirect sum of A and B with kernels $\operatorname{Ker}\alpha, \operatorname{Ker}\beta$.
(4) The pull-back diagram above is a push-out diagram (for γ, δ) exactly if the map $\nabla(\alpha \oplus \beta)\colon A \oplus B \to C$ is surjective.
(5) If in the diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[d] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \end{tikzcd}$$

each of the two squares is a pull-back, then the outer rectangle is also a pull-back.
(6) Formulate and prove the dual of the preceding exercise for push-outs.
(7) Using the notations of the above pull-back and push-out diagrams, the sequences $0 \to G \to A \oplus B \to C \to 0$ and $0 \to C \to A \oplus B \to H \to 0$ (with the obvious maps) are exact.

4 Direct Limits

Direct Systems Let $\{A_i\ (i \in I)\}$ be a system of groups where the index set I is partially ordered and **directed (upwards)** in the sense that to each pair $i, j \in I$, there is a $k \in I$ such that both $i \le k$ and $j \le k$. Suppose that for every pair $i, j \in I$ with $i \le j$, there is a homomorphism $\pi_i^j : A_i \to A_j$ (called **connecting map**) subject to the conditions:

(i) π_i^i is the identity map of A_i for all $i \in I$; and
(ii) if $i \le j \le k$ in I, then $\pi_j^k \pi_i^j = \pi_i^k$.

In this case, $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ is called a **direct system**. (If the index set is ω, then it suffices to specify only π_n^{n+1} for all $n < \omega$, because the other π_n^m are then determined by rule (ii).) By the **direct** or **injective limit**, or **colimit** of $\mathfrak{A}$ is meant a group A_* such that

(a) there are maps $\pi_i \colon A_i \to A_*$ such that $\pi_i = \pi_j \pi_i^j$ holds for all $i \leq j$;
(b) if G is any group, and $\rho_i \colon A_i \to G\ (i \in I)$ are maps satisfying $\rho_i = \rho_j \pi_i^j$ for all $i \leq j$, then there is a *unique* map $\alpha \colon A_* \to G$ such that $\rho_i = \alpha \pi_i$ for all $i \in I$.

We write: $A_* = \varinjlim_{i \in I} A_i$, and call the maps $\pi_i \colon A_i \to A_*$ **canonical**.

Theorem 4.1. *A direct system $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ of groups has a limit, unique up to isomorphism.*

Proof. We form the direct sum $A = \oplus_i A_i$, and consider the subgroup $B \leq A$ generated by the elements $a_i - \pi_i^j a_i$ for all $a_i \in A_i$ and for all $i \leq j$ in I. Our claim is that $A/B = A_*$ is the direct limit of $\mathfrak{A}$.

The elements of A/B are cosets of the form $a_{i_1} + \cdots + a_{i_n} + B$ with $a_{i_j} \in A_{i_j}$. If $i \in I$ is such that $i_1, \ldots, i_n$ are all $\leq i$, then this coset is $\pi_{i_1}^i a_{i_1} + \cdots + \pi_{i_n}^i a_{i_n} + B$, since $a_{i_1} + \cdots + a_{i_n} - \pi_{i_1}^i a_{i_1} - \cdots - \pi_{i_n}^i a_{i_n} = (a_{i_1} - \pi_{i_1}^i a_{i_1}) + \cdots + (a_{i_n} - \pi_{i_n}^i a_{i_n}) \in B$. Thus every element in A/B can be written as $a_i + B$ for some $a_i \in A_i$. In particular, B consists of all finite sums of the form $b = a_{i_1} + \cdots + a_{i_n}$ with $a_{i_j} \in A_{i_n}$ for which there is an $i \in I$ such that $i_1, \ldots, i_n \leq i$ and $\pi_{i_1}^i a_{i_1} + \cdots + \pi_{i_n}^i a_{i_n} = 0$.

Consider the maps $\pi_i \colon A_i \to A/B$ acting as $a_i \mapsto a_i + B$. They obviously satisfy $\pi_i = \pi_j \pi_i^j\ (i \leq j)$. If G is any group as stated in (b), then define $\alpha : A/B \to G$ by $\alpha(a_i + B) = \rho_i a_i$. Owing to $\rho_i = \rho_j \pi_i^j$, this definition is independent of the choice of the coset representative, and since α is evidently additive, α is a genuine homomorphism. It satisfies $\rho_i = \alpha \pi_i$ for all $i \in I$, as required. If $\alpha' \colon A/B \to G$ also satisfies $\rho_i = \alpha' \pi_i$ for all $i \in I$, then $(\alpha - \alpha')\pi_i = 0$ for each $i \in I$, thus $\alpha - \alpha'$ sends every $a_i + B = \pi_i a_i$ to 0, i.e. $\alpha = \alpha'$. It follows that A/B is a limit of the given direct system, so we can write $A_* = A/B$.

To show that A_* is unique up to isomorphism, suppose that also A_0 shares properties (a)–(b). Then there exist unique maps $\alpha \colon A_* \to A_0$ and $\alpha_0 \colon A_0 \to A_*$ as required by (b). Also, $\alpha_0 \alpha \colon A \to A$ and $\alpha \alpha_0 \colon A_0 \to A_0$ are unique, so they are the identity maps. Consequently, $A_0 \cong A_*$. □

We now list some of the most useful properties of direct limits.

(A) *A_* is the set union of the subgroups $\pi_i A_i\ (i \in I)$.*
(B) *If $\pi_i a_i = 0$ for some $a_i \in A_i$, then there is a $j \geq i$ such that $\pi_i^j a_i = 0$.* Indeed, if $\pi_i a_i = 0$, then $a_i \in B$, and the proof above establishes this claim.
(C) *If every π_i^j is a monic map, then all the π_i are monomorphisms.* This follows from (B).
(D) *If J is a cofinal subset of I, then the system restricted to J has the same direct limit: $\varinjlim_J A_i \cong \varinjlim_I A_i$.* In fact, if the first group is A'/B', then $a_j + B' \mapsto a_j + B$ is an isomorphism of A'/B' with A/B.

Example 4.2. Let $\{A_i \ (i \in I)\}$ be the collection of all subgroups of a group A where the index set I is partially ordered by the rule: $i \leq j$ if and only if $A_i \leq A_j$. Let $\pi_i^j : A_i \to A_j$ denote the injection map for $i \leq j$. Then $\mathfrak{A} = \{A_i \ (i \in I); \pi_i^j\}$ is a direct system with limit A.

Example 4.3. If we admit only finitely generated subgroups of A in the direct system $\mathfrak{A} = \{A_i \ (i \in I); \pi_i^j\}$ with the injection maps π_i^j, the direct limit is still A. In the special case where A is arbitrary torsion-free, we get A as the direct limit of finitely generated free groups.

Example 4.4. Let $A = \oplus_{j \in J} C_j$ be a direct sum. Let i range over the set I of finite subsets of J, so that $i \leq k$ in I means that i is a subset of k. If we define $A_i = \oplus_{j \in i} C_j$ for all $i \in I$, and $\pi_i^k : A_i \to A_k$ to be the obvious inclusion map, then we get a direct system whose limit is A.

Maps Between Direct Systems We consider homomorphisms between direct limits that are induced by homomorphisms between direct systems. If $\mathfrak{A} = \{A_i \ (i \in I); \ \pi_i^j\}$ and $\mathfrak{B} = \{B_i \ (i \in I); \ \rho_i^j\}$ are direct systems with the same index set I, then by a **homomorphism** $\Phi : \mathfrak{A} \to \mathfrak{B}$ we mean a set of homomorphisms $\Phi = \{\phi_i : A_i \to B_i \mid i \in I\}$ such that the diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{\pi_i^j} & A_j \\ \phi_i \downarrow & & \downarrow \phi_j \\ B_i & \xrightarrow{\rho_i^j} & B_j \end{array}$$

commute for all $i \leq j$ in I.

Lemma 4.5. *If Φ is a homomorphism between the direct systems $\mathfrak{A}$ and $\mathfrak{B}$, then there exists a unique morphism $\Phi_* : A_* = \varinjlim A_i \to B_* = \varinjlim B_i$ making all the diagrams*

$$\begin{array}{ccc} A_i & \xrightarrow{\pi_i} & A_* \\ \phi_i \downarrow & & \downarrow \Phi_* \\ B_i & \xrightarrow{\rho_i} & B_* \end{array}$$

commute (π_i, ρ_i *denote the canonical morphisms*). Φ_* *is an epimorphism* (*monomorphism*) *if all the* ϕ_i *are epimorphisms* (*monomorphisms*).

Proof. Since the maps $\rho_i \phi_i : A_i \to B_*$ satisfy the condition $\rho_j \phi_j \pi_i^j = \rho_j \rho_i^j \phi_i = \rho_i \phi_i$ for every pair $i \leq j$, the existence of a unique homomorphism $\Phi_* : A_* \to B_*$ such that $\rho_i \phi_i = \Phi_* \pi_i$ for each $i \in I$ is guaranteed. This proves the first assertion.

If all the ϕ_i are epic, then the subgroups $\rho_i B_i = \rho_i \phi_i A_i$ cover B_*, so Φ_* must be epic. If all the ϕ_i are monic, then pick an $a \in \operatorname{Ker} \Phi_*$. There is $j \in I$ such that $a = \pi_j a_j$ for some $a_j \in A_j$. Hence $\rho_j \phi_j a_j = \Phi_* \pi_j a_j = \Phi_* a = 0$, and so by (B) we have a $k \geq j$ with $\rho_j^k \phi_j a_j = 0$. But $\rho_j^k \phi_j = \phi_k \pi_j^k$ and ϕ_k is monic, so $\pi_j^k a_j = 0$, whence $\pi_j a_j = 0$ and $a = 0$. $\square$

We now move to three direct systems: $\mathfrak{A}$, $\mathfrak{B}$ as above, and a third one, $\mathfrak{C} = \{C_i\ (i \in I);\ \sigma_i^j\}$, all with the same directed index set I. If $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ and $\Psi\colon \mathfrak{B} \to \mathfrak{C}$ are homomorphisms between them such that the sequence $0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$ is exact for each $i \in I$, then we say that the sequence

$$0 \to \mathfrak{A} \xrightarrow{\Phi} \mathfrak{B} \xrightarrow{\Psi} \mathfrak{C} \to 0 \tag{2.3}$$

is exact. It is an important fact that direct limits of exact sequences is exact. More precisely,

Theorem 4.6. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be direct systems over the same index set I, and $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ and $\Psi : \mathfrak{B} \to \mathfrak{C}$ homomorphisms between them. If the sequence (2.3) is exact, then the sequence*

$$0 \to A_* = \varinjlim_i A_i \xrightarrow{\Phi_*} B_* = \varinjlim_i B_i \xrightarrow{\Psi_*} C_* = \varinjlim_i C_i \to 0$$

of direct limits is likewise exact.

Proof. Exactness at A_* and C_* is guaranteed by Lemma 4.5, so we prove exactness only at B_*. By Lemma 4.5, the diagram

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\pi_i"'] & B_i \arrow[r, "\psi_i"] \arrow[d, "\rho_i"'] & C_i \arrow[r] \arrow[d, "\sigma_i"] & 0 \\
0 \arrow[r] & A_* \arrow[r, "\Phi_*"] & B_* \arrow[r, "\Psi_*"] & C_* \arrow[r] & 0
\end{tikzcd}$$

is commutative for all $i \in I$. If $a \in A_*$, then $\pi_i a_i = a$ for some $a_i \in A_i$, so $\Psi_*\Phi_* a = \Psi_*\Phi_*\pi_i a_i = \Psi_*\rho_i\phi_i a_i = \sigma_i\psi_i\phi_i a_i = 0$. Next let $b \in \operatorname{Ker}\Psi_*$. For some $b_i \in B_i$, we have $\rho_i b_i = b$, whence $\sigma_i\psi_i b_i = \Psi_*\rho_i b_i = \Psi_* b = 0$. There exists $j \in I$ with $\sigma_i^j\psi_i b_i = 0$, thus $\psi_j b_j = \psi_j\rho_i^j b_i = 0$. Since the top row in the diagram is exact, there is an $a_j \in A_j$ with $\phi_j a_j = b_j$. Setting $a = \pi_j a_j$, we arrive at $\Phi_* a = \Phi_*\pi_j a_j = \rho_j\phi_j a_j = \rho_j\rho_i^j b_i = \rho_i b_i = b$, i.e. $b \in \operatorname{Im}\Phi_*$, and the bottom row is exact at B_*. $\square$

Exercises

(1) Show that $\varinjlim_n \mathbb{Z}(p^n) = \mathbb{Z}(p^\infty)$, using inclusion maps.
(2) Let $A_n \cong \mathbb{Z}$ $(n < \omega)$ with $\pi_n^{n+1} : A_n \to A_{n+1}$ multiplication by n. Prove that $\varinjlim_n A_n \cong \mathbb{Q}$.
(3) If every π_i^j is an onto map, then all the π_i are epimorphisms.
(4) A group is locally cyclic if and only if it is a direct limit of cyclic groups.

(5) (a) Let A_* be the limit of the direct system $\mathfrak{A} = \{A_i\ (i \in I); \pi_i^j\}$, and $a \in A_*$. There exist a $j \in I$ and an $a_j \in A_j$ such that $\pi_j a_j = a$ and $o(a_j) = o(a)$.
(b) Direct limit of torsion (torsion-free) groups is again torsion (torsion-free).
(6) If G is finitely generated, and $\alpha : G \to A_*$ (notations as above), then there exist a $j \in I$ and an $\alpha_j : G \to A_j$ such that $\alpha = \pi_j \alpha_j$.
(7) If $\mathfrak{A} = \{A_i\ (i \in I); \pi_i^j\}$ and $\mathfrak{B} = \{B_i\ (i \in I); \rho_i^j\}$ are direct systems of groups, then $\mathfrak{A} \oplus \mathfrak{B} = \{A_i \oplus B_i\ (i \in I); \pi_i^j \oplus \rho_i^j\}$ is likewise a direct system whose direct limit is the direct sum of the direct limits of $\mathfrak{A}$ and $\mathfrak{B}$.
(8) What is wrong with the following argument? Because of Theorem 4.6, the sequence $0 \to \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty) \to 0$ must be exact, since it can be obtained as the direct limit of the exact sequences $0 \to \mathbb{Z}(p^m) \to \mathbb{Z}(p^{2m}) \to \mathbb{Z}(p^m) \to 0$ $(m \in \mathbb{N})$.

5 Inverse Limits

Inverse Systems Inverse limits are dual to direct limits: we just reverse the arrows.

Assume $\{A_i \mid i \in I\}$ is a collection of groups, indexed by a poset I, and for each pair $i, j \in I$ of indices with $i \leq j$ there is given a **connecting** homomorphism $\pi_i^j : A_j \to A_i$ such that

(i) π_i^i is the identity map of A_i for all $i \in I$; and
(ii) if $i \leq j \leq k$ in I, then $\pi_i^j \pi_j^k = \pi_i^k$.

In this case, $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ is called an **inverse system**. By the **inverse** or **projective limit**, or simply **limit**, of this inverse system is meant a group A^* such that

(a) there are maps $\pi_i : A^* \to A_i$ such that $\pi_i = \pi_i^j \pi_j$ for all $i \leq j$; and
(b) if G is any group with maps $\rho_i : G \to A_i$ $(i \in I)$ subject to $\rho_i = \pi_i^j \rho_j$ for $i \leq j$, then there is a *unique* map $\phi : G \to A^*$ satisfying $\rho_i = \pi_i \phi$ for all $i \in I$.

We write: $A^* = \varprojlim_{i \in I} A_i$, and call the maps $\pi_i : A^* \to A_i$ **canonical**.

Theorem 5.1. *An inverse system $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ of groups has a limit, unique up to isomorphism.*

Proof. Consider the subgroup A^* in the direct product $A = \prod_{i \in I} A_i$ that consists of all vectors $a = (\dots, a_i, \dots)$ whose coordinates satisfy $\pi_i^j a_j = a_i$ for all $i \leq j$. This is in fact a subgroup as is seen immediately. The projection maps $\pi_i : a \mapsto a_i$ satisfy $\pi_i = \pi_i^j \pi_j$, so (a) holds for A^*.

To verify (b), let G be a group as stated in (b), and for $g \in G$ define $\phi : g \mapsto (\dots, \rho_i g, \dots) \in \prod_i A_i$. Owing to the condition $\rho_i = \pi_i^j \rho_j$, we have $\phi g \in A^*$. Clearly, $\phi : G \to A^*$ satisfies $\rho_i = \pi_i \phi$ for all $i \in I$. If $\rho_i = \pi_i \phi'$ holds also for $\phi' : G \to A^*$, then $\pi_i(\phi - \phi') = 0$ for all i, so every coordinate projection of $(\phi - \phi')G$ is 0, hence $\phi = \phi'$.

In order to establish the uniqueness of A^*, we can mimic the argument at the end of the last paragraph almost word-by-word. □

It is worthwhile noting the following properties of inverse limits.

(A) *If I is directed, and if in the inverse system $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ all connecting maps π_i^j are monomorphisms, then so are all the π_i.* In fact, assume $a \in A^*$ is such that $\pi_i a = 0$. Given $j \in I$, there is a $k \in I$ with $i, j \leq k$. Then $\pi_i^k \pi_k a = \pi_i a = 0$, whence π_i^k monic implies $\pi_k a = 0$. Therefore, $\pi_j a = \pi_j^k \pi_k a = 0$ for all $j \in I$, and so $a = 0$. (Exercise 5 will show that, in general, the same fails for epimorphisms.)

(B) *If I is directed, and if J is a cofinal directed subset in I, then we have $\varprojlim_{i \in I} A_i = \varprojlim_{j \in J} A_j$.*

(C) *A^* is the intersection of kernels of certain endomorphisms of $\prod_i A_i$.* For, every pair $i \leq j$ in I defines an endomorphism

$$\theta_{ij} : (\dots, a_i, \dots, a_j, \dots) \mapsto (\dots, a_i - \pi_i^j a_j, \dots, a_j, \dots).$$

Comparing this with the proof of Theorem 5.1, it becomes evident that $A^* = \cap_{i \leq j} \operatorname{Ker} \theta_{ij}$.

(D) *If all the groups in the inverse system $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ are Hausdorff topological groups and the connecting maps π_i^j are continuous homomorphisms, then the inverse limit A^* is a closed subgroup of $\prod_i A_i$ (which is equipped with the product topology), and the canonical maps $\pi_i : A^* \to A_i$ are continuous.* Indeed, then the endomorphisms θ_{ij} in (C) are continuous, so their kernels as well as the intersection of the kernels are closed subgroups. A^* carries the topology inherited from $\prod_i A_i$, so the continuity of the π_i is obvious.

Example 5.2. Let $A = \prod_{\alpha \in J} B_\alpha$ be the direct product of the groups B_α. Let I denote the set of all finite subsets of J, partially ordered by inclusion. For $i \in I$, set $A_i = \oplus_{\alpha \in i} B_\alpha$, and for $i \leq j$ in I let π_i^j be the projection map $A_j \to A_i$. This gives rise to an inverse system $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$. We now claim: $A^* = \varprojlim_{i \in I} A_i \cong A$. To prove this, let $\pi_i : A^* \to A_i$ be the ith canonical map, and $\rho_i : A \to A_i$ the ith projection map. By definition, there is a unique map $\phi : A \to A^*$ such that $\pi_i \phi = \rho_i$. If $\phi a = 0$ for some $a \in A$, then $\rho_i a = \pi_i \phi a = 0$ for all $i \in I$, so ϕ is monic. If $a^* = (\dots, a_i, \dots, a_j, \dots) \in A^*$, then write $a_i = b_{\alpha_1} + \dots + b_{\alpha_k}$ with $b_{\alpha_\ell} \in B_{\alpha_\ell}$ if $i = \{\alpha_1, \dots, \alpha_k\}$. If $i \leq j$, then by the choice of π_i^j, the B_α-coordinates of a_i are identical with the corresponding coordinates of a_j, so a^* defines a unique $(\dots, b_\alpha, \dots) \in A$. A glance at the definition of ϕ in the proof of Theorem 5.1 shows that $\phi(\dots, b_\alpha, \dots) = a^*$, so ϕ is epic as well.

Example 5.3. Let $C_n = \langle c_n \rangle$ be cyclic groups of order p^n $(n \in \mathbb{N})$, and define maps $\pi_n^{n+1} : C_{n+1} \to C_n$ induced by $c_{n+1} \mapsto c_n$. Now $\mathfrak{C} = \{C_n\ (n \in \mathbb{N});\ \pi_n^m\}$ is an inverse system, and our claim is that $C^* = \varprojlim_{n \in \mathbb{N}} C_n \cong J_p$. If $\pi_n : C^* \to C_n$ is the canonical map, and if we define $\rho_n : J_p \to C_n$ via $\rho_n(1) = c_n$, then by definition there is a unique map $\phi : J_p \to C^*$ such that $\pi_n \rho = \rho_n$ for all $n \in \mathbb{N}$. Since only $0 \in J_p$ can belong to all $\operatorname{Ker} \rho_n$, $\operatorname{Ker} \phi = 0$ is clear. Now let $c = (b_1, \dots, b_n, \dots) \in C^*$ with $b_n = k_n c_n$ $(k_n \in \mathbb{Z})$; by the choice of π_n^{n+1} we have $k_{n+1} \equiv k_n \bmod p^n$, so there is a p-adic integer σ such that $\sigma \equiv k_n \bmod p^n$ for all n. We conclude that $\rho_n \sigma = b_n$, and ϕ must be epic.

Example 5.4 (The Intersection of Subgroups is an Inverse Limit). Let $\{A_i \mid i \in I\}$ denote a set of subgroups of a group A closed under finite intersections. We partially order I by reverse inclusion. The groups A_i, along with the injection maps $\pi_i^j : A_j \to A_i$ $(i \leq j)$, form an inverse system. Its inverse limit will be $\cap_{i\in I} A_i$, because only the constant vectors in $\prod_{i\in I} A_i$ can belong to the inverse limit A^*.

Maps Between Inverse Systems Assume $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ and $\mathfrak{B} = \{B_i\ (i \in I);\ \rho_i^j\}$ are inverse systems, indexed by the same poset I. A **homomorphism** $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a set $\{\phi_i : A_i \to B_i\ (i \in I)\}$ of homomorphisms subject to the requirement that all diagrams of the form

$$\begin{array}{ccc} A_j & \xrightarrow{\pi_i^j} & A_i \\ \phi_j \downarrow & & \downarrow \phi_i \\ B_j & \xrightarrow{\rho_i^j} & B_i \end{array}$$

be commutative for all $i \leq j$.

Lemma 5.5. *If $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism between inverse systems, then there exists a unique map $\Phi^* : A^* = \varprojlim_{i\in I} A_i \to B^* = \varprojlim_{i\in I} B_i$ such that, for every $i \in I$, the diagram*

$$\begin{array}{ccc} A^* & \xrightarrow{\pi_i} & A_i \\ \Phi^* \downarrow & & \downarrow \phi_i \\ B^* & \xrightarrow{\rho_i} & B_i \end{array}$$

commutes (with canonical maps π_i, ρ_i). Φ^* is monic, if so are the ϕ_i.

Proof. The homomorphisms ϕ_i $(i \in I)$ induce a homomorphism $\bar{\phi} = \prod_i \phi_i : \prod_i A_i \to \prod_i B_i$. The commutativity of the diagram before the lemma shows that if $a = (\dots, a_i, \dots) \in A^*$, then $\bar{\phi}a \in B^*$, hence we can define $\Phi^* : A^* \to B^*$ as the restriction of $\bar{\phi}$. With this Φ^* we have $\phi_i\pi_i a = \phi_i a_i = \rho_i \Phi^* a$, establishing the commutativity of the diagram. If also $\Phi_0 : A^* \to B^*$ makes the diagram commute for every i, then $\rho_i(\Phi^* - \Phi_0) = 0$ for every i, thus $\Phi^* = \Phi_0$.

Finally, if all the ϕ_i are monic, and if $\Phi^* a = 0$ for some $a \in A^*$, then $\phi_i\pi_i a = \rho_i\Phi^* a = 0$ implies $\pi_i a = 0$ for every i, whence $a = 0$. ☐

For the inverse limits of exact sequences, we have a somewhat weaker result than for direct limits.

Theorem 5.6. *Assume $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}, \mathfrak{B} = \{B_i\ (i \in I);\ \rho_i^j\}$, and $\mathfrak{C} = \{C_i\ (i \in I);\ \sigma_i^j\}$ are inverse systems over the same index set I. Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ and $\Psi : \mathfrak{B} \to \mathfrak{C}$ be homomorphisms. If the sequence $0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$ is exact* *for every $i \in I$, then the sequence*

$$0 \to A^* = \varprojlim_i A_i \xrightarrow{\Phi^*} B^* = \varprojlim_i B_i \xrightarrow{\Psi^*} C^* = \varprojlim_i C_i \tag{2.4}$$

of inverse limits is likewise exact.

Proof. Exactness at A^* follows from Lemma 5.5. From the definition of Φ^*, Ψ^* it is evident that $\Psi^*\Phi^* = 0$. If π_i, ρ_i, σ_i denote the canonical maps, then by Lemma 5.5 the diagram

$$\begin{tikzcd}
0 \arrow[r] & A^* \arrow[r, "\Phi^*"] \arrow[d, "\pi_i"'] & B^* \arrow[r, "\Psi^*"] \arrow[d, "\rho_i"'] & C^* \arrow[d, "\sigma_i"] & \\
0 \arrow[r] & A_i \arrow[r, "\phi_i"] & B_i \arrow[r, "\psi_i"] & C_i \arrow[r] & 0
\end{tikzcd}$$

is commutative for each $i \in I$. In order to show the exactness of the top row at B^*, let $b \in \operatorname{Ker}\Psi^*$. In view of $\psi_i\rho_i b = \sigma_i\Psi^* b = 0$ and the exactness of the bottom row, for every $i \in I$ there is an $a_i \in A_i$ satisfying $\phi_i a_i = \rho_i b$. For $j > i$, $\phi_i\pi_i^j a_j = \rho_i^j\phi_j a_j = \rho_i^j\rho_j b = \rho_i b = \phi_i a_i$, whence $\pi_i^j a_j = a_i$ as ϕ_i is monic. We infer that $a = (\dots, a_i, \dots, a_j, \dots) \in A^*$. For this a we have $\rho_i\Phi^* a = \phi_i\pi_i a = \phi_i a_i = \rho_i b$ for every i, so $\Phi^* a = b$. □

Exercise 5 will show that, in general, Theorem 5.6 cannot be improved by putting $\to 0$ at the end of the exact sequence (2.4). A noteworthy special case when the exact sequence of inverse limits is exact is as follows.

Proposition 5.7. *If in Theorem 5.6 we specialize: $\mathfrak{A} = \{A_n\ (n < \omega);\ \pi_n^{n+1}\}$, $\mathfrak{B} = \{B_n\ (n < \omega);\ \rho_n^{n+1}\}$, $\mathfrak{C} = \{C_n\ (n < \omega);\ \sigma_n^{n+1}\}$, and assume that all the maps π_n^{n+1} are epic, then the sequence of inverse limits is exact:*

$$0 \to A^* = \varprojlim_n A_n \xrightarrow{\Phi^*} B^* = \varprojlim_n B_n \xrightarrow{\Psi^*} C^* = \varprojlim_n C_n \to 0.$$

Proof. Let $c^* = (c_0, \dots, c_n, \dots)$ represent an element of C^*. We now construct by induction an element $b^* = (b_0, \dots, b_n, \dots) \in B^*$ such that $\Psi^* b^* = c^*$. As ψ_0 is surjective, there is $b_0 \in B_0$ with $\psi_0 b_0 = c_0$. Suppose that, for some $n < \omega$, we have found $b_i \in B_i$ for all $i \le n$ such that $\psi_i b_i = c_i$ and $\rho_{i-1}^i b_i = b_{i-1}$. Choose any $b'_{n+1} \in B_{n+1}$ mapped upon c_{n+1} by ψ_{n+1}. Then $b_n - \rho_n^{n+1} b'_{n+1} = \phi_n a_n$ for some $a_n \in A_n$. If $a_{n+1} \in A_{n+1}$ is such that $\pi_n^{n+1} a_{n+1} = a_n$ (which exists by hypothesis), then $b'_{n+1} + \phi_{n+1} a_{n+1} \in B_{n+1}$ is our choice for the next coordinate b_{n+1} in b^*. It is clear that then $b^* \in B^*$ is as desired. □

Derived Functor of $\mathcal{I}nv$ The inverse systems of abelian groups (with a fixed index set I) and the morphisms between them form a category $\mathcal{I}nv(I)$. The functor

$\mathcal{I}nv(I) \mapsto \mathcal{A}b$ assigning to an inverse system its inverse limit is left-exact. Since its right-exactness fails in general, the inverse limit functor has a derived functor, denoted $\lim^1$. This is especially interesting in case the index set is ω when the inverse system looks like

$$C_0 \xleftarrow{\gamma_1} C_1 \xleftarrow{\gamma_2} \dots \xleftarrow{\gamma_n} C_n \xleftarrow{\gamma_{n+1}} C_{n+1} \xleftarrow{\gamma_{n+2}} \dots .$$

Then $\varprojlim_n^1 C_n \cong \operatorname{Coker} \psi$ where

$$\psi : (\dots, c_n, \dots) \mapsto (\dots, c_n - \gamma_{n+1} c_{n+1}, \dots).$$

denotes the **Eilenberg map** $\psi : \prod_{n<\omega} C_n \to \prod_{n<\omega} C_n$; see Jensen [Je], as well as Schochet [1]. The functor $\varprojlim^1$ will be discussed later in Proposition 6.9 in Chapter 9. We just point out here that, for an exact sequence of inverse systems in Proposition 5.7, there is an exact sequence

$$0 \to \varprojlim_n A_n \to \varprojlim_n B_n \to \varprojlim_n C_n \to \varprojlim_n{}^1 A_n \to \varprojlim_n{}^1 B_n \to \varprojlim_n{}^1 C_n \to 0.$$

Example 5.8. Consider the inverse system $\{\mathbb{Z}; n!\}$: $\mathbb{Z} \xleftarrow{1!} \mathbb{Z} \xleftarrow{2!} \dots \xleftarrow{n!} \mathbb{Z} \xleftarrow{(n+1)!} \mathbb{Z} \longleftarrow \dots$ The Eilenberg map ψ carries the vector $(k_0, \dots, k_n, \dots)$ ($k_n \in \mathbb{Z}$) to the vector $(k_0 - k_1, k_1 - 2!k_2, \dots, k_n - (n+1)!k_{n+1}, \dots)$. This vector is divisible modulo $\oplus\,\mathbb{Z}$ (which group is in $\operatorname{Ker} \psi$), so $\operatorname{Coker} \psi$ is divisible. An easy argument shows that the cokernel is torsion-free and its cardinality is the continuum, hence $\varprojlim_n^1 \{\mathbb{Z}; n!\} \cong \mathbb{Q}^{\aleph_0}$.

Example 5.9. We now consider three inverse systems: $\{\mathbb{Z}, p\}$: $\mathbb{Z} \xleftarrow{\dot{p}} \mathbb{Z} \xleftarrow{\dot{p}} \mathbb{Z} \xleftarrow{\dot{p}} \dots$, $\{\mathbb{Z}, 1\}$: $\mathbb{Z} \xleftarrow{\mathbf{1}} \mathbb{Z} \xleftarrow{\mathbf{1}} \mathbb{Z} \xleftarrow{\mathbf{1}} \dots$, and $\{\mathbb{Z}/p^n\mathbb{Z}, \pi\}$: $0 \longleftarrow \mathbb{Z}/p\mathbb{Z} \xleftarrow{\pi} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{\pi} \dots$ (with canonical maps π). They fit into the exact sequence

$$0 \to \{\mathbb{Z}, p\} \to \{\mathbb{Z}, 1\} \to \{\mathbb{Z}/p^n\mathbb{Z}, \pi\} \to 0$$

of inverse systems. The $\varprojlim$-$\varprojlim^1$ exact sequence (see above) yields the exact sequence $0 \to \mathbb{Z} \to \varprojlim \mathbb{Z}/p^n\mathbb{Z} \to \varprojlim^1 \{\mathbb{Z}, p\} \to 0$, whence

$$\varprojlim{}^1 \{\mathbb{Z}, p\} \cong J_p/\mathbb{Z} \cong \mathbb{Q}^{\aleph_0}.$$

★ **Notes.** The so-called Mittag-Leffler condition (not stated) is a most useful sufficient criterion to guarantee that $\to 0$ can be put at the end of (2.4). See Jensen [Je].

Exercises

(1) If $\mathfrak{A} = \{A_i\ (i \in I);\ \pi_i^j\}$ and $\mathfrak{B} = \{B_i\ (i \in I);\ \rho_i^j\}$ are inverse systems, then $\mathfrak{A} \oplus \mathfrak{B} = \{A_i \oplus B_i\ (i \in I);\ \pi_i^j \oplus \rho_i^j\}$ is again an inverse system. Its limit is the direct sum of the limits of $\mathfrak{A}$ and $\mathfrak{B}$.

(2) Let $C_n = \langle c_n \rangle$ be cyclic of order n, and for $n|m$ let $\pi_n^m : C_m \to C_n$ be the homomorphism induced by $c_m \mapsto c_n$. Then $\mathfrak{C} = \{C_n\ (n \in \mathbb{N});\ \pi_n^m\}$ is an inverse system where $\mathbb{N}$ is partially ordered by the divisibility relation. Show that $\varprojlim_n \mathfrak{C} \cong \prod_p J_p$.

(3) Let $A_n \cong \mathbb{Z}(p^\infty)$, and let $\pi_n^{n+1} : \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$ be the multiplication by p. Then the inverse limit of the inverse system $\mathfrak{A} = \{A_n\ (n \in \mathbb{N});\ \pi_n^{n+1}\}$ is isomorphic to the group of all p-adic numbers.

(4) The inverse limit of torsion-free groups is torsion-free, but the inverse limit of torsion groups need not be torsion.

(5) Let $B_n = \langle b_n \rangle \cong \mathbb{Z}$ and $\pi_n^m : b_m \mapsto b_n$ for all $n \leq m$ in $\mathbb{N}$. Let $C_n = \langle c_n \rangle \cong \mathbb{Z}(p^n)$ and $\rho_n^m : c_m \mapsto c_n$ for $n \leq m$. Show that

   (a) $\mathfrak{B} = \{B_n\ (n \in \mathbb{N});\ \pi_n^m\}$ and $\mathfrak{C} = \{C_n\ (n \in \mathbb{N});\ \rho_n^m\}$ are inverse systems, and the epimorphisms $\phi_n : b_n \to c_n (n \in \mathbb{N})$ define a map $\Phi : \mathfrak{B} \to \mathfrak{C}$.

   (b) The induced homomorphism $\Phi^* : B^* \to C^*$ between the inverse limits is not epic. [Hint: $\mathbb{Z} \to J_p$.]

(6) The inverse limit of splitting exact sequences need not be exact.

(7) Let $\mathfrak{A} = \{A_n\ (n < \omega);\ \pi_n^{n+1}\}$ be an inverse system where the maps π_n^{n+1} are epimorphisms, but not isomorphisms. Then the inverse limit A^* has cardinality at least the continuum.

6 Direct Products vs. Direct Sums

One aspect of direct products that deserves special attention is related to their homomorphisms. There is a remarkable contrast between homomorphisms from a direct sum and from a direct product: those from direct sums are completely determined by their restrictions to the components, but not much can be said about homomorphisms from a direct product, except when either the components or the target groups satisfy restrictive conditions. A most fascinating result is concerned with homomorphisms of direct products into direct sums—this is the case that we wish to explore here. What is a surprising, if not recondite, phenomenon about it is that it works only up to the first measurable cardinal.

Before entering into the discussion, a simple remark might be helpful on infinite sums in direct products $A = \prod_{i \in I} A_i$. Infinite sums $\sum_{j \in J} x_j$ do make sense when the terms are vectors $x_j = (\ldots, a_{ji}, \ldots)$ $(a_{ji} \in A_i)$ such that, for each $i \in I$, only a finite number of ith coordinates $a_{ji} \neq 0$. (Actually, $\sum_{j \in J} x_j$ is then a convergent sum in the product topology.)

Example 6.1. Let $A = \prod_{n<\omega} A_n$ be a countable product. Then $x = \sum_{n<\omega} x_n$ is a well-defined element of A if $x_n = (0, \ldots, 0, a_{nn}, a_{n,n+1}, \ldots)$ $(a_{ni} \in A_i)$ (n zeros).

Maps from Direct Product into Direct Sum We start with a special case which has independent interest. ('Reduced' means no divisible subgroup $\neq 0$, and $C^1 = \cap_{n \in \mathbb{N}} nC$ denotes the first Ulm subgroup of C.)

Theorem 6.2 (Chase [1], Ivanov [5]). *Let $A = \prod_{i<\omega} A_i$ denote a countable product of groups, and $\phi : A \to C = \oplus_{j\in J} C_j$ a homomorphism into the direct sum of reduced groups C_j. Then there exist integers $m > 0$, k, as well as a finite subset $J_0 \subseteq J$ such that*

$$\phi(mB_k) \leq (\oplus_{j\in J_0} C_j) + (\oplus_{j\in J} C_j^1),$$

where $B_k = \prod_{k\leq i<\omega} A_i$ (summand of A).

Proof. Let $\phi_j : A \to C_j$ denote the map ϕ followed by the jth coordinate projection. Assume the claim is false. Then we can find inductively an increasing sequence $1 = m_0 < m_1 < \cdots < m_k < \dots$ of integers, a sequence of elements $b_k \in m_k B_k$, and indices $j_k \in J$ such that

$$m_k | m_{k+1}, \quad \phi_{j_k}(b_\ell) = 0 \text{ for } \ell < k \quad \text{and} \quad \phi_{j_k}(b_k) \notin m_{k+1} C_{j_k}$$

for all $k < \omega$. Indeed, if, for some $k < \omega$, we have b_ℓ, m_ℓ and j_ℓ for all $\ell \leq k$ at hand as required, then j_{k+1} will be selected as an index not in $\cup_{\ell\leq k}(\operatorname{supp} \phi(b_\ell))$ such that $\phi_{j_{k+1}}(m_{k+1}B_{k+1}) \not\leq \cap_{n\in\mathbb{N}} nC_{j_{k+1}}$ for some proper multiple m_{k+1} of m_k; this can be done, since otherwise the claim would be true. Only a finite number of b_k have non-zero coordinates in any A_i, therefore, the infinite sum $a = \sum_{k<\omega} b_k$ is a well-defined element in A. Consider

$$\phi_{j_k} a = \phi_{j_k}(\sum_{\ell<\omega} b_\ell) = \phi_{j_k}(\sum_{\ell<k} b_\ell) + \phi_{j_k}(b_k) + \phi_{j_k}(\sum_{k<\ell<\omega} b_\ell),$$

and observe that in the last sum the first term is 0, and the third term is contained in $m_{k+1}C_{j_k}$, but the second term is not. Since ϕa has a finite support in C, this equation can hold only for a finite number of indices k—an obvious contradiction. $\square$

We state the following theorem for p-groups that involves transfinite heights; its proof runs parallel to the preceding one.

Theorem 6.3 (Zimmermann-Huisgen [1]). *Let $A = \prod_{i<\omega} A_i$ be a countable product of p-groups, and $\phi : A \to C = \oplus_{j\in J} C_j$ a homomorphism into the direct sum of reduced p-groups C_j. Given a limit ordinal τ, there exist an integer $k < \omega$, an ordinal $\sigma < \tau$, and a finite subset $J_0 \subseteq J$ such that*

$$\phi(p^\sigma \prod_{k\leq i<\omega} A_i) \leq (\oplus_{j\in J_0} C_j) + (\oplus_{j\in J} p^\tau C_j). \qquad \square$$

The Measurable Cardinal Phenomenon If we wish to extend the preceding results to uncountable direct products, then we are confronted with an unusual phenomenon. There is a natural boundary to the extension: the first measurable cardinal. The reader who wishes to avoid the following delicate set-theoretical arguments can safely assume that there are no such cardinals in our model of ZFC, and jump to Theorem 6.5.

Recall that a cardinal κ is **measurable** if a set X of cardinality κ admits a countably additive measure μ such that μ assumes only two values: 0 and 1, and satisfies $\mu(X) = 1$, while $\mu(\{x\}) = 0$ for all $x \in X$. Here 'countably additive' means that if X_i $(i < \omega)$ are pairwise disjoint subsets of X, then $\mu(\cup_{i<\omega} X_i) = \sum_{i<\omega} \mu(X_i)$.

Let f be a function $\mathbf{B} \to G$ where $\mathbf{B} = \mathbf{2}^X$ is the Boolean lattice of all subsets of a set X, and $G \neq 0$ is a group. We will say f is a **G-valued measure on** X if it satisfies the following conditions:

(i) $f(\{x\}) = 0$ for every singleton $\{x\} \in \mathbf{B}$;
(ii) if $V \subset U$ are subsets of X, then $f(U) = 0$ implies $f(V) = 0$;
(iii) if U, V are disjoint subsets of X, then $f(U \cup V) = f(U) + f(V)$;
(iv) if U_i $(i < \omega)$ are pairwise disjoint subsets of X, then there is $n \in \mathbb{N}$ such that $f(\cup_{i<\omega} U_i) = f(U_0) + \cdots + f(U_n)$ and $f(U_i) = 0$ for all $i > n$.

We call f *non-trivial* if $f(X) \neq 0$. The following striking argument is due to J. Łoś.

Lemma 6.4. *If a non-trivial group-valued measure exists on the subsets of the set X, then $|X|$ is a measurable cardinal.*

Proof. Assume $f : \mathbf{2}^X \to G$ is a non-trivial G-valued measure on X, $G \neq 0$ any group. We show that then there exists a non-trivial countably additive $\{0, 1\}$-valued measure on X.

Consider all subsets $U \subset X$ such that $f(U) = 0$. From (i)–(iv) we conclude that these U form a countably additive ideal $\mathbf{I}$ in the Boolean lattice $\mathbf{B}$ of all subsets of X. It is readily checked that f induces a countably additive G-valued measure $\bar{f}$ on the Boolean quotient $\mathbf{B}/\mathbf{I}$. Let $\bar{U}_0, \dots, \bar{U}_i, \dots$ be pairwise disjoint elements in $\mathbf{B}/\mathbf{I}$. We can find representatives $U_i \subseteq X$ of the $\bar{U}_i$ which are still pairwise disjoint. By condition (iv), $f(U_i) \neq 0$ can hold only for a finite set of indices i; in other words, $\mathbf{B}/\mathbf{I}$ is a finite Boolean lattice. Thus $\mathbf{B}/\mathbf{I}$ has but a finite number of atoms, and on them $\bar{f}$ is not 0. Hence we derive a $\{0, 1\}$-valued measure μ' on $\mathbf{B}/\mathbf{I}$ by selecting an atom in $\mathbf{B}/\mathbf{I}$ and define $\mu'(\bar{U})$ to be 1 or 0 according as $\bar{U}$ does or does not contain the selected atom. In the obvious manner, μ' gives rise to a $\{0, 1\}$-valued measure μ on $\mathbf{B}$, showing that the set X is measurable. □

It is remarkable that Theorems 6.2 and 6.3 generalize to larger products provided that the cardinality of the set of components is not measurable.

Theorem 6.5 (Dugas–Zimmermann-Huisgen [1]). *Let $A = \prod_{i \in I} A_i$ be a direct product, and $\phi : A \to C = \oplus_{j \in J} C_j$ a homomorphism where the C_j are reduced groups. If $|I|$ is not a measurable cardinal, then there are an integer $m \neq 0$ and finite subsets $I_0 \subseteq I, J_0 \subseteq J$ such that*

$$\phi(m \prod_{i \in I \setminus I_0} A_i) \leq (\oplus_{j \in J_0} C_j) + (\oplus_{j \in J} C_j^1).$$

Proof. Consider the set $\mathcal{S}$ of all subsets S of I such that for the product $A_S = \prod_{i \in S} A_i$ the statement of the theorem holds (i.e., if I is replaced by S). Evidently, if $S \in \mathcal{S}$, then all subsets of S also belong to $\mathcal{S}$. Furthermore, $\mathcal{S}$ is not only closed under finite unions (which is evident), but also under countable unions. In fact, if $S_k \in \mathcal{S}$ $(k < \omega)$ are pairwise disjoint subsets, then, for some $n < \omega$ we have $\cup_{n<k<\omega} S_k \in \mathcal{S}$—this follows by applying Theorem 6.2 to the countable product $\phi : \prod_{k<\omega}(\prod_{i \in S_k} A_i) \to C$.

Once this has been established, in order to complete the proof it suffices to repeat the arguments in Łoś' theorem to conclude that if the claim fails, then I must be measurable. □

Example 6.6. To show that the last theorem may indeed fail for a measurable index set I, let each A_i denote a copy of the Σ-cyclic p-group $B = \oplus_{k<\omega} \mathbb{Z}(p^k)$, and let $C = B$. To define $\phi : A \to B$, pick an $a = (\dots, a_i, \dots) \in A = \prod_{i \in I} A_i$. a has only countably many different coordinates (as elements of B), so the supports of the equal ones give rise to a countable partition of I into disjoint subsets, exactly one of which has measure 1, and the rest have measure 0. If $b \in B$ is the element for which the support is of measure 1, then we set $\phi(a) = b$. It is easy to see that this gives rise to a well-defined homomorphism. It violates the conclusion of Theorem 6.5: $m \operatorname{Im} \phi$ is not contained in any finite direct sum of cyclic groups in C, for any integer $m > 0$.

The proof of Theorem 6.5 also applies to verify:

Theorem 6.7 (Zimmermann-Huisgen [1]). *Let $A = \prod_{i \in I} A_i$ be a product of p-groups, and $\phi : A \to C = \oplus_{j \in J} C_j$ a homomorphism where the C_j are reduced p-groups. Given a limit ordinal ρ, if $|I| = \kappa$ is not measurable, then there exist an ordinal $\sigma < \rho$, as well as finite subsets $I_0 \subseteq I, J_0 \subseteq J$, such that*

$$\phi(p^\sigma \prod_{i \in I \setminus I_0} A_i) \leq (\oplus_{j \in J_0} C_j) + (\oplus_{j \in J} p^\rho C_j). \qquad \square$$

★ **Notes.** The peculiar behavior of homomorphisms from a countable direct product into an infinite direct sum was noticed by Chase [1]. The same phenomenon of larger direct products was observed by Dugas–Zimmermann-Huisgen [1] up to the first measurable cardinal (just as in the case of slender groups). By using $\aleph_1$-complete ultrafilters, Eda [1] gave a generalization to all cardinals; see Lemma 2.13 in Chapter 13.

Ivanov [1] proves various theorems on so-called **Fuchs-44 groups** with respect to a class $\mathfrak{A}$, which is closed under extensions, submodules, and direct products. G is such a group if for every $\phi : G \to \oplus_{i \in I} A_i$ with $A_i \in \mathfrak{A}$, there are $m \in \mathbb{N}$ and a finite subset $J \subset I$ such that $\phi(mG) \leq \oplus_{i \in J} A_i$.

Exercises

(1) The group $A = \prod_{k \in \mathbb{N}} \mathbb{Z}(p^k)$ has no unbounded Σ-cyclic p-group as an epimorphic image.

(2) Let $A = \mathbb{Z}^\kappa$ where κ is not measurable, and F a free group. Show that the image of every homomorphism $A \to F$ is finitely generated.
(3) (Keef) Let A_i $(i \in I)$ be an infinite set of unbounded separable p-groups. There is no epimorphism $\prod_{i \in I} A_i \to \oplus_{i \in I} A_i$.

7 Completeness in Linear Topologies

Groups that are complete in some linear topology are very special. Therefore, we examine completeness and the completion processes.

Linear Topologies Assume that a linear topology is defined on the group A in terms of a filter $\mathbf{u}$ in the lattice $\mathbf{L}(A)$ of subgroups of A. The subgroups $U \in \mathbf{u}$ form a base of open neighborhoods about 0; we label them by a directed index set I, so that $i \leq j$ for $i, j \in I$ means that $U_i \geq U_j$. Thus I as a (directed) poset is dual-isomorphic to a subset of $\mathbf{u}$ (which has the natural order relation by inclusion).

By a **net** in A we mean a set $\{a_i\}_{i \in I}$ of elements in A, indexed always by I. A net is said to **converge** to a **limit** $a \in A$ if to every $i \in I$ there is a $j \in I$ such that

$$a_k - a \in U_i \qquad \text{for all } k \geq j.$$

If A is Hausdorff in the topology, then limits are unique; if, however, A fails to be Hausdorff, then limits are determined only up to mod $\cap_i U_i$. The classical proof applies to show that a subgroup B of A is closed in the topology if and only if it contains the limits of convergent nets whose elements belong to B.

A net $\{a_i\}_{i \in I}$ is a **Cauchy net** if to any given $i \in I$, there is a $j \in I$ such that

$$a_k - a_\ell \in U_i \qquad \text{whenever } k, \ell \geq j.$$

Since the U_i are subgroups, $a_k - a_j, a_\ell - a_j \in U_i$ implies $a_k - a_\ell \in U_i$, for the Cauchy character of a net it suffices to require that $a_k - a_j \in U_i$ for all $k \geq j$. Clearly, cofinal subnets of a Cauchy net are again Cauchy nets, and such a subnet converges if and only if the larger net also converges; moreover, the limits are then the same. To facilitate discussion and to simplify notation, we shall concentrate without loss of generality on Cauchy nets $\{b_i\}_{i \in I}$ which are **neat** in the sense that, for every $i \in I$, $b_k - b_i \in U_i$ holds for $k \geq i$ (i.e., $j = i$ can be chosen). If a neat Cauchy net $\{b_i\}_{i \in I}$ converges to a limit $b \in A$, then it converges **neatly**: $b_k - b \in U_i$ for all $k \geq i$. In a group whose topology satisfies the first axiom of countability, Cauchy sequences $\{a_n \mid n < \omega\}$ satisfying $a_{n+1} - a_n \in U_n$ for all $n \in \mathbb{N}$ are neat.

Topological Completeness A group A is said to be **complete** in a topology if it is Hausdorff, and every (neat) Cauchy net in A has a limit in A. Observe that we mean by complete groups only Hausdorff groups.

Lemma 7.1. *A subgroup of a complete group is closed if and only if it is complete in the induced topology.*

Proof. Let G be a subgroup in the complete group C. First assume G is closed in C, and $\{g_i\}_{i\in I}$ is a Cauchy net in G (in the inherited topology). The net is Cauchy in C too, so it converges to a limit $c \in C$ which must be in G, since G is closed. Thus G is complete. Conversely, suppose G is complete in the induced topology, and $c \in C$ is the limit of a Cauchy net $\{g_i\}_{i\in I}$ with $g_i \in G$. It is a Cauchy net in G as well, so has a limit in G, which cannot be anything else than c. Thus G is closed in C. $\square$

In the next result the countability hypothesis is essential.

Lemma 7.2. *Let B be a closed subgroup of a complete group A that satisfies the first axiom of countability. Then the factor group A/B is complete in the induced topology.*

Proof. Since B is closed, A/B is Hausdorff. Consider a base of neighborhoods about 0 in A such that $U_1 \geq \cdots \geq U_m \geq \ldots$ with $\cap_{m\in\mathbb{N}} U_m = 0$. Let $\{a_m + B \mid m \in \mathbb{N}\}$ be a Cauchy sequence in A/B; without loss of generality, we assume that it is neat, i.e. $a_{m+1} - a_m + B \subseteq U_m + B$. We want to lift this Cauchy sequence to a Cauchy sequence $\{c_m \mid m \in \mathbb{N}\}$ in A. Let $c_1 = a_1$, and assume that $c_1, \ldots, c_m \in A$ have already been chosen such that $c_i \in a_i + B$ and $c_i - c_{i-1} \in U_{i-1}$ for $i = 2, \ldots, m$. Then $a_{m+1} - c_m = u_m + b_m$ for some $u_m \in U_m,\ b_m \in B$, and set $c_{m+1} = a_{m+1} - b_m \in a_{m+1} + B$ to have $c_{m+1} - c_m \in U_m$. If $\lim\{c_m\} = a \in A$, then $a + B$ is the limit of the sequence $\{a_m + B \mid m \in \mathbb{N}\}$ in A/B. $\square$

Recall that if $\{A_j \mid j \in J\}$ is a family of groups, each equipped with a linear topology, say, defined by the filter $\mathbf{u}_j$ in $\mathbf{L}(A_j)$, then the direct product $A^* = \prod_{j\in J} A_j$ is given the **product (Tychonoff) topology**: a subbase of neighborhoods of 0 consists of the subgroups $\pi_j^{-1} U_{ji}$ where $\pi_j \colon A^* \to A_j$ is the jth coordinate projection, and $U_{ji} \in \mathbf{u}_j$. The product topology is again linear, and the π_j are continuous, open homomorphisms. The direct sum $A = \oplus_{j\in J} A_j$ is a dense subgroup of A^*.

We should also mention the box topology on the direct product; this topology is used, e.g., when the components are viewed in the $\mathbb{Z}$-adic topology, and we want to have this topology on their direct product. We now assume that the same poset I serves to index a base of neighborhoods about 0 in each A_j. If $\{U_{ji} \leq A_j \mid i \in I\}$ is a base in the topology of A_j (where $U_{ji} \leq U_{jk}$ whenever $k \leq i$ in I), then the **box topology** on $A^* = \prod_{j\in J} A_j$ is defined to have the subgroups

$$U_i = \prod_{j\in J} U_{ji} \qquad (i \in I)$$

as a base of neighborhoods about 0. The box topology on A^* satisfies the first countability hypothesis if all the A_j do. The inclusion $U_i \leq \pi_j^{-1} U_{ji}$ for all j shows that the box topology is finer than the product topology. Hence the projections π_j are continuous in the box topology as well.

Example 7.3. Actually, there are several methods of furnishing a direct product with a linear topology. E.g., let $G = \prod_{j \in I} A_j$ be a product, and $\mathcal{F}$ a filter on the index set I. For each $X \in \mathcal{F}$ we form the subgroup

$$V_X = \{g = (\ldots, a_j, \ldots) \in \prod_{j \in I} A_j \mid n(g) \in X\}$$

(where $n(g) = \{j \in I \mid a_j = 0\}$ denotes the null-set of g), and declare the subgroups V_X $(X \in \mathcal{F})$ as a base of neighborhoods about 0. This linear topology is Hausdorff if and only if $\mathcal{F}$ is a **free filter**, i.e. $\cap_{X \in \mathcal{F}} X = \varnothing$.

Example 7.4. Choose the filter of subsets of I with finite complements. Then the topology defined in Example 7.3 is the product topology. If A_i $(i \in I)$ are non-trivial groups in the discrete topology, then $\prod_i A_i$ is complete in the product topology. The direct sum $\oplus_i A_i$ is dense in the direct product.

Completions The rest of this section is devoted to the completion of groups in linear topologies. There are two important completion processes: one is *via* Cauchy nets, and another is by using inverse limits. We will employ the second method which fits better to linear topologies.

Let A be a group with linear topology (not necessarily Hausdorff), and $\{U_i \mid i \in I\}$ a base of neighborhoods of 0, with I a directed index set: $i \leq j$ in I if and only if $U_i \geq U_j$. Define the groups $C_i = A/U_i$, and for $j \geq i$ in I, the homomorphisms $\pi_i^j : C_j \to C_i$ via $\pi_i^j : a + U_j \mapsto a + U_i$. The limit of the arising inverse system $\mathfrak{C} = \{C_i \ (i \in I); \pi_i^j\}$ will be denoted by $\breve{A}$: it is furnished with the topology inherited from the product topology of $\prod C_i$. Thus, if π_i denotes the ith projection $\prod C_i \to C_i$, then a subbase of neighborhoods of 0 in $\breve{A}$ is given by the subgroups $\breve{U}_i = \breve{A} \cap \pi_i^{-1} 0$. Evidently, $\theta_A : a \mapsto (\ldots, a + U_i, \ldots) \in \breve{A}$ is a homomorphism $A \to \breve{A}$ which is continuous and open, and $\theta_A U_i = \theta_A A \cap \breve{U}_i$ holds for each $i \in I$. It is clear that $\operatorname{Ker} \theta_A$ is the intersection of all U_i.

Lemma 7.5. *For every group A with a linear topology, the group $\breve{A}$ is complete in the induced topology, and the image of the map $\theta : A \to \breve{A}$ is a dense subgroup of $\breve{A}$.*

Proof. Let $\breve{a} = (\ldots, a_i + U_i, \ldots) \in \breve{A}$, and let $\breve{U}_i \subset \breve{A}$ be an open set. As $\theta_A a_i$ lies in the $\breve{U}_i$-neighborhood of $\breve{a}$, $\theta_A A$ is dense in $\breve{A}$. Therefore, to prove completeness, we need only verify the convergence of Cauchy nets in $\theta_A A$ to elements of $\breve{A}$. A neat Cauchy net in $\theta_A A$ is the image of a neat Cauchy net $\{b_i\}_{i \in I}$ in A. We claim that $\breve{b} = (\ldots, b_i + U_i, \ldots)$ is the limit of $\{\theta_A b_i\}_{i \in I}$. First, $\breve{b} \in \breve{A}$, since $\pi_i^j (b_j + U_j) = b_j + U_i = b_i + U_i$ for $j \geq i$. Secondly, the ith coordinate of $\theta_A b_i - \breve{b}$ is 0, so it belongs to the open set $\breve{U}_i$. □

Observe that the completion is always Hausdorff, and $\theta_A : A \to \breve{A}$ is monic if and only if A had a Hausdorff topology to start with.

Lemma 7.6. *If ϕ is a continuous homomorphism of the group A into a complete group C, then there is a unique continuous homomorphism $\breve{\phi} : \breve{A} \to C$ such that $\breve{\phi} \theta_A = \phi$.*

Proof. Let $\{a_i \mid i \in I\}$ be a Cauchy net in A converging to the element $\breve{a} \in \breve{A}$. Continuity implies that $\{\phi a_i \mid i \in I\}$ is a Cauchy net in C. If $c \in C$ is its limit, then the only possible way of defining a continuous $\breve{\phi}$ is to let $\breve{\phi} : \breve{a} \mapsto c$. The rest of the claim is straightforward. □

From this lemma it also follows that the completion $\breve{A}$ of A is unique up to topological isomorphism. Also, $\theta_A : A \to \breve{A}$ is a natural map, for if $\phi : A \to C$ is a continuous homomorphism, then the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\theta_A"'] & C \arrow[d, "\theta_C"] \\ \breve{A} \arrow[r, "\breve{\phi}"] & \breve{C} \end{tikzcd}$$

commutes where $\breve{\phi}$ is the map whose existence was established in Lemma 7.6.

Our main interest lies in the $\mathbb{Z}$-adic topology, and in completions in that topology. Therefore, if we say that '*a group is complete*,' then we always mean completeness in the $\mathbb{Z}$-adic or p-adic topology (whichever is obvious), unless stated otherwise. Furthermore, we shall use the special notation $\tilde{A}$ for the completion of A in the $\mathbb{Z}$-adic topology.

In the next theorem we refer to linear compactness; see Sect. 3 in Chapter 6.

Theorem 7.7. *Let A be any group.*

(i) *Its completion in the $\mathbb{Z}$-adic (p-adic) topology carries the $\mathbb{Z}$-adic (p-adic) topology.*
(ii) *Its completion in the finite index topology has a compact topology.*
(iii) *Its completion in the Prüfer topology carries a linearly compact topology.*

Proof. (i) Let $\tilde{A} = \varprojlim_{n\in\mathbb{N}} A/nA$, or, equivalently, $\tilde{A} = \varprojlim_{n\in\mathbb{N}} A/n!A$ whenever we consider the collection of subgroups $U_n = n!A$ ($n \in \mathbb{N}$) as a decreasing sequence of neighborhoods about 0. The elements in the induced $\tilde{U}_n$ have nth coordinates 0, and it is easy to see that the conditions on the coordinates of elements on $\tilde{A}$ imply that all the ith coordinates in $\tilde{U}_n$ are 0 for $i < n$, while all those for $i > n$ are divisible by $n!$. This means that $\tilde{U}_n = n!\tilde{A}$.

(ii) In the finite index topology, the groups A/U_i are finite, so they are compact. Thus the product $\prod_i A/U_i$ is compact, and the inverse limit $\breve{A} = \varprojlim_{i\in I} A/U_i$ as a closed subgroup is also compact.

(iii) The proof is similar to the one in (ii), using the linear compactness of A/U_i in the Prüfer case.

Example 7.8. Let $A = \oplus_{n<\omega} A_n$ be furnished with the topology where the subgroups $U_k = \oplus_{k\le n<\omega} A_n$ form a base of neighborhoods about 0. The completion of A in this topology is the direct product $\prod_{n<\omega} A_n$.

If the topology fails to satisfy the first axiom of countability, then completeness may occur in an unexpected situation. This is demonstrated by the following example where, for a limit ordinal λ, the p^λ-topology of a p-group A is defined by declaring the subgroups $p^\sigma A$ $(\sigma < \lambda)$ as a base of neighborhoods of 0.

Example 7.9. Suppose λ is a limit ordinal not cofinal with ω, and let A_σ $(\sigma < \lambda)$ be p-groups such that A_σ has length σ. Then the A_σ are discrete (and hence complete) in the p^λ-topology. Consequently, $A^* = \prod_{\sigma<\lambda} A_\sigma$ is complete in the p^λ-topology which is now the box topology on A^* (cf. Exercise 2).

Strangely enough, $A = \oplus_{\sigma<\lambda} A_\sigma$ is complete in the p^λ-topology. To prove this, we show that A is closed in A^*. Assume the contrary, i.e. there is $x = (\dots, a_\sigma, \dots) \in A^* \setminus A$ in the closure of A. We can find a sequence $\sigma_1 < \cdots < \sigma_n < \dots$ of ordinals with $a_{\sigma_n} \neq 0$. Let $\sup \sigma_n = \sigma' < \lambda$ and $y \in A$ such that $x - y \in p^{\sigma'} \prod_{\sigma<\lambda} A_\sigma$. Then x and y have equal coordinates in every A_ρ with $\rho < \sigma'$ which contradicts the fact that x has infinitely many and y only finitely many non-zero coordinates for $\rho < \sigma'$.

$\mathbb{Z}$-adic Completeness Direct products of complete groups are complete in the product topology. We wish to point out the following result on the $\mathbb{Z}$-adic topology.

Lemma 7.10. *A direct product is complete in the $\mathbb{Z}$-adic topology if and only if every component is complete in its $\mathbb{Z}$-adic topology.*

Proof. Summands inherit $\mathbb{Z}$-adic topology and completeness. Conversely, assume every A_j in $G = \prod_{j\in J} A_j$ is $\mathbb{Z}$-adically complete and $\{g_i \mid i \in I\}$ is a neat Cauchy net in G. Then $\{\pi_j g_i \mid i \in I\}$ is a neat Cauchy net in A_j, and if $a_j \in A_j$ is the limit of this net, then $g \in G$ with $\pi_j g = a_j$ is the limit of $\{g_i \mid i \in I\}$. □

★ **Notes.** While the completion in the Prüfer topology may be viewed as a 'linear compactification,' completion in the finite index topology is not at all compactification. The latter process kills the first Ulm subgroup of the group, so it is an embedding only for groups that are Hausdorff in the finite index topology. A genuine 'compactification' can be accomplished by the so-called Stone compactification. This is the process of embedding A in the group $\mathrm{Hom}(\mathrm{Hom}(A, \mathbb{T}), \mathbb{T})$, where $\mathbb{T}$ denotes the circle group $\mathbb{R}/\mathbb{Z}$ (the inner Hom is furnished with the discrete topology, and the outer with the compact-open topology).

Exercises

(1) (a) The completions of the groups A and $A/\cap U_i$ are the same.
    (b) A and A/A^1 have the same $\mathbb{Z}$-adic completion.
(2) A direct product is complete in the box topology if and only if every component is complete.
(3) Every compact (linearly compact) group is complete in its topology.
(4) The direct product of discrete groups is Hausdorff and complete in every **u**-topology where **u** is a free filter.
(5) The inverse limit of complete groups is complete. (Careful with the topology.)
(6) Compare the completions of a group in the finite index and in the Prüfer topologies.

Problems to Chapter 2

PROBLEM 2.1 (J. Dauns). Suppose A has the property that every summand B of A has a decomposition $B = B_1 \oplus B_2$ with $B_1 \cong B_2$. Is then $A \cong A \oplus A$?

PROBLEM 2.2. Study the Boolean powers $A^{(\mathbf{B})}$ of a group A.

Cf. Balcerzyk [3], and especially, Eda [2].

PROBLEM 2.3. Represent a p-group A as a direct limit $A \cong \varinjlim_n A[p^n]$. How does the structure of A change if the connecting monomorphisms $A[p^n] \to A[p^{n+1}]$ are modified?

PROBLEM 2.4. Suppose $\phi : A = \prod A_i \to C = \prod^{<\aleph_1} C_j$ is a homomorphism of a product into an $\aleph_1$-product. Can we say something about where the image must be contained (like Theorem 6.5)?

Chapter 3
Direct Sums of Cyclic Groups

Abstract The study of important classes of abelian groups begins in this chapter. Not counting the finite and finitely generated groups, the class of direct sums of cyclic groups is perhaps the best understood class.

We give a fairly detailed account of free abelian groups, and discuss the presentation of groups *via* generators and defining relations. Several sections are devoted to direct sums of cyclic groups (called Σ-cyclic groups); these groups share most useful properties, and can easily be characterized by cardinal invariants. We present a few criteria for such groups, and establish several remarkable results, e.g. Kulikov's theorem that passage to subgroups preserves Σ-cyclicity. We draw attention to the method of smooth chains, which became the most important tool in the theory, and provides basic machinery for several results to come.

We shall cover some of the aspects of almost free groups, but shall not pursue their theory farther, due to the sophisticated set-theoretical arguments required.

In this chapter, in a number of proofs we have to use purity, so readers should be familiar with the fundamental results on pure subgroups (in Chapter 5) before studying the second part of this chapter.

1 Freeness and Projectivity

Free Abelian Groups By a **free (abelian) group** is meant a direct sum of infinite cyclic groups. If these cyclic groups are generated by the elements x_i $(i \in I)$, then the free group will be

$$F = \oplus_{i \in I} \langle x_i \rangle.$$

The set $\{x_i\}_{i \in I}$ is a **basis** of F. The elements of F are linear combinations

$$g = n_1 x_{i_1} + \cdots + n_k x_{i_k} \qquad (k \geq 0) \tag{3.1}$$

with different x_i and non-zero integers n_i. In view of the definition of direct sums, two such linear combinations represent the same element of F exactly if they differ at most in the order of the terms. Addition is performed in the obvious way by adding the coefficients of the same x_i.

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-319-19422-6_3

We can define F formally by starting with a set $X = \{x_i\}_{i\in I}$ of symbols, called **a free set of generators**, and declaring F as the set of all formal expressions (3.1) under the mentioned equality and addition. We say that F is **the free group on the set** X.

Example 1.1. An immediate example for a free group is the multiplicative group of positive rational numbers. The prime numbers form a free set of generators.

Needless to say, F is, up to isomorphism, uniquely determined by the cardinal number $\kappa = |I|$ of the index set I. Thus we are justified to write F_κ for the free group with κ free generators. κ is also called the **rank** of the free group F, in symbols, $\operatorname{rk} F = \kappa$ (for the discussion of rank, see Sect. 4).

Theorem 1.2. *The free groups F_κ and F_λ are isomorphic exactly if the cardinals κ and λ are equal.*

Proof. We need only verify the 'only if' part of the assertion. Observe that if F is a free group with free generators x_i $(i \in I)$, then an element (3.1) of F belongs to pF if and only if $p|n_1,\dots,p|n_k$. Hence, if p is a prime, then F/pF is a vector space over the prime field $\mathbb{Z}/p\mathbb{Z}$ of characteristic p with basis $\{x_i + pF\}_{i\in I}$, and so its cardinality is $p^{|I|}$ or $|I|$ according as I is finite or infinite. Thus $|F/pF|$ completely determines $|I|$. $\square$

The Universal Property Free groups enjoy a universal property formulated in the next theorem which is frequently used for the definition of free groups.

Theorem 1.3 (Universal Property of Free Groups). *Let X be a free set of generators of the free group F. Any function $f : X \to A$ of X into any group A extends uniquely to a homomorphism $\phi : F \to A$. This property characterizes free sets of generators, and hence free groups.*

Proof. Write $X = \{x_i\}_{i\in I}$, and $f(x_i) = a_i \in A$. There is only one way f can be extended to a homomorphism $\phi : F \to A$, namely, by letting

$$\phi g = \phi(n_1x_{i_1} + \cdots + n_kx_{i_k}) = n_1a_{i_1} + \cdots + n_ka_{i_k}.$$

(The main point is that the uniqueness of (3.1) guarantees that ϕ is well defined.) It is immediate that ϕ preserves addition.

To verify the second part, assume that a subset X of a group F has the stated property. Let G be a free group with a free set $Y = \{y_i\}_{i\in I}$ of generators, where the index set is the same as for X. By hypothesis, the correspondence $f : x_i \mapsto y_i$ extends to a homomorphism $\phi : F \to G$; this cannot be anything else than the map $n_1x_{i_1} + \cdots + n_kx_{i_k} \mapsto n_1y_{i_1} + \cdots + n_ky_{i_k}$. ϕ is injective, because the linear combination of the y_i is 0 only in the trivial case. ϕ is obviously surjective, and so it is an isomorphism. $\square$

Mapping X onto a generating system of a given group, we arrive at the following result which indicates that *the group $\mathbb{Z}$ is a generator of the category $\mathcal{A}b$* ('generator' in the sense used in category theory).

Corollary 1.4. *Every group with at most κ generators is an epimorphic image of a free group with κ generators.* $\square$

Consequently, every group A can be embedded in a short exact sequence

$$0 \to H \to F \xrightarrow{\phi} A \to 0,$$

where F is free group, and $H = \operatorname{Ker}\phi$. (We will see shortly that H is likewise free.) This is called a **free resolution** of A. It is far from being unique, because both F and ϕ can be chosen in many ways.

If κ is an infinite cardinal, then F_κ has 2^κ subsets, and hence at most 2^κ subgroups and factor groups. We conclude that *there exist at most 2^κ pairwise non-isomorphic groups of cardinality $\leq \kappa$*. (We will learn in Corollary 3.8 in Chapter 11, that 2^κ is the precise number.)

The next two theorems are fundamental, they are quoted most frequently.

Theorem 1.5. *Suppose that B is a subgroup of a group A such that A/B is a free group. Then B is a summand of A, i.e., $A = B \oplus C$ for a subgroup $C \cong A/B$.*

Proof. That only free factor groups can share the stated property will follow from Theorem 1.7. In order to show that free groups do have this property, by Lemma 2.4 in Chapter 2, it suffices to verify the claim for $A/B \cong \mathbb{Z}$ only, say $A/B = \langle a + B\rangle$ with $a \in A$. The elements of A/B are the cosets $n(a + B) = na + B$ ($n \in \mathbb{Z}$) (all different). Hence $A = B \oplus \langle a\rangle$ is immediate. $\square$

This theorem can also be phrased by saying that an exact sequence $0 \to B \to A \to F \to 0$ with a free group F is necessarily splitting.

Subgroups of Free Groups In the next theorem we study the subgroups of free abelian groups. Recall the famous result in group theory that subgroups of (non-commutative) free groups are again free. For abelian groups the situation is the same. To prove this, we use a well ordering of the index set.

Theorem 1.6. *Subgroups of free groups are free.*

Proof. Let F be a free group on the set X, which we now assume to be well ordered, say $X = \{x_\sigma\}_{\sigma<\tau}$ for some ordinal τ. Thus $F = \oplus_{\sigma<\tau} \langle x_\sigma\rangle$. For $\sigma < \tau$, define $F_\sigma = \oplus_{\rho<\sigma} \langle x_\rho\rangle$, and set $G_\sigma = G \cap F_\sigma$ for a subgroup $G < F$. Clearly, $G_\sigma = G_{\sigma+1} \cap F_\sigma$, so $G_{\sigma+1}/G_\sigma \cong (G_{\sigma+1} + F_\sigma)/F_\sigma$. The last factor group is a subgroup of $F_{\sigma+1}/F_\sigma \cong \langle x_\sigma\rangle$, thus either $G_{\sigma+1} = G_\sigma$ or $G_{\sigma+1}/G_\sigma$ is an infinite cyclic group. From Theorem 1.5 we conclude that $G_{\sigma+1} = G_\sigma \oplus \langle g_\sigma\rangle$ for some $g_\sigma \in G_{\sigma+1}$ (which is 0 if $G_{\sigma+1} = G_\sigma$). It follows that the elements g_σ generate the direct sum $\oplus_{\sigma<\tau} \langle g_\sigma\rangle$ in G. This must be all of G, since G is the union of the G_σ ($\sigma < \tau$). $\square$

Projectivity Call a group P **projective** if every diagram

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact row can be completed by a suitable homomorphism $\psi : P \to B$ to a commutative diagram, i.e. $\beta\psi = \phi$. We then say: ϕ is **lifted** to ψ.

Theorem 1.7. *A group is projective if and only if it is free.*

Proof. Let $\beta : B \to C$ be a surjective map, and F a free group with a homomorphism $\phi : F \to C$. For each x_i in a free set $X = \{x_i\}_{i\in I}$ of generators of F, we pick an element $b_i \in B$ such that $\beta b_i = \phi x_i$—this is possible, β being epic. Owing to Theorem 1.3, the correspondence $x_i \mapsto b_i$ $(i \in I)$ extends to a homomorphism $\psi : F \to B$. The maps $\beta\psi$ and ϕ are equal on the generators of F, so $\beta\psi = \phi$, and F is projective.

Next, let P be a projective group, and $\beta : F \to P$ an epimorphism, F a free group. By definition, the identity map $\mathbf{1}_P : P \to P$ can be lifted to a map $\psi : P \to F$, i.e. $\beta\psi = \mathbf{1}_P$. Thus ψP is a summand of F, so a free group by Theorem 1.6. The isomorphism $P \cong \psi P$ completes the proof. $\square$

Thus 'free' and 'projective' have the same meaning for abelian groups. Therefore, free resolutions may also be called **projective resolutions**.

Projective Cover The **projective cover** of a group A is defined as a projective group P with a surjective map $\pi : P \to A$ such that $\operatorname{Ker}\pi$ is a superfluous subgroup of P. Projective covers are duals of injective hulls (to be discussed in Chapter 4), but in contrast to their dual counterparts, they rarely exist.

Example 1.8. (a) The cyclic group $\mathbb{Z}(p)$ has no projective cover. If it had one, $\mathbb{Z}$ would be a good candidate, but then the kernel would not be superfluous.

(b) However, $\mathbb{Z}(p)$ regarded as a $\mathbb{Z}_{(p)}$-module does have a projective cover, since $p\mathbb{Z}_{(p)}$ is superfluous in $\mathbb{Z}_{(p)}$.

Theorem 1.9. *A group has a projective cover if and only if it is free.*

Proof. We show that the zero-group is the only superfluous subgroup of a free group F. If $H \neq 0$ is a subgroup in F, then there is a prime p with $H \not\leq pF$ (since $\cap_p pF = 0$). Evidently, $(H + pF)/pF \neq 0$ is a summand of the $\mathbb{Z}/p\mathbb{Z}$-vector space F/pF, say, with complement G/pF for some $pF < G < F$. Then $G + H = F$ where G is a proper subgroup of F, so H cannot be superfluous. $\square$

Defining Relations We shall discuss briefly the method of defining a group in terms of generators and relations. Though this is well known from general group theory, in the commutative case there are simplifications worthwhile to be pointed out.

Let $\{a_i\}_{i\in I}$ be a set of generators of a group A, and $\theta : F \to A$ an epimorphism from a free group $F = \oplus_{i\in I} \langle x_i \rangle$ such that $\theta x_i = a_i$ for each $i \in I$. Ker θ consists of those linear combinations $m_1 x_{i_1} + \cdots + m_k x_{i_k} \in F$ with integral coefficients m_i for which $m_1 a_{i_1} + \cdots + m_k a_{i_k} = 0$ holds in A. These equalities are called the **defining relations** relative to the generating system $\{a_i\}_{i\in I}$.

It follows that the group A is completely determined by giving a set $\{a_i\}_{i\in I}$ of generators along with the set of all defining relations:

$$A = \langle a_i \ (i \in I) | \ m_{j1} a_{i_1} + \cdots + m_{jk} a_{i_k} = 0 \ (j \in J) \rangle \tag{3.2}$$

(since we are dealing exclusively with abelian groups, the commutativity relations are not listed). Indeed, if (3.2) is given, then A is defined as the factor group F/H, where F is a free group on the free set $\{x_i\}_{i\in I}$ of generators, and H is the subgroup of F, generated by the elements $m_{j1} x_{i_1} + \cdots + m_{jk} x_{i_k}$ for all $j \in J$. The relations between the given generators of A are exactly those which are listed in (3.2), and their consequences. (The emphasis is on the non-existence of more relations.) Equation (3.2) is said to be a **presentation** of A.

Example 1.10. A presentation of a free group F with free generators $\{x_i\}_{i\in I}$ is given as $F = \langle x_i \ (i \in I) \mid \varnothing \rangle$ (there are no relations between the generators). Of course, there are numerous other presentations; e.g. $\mathbb{Z} = \langle x, y \mid 2x - 3y = 0 \rangle$.

Example 1.11. The group $C = \langle x \mid nx = 0 \rangle$ for $n \in \mathbb{N}$ is cyclic of order n.

★ **Notes.** The material on free groups is fundamental, and will be used in the future without explicit reference. Though in homological algebra, projectivity is predominant, in abelian group theory freeness seems to prevail. Fortunately, for abelian groups, freeness and projectivity are equivalent, while for modules, the projectives are exactly the direct summands of free modules. Projective modules are rarely free; they are free over principal ideal domains (but not even over Dedekind domains that are not PID), and over local rings (Kaplansky [2]).

Theorem 1.6 holds for modules over left principal ideal domains. Submodules of projectives are again projective if and only if the ring is left hereditary, i.e., all left ideals are projective. Theorem 1.2 holds over commutative rings or under the hypothesis that at least one of κ and λ is infinite. There exist, however, rings R such that all free R-modules $\neq 0$ with finite sets of generators are isomorphic. It is perhaps worthwhile pointing out that every R-module is free if and only if R is a field, and every R-module is projective exactly if R is a semi-simple artinian ring. The property that all R-modules have projective covers characterizes the perfect rings, introduced by H. Bass.

Hausen [6] defines a group P **κ-projective** for an infinite cardinal κ if it has the projective property with respect to all exact sequences $0 \to A \to B \to C \to 0$ with $|C| < \kappa$. She establishes various properties of κ-projective groups, e.g. P is κ-projective if and only if, for every subgroup G with $|P/G| < \kappa$, there is a summand H of P such that $G \leq H$ and G/H is a free group.

Exercises

(1) Let F be a free group on n free generators. If n elements $a_1, \ldots, a_n \in F$ generate F, then this set is a basis of F.

(2) Prove the following converse of Theorem 1.5: a group F is free if it has the property that whenever $B < A$ and $A/B \cong F$, then B is a summand of A.
(3) Give a presentation of $\mathbb{Z}(p^\infty)$, and one of $\mathbb{Q}$.
(4) Let A be presented by a set of generators and defining relations, and assume that the set of generators is the union of two disjoint subsets, $\{b_i\}_{i\in I}$ and $\{c_j\}_{j\in J}$, such that each of the defining relations contains only generators from the same subset. Then $A = B \oplus C$, where B is generated by the b_i, and C by the c_j.
(5) Let A be presented by a set of generators and defining relations, and B by a subset of these generators and defining relations. Show that letting the generators of B correspond to themselves *qua* generators of A induces a homomorphism $B \to A$.
(6) For every set of generators, there is a minimal set of defining relations relative to these generators (i.e., no relation can be omitted). [Hint: Theorem 1.6.]
(7) Let $0 \to A_1 \to A_2 \xrightarrow{\alpha} A_3 \to 0$ be an exact sequence, and $\phi_i : F_i \to A_i$ $(i = 1, 3)$ epimorphisms where F_i are free. If $\psi : F_3 \to A_2$ is such that $\alpha\psi = \phi_3$, then $\phi_1 \oplus \psi : F_1 \oplus F_3 \to A_2$ is epic, and its kernel is $\operatorname{Ker}\phi_1 \oplus \operatorname{Ker}\phi_3$.
(8) Let $0 \to F_1 \to F_2 \to \cdots \to F_n \to 0$ be an exact sequence of finitely generated free groups. Prove the equality $\sum_{k=1}^{n}(-1)^k \operatorname{rk} F_k = 0$.
(9) Assume $\{A_n \mid n \in \mathbb{Z}\}$ is a set of groups. Verify the existence of free groups F_n $(n \in \mathbb{Z})$ and a long sequence

$$\dots \xrightarrow{\alpha_{n-2}} F_{n-1} \xrightarrow{\alpha_{n-1}} F_n \xrightarrow{\alpha_n} F_{n+1} \xrightarrow{\alpha_{n+1}} \dots$$

such that $\alpha_{n-1}\alpha_n = 0$ and $\operatorname{Ker}\alpha_n / \operatorname{Im}\alpha_{n-1} \cong A_n$ for every $n \in \mathbb{Z}$.

2 Finite and Finitely Generated Groups

We turn our attention to groups with a finite number of generators. First, we discuss finite groups separately. Though this is a special case of the general theory of finitely generated groups (to be developed independently), a short, direct approach to the theory of finite groups is not without merit.

Finite Groups We start with a simple lemma.

Lemma 2.1. *Let A be a p-group that contains an element g of maximal order p^k for an integer $k > 0$. Then $\langle g\rangle$ is a direct summand of A.*

Proof. If A is infinite, then use Zorn's lemma to argue that there is a subgroup B of A maximal with respect to the property $B \cap \langle g\rangle = 0$. To show that $A^* = \langle g\rangle \oplus B$ equals A, by way of contradiction assume that some $a \in A$ does not belong to A^*. Replacing a by $p^i a$ if necessary, we may also suppose that $pa \in A^*$, i.e. $pa = mg + b$ for some $m \in \mathbb{Z}, b \in B$. By the maximality of the order of g, we have $p^{k-1}mg + p^{k-1}b = p^k a = 0$. Hence $p^{k-1}mg = 0$, so m must be divisible by p, say, $m = pm'$. Then $a' = a - m'g \notin A^*$ satisfies $pa' = b$. By the maximal choice of B, $\langle B, a'\rangle \cap \langle g\rangle \neq 0$, thus $0 \neq ra' + b' = sg$ for some $r, s \in \mathbb{Z}, b' \in B$. This can happen only if $(r, p) = 1$, since $pa' \in B$. But then $pa', ra' \in A^*$ implies $a' \in A^*$, a contradiction. □

Fundamental Theorem on Finite Abelian Groups The first structure theorem in the history of group theory was the famous Basis Theorem on finite abelian groups.

Theorem 2.2 (Frobenius–Stickelberger [1]). *A finite group is the direct sum of a finite number of cyclic groups of prime power orders.*

Proof. Thanks to Theorem 1.2 in Chapter 2, the proof reduces at once to p-groups. In a finite p-group $A \neq 0$, we select an element g of maximal order. By the preceding lemma, $A = \langle g \rangle \oplus B$ for some subgroup B. Since B has a smaller order than A, a trivial induction completes the proof. □

There is a uniqueness theorem attached to the preceding result. Again, it suffices to state it for p-groups.

Theorem 2.3. *Two direct decompositions of a finite p-group A into cyclic groups are isomorphic.*

Proof. In a direct decomposition of A collect the cyclic summands of equal orders into a single summand to obtain a courser decomposition $A = B_1 \oplus \cdots \oplus B_k$ where each B_i is 0 or a direct sum of cyclic groups of fixed order p^i. Evidently, $p^{k-1}A = p^{k-1}B_k$ is the socle of B_k, it is an elementary p-group, its dimension (as a $\mathbb{Z}/p\mathbb{Z}$-vector space) tells us the number of cyclic components in B_k. As this socle depends only on A, the number of cyclic summands of order p^k is independent of the choice of the direct sum representation of A. In general, $p^{i-1}A[p] = p^{i-1}B_i[p] \oplus \cdots \oplus p^{i-1}B_k[p]$ modulo $p^iA[p] = p^iB_{i+1}[p] \oplus \cdots \oplus p^iB_k[p]$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space $\cong p^{i-1}B_i[p]$ whose dimension is equal to the number of cyclic summands (of order p^i) in B_i. The same argument shows that this dimension is independent of the choice of the selected direct decomposition of A. □

Finitely Generated Groups We proceed to the discussion of finitely generated groups. We start with a preliminary lemma.

Lemma 2.4 (Rado [1]). *Assume $A = \langle a_1, \dots, a_k \rangle$, and $n_1, \dots, n_k$ are integers such that* $\gcd\{n_1, \dots, n_k\} = 1$. *Then there exist elements $b_1, \dots, b_k \in A$ such that*

$$A = \langle b_1, \dots, b_k \rangle \qquad \textit{with } b_1 = n_1a_1 + \cdots + n_ka_k.$$

Proof. We induct on $n = |n_1| + \cdots + |n_k|$. If $n = 1$, then let $b_1 = \pm a_i$ for any i, and the claim is evident. Next let $n > 1$. Then at least two of the n_i are different from 0, say, $|n_1| \geq |n_2| > 0$. Since either $|n_1 + n_2| < |n_1|$ or $|n_1 - n_2| < |n_1|$, we have $|n_1 \pm n_2| + |n_2| + \cdots + |n_k| < n$ for one of the two signs. $\gcd\{n_1 \pm n_2, n_2, \dots, n_k\} = 1$ and the induction hypothesis imply that $A = \langle a_1, \dots, a_k \rangle = \langle a_1, a_2 \mp a_1, \dots, a_k \rangle = \langle b_1, \dots, b_k \rangle$ with $b_1 = (n_1 \pm n_2)a_1 + n_2(a_2 \mp a_1) + n_3a_3 + \cdots + n_ka_k = n_1a_1 + \cdots + n_ka_k$. □

The main result on finitely generated groups is our next theorem which is regarded as the first major result in the abstract structure theory of infinite abelian groups. It plays an important role in several applications.

Theorem 2.5. *The following conditions on a group A are equivalent:*

(i) *A is finitely generated;*
(ii) *A is the direct sum of a finite number of cyclic groups;*
(iii) *the subgroups of A satisfy the maximum condition.*

Proof. (i) $\Rightarrow$ (ii) assume A is finitely generated, and a minimal generating set of A contains k elements. Pick such a set with k generators, say, $a_1, \dots, a_k$, with the additional property that a_1 has minimal order, i.e. no other set of k generators contains an element of smaller order. If $k = 1$, then $A = \langle a_1 \rangle$, and we are done. So let $k > 1$, and as a basis of induction, assume that $B = \langle a_2, \dots, a_k \rangle$ is a direct sum of cyclic groups. Thus it suffices to verify that $A = \langle a_1 \rangle \oplus B$, which will follow if we can show that $\langle a_1 \rangle \cap B = 0$.

By the choice of k, we have $o(a_1) > 1$. Working toward a contradiction, suppose that $\langle a_1 \rangle \cap B \neq 0$, i.e. $m_1 a_1 = m_2 a_2 + \dots + m_k a_k \neq 0$ with $0 < m_1 < o(a_1)$. Let $d = \gcd\{m_1, \dots, m_k\}$, and write $m_i = dn_i$. Then $\gcd\{n_1, \dots, n_k\} = 1$, and from Lemma 2.4 we conclude that $A = \langle a_1, \dots, a_k \rangle = \langle b_1, \dots, b_k \rangle$ with $b_1 = -n_1 a_1 + n_2 a_2 + \dots + n_k a_k$. Here $db_1 = 0$, thus $o(b_1) < o(a_1)$, contradicting the choice of a_1. Thus $\langle a_1 \rangle \cap B = 0$.

(ii) $\Rightarrow$ (iii) Let $A = \langle a_1 \rangle \oplus \dots \oplus \langle a_k \rangle$. If $k = 1$, then A is cyclic, and every non-zero subgroup is of finite index in A. Hence the subgroups satisfy the maximum condition. (iii) will follow by a trivial induction if we can show that $A = B \oplus C$ has the maximum condition on subgroups whenever both B and C share this property. If $A_1 \leq \dots \leq A_n \leq \dots$ is an ascending chain of subgroups in A, then $A_1 \cap B \leq \dots \leq A_n \cap B \leq \dots$ is one in B, so from some index m on, all $A_n \cap B$ are equal to $A_m \cap B$. For $n > m$ we have $A_n/(A_m \cap B) = A_n/(A_n \cap B) \cong (A_n + B)/B \leq A/B \cong C$, whence we conclude that from a certain index $t > m$ on all factor groups $A_t/(A_m \cap B)$, and hence all subgroups A_t, are equal.

(iii) $\Rightarrow$ (i) The set $\mathcal{S}$ of all finitely generated subgroups of A is not empty, so by hypothesis (iii) A contains a maximal finitely generated subgroup G. For any $a \in A$, $\langle G, a \rangle$ is still finitely generated. Hence $\langle G, a \rangle = G$, thus $A = G$, and A is finitely generated. $\square$

Let us point out two immediate consequences of Theorem 2.5. First, *every finitely generated group is the direct sum of a finite group and a finitely generated free group* (follows from (ii)). Secondly, *subgroups of finitely generated groups are again finitely generated* (follows from (iii)).

The most essential part of the preceding theorem is the first implication. We give another quick proof, reducing it to Theorem 2.2. If we can show that A/T is free ($T = t(A)$), then $A \cong T \oplus A/T$ by Theorem 1.5, and we are done. Thus, it is enough to consider $A = \langle a_1, \dots, a_n \rangle$ torsion-free. To start the induction on n, there is nothing to prove if $n = 1$, since then $A \cong \mathbb{Z}$ trivially. Let $U/\langle a_n \rangle$ denote the torsion subgroup of $A/\langle a_n \rangle$. Then A/U is torsion-free and has a smaller number of generators, so it is free. Hence $A \cong U \oplus A/U$ (again by Theorem 1.5), where U is a finitely generated group isomorphic to a subgroup of $\mathbb{Q}$, so it is cyclic.

Stacked Basis Theorem A third proof of Theorem 2.5 is based on the following theorem which is of considerable interest in its own right (see the more general Theorem 6.5). We say $\{a_i\}_{i \in I}$ is a **basis** of A if $A = \oplus_{i \in I} \langle a_i \rangle$.

Theorem 2.6. *If H is a subgroup of the free group F of finite rank k, then F and H have 'stacked bases:'*

$$F = \langle a_1 \rangle \oplus \cdots \oplus \langle a_k \rangle \quad \textit{and} \quad H = \langle b_1 \rangle \oplus \cdots \oplus \langle b_k \rangle$$

such that there are non-negative integers $m_1, \dots, m_k$ satisfying

$$b_i = m_i a_i \quad (i = 1, \dots, k) \quad \textit{and} \quad m_{i-1} | m_i \quad (i = 2, \dots, k).$$

Proof. We select a free basis $\{x_1, \dots, x_k\}$ of F with the following extremal property: H contains an element $b_1 = n_1 x_1 + \cdots + n_k x_k$ with a minimal positive coefficient n_1. In other words, for another basis of F, or for another permutation of the basis elements, or for other elements of H, the leading positive coefficient is never less than n_1.

The first observation is that $n_1 | n_i$ $(i = 2, \dots, k)$. For, if $n_i = q_i n_1 + r_i$ $(q_i, r_i \in \mathbb{Z}, 0 \leq r_i < n_1)$, then we can write $b_1 = n_1 a_1 + r_2 x_2 + \cdots + r_k x_k$ where $\{a_1 = x_1 + q_2 x_2 + \cdots + q_k x_k, x_2, \dots, x_k\}$ is a new basis of F. By the special choice of $\{x_1, \dots, x_k\}$, we must have $r_2 = \cdots = r_k = 0$. The same argument shows that if $b = s_1 x_1 + \cdots + s_k x_k$ $(s_i \in \mathbb{Z})$ is any element of H, then $s_1 = q n_1$ for some $q \in \mathbb{Z}$. Hence $b - q b_1 \in \langle x_2 \rangle \oplus \cdots \oplus \langle x_k \rangle = F_1$. We conclude that F has a decomposition $F = \langle a_1 \rangle \oplus F_1$ such that $H = \langle b_1 \rangle \oplus H_1$, where $b_1 = n_1 a_1$ and $H_1 \leq F_1$. Using induction hypothesis for the pair H_1, F_1, we infer that F has a basis $\{a_1, \dots, a_k\}$ and H has a basis $\{b_1, \dots, b_k\}$ such that $b_i = m_i a_i$ for some non-negative integers m_i.

It remains to establish the divisibility relation $m_1 | m_2$ (the others will follow by induction). Write $m_2 = t m_1 + r$ with $t, r \in \mathbb{Z}, 0 \leq r < m_1$. Then $\{a = a_1 + t a_2, a_2, \dots, a_k\}$ is a new basis of F, in terms of which we have $b_1 + b_2 = m_1 a_1 + (t m_1 + r) a_2 = m_1 a + r a_2 \in H$. The minimality of $m_1 = n_1$ implies $r = 0$. $\square$

With the aid of Theorem 2.6, we can reprove the implication (i) $\Rightarrow$ (ii) in Theorem 2.5. If A is generated by k elements, then $A \cong F/H$, where F is a free group on a set of k elements. Choosing stacked bases for F and H, as described in Theorem 2.6, we obtain

$$A \cong \langle a_1 \rangle / \langle m_1 a_1 \rangle \oplus \cdots \oplus \langle a_k \rangle / \langle m_k a_k \rangle.$$

Consequently, A is the direct sum of cyclic groups: the ith summand is cyclic of order m_i if $m_i > 0$, and infinite cyclic if $m_i = 0$. The numbers m_i are called **elementary divisors**.

Fundamental Theorem on Finitely Generated Groups Of course, the numbers m_i in Theorem 2.6 are not necessarily prime powers, but we can decompose the finite summands into direct sums of cyclic groups of prime power orders. Cyclic groups of prime power orders are indecomposable (and so are the infinite cyclic groups), so we can claim the fundamental theorem:

Theorem 2.7 (Fundamental Theorem on Finitely Generated Abelian Groups). *A finitely generated group is the direct sum of finitely many indecomposable cyclic groups, each of which is of prime power order or infinite cyclic.* □

Whenever one has a direct decomposition, then the standard question is: to what extent is the decomposition unique? This question is fully answered in the following theorem.

Theorem 2.8. *Any two direct decompositions of a finitely generated group into indecomposable cyclic groups are isomorphic.*

Proof. If A is finitely generated, then by Theorem 2.7 $A = tA \oplus F$ where $F \cong A/tA$ is finitely generated free. Both summands are uniquely determined by A up to isomorphism. Theorems 2.3 and 1.2 guarantee the uniqueness of the decompositions of the summands, whence the claim is evident. □

Invariants Thus in the decompositions of a finitely generated group A, the orders of the indecomposable cyclic summands (but not the summands themselves) are uniquely determined. These orders are referred to as the **invariants** of A. For instance, the invariants of $A \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}(p) \oplus \mathbb{Z}(p^2) \oplus \mathbb{Z}(q^3) \oplus \mathbb{Z}(q^3)$ (with primes p, q) are: $\infty, \infty, \infty, p, p^2, q^3, q^3$. We also say: A is **of type** $(\infty, \infty, \infty, p, p^2, q^3, q^3)$.

Consequently, with every finitely generated group A, a finite system of symbols ∞ and prime powers is associated. Not only is it uniquely determined by A, but it also determines A up to isomorphism, i.e. two finitely generated groups are isomorphic if and only if they have the same system of invariants (maybe in different orders)—this fact is expressed by saying that this is a **complete system of invariants**. Moreover, these invariants are **independent** in the sense that, for an arbitrary choice of a finite system of symbols ∞ and prime powers, there exists a finitely generated group exactly with this system of invariants (this is obvious).

Example 2.9. Let $C(m)$ denote the multiplicative group of those residue classes of integers modulo the integer $m = p_1^{r_1} \cdots p_k^{r_k}$ (canonical form) which are relatively prime to m. Its order is given by Euler's totient function $\varphi(m)$. Elementary number theory tells us that

(a) $C(m)$ is the direct product of the groups $C(p_i^{r_i})$ for $i = 1, \ldots, k$;
(b) for odd primes p, $C(p^r)$ is cyclic of order $\varphi(p^r) = p^r - p^{r-1}$;
(c) $C(4)$ is cyclic of order 2, while $C(2^r)$ $(r \geq 3)$ is of type $(2, 2^{r-2})$.

Kaplansky's Test Problems In his famous little red book [K], Kaplansky raises the question about criteria for satisfactory structure theorems. He lists two test problems that such theorems must pass in order to qualify 'satisfactory.' These are:

Test Problem I. If the group G is isomorphic to a direct summand of H, and H is isomorphic to a direct summand of G, are then G and H isomorphic?
Test Problem II. If $G \oplus G \cong H \oplus H$, are G and H isomorphic?

Evidently, the structure theorem on finitely generated groups passes the test with flying colors: both answers are easy 'yes.' However, some of the theorems that will be discussed later on will fail one or both test problems.

★ **Notes.** Whenever it seems instructive or interesting, we shall make historical remarks that are intended to give a sense of the way in which the subject has developed, but are in no way a comprehensive survey of the relevant contributions. As far as the fundamental theorem on finite abelian groups is concerned, it is not clear how far back in time one needs to go to trace its origin. It was F.C. Gauss who established a decomposition in number theory reminiscent to it. That time the concept of a group was unknown, it took a long time to formulate and to prove the fundamental theorem in the present form; see Frobenius–Stickelberger [1]. The theorem on finitely generated groups may be credited to H.J.S. Smith [Phil. Trans. **151**, 293–326 (1861)]. He reduced matrices with integral entries to canonical form that bears his name.

This is the first time we encounter a structure theorem, so a few comments are in order. Such a theorem (on any class of algebraic systems) is supposed to be in terms of easily recognizable invariants, like natural numbers, cardinal or ordinal numbers, but they can be matrices with integral entries, etc. '*Invariants*' mean by definition that they are exactly the same for isomorphic objects. A set of invariants is *complete* if we can reconstruct from it the object within the class by using a method typical for the class (for finitely generated groups, this method consists in forming the direct sum of cyclic groups with the given invariants as orders). Finally, *independence* means that the system of invariants can be chosen arbitrarily, i.e. without additional restriction (in this case, arbitrary prime powers and the sign ∞, each with arbitrary multiplicities). The system of invariants for finitely generated groups is most satisfactory, it has served as a prototype for structure theorems in algebra.

The Kaplansky test problems have been discussed for various classes, mostly with negative answers. de Groot modified Test Problem I by asking the isomorphy of G and H if G has a summand $G_1 \cong H$ and H has a summand $H_1 \cong G$ such that, in addition, $G/G_1 \cong H/H_1$ is also satisfied.

There are numerous generalizations of the theorems in this section. Kaplansky [J. Indian Math. Soc. **24**, 279–281 (1960)] proved that, for integral domains R, the torsion parts of finitely generated R-modules are summands exactly if R is a Prüfer domain. There is an extensive literature on commutative rings over which finitely generated torsion modules are Σ-cyclic. Unless the ring is left noetherian, finitely generated left modules are different from finitely presented ones which are somewhat better manageable. Finitely presented R-modules are Σ-cyclic if and only if R is an elementary divisor ring, i.e., every matrix over R can be brought to a diagonal form by left and right multiplications by unimodular matrices (Kaplansky [Trans. Amer. Math. Soc. **66**, 464–491 (1949)]). In this case, Theorem 2.6 still holds true.

Exercises

(1) A group is finite if and only if its subgroups satisfy both the maximum and the minimum conditions.
(2) A finite group A is cyclic exactly if $|A[p]| \leq p$ for every prime p.

(3) (a) If the integer m divides the order of the finite group A, then A has both a subgroup and a factor group of order m.
(b) (G. Frobenius) In a finite p-group, the number of subgroups of a fixed order (dividing the order of the group) is $\equiv 1 \bmod p$.
(4) A group is isomorphic to a subgroup of the finite group A if and only if it is isomorphic to a factor group of A.
(5) The number of non-isomorphic groups of order $m = p_1^{r_1} \cdots p_k^{r_k}$ (canonical form of m) is equal to $P(r_1) \cdots P(r_k)$, where $P(r)$ stands for the number of partitions of r into positive integers.
(6) If A, B are finite groups such that, for every integer m, they contain the same number of elements of order m, then $A \cong B$.
(7) A set $\{a_1, \dots, a_k\}$ of generators of a finite group is a basis if and only if the product $o(a_1) \cdots o(a_k)$ is minimal among the products of orders for all generating sets.
(8) In a finitely generated group, every generating set contains a finite set of generators.
(9) (a) The sum of all the elements of a finite group A is 0, unless A contains just one element a of order 2, in which case the sum is equal to this a.
(b) From (a) derive Wilson's congruence $(p-1)! \equiv -1 \bmod p$, p a prime.
(10) Let A, B be finitely generated groups. There is a group C such that both A and B have summands isomorphic to C, and every group that is isomorphic to summands of both A and B is isomorphic to a summand of C.
(11) Any set of pairwise non-isomorphic finite (finitely generated) groups has cardinality $\leq \aleph_0$.
(12) (Cohn, Honda, E. Walker) Finitely generated groups A have the cancellation property: $A \oplus B \cong A \oplus C$ implies $B \cong C$, or equivalently, if $G = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A \cong A_2$, then $B \cong C$. [Hint: enough for $A_1 = \langle a \rangle$ cyclic of order ∞ or prime power p^r.]
(13) If A and B are finitely generated groups, and if each is isomorphic to a subgroup of the other, then $A \cong B$.
(14) A surjective endomorphism of a finitely generated group is an automorphism.

3 Factorization of Finite Groups

In most cases, the fundamental theorem is instrumental in solving problems related to finite abelian groups. However, there are notable exceptions where it seems the fundamental theorem is totally irrelevant. One of these is Hajós' theorem on the 'factorization' of finite abelian groups.

The problem goes back to a famous conjecture by H. Minkowski in 1896 on tiling the n-dimensional Euclidean space by n-dimensional cubes. If the space is filled gapless such that no two cubes have common interior points, then it was conjectured that there exist cubes sharing $n-1$-dimensional faces. The conjecture was rephrased as an abelian group-theoretical problem, and solved in this form by G. Hajós. We discuss briefly this celebrated result. The proof involves group rings, and therefore at some point we have to switch to the multiplicative notation. It is reasonable to do this right away.

Thus in this section, all groups are finite, written multiplicatively. Accordingly, 1 will denote the identity element of groups.

Direct Products of Subsets If $S_1, \dots, S_k$ are non-empty subsets of a multiplicative group G, then we say that G is **a direct product** of these subsets, in notation,

$$G = S_1 \dot{\times} \dots \dot{\times} S_k, \tag{3.3}$$

if each element $g \in G$ can be written uniquely as $g = s_1 \cdot \ldots \cdot s_k$ with $s_i \in S_i$. This definition is in line with the definition of direct sum of subgroups. We will call the components S_i **factors** of G, and (3.3) **a factorization of** G. We obviously have

(A) *The cardinality of a factor is a divisor of the group order.*
(B) *Every subgroup H of G is a factor: $G = H \dot{\times} S$ if S is a complete set of representatives* mod H.
(C) *A factor S_i can be replaced by gS_i with any $g \in G$.* For this reason, there is no loss of generality in assuming that each factor contains $1 \in G$.

Periodic and Cyclic Subsets A subset P is called **periodic** and a non-unit $g \in G$ a **period** of P if $gP = P$. Subgroups are trivially periodic. If g is a period, then so are the elements $\neq 1$ of $\langle g \rangle$. In this case, P is the set union of certain cosets mod $\langle g \rangle$.

Lemma 3.1. *If $G = \langle a \rangle$ is cyclic of order p^n, and $G = S \dot{\times} T$, then either S or T is periodic.*

Proof. Set $S = \{a^{n_1}, \dots, a^{n_k}\}$, $T = \{a^{m_1}, \dots, a^{m_\ell}\}$ $(n_i, m_j \geq 0)$ and form the polynomials $S(z) = z^{n_1} + \cdots + z^{n_k}$, $T(z) = z^{m_1} + \cdots + z^{m_\ell}$ (with indeterminate z). Hypothesis implies

$$S(z)T(z) \equiv 1 + z + z^2 + \cdots + z^{p^n - 1} \mod z^{p^n} - 1.$$

It follows that $S(z)T(z)$ is divisible by the p^nth cyclotomic polynomial $\Phi_n(z) = 1 + z^{p^{n-1}} + \cdots + z^{(p-1)p^{n-1}}$. This polynomial is known to be irreducible over $\mathbb{Q}$, so one of the factors, say, $S(z)$ is divisible by $\Phi_n(z)$. Hence we conclude that $a^{p^{n-1}}$ is a period of S. $\square$

Our main concern is with factors that are **cyclic subsets** in the sense that they are of the form

$$[a]_n = \{1, a, \dots, a^{n-1}\} \qquad (2 \leq n \leq o(a))$$

for some $a \in G$. We need two preliminary lemmas.

Lemma 3.2 (Hajós [1]). *A cyclic subset is periodic if and only if it is a group.*

Proof. Let $P = [a]_n$ be periodic with period $g \in G$, so $P = \langle g \rangle \dot{\times} S$ for some $S \subset G$ where $1 \in S$ may be assumed. Evidently, $g = a^t$ for some $t \in \mathbb{N}$, and P contains $1, a, \dots, a^{t-1}$, the powers of a^t, as well as their cosets mod $\langle a^t \rangle$. This means $\langle a \rangle \subseteq P$, so P is a (cyclic) group. □

Lemma 3.3 (Hajós [1]). *A cyclic subset C can be written as a direct product of cyclic subsets of prime orders such that C is a subgroup if and only if one of the factor cyclic subsets is a subgroup.*

Proof. Suppose $C = [c]_n$, and let $n = p_1 \cdots p_k$, a product of primes. It is an easy computation to show that

$$[c]_n = [c]_{p_1} \dot{\times} [c^{p_1}]_{p_2} \dot{\times} \dots \dot{\times} [c^{p_1 \cdots p_{k-1}}]_{p_k}.$$

If C is a subgroup, i.e. if $c^n = 1$, then the last factor is also a subgroup. For the converse, we show that if $C = \langle a \rangle \dot{\times} S$ for some $1 \neq a \in C$, $S \subset C$, then C has to be a subgroup. In fact, a is then a period of C, and the claim follows from Lemma 3.2. □

Hajós' Theorem We can now state the main theorem.

Theorem 3.4 (Hajós [1]). *If a finite group G is the direct product of cyclic subsets,*

$$G = [a_1]_{n_1} \dot{\times} \dots \dot{\times} [a_k]_{n_k},$$

then one of the factors is a subgroup.

Proof. In view of Lemma 3.3, for the proof we may assume that the orders n_i of the factors are primes p_i. Suppose $[a_k]_{p_k}$ is not a subgroup, i.e. $a_k^{p_k} \neq 1$. Then from $a_k G = G$ we derive that

$$[a_1]_{p_1} \dot{\times} \dots \dot{\times} [a_{k-1}]_{p_{k-1}} \cdot a_k^{p_k} = [a_1]_{p_1} \dot{\times} \dots \dot{\times} [a_{k-1}]_{p_{k-1}}, \tag{3.4}$$

that is, the product on the right is periodic with period $a_k^{p_k}$. Delete as many factors as possible until no more factor can be omitted without violating the periodicity of the product. Let $a \in G$ denote a period of a shortest periodic subset $P = [a_1]_{p_1} \dot{\times} \dots \dot{\times} [a_h]_{p_h}$.

Consider the subgroup $H = \langle a_1, \dots, a_h \rangle$ of G. As P is a factor of G, it is also a factor of H, thus $|P|$ divides $|H|$, i.e. $p_1 \cdots p_h \mid |H|$. If we can show that $|H|$ is the product of not more than h primes, then $P = H$ will follow. We will then have a similar direct product decomposition for H, a group of smaller order, so observing that the case $h = 1$ is trivial, an obvious induction will complete the proof.

It remains to substantiate the claim concerning the order of the subgroup H. We interrupt the proof to verify a lemma that will do the job.

The Crucial Lemma The crux of the problem is to find a proper statement, more general than actually needed for the proof, that will allow an induction to complete the proof. We need the group ring $\mathbb{Z}[G]$ to formulate such a lemma. $\mathbb{Z}[G]$ consists of elements of the form

$$\mathfrak{x} = m_1 g_1 + \cdots + m_\ell g_\ell \quad (g_i \in G, m_i \in \mathbb{Z}) \tag{3.5}$$

which we add and multiply according to the usual rules, respecting the multiplication rules in G.

In what follows we will assume that the expression of $\mathfrak{x}$ is canonical, i.e. all the g_i are different, and all $m_i \neq 0$. With this in mind, we go on to define $\langle \mathfrak{x} \rangle$ as the subgroup of G generated by the elements g_i in (3.5), and denote by $\pi(\mathfrak{x})$ the number of prime factors in the order of $\langle \mathfrak{x} \rangle$. Finally, the symbol $\mathfrak{a}$ will have double meaning: for an $a \in G$, it is either $1 + a + \cdots + a^{p-1}$ for a prime p, or $1 - a$. Thus $\mathfrak{a} \in \mathbb{Z}[G]$ and $\langle \mathfrak{a} \rangle = \langle a \rangle$.

Lemma 3.5 (Hajós [1]). *Assume that the equation*

$$\mathfrak{x}\, \mathfrak{a}_1 \cdots \mathfrak{a}_k = 0 \tag{3.6}$$

holds in the group ring $\mathbb{Z}[G]$, where $a_i \in G$, $\mathfrak{x} \in \mathbb{Z}[G]$. If no factor $\mathfrak{a}_i$ can be deleted without violating the validity of the equation, then

$$\pi(\mathfrak{x}, \mathfrak{a}_1, \cdots, \mathfrak{a}_k) - \pi(\mathfrak{x}) < k. \tag{3.7}$$

Proof. We begin with the verification in case $k = 1$. Thus we have $\pi(\mathfrak{x}\,\mathfrak{a}) = 0$ with non-zero factors, and what we wish to prove amounts to $a \in \langle \mathfrak{x} \rangle$. If $\mathfrak{a} = 1 - a$, then $\mathfrak{x} = a\mathfrak{x}$, which implies that there are $b_1, b_2 \in G$ in the normal form of $\mathfrak{x}$ such that $b_1 = ab_2$. Hence $a \in \langle \mathfrak{x} \rangle$ in this case. If $\mathfrak{a} = 1 + a + \cdots + a^{p-1}$ for some prime p, then by multiplication by $1 - a$ we get $\mathfrak{x}(1 - a^p) = 0$, whence $a^p \in \langle \mathfrak{x} \rangle$. On the other hand, from $\mathfrak{x}(a + \cdots + a^{p-1}) = -\mathfrak{x}$ we conclude that $b_1 = a^i b_2$ for some $b_1, b_2 \in \langle \mathfrak{x} \rangle$ and $1 \leq i \leq p - 1$. Thus also $a^i \in \langle \mathfrak{x} \rangle$, and therefore $a \in \langle \mathfrak{x} \rangle$.

We continue with induction on $n = \pi(\mathfrak{a}_1) + \cdots + \pi(\mathfrak{a}_k)$. If $n = 1$, then $k = 1$, and we are done. Assuming $k \geq 2$, we rewrite (3.6) in the form $(\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j)\mathfrak{a}_{j+1} \cdots \mathfrak{a}_k = 0$ for $j < k$, and apply the induction hypothesis to obtain

$$\pi((\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j), \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_k) - \pi(\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j) < k - j \qquad (1 \leq j < k).$$

The index of the subgroup $\langle (\mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j), \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_k \rangle$ in $\langle \mathfrak{x}, \mathfrak{a}_1 \ldots, \mathfrak{a}_j, \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_k \rangle$ evidently divides the index of $\langle \mathfrak{x}\,\mathfrak{a}_1 \cdots \mathfrak{a}_j \rangle$ in $\langle \mathfrak{x}, \mathfrak{a}_1, \ldots \mathfrak{a}_k \rangle$ (cf. Exercise 1). Hence, from the last inequality we get

$$\pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_k) - \pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_j) < k - j \qquad (1 \leq j < k). \tag{3.8}$$

If $\pi(\mathfrak{a}_j) = 1$ for all $j \leq k$, then clearly $\pi(\mathfrak{x}, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k-1}) - \pi(\mathfrak{x}) \leq k - 1$, along with (3.8) for $j = k - 1$ yields (3.7). If, e.g., $\pi(\mathfrak{a}_k) \geq 2$, then by multiplication by $1 - a_k$ or by $1 + a_k + \cdots + a_k^{p-1}$ for some prime p, we can replace the factor $\mathfrak{a}_k$ by

$\mathfrak{a}_0 = 1 - a_0$ with $1 \leq \pi(\mathfrak{a}_0) = \pi(\mathfrak{a}_k) - 1$. After deleting superfluous factors $\mathfrak{a}_i$, and renumbering, we get

$$\mathfrak{x}\, \mathfrak{a}_1, \dots \mathfrak{a}_\ell \mathfrak{a}_0 = 0 \qquad (0 \leq \ell \leq k-1)$$

where no factor can be omitted, not even the last one. By induction hypothesis, $\pi(\mathfrak{x}, \mathfrak{a}_1, \dots, \mathfrak{a}_\ell, \mathfrak{a}_0) - \pi(\mathfrak{x}) \leq \ell$. In case $\ell = 0$, we have $\pi(\mathfrak{x}, \mathfrak{a}_0) - \pi(\mathfrak{x}) = 0$, and $a_0 \in \langle \mathfrak{x} \rangle$, thus $\pi(\mathfrak{x}, \mathfrak{a}_k) - \pi(\mathfrak{x}) \leq 1$. This, together with (3.8) for $j = k-1$, leads to (3.7). If $\ell \geq 1$, then manifestly $\pi(\mathfrak{x}, \mathfrak{a}_1, \dots, \mathfrak{a}_\ell) - \pi(\mathfrak{x}) \leq \ell$, hence along with (3.8) for $j = \ell$ it yields the desired (3.7). □

Resuming the proof of Theorem 3.4, we rewrite (3.4) (after deleting superfluous factors) as an equation in $\mathbb{Z}[G]$:

$$\mathfrak{a}_1 \cdots \mathfrak{a}_h \cdot (1 - a) = 0$$

where $\mathfrak{a}_i = 1 + a_i + \cdots + a_i^{p_i - 1}$ $(i = 1, \dots, h)$. Applying Lemma 3.5 to the case $\mathfrak{x} = 1$, we obtain $\pi(a_1, \dots, a_h, a) = \pi(\mathfrak{a}_1, \dots, \mathfrak{a}_h, a) \leq h$, and *a fortiori* $\pi(a_1, \dots, a_h) \leq h$. As pointed out above, this completes the proof. □

Example 3.6 (Hajós). Theorem 3.4 may fail if the factors are not cyclic. This is shown by the following examples.

(a) Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ be a direct product of cyclic groups where a, b, c are generators of orders $4, 4, 2$, respectively. Then

$$G = \{1, a\} \dot{\times} \{1, b\} \dot{\times} \{1, a^2, ab^2, a^3b^2, c, a^2bc, a^2b^3c, b^2c\}.$$

(b) Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ where all the generators a, b, c are of order 4. Then

$$G = \{1, a\} \dot{\times} \{1, b\} \dot{\times} \{1, c\} \dot{\times} \{1, a^2b, b^2c, c^2a, a^2b^3, b^2c^3, c^2a^3, a^2b^2c^2\}.$$

★ **Notes.** The proof above is based on the original proof by Hajós [1] with essential simplifications due to L. Rédei and T. Szele. Various modified versions of the problem have been considered. One version requires the factors to be simulated subsets: a subset S of a group is *simulated* if it is obtainable from a subgroup by replacing an element by an arbitrary group element. There is an extensive literature on this difficult subject, most advanced papers are written recently by A.D. Sands and S. Szabó. There are remarkable connections to tessellations.

It is hard to understand why so far no evidence of a link has been found between the fundamental theorem on finite abelian groups and the Hajós theorem. Such a link would probably avoid group rings, but it seems doubtful we could have found our way through without making use of them.

A generalized, still unsolved version of Minkowski's conjecture was formulated by O.H. Keller. Its algebraized version says that if $G = S \dot{\times} [a_1]_{n_1} \dot{\times} \cdots \dot{\times} [a_k]_{n_k}$ with a subset $S \subset G$, then one of the elements $a_i^{n_i}$ equals $s_1 s_2^{-1}$ for some $s_1, s_2 \in S$.

Exercises

All groups are finite, written multiplicatively.

(1) If A, B, C are finite index subgroups of the group G, and $B \leq C$, then $[AC : AB]$ divides $[C : B]$.
(2) If in a group G, the subset $P = [a]_p \dot{\times} [b]_q$ $(a, b \in G)$ is periodic with different primes p, q, then one of the factors is a subgroup.
(3) (Sands) Let G be cyclic of order 8. Find $G = S \dot{\times} T$ such that none of S, T can be replaced by a subgroup. [Hint: $\langle a \rangle = \{1, a^2\} \dot{\times} \{1, a, a^4, a^5\}$.]
(4) (de Bruijn) A subset S of a cyclic group of order n is periodic if and only if there is a proper divisor d of n such that $S(z)$ (defined above in Lemma 3.1) is divisible by the polynomial $f(z) = (z^n - 1)(z^d - 1)^{-1}$.
(5) Assume G is a finite group of one of the types $(2, 2, 2), (2, 2^2), (2, 2, 3), (2, 3, 3), (3, 3^2), (3, 3, 3)$. If $G = S \dot{\times} T$ for subsets S, T, then S or T is periodic. [Hint: S or T contains 2 or 3 elements.]
(6) (de Bruijn) Let G be an elementary 2-group with generators $a_1, a_2, a_3, b_1, b_2, b_3$. None of the factors is periodic in the factorization

$$G = \{1, a_1a_3b_1, a_2a_3, a_1a_2b_1, b_2, a_1a_2a_3b_2, a_1b_1b_2, a_2a_3b_1b_2\} \cdot$$
$$\cdot \{1, a_1, a_2, a_1a_2, b_3, a_3b_3, b_1b_3, a_3b_1b_3\}.$$

(7) (de Bruijn) Let $G = \langle a \rangle$ be cyclic of order 72. It factorizes into two non-periodic subsets: $\{1, a^8, a^{16}, a^{18}, a^{26}, a^{34}\}$ and $\{a^{18}, a^{54}, a^{24}, a^{60}, a^{48}, a^{12}, a^{17}, a^{41}, a^{65}, a^{45}, a^{69}, a^{21}\}$.

4 Linear Independence and Rank

Motivated by linear independence and dimension in vector spaces, we are in search for corresponding notions in groups.

Linear Independence Linear independence in groups can be defined in two inequivalent ways: one permits only elements of infinite order to be in the system, while the other makes no such restriction, and as a result, it is useful for torsion and mixed groups as well. With that said, we proceed to introduce the more useful version.

A set $\{a_1, \dots, a_k\}$ of non-zero elements in a group is called **linearly independent**, or briefly, **independent** if

$$n_1a_1 + \cdots + n_ka_k = 0 \;\; (n_i \in \mathbb{Z}) \;\text{ implies }\; n_1a_1 = \cdots = n_ka_k = 0. \tag{3.9}$$

More explicitly, this means that $n_i = 0$ if $o(a_i) = \infty$ and $o(a_i)|n_i$ if $o(a_i)$ is finite. By definition, 0 is not allowed to be in an independent system. An infinite family L of group elements is **(linearly) independent** if every finite subset of L is independent. Thus independence is by definition a property of finite character.

Lemma 4.1. *A subset $L = \{a_i\}_{i\in I}$ $(0 \notin L)$ of a group is independent if and only if*

$$\langle L\rangle = \oplus_{i\in I} \langle a_i\rangle. \tag{3.10}$$

Proof. If L is independent, then the intersection of the cyclic group $\langle a_i\rangle$ with the subgroup generated by $L \setminus \{a_i\}$ is necessarily 0; hence, $\langle L\rangle$ is the direct sum of the $\langle a_i\rangle$ for $i \in I$. Conversely, if (3.10) holds, then a linear combination $n_1a_{i_1} + \dots + n_ka_{i_k} = 0$ (with different $i_1, \dots, i_k \in I$) can hold only in the trivial way: $n_1a_{i_1} = \dots = n_ka_{i_k} = 0$. □

An element $g \in A$ is said to **depend on** a subset L of A if there is a **dependence relation**

$$0 \neq ng = n_1a_1 + \dots + n_ka_k \qquad (n, n_i \in \mathbb{Z}) \tag{3.11}$$

for some elements $a_i \in L$. Thus g depends on L exactly if $\langle g\rangle \cap \langle L\rangle \neq 0$. A subset K depends on L if every element of K depends on L.

Every element a in an independent system can be replaced, without violating independence, by a non-zero multiple ma. Therefore, by replacing elements of finite order by multiples of prime power order, from every independent system we can get one in which each element is either of infinite or of prime power order.

An independent system M in A is **maximal** if there is no independent system in A that properly contains M. Every element $\neq 0$ of A depends on a maximal independent system. By Zorn's Lemma, *every independent system is contained in a maximal one.* Moreover, if the original system contained only elements of infinite or prime power orders, then a maximal one containing it can also be chosen to have this property.

Lemma 4.2. *An independent system is maximal if and only if it generates an essential subgroup.*

Proof. It suffices to observe that a non-zero element $a \in A$ depends on an independent system M if and only if $\langle a\rangle \cap \langle M\rangle \neq 0$. □

Rank of a Group By the **rank** $\mathrm{rk}(A)$ of a group A is meant the cardinal number of a maximal independent system containing only elements of infinite and prime power orders. If we consider only independent systems with elements of infinite order (of orders that are powers of a fixed prime p) which are maximal with respect to this property, then the cardinality of the system is called the **torsion-free rank** $\mathrm{rk}_0(A)$ (**p-rank** $\mathrm{rk}_p(A)$) of A. From the definitions it is evident that the equation

$$\mathrm{rk}(A) = \mathrm{rk}_0(A) + \sum_p \mathrm{rk}_p(A) \tag{3.12}$$

holds with p running over all primes. Obviously, $\mathrm{rk}(A) = 0$ means $A = 0$.

At this point the natural question is: how unique are these various ranks? In order to legitimize them, we need to show:

Theorem 4.3. *The ranks* $\mathrm{rk}(A), \mathrm{rk}_0(A), \mathrm{rk}_p(A)$ *of a group A are invariants of A.*

Proof. It suffices to prove that $\mathrm{rk}_0(A)$ and $\mathrm{rk}_p(A)$ are independent of the choice of the maximal independent system defining them.

It is routine to check that $\mathrm{rk}_0(A) = \mathrm{rk}(A/tA)$. As a consequence, in proving the invariance of $\mathrm{rk}_0(A)$, we may assume without loss of generality that A is a torsion-free group. Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_\ell\}$ be two maximal independent systems in A. Then there are integers m, m_i, n, n_j with $m, n \neq 0$ such that $ma_i = \sum_{j=1}^{\ell} m_{ij} b_j$ and $nb_j = \sum_{i=1}^{k} n_{ji} a_i$. Hence

$$mna_i = \sum_{h=1}^{k} \sum_{j=1}^{\ell} n_{ij} m_{jh} a_h$$

where the corresponding coefficients on both sides must be equal. This means that the product of matrices $\|n_{ij}\| \cdot \|m_{jh}\|$ is a scalar matrix mnE_k (E_k denotes the $k \times k$ identity matrix). This is impossible if $k < \ell$, thus $k \geq \ell$ must hold. For reasons of symmetry, $k = \ell$ follows, i.e. equivalent finite independent systems contain the same number of elements. This tells us that $\mathrm{rk}_0(A)$ is well defined whenever it is finite.

If $\mathrm{rk}_0(A)$ is infinite, then we show that $\mathrm{rk}_0(A) = |A|$ (A is still torsion-free). The inequality $\leq$ is obvious. To prove the converse, we choose a maximal independent system $L = \{a_i\}_{i \in I}$. For every $0 \neq g \in A$, there is $n \in \mathbb{N}$ such that $ng \in \langle L \rangle$, and if $ng = ng'$ ($g' \in A$), then $g = g'$. Hence we conclude that $|A| \leq |L|\aleph_0 = |L|$.

Turning to the ranks $\mathrm{rk}_p(A)$, it is clear that $\mathrm{rk}_p(A) = \mathrm{rk}(T_p)$ where T_p denotes the p-component of $T = tA$. Hence it is enough to verify the claim for p-groups A. Now if $\{a_i\}_{i \in I}$ is a maximal independent system, then so is $\{p^{m_i - 1} a_i\}_{i \in I}$ where $p^{m_i} = o(a_i)$. Therefore, $\mathrm{rk}_p(A)$ is the same as the rank of the socle $s(A)$. The socle is a $\mathbb{Z}/p\mathbb{Z}$-vector space, its dimension is obviously the same as its rank as a group. The uniqueness of the vector space dimension implies the uniqueness of $\mathrm{rk}(s(A)) = \mathrm{rk}_p(A)$. $\square$

There is another important cardinal invariant associated with groups. This is the dimension of the $\mathbb{Z}/p\mathbb{Z}$-vector space $A/(tA + pA)$ which we shall call the **p-corank** of A, and will be denoted as

$$\mathrm{rk}^p(A) = \dim A/(tA + pA).$$

We will see later that this is the rank of the torsion-free part of p-basic subgroups of A.

★ **Notes.** The torsion-free rank of A is often defined as the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes A$ (then the uniqueness of $\mathrm{rk}_0(A)$ follows from that of the vector space dimension). The rank as we use here has been generalized to modules, called *Goldie dimension*.

Exercises

(1) Show that $\mathrm{rk}(\mathbb{Q}) = 1, \mathrm{rk}(\mathbb{Q}/\mathbb{Z}) = \aleph_0$, and $\mathrm{rk}(J_p) = 2^{\aleph_0}$ for each prime p.
(2) Prove that $\mathrm{rk}(A) = 1$ exactly if A is isomorphic to a subgroup of $\mathbb{Q}$ or to a subgroup of $\mathbb{Z}(p^\infty)$ for some prime p.
(3) Let B be a subgroup of A. Prove that: (i) $\mathrm{rk}(B) \leq \mathrm{rk}(A)$; (ii) $\mathrm{rk}(A) \leq \mathrm{rk}(B) + \mathrm{rk}(A/B)$; (iii) $\mathrm{rk}_0(A) = \mathrm{rk}_0(B) + \mathrm{rk}_0(A/B)$.
(4) The non-zero subgroups B_i $(i \in I)$ of A generate their direct sum in A if and only if every subset $L = \{b_i\}_{i \in I}$ with one $b_i \neq 0$ from each B_i is independent.
(5) A group of rank $\kappa \geq \aleph_0$ has 2^κ different subgroups.

5 Direct Sums of Cyclic Groups

The simplest kinds of infinitely generated groups are the direct sums of cyclic groups. These groups admit a satisfactory classification as we shall see below. We will feel fortunate if we are able to prove that certain groups under consideration are direct sums of cyclic groups.

For brevity, a direct sum of cyclic groups will be called a **Σ-cyclic group**.

Kulikov's Theorem A Σ-cyclic p-group contains no elements $\neq 0$ of infinite height. However, the absence of elements of infinite height does not ensure that a p-group is Σ-cyclic. We are looking for criteria under which a p-group is Σ-cyclic.

Theorem 5.1 (Kulikov [1]). *A p-group A is Σ-cyclic if and only if it is the union of a countable ascending chain of subgroups,*

$$A_0 \leq A_1 \leq \cdots \leq A_n \leq \ldots, \tag{3.13}$$

such that the heights of elements $\neq 0$ in A_n (computed in A) are bounded.

Proof. The stated condition is necessary: if A is a Σ-cyclic p-group, then in a decomposition, collect the cyclic summands of the same order p^n, for every n. If we denote their direct sum by B_n, then the subgroups $A_n = B_1 \oplus \cdots \oplus B_n$ $(n < \omega)$ (with bound $n-1$ on the heights) satisfy the stated condition.

For the proof of sufficiency, suppose that the chain (3.13) is as stated. Since we may adjoin the trivial subgroup to the beginning of the chain (3.13) and repeat subgroups without violating the hypothesis, it is clear that there is no loss of generality in assuming that $n-1$ is a bound for the heights in A_n, that is, $A_n \cap p^n A = 0$ for every $n < \omega$.

Accordingly, we consider the set of all chains $0 = C_0 \leq C_1 \leq \cdots \leq C_n \leq \ldots$ of subgroups of A such that

$$A_n \leq C_n \quad \text{and} \quad C_n \cap p^n A = 0 \quad \text{for every } n < \omega.$$

Define the chain of the C_n to be less than or equal to the chain of the B_n if and only if $C_n \leq B_n$ for all $n < \omega$. The set of all such chains in A is non-empty and is easily seen to be inductive, so Zorn's lemma applies to conclude that there exists a chain $0 = G_0 \leq G_1 \leq \cdots \leq G_n \leq \ldots$ that is maximal in the sense defined. Needless to say, $A = \bigcup_{n<\omega} G_n$.

The group G_n contains only elements of order $\leq p^n$, so $G_n \cap p^{n-1}A$ is in the socle of G_n. Select a $\mathbb{Z}/p\mathbb{Z}$-vector space basis L_n of $G_n \cap p^{n-1}A$, and set $L = \cup_{n<\omega} L_n$. For every $c_i \in L$ of $h(c_i) = n_i$ choose an $a_i \in A$ such that $p^{n_i} a_i = c_i$. The claim is that $A' = \langle \ldots, a_i, \ldots \rangle = \oplus_i \langle a_i \rangle$ is equal to A.

First we show that $\langle L \rangle = A[p]$. Since evidently $\langle L_n \rangle = G_n \cap p^{n-1}A$, all the elements $\neq 0$ in $\langle L_n \rangle$ are of height exactly $n-1$, so the $\langle L_n \rangle$ generate their direct sum, $\langle L \rangle = \oplus_{n<\omega} \langle L_n \rangle$. Assume, as a basis of induction on k, that $G_k[p] = \langle L_1 \rangle \oplus \cdots \oplus \langle L_k \rangle$. Let $a \in G_{k+1}[p] \setminus G_k$. By maximality, $\langle G_k, a \rangle \cap p^k A \neq 0$, thus $0 \neq g + ra = b \in p^k A$ with some $g \in G_k, r \in \mathbb{Z}$, where $r = 1$ may be assumed. Therefore, $g + a \in G_{k+1} \cap p^k A = \langle L_{k+1} \rangle$, thus a and hence $G_{k+1}[p]$ is contained in $\langle L_1 \rangle \oplus \cdots \oplus \langle L_{k+1} \rangle$. Consequently, $\langle L \rangle = A[p]$ follows.

Assume that, for some $n \in \mathbb{N}$, we have proved that every element of A of order $\leq p^n$ belongs to A'; for $n = 1$, this was done in the preceding paragraph. Pick an $a \in A$ of order p^{n+1} $(n \geq 1)$. Then $p^n a \in \langle L \rangle$, so we have $p^n a = m_1 c_1 + \cdots + m_\ell c_\ell$ with some $c_j \in L, m_j \in \mathbb{Z}$. Let $c_1, \ldots, c_r$ be of height $\geq n$, and $c_{r+1}, \ldots, c_\ell$ of height $< n$. Then in the equation

$$p^n a - m_1 c_1 - \cdots - m_r c_r = m_{r+1} c_{r+1} + \cdots + m_\ell c_\ell$$

the left-hand side is of height $\geq n$, while the right-hand side is contained in G_{n-1}; so both sides are 0. If we write $m_j c_j = p^n m_j' a_j$ $(j \leq r)$, then $a - m_1' a_1 - \cdots - m_r' a_r$ is of order $\leq p^n$, so it is contained in A' by induction hypothesis. Hence $a \in A'$ as well. $\square$

Prüfer's Theorems As corollaries we obtain the following two important, frequently quoted results.

Theorem 5.2 (Prüfer [1], Baer [1]). *A bounded group is* Σ*-cyclic.*

Proof. If A is bounded, then it can have but a finite number of non-zero p-components A_p. These components are also bounded, so we can apply Theorem 5.1 with all subgroups in (3.13) equal A_p, to conclude that each A_p is Σ-cyclic. □

Theorem 5.3 (Prüfer [1]). *A countable* p*-group is* Σ*-cyclic if and only if it contains no elements* $\neq 0$ *of infinite height.*

Proof. Only the 'if' part requires a verification. Suppose A is a countable p-group without elements of infinite height. If $\{a_0, \dots, a_n, \dots\}$ is a generating set of A, then A is the union of its finite subgroups $A_n = \langle a_0, \dots, a_n\rangle$ $(n < \omega)$, where the heights of the elements are obviously bounded. The claim follows from Theorem 5.1. □

The following example shows that countability is an essential hypothesis in Theorem 5.3.

Example 5.4 (Kurosh). Let A be the torsion part of the direct product of the cyclic groups $\mathbb{Z}(p), \dots, \mathbb{Z}(p^n), \dots$. Then A is a p-group of the power of the continuum, without elements of infinite height (by the way, each $\mathbb{Z}(p^n)$ is a summand of A). Assume, by way of contradiction, that A is Σ-cyclic, say, $A = \oplus_{n<\omega} B_n$ where B_n is a direct sum of cyclic groups of fixed order p^n. Consider the socles $S_n = \oplus_{n \le i<\omega} p^{i-1}B_i$; they form, with increasing n, an infinite properly descending chain such that S_n consists of those elements of $A[p]$ which are of heights $\geq n-1$. Clearly,

$$a = (c_1, \dots, c_n, \dots) \in A[p] \qquad (c_n \in \mathbb{Z}(p^n))$$

is of height $\geq n-1$ if and only if $c_1 = \dots = c_{n-1} = 0$. This shows that each factor group S_n/S_{n+1} $(n = 1, 2, \dots)$ is of order p. Hence $B_n[p] \cong S_n/S_{n+1}$ implies that the B_n are finite, and so A is countable, a contradiction. An $\aleph_1$-generated pure subgroup of A containing the direct sum $\oplus_n \mathbb{Z}(n)$ yields an example of smallest cardinality.

A quicker counterexample is available if we make use of the isomorphism of basic subgroups: no uncountable p-group with countable basic subgroup is Σ-cyclic.

Kulikov's criterion can be generalized to arbitrary cardinalities as follows (we make use of purity which will be discussed in Chapter 5).

Theorem 5.5 (Hill [13]). *A* p*-group* A *is* Σ*-cyclic if it is the union of an ascending chain* (3.13) *of* Σ*-cyclic pure subgroups* A_n $(n < \omega)$.

Proof. For countable A, sufficiency is easy: list the generators in a sequence: $a_1, a_2, \dots, a_i, \dots$. If $0 = B_0 < B_1 \leq \dots \leq B_i$ is a chain of finite pure subgroups of A such that $a_1, \dots, a_j \in B_j$ for all $j \leq i$, then choose a finite summand B_{i+1} of an A_n containing both B_i and a_{i+1}; such an n must exist. Then B_i as a bounded pure subgroup is a summand of A, and $A = \cup_{i<\omega} B_i$. Since $B_{i+1} = B_i \oplus C_i$ for some $C_i \leq A$, we get $A = \oplus_{i\in\mathbb{N}} C_i$.

The proof for the uncountable case is the exact analog of Theorem 7.5; we leave the details to the reader. □

Isomorphy of Decompositions Though a group may have several decompositions into a direct sum of cyclic groups, one can establish a strong uniqueness statement, just as in the finitely generated case. (Actually, one can prove more: the Krull-Schmidt property holds for Σ-cyclic groups.)

Theorem 5.6. *Any two direct decompositions of a group into direct sums of infinite cyclic groups and cyclic groups of prime power orders are isomorphic.*

Proof. First assume that A is a p-group. Collecting the cyclic summands of the same order, we get a decomposition $A = \oplus_{n<\omega} B_n$ where B_n is a direct sum of cyclic groups of the same order p^n. As in Example 5.4, we can argue that $B_n[p] \cong S_n/S_{n+1}$ where S_n is the set of elements of heights $\geq n-1$ in $A[p]$. The latter group is independent of the representation of A as direct sum of cyclic p-groups, and the dimension of S_n/S_{n+1} as a $\mathbb{Z}/p\mathbb{Z}$-vector space determines the number of cyclic summands of order p^n in any decomposition of A as a Σ-cyclic group.

In the general case, $A = B \oplus C$ where B is a Σ-cyclic torsion group and C is a free group. Then both B and C have unique decompositions (rk C being well defined), so the same holds for A. □

Subgroups of Σ-Cyclic Groups It is extremely important and most useful that the property of being Σ-cyclic is inherited by subgroups.

Theorem 5.7 (Kulikov [2]). *Subgroups of Σ-cyclic groups are again Σ-cyclic.*

Proof. First we dispose of the case when the group A is a p-group. By Theorem 5.1, A is the union of an ascending chain $A_0 \leq A_1 \leq \cdots \leq A_n \leq \ldots$ of subgroups, where the heights of elements of A_n are bounded, say, k_n is a bound in A_n. A subgroup B is the union of the chain

$$A_0 \cap B \leq A_1 \cap B \leq \cdots \leq A_n \cap B \leq \ldots$$

where the heights of elements of $A_n \cap B$, computed in B, do not exceed k_n. By virtue of Theorem 5.1, B is Σ-cyclic.

Turning to the general case, let A be an arbitrary Σ-cyclic group, and B a subgroup of A. Clearly, $tB = B \cap tA$, and so

$$B/tB = B/(B \cap tA) \cong (B + tA)/tA \leq A/tA,$$

where A/tA is a free group. By Theorem 1.6, B/tB is free, whence Theorem 1.5 implies that $B = tB \oplus C$ for some free subgroup C of B. By what has been shown in the preceding paragraph, tB is a direct sum of cyclic p-groups. Thus B is Σ-cyclic. □

Corollary 5.8 (Kulikov [2]). *Any two direct decompositions of a Σ-cyclic group have isomorphic refinements.*

Proof. In view of Theorem 5.7, each summand is Σ-cyclic. Replacing each summand by a direct sum of cyclic groups of orders ∞ or prime power, we arrive at refinements that are isomorphic, as is guaranteed by Theorem 5.6. □

The next lemma provides information about pure subgroups in free groups.

Lemma 5.9. (a) *A finite rank pure subgroup of a free group is a summand.*
(b) (Erdős [1]) *A pure subgroup H of a free group F contains a summand of F whose rank is the same as the rank of H.*

Proof. (a) A finite rank pure subgroup H is contained in a finitely generated summand of the free group F. Then F/H is finitely generated and torsion-free, so a free group. Therefore, H is a summand of F.
(b) If H is of finite rank, then it is a summand of F, and we are done. So assume that H is of infinite rank κ. Let $B = \{b_\alpha \mid \alpha < \kappa\}$ be a basis of F, and consider finite subsets B_i of B such that $\langle B_i \rangle \cap H \neq 0$. Select a maximal pairwise disjoint set S of such subsets B_i, and a non-zero h_i in each $\langle B_i \rangle \cap H$. Then the pure subgroup $\langle h_i \rangle_\star$ is a summand of $\langle B_i \rangle$, and hence $K = \oplus \langle h_i \rangle_\star$ is a summand of F, and so of H. Write $F = \langle S \rangle \oplus G$ where G is generated by the basis elements not in any member of S. Now $G \cap H \neq 0$ is impossible, because then the basis elements b_α occurring in a linear combination of a non-zero element in this intersection form a finite subset disjoint from every finite subset in S—this contradicts the maximality of S. Therefore, $G \cap H = 0$. Manifestly, the cardinality of the set of all basis elements b_α occurring in members of S is the same as the cardinality of S. Hence $G \cap H = 0$ implies that $\operatorname{rk} K = \operatorname{rk}\langle S \rangle = \operatorname{rk} F/G \geq \operatorname{rk} H = \kappa$.

□

★ **Notes.** Various properties of Σ-cyclic groups have been investigated that are shared by larger classes of groups. The name of *Fuchs-5-group* is used in the literature for a group in which every infinite set is contained in a direct summand of the same cardinality. Trivial examples for such groups are direct sums of countable groups. Hill [8] proved that for every uncountable cardinal κ there exist p-groups with this property that need not be direct sums of countable groups. The existence of non-free $\aleph_1$-separable torsion-free groups shows that not all torsion-free Fuchs-5-groups are direct sums of countable subgroups.

Exercises

(1) For a group A, the following conditions are equivalent: (a) A is elementary; (b) every subgroup of A is a summand; (c) A is torsion with trivial Frattini subgroup; (d) A contains no proper essential subgroup.
(2) The direct product of $\kappa \geq \aleph_0$ copies of the cyclic group $\mathbb{Z}(p^k)$ is a direct sum of 2^κ copies of $\mathbb{Z}(p^k)$.
(3) Let A, B be Σ-cyclic groups.

(a) $A \oplus A \cong B \oplus B$ implies $A \cong B$.
(b) $A^{(\aleph_0)} \cong B^{(\aleph_0)}$ fails to imply $A \cong B$ even if A, B are finitely generated.

(4) Let A be a countable direct sum of cyclic groups of order p^2, and $B \cong A \oplus \mathbb{Z}(p)$. The isomorphy classes of subgroups (and factor groups) of A are equal to those of B, but $A \ncong B$.

(5) (Dlab)

(a) Let A be a bounded p-group, and $S = \{a_i\}_{i \in I}$ a subset of A such that the cosets $a_i + pA$ $(i \in I)$ generate A/pA. Then S generates A.
(b) Every generating set of a bounded p-group contains a minimal generating set (i.e. no generator can be omitted).

(6) (Szele) Improve on Example 5.4 by exhibiting an example of cardinality $\aleph_1$.
(7) Let $B = \oplus_{k \in \mathbb{N}} \mathbb{Z}(p^k)$.

(a) Every countable p-group is an epimorphic image of B.
(b) Each p-group of infinite cardinality κ is an epic image of $B^{(\kappa)}$.

(8) A is Σ-cyclic if it contains a Σ-cyclic subgroup G such that A/G is bounded.
(9) (Dieudonné [1]) Let G be a p-group that contains a subgroup A such that G/A is Σ-cyclic. Suppose that A is the union of a chain $A_0 \leq A_1 \leq \cdots \leq A_n \leq \ldots$ such that the heights of elements of A_n, computed in G, are bounded. Then G is Σ-cyclic.
(10) Let A, G be p-groups, and assume $C < A$ with Σ-cyclic A/C. If the homomorphism $\phi : C \to G$ does not decrease heights, then it extends to a homomorphism $A \to G$. [Hint: if $p^n a \in C$, there is $g \in G$ with $\phi(p^n a) = p^n g$.]
(11) An **equational class** or **variety** of groups is a class of groups that is closed under isomorphism, the formations of subgroups, epic images, and direct products. Prove that the following is a complete list of equational classes of abelian groups:

(a) the class of all abelian groups;
(b) for every positive integer n, the class of n-bounded abelian groups.

6 Equivalent Presentations

This section is concerned with special kind of presentations. First, Σ-cyclic groups will be considered.

Presentation with Stacked Basis We say that the group A has a **presentation with stacked bases** if there is a short exact sequence $0 \to H \to F \xrightarrow{\phi} A \to 0$ where $F = \oplus_{i \in I} \langle x_i \rangle$ is a free group and $H = \oplus_{i \in I} \langle n_i x_i \rangle$ is a free subgroup with $n_i \geq 0$ (see Theorem 2.6).

An obvious necessary condition for a group to be presented with stacked bases is that it be a Σ-cyclic group. Kaplansky raised the question whether or not every presentation of a Σ-cyclic group is with stacked bases. The affirmative answer was given by Cohen–Gluck [1]. In our treatment we follow closely their argument.

As a first step, we reduce the proof of the theorem to the torsion case. This is one of the rare situations when the discussion for torsion groups cannot be delegated to p-groups.

Lemma 6.1 (Cohen–Gluck [1]). *Let F be a free group and $A = B \oplus C$ any group with a free summand C. Given an epimorphism $\phi\colon F \to A$, F admits a decomposition $F = F_1 \oplus F_2$ such that $\phi(F_1) = B$ and $F_2 \cong C$.*

Proof. Let $\gamma : A \to C$ denote the projection along B. Then $F = F_1 \oplus F_2$ with $F_1 = \operatorname{Ker} \gamma\phi$ and $F_2 \cong \operatorname{Im} \gamma\phi = C$. The inclusion $B \leq \phi F_1$ cannot be proper. □

Next, we reduce the proof to the countable case; this is a main step, supported primarily by a straightforward back-and-forth argument.

Lemma 6.2 (Cohen–Gluck [1]). *Suppose $0 \to H \to F \xrightarrow{\phi} A \to 0$ is an exact sequence, and both F and A are direct sums of countable groups. Then there exist ('matching') direct decompositions*

$$F = \oplus_{\sigma<\tau} F_\sigma \qquad \textit{and} \qquad A = \oplus_{\sigma<\tau} A_\sigma \tag{3.14}$$

for some ordinal τ such that, for each $\sigma < \tau$,

(i) *F_σ is countable; and*
(ii) *$\phi F_\sigma = A_\sigma$.*

Proof. There is nothing to prove if A is countable, so suppose A is uncountable. Let $F = \oplus_{i\in I} G_i$ and $A = \oplus_{j\in J} B_j$ be decompositions with countable summands. For any $k \in I$, there is a countable subset Y_0 of J such that $\phi G_k \leq \oplus_{j\in Y_0} B_j$ and a countable subset X_0 of I such that $\oplus_{j\in Y_0} B_j \leq \phi(\oplus_{i\in X_0} G_i)$. Arguing the same way repeatedly, we obtain countable ascending chains of countable subsets $X_0 \subset \cdots \subset X_n \subset \ldots$ and $Y_0 \subset \cdots \subset Y_n \subset \ldots$ of I and J, respectively, satisfying

$$\oplus_{j\in Y_n} B_j \leq \phi(\oplus_{i\in X_n} G_i) \leq \oplus_{j\in Y_{n+1}} B_j \qquad (n < \omega).$$

If I_0 and J_0 denote the unions of the X_n and the Y_n, respectively, then let $F_0 = \oplus_{i\in I_0} G_i$ and $A_0 = \oplus_{j\in J_0} B_j$. They are clearly countably generated summands of F and A, respectively, such that $\phi F_0 = A_0$.

Assume that we have already found, for some ordinal σ, smooth chains of subsets $I_0 \subset \cdots \subset I_\rho \subset \cdots \subset I_\sigma$ and $J_0 \subset \cdots \subset J_\rho \subset \cdots \subset J_\sigma$ $(\rho \leq \sigma)$ of I and J, respectively, such that for all $\rho+1 \leq \sigma$, the sets $I_{\rho+1} \setminus I_\rho$ and $J_{\rho+1} \setminus J_\rho$ are countable, and the groups $F'_\rho = \oplus_{i\in I_{\rho+1}\setminus I_\rho} G_i$, $A_\rho = \oplus_{j\in J_{\rho+1}\setminus J_\rho} B_j$ satisfy $\phi(\oplus_{\rho<\sigma} F'_\rho) = \oplus_{\rho<\sigma} A_\rho$. Using a back-and-forth argument, we adjoin to I_σ and J_σ countable subsets U and V, respectively, such that putting $I_{\sigma+1} = I_\sigma \cup U$ and $J_{\sigma+1} = J_\sigma \cup V$, condition (ii) will be satisfied for $F'_{\sigma+1} = \oplus_{i\in U} G_i$, $A_{\sigma+1} = \oplus_{j\in V} B_j$. We repeat this argument transfinitely until the index sets I and J are exhausted, where—as usual—at limit ordinals we take unions of the previously selected subsets. Finally, we get decompositions satisfying $\phi(\oplus_{\rho<\sigma} F'_\rho) = \oplus_{\rho<\sigma} A_\rho$ for all $\sigma < \tau$.

These are not yet decompositions we are looking for, we still have to modify them to obtain ones satisfying (i)–(ii). Suppose that we have found F_ρ to satisfy $\phi F_\rho = A_\rho$ for all $\rho < \sigma$. Consider the diagram

$$\begin{tikzcd} F'_\sigma \arrow[r, "\phi"] \arrow[d, dashed, "\psi"'] & \oplus_{\rho\le\sigma} A_\rho \arrow[d, "\delta"] \\ \oplus_{\rho<\sigma} F_\rho \arrow[r, "\phi"] & \oplus_{\rho<\sigma} A_\rho \end{tikzcd}$$

where δ denotes the projection with kernel A_σ. Since the map in the bottom row is surjective and F'_σ is a free group, we can find a map ψ making the diagram commute. Clearly, $F_\sigma = \{x - \psi x \mid x \in F'_\sigma\}$ is isomorphic to F'_σ. Furthermore, $\phi(x - \psi x) = \phi x - \phi\psi x = \phi x - \delta\phi x \in A_\sigma$ shows that $\phi F_\sigma \le A_\sigma$. This inclusion cannot be proper, thus $\phi F_\sigma = A_\sigma$. As $\oplus_{\rho<\sigma} F_\rho \oplus F'_\sigma = \oplus_{\rho\le\sigma} F_\rho$, we may replace F'_σ by F_σ for each $\sigma < \tau$ inductively, to obtain $\phi F_\sigma = A_\sigma$ for all $\sigma < \tau$. □

The Torsion Case We are now prepared to tackle the torsion case. The starting point is a preliminary lemma (valid for arbitrary groups).

Lemma 6.3. *Let $F = F_1 \oplus F_2$ be a free group, and $\phi\colon F \to A = A_1 \oplus A_2$ an epimorphism such that $A_1 \le \phi F_1$. Then in the given direct decomposition, F_2 can be replaced by some $G \le F$ satisfying $\phi G \le A_2$.*

Moreover, if F' is a summand of F_2 with $\phi F' \le A_2$, then G can be chosen so as to contain F'.

Proof. Let $\pi\colon A \to A_1$ denote the projection with kernel A_2. The projectivity of F_2 guarantees the existence of ρ making the square

$$\begin{tikzcd} F_2 \arrow[r, dashed, "\rho"] \arrow[d, "\phi"'] & F_1 \arrow[d, "\pi\phi"] \\ A_1 \oplus A_2 \arrow[r, "\pi"] & A_1 \end{tikzcd}$$

commutative. Setting $G = (1 - \rho)F_2$, evidently $\pi\phi G = \pi(\phi - \phi\rho)F_2 = 0$. We conclude that $F_1 \oplus F_2 = F_1 \oplus G$, establishing the first claim. For the rest, it is enough to observe that the map ρ can be chosen so as to act trivially on the summand F'. □

The following lemma is a crucial ingredient in the proof of Theorem 6.5 to guarantee that no generator of F is left out in the successive decompositions. Theorem 6.2 permits us to confine ourselves to countable groups.

Lemma 6.4. *Let* $0 \to H \to F \xrightarrow{\phi} A \to 0$ *be a presentation of a* Σ*-cyclic torsion group* A*,* F *a countable free group. If* A_0 *is a finitely generated summand of* A*, then there are direct decompositions*

$$F = F_1 \oplus F_2 \quad \textit{and} \quad H = (H \cap F_1) \oplus (H \cap F_2)$$

such that

(a) F_1 *is finitely generated and* ϕF_1 *contains* A_0*; and*
(b) $A = \phi F_1 \oplus \phi F_2$.

Proof. Write $A = A_0 \oplus A_0'$ where A_0' is the complement of A_0 in a direct decomposition of A into cyclic groups of prime power orders. Apply Lemma 6.3 to this decomposition to get $F = F_0 \oplus G_2$ with $\phi G_2 \leq A_0'$.

Choose a summand $A_2 \leq \phi G_2$ of A_0', say $A_0' = A_1 \oplus A_2$ with finitely generated A_1. Again by Lemma 6.3, we argue that $\phi F_0 \leq A_0 \oplus A_1$ may be assumed. In this way, we obtain a decomposition $A = A_0 \oplus A_1 \oplus A_2$, where $A_0 \leq \phi F_0$ and $A_2 \leq \phi G_2$. That $A_1 = (A_1 \cap \phi F_0) + (A_1 \cap \phi G_2)$ should be clear. Assuming that the cyclic summands in A are decomposed into their p-components, for any p, either ϕF_0 or ϕG_2 contains an element of A_1 of maximal p-power order; this generates a summand C of A_1. If C is contained in ϕF_0, then write $A_1 = C \oplus B_1$, and with the aid of Lemma 6.3 we can change G_2 to a summand G such that ϕG has trivial projection on $A_0 \oplus C = B_0$, and at the same time replace A_0 by B_0, and A_1 by B_1 to obtain $A = B_0 \oplus B_1 \oplus A_2$.

We continue in a similar fashion, next adjoining a summand of B_1 to A_2, etc. After a finite number of steps, we arrive at a decomposition $F = F_1 \oplus F_2$, satisfying (i) and (ii). □

The Stacked Basis Theorem Equipped with these lemmas, we are well prepared for the proof of the main result. We keep the same notation.

Theorem 6.5 (Cohen–Gluck [1]). *Every presentation of a* Σ*-cyclic group has stacked bases.*

Proof. In view of Lemma 6.1 and 6.2, the proof can be reduced to the case, in which A is a countable Σ-cyclic torsion group. Then F can also be assumed to be countable, say $F = \oplus_{i \in \mathbb{N}} \langle x_i \rangle$. We will be done if we reduce the problem to the finitely generated case, because then a simple reference to Theorem 2.6 will complete the proof.

By the preceding lemma, there is a decomposition $F = F_{11} \oplus F_{12}$ such that F_{11} is finitely generated, $x_0 \in F_{11}$, and $H = (H \cap F_{11}) \oplus (H \cap F_{12})$. Next, F admits a decomposition $F = F_{21} \oplus F_{22}$ where F_{21} is finitely generated, contains F_{11} and x_1, and H splits accordingly. Continuing in the same way, we obtain an ascending chain $F_{11} \leq F_{21} \leq \dots$ of summands of F, for which $H \cap F_{n1}$ is a summand of H. The union of the F_{n1} must be all of F. If we define $A_n (n < \omega)$ *via* $A_0 = 0$, $F_{n1} = F_{n-1,1} \oplus A_n$, and let $B_n = H \cap A_n$, then $F = \oplus_{n<\omega} A_n$ and $H = \oplus_{n<\omega} B_n$ are decompositions into finitely generated summands such that A_n and B_n are stacked. The reduction to the finitely generated case has been accomplished, and the proof is completed. □

Equivalent Presentations of Torsion-Free Groups The last theorem asserts that every presentation of a Σ-cyclic group is equivalent to one with stacked bases in the sense made precise by the following definition.

Let F, F' be free groups, H, H' subgroups such that $F/H \cong F'/H'$. We say that F/H and F'/H' are **equivalent presentations** of $A \cong F/H$ if there is an isomorphism $\xi : F \to F'$ carrying H onto H'.

In general, not much can be said about the situations when two presentations of a group have to be equivalent. However, the case of torsion-free groups provides an interesting, though not so easy positive example.

First of all, note that an obvious necessary condition for the equivalence of two presentations F/H and F'/H' of A is that $\operatorname{rk} F = \operatorname{rk} F'$ and $\operatorname{rk} H = \operatorname{rk} H'$. Our next purpose will be to show that, if A is torsion-free, then the sole condition $\operatorname{rk} H = \operatorname{rk} H'$ will be enough to ensure the equivalence of the presentations F/H and F'/H'.

We require an interesting preliminary lemma.

Lemma 6.6 (Erdős [1]). *Let F be a free group, and H a pure subgroup of F. F has a basis which is a complete set of representatives* mod H *if and only if* $|F/H| = \operatorname{rk} H$.

Proof. If F has such a basis $B = \{b_\alpha \mid \alpha < \kappa\}$ with (only) b_0 contained in H, then by the purity of H, B must be infinite, and obviously $|F/H| = |B| = \operatorname{rk} F$. For each $b_\alpha \in B$ there is a unique $b_\beta \in B$ such that $b_\alpha + b_\beta \in H$. If $b_\alpha = b_\beta$, then $2b_\alpha \in H$, so $b_\alpha = b_0$, by purity. The elements $b_\alpha + b_\beta$ $(b_\alpha \neq b_\beta)$ and b_0 form a basis of a summand of F contained in H. Hence $|B| \le \operatorname{rk} H$, and necessity is established.

Turning to the proof of the sufficiency, suppose $|F/H| = \operatorname{rk} H$. From Lemma 5.9 we derive that H contains a summand G of F such that $\operatorname{rk} G = |F/H|$. Choose a basis $Y = \{y_j\}$ of G, and extend it to a basis $B = \{b_\alpha \mid \alpha < \kappa\}$ of F. Well-order B in such a way that the elements of Y precede the rest of the basis elements in B. Each element $h \in H$ can be written uniquely as a linear combination $h = t_1 b_{\alpha_1} + \cdots + t_s b_{\alpha_s}$ $(t_i \in \mathbb{Z})$ with non-zero terms such that $\alpha_1 < \cdots < \alpha_s$. To simplify our wording, we will say that the ordinal α_s is associated with h. If among the elements $h \in H$ associated with the same α_s there is one, say h', with $|t_s| = 1$, then in the basis B the element b_{α_s} can be replaced by h', without violating the basis character of the set. In doing so for all possible ordinals α_s inductively, Y remains unchanged, and the new basis (which we continue denoting by B) will have the additional property that if $h \in H$ is associated with α_s, and in the expression for h the coefficient of b_{α_s} is 1 in absolute value, then necessarily $b_{\alpha_s} = \pm h \in H$.

Split the basis B into two disjoint subsets, $B = B' \cup B''$, such that $Y \le B' = B \cap H$. We keep B'', but change B' in order to obtain a basis $B^\star$ of F which is a complete set of representatives mod H, as desired.

First, observe that different elements b_α and b_β of B'' must belong to different cosets mod H. Indeed, otherwise $h = b_\alpha - b_\beta \in H$ is associated with either b_α or b_β, so either $b_\alpha \in H$ or $b_\beta \in H$, which is impossible, B'' being disjoint from H. Of course, there are cosets mod H which do not intersect B''. Since $B' \subset H$ implies that each coset mod H is represented by an element of the subgroup $\langle B'' \rangle$, for each coset mod H disjoint from B'' we can choose a representative $x_i \in \langle B'' \rangle$. Thus $B'' \cup X$ (with $X = \{x_i \mid i \in I\}$ for some index set I) is a complete set of representatives mod H.

Next we show that $|X| = |B'|$. On one hand, $\operatorname{rk} H = |F/H| = |Y| \leq |B'| \leq \operatorname{rk} F$ implies $|B'| = \operatorname{rk} F = |F/H|$. On the other hand, let b_γ be the first element of B'' in the chosen well-ordering of B. No two of the elements of the form $b_\alpha - b_\gamma$ $(b_\alpha \in B'')$ belong to the same coset mod H, and none of these is congruent mod H to a $b_\beta \in B''$ (again, otherwise $b_\alpha - b_\gamma - b_\beta \in H$ would be associated with either b_α or b_β, etc.). Thus there are at least $|B''|$ many cosets of H which do not intersect B''; hence, $|B''| \leq |X|$ follows. This together with $|B''| + |X| = \operatorname{rk} F$ yields $|X| = \operatorname{rk} F$. Hence $|B'| = |X|$, so there is a bijection between the set of elements $\{b_i\}$ of B' and the set of cosets $\{x_i + H\}$ (where we have the corresponding elements carrying the same index i). If in the basis B, $b_i \in B'$ will be replaced by $b_i + x_i$, then we obtain a new basis $B^\star$ of F which is at the same time a complete set of representatives mod H. $\square$

We are now able to verify the main result mentioned earlier.

Theorem 6.7 (Erdős [1]). *Two presentations, F/H and F'/H', of a torsion-free group are equivalent if and only if* $\operatorname{rk} H = \operatorname{rk} H'$.

Proof. To verify sufficiency, suppose $\operatorname{rk} H = \operatorname{rk} H'$; as noted above, this implies $\operatorname{rk} F = \operatorname{rk} F'$. We prove more than stated, viz. we show that every isomorphism $\psi : F/H \to F'/H'$ is induced by an isomorphism $\phi : F \to F'$ carrying H onto H'.

Since A is torsion-free, both H and H' are pure. Ignoring the trivial case, we may suppose that $\operatorname{rk} H$ is infinite. We distinguish three cases.

Case I: $\operatorname{rk} H = |A|$. Then the same is true for $\operatorname{rk} H'$. In view of Lemma 6.6, there exist a basis B of F and a basis B' of F' which are complete sets of representatives mod H and mod H', respectively. The correspondence $B \to B'$ which maps $b \in B$ upon $b' \in B'$ if and only if ψ maps the coset $b + H$ upon $b' + H'$ extends uniquely to an isomorphism $\phi : F \to F'$ under which H' is the image of H. Thus the two presentations are equivalent.

Case II: $\operatorname{rk} H > |A|$. Let G be a free group whose rank is $\operatorname{rk} H$. Replace F by $F \oplus G$ and F' by $F' \oplus G$, but keep H and H'. Application of Case I to $A \oplus G$ implies the existence of an isomorphism $\phi : F \oplus G \to F' \oplus G$ with $\phi H = H'$ inducing ψ. It is self-evident that $\phi F = F'$.

Case III: $\operatorname{rk} H < |A|$. There is a decomposition $F = F_1 \oplus F_2$ such that $H \leq F_1$ and $\operatorname{rk} H = \operatorname{rk} F_1 < \operatorname{rk} F_2 = |A|$. Thus $A = F_1/H \oplus F_2$, and ψ yields a similar decomposition $A' = F_1'/H' \oplus F_2'$. Case I guarantees the existence of an isomorphism $F_1 \to F_1'$ mapping H upon H'; this along with $F_2 \to F_2'$ (restriction of ψ) yields an isomorphism $\phi : F \to F'$. $\square$

★ **Notes.** Hill–Megibben [4] furnished another proof of Theorem 6.5 as a corollary to a more general result which they proved on equivalent presentations of arbitrary abelian groups. F/H and F'/H' are equivalent presentations if and only if, for each prime p,

$$\dim(H + pF)/pF = \dim(H' + pF')/pF'.$$

For Dedekind domain, A.I. Generalov and M.V. Zheludev [St. Petersburg Math. J. **7**, 619–661 (1996)] characterized equivalent presentations. No such study is available for larger classes of domains, but several special cases have been investigated.

Various generalizations of the stacked basis theorem may be found in the literature. Let us mention Ould-Beddi–Strüngmann [1] where homogeneous completely decomposable groups are considered. Osofsky [1] studied a kind of dual to the stacked basis theorem. She proved that if H is a subgroup of a free group F such that F/H is p^n-bounded, then for every decomposition $F/H = \oplus C_i$ with cyclic groups C_i there is a decomposition $F = \oplus F_i$ such that $C_i = F_i/(H \cap F_i)$.

Cutler–Irwin–Pfaendtner–Snabb [1] have a nice generalization of Lemma 6.6. They show that a pure subgroup H in a Σ-cyclic group G contains a summand K of G such that $\mathrm{rk}_0(K) = \mathrm{rk}_0(H)$ and $\mathrm{rk}_p(K) = \mathrm{rk}_p(H)$ for each p. See Lemma 6.12 in Chapter 5, for the torsion case.

Exercises

(1) (Erdős) Let H be a subgroup of a group G such that G/H is torsion-free. There is a generating system of G which is a complete set of representatives mod H if and only if $|H| \leq |G/H|$. [Hint: Lemma 6.2 with a presentation of G.]

(2) (Hill–Megibben) If $A = F/H$ is a presentation of an infinite group such that F is free and $\mathrm{rk}\, F > |A|$, then there is a direct decomposition $F = F_1 \oplus F_2$ such that $\mathrm{rk}\, F_1 = |A|$ and $F_2 \leq H$.

(3) Let $H_0 < \cdots < H_n < \ldots$ be a countable ascending chain of summands of a free group F.

   (a) The union $H = \cup_{n<\omega} H_n$ need not be a summand of F.
   (b) H contains a summand of F whose rank is $\sum_{n<\omega} \mathrm{rk}(H_n)$. [Hint: H is pure in F, and apply Exercise 1.]

(4) (Erdős) Let $A = \oplus_{i \in I} A_i$ be a direct sum of torsion-free groups. If F is a free group and $\phi : F \to A$ is an epimorphism, then there is a decomposition $F = \oplus_{i \in I} F_i$ such that $\phi : F_i \to A_i$ for each $i \in I$. [Hint: represent $A_i = F_i'/H_i'$ such that $\sum_{i \in I} \mathrm{rk}(H_i') \leq \mathrm{rk}(\mathrm{Ker}\, \phi)$, and apply Lemma 6.6.]

7 Chains of Free Groups

We are looking for criteria for a group to be free, especially when the union of a chain of free subgroups is again free. In this section and in the next one, we have to use frequently purity to be discussed in Chapter 5.

Pontryagin's Criterion In a few cases useful criteria for freeness can be established. The one which is most often used works for countable torsion-free groups.

Theorem 7.1 (Pontryagin [1]). *A countable torsion-free group is free if and only if each of its finite rank subgroups is free. Equivalently, for every $n \in \mathbb{N}$, the subgroups of rank $\leq n$ satisfy the maximum condition.*

Proof. Because of Theorem 1.6, necessity is evident. For sufficiency, let $A = \langle a_0, \dots, a_n, \dots \rangle$ be a countable torsion-free group all of whose subgroups of finite rank are free. Define $A_0 = 0, A_n = \langle a_0, \dots, a_{n-1} \rangle_*$ ($n \in \mathbb{N}$) (the purification of $\langle a_0, \dots, a_{n-1} \rangle$ in A). Then $\operatorname{rk} A_n \leq n$ and $\operatorname{rk} A_{n+1} \leq \operatorname{rk} A_n + 1$. Therefore, either A is of finite rank—in which case there is nothing to prove—or there is a subsequence B_n of the A_n, such that $\operatorname{rk} B_n = n$, and A is the union of the strictly ascending chain $0 = B_0 < B_1 < \dots < B_n < \dots$. Now B_{n+1}/B_n is torsion-free of rank 1 and finitely generated, thus $B_{n+1}/B_n \cong \mathbb{Z}$. From Theorem 1.5 we obtain $B_{n+1} = B_n \oplus \langle b_n \rangle$ for some $b_n \in A$. This shows that the elements $b_0, b_1, \dots, b_n, \dots$ generate the direct sum $\oplus_{n<\omega} \langle b_n \rangle$, whence $A = \oplus_{n<\omega} \langle b_n \rangle$ is immediate.

By Theorem 1.6, the second formulation is equivalent to the first one. □

Corollary 7.2. *Suppose $0 = G_0 < G_1 < \dots < G_n < \dots$ is a chain of countable free groups such that each G_n is pure in the union G of the chain. Then G is free.*

Proof. A finite rank subgroup of G is contained in some G_n, so it is free. The claim is immediate from Theorem 7.1. □

If we have a chain like in Corollary 7.2 with the G_n as summands in a larger group F, the union G need not be a summand of F.

Example 7.3. Let G be a free group that is the union of a countable chain of infinite rank summands $G_0 < G_1 < \dots < G_i < \dots$. Our claim is that there exists a countable free group F containing G such that each G_i is, but G is not a summand of F.

Let $0 \to H \to F' \to \mathbb{Q} \to 0$ be a presentation of $\mathbb{Q}$ with countable free F'. Let H_n ($n < \omega$) be a chain of finite rank summands of the free group H with union H. Then F'/H_n is free for all $n < \omega$. Next, pick free groups $F_0 \cong G_0$ and $F_i \cong G_i/G_{i-1}$ ($i \geq 1$). It is evident that

$$G \cong H \oplus \oplus_{i<\omega} F_i \quad \text{and} \quad G_n \cong H_n \oplus \oplus_{n \leq i < \omega} F_i \quad (n < \omega).$$

Finally, we embed G in a free group $F \cong F' \oplus \oplus_{i<\omega} F_i$ imitating the embedding of H in F' and keeping the G_i fixed. This F is as desired.

The Eklof–Shelah Criterion The following lemmas provide us with versatile criteria for a group to be free.

Lemma 7.4. *Let, for some ordinal τ,*

$$0 = A_0 < A_1 < \dots < A_\sigma < \dots \quad (\sigma < \tau) \tag{3.15}$$

be a smooth chain of pure subgroups of a group A such that $A = \bigcup_{\sigma<\tau} A_\sigma$. If, for each $\sigma < \tau$, the factor group $A_{\sigma+1}/A_\sigma$ is free, then A is free.

Proof. In view of the stated condition, $A_{\sigma+1} = A_\sigma \oplus B_\sigma$ for each $\sigma < \tau$, for a suitable subgroup B_σ of $A_{\sigma+1}$ Theorem 1.5. If X_σ denotes a basis of B_σ, then the set union $X = \bigcup_{\sigma<\tau} X_\sigma$ is a basis for A. □

We can now verify the Eklof–Shelah criterion which provides a necessary and sufficient condition for freeness.

Theorem 7.5 (Eklof–Shelah). *Let κ be an uncountable regular cardinal, and assume $0 = A_0 < A_1 < \cdots < A_\sigma < \ldots (\sigma < \kappa)$ is a smooth chain of pure subgroups of a group A such that*

(i) *all the A_σ are free groups of cardinality $< \kappa$, and*
(ii) $A = \bigcup_{\sigma<\kappa} A_\sigma$.

Then A is free if and only if the set

$$E = \{\sigma < \kappa \mid \exists\, \rho > \sigma \text{ such that } A_\rho/A_\sigma \text{ is not free}\}$$

is not stationary in κ.

Proof. Suppose A is free. Consider a filtration $\{B_\sigma\}_{\sigma<\kappa}$ of A with summands. The set C of indices σ of those subgroups A_σ which appear in the filtration $\{B_\sigma\}_{\sigma<\kappa}$ is a cub in κ, so $\{A_\sigma\}_{\sigma\in C}$ provides a filtration of A with summands. We see that A/A_σ is free for all $\sigma \in C$, and so C does not intersect the set E. This proves that E is not stationary.

Conversely, assume that E is not stationary. Then there is a cub $C \subset \kappa$ which does not intersect E. Evidently, $\{A_\sigma\}_{\sigma\in C}$ is still a filtration of A. Relabeling, we have a filtration $\{A_\sigma\}_{\sigma<\kappa}$ where all factor groups $A_{\sigma+1}/A_\sigma$ are free. By Lemma 7.4, A is free. □

Remark. For future applications we point out that both Lemma 7.4 and Theorem 7.5 hold for p-groups A_σ if 'free' is replaced throughout by 'Σ-cyclic.' The proofs are the same with obvious changes.

The next lemma teaches us how to create from a short chain of direct sums with large factor groups a long chain with small factor groups. (The main interest is in the torsion-free case, but no such restriction is needed.)

Lemma 7.6. *Assume*

$$0 = G_0 < G_1 < \cdots < G_n < \ldots$$

is a chain of groups that are pure in the union $G = \cup_{n<\omega} G_n$, where each G_n is a direct sum of countable groups. Then there is a smooth chain

$$0 = A_0 < A_1 < \cdots < A_\sigma < \ldots \quad (\sigma < \tau) \tag{3.16}$$

of pure subgroups A_σ of G such that

(i) *$A_\sigma \cap G_n$ is a summand of G_n, for every $n < \omega$ and $\sigma < \tau$; and*
(ii) *$A_{\sigma+1}/A_\sigma$ $(\sigma + 1 < \tau)$ is countable and the union of an ascending chain of pure subgroups, isomorphic to summands of G_n $(n < \omega)$.*

Proof. We start choosing a fixed direct decomposition of each G_n into countable summands, and define an $H(\aleph_0)$-family $\mathcal{H}_n$ of summands of G_n to consist of all direct sums of subsets of components in the chosen decomposition. We break the proof into three steps.

Step 1. The collection

$$\mathcal{G}_n = \{A \in \mathcal{H}_n \mid A + G_i \textit{ is pure in } G_n \textit{ for each } i < n\}$$

is a $G(\aleph_0)$*-family of subgroups of* G_n.

All that we have to check is that $\mathcal{G}_n$ satisfies the countability condition for a $G(\aleph_0)$-family, since the other conditions are obvious. Let $A \in \mathcal{G}_n$, and H_0 a countable subgroup of G_n. Suppose that we already have a chain $A = B_0 < B_1 < \cdots < B_m$ of subgroups in $\mathcal{H}_n$ such that

1. $A + H_0 \leq B_1$;
2. B_{j+1}/B_j is countable for all $j < m$; and in addition,
3. for each $j < m$ and for each $i < n$, $(B_{j+1} + G_i)/(A + G_i)$ contains a purification of $(B_j + G_i)/(A + G_i)$ in $G_n/(A + G_i)$.

To find a next member B_{m+1} of the chain, for each $i < n$, let $V_i \subset G_n$ be a countable set that—along with $(B_m + G_i)/(A + G_i)$—generates a pure subgroup in $G_n/(A + G_i)$. Thus $H_{m+1} = \bigcup_{i<n} V_i$ is likewise a countable set. Consequently, there is a $B_{m+1} \in \mathcal{H}_n$ such that $B_m + H_{m+1} \subset B_{m+1}$ and B_{m+1}/B_m is countable. Then, for each $i < n$, $(B_{m+1} + G_i)/(A + G_i)$ contains the purification of $(B_m + G_i)/(A + G_i)$ in $G_n/(A + G_i)$. The union B of the chain of the B_m for all $m < \omega$ is a member of $\mathcal{H}_n$, B/A is evidently countable, and our construction guarantees that $(B + G_i)/(A + G_i)$ is pure in $G_n/(A + G_i)$. Thus $B + G_i$ is pure in G_n, i.e. $B \in \mathcal{G}_n$.

Step 2. The family

$$\mathcal{B} = \{A \leq G \mid A \cap G_n \in \mathcal{G}_n \textit{ for each } n < \omega\}$$

is a $G(\aleph_0)$*-family of subgroups in* G.

Again, only the countability condition requires a proof. Since there are but countably many indices n to deal with, a similar back-and-forth argument (ω times) suffices to ensure that for each $A \in \mathcal{B}$, there exists an $A' \in \mathcal{B}$ such that A'/A is countable, as needed.

Step 3. At this point we know that $\mathcal{B}$ is a $G(\aleph_0)$-family satisfying (i), so we can extract a smooth chain (3.16) of pure subgroups with countable factor groups. Evidently, the group $A_{\sigma+1}/A_\sigma$ is the union of the ascending chain of groups $[A_\sigma + (G_n \cap A_{\sigma+1})]/A_\sigma \cong (G_n \cap A_{\sigma+1})/(G_n \cap A_\sigma)$ $(n < \omega)$, all summands of G_n in the chosen direct decomposition. □

Hill's Criterion The following result is a far-reaching generalization of Pontryagin's theorem.

Theorem 7.7 (Hill [9]). *The union G of a countable ascending chain*

$$0 = G_0 < G_1 < \cdots < G_n < \dots$$

of pure subgroups, each of which is free, is a free group.

Proof. The given chain can be replaced by a chain of the A_σ as stated in Lemma 7.6. Apply Corollary 7.2 to the factor groups $A_{\sigma+1}/A_\sigma$ to conclude that they are free. A simple reference to Theorem 7.5 completes the proof. □

It should be pointed out that this theorem fails to hold for longer chains, as is shown by Theorem 8.6 below.

Before we go on, we would like to mention an important consequence of Hill's theorem. This is a special case of Shelah's compactness theorem 9.2 for limit ordinals cofinal with ω.

Corollary 7.8 (Hill [21]). *Suppose λ is an infinite cardinal whose cofinality is ω. A group of cardinality λ is free provided that all of its subgroups of cardinalities $< \lambda$ are free.*

Proof. Evidently, a group of cardinality λ is the union of a countable ascending chain of pure subgroups whose cardinalities are $< \lambda$. By hypothesis, each of these is free, so the claim follows right away from Theorem 7.7. □

Another criterion worthwhile recording is the following. (Observe the difference between Theorem 7.5 and Lemma 7.9.)

Lemma 7.9 (Eklof [5]). *Let κ be a regular cardinal, and*

$$0 = A_0 < A_1 < \cdots < A_\sigma < \cdots < A_\kappa = A \tag{3.17}$$

a smooth chain of free groups such that A_σ/A_ρ is free whenever ρ is a successor ordinal and $\rho < \sigma < \kappa$. If the set

$$E = \{\lambda < \kappa \mid \lambda \textit{ limit ordinal}, \ A_{\lambda+1}/A_\lambda \textit{ not free}\}$$

is not stationary in κ, then A is a free group, and so is A/A_ρ for every successor ordinal ρ.

Proof. Suppose E is not stationary, i.e. there is a cub $C \subset \kappa$ that does not intersect E. Those A_σ whose indices belong to C form a chain like (3.17); we may assume that (3.17) is this subchain. In this chain, $A_{\sigma+1}/A_\sigma$ is free for all $\sigma < \kappa$, so Theorem 7.5 implies that A is free. The second claim follows by applying the result to A/A_ρ. □

When Torsion-Free has to Be Free A rather remarkable feature of free groups was discovered by Griffith [5]. A torsion-free group containing a free subgroup with bounded factor group is easily seen to be again free, and interestingly, the same conclusion can be reached under much weaker conditions on the factor group. The following theorem is a slightly modified version of Griffith's theorem.

Theorem 7.10. *Let A be a torsion-free group, and F a free subgroup of A. If A/F is a p-group that admits an $H(\aleph_0)$-family of subgroups such that all the factor groups in $H(\aleph_0)$ are reduced, then A is free* (*and* $\cong F$).

Proof. If A is of finite rank, then hypothesis implies A/F is a reduced p-group of finite rank, so it is finite (cp. Theorem 5.3). As a finitely generated torsion-free group, A is free.

Next assume F is of countable rank, and write $F = \oplus_{n<\omega} Z_n$ with $Z_n \cong \mathbb{Z}$. For $n < \omega$, set $F_n = \oplus_{i\leq n} Z_i$ and $A_n = \langle F_n \rangle_*$. Manifestly, $A_n/F_n = A_n/(A_n \cap F) \cong (A_n+F)/F \leq A/F$, which shows that A_n/F_n is reduced, and hence it must be a finite p-group. Therefore, A_n is free of finite rank by the preceding paragraph. Hence A is the union of a countable ascending chain $\{A_n\}_{n<\omega}$ of pure free subgroups, and by Corollary 7.2 we conclude that A itself is free.

Turning to the uncountable case, set $F = \oplus_{i\in I} Z_i$ with $Z_i \cong \mathbb{Z}$, and for the p-group $T = A/F$, select an $H(\aleph_0)$-family $\mathcal{H}$ as stated. As T is reduced, and A is torsion-free, every non-zero element of A is divisible but by a finite number of integers. We are going to define by transfinite induction a smooth chain $\varnothing = I_0 \subset I_1 \subset \cdots \subset I_\sigma \subset \cdots \subset I_\tau = I$ of subsets of I such that for all $\sigma < \tau$, we have

(a) $|I_{\sigma+1} \setminus I_\sigma| \leq \aleph_0$, and
(b) $(A_\sigma + F)/F$ (which is $\cong A_\sigma/F_\sigma$) is a subgroup $T_\sigma \in \mathcal{H}$, where $F_\sigma = \oplus_{i\in I_\sigma} Z_i$ and $A_\sigma = \langle F_\sigma \rangle_*$.

Suppose σ is an ordinal such that, for all $\rho < \sigma$, the subsets I_ρ have been selected as required. If σ is a limit ordinal, then we set $I_\sigma = \cup_{\rho<\sigma} I_\rho$, as is forced by continuity. In this case, (b) will be satisfied, since $\mathcal{H}$ is closed under unions. If $\sigma = \rho + 1$ and $I_\rho \neq I$, pick any $i \in I \setminus I_\rho$. Note that $(A_\rho + F)/F = T_\rho$ is a subgroup of countable index in C_1/F where $C_1 = \langle A_\rho + Z_i \rangle_* + F$, so there is a subgroup $B_1 \leq A$ such that $C_1/F \leq B_1/F \in \mathcal{H}$ and $|B_1/C_1| \leq \aleph_0$. There is a countable subset $J_1 \subseteq I \setminus I_\rho$ for which $\langle A_\rho \oplus \oplus_{j\in J_1} Z_j \rangle_* + F$ contains B_1. We keep repeating this process, to define an ascending chain of countable subsets J_n of $I \setminus I_\rho$, along with subgroups C_n and B_n of A (for $n < \omega$) such that $C_1 \leq B_1 \leq C_2 \leq B_2 \leq \ldots$. If we set

$$I_\sigma = I_\rho \cup \cup_{n<\omega} J_n, \quad F_\sigma = \oplus_{j\in I_\sigma} Z_j, \quad \text{and} \quad A_\sigma = \langle F_\sigma \rangle_*,$$

then $\cup_{n<\omega} C_n/F = \cup_{n<\omega} B_n/F$ will be a subgroup $T_\sigma \in \mathcal{H}$, and (a)-(b) will be satisfied for this σ. The factor group A_σ/A_ρ is torsion-free and countable; it contains $(A_\rho + F_\sigma)/A_\rho$ as a free subgroup such that the factor group is isomorphic to

$$A_\sigma/(A_\rho + F_\sigma) = A_\sigma/[A_\sigma \cap (A_\rho + F)] \cong (A_\sigma + F)/(A_\rho + F) \cong T_\sigma/T_\rho$$

which is a countable reduced p-group. Therefore, we can apply the countable case to derive that A_σ/A_ρ is a free group. Hence the chain of the A_σ $(\sigma < \tau)$ has free factor groups, thus their union A is a free group. □

Example 7.11. Reduced totally projective p-groups (Sect. 6 in Chapter 11) admit an $H(\aleph_0)$-family as stated in the theorem. However, no uncountable p-group with countable basic subgroups has such a family.

The Summand Intersection Property We say that a group A has the **summand intersection property** if the intersection of two summands in A is likewise a summand of A. If the same holds for infinite intersections as well, then we refer to it as the **strong summand intersection property**. Needless to say, this property is shared by very special groups only. We are looking for free groups with this property.

Proposition 7.12 (Kaplansky [K], Wilson [1]). *All free groups have the summand intersection property. A free group has the strong summand intersection property if and only if it is countable.*

Proof. Let F be a free group, and $F = B_i \oplus C_i$ $(i = 1, 2)$ direct decompositions. Then $F/(B_1 \cap B_2)$ is isomorphic to a subgroup of the free group $F/B_1 \oplus F/B_2$, so is itself free. Hence $B_1 \cap B_2$ is a summand of F.

If F is a countable free group, then the same argument with countable summands leads to a countable factor group contained in a product of countable free groups F/B_i. Theorem 8.2 below implies that this factor group is free, so the intersection of countably many summands is a summand.

Finally, suppose $|F| \geq \aleph_1$. Let A be a torsion-free, non-free group of cardinality $\aleph_1$, and $\phi \colon F \to A$ a homomorphism that is a bijection between a basis $\{b_\sigma \mid \sigma < \kappa\}$ of F and the elements of A. Select homomorphisms $\phi_i \colon F \to A$ with cyclic images whose kernels C_i are summands of F containing $K = \operatorname{Ker} \phi$. This can be done such that the intersection $\cap_{i \in I} C_i = K$. But K cannot be a summand, since F/K is not free. □

★ **Notes.** In this section, we have collected the most useful results on free groups. They have fascinating features, no wonder that their theory attracted so many researchers. For criteria on the existence of a basis, we refer to Kertész [1], Fuchs [1]. In view of the very useful chain criteria of freeness, basis criteria are hardly used.

The summand intersection property for free groups was observed by Kaplansky [K]. More on this property can be found in Wilson [1], Arnold–Hausen [1], Albrecht–Hausen [1]. Hausen [9] proved that $A^{(I)}$ has the summand intersection property if End A is a PID. This property was investigated by Kamalov [1] for non-free groups, and by Chekhlov [2] for torsion groups.

Exercises

(1) A countable group is Σ-cyclic if and only if every finite set of its elements is contained in a finitely generated direct summand.

(2) (a) A subset $\{a_i\}_{i\in I}$ of a torsion-free group A is a basis of A if and only if it is a minimal generating system such that, for every finite subset $\{a_1, \dots, a_n\}$, if $a \in A$ depends on $\{a_1, \dots, a_n\}$, then $a \in \langle a_1, \dots, a_n \rangle$.
(b) The same with "minimal generating system" replaced by "maximal independent subset."
(3) In any presentation of $\mathbb{Z}^{\aleph_0}$, there are continuously many generators and continuously many relations.
(4) (Danchev) In a p-group A, the $p^{\omega}A$-high subgroups are Σ-cyclic if and only if $A[p]$ is the union of an ascending chain T_n $(n < \omega)$ such that the finite heights in T_n are bounded.
(5) The summand intersection property is inherited by summands.
(6) (Wilson) A torsion group has the summand intersection property if and only if each of its p-components is either cocyclic or elementary.
(7) (Wilson, Hausen) A has the summand intersection property if and only if for every direct decomposition $A = B \oplus C$, the kernel of any map $B \to C$ is a summand of A.

8 Almost Free Groups

Almost free groups are those (necessarily) torsion-free groups in which all subgroups of smaller sizes are free. More precisely, for an infinite cardinal κ, we say that a group A is κ-**free** if every subgroup of A whose rank is $< \kappa$ is free [AG]. The problem of finding the cardinals κ for which there exist κ-free groups that fail to be κ^+-free was raised in [IAG]. As it turns out, it is an intricate problem, requiring sophisticated machinery from set theory. It has been studied extensively, and a significant amount of information has already been gained, but still much remains to be done. Here we aim simply at giving a taste of the subject. The objective is to understand how close almost free groups are to being free.

κ-Free Groups Since purification does not increase rank, it is clear that if A is κ-free, then every subgroup of rank $< \kappa$ is contained in a pure free subgroup of the same rank. Thus the collection $\mathfrak{C}$ of pure free subgroups of rank $< \kappa$ is witness for κ-freeness. In view of Theorem 7.7, $\mathfrak{C}$ is closed under taking unions of countable chains.

Example 8.1. In this new terminology, Pontryagin's theorem 7.1 can be rephrased by saying that a countable $\aleph_0$-free group is free.

(A) *If $\kappa < \lambda$ are infinite cardinals, then λ-free implies κ-free.* In particular, free groups are trivially κ-free for every cardinal κ.
(B) *Subgroups and direct sums of κ-free groups are κ-free.*
(C) *Extension of a κ-free group by a κ-free group is κ-free.* More generally, we have:

(D) *Let* $0 = A_0 < \cdots < A_\sigma < \cdots < A_\tau = A$ *be a smooth chain of groups such that all the factor groups* $A_{\sigma+1}/A_\sigma$ *are* κ*-free. Then* A *is also* κ*-free.* In fact, let X be a pure subgroup of A with $|X| < \kappa$. Then in the smooth chain $X \cap A_\sigma$ $(\sigma < \tau)$ each factor group $(X \cap A_{\sigma+1})/(X \cap A_\sigma)$ is torsion-free of cardinality $< \kappa$, and therefore it is isomorphic to the free subgroup $(A_\sigma + (X \cap A_{\sigma+1}))/A_\sigma \leq A_{\sigma+1}/A_\sigma$. An appeal to Lemma 7.4 completes the proof.

The Baer–Specker Group The next theorem is concerned with a prototype for $\aleph_1$-freeness; actually, the group is of major interest.

Theorem 8.2 (Baer [6], Specker [1]). *The direct product of infinitely many infinite cyclic groups is* $\aleph_1$*-free, but not free.*

Proof. Write $A = \prod_{i \in I} \langle a_i \rangle$, where I is an infinite set, and $\langle a_i \rangle \cong \mathbb{Z}$ for each i. The first step in the proof is to show that every finite subset $\{x_1, \ldots, x_m\} \subset A$ is contained in a finitely generated direct summand of A whose complement is a direct product of infinite cyclic groups.

We induct on m. If $m = 1$ and $x_1 \neq 0$, then $x_1 = (\ldots, n_i a_i, \ldots)$ with $n_i \in \mathbb{Z}$. If there is an index $j \in I$ such that $|n_j| = 1$, then the jth component $\langle a_j \rangle$ in the direct product can be replaced by $\langle x_1 \rangle$, i.e. $A = \langle x_1 \rangle \oplus A_j$, where A_j is the set of elements with vanishing jth coordinate, so it is also a product of infinite cyclic groups. If the minimum n of the $|n_i|$ with $n_i \neq 0$ is greater than 1, then setting $n_i = q_i n + r_i$ with $q_i, r_i \in \mathbb{Z}, 0 \leq r_i < n$, define $y_1 = (\ldots, q_i a_i, \ldots), y_2 = (\ldots, r_i a_i, \ldots) \in A$ so that $x_1 = n y_1 + y_2$. There must be an index $j \in I$ with $|q_j| = 1$ and $r_j = 0$, thus $A = \langle y_1 \rangle \oplus A_j$, where $y_2 \in A_j$ with coefficients $0 \leq r_i < n$. By induction on n, A_j has a finitely generated summand B' containing y_2, and so $\langle y_1 \rangle \oplus B'$ is a finitely generated summand of A containing x_1 such that it has a direct product of infinite cyclic groups as a complement.

Assume that $m > 1$, and $A = B \oplus C$ where B is finitely generated containing $\{x_1, \ldots, x_{m-1}\}$, and C is a direct product of copies of $\mathbb{Z}$. Setting $x_m = b + c$ $(b \in B, c \in C)$ and embedding c in a finitely generated summand C' of C, we obtain a finitely generated summand $B \oplus C'$ of A, containing $\{x_1, \ldots, x_m\}$, again with a complement that is a direct product of infinite cyclic groups.

The next step is to show that A is $\aleph_1$-free. Let G be a countable subgroup of A. A maximal independent set of a finite rank subgroup G' of G is contained in a finitely generated summand B of A, so by torsion-freeness, $G' \leq B$. Thus G' is free, and Theorem 7.1 implies that G is free.

It remains to prove that A itself is not free. We exhibit a non-free subgroup of A. Let p be any prime, and H the subgroup of $A' = \prod_{i<\omega} \langle a_i \rangle$ (a summand of A) that consists of all vectors $b = (n_0 a_0, n_1 a_1, \ldots, n_i a_i, \ldots)$ such that, for every integer $k > 0$, almost all coefficients n_i are divisible by p^k. Manifestly, H contains the direct sum $S = \oplus_{i<\omega} \langle a_i \rangle$, and has cardinality $2^{\aleph_0}$. Since each coset of H mod pH can be represented by some element of S, H/pH cannot be uncountable. If H were free, we would have $|H/pH| = |H|$, so H cannot be free. $\square$

The countable direct product of infinite cyclic groups, i.e. the group $\mathbb{Z}^{\aleph_0}$, is often called the **Baer-Specker group**.

An immediate consequence is the following result that shows that in countable groups free summands can be collected into a single summand.

Corollary 8.3 (K. Stein). *A countable torsion-free group A can be decomposed as $A = F \oplus N$ where F is a free group, and N has no free factor group. N is uniquely determined by A.*

Proof. Define N as the intersection of the kernels of all homomorphisms $\eta\colon A \to \mathbb{Z}$. Then A/N is isomorphic to a countable subgroup of the direct product $\prod_\eta \mathbb{Z}$, so it is free in view of Theorem 8.2. N is then a summand of A, and we have $A = F \oplus N$ with F free. From the definition of N it is evident that N cannot have a non-trivial map into $\mathbb{Z}$. □

The next two examples show that there exist very large $\aleph_1$-free groups A such that $\mathrm{Hom}(A, \mathbb{Z}) = 0$, and it may also happen that an $\aleph_1$-free group is isomorphic to the countable direct sum and to the countable direct product of itself.

Example 8.4 (G. Reid [1]). Let $\kappa \geq \aleph_1$ be a non-measurable cardinal, and N the subgroup of $\mathbb{Z}^\kappa$ consisting of vectors with countable support. Then $A = \mathbb{Z}^\kappa/N$ is $\aleph_1$-free. To see this, let $b_n + N$ $(n < \omega)$ be a list of elements in a countable subgroup F of A. Clearly, each b_n has uncountable support, and each sum $b_i + b_j$ is equal to some b_k modulo a countable index set. Thus if we change the representatives b_n by dropping all the indices in these countably many index sets, then the new representatives form a subgroup $F' \cong F$. By Theorem 8.2, F' is free, so A is $\aleph_1$-free.

A homomorphism $\phi : A \to \mathbb{Z}$ may be viewed as a map $\phi^* : \mathbb{Z}^\kappa \to \mathbb{Z}$ such that $\phi^*(N) = 0$. By Theorem 2.8 in Chapter 13, $\phi^* = 0$, which means $\mathrm{Hom}(A, \mathbb{Z}) = 0$.

Example 8.5. There exists an $\aleph_1$-free group which is not free and isomorphic both to the direct sum of countably many copies of itself, and to the direct product of countable many copies of itself. See Proposition 4.9 in Chapter 13.

Strongly κ-Free The study of almost free groups brings a stronger version of κ-freeness into the picture. Let κ be a regular cardinal. A group A is said to be **strongly κ-free** if every subgroup of cardinality $< \kappa$ is contained in a free subgroup C of cardinality $< \kappa$ such that A/C is κ-free. Evidently, free groups are strongly κ-free for any cardinal κ.

It is not obvious that if $\kappa < \lambda$ are infinite cardinals, then strongly λ-free implies strongly κ-free, but it is true. In fact, if a subgroup B of cardinality $< \kappa$ is contained in a free subgroup C of cardinality $< \lambda$ with λ-free A/C, then B is contained in a summand C' of C of cardinality $< \kappa$. Because of (C), A/C' is κ-free, being an extension of the free group C/C' by the κ-free group A/C.

The fine nuance between strongly κ-free and just plainly κ-free groups can be better understood if we compare filtrations.

Lemma 8.6 (Eklof–Mekler [EM]). *Let A be a group of cardinality κ, where κ is an uncountable regular cardinal.*

(a) *A is κ-free exactly if it has a filtration $\{A_\sigma \mid \sigma < \kappa\}$ with free subgroups A_σ of cardinality $< \kappa$.*
(b) *A is strongly κ-free if and only if it admits a filtration $\{A_\sigma \mid \sigma < \kappa\}$ with free subgroups A_σ of cardinality $< \kappa$ such that, for all $\sigma < \tau < \kappa$, the factor groups $A_{\tau+1}/A_{\sigma+1}$ are free.*

Proof. (a) Since every subgroup of cardinality $< \kappa$ is contained in some member of a κ-filtration, the stated condition evidently implies the κ-freeness of A. Conversely, if A is κ-free, then the subgroups in any κ-filtration of A are free.

(b) For sufficiency, it is enough to observe that the stated condition is equivalent to that A is κ-free, and for every $A_{\sigma+1}$, the factor group $A/A_{\sigma+1}$ is κ-free. To prove the converse, assume A is strongly κ-free, and $\{a_\sigma \mid \sigma < \kappa\}$ is a well-ordered list of elements of A. We construct a filtration $\{A_\sigma \mid \sigma < \kappa\}$ of A as desired, with the additional property that $a_\rho \in A_\sigma$ for all $\rho < \sigma < \kappa$. Suppose that, for some $\sigma < \kappa$, we have a chain $\{A_\rho \mid \rho \leq \sigma\}$ satisfying the requisite properties. Choose for $A_{\sigma+1}$ a subgroup of cardinality $< \kappa$ that contains both A_σ and a_σ such that $A/A_{\sigma+1}$ is κ-free. Then the factor groups $A_{\tau+1}/A_{\sigma+1}$ are free for all $\tau > \sigma$, and by the κ-freeness of A, $A_{\sigma+1}$ is free. ◻

Observe that in (b) we have not said anything about the freeness of the factor groups A_τ/A_σ at limit ordinals σ.

Lemma 8.7. *The Baer-Specker group $P = \mathbb{Z}^{\aleph_0}$ is not strongly $\aleph_1$-free.*

Proof. We prove that the direct sum $S = \mathbb{Z}^{(\aleph_0)}$ is not contained in any countable subgroup G with $\aleph_1$-free P/G. Anticipating theorems that we will prove later on, the proof is quick. Corollary 1.12 in Chapter 6 asserts that P/S is algebraically compact, thus for every intermediate pure subgroup $S \leq G < P$, the factor group P/G is torsion-free and algebraically compact (see Lemma 8.1 in Chapter 9). Therefore, it contains a subgroup isomorphic to either $\mathbb{Q}$ or J_p for some prime p, and consequently, P/G can never be $\aleph_1$-free. ◻

Uncountable Chains If we wish to consider chains of free groups of cofinality exceeding ω, then we are confronted with a more complicated situation. In order to guarantee that the union of long chains of free groups will again be free, it is necessary to impose restrictions on the factors in the chain. A typical result is as follows.

Theorem 8.8 (Fuchs–Rangaswamy [4]). *Suppose κ is an uncountable regular cardinal, and $0 = F_0 < F_1 < \cdots < F_\sigma < \ldots$ $(\sigma < \kappa)$ is a smooth chain of groups such that, for every $\sigma < \kappa$,*

(a) *F_σ is free of cardinality $\leq \kappa$, and*
(b) *F_σ is a pure subgroup of $F_{\sigma+1}$.*

(i) *The union F of the chain is free provided the set*

$$S = \{\sigma < \kappa \mid \exists\, \rho > \sigma \text{ such that } F_\rho/F_\sigma \text{ is not } \kappa\text{-free}\}$$

is not stationary in κ.

(ii) *If all $|F_\sigma| < \kappa$, and S is stationary in κ, then F is κ-free, but not free.*

Proof. The proof is similar to the one in Lemma 7.6, but the construction of the $G(\kappa)$-families becomes complicated at limit ordinals. The details are too long to be reproduced here. ◻

Large Almost Free Non-free Groups We next prove that in the constructible universe, there exist large κ-free groups that are not free.

Theorem 8.9 (Gregory [1]). *Assume* $\mathrm{V} = \mathrm{L}$. *For every uncountable regular cardinal* κ *that is not weakly compact, there exists a* κ*-free group of cardinality* κ *which is not free.*

Proof. Let E be a stationary subset of κ that consists of limit ordinals cofinal with ω; see Lemma 4.5 in Chapter 1. Suppose for a moment that we have succeeded in constructing a group F as the union of a smooth chain of subgroups F_σ $(\sigma < \kappa)$ satisfying the following conditions for all $\sigma < \rho < \kappa$:

(i) F_σ is free of cardinality $|\sigma| \cdot \aleph_0$;
(ii) if $\sigma \in E$, then the quotient $F_{\sigma+1}/F_\sigma$ is not free;
(iii) if $\sigma \notin E$, then F_ρ/F_σ is free of cardinality $|\rho| \cdot \aleph_0$.

Then F is of cardinality κ and κ-free. Working toward contradiction, suppose F is free. Then there exists a cub $C \subset \kappa$ such that F_ρ/F_σ is free for each pair $\sigma < \rho$ in C. For such a pair of indices, the exact sequence $0 \to F_{\sigma+1}/F_\sigma \to F_\rho/F_\sigma \to F_\rho/F_{\sigma+1} \to 0$ must split because of (iii). This means that $F_{\sigma+1}/F_\sigma$ must be free for $\sigma \in E \cap C$, contrary to (ii).

It remains to construct a smooth chain of groups F_σ $(\sigma < \kappa)$ with the listed properties. Starting with F_0 free of rank $\aleph_0$, we proceed to define F_σ $(\sigma > 0)$ *via* transfinite induction as follows. Assume that σ is an ordinal $< \kappa$ such that the groups F_ρ $(\rho < \sigma)$ have already been constructed, and they satisfy conditions (i)–(iii) up to σ. To define F_σ we distinguish three cases.

Case 1. If σ is a limit ordinal, then we have no choice: $F_\sigma = \bigcup_{\rho<\sigma} F_\rho$. Since $E \cap \sigma$ is not stationary in σ Lemma 4.5 in Chapter 1, (iii) allows us to apply Theorem 7.5 to claim that F_σ is free. Hence conditions (i)–(iii) hold for all ordinals $\leq \sigma$.

Case 2. If $\sigma = \rho + 1$ and $\rho \notin E$, then we simply let $F_\sigma = F_\rho \oplus X$ where X is a countable free group.

Case 3. The critical case is when $\sigma = \rho + 1$ and $\rho \in E$. In view of the choice of E, we have $\operatorname{cf} \rho = \omega$, so ρ is the supremum of an increasing sequence of non-limit ordinals $\rho_0 < \rho_1 < \cdots < \rho_n < \dots$ $(n < \omega)$. Consider the chain $F_{\rho_0} < \cdots < F_{\rho_n} < \dots$ of free groups whose union is the free group F_ρ. We are in the situation of Example 7.3, and so we can define $F_{\sigma+1}$ such that the F_{ρ_n} $(n < \omega)$ are, but F_ρ is not a summand of $F_{\sigma+1}$. With this choice, (i)–(iii) will be satisfied by all ordinals $\leq \sigma$. □

For cardinals $\aleph_n$ $(n \geq 1)$, the existence of a stationary E of property Lemma 4.5 in Chapter 1 can be established without the hypothesis $\mathrm{V} = \mathrm{L}$, therefore we can state:

Corollary 8.10 (Eklof [2], Griffith [7], Hill [13]). *For every integer* $n > 1$, *there is a non-free* $\aleph_n$*-free group of cardinality* $\aleph_n$. □

The Σ-Cyclic Case Several results proved above carry over to torsion and mixed groups provided we can interpret freeness in an appropriate way. This can be done by introducing κ**-cyclic** groups meaning that every subgroup of cardinality $< \kappa$ is Σ-cyclic.

A proof similar to Theorem 8.9 applies to verify:

Corollary 8.11 (Eklof [2]). *If κ is an uncountable regular cardinal that is not weakly compact, then there exist κ-cyclic torsion groups of cardinality κ that are not Σ-cyclic.*

For every $n > 0$, there are $\aleph_n$-cyclic torsion groups of cardinality $\aleph_n$ which are not Σ-cyclic.

Proof. Obvious modification to Theorem 8.9 is that 'free' should be replaced by 'Σ-cyclic,' 'κ-free' by 'κ-cyclic,' and purity should be assumed throughout. In place of Example 7.3, a modified example should be referred to where a pure-projective resolution of $\mathbb{Z}(p^\infty)$ is used. $\square$

Mittag-Leffler Groups Most recently, a lot of attention has been devoted to Mittag-Leffler modules. In the group case, a satisfactory characterization is available. In the definition, we need tensor products: M is a **Mittag-Leffler group** if for every collection $\{A_i\}_{i \in I}$ of groups, the natural map

$$\phi : M \otimes \prod_{i \in I} A_i \to \prod_{i \in I} (M \otimes A_i)$$

given by $\phi(x \otimes (\ldots, a_i, \ldots)) \mapsto (\ldots, x \otimes a_i, \ldots)$ is monic ($x \in M, a_i \in A_i$).

Example 8.12. (a) Cyclic groups are Mittag-Leffler. This is trivial for $\mathbb{Z}$, and follows for $M = \mathbb{Z}(n)$ from the fact that both the domain and the image of ϕ are then isomorphic to $\prod_{i \in I} (A_i / nA_i)$.

(b) The Prüfer group $H_{\omega+1}$ (of length $\omega + 1$) is not Mittag-Leffler. The natural map $H_{\omega+1} \otimes \prod_{n<\omega} \mathbb{Z}(p^n) \to \prod_{n<\omega} H_{\omega+1} \otimes \mathbb{Z}(p^n)$ is not monic. (See the proof of Theorem 8.14.)

Lemma 8.13 (M. Raynaud, L. Gruson). *The class of Mittag-Leffler groups is closed under taking pure subgroups, pure extensions and arbitrary direct products.*

Proof. Starting with a pure-exact sequence $0 \to M' \to M \to M'' \to 0$, we form the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' \otimes \prod A_i & \longrightarrow & M \otimes \prod A_i & \longrightarrow & M'' \otimes \prod A_i & \longrightarrow & 0 \\
 & & \phi' \downarrow & & \phi \downarrow & & \downarrow \phi'' & & \\
0 & \longrightarrow & \prod (M' \otimes A_i) & \longrightarrow & \prod (M \otimes A_i) & \longrightarrow & \prod (M'' \otimes A_i) & \longrightarrow & 0
\end{array}$$

with pure-exact rows (see Corollary 3.7 in Chapter 5). Evidently, if ϕ is monic, then so is ϕ'. If both ϕ' and ϕ'' are monic, then Lemma 2.6 in Chapter 1 (or a simple diagram-chasing) shows that ϕ has to be monic as well. Finally, for direct products, the claim will be a simple consequence of Theorem 8.14 and Exercise 1. $\square$

It is not difficult to characterize Mittag-Leffler groups.

Theorem 8.14 (Raynaud–Gruson). *A group is Mittag-Leffler if and only if it is $\aleph_1$-cyclic.*

Proof. We start the proof by showing that M is Mittag-Leffler if and only if each of its countable pure subgroups is Mittag-Leffler. One direction the claim follows from Lemma 8.13.

For the converse, assume ϕ maps $\sum_{j=1}^{n}(x_j \otimes b_j)$ $(x_j \in M,\ b_j \in \prod A_i)$ to 0. Then by Lemma 1.12 in Chapter 8 and by the remark after it, the same sum vanishes in $\phi(M' \otimes \prod A_i)$ for a countable pure subgroup $M' \leq M$ containing the x_j's. Hence M cannot be Mittag-Leffler if its countable pure subgroups are not, but it is if its countable pure subgroups are Mittag-Leffler.

It remains to prove that countable Mittag-Leffler groups are Σ-cyclic. Suppose N is a countable p-group which has elements $\neq 0$ of infinite p-heights. Clearly, all non-zero elements of $\prod_{n\in\mathbb{N}}(N \otimes \mathbb{Z}(p^n))$ have finite p-heights. However, $\prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$ has summands $\cong J_p$, so $N \otimes \prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$ has summands $\cong N \otimes J_p \cong N$ with elements of infinite heights. Thus N is not Mittag-Leffler. A countable Mittag-Leffler group must therefore have separable p-components, so its torsion subgroup is Σ-cyclic.

Next, let M be of finite torsion-free rank $n > 0$ such that M/tM is not finitely generated. Then M contains a subgroup N such that $N/tM \cong \mathbb{Z}^n$ and M/N is an infinite torsion group. First assume M/N is reduced. Then it is Σ-cyclic of the form $\oplus_{i<\omega} \mathbb{Z}(p_i^{k_i})$ with $k_i \in \mathbb{N}$, where the p_i are not necessarily different primes, but each prime may occur at most n times. We tensor the exact sequence

$$0 \to N \cong tM \oplus \mathbb{Z}^n \to M \to \oplus_{i<\omega} \mathbb{Z}(p_i^{k_i}) \to 0$$

with $A = \prod_j \mathbb{Z}(p_j)$ where p_j varies over the (infinitely many) different primes in the set of the p_i. In the long exact sequence connecting Tor and $\otimes$, the map $\mathrm{Tor}(N, A) \to \mathrm{Tor}(M, A)$ is an isomorphism as N, M share the same torsion subgroup, thus the induced sequence

$$0 \to \mathrm{Tor}(\oplus_{i<\omega} \mathbb{Z}(p_i^{k_i}), A) \xrightarrow{\delta} (tM \oplus \mathbb{Z}^n) \otimes A \to M \otimes A \to \dots$$

is exact. We calculate: $\mathrm{Tor}(\oplus_{i<\omega} \mathbb{Z}(p_i^{k_i}), A) \cong \oplus_i \mathbb{Z}(p_i)$, and note that this Tor is sent by the connecting map δ into $\mathbb{Z}^n \otimes A$. Therefore, $M \otimes A$ must contain an image of the divisible group $A/\oplus_j \mathbb{Z}(p_j)$. But $\prod_j (M \otimes \mathbb{Z}(p_j))$ is reduced, thus $M \otimes \prod_j \mathbb{Z}(p_j) \to \prod_j (M \otimes \mathbb{Z}(p_j))$ is not monic. Such an M cannot be Mittag-Leffler.

A similar proof applies to show that M cannot be Mittag-Leffler if M/tM contains a rank 1 pure subgroup that is p-divisible for some prime p (in this case, we tensor with $A = \prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$). The conclusion is that if M is Mittag-Leffler, then the finite rank pure subgroups of M/tM are free, i.e. M/tM is free if it is countable by Theorem 7.1. Therefore, a countable Mittag-Leffler group is Σ-cyclic. □

★ **Notes.** The Baer–Specker group has been investigated from various points of view, it is an excellent source of ideas. We point out that, among others, Blass–Irwin have several interesting papers on this group and its subgroups. In their paper [2], several interesting subgroups are dealt with. In the other paper [1], a core class for $\aleph_1$-freeness is discussed: a well-defined class of non-free $\aleph_1$-free groups of cardinality $\aleph_1$ such that every non-free $\aleph_1$-free group of cardinality $\aleph_1$ contains a subgroup from the class. Another interesting result is the existence of indecomposable $\aleph_1$-free groups by Palyutin [1] (under CH) which was generalized to rigid $\aleph_1$-free groups of cardinality $\aleph_1$ by Göbel–Shelah [2].

Eda [4] shows that a group is $\aleph_1$-free if and only if it is contained in $\mathbb{Z}^{(\mathbf{B})}$ for some Boolean lattice **B**. To illustrate the importance of $\aleph_1$-freeness, we also mention several topological connections. L. Pontryagin proved that a connected compact abelian group G is locally connected exactly if its character group Char G is $\aleph_1$-free, and J. Dixmier showed that it is arcwise connected if and only if $\operatorname{Ext}(\operatorname{Char} G, \mathbb{Z}) = 0$ (which is stronger than $\aleph_1$-freeness). We also point out that for a compact connected group G, the nth homotopy group $\pi_n(G) = 0$ for all $n > 1$, while $\pi_1(G) = \operatorname{Hom}(\operatorname{Char} G, \mathbb{Z})$ is always $\aleph_1$-free.

That $\aleph_n$-free groups need not be $\aleph_{n+1}$-free was proved by Hill, Griffith, and then by Eklof. Mekler–Shelah [2] study regular cardinals κ for which κ-free implies strongly κ-free or κ^+-free. Gregory [1] proved in L the most interesting Theorem 8.9. Assuming V = L, Rychkov [3] proves that for each uncountable regular, not weakly compact cardinal κ, there exist p-groups A of final rank κ such that every subgroup C of cardinality $< \kappa$ is contained in a Σ-cyclic direct summand of cardinality $|C|\aleph_0$, but A itself is not Σ-cyclic, not even the direct sum of two subgroups of final ranks κ.

Mittag-Leffler modules were introduced by M. Raynaud and L. Gruson [Invent. Math. **13**, 1–89 (1971)].

Exercises

(1) (a) A direct product of $\aleph_1$-free groups is $\aleph_1$-free.
 (b) The same may fail for larger cardinals.
 (c) Derive from Theorem 8.14 that a direct product of Mittag-Leffler groups is Mittag-Leffler.

(2) In a free group F, a subgroup G of cardinality $< \kappa$ for which F/G is κ-free is a summand.

(3) An extension of a free group by a strongly κ-free group is strongly κ-free.

(4) Let A be a direct product of infinite cyclic groups, and B the subgroup of A whose elements are the vectors with countable support. B is $\aleph_1$-free, but not free.

(5) In the Baer–Specker group $A = \prod_{k\in\mathbb{N}} \langle e_k \rangle$, let D denote the $\mathbb{Z}$-adic closure of $S = \oplus_k \langle e_k \rangle$. Prove that D consists of all vectors $x = \sum m_k e_k$ such that, for every $n \in \mathbb{N}$, n divides almost all m_k.

(6) Is it possible to define Mittag-Leffler groups by using only countable index sets I?

(7) If M is Mittag-Leffler, then so is M/N for every finitely generated subgroup N of M.

9 Shelah's Singular Compactness Theorem

The question as to when κ-free implies κ^+-free turns out to be extremely complicated for regular cardinals κ (see Magidor–Shelah [1]). As far as singular cardinals are concerned, the same question can be fully answered; this is shown by the next theorem, a most powerful result.

The following lemma will be required in the proof of Theorem 9.2.

Lemma 9.1 (Eklof–Mekler [EM]). *If κ is a regular cardinal, then a κ^+-free group is strongly κ-free.*

Proof. By way of contradiction, assume that A is κ^+-free, but not strongly κ-free. This means that A contains a subgroup B of cardinality $< \kappa$ which is not contained in any subgroup of A of cardinality $< \kappa$ with κ-free factor group. Set $C_0 = B$, and let C_1 be a pure subgroup of A of cardinality $< \kappa$ that contains C_0 such that C_1/C_0 is not free. Repeat this with C_1 in the role of C_0 to obtain C_2, and continue this process transfinitely up to κ steps, taking unions at limit ordinals. We get a chain $C_0 < C_1 < \cdots < C_\sigma < \dots$ $(\sigma < \kappa)$ where none of the factor groups $C_{\sigma+1}/C_\sigma$ is free. The union $C = \bigcup_{\sigma<\kappa} C_\sigma$ has cardinality κ, and is not free because of Theorem 7.5. This contradicts the κ^+-freeness of A. $\square$

Theorem 9.2 (Shelah [1]). *For a singular cardinal λ, a λ-free group of cardinality λ is free.*

Proof. Suppose A is λ-free of cardinality λ. Let $\{\kappa_\nu \mid \nu < \mathrm{cf}(\lambda)\}$ be a smooth increasing sequence of cardinals $> \mathrm{cf}(\lambda)$ with λ as supremum, and $\{A_\nu \mid \nu < \mathrm{cf}(\lambda)\}$ a smooth chain of pure subgroups of A with union A such that $|A_\nu| = \kappa_\nu$. Set

$$\mathcal{P}_\nu = \{B < A \mid |B| \leq \kappa_\nu \text{ and } A/B \text{ is } \kappa_\nu^+\text{−free}\}.$$

Since A is λ-free for all $\kappa < \lambda$, by Lemma 9.1 it is strongly λ-free for all $\kappa < \lambda$ (including limit ordinals $< \lambda$); thus, every subgroup of A of cardinality $\leq \kappa_\nu$ is contained in a member of $\mathcal{P}_\nu$. For all $\nu < \mathrm{cf}(\lambda)$, define subgroups $B_{\nu k}$ $(k < \omega)$ and subsets $X_{\nu k}$ $(k < \omega)$ such that

(i) $B_{\nu k} \in \mathcal{P}_\nu$ $(\nu < \mathrm{cf}(\lambda),\ k < \omega)$;
(ii) $X_{\nu k}$ is a basis of $B_{\nu k}$ $(\nu < \mathrm{cf}(\lambda),\ k < \omega)$;
(iii) $A_\nu < B_{\nu 0} < B_{\nu 1} < \cdots < B_{\nu k} < \dots$ and $X_{\nu 0} \subset X_{\nu 1} \subset \cdots \subset X_{\nu k} \subset \dots$ for each $\nu < \mathrm{cf}(\lambda)$;
(iv) $B_{\nu,k-1} \leq \langle B_{\nu k} \cap X_{\nu+1,k-1}\rangle$ for each $\nu < \mathrm{cf}(\lambda)$, $0 < k < \omega$;
(v) for a limit ordinal $\mu < \mathrm{cf}(\lambda)$, $X_{\mu k}$ is the union of a chain of subsets $Y_{\mu k}(\nu)$ where $|Y_{\mu k}(\nu)| = \kappa_\nu$ $(\nu < \mu)$, and $Y_{\mu k}(\nu) \subset B_{\nu k+1}$ for all $\nu < \mu$.

The construction is by induction on κ. In the first step, we define the subgroups $B_{\nu 0}$ $(\nu < \mathrm{cf}(\lambda))$ recursively on ν. Let B_{00} be any member of $\mathcal{P}_0$ that contains A_0. If, for some $\mu < \mathrm{cf}(\lambda)$, the $B_{\nu 0}$ have been defined for all $\nu < \mu$, then pick $B_{\mu 0} \in \mathcal{P}_\mu$ such that it contains $A_\nu + \sum_{\nu<\mu} B_{\nu 0}$; this can be done in view of the cardinality hypotheses. We are led to a well-ordered ascending chain $B_{00} < B_{10} < \cdots < B_{\nu 0} < \ldots$ (that need not be smooth) where $B_{\nu 0}$ has cardinality κ_ν. Choose any basis $X_{\nu 0}$ for $B_{\nu 0}$. For limit ordinals ν, represent $X_{\nu 0}$ as the union of a chain of subsets $Y_{\nu 0}(\sigma)$ where $Y_{\nu 0}(\sigma)$ has cardinality κ_σ $(\sigma < \nu)$.

The next step is to define $B_{\mu k}$ along with $X_{\mu k}$ after all $B_{\nu j}, X_{\nu j}$ (and $Y_{\nu j}(\sigma)$ only for limit ordinals ν) have been defined for all $j < k$ and for all $\nu < \mathrm{cf}(\lambda)$, and $B_{\nu k}, X_{\nu k}$, and $Y_{\nu k}(\sigma)$ for all $\nu < \mu$. Choose $B_{\mu k} \in \mathcal{P}_\mu$ so as to satisfy (iv), and to contain all of the following: (a) $B_{\mu,k-1}$; (b) $B_{\nu k}$ for all $\nu < \mu$; and (c) the sets $Y_{\nu,k-1}(\mu)$ for limit ordinals $\nu > \mu$. As $B_{\mu,k-1}$ is a summand of $B_{\mu k}$, we can select a basis $X_{\mu k}$ of $B_{\mu k}$ that contains $X_{\mu,k-1}$. If μ happens to be a limit ordinal, we choose the $Y_{\mu k}(\nu)$ $(\nu < \mu)$ so as to satisfy (v). An easy cardinality argument convinces us that this can be done in view of the hypothesis that $\kappa_\mu > \mathrm{cf}(\lambda)$. It is obvious that conditions (i)–(v) are satisfied.

We claim that the subgroups $B_\nu = \bigcup_{k<\omega} B_{\nu k}$ $(\nu < \mathrm{cf}(\lambda))$ form a smooth chain $B_0 < B_1 < \cdots < B_\nu < \ldots$ $(\nu < \mathrm{cf}(\lambda))$ with free factor groups $B_{\nu+1}/B_\nu$. Observe that if μ is a limit ordinal, then in view of

$$B_\mu = \bigcup_{k<\omega} B_{\mu k} = \bigcup_{k<\omega} \langle X_{\mu k}\rangle = \bigcup_{k<\omega}\bigcup_{\nu<\mu} \langle Y_{\mu k}(\nu)\rangle \le \bigcup_{k<\omega}\bigcup_{\nu<\mu} B_{\mu k+1} = \bigcup_{\nu<\mu} B_\nu,$$

the chain of the B_ν is continuous. Since (iv) implies that B_ν is generated by $B_\nu \cap X_{\nu+1}$ (where $X_\nu = \bigcup_{k<\omega} X_{\nu k}$), $B_{\nu+1}/B_\nu$ is indeed free. By Theorem 7.5, the group $A = \bigcup_\nu B_\nu$ is free. $\square$

In Chapter 14, a more general form of the Singular Compactness Theorem will be needed (for Butler groups); we state it here for groups without proof. This axiomatic form is due to Eklof–Mekler [EM], generalizing W. Hodges' version [Algebra Universalis **12**, 205–220 (1981)].

Assume $\mathcal{F}$ is a class of groups such that $0 \in \mathcal{F}$, and for each $G \in \mathcal{F}$, there is given a family $\mathcal{B}(G)$ of sets of subgroups of G. We say that G is **'free'** if $G \in \mathcal{F}$ and $\mathfrak{B}$ is a **'basis'** of G if $\mathfrak{B} \in \mathcal{B}(G)$. The subgroups $B \in \mathfrak{B}$ are called 'free' factors of G.

For a fixed infinite cardinal μ, the following properties (i)-(v) are required for every 'free' group G, and for every 'basis' $\mathfrak{B}$ of G.

- (i) $\mathfrak{B}$ is closed under unions of chains.
- (ii) If $B \in \mathfrak{B}$ and $g \in G$, then there is a $C \in \mathfrak{B}$ that contains both B and g, and is such that $|C| \le |B| + \mu$.
- (iii) Every $B \in \mathfrak{B}$ is 'free' (i.e., 'free' factors are 'free'); and moreover, the set $\{C \in \mathfrak{B} \mid C \le B\} = \mathfrak{B} \restriction B$ is a 'basis' for B.
- (iv) If B is a 'free' factor of G, then for every 'basis' $\mathfrak{B}'$ of B, there exists a 'basis' $\mathfrak{B}$ of G such that $\mathfrak{B}' = \mathfrak{B} \restriction B$.
- (v) Suppose B_σ $(\sigma < \kappa)$ is a smooth chain of 'free' subgroups of G with 'bases' $\mathfrak{B}_\sigma$ satisfying $\mathfrak{B}_\rho \restriction B_\sigma = \mathfrak{B}_\sigma$ for all $\sigma < \rho < \kappa$ (in particular, $B_\sigma \in \mathfrak{B}_\rho$). Then the union $B = \bigcup_{\sigma<\kappa} B_\sigma$ is a 'free' subgroup of G such that $\bigcup_{\sigma<\kappa} \mathfrak{B}_\sigma$ is a 'basis' of B.

Theorem 9.3. *Suppose that the class $\mathcal{F}$ of groups satisfies conditions* (*i*)-(*v*) *for cardinal μ, and the cardinality λ of the group $G \in \mathcal{F}$ is a singular cardinal $> \mu$. G is 'free' if, for every cardinal $\kappa < \lambda$, there is a family $\mathcal{C}_\kappa$ of subgroups of G of cardinality κ satisfying the following conditions:*

(a) *$\mathcal{C}_\kappa$ is a subclass of $\mathcal{F}$;*
(b) *$\mathcal{C}_\kappa$ is closed under unions of chains of lengths $\leq \kappa$;*
(c) *every subset of G of cardinality $\leq \kappa$ is contained in a subgroup that belongs to $\mathcal{C}_\kappa$.* □

★ **Notes.** Hill [13] showed that $\aleph_\omega$-free groups of cardinality $\aleph_\omega$ are free, defeating the conjecture that κ-free never implies κ^+-free. In a subsequent paper, he proved the same for $\aleph_{\omega_1}$-free groups. Based on these results, Shelah conjectured and proved the general theorem on singular cardinals. (The term 'compact' is designated in the sense used in logic, not as in topology: properties of small substructures imply the same for the entire structure.)

Various generalizations of the compactness theorem are available in the literature which we do not wish to review here. Let us point out that Hodges [loc.cit.] published an interesting proof of the singular compactness theorem, based on Shelah's ideas. The κ-Shelah game on a group A (for a regular cardinal κ) is introduced; it is played by two players. The players take turns to choose subgroups of A of cardinalities $< \kappa$ to build an increasing chain $\{B_n\}_{n<\omega}$ of subgroups. The players know what subgroups have been chosen at previous steps. B_n is chosen by player I if n is even and by player II if n is odd. Player II wins if for every odd integer n, B_n is a free summand of B_{n+2}, otherwise player I is the winner. The κ-Shelah game is determinate in the sense that one of the players has a winning strategy. It is then shown that player I has no winning strategy, so player II wins. Being a 'free summand' is used in a more general sense in order to obtain a singular compactness result more general than our Theorem 9.3.

Exercises

(1) Let A be a p-group of singular cardinality λ. If all subgroups of A of cardinalities $< \lambda$ are Σ-cyclic, then A too is Σ-cyclic. [Hint: the λ-free vector space $A[p]$ is free.]
(2) Let A be a group of singular cardinality λ. If A is λ-cyclic, then it is Σ-cyclic.

10 Groups with Discrete Norm

Normed vector spaces play a most important role in functional analysis. In abelian group theory, the idea of an integer-valued (more generally, a discrete) norm leads to an interesting characterization of free groups—a result that has several important applications.

Discrete Norm A **norm** on a group A is a function $\| \dots \| : A \to \mathbb{R}$ such that

(i) $\| a \| \geq 0$ for all $a \in A$; and $\| a \| = 0$ exactly if $a = 0$;
(ii) $\| a + b \| \leq \| a \| + \| b \|$ for all $a, b \in A$;
(iii) $\| ma \| = |m|.\| a \|$ for each $m \in \mathbb{Z}$ and $a \in A$.
A norm $\| \dots \|$ is called **discrete** if it also satisfies:
(iv) there is a real number $\epsilon > 0$ such that $\| a \| \geq \epsilon$ for all $0 \neq a \in A$.
(Requirement (iv) is *a priori* less than demanding that the norms be always integers.) We record the following elementary facts.

(A) *A group with a norm has to be torsion-free.* This follows at once from properties (i) and (iii).
(B) *If* $\| \dots \|$ *is a (discrete) norm, then so is* $r\| \dots \|$ *for every positive* $r \in \mathbb{R}$.
(C) *Subgroups inherit the norm function. Discreteness is inherited as well.*
(D) *A norm* $\| \dots \|$ *on a torsion-free group extends uniquely to a norm on its divisible hull* (for divisible hull, see Sect. 2 in Chapter 4). Needless to say, an extended norm is never discrete.

Example 10.1. A free abelian group F admits a discrete norm. In fact, if $\{e_i\}_{i \in I}$ is a free basis of F, then

$$\| \sum n_i e_i \| = \sum |n_i| \qquad (n_i \in \mathbb{Z})$$

defines a discrete norm on F. Another way of furnishing F with a discrete norm is by setting

$$\| \sum n_i e_i \| = \max |n_i|.$$

It would be futile to look for other groups as examples, because—as is shown by the theorem below—only the free groups admit discrete norms.

The discussion starts with the finite rank case.

Lemma 10.2 (Lawrence [1], Zorzitto [1]). *A finite rank torsion-free group with discrete norm is free.*

Proof. Let A be torsion-free of finite rank with a discrete norm $\| \dots \|$. By induction on the rank, we prove that A is free.

Without loss of generality, we may assume that $\| a \| \geq 1$ for all non-zero $a \in A$, and that there is an $x_0 \in A$ whose norm is $< 3/2$. Under this hypothesis on the norm, x_0 is evidently not divisible in A by any integer > 1, hence the cyclic subgroup $\langle x_0 \rangle$ must be pure in A. Therefore, if A is of rank 1, then $\langle x_0 \rangle$ is all of A.

Let A be of rank $n + 1$, and assume the claim holds for groups of rank $n \geq 1$. Starting with x_0, pick a maximal independent set $\{x_0, x_1, \dots, x_n\}$ in A. The factor group $A^\star = A/\langle x_0 \rangle$ is torsion-free of rank n. It is straightforward to check that one can define a norm μ in $A^\star$ by setting

$$\mu(r_1 x_1^\star + \dots + r_n x_n^\star) = |r_1| \cdot \| x_1 \| + \dots + |r_n| \cdot \| x_n \|$$

where the coefficients r_i are rational numbers, and stars indicate cosets mod $\langle x_0 \rangle$. Supposing $A^\star$ is not free, induction hypothesis implies that $A^\star$ cannot have a discrete norm, so some coset $y^\star = s_1 x_1^\star + \cdots + s_n x_n^\star$ ($s_i \in \mathbb{Q}$) has a norm $< 1/4$. There is an $a \in A$ such that $a = s_0 x_0 + s_1 x_1 + \cdots + s_n x_n$ for some $s_0 \in \mathbb{Q}$. By adding to a an integral multiple of x_0 if necessary, we can assume that $|s_0| \leq 1/2$. But then

$$\| a \| \leq |s_0| \cdot \| x_0 \| + |s_1| \cdot \| x_1 \| + \cdots + |s_n| \cdot \| x_n \| < 1/2 \cdot 3/2 + 1/4 = 1,$$

a contradiction. Thus A^*, and hence A, is free. □

Free Groups and Discrete Norm We can now verify the main result.

Theorem 10.3 (Steprāns [1]). *A group admits a discrete norm if and only if it is free.*

Proof. In view of our example above, it is enough to show that a group A with a discrete norm $\| \dots \|$ is free. We induct on the rank κ of A. The preceding lemma settles the case if κ is finite, so assume that κ is an infinite cardinal, and that the claim holds for groups of rank $< \kappa$. If $\kappa = \aleph_0$, then finite rank subgroups are free, so Pontryagin's theorem 7.1 implies that A is free.

Next, let κ be an uncountable regular cardinal, and $0 = A_0 < A_1 < \cdots < A_\sigma < \dots$ ($\sigma < \kappa$) a smooth chain of pure subgroups of the group A such that the A_σ are of cardinality $< \kappa$, and $A = \bigcup_{\sigma < \kappa} A_\sigma$. By induction hypothesis, the subgroups A_σ are free. Consider the set

$$E = \{\sigma < \kappa \mid \exists\, \rho > \sigma \text{ such that } A_\rho / A_\sigma \text{ is not free}\},$$

and suppose E is a stationary set in κ. Without loss of generality, we may assume that $\rho = \sigma + 1$ in the definition of E by thinning out the chain. For each $\sigma \in E$, pick elements $x_{\sigma\tau}$ (where τ runs over a suitable index set) such that $\{x_{\sigma\tau} + A_\sigma\}_\tau$ is a maximal independent set of $A_{\sigma+1}/A_\sigma$. As above, define a norm μ_σ in $A_{\sigma+1}/A_\sigma$ by setting

$$\mu_\sigma\Big(\sum_\tau r_{\sigma\tau}(x_{\sigma\tau} + A_\sigma)\Big) = \sum_\tau |r_{\sigma\tau}| \cdot \| x_{\sigma\tau} \|$$

where the coefficients $r_{\sigma\tau}$ are rational numbers, and of course, all sums are finite. Since $A_{\sigma+1}/A_\sigma$ has cardinality $< \kappa$ and is not free, the norm μ_σ cannot be discrete. Thus there is a coset $y_\sigma + A_\sigma$ with norm $< \frac{1}{2}$, say, $y_\sigma = \sum_\tau s_{\sigma\tau} x_{\sigma\tau} + z_\sigma$ for some z_σ in A_σ.

For convenience, we assume that the underlying set of A consists of all ordinals $< \kappa$, and A_σ ($\sigma \in E$) is just the set of ordinals $< \sigma$. Then the correspondence $\psi : \sigma \mapsto z_\sigma$ is a regressive function from E into κ. Fodor's theorem (Jech [J]) implies that there exist a $z \in A$ and a stationary subset E' of E such that $\psi(\sigma) = z$ for all $\sigma \in E'$.

Choose different $\sigma, \rho \in E'$ such that $y_\sigma \neq y_\rho$ whose cosets have norm $< \frac{1}{2}$. We then have

$$\|y_\sigma - y_\rho\| = \|(\sum_\tau s_{\sigma\tau}x_{\sigma\tau} + z) - (\sum_\nu s_{\rho\nu}x_{\rho\nu} + z)\| \leq$$

$$\leq \sum_\tau |s_{\sigma\tau}| \cdot \| x_{\sigma\tau} \| + \sum_\nu |s_{\rho\nu}| \cdot \| x_{\rho\nu} \| = \mu_\sigma(y_\sigma + A_\sigma) + \mu_\rho(y_\rho + A_\rho) < 1,$$

a contradiction. We conclude that E is not stationary, and hence Theorem 7.5 implies that A is free.

To complete the proof for singular cardinals κ, it suffices to refer to Shelah's singular compactness theorem 9.2. □

Corollaries To underscore the significance of this result, we record a few applications of this theorem.

Let A be an arbitrary group, and X an index set. The set of all functions $f: X \to A$ such that f assumes but a finite number of distinct values in A is a subgroup $B(X, A)$ of the cartesian power A^X. In case $A = \mathbb{Z}$, this subgroup consists of the bounded integer-valued functions on X.

Theorem 10.4 (Specker [1], Nöbeling [1]). *The group $B(X, \mathbb{Z})$ of bounded functions on any set X into $\mathbb{Z}$ is a free abelian group.*

Proof. For the application of Theorem 10.3 all that we have to note is that the group $B(X, \mathbb{Z})$ carries a discrete norm. In fact, the norm of a function $f \in B(X, \mathbb{Z})$ is defined as the maximum of the absolute values of integers in the range of f. □

An immediate corollary is a far-reaching generalization.

Corollary 10.5 (Kaup–Kleane [1]). *The group of all finite-valued functions on a set X into any group A is a direct sum of copies of A.*

Proof. In view of the last theorem, it suffices to verify the isomorphism $B(X, A) \cong A \otimes B(X, \mathbb{Z})$. Let h_Y denote the characteristic function of the subset Y of X, i.e. $h_Y(x) = 1$ or 0 according as $x \in Y$ or not. Every $f \in B(X, A)$ can be written as

$$f = a_1 h_{Y_1} + \cdots + a_k h_{Y_k} \qquad (a_i \in A)$$

for some k and disjoint subsets $Y_1, \ldots, Y_k$ of X. If the characteristic functions h_Y are viewed as elements of $B(X, \mathbb{Z})$, then f can be identified with the element $a_1 \otimes h_{Y_1} + \cdots + a_k \otimes h_{Y_k}$ of $A \otimes B(X, \mathbb{Z})$. □

An interesting corollary is concerned with continuous functions on a compact space. J. de Groot considered the group $C(X, \mathbb{Z})$ of all continuous functions from a topological space X into the discrete group of the integers $\mathbb{Z}$. Of special interest is the case in which X is a compact space. In this case, a continuous function from X to $\mathbb{Z}$ is finite-valued, i.e. $C(X, \mathbb{Z})$ is a subgroup of $B(X, \mathbb{Z})$. As such it is free:

Corollary 10.6 (de Groot). *The group of all continuous functions from a compact space into the discrete group of the integers is free.* □

Yamabe [1] considered, for groups A, bilinear, positive definite functions $f : A \times A \to \mathbb{Z}$. Note that such a function f defines a discrete norm as usual *via* $\| a \| = \sqrt{f(a,a)}$ for $a \in A$. This leads us to

Corollary 10.7. *If A is a group such that there is a bilinear, positive definite function f from $A \times A$ into the integers $\mathbb{Z}$, then A has to be a free group.* □

★ **Notes.** This section is a typical example how a difficult question can sometimes be rephrased to an easier one by making it more general. Specker [1] could prove only under the CH that the group of bounded sequences of the integers is free. Nöbeling [1] succeeded in solving the more general problem on bounded functions of integers by induction on what he called Specker groups. Bergman [1] provided another proof by establishing an even more general theorem on commutative torsion-free rings generated by idempotents. Finally, the powerful theorem on groups with discrete norm was proved. It is due to Steprāns [1] who proved it after Lawrence [1], Zorzitto [1] settled the countable case. As shown above, this result has important applications.

Hill [14] found an interesting generalization of Bergman's version by dropping the condition of torsion-freeness: the additive group of a commutative ring generated by idempotents is Σ-cyclic.

Exercises

(1) Find a discrete norm on a free group of rank $\aleph_0$ that is not a multiple of any of examples in Example 10.1.

(2) Let $P = \mathbb{Z}^X$ and $B = B(X, \mathbb{Z})$ for an infinite set X. Show that P/B is divisible. [Hint: for $a \in P, n \in \mathbb{N}$ find $c \in P$ with $a = nc + b$ with $b \in B$.]

(3) (Nöbeling) Recalling that every element $f \in B(X, A)$ can be written as $f = a_1 h_{Y_1} + \cdots + a_k h_{Y_k} (a_i \in A)$ for some k and disjoint subsets $Y_1, \ldots, Y_k$ of X, call a subgroup S of $B(X, A)$ a **Specker group** if $f \in S$ implies that $Ah_{Y_1}, \ldots, Ah_{Y_k}$ are contained in S. Prove that the following conditions are equivalent for a subgroup S of $B(X, \mathbb{Z})$:

   (a) S is a Specker group;
   (b) $f \in S$ implies h_Y where Y denotes the support of f;
   (c) S is a pure subgroup and a subring in $\mathbb{Z}^X$. [Hint: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).]

(4) The intersection of Specker subgroups in $B(X, \mathbb{Z})$ is again a Specker group.

(5) If S is a Specker subgroup in $B(X, \mathbb{Z})$ and $Y \subseteq X$, then Sh_Y is also a Specker group.

(6) (Bergman) Let R be a commutative ring with identity whose additive group R^+ is torsion-free. If R is generated as a ring by a set E of idempotents, then R^+ is a free group. It can be freely generated by idempotents that are products of elements of E. [Hint: assume E is a multiplicative semigroup, well-order its elements and show that the elements of E which are not linear combinations of preceding elements of E in the ordering form a basis of R^+.]

11 Quasi-Projectivity

Another fundamental concept in this circle of ideas is that of quasi-projectivity. It is a natural generalization of projectivity where the projective property is required only with respect to the group itself (so 'self-projective' would probably be a better name).

Quasi-Projective Groups Thus, a group P is called **quasi-projective** if for every exact sequence with P in the middle and for every homomorphism $\phi : P \to P/G$

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "\theta"'] \arrow[d, "\phi"] & \\ 0 \arrow[r] & G \arrow[r, "\alpha"] & P \arrow[r, "\beta"] & P/G \arrow[r] & 0 \end{tikzcd}$$

there exists an endomorphism θ of P making the triangle commute: $\beta\theta = \phi$. Free groups are quasi-projective, but not only these.

Example 11.1. (a) All cyclic groups are quasi-projective.
(b) Elementary groups are quasi-projective.

A few properties that are worth noting are as follows.

(A) *Summands of quasi-projective groups are quasi-projective.* If $P = X \oplus Y$ and $G \leq X$, then for any homomorphism $\phi : X \to X/G$, the map $\phi \oplus \mathbf{1}_Y : P \to X/G \oplus Y = P/G$ lifts to a $\theta : P \to P$, and $\theta \restriction X$ is a desired endomorphism of X.

(B) *A torsion group $P = \oplus_p P_p$ is quasi-projective if and only if its p-components P_p are.* Necessity follows from (A), and sufficiency is straightforward.

(C) *Factor groups modulo fully invariant subgroups inherit quasi-projectivity.* To see this, let S be fully invariant in the quasi-projective group P, and $\beta : P/S \to X$ an epimorphism. If $\phi : P/S \to X$ is any map and $\gamma : P \to P/S$ is the canonical homomorphism, then by the quasi-projectivity of P, there is a map $\theta : P \to P$ such that $\beta\gamma\theta = \phi\gamma$.

$$\begin{tikzcd} & P \arrow[r, "\gamma"] \arrow[dl, dashed, "\theta"'] & P/S \arrow[d, "\phi"] \\ P \arrow[r, "\gamma"] & P/S \arrow[r, "\beta"] & X \end{tikzcd}$$

Since S is fully invariant in P, θ induces a map $\theta' : P/S \to P/S$ such that $\gamma\theta = \theta'\gamma$. γ can be canceled in $\beta\theta'\gamma = \phi\gamma$, thus $\beta\theta' = \phi$.

(D) *Let G be a subgroup of a quasi-projective P such that P/G is isomorphic to a summand A of P. Then G is a summand of P.* Let $\alpha : A \to P$ and $\rho : P \to A$ be the canonical injection and projection maps, respectively. If $\beta: A \to P/G$ is an isomorphism, and $\gamma: P \to P/G$ is the canonical map, then by quasi-projectivity there is a $\theta: P \to P$ rendering the diagram

$$\begin{tikzcd} P \arrow[r, "\rho"] \arrow[d, dashed, "\theta"'] & A \arrow[d, "\beta"] \\ P \arrow[r, "\gamma"] & P/G \end{tikzcd}$$

commutative. Define the homomorphism $\delta: P/G \to P$ as $\delta = \theta\alpha\beta^{-1}$; then $\gamma\delta = \gamma\theta\alpha\beta^{-1} = \beta\rho\alpha\beta^{-1} = \mathbf{1}_{P/G}$. This means that the exact sequence $0 \to G \to P \xrightarrow{\gamma} P/G \to 0$ is splitting.

(E) *Let G be a subgroup of the quasi-projective group P such that there is an epimorphism $\rho: G \to P$. Then $K = \operatorname{Ker}\rho$ is a summand of G.* Let $\alpha: G/K \to P$ be the isomorphism induced by ρ. We have an injection $\beta: P \to P/K$ with $\operatorname{Im}\beta = G/K$ and $\beta\alpha = \mathbf{1}_{G/K}$. By quasi-projectivity, there is a $\theta: P \to P$ such that $\gamma\theta = \beta$ where $\gamma: P \to P/K$ denotes the canonical map. We argue that $\theta(P) \leq \gamma^{-1}(G/K) = G$. Now $\delta = \theta\alpha: G/K \to G$ satisfies $\gamma\delta = \mathbf{1}_{G/K}$, thus $0 \to K \to G \xrightarrow{\gamma} G/K \cong P \to 0$ is a splitting exact sequence.

Structure of Quasi-Projective Groups A complete classification of quasi-projective groups can be given in terms of cardinal invariants, based on the following theorem.

Theorem 11.2 (Fuchs–Rangaswamy [2]). *A group is quasi-projective if and only if either*

(i) *it is a free group; or*
(ii) *it is a torsion group such that each of its p-components is a direct sum of cyclic groups of fixed order p^{k_p}.*

Proof. Free groups F are obviously quasi-projective, and (C) implies that the groups F/nF are also quasi-projective for every $n \in \mathbb{N}$. By (B), the same holds for the direct sum $\oplus(F/p^{k_p}F)$ with different primes p. As $F/p^{k_p}F$ is a direct sum of cyclic groups of fixed order p^{k_p}, the sufficiency follows.

Conversely, assume P is quasi-projective. If P is torsion, then it cannot have a summand $\mathbb{Z}(p^\infty)$, because by (A) this summand would be quasi-projective, so by (D) it would contain every $\mathbb{Z}(p^n)$ as a summand—this is impossible. Thus P is reduced. It cannot have a summand of the form $C = \mathbb{Z}(p^n) \oplus \mathbb{Z}(p^m)$ with $n > m$, since there is an epimorphism $\mathbb{Z}(p^n) \to \mathbb{Z}(p^m)$ whose kernel is not a summand of C (cp. (D)). Therefore, the p-components of P are bounded by some p^k with no cyclic summands of different orders. Hence (ii) holds for P if torsion.

If P is torsion-free, then let F be a free subgroup of P generated by a maximal independent set, so that P/F is a torsion group. Let $\gamma: P \to P/F$ denote the natural map. We distinguish two cases according as P is of finite or infinite rank. If $\operatorname{rk} P$ is finite, then for every map $\phi: P \to P/F$ there is a $\theta: P \to P$ with $\phi = \theta\gamma$, and for different ϕ we have different θ. If P/F were infinite, then it had continuously many automorphisms (see Sect. 2 in Chapter 17, Exercise 3), so there would be this many choices for ϕ. But a finite rank P has only countably many endomorphisms. Thus P/F must be finite, P is finitely generated, so P is finitely generated free. If $\operatorname{rk} P$ is infinite, then we can find a surjective map $F \to P$, and (E) shows that P is isomorphic to a summand of F, so it is free.

Finally, suppose P is mixed. (C) implies that P/tP is quasi-projective, so free: $P = tP \oplus F$ with F a free group. If none of the summands is 0, then there is an epimorphism $F \to C \leq tP$, C cyclic, whose kernel is not a summand of F, contradicting (D). Thus P cannot be mixed. □

★ **Notes.** The listed properties of quasi-projectivity were borrowed from the pioneering paper L. Wu–J.P. Jans [Ill. J. Math. **11**, 439–448 (1967)], and from Fuchs–Rangaswamy [2]. The majority of the results (e.g., Theorem 11.2 is an exception) are valid for modules as well.

Exercises

(1) Describe the complete set of cardinal invariants attached to a quasi-projective group.
(2) $P^{(\kappa)}$ is quasi-projective for every cardinal κ whenever P is quasi-projective.
(3) (a) Suppose $P = \oplus_{n<\omega} P_n$ where the P_n are fully invariant in P. P is quasi-projective if and only if every P_n is quasi-projective.
    (b) Claim (a) may fail if the summands are not fully invariant.
(4) Fully invariant subgroups inherit quasi-projectivity.
(5) Only quasi-projective groups admit quasi-projective covers.
(6) Let $G < P$, P a quasi-projective group. Then $|\operatorname{End} P/G| \leq |\operatorname{End} P|$.

Problems to Chapter 3

PROBLEM 3.1. Characterize almost free groups in which the intersection of two direct summands is again a summand.

PROBLEM 3.2. For which ordinals σ do there exist (strongly) $\aleph_\sigma$-free groups that are not (strongly) $\aleph_{\sigma+1}$-free?

PROBLEM 3.3 (IRWIN). Is there a core class of $\aleph_1$-free groups? That is, a small collection of $\aleph_1$-free groups, each of cardinality $\aleph_1$, such that every $\aleph_1$-free group contains a member of this class.

Cf. Blass–Irwin [1].

PROBLEM 3.4. Let A be the free lattice-ordered group generated by the partially ordered group G. Relate A to G as groups.

Chapter 4
Divisibility and Injectivity

Abstract Most perfect objects in the category of abelian groups are those groups in which we can also 'divide:' for every element a and for every positive integer n, the equation $nx = a$ has a (not necessarily unique) solution for x in the group. These objects are the divisible groups which are universal in the sense that every group can be embedded as a subgroup in a suitable divisible group.

The divisible groups form one of the most important classes of abelian groups. In our presentation, we focus on their most prominent properties, many of them may serve as their characterization. Their outstanding feature is that they coincide with the injective groups, and as such they are direct summands in every group containing them as subgroups. Moreover, they constitute a class in which the groups admit a satisfactory characterization in terms of cardinal invariants.

The concluding topic for this chapter is concerned with a remarkable duality between maximum and minimum conditions on subgroups.

1 Divisibility

Since multiplication of group elements by integers makes sense, it is natural to consider divisibility of group elements by integers. Divisibility offers a great deal of information on how an element fits in the group.

Divisibility of Elements We shall say that the element a of the group A is **divisible** by $n \in \mathbb{N}$, in symbols: $n|a$, if the equation

$$nx = a \quad (a \in A) \tag{4.1}$$

is solvable for x in A, i.e., there exists a $b \in A$ such that $nb = a$. Evidently, (4.1) is solvable if and only if $a \in nA$.

We list some elementary consequences of the definition.

(a) If $x = b$ is a solution to (4.1), then the coset $b + A[n]$ is the set of all solutions of (4.1).
(b) If A is torsion-free, then (4.1) has at most one solution.
(c) If $\gcd\{n, o(a)\} = 1$, then (4.1) is solvable. For if $r, s \in \mathbb{Z}$ are such that $nr + o(a)s = 1$, then $x = ra$ satisfies $nx = nra = nra + o(a)sa = a$.

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(d) $m|a$ and $n|a$ imply $\text{lcm}\{m, n\}|a$. Indeed, if r, s satisfy $mr + ns = d = \gcd\{m, n\}$, and if $b, c \in A$ are such that $mb = a = nc$, then $(\text{lcm}\{m, n\})(rc + sb) = mnd^{-1}(rc + sb) = md^{-1}ra + nd^{-1}sa = a$.
(e) $n|a$ and $n|b$ with $a, b \in A$ imply $n|(a \pm b)$.
(f) If $A = B \oplus C$ is a direct sum, then $n|a = b + c$ $(b \in B, c \in C)$ if and only if both $n|b$ in B and $n|c$ in C. The same holds for infinite direct sums and direct products.
(g) If $\alpha : A \to B$ is a homomorphism, then $n|a$ in A implies $n|\alpha b$ in B.
(h) If p is a prime, then $p^k|a$ is equivalent to $k \le h_p(a)$.

Divisibility of Groups A group D is called **divisible** if

$$n|d \quad \text{for all } d \in D \text{ and all } 0 \neq n \in \mathbb{Z}.$$

Thus D is divisible exactly if $nD = D$ for every integer $n \neq 0$.

Example 1.1. The groups 0, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}(p^\infty)$, $\mathbb{R}$ are divisible groups, but no non-zero cyclic group is divisible.

A most useful criterion of divisibility is our next lemma.

Lemma 1.2. *A group D is divisible if and only if every homomorphism $\xi : \mathbb{Z} \to D$ can be extended to a homomorphism $\eta : \mathbb{Q} \to D$.*

Proof. Let ξ be a homomorphism of $\mathbb{Z}$ into the divisible group D. We think of $\mathbb{Q}$ as the union of the chain $\langle 1 \rangle < \langle (2!)^{-1} \rangle < \cdots < \langle (n!)^{-1} \rangle < \ldots$ of infinite cyclic groups. The element $\xi(1) = d_1 \in D$ is divisible by 2, so there is a $d_2 \in D$ with $2d_2 = d_1$, and we can extend ξ to $\xi_2 : \langle (2!)^{-1} \rangle \to D$ by letting $\xi_2((2!)^{-1}) = d_2$. If we have an extension $\xi_n : \langle (n!)^{-1} \rangle \to D$ with $\xi_n((n!)^{-1}) = d_n$, then we select a $d_{n+1} \in D$ satisfying $(n+1)d_{n+1} = d_n$, and define $\xi_{n+1}(((n+1)!)^{-1}) = d_{n+1}$ to extend ξ_n. At the end of this stepwise process we arrive at a desired homomorphism $\eta : \mathbb{Q} \to D$ (whose restriction to $\langle (n!)^{-1} \rangle$ equals ξ_n).

Conversely, assume that the group D has the indicated property, and pick any $d \in D$. There is a homomorphism $\xi : \mathbb{Z} \to D$ with $\xi(1) = d$. By hypothesis, ξ can be extended to a map $\eta : \mathbb{Q} \to D$. Then the element $\eta(n^{-1})$ satisfies $n\eta(n^{-1}) = d$, establishing the divisibility of d. □

A group D is said to be p-**divisible** (p a prime) if $p^k D = D$ for every positive integer k. Since $p^k D = p \cdots pD$, it is obvious that p-divisibility is implied by $pD = D$, i.e. every element of D is divisible by p. Then every element is of infinite p-height in D.

(A) *A group is divisible if and only if it is p-divisible for every prime p.* Indeed, if $pD = D$ for every prime p and $n = p_1 p_2 \cdots p_k$ with primes p_i, then $nD = p_1 p_2 \cdots p_k D = p_1 \cdots p_{k-1} D = \cdots = p_1 D = D$.
(B) *A p-group is divisible if and only if it is p-divisible.* In view of (c), for a p-group A we always have $qA = A$ whenever the primes p, q are different.
(C) *A p-group D is divisible exactly if every element of order p is of infinite height.* Only sufficiency requires a proof. So assume every element of order p is of

infinite height, and let $a \in D$ be of order p^k. We induct on k to prove that $p|a$. For $k = 1$, the claim is included in the hypothesis, so assume $k > 1$. By hypothesis, $p^{k-1}a$ has infinite height, thus $p^{k-1}a = p^k b$ for some $b \in D$. Since $o(a - pb) \leq p^{k-1}$, we have $p|a - pb$ by induction hypothesis, whence $p|a$, indeed.

(D) *Epimorphic images of a divisible group are divisible.* This is an immediate consequence of (g) above.

(E) *A direct sum* (*direct product*) *of groups is divisible if and only if each component is divisible.* This follows at once from (f).

(F) *If D_i ($i \in I$) are divisible subgroups of A, then so is their sum $\sum_{i \in I} D_i$.* This is evident in view of (e).

An immediate consequence is that the sum of all divisible subgroups of a group is again divisible, so we have the following result:

Lemma 1.3. *Every group A contains a maximal divisible subgroup D. D contains all divisible subgroups of A.* □

Groups that contain no divisible subgroups other than 0 are called **reduced**.

Embedding in Divisible Groups Recall that free groups are universal in the sense that every group is an epic image of a suitable free group. The next result shows that divisible groups have the dual universal property.

Theorem 1.4. *Every group can be embedded as a subgroup in a divisible group.*

Proof. The infinite cyclic group $\mathbb{Z}$ can be embedded in a divisible group, namely, in $\mathbb{Q}$. Hence every free group can be embedded in a direct sum of copies of $\mathbb{Q}$, which is a divisible group. Now if A is an arbitrary group, then $A \cong F/H$ for some free group F and a subgroup H of F. If we embed F in a divisible group D, then A will be isomorphic to the subgroup F/H of the divisible group D/H. □

It follows that, for every group A, there is an exact sequence

$$0 \to A \to D \to E \to 0$$

with D and (hence) E divisible.

★ **Notes.** There is no need to emphasize the relevance of divisibility in the theory of abelian groups: the reader will soon observe that a very large number of theorems rely on this concept. (In early literature, divisible groups were called *complete* groups.)

Exercises

(1) The additive group of any field of characteristic 0 is divisible.

(2) The factor group $J_p/\mathbb{Z}$ is divisible.

(3) A group is divisible exactly if it satisfies one of the following conditions:

   (a) it has no finite epimorphic image $\neq 0$;
   (b) it has no maximal subgroups (it coincides with its own Frattini subgroup).

(4) If $\{a_i\}_{i \in I}$ is a generating set (or a maximal independent set) in a group D, and if $n | a_i$ in D for every $i \in I$ and every $n \in \mathbb{N}$, then D is divisible.
(5) A direct sum (direct product) of groups is reduced if and only if every component is reduced.
(6) Let $0 \to A \to B \to C \to 0$ be an exact sequence. If both A and C are divisible (p-divisible), then so is B; if both are reduced, then B is also reduced.
(7) The maximal divisible subgroup of a torsion-free group coincides with the first Ulm subgroup of the group.
(8) Let A be the direct product, and B the direct sum of the groups B_n $(n < \omega)$. A/B is a divisible group if and only if, for every prime p, $pB_n = B_n$ holds for almost all n.
(9) Direct limits of divisible groups are divisible.
(10) (Szélpál) Assume A is a group such that all non-zero factor groups of A are isomorphic to A. Show that $A \cong \mathbb{Z}(p)$ or $A \cong \mathbb{Z}(p^\infty)$ for some p.
(11) Using Hom and Ext, show that (a) D is divisible if and only if $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, D) = 0$; (b) A is reduced if and only if $\mathrm{Hom}(\mathbb{Q}, A) = 0$.

2 Injective Groups

Injective groups are dual to projective groups; they are defined by dualizing the definition of projectivity.

Injectivity A group D is said to be **injective** if, for every diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\alpha"] \arrow[d, "\xi"] & A \arrow[r] \arrow[dl, dashed, "\eta"] & C \arrow[r] & 0 \\
& D & & &
\end{tikzcd}$$

with exact rows and a homomorphism $\xi : B \to D$, there is a homomorphism $\eta \colon A \to D$ making the triangle commute: $\eta\alpha = \xi$. If B is identified with its image in A, then the injectivity of D can be interpreted as the extensibility of any homomorphism $\xi \colon B \to D$ to a homomorphism of any group A containing B into D.

Our next purpose is to show that the injective groups are precisely the divisible groups.

Theorem 2.1 (Baer [8]). *A group is injective if and only if it is divisible.*

Proof. That an injective group is divisible follows at once from Lemma 1.2. In order to verify the converse, let D be a divisible group, and $\xi : B \to D$ a homomorphism from a subgroup B of the group A. Consider all groups G between B and A, such that ξ has an extension $\theta : G \to D$. The set S of all pairs (G, θ) is partially ordered by setting

$$(G, \theta) \leq (G', \theta') \quad \text{if and only if} \quad G \leq G' \text{ and } \theta = \theta' \restriction G.$$

S is not empty, since $(B, \xi) \in S$, and is inductive, since every chain (G_i, θ_i) $(i \in I)$ has an upper bound in S, *viz.* (G, θ) where $G = \bigcup_{i \in I} G_i$ and $\theta = \bigcup_{i \in I} \theta_i$. By Zorn's lemma, there exists a maximal pair (G_0, θ_0) in S. We claim: $G_0 = A$.

By way of contradiction, suppose $G_0 < A$. If $a \in A \setminus G_0$ is such that $na = g \in G_0$ for some $n \in \mathbb{N}$, then choose a minimal such n. By the divisibility of D, some $d \in D$ satisfies $nd = \theta_0 g$. It is straightforward to check that

$$x + ka \mapsto \theta_0 x + kd \qquad (x \in G_0,\ 0 \leq k < n)$$

is a genuine homomorphism of $\langle G_0, a \rangle$ into D. If $na \notin G_0$ for all $n \in \mathbb{N}$, then the correspondence $x + ka \mapsto \theta_0 x + kd$ $(x \in G_0)$ is a homomorphism for any choice of $d \in D$ (no restriction on $k \in \mathbb{Z}$). In either case, $G_0 < A$ contradicts the maximality of (G_0, θ_0). Hence $\theta_0 : A \to D$ is a desired extension of ξ. □

Baer's Criterion From the proof we derive the famous **Baer criterion for injectivity** (whose real importance lies in the fact that its analogue holds for modules over any ring):

Corollary 2.2. *A group D is injective if and only if, for each $n \in \mathbb{N}$, every homomorphism $\mathbb{Z}n \to D$ extends to a homomorphism $\mathbb{Z} \to D$.*

Proof. A careful analysis of the preceding proof shows that the only place where the injectivity of D was needed was to assure the existence of a $d \in D$. The same conclusion can be reached if we extend the map $\mathbb{Z}n \to D$ given by $n \mapsto \theta_0 g$ to $\mathbb{Z} \to D$, and pick d as the image of 1. □

Another simple result is a trivial consequence of Theorem 2.1 and Corollary 2.2, but it is important enough to record it as a corollary.

Corollary 2.3. *Epic images of injective groups are injective.* □

The Summand Property We are now able to show that injective (divisible) subgroups are always summands.

Corollary 2.4 (Baer [8]). *A divisible subgroup D of a group A is a summand, $A = D \oplus C$ for some subgroup C of A. Here C can be chosen so as to contain any preassigned subgroup B of A with $D \cap B = 0$. (Thus D-high subgroups are always summands.)*

Proof. By Theorem 2.1, the identity map $\mathbf{1}_D : D \to D$ extends to a homomorphism $\eta : A \to D$. Therefore, $A = D \oplus \operatorname{Ker} \eta$. If $D \cap B = 0$, then the same argument implies that the map $D \oplus B \to D$ which is the identity on D and trivial on B extends to an $\eta : A \to D$. Then evidently, $B \leq \operatorname{Ker} \eta$. □

Given a group A, consider the subgroup generated by all divisible subgroups of A. From Sect. 1(F) we know that D is divisible, it is the **maximal divisible subgroup** of A. By Corollary 2.4, $A = D \oplus C$, where evidently, the summand C has to be reduced. We thus have the first part of

Theorem 2.5. *Every group A is the direct sum of a divisible group D and a reduced group C,*

$$A = D \oplus C.$$

D is a uniquely determined subgroup of A, C is unique up to isomorphism.

Proof. To verify the second claim, it is clear that if $A = D \oplus C$ with D divisible and C reduced, then D ought to be the unique maximal divisible subgroup of A. Hence D is unique, and a complement is as always unique up to isomorphism. □

A consequence of the last theorem is that a problem on abelian groups can be often reduced to those on divisible and reduced groups.

We now summarize as a main result:

Theorem 2.6. *For a group, the following conditions are equivalent:*

(i) *it is divisible;*
(ii) *it is injective;*
(iii) *it is a direct summand in every group containing it.*

Proof. We had proved earlier the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), so only (iii) $\Rightarrow$ (i) is needed to complete the proof. Let D satisfy (iii). Theorem 1.4 shows that $D \leq E$ for a divisible group E. Then (iii) implies D is a summand of E, so (i) follows. □

Injective Hulls Theorems 1.4 and 2.1 guarantee that every group can be embedded in an injective group. This important fact can be considerably improved by establishing the existence of a *minimal* embedding.

A minimal divisible (injective) group containing the group A is called a **divisible hull** or **injective hull** of A, it will be denoted as $E(A)$. The main result on injective hulls is as follows.

Theorem 2.7. *Any injective group containing A contains an injective hull of A. The injective hull of A is unique up to isomorphism over A.*

Proof. Let E be an injective group containing A. There exists a subgroup C of E maximal with respect to the property $A \cap C = 0$. For each $c_0 \in C$ and prime p, there is $x \in E$ such that $px = c_0 \in C$. If $x \notin C$, then $c + kx = a \neq 0$ $(c \in C, a \in A)$ for some $k \in \mathbb{Z}$ prime to p. Then $pc + kc_0 = pa$ must be 0, which shows that kc_0, and hence c_0 is divisible by p in C. That is, C is divisible. By Corollary 2.4, we can write $E = C \oplus D$ with $A \leq D$. Clearly, D is divisible, and by the maximality of C, D cannot have any proper summand still containing A. Therefore, D is minimal divisible containing A, and by the choice of C, A is essential in D.

If D_1 and D_2 are two minimal injective groups containing A, then because of Theorem 2.1, the identity map $\mathbf{1}_A$ of A extends to a homomorphism $\phi : D_1 \to D_2$. Since ϕD_1 is divisible containing A, we have $A \leq \phi D_1 \leq D_2$. By the minimality of D_2, the last $\leq$ must be equality. Now $\phi|_A = \mathbf{1}_A$ implies $\operatorname{Ker}\phi \cap A = 0$ which means that ϕ is monic. Hence ϕ is an isomorphism between D_1 and D_2 leaving A element-wise fixed. □

An important consequence of the preceding proof is

Corollary 2.8. *An injective group E containing A is an injective hull of A if and only if A is an essential subgroup in E.* □

Example 2.9.

(a) The injective hull of a torsion-free group A is its divisible hull; it can also be obtained as $\mathbb{Q} \otimes A$.
(b) In order to get the injective hull of a p-group A with basic subgroup $B = \oplus_{i \in I} \langle b_i \rangle$, embed each $\langle b_i \rangle$ in a quasi-cyclic group C_i, and form $A + \sum_{i \in I} C_i$. (Check that this group is divisible, and A is essential in it.)

Quasi-Injectivity We now embark upon a noteworthy generalization of injectivity. A group A is called **quasi-injective** if every homomorphism of every subgroup into A extends to an endomorphism of A.

(A) Summands of quasi-injective groups are again quasi-injective.
(B) Powers of a quasi-injective group are quasi-injective.
(C) A torsion group A is quasi-injective if and only if all of its p-components A_p are quasi-injective.
(D) Neither direct sums nor direct products of quasi-injective groups are necessarily quasi-injective (this is trivial from Theorem 2.11).

Theorem 2.10 (Johnson–Wong). *A group is quasi-injective if and only if it is a fully invariant subgroup of its injective hull.*

Proof. First assume A is fully invariant in an injective group E, and let $\phi\colon B \to A$ where $B \leq A$. By injectivity, ϕ extends to an endomorphism $\psi\colon E \to E$ which—by full invariance—must map A into itself.

Conversely, let A be quasi-injective, and η an endomorphism of the injective hull E of A. The subgroup $B = \{a \in A \mid \eta a \in A\}$ is mapped by η into A, so η extends to a map $\psi : A \to A$. If $(\eta - \psi)a \in A$ for an $a \in A$, then $\eta a \in A$, so $a \in B$. Thus $(\eta - \psi)A \cap A = 0$, whence $(\eta - \psi)A = 0$ by the essential character of A. This means that $\eta \restriction A = \psi$, and so $\eta(A) \leq A$, indeed. □

From full invariance it follows that if $E = E_1 \oplus E_2$ is a direct decomposition of the injective hull of a quasi-injective group A, then A has a corresponding decomposition: $A = (A \cap E_1) \oplus (A \cap E_2)$.

A consequence of the last lemma is that every group A admits a **quasi-injective hull**: a smallest quasi-injective group containing A as a subgroup. This is simply the fully invariant subgroup generated by A in the injective hull $E(A)$ of A. This characterization of quasi-injective groups enables us to prove the following structural result on them.

Theorem 2.11 (Kil'p [1]). *A group is quasi-injective exactly if it either injective or is a torsion group whose p-components are direct sums of isomorphic cocyclic groups.*

Proof. If A is a group as stated, then it is immediately seen that it is fully invariant in its injective hull.

Conversely, let A be a quasi-injective group, and E its injective hull. If A contains an element a of infinite order, then for every $b \in E$ there is a map $\phi : \langle a \rangle \to E$ with $\phi a = b$, so the only fully invariant subgroup of E containing a is E itself. Thus $A = E$ in this case. If A is a torsion group, then its p-components A_p are likewise quasi-injective, so fully invariant in their injective hulls E_p. The latter group is a direct sum of copies of $\mathbb{Z}(p^\infty)$ (see Theorem 3.1), and its non-zero fully invariant subgroups are the direct sums of copies of a fixed subgroup $\mathbb{Z}(p^k)$ of $\mathbb{Z}(p^\infty)$, where $k \in \mathbb{N}$ or $k = \infty$. □

More on Quasi-Injectivity Quasi-injective groups have several remarkable properties which led to various generalizations of quasi-injectivity in module categories (for abelian groups some of them coincide with quasi-injectivity). We mention a few interesting facts for illustration.

Proposition 2.12. *A quasi-injective group has the following properties:*

(i) *it is a* CS-*group: high subgroups are summands;*
(ii) *it is an extending group: every subgroup is contained as an essential subgroup in a direct summand;*
(iii) *a subgroup that is isomorphic to a summand is itself a summand;*
(iv) *if B, C are summands and $B \cap C = 0$, then $B \oplus C$ is also a summand.*

Proof. (i) $\Leftrightarrow$ (ii) is routine.

(ii) Let G be a subgroup of the quasi-injective group A, and let E_1 denote the injective hull of G in the injective hull E of A. Then $E = E_1 \oplus E_2$ holds for some $E_2 \leq E$, and $A = (A \cap E_1) \oplus (A \cap E_2)$. Clearly, G is essential in the first summand of A.
(iii) Let H be a subgroup, and G a summand of A with inverse isomorphisms $\gamma : G \to H$, $\beta : H \to G$. Now β followed by the injection map $G \to A$ extends to an endomorphism $\alpha : A \to A$. If this is followed by the projection $A \to G$ and then by γ, then the composite is $\mathbf{1}_H$. As this extends to $A \to H$, H is a summand.
(iv) Let $A = B \oplus B'$ and $\pi : A \to B'$ the projection. Then $B \oplus C = B \oplus \pi C$ where $\pi \restriction C$ is an isomorphism. By (iii), πC is a summand of A and hence of B'. It follows $B \oplus C$ is a summand of A. □

★ **Notes.** The true significance of injectivity of groups lies not only in its extremely important role in the theory of abelian groups, but also in the fact that it admits generalizations to modules over any ring such that most of its relevant features carry over to the general case. The injective property was discovered by Baer [8]. He also proved that for the injectivity of a left R-module M, it is necessary and sufficient that every homomorphism from every left ideal L of R into M extends to an R-homomorphism $\mathsf{R} \to M$. This extensibility property, with L restricted to principal left ideals generated by non-zero divisors, is perhaps the most convenient way to define divisible R-modules. It is then immediately clear that an injective module is necessarily divisible. For modules over integral domains, the coincidence of injectivity and divisibility characterizes the Dedekind domains (see Cartan–Eilenberg [CE]). For torsion-free modules over Ore domains divisibility always implies injectivity.

Mishina [3] calls a group A **weakly injective** if every endomorphism of every subgroup extends to an endomorphism of A. Besides quasi-injective groups, only the groups of the form $A = D \oplus R$ have this property where D is torsion divisible and R is a rational group.

Epimorphic images of injective left R-modules are again injective if and only if R is left hereditary (left ideals are projective); an integral domain is hereditary exactly if it is Dedekind. Note that the semi-simple artinian rings are characterized by the property that all modules over them are injective.

It is an easy exercise to show that over left noetherian rings every left module contains a maximal injective submodule. This is not necessarily a uniquely determined submodule, unless the ring is, in addition, left hereditary. E. Matlis [Pac. J. Math. **8**, 511–528 (1958)] and Z. Papp [Publ. Math. Debrecen **6**, 311–327 (1959)] proved that every injective module over a left noetherian ring R is a direct sum of directly indecomposable ones. If, in addition, R is commutative, then the indecomposable injective R-modules are in a one-to-one correspondence with the prime ideals P of R, namely, they are the injective hulls of R/P (for $\mathsf{R} = \mathbb{Z}$ take $\mathsf{P} = (0)$ or (p)); cf. Matlis [loc. cit.]. It is remarkable that direct sums (and direct limits) of injective left modules are again injective if and only if the ring is left noetherian.

Before the term 'essential extension' was generally accepted, some authors were using 'algebraic extension' instead. Szele [3] developed a theory of 'algebraic' and 'transcendental' extensions of groups, modeled after field theory, where 'algebraic' meant 'essential,' while 'transcendental' was used for 'non-essential' extensions. He established the analogue of algebraic closure (injective hull) a few years before the Eckman–Schopf paper on the existence of injective hulls was published.

Quasi-injectivity was introduced by R.E. Johnson and E.T. Wong [J. London Math. Soc. **36**, 260–268 (1961)]. As far as generalizations of quasi-injectivity are concerned, interested readers are referred to the monographs S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Notes 17 (1990), and N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, Pitman Research Notes 313 (1994).

Exercises

(1) (Kertész) A group is divisible if and only if it is the endomorphic image of every group containing it.

(2) If $A = \oplus_{i \in I} B_i$, then $E(A) = \oplus_{i \in I} E(B_i)$ where $E(*)$ stands for the injective hull.

(3) Every automorphism of a subgroup A of an injective group D is induced by an automorphism of D.

(4) A direct sum of copies of $\mathbb{Z}(p^\infty)$ is injective as a J_p-module as well.

(5) Let A be a torsion-free group, and E the set of all pairs $(a, m) \in A \times \mathbb{Z}$ with $m \neq 0$ subject to

(a) $(a, m) = (b, n)$ if and only if $mb = na$,
(b) $(a, m) + (b, n) = (na + mb, mn)$.

Show that E is a divisible hull of the image of A under the map $a \mapsto (a, 1)$.

(6) If C is a subgroup of the group B such that B/C is isomorphic to a subgroup H of G, then there exists a group A containing B such that $A/C \cong G$.

(7) Given A and integer $n > 0$, there exists an essential extension C of A such that $A = nC$. Is C unique up to isomorphism (over A)?

(8) (a) (Charles, Khabbaz) A subgroup A of a divisible group D is the intersection of divisible subgroups of D if and only if, for every prime p, $A[p] = D[p]$ implies $pA = A$.
(b) (Bergman) Every group is the intersection of divisible subgroups in a suitable divisible group. [Hint: push-out of two different injective hulls.]

(9) (Szele) Let B be a subgroup of A. Call an $a \in A$ of infinite or prime power order **algebraic over** B if $a = 0$ or $\langle a \rangle \cap B \neq 0$. A is **algebraic over** B if every $a \in A$ is algebraic over B.

(a) A is algebraic over B if and only if B is an essential subgroup of A.
(b) A is a maximal algebraic extension of B exactly if $A = E(B)$.
(c) Derive Theorem 2.1 from the existence of maximal algebraic extensions in E.

(10) The group $A \cong \mathbb{Z}(p^2) \oplus \mathbb{Z}(p)$ is a CS-group, but not quasi-injective.

3 Structure Theorem on Divisible Groups

Structure of Divisible Groups The groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ were among our first examples for divisible groups. The main theorem of this section will show that there are no divisible groups other than the direct sums of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ with various primes p.

Theorem 3.1. *A divisible group D is the direct sum of groups each of which is isomorphic either to the additive group $\mathbb{Q}$ of rational numbers and or to a quasi-cyclic group $\mathbb{Z}(p^\infty)$:*

$$D \cong (\oplus_\kappa \mathbb{Q}) \oplus \oplus_p (\oplus_{\kappa_p} \mathbb{Z}(p^\infty)).$$

The cardinal numbers κ, κ_p (for every prime p) form a complete and independent system of invariants for D.

Proof. The torsion part $T = tD$ of D is divisible, so by Corollary 2.4 it is a summand: $D = T \oplus E$, where E is torsion-free and divisible. The p-components

T_p of $T = \oplus_p T_p$ are divisible, so it suffices to show that T_p is a direct sum of copies of $\mathbb{Z}(p^\infty)$, and E is a direct sum of copies of $\mathbb{Q}$.

Owing to divisibility, for each $a \in T_p$ we can find a sequence $a = a_0, a_1, a_2, \ldots, a_n, \ldots$ in T_p such that $pa_{n+1} = a_n$ for $n = 0, 1, \ldots$. Thus every element embeds in a subgroup $\cong \mathbb{Z}(p^\infty)$. Consider the set S of subgroups B_i in T_p that are direct sums of subgroups $\cong \mathbb{Z}(p^\infty)$, and partially order S by declaring $B_i \leq B_j$ if B_i is a summand of B_j. Use Zorn's lemma to argue that T_p contains a maximal $B \in S$. Such a B is injective, so $T_p = B \oplus C$. If $C \neq 0$, then it must contain a subgroup $\cong \mathbb{Z}(p^\infty)$, contradicting the maximal choice of B. Hence $T_p = B$, and T_p is a direct sum of copies of $\mathbb{Z}(p^\infty)$. The proof for E is similar, making use of the embeddability of every element in a subgroup $\cong \mathbb{Q}$.

To show that the cardinal numbers of the summands $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ do not depend on the special choice of the decompositions, it is enough to note that these cardinals are exactly the ranks $\mathrm{rk}_0(D)$ and $\mathrm{rk}_p(D)$, which are uniquely determined by D. They do form a complete system of invariants for D, since if given $\mathrm{rk}_0(D) = \kappa$ and $\mathrm{rk}_p(D) = \kappa_p$, we can uniquely reconstruct D as a direct sum of κ copies of $\mathbb{Q}$ and κ_p copies of $\mathbb{Z}(p^\infty)$ for each prime p. Moreover, that these cardinals can be chosen arbitrarily is obvious. □

Note that Corollary 2.8 implies that

$$\mathrm{rk}_0(E(A)) = \mathrm{rk}_0(A) \quad \text{and} \quad \mathrm{rk}_p(E(A)) = \mathrm{rk}_p(A) \text{ for every prime } p.$$

Consequently, the structure of the divisible hull $E(A)$ of A is completely determined by the ranks of A.

Example 3.2. The additive group $\mathbb{R}$ of the real numbers is a torsion-free divisible group of the power of the continuum $2^{\aleph_0}$. Hence $\mathbb{R} \cong \oplus_\kappa \mathbb{Q}$, where $\kappa = 2^{\aleph_0}$.

Example 3.3. The multiplicative group of the positive real numbers is a torsion-free divisible group of the power of the continuum $2^{\aleph_0}$. It is isomorphic to $\mathbb{R}$ under the correspondence $r \mapsto \log r$.

Example 3.4. The multiplicative group $\mathbb{R}^\times$ of the non-zero real numbers is the direct product of the group in Example 3.3 and the multiplicative cyclic group $\langle -1 \rangle \cong \mathbb{Z}(2)$.

Example 3.5. The multiplicative group $\mathbb{T}$ of complex numbers of absolute value 1, the **circle group**, is isomorphic to $\mathbb{R}/\mathbb{Z}$, the (additive) group of reals mod 1. The torsion subgroup is $\cong \mathbb{Q}/\mathbb{Z}$, so the p-components are quasi-cyclic p-groups. Therefore,

$$\mathbb{R}/\mathbb{Z} \cong \oplus_p \mathbb{Z}(p^\infty) \oplus (\oplus_\kappa \mathbb{Q})$$

where again $\kappa = 2^{\aleph_0}$.

Example 3.6. The multiplicative group $\mathbb{C}^\times$ of all complex numbers $\neq 0$ is the direct product of the circle group and the multiplicative group of the positive real numbers. It is isomorphic to its subgroup: the circle group.

Cogenerators A group C is called a **cogenerator** of the category $\mathcal{A}b$ of abelian groups if every abelian group is contained in a suitable direct product of copies of C, or, equivalently, every non-zero abelian group has a non-trivial homomorphism into C.

Theorem 3.7. *A group is a cogenerator of $\mathcal{A}b$ if and only if it has a summand isomorphic to $\mathbb{Q}/\mathbb{Z}$. Thus $\mathbb{Q}/\mathbb{Z}$ is the minimal cogenerator of the category $\mathcal{A}b$.*

Proof. Every non-trivial homomorphic image of a quasi-cyclic group is isomorphic to the group itself, so any cogenerator C must contain $\mathbb{Z}(p^\infty)$ for each prime p. These $\mathbb{Z}(p^\infty)$ generate their direct sum in any group. Such a direct sum is $\cong \mathbb{Q}/\mathbb{Z}$, an injective group, so a summand of C.

Conversely, any group $A \neq 0$ has a non-trivial homomorphism into $\mathbb{Q}/\mathbb{Z}$. For, $0 \neq a \in A$ implies $\langle pa \rangle < \langle a \rangle$ for some prime p, and then the cyclic group $\langle a \rangle$ can be mapped upon the cyclic subgroup $\langle c \rangle$ of order p in $\mathbb{Z}(p^\infty)$ *via* $a \mapsto c$. This map extends to a homomorphism of $A \to \mathbb{Q}/\mathbb{Z}$. □

It is an important fact that the endomorphism rings of injective groups are very special. We do not discuss them here, because later we will learn more about these rings. We refer to Theorem 4.3 in Chapter 16.

Exercises

(1) Find the cardinal invariants for the following groups: (a) the direct product of κ copies of $\mathbb{Z}(p^\infty)$; (b) the direct product of κ copies of $\mathbb{Q}/\mathbb{Z}$; (c) the direct product of κ copies of $\mathbb{R}/\mathbb{Z}$; here κ denotes an infinite cardinal.
(2) Any two direct decompositions of a divisible group have isomorphic refinements.
(3) If A, B are divisible groups, each containing a subgroup isomorphic to the other, then $A \cong B$.
(4) If A is divisible, and B is a group such that $A \oplus A \cong B \oplus B$, then $B \cong A$.
(5) Find minimal cogenerators for the following categories: (a) torsion-free groups; (b) torsion groups; (c) p-groups.
(6) (Szele) A group contains no two distinct isomorphic subgroups if and only if it is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$.
(7) (Kertész) Let A be a p-group in which the heights of elements of finite heights are bounded by an integer $m > 0$. Then A is the direct sum of cocyclic groups. [Hint: $p^m A$ is divisible.]
(8) (a) (E. Walker) Any torsion-free group of infinite rank is a subdirect sum of copies of the group $\mathbb{Q}$.
    (b) An unbounded p-group is a subdirect sum of quasi-cyclic groups.
(9) (a) For every infinite cardinal κ, there is a group U_κ of cardinality κ which contains an isomorphic copy of every group of cardinality $\leq \kappa$.
    (b) In the set of groups U_κ with the indicated property there is one that is isomorphic to a summand of every other one.
(10) (W.R. Scott) An infinite group A is a **Jónsson group** if every proper subgroup has cardinality $< |A|$. Prove that A is a Jónsson group if and only if $A \cong \mathbb{Z}(p^\infty)$ for some prime p. [Hint: it is indecomposable, divisible, torsion.]

(11) A group is **hopfian** if its surjective endomorphisms are automorphisms; it is **cohopfian** if its injective endomorphisms are automorphisms. Show that the only hopfian-cohopfian torsion-free groups are the finite direct sums of $\mathbb{Q}$. (It is difficult to construct an infinite hopfian-cohopfian p-group.)

4 Systems of Equations

By the definition of divisible groups D, all 'linear' equations of the form $nx = d \in D$ with positive integers n are solvable for x in D. It is natural to raise the question of solvability of *systems* of linear equations in D. We are going to show that all consistent systems of linear equations are solvable in any divisible group.

Systems of Linear Equations By a **system of equations** over a group A is meant a set of equations

$$\sum_{j\in J} n_{ij}x_j = a_i \qquad (a_i \in A,\ i \in I) \tag{4.2}$$

where the coefficients n_{ij} are integers such that, for any fixed $i \in I$, almost all $n_{ij} = 0$; here, $\{x_j\}_{j\in J}$ is a set of unknowns, while I, J are index sets of arbitrary cardinalities. Equation (4.2) is a **homogeneous system** if $a_i = 0$ for all $i \in I$. We say that $x_j = g_j \in A$ $(j \in J)$ is a **solution** to (4.2) if Eq. (4.2) are satisfied in A whenever the x_j are replaced by the g_j. Sometimes, it is convenient to view a solution $x_j = g_j \in A$ $(j \in J)$ as an element $(\dots, g_j, \dots)$ in the direct product A^J.

For the solvability of the system (4.2), a trivial necessary condition is that it be **consistent** in the sense that, if a linear combination of the left sides of some equations vanishes (i.e., the coefficients of all the unknowns are 0), then it equals $0 \in A$ when the corresponding right-hand sides are substituted. Following Kertész, we give another, more versatile interpretation of the consistency and solvability of systems of equations.

The left members of the equations in (4.2) may be thought of as elements of the free group F on the set $\{x_j\}_{j\in J}$ of unknowns. Let H denote the subgroup of F generated by the left hand sides of the equations in (4.2). It is readily checked that the correspondence

$$\sum_{j\in J} n_{ij}x_j \mapsto a_i \qquad (i \in I) \tag{4.3}$$

induces a homomorphism $\eta : H \to A$ if and only if every representation of 0 as a linear combination of the left-hand sides is mapped by η upon 0, i.e. if the system is consistent in the sense above. Accordingly, we call (4.2) a **consistent system** if (4.3) extends to a homomorphism $\eta : H \to A$.

Clearly, two consistent systems define the same pair (H, η) exactly if the equations of either system are linear combinations of the equations of the other system, i.e. if the two systems of equations are **equivalent.** Thus a consistent system (or any of its equivalent systems) may be viewed as a pair (H, η), where H is a subgroup of the free group F on the set of unknowns, and η is a homomorphism $H \to A$.

Solvability of Systems of Equations Manifestly, $x_j = g_j \in A$ $(j \in J)$ is a solution of (4.2) if and only if the correspondence

$$x_j \mapsto g_j \qquad (j \in J) \tag{4.4}$$

extends to a homomorphism $\chi : F \to A$ whose restriction to H is η. Moreover, the extensions $\chi : F \to A$ of $\eta : H \to A$ are in a bijective correspondence with the solutions of (4.2), so we may use the notation (F, χ) for a solution of (4.2).

Theorem 4.1 (Gacsályi [1]). *Every consistent system of equations over A is solvable in A if and only if A is an injective group.*

Proof. The necessity is evident, since a single equation $nx = a \in A$ with $n \neq 0$ is a consistent system. Turning to the proof of sufficiency, let (H, η) be a consistent system of equations over a divisible group A. By Theorem 2.1, η extends to a homomorphism $\chi : F \to A$, that is, a solution exists. □

Consistency being a property of finite character, we conclude at once:

Corollary 4.2 (Gacsályi [1]). *A system of equations over an injective group D is solvable in D if and only if every finite subsystem has a solution in D.* □

It is worthwhile mentioning the following characterization of summands in terms of solvability of equations.

Proposition 4.3 (Gacsályi [1]). *A subgroup B of a group A is a direct summand exactly if every system of equations over B that is solvable in A can also be solved in B.*

Proof. If B is a summand, say, $A = B \oplus C$, then the B-coordinates of a solution in A provide a solution in B.

Conversely, assume that any system over B is solvable in B whenever it has a solution in A. For each coset u of A mod B, select a representative $a(u) \in A$, and consider the system

$$x_u + x_v - x_{u+v} = a(u) + a(v) - a(u + v) \in B \qquad \text{for all } u, v \in A/B.$$

By hypothesis, it has a solution $x_u = b(u) \in B$. Then the representatives $a(u) - b(u)$ of the cosets u form a subgroup C of A, and $A = B \oplus C$. □

★ **Notes.** The idea of considering systems of equations over a group was suggested by Szele whose student Gacsályi developed the theory in two papers. Generalizations to modules are due to Kertész who published several papers on the subject, starting with [Publ. Math. Debrecen **4**, 79–86 (1955)].

Exercises

(1) A system of equations over a group A is consistent if and only if it is solvable in some group containing A.
(2) A system of equations over an injective group contains maximal solvable subsystems.
(3) Prove that, for any prime p, the equation system

$$x_1 - px_2 = 1,\ x_2 - p^2x_3 = 1,\ \dots,\ x_n - p^n x_{n+1} = 1, \dots$$

over $\mathbb{Z}$ is not solvable in $\mathbb{Z}$, though each of its finite subsystems is solvable.
(4) A homogeneous system $(H, 0)$ over an arbitrary group A admits a non-trivial solution in A if and only if there exists a non-zero homomorphism $\phi : F/H \to A$ (notation as above). The maps ϕ are in a bijective correspondence with the non-trivial solutions of the system.
(5) A homogeneous system of n equations with $n + 1$ unknowns over any group $A \neq 0$ always has a non-trivial solution in A.

5 Finitely Cogenerated Groups

We turn our attention to a concept dual to finite generation.

Finite Cogeneration A set C of non-zero elements in a group A is called a set of **cogenerators** if, every non-zero subgroup of A contains an element of C. Equivalently, for any group G, and for any homomorphism $\phi : A \to G$, $C \cap \operatorname{Ker} \phi = \varnothing$ implies that ϕ is monic.

Example 5.1. For a set C of cogenerators, the subgroup $\langle C \rangle$ is an essential subgroup in A, and the set of elements in an essential subgroup with 0 omitted is always a set of cogenerators.

Example 5.2. In a cocyclic group $\mathbb{Z}(p^k)$ $(k \in \mathbb{N} \cup \infty)$, a generator of its minimal subgroup $\mathbb{Z}(p)$ is a singleton cogenerator of this group.

A group is **finitely cogenerated** if it has a finite set of cogenerators. The following theorem is an analogue of Theorem 2.5 in Chapter 3, and points out a beautiful duality between maximum and minimum conditions.

Theorem 5.3 (Prüfer [1], Kurosh [1], Yahya [1]). *For a group A, the following conditions are equivalent:*

(i) *A is finitely cogenerated;*
(ii) *A is an essential extension of a finite group;*
(iii) *A is torsion of finite rank;*
(iv) *A is a direct sum of a finite number of cocyclic groups;*
(v) *the subgroups of A satisfy the minimum condition.*

Proof. (i) $\Rightarrow$ (ii) By hypothesis, A has a finite set C of cogenerators. A cannot have elements a of infinite order, for otherwise we could select a cyclic subgroup in $\langle a\rangle$ disjoint from C. Thus the elements in C are of finite order, whence $\langle C\rangle$ is finite. A must be an essential extension of $\langle C\rangle$, so (ii) follows.

(ii) $\Leftrightarrow$ (iii) is straightforward.

(ii) $\Rightarrow$ (iv) Let A be an essential extension of a finite subgroup B. It follows that A is a torsion group with a finite number of non-zero p-components, and in order to prove (iv), we may without loss of generality assume that A is a p-group. Write $A = D \oplus F$ where D is divisible and F reduced. As $A[p] = B[p]$ is finite, there is a bound p^m for the heights of elements in $F[p]$, whence $p^{m+1}F = 0$ follows, i.e. F is finite. Both D and F are direct sums of cocyclic groups, their number ought to be finite, due to the finiteness of the socle.

(iv) $\Rightarrow$ (v) Observe that if A is quasi-cyclic, then it enjoys the minimum condition on subgroups. To complete the proof, we show that if $A = U \oplus V$ and both of U, V have the minimum condition of subgroups, then the same holds for A. If $B_1 \geq B_2 \geq \cdots \geq B_n \geq \ldots$ is a descending chain of subgroups in A, then in the chain $B_1 \cap U \geq B_2 \cap U \geq \cdots \geq B_n \cap U \geq \ldots$ there is a minimal member, say $B_m \cap U$. Then the chain $B_n/(B_m \cap U) = B_n/(B_n \cap U) \cong (B_n + U)/U \leq A/U \cong V$ for $n \geq m$ also contains a minimal member, say $B_t/(B_m \cap U)$. Then B_t is minimal in the given chain.

Finally, (v) $\Rightarrow$ (i). The minimum condition is inherited by subgroups, so A cannot contain elements of infinite order, neither can the socle of A be infinite. Thus the socle contains a finite set of cogenerators. $\square$

From the equivalence of (i) and (v) we infer that factor groups of finitely cogenerated groups are again finitely cogenerated. Observe that (ii) is equivalent to the finiteness of the socle in a torsion group.

Countable Number of Subgroups The groups in Theorem 5.3 have but countably many subgroups. There are only few other groups with this special property.

Proposition 5.4 (Rychkov–Fomin [1]). *A group A has fewer than continuously many subgroups if and only if it is an extension of a finitely generated group by a finite rank divisible subgroup of $\mathbb{Q}/\mathbb{Z}$. Then the set of subgroups is countable.*

Proof. It is an easy exercise to show that finitely generated groups and finite rank subgroups of $\mathbb{Q}/\mathbb{Z}$ have but a countable number of subgroups. A proof like (iv) $\Rightarrow$ (v) above shows that this remains true for their extensions.

For the converse, observe that a group of infinite rank has at least $2^{\aleph_0}$ subgroups, since different subsets of a maximal independent set generate different subgroups. Thus, if A has less than $2^{\aleph_0}$ subgroups, then $\operatorname{rk} A$ is finite. A maximal independent set in such an A generates a finitely generated subgroup F. The factor group A/F is torsion, it must also be of finite rank, so it satisfies the minimum condition on subgroups (Theorem 5.3); if its finite part is included in F, then A/F is divisible of finite rank. The group $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ has $2^{\aleph_0}$ subgroups, so $A/F < \mathbb{Q}/\mathbb{Z}$. □

The Prüfer Topology As another application of Theorem 5.3, we show:

Proposition 5.5. *Let S be a finite subset in the group A, and B a subgroup of A maximal disjoint from S. Then A/B satisfies the minimum condition for subgroups. If $|S| = 1$, then A/B is cocyclic.*

Proof. Every subgroup of A that contains B properly must intersect S, i.e., every non-zero subgroup of A/B contains one of the cosets $s + B$ with $s \in S$. Hence A/B is finitely cogenerated, and a reference to Theorem 5.3 completes the proof. □

Recall that the **Prüfer topology** of a group A is defined by declaring those subgroups U of A as a base of open neighborhoods of 0 for which A/U satisfies the minimum condition on subgroups. The preceding proposition is nothing else than asserting that the Prüfer topology is always Hausdorff.

★ **Notes.** While groups with maximum condition on subgroups are quite familiar to many mathematicians, because they have numerous applications, groups with minimum condition are often ignored due to their limited occurrence.

Exercises

(1) Let $0 \to A \to B \to C \to 0$ be an exact sequence of groups. B satisfies the minimum condition on subgroups if and only if so do A and C.
(2) An endomorphism η of a group with minimum condition is an automorphism if and only if $\operatorname{Ker} \eta = 0$.
(3) If A has minimum condition on subgroups, and if $A \oplus B \cong A \oplus C$, then $B \cong C$.
(4) If A has the minimum condition on subgroups and the group B satisfies $A \oplus A \cong B \oplus B$, then $B \cong A$.
(5) If A is finitely cogenerated, then a minimal set of cogenerators is contained in the socle of A.
(6) (Kulikov) Suppose A is a direct sum of cocyclic groups.
    (a) Every summand of A is likewise a direct sum of cocyclic groups.
    (b) Any two direct decompositions of A have isomorphic refinements.
(7) (de Groot) Let A and B be direct sums of cocyclic groups. If each is isomorphic to a pure subgroup of the other, then $A \cong B$.

(8) A group satisfies the minimum condition on fully invariant subgroups exactly if it is a direct sum of groups $\mathbb{Q}$, $\mathbb{Z}(p^{\infty})$ for finitely many different primes p, and a bounded group.

Problems to Chapter 4

PROBLEM 4.1. Call A *endo-divisible* if every E-homomorphism $\mathsf{L} \to A$ from a principal left ideal L of $\mathsf{E} = \operatorname{End} A$ extends to $\mathsf{E} \to A$. Which groups are endo-divisible?

PROBLEM 4.2. Characterize the quasi-injective hull of a p-group over its endomorphism ring.

Chapter 5
Purity and Basic Subgroups

Abstract In this chapter, we are going to discuss a basic concept: pure subgroup. This concept has been one of the most fertile notions in the theory since its inception in a paper by the pioneer H. Prüfer. The relevance of purity in abelian group theory, and later in module theory, has tremendously grown with time. While abelian groups have been major motivation for a number of theorems in category theory, purity has served as a prototype for relative homological algebra, and has played a significant role in model theory as well.

Pure subgroups, and their localized version: p-pure subgroups, are often used as a weakened notion of summands. In contrast to summands, most groups admit a sufficient supply of pure subgroups: every infinite set of elements embeds in a pure subgroup of the same cardinality. They are instrumental in several results that furnish us with criteria for a summand.

Every group contains, for every prime p, a p-pure subgroup, called p-basic subgroup, that is (if not zero) a direct sum of infinite cyclic groups and cyclic p-groups. Basic subgroups are unique up to isomorphism, and store relevant information about the containing group. Basic subgroups were introduced by Kulikov for p-groups, and occupy a center stage in the theory of these groups.

1 Purity

Pure Subgroups A subgroup G of a group A is called **pure** if the equation $nx = g \in G$ ($n \in \mathbb{N}$) is solvable for x in G whenever it is solvable in A. This amounts to saying that G is pure in A if, for any $g \in G$, $n|g$ in A implies $n|g$ in G. As $n|g$ in G means $g \in nG$, we see that G is pure in A if and only if

$$nG = G \cap nA \qquad \text{for every } n \in \mathbb{N}. \tag{5.1}$$

Thus purity means that the divisibility properties of the elements in G by integers are the same whether computed in A or in G.

If we equip A and a pure subgroup G with their $\mathbb{Z}$-adic topologies, then (5.1) implies that the topology of G inherited from A is equal to its own $\mathbb{Z}$-adic topology (but the converse fails).

We will often need the concept of p-purity for a prime p. A subgroup G of A is p-**pure** if

$$p^kG = G \cap p^kA \qquad \text{for every } k \in \mathbb{N}, \tag{5.2}$$

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
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or, in other words, the finite p-heights of elements in G are the same as (i.e., not less than) in A. Again, it follows that the induced p-adic topology on G coincides with its own p-adic topology.

We pause for a moment to clarify the connection between purity and p-purity. Our claim is that G is pure in A if and only if it is p-pure in A for every prime p. This is trivial one way, so assume that G is p-pure for every p. If $n = p_1^{r_1} \cdots p_k^{r_k}$ is the canonical representation of $n \in \mathbb{N}$, then $nG = p_1^{r_1} G \cap \cdots \cap p_k^{r_k} G = (G \cap p_1^{r_1} A) \cap \cdots \cap (G \cap p_k^{r_k} A) = G \cap nA$. Hence G is pure in A if and only if $G_{(p)}$ pure in $A_{(p)}$ holds for every localization. Thus in a p-group, purity and p-purity are equivalent.

Example 1.1. The direct sum $A = \oplus_{i \in I} A_i$ is always pure in the direct product $\bar{A} = \prod_{i \in I} A_i$.

Example 1.2. Rational groups (i.e., subgroups of $\mathbb{Q}$) and cocyclic groups contain no pure subgroups other than 0 and themselves.

Basic Properties of Purity Next, we assemble several useful facts concerning pure subgroups.

(A) *Direct summands are pure subgroups.* In particular, 0 and A itself are pure subgroups of A.

(B) *The torsion part of a mixed group and its p-components are pure subgroups.*

(C) *A subgroup of a divisible group is pure if and only if it is divisible* (*and hence a summand*).

(D) *If A/G is torsion-free, then G is pure in A.* In fact, $na = g \in G$ $(n \in \mathbb{N})$ with $a \in A$ implies $a \in G$.

(E) *If A is a p-group, and if the elements of order p of a subgroup G have the same finite heights in G as in A, then G is pure in A.* We use induction on the order to verify that if $g \in G$ is divisible by p^k in A, then also $p^k | g$ in G. For $g \in G$ of order p, this being true by hypothesis, assume that the claim holds for elements $g \in G$ of orders $< p^n$ where $n \geq 2$. If $o(g) = p^n$, and if some $a \in A$ satisfies $p^k a = g$, then by induction hypothesis there is an $h \in G$ such that $p^{k+1} h = pg$. Now $p^k h - g$ is either 0 or of order p, and $p^k(h - a) = p^k h - g$. By induction hypothesis, $p^k g' = p^k h - g$ for some $g' \in G$, whence $p^k(h - g') = g$ with $h - g' \in G$. Thus the height of g is not smaller in G than in A.

(F) *If G is a pure subgroup of a p-group A such that $G[p] = A[p]$, then $G = A$.* Again, we use induction on the order to prove that every $a \in A$ belongs to G. Let $o(a) = p^n$ $(n \geq 2)$, thus $p^{n-1} a \in A[p] = G[p]$. Owing to purity, $p^{n-1} g = p^{n-1} a$ for some $g \in G$. Here $o(g - a) \leq n - 1$, so by induction $g - a \in G$, so also $a \in G$.

(G) *In torsion-free groups, intersection of pure subgroups is again pure.* As a matter of fact, an equation $nx = g$ has at most one solution in A; therefore, if it is solvable in A, then its unique solution belongs to every pure subgroup containing g.

In view of this, in a torsion-free group A, for every subset $S \subset A$, there exists a minimal pure subgroup containing S. This is the intersection of all pure subgroups containing S; our notation for this subgroup is $\langle S \rangle_*$. It may be called **the pure subgroup generated by** S. It is easy to check that $\langle S\rangle_*/\langle S\rangle$ is precisely the torsion subgroup in $A/\langle S\rangle$.

(H) *Purity is an inductive property: the union of a chain of pure subgroups is pure.* For, if G is the union of a chain $G_1 \leq \dots \leq G_\sigma \leq \dots$ of pure subgroups, and if $nx = g \in G$ is solvable in A, then it is solvable in G_σ for every index σ with $g \in G_\sigma$. It is *a fortiori* solvable in G.

The following theorem is of utmost importance, it lists the most frequently needed properties of purity.

Theorem 1.3 (Prüfer [2]). *Let B, C be subgroups of the group A such that $C \leq B \leq A$. We then have:*

(i) *if C is pure in A, then it is pure in B;*
(ii) *if C is pure in B and B is pure in A, then C is pure in A;*
(iii) *if B is pure in A, then B/C is pure in A/C;*
(iv) *if C is pure in A and B/C is pure in A/C, then B is pure in A.*

Proof.

(i) is obvious.
(ii) Under the stated hypotheses, $nC = C \cap nB = C \cap (B \cap nA) = (C \cap B) \cap nA = C \cap nA$ for every $n \in \mathbb{N}$.
(iii) follows from the equalities $n(B/C) = (nB + C)/C = [(B \cap nA) + C]/C = [B \cap (nA + C)]/C = B/C \cap n(A/C)$ (we used the modular law).
(iv) Assuming the stated hypotheses, let $na = b \in B$ for some $a \in A$ and $n \in \mathbb{N}$. Then $n(a + C) = b + C$ whence hypothesis implies that there is a $b' \in B$ such that $n(b' + C) = b + C$, i.e. $nb' = b + c$ for a suitable $c \in C$. In view of the purity of C, from $n(b' - a) = c$ we get a $c' \in C$ satisfying $nc' = c$. It only remains to check that $b' - c' \in B$ and $n(b' - c') = b$. □

Thus (iii) and (iv) combined claim that *the natural correspondence between subgroups of A/C and subgroups of A containing the pure subgroup C preserves purity.*

Lemma 1.4. *Let B be a pure subgroup of A. If B is torsion-free and A/B is torsion, then B is a summand of A.*

Proof. If T denotes the torsion subgroup of A, then $B \oplus T$ is an essential subgroup in A. We claim that it is all of A. For every $a \in A \setminus T$ there is an integer $n > 0$ such that $na = b + t$ with $b \in B, t \in T$. For some $m > 0$ we have $mt = 0$, so $mna = mb$. By purity, some $b' \in B$ satisfies $mna = mnb'$. Then $a - b' \in T$ and $a \in B + T$. □

Embedding in Pure Subgroup A fundamental property of purity is stated in the following theorem.

Theorem 1.5 (Szele). *Every finite subgroup can be embedded in a countable pure subgroup, and every subgroup of infinite cardinality in a pure subgroup of the same power.*

Proof. Let B be a subgroup of A of cardinality κ. Consider all equations $nx = b$ for all $n \in \mathbb{N}$ and all $b \in B$ that are solvable in x $\in A$. For each such equation, we adjoin a solution $a_{n,b} \in A$ to B in order to obtain a subgroup $B_1 \leq A$ in which all these equations are solvable; thus, $B_1 = \langle B, a_{n,b} \, (\forall\, n, b)\rangle$. We repeat this precess with B_1 in place of B to get a subgroup B_2 in which all equations with right members in B_1 are solvable whenever they admit a solution in A. Thus proceeding, we form the union G of the chain $B \leq B_1 \leq \cdots \leq B_m \leq \ldots \; (m < \omega)$. This G must be pure in A, since every equation $nx = g \in G$ that is solvable in A is solvable in B_{m+1} if $g \in B_m$. As far as the size of G is concerned, it is clear that $|G| \leq \kappa\aleph_0$, whence both parts of our claim follow. □

For torsion-free groups, (G) above gives a more powerful statement: *there is a minimal pure subgroup containing a given subgroup*—a pure subgroup of the same rank. However, in torsion and mixed groups, it is unpredictable if there is a minimal pure subgroup containing a subgroup.

Example 1.6. Let $A = \oplus_{n<\omega} \langle a_n \rangle$ where $o(a_n) = p^{2n}$, and consider the subgroup $L = \oplus_{n<\omega} \langle a_{2n} + pa_{2n+1}\rangle$. Suppose there is a minimal pure subgroup C containing L. Clearly, $p^{4n+1}a_{2n+1} \in L[p]$, thus $p^{4n+1}c_{2n+1} = p^{4n+1}a_{2n+1}$ for some $c_{2n+1} \in C$, so we have $a_{2n}+p(a_{2n+1}-c_{2n+1}) \in C$. Hill–Megibben [1] prove that any minimal pure subgroup C containing a subgroup L must satisfy $p^mC[p] \leq L[p]$ for some $m \in \mathbb{N}$. Therefore, if $4n - 1 > m$, then $p^{4n-1}a_{2n} + p^{4n}(a_{2n+1} - c_{2n+1}) \in p^mC[p] \leq L[p]$. It follows that $p^{4n-1}a_{2n} \in L[p]$ which is clearly impossible. Thus no such C exists.

A subgroup C of A is **purifiable** if there exists a pure subgroup $G \leq A$ containing C such that A has no pure subgroup containing C and properly contained in G. There is no satisfactory characterization of purifiable subgroups. We state a relevant result without proof.

Theorem 1.7 (Hill–Megibben [1]). *A p-group A has the property that all of its subgroups are purifiable if and only if $A = B \oplus D$, where B is a bounded and D is a divisible group.* □

It is useful to keep in mind that groups admit **pure $\aleph_0$-filtrations**:

Proposition 1.8. *For every pure subgroup G of A, there is a smooth chain $G = G_0 < G_1 < \cdots < G_\sigma < \cdots < G_\tau = A$ of pure subgroups for some ordinal τ such that each $G_{\sigma+1}/G_\sigma$ is countable.*

Proof. To obtain $G_{\sigma+1}$ from G_σ, choose $G_{\sigma+1}/G_\sigma$ as a countable pure subgroup of A/G_σ (Theorem 1.5). □

A word on purity in p-adic modules. Since p-adic modules are q-divisible for every prime $q \neq p$, purity is the same as p-purity. (p-purity in the p-adic sense is the same as in groups.)

Neat Subgroups There exist several concepts related to purity that deserve mentioning, besides the isotype subgroups that will be discussed in Chapters 11 and 15.

Honda [1] introduced a widely used generalization of purity. He called a subgroup N of A a **neat** subgroup if, for every prime p, the solvability of the equation $px = b \in N$ in A implies that it is also solvable in N. Equivalently, if

$$pN = N \cap pA \qquad \text{for every prime } p.$$

In torsion-free groups neatness does not offer anything new: it coincides with purity (Exercise 12). However, for torsion groups the situation is different.

Example 1.9. Here is a simple example of a neat subgroup that is not pure. Let $A = \langle u \rangle \oplus \langle v \rangle$ where $o(u) = p^3, o(v) = p$. The subgroup $N = \langle pu + v \rangle$ satisfies $pN = N \cap pA$, however, $0 = p^2N < N \cap p^2A = \langle p^2u \rangle$.

The importance of neatness lies in the following basic fact.

Lemma 1.10 (Honda [3]). *High subgroups are neat.*

Proof. Let B be a C-high subgroup of A, and $pa = b \in B$ $(a \in A)$. If $a \notin B$, then $\langle B, a \rangle \cap C \neq 0$, so there is a non-zero $c \in C$ with $b' + ka = c$ for some $0 < k < p$ and $b' \in B$. Then both pa and ka belong to $B \oplus C$, thus $a = b_0 + c_0$ for some $b_0 \in B, c_0 \in C$. Hence $b = pa = pb_0 + pc_0$ along with $B \cap C = 0$ implies $pa = pb_0$, so $pb_0 = b$. □

***κ*-Purity** We mention quickly another generalization of purity that involves higher cardinalities. Let κ be an infinite cardinal. A subgroup B of A is called **κ-pure** if B is a summand in every subgroup C of A that contains B such that $|C/B| < \kappa$. Using Theorem 2.12 below, it is an easy exercise to check that $\aleph_0$-purity is the same as purity.

Example 1.11. Let F be a free group with an epimorphism $\phi : F \to \mathbb{Q}$. Then $\operatorname{Ker}\phi$ is $\aleph_0$-pure, but not $\aleph_1$-pure in F.

Example 1.12. Let F be a free group, and H a subgroup such that F/H is an $\aleph_1$-free, but not a free group (e.g., the Baer–Specker group). Then H is $\aleph_1$-pure in F, but it is not a summand.

Example 1.13. Let H be a pure subgroup of a Σ-cyclic p-group F. Assume that F/H is of cardinality $\aleph_1$ and has no elements of infinite heights, but it is not Σ-cyclic. Then H is $\aleph_1$-pure in F, but not a summand. (Use Theorem 2.9.)

★ **Notes.** Purity was introduced by Prüfer in his seminal paper [2] under the name 'Servanzuntergruppe' (serving subgroup); it played an important role already in his papers. The early literature followed this terminology, until the term 'pure' was firmly established on the initiative of Kaplansky who adopted this simpler term from a paper by J. Braconnier.

The question of finding a minimal pure subgroup containing a given subgroup attracted several researchers. Benabdallah–Okuyama [1] introduce new invariants all of which vanish for purifiable subgroups. Both purifiability and its generalizations have been studied extensively by Okuyama; interested readers are advised to consult his papers [1], [2], where several new concepts were introduced. Pierce [2] characterizes and classifies **centers of purity** in p-groups; these are defined as subgroups C such that every C-high subgroup is pure.

The idea of κ-purity is credited to Gacsályi [1]. The search for cardinals κ for which κ-purity implies κ^+-purity is still on. Janakiraman–Rangaswamy [1] discuss strong purity: C is *strongly pure* in A if for every $c \in C$, there is a homomorphism $\phi : A \to C$ such that $\phi(c) = c$. Finitely generated and finite rank torsion-free strongly pure subgroups are summands. For a further generalization of purity, we refer to C. Walker [1], Nunke [5], B. Stenström [J. Algebra **8**, 352–361 (1968)]. There are a number of papers in the Russian literature on so-called ω-purity, introduced by Mishina and Skornyakov. Honda's neat subgroups generated a great deal of interest, even their homological aspects were explored.

Exercises

(1) Neither the sum nor the intersection of direct summands need be a pure subgroup.
(2) If G is pure in A, then nG is pure in nA for every $n \in \mathbb{N}$.
(3) If G is a pure subgroup in each member of a chain $B_0 \leq \cdots \leq B_i \leq \ldots$, then it is pure in their union $B = \cup_i B_i$.
(4) If $B \cap C$ and $B + C$ are pure subgroups of A, then so are B and C.
(5) A pure subgroup that is essential in A cannot be a proper subgroup.
(6) If G is pure in A, then so is $tA + G$.
(7) Let G be a pure subgroup of A. Then:

   (a) $G^1 = G \cap A^1$ (first Ulm subgroups);
   (b) $(G + A^1)/A^1$ is pure in A/A^1;
   (c) $G \leq A^1$ if and only if G is divisible.

(8) Let B be a pure subgroup of A, and S a subset of A such that $B \cap S = \varnothing$. There exists a pure subgroup C of A containing B and maximal with respect to the property $C \cap S = \varnothing$.
(9) A group is pure in every group containing it as a subgroup if and only if it is divisible.
(10) (a) Describe the groups in which every subgroup is pure.
   (b) Find the torsion groups in which every pure subgroup is a summand.
(11) Give an example of a non-pure subgroup such that its own $\mathbb{Z}$-adic topology is the same as the relative $\mathbb{Z}$-adic topology.
(12) (Honda)

   (a) In torsion-free groups, neatness coincides with purity.
   (b) Neatness is an inductive property.
   (c) If N is neat in A, and if either N or A/N is an elementary p-group, then N is a summand of A.

(13) A group A has no neat subgroups other than 0 and A if and only if $\mathrm{rk}(A) = 1$.
(14) (a) Let E denote an injective hull of A. A subgroup B is neat in A if and only if $B = A \cap D$ for a divisible subgroup D of E.
   (b) Every subgroup C of A can be embedded in a neat subgroup of A which is minimal neat containing C.
(15) (Rangaswamy) A subgroup G in A is the intersection of neat subgroups of A if and only if $A[p] \not\leq G$ whenever $(A/G)[p] \neq 0$.

(16) (Irwin) Let H be an A^1-high subgroup of A. Prove that:

   (a) H is pure in A. [Hint: choose n minimal with $p^n x = b \in H$ solvable in A, but not in H, and consider $A^1 \cap \langle H, p^{n-1}a \rangle$ where $p^n a = b$.]
   (b) A/H is the divisible hull of $(A^1 \oplus H)/H \cong A^1$.

(17) (V = L) For every infinite cardinal $\aleph_\sigma$ that is not compact, there is an example of an $\aleph_\sigma$-pure subgroup that is not $\aleph_{\sigma+1}$-pure. [Hint: Theorem 8.9 in Chapter 3.]

2 Theorems on Pure Subgroups

The results of this section are fundamental, especially for p-groups.

The Summand Property We find important criteria under which a pure subgroup becomes a summand.

Lemma 2.1 (Szele [4]). *Suppose that B is a subgroup of A such that B is a direct sum of cyclic groups of the same order p^k. The following are equivalent:*

(a) *B is a pure (p-pure) subgroup of A;*
(b) *B satisfies $B \cap p^k A = 0$;*
(c) *B is a direct summand of A.*

Proof. (a) $\Rightarrow$ (b) If B is p-pure in A, then $B \cap p^m A = p^m B$ for every $m \in \mathbb{N}$. The choice $m = k$ yields $p^k B = 0$. Thus (b) follows from (a).

(b) $\Rightarrow$ (c) Assuming (b), let C be maximal in A with respect to the properties $p^k A \leq C$ and $C \cap B = 0$. We show that $A = B \oplus C$. If there is an $a \in A$ not in $B \oplus C$, then there is also one with $pa \in B \oplus C$, so $pa = b + c$ with $b \in B, c \in C$. Then $p^{k-1}b + p^{k-1}c = p^k a \in C$ implies $p^{k-1}b = 0$. By the assumption on the structure of B, there is a $b' \in B$ satisfying $pb' = b$. The maximal choice of C guarantees that the subgroup $\langle C, a - b' \rangle$ must contain a non-zero $b_0 \in B$, thus $b_0 = c' + m(a - b')$ for some $c' \in C, m \in \mathbb{Z}$. Since $B \cap C = 0$ and $p(a - b') = c \in C$, we must have $(m, p) = 1$. But then both $m(a - b') = b_0 - c'$ and $p(a - b') = c$ are in $B \oplus C$, so $a - b' \in B \oplus C$, and $a \in B \oplus C$ follows.

(c) $\Rightarrow$ (a) is trivial. ☐

The following two corollaries are used frequently.

Corollary 2.2 (Prüfer [2], Kulikov [1]). *Every element of order p and of finite p-height can be embedded in a cyclic summand of the group.*

Proof. If $a \in A$ is of order p and of height $k < \infty$, then let $p^k b = a$ for some $b \in A$. Then $\langle b \rangle$ is pure of order p^{k+1}, and Lemma 2.1 applies. ☐

Corollary 2.3 (Kulikov [1]). *If a group contains non-zero elements of finite order, then it has a cocylic direct summand.*

Proof. If the group contains a quasi-cyclic group $\mathbb{Z}(p^\infty)$ for some prime p, then this is a direct summand. If it does not contain any quasi-cyclic subgroup, but it contains elements of order p, then it must also contain one of finite height; cf. (E). The claim follows from Corollary 2.2. □

Another corollary worthwhile recording is as follows.

Corollary 2.4 (Kulikov [1]). *A directly indecomposable group is either cocyclic or torsion-free.*

Proof. By Corollary 2.3 there are no indecomposable mixed groups, and the only indecomposable torsion groups are the cocyclic groups. □

Bounded Pure Subgroups We now proceed to show that certain pure subgroups are necessarily summands.

Theorem 2.5 (Prüfer [2], Kulikov [1]). *A bounded pure subgroup is a direct summand.*

Proof. If B is a bounded group, then Lemma 2.1 allows us to decompose $B = B_1 \oplus C$ where B_1 is a direct sum of cyclic groups of the same order p^k, and the bound for C is less than the bound for B. If B is pure in A, then so is B_1, and Lemma 2.1 implies that $A = B_1 \oplus A_1$ for some $A_1 \leq A$. Then $B = B_1 \oplus C_1$ with $C_1 = B \cap A_1 \cong C$. Here C_1 is pure in A_1, and by induction, C_1 is a summand of A_1, and hence B is one of A. □

Corollary 2.6 (Khabbaz [2]). *p^nA-high subgroups* ($n \in \mathbb{N}$) *of a group A are summands.*

Proof. We show that a p^nA-high subgroup B is a bounded pure subgroup. Since $p^nB \leq B \cap p^nA = 0$, B is a bounded p-group. By induction, we prove that $B \cap p^kA \leq p^kB$ for integers k with $0 \leq k \leq n$. This being trivially true for $k = 0$, assume it is true for some k where $0 \leq k < n$. Let $b = p^{k+1}a \neq 0$ ($b \in B, a \in A$). If $p^ka \in B$, then by induction hypothesis $p^ka = p^kb'$ for some $b' \in B$, and then $b = p^{k+1}b'$. If $p^ka \notin B$, then $\langle B, p^ka\rangle$ contains a non-zero element $b' + p^ka \in p^nA$, where $b' \in B$ is also in p^kA, so $b' \in p^kB$ as well. Now $b' = p^na' - p^ka$ for an $a' \in A$, and hence $b = p^{k+1}a = p^{n+1}a' - pb'$, where the term $p^{n+1}a'$ has to vanish because of $B \cap p^nA = 0$. Thus $b = -pb' \in p^{k+1}B$. □

Corollary 2.7 (Erdélyi [1]). *A p-subgroup embeds in a bounded summand if and only if the heights of its elements* (*computed in the containing group*) *are bounded.*

Proof. The necessity being obvious, suppose that p^m is an upper bound for the heights in the subgroup B of A. There is a $p^{m+1}A$-high subgroup C such that $B \leq C$. Invoke the preceding corollary to conclude that C is a summand of A. □

Cosets of Pure Subgroups Here is another characterization of purity.

Lemma 2.8 (Prüfer [1]). *A subgroup B of A is pure if and only if every coset of A modulo B can be represented by an element that has the same order as the coset.*

Proof. Let B be pure in A, and $a \in A \setminus B$. If the order of the coset $a + B$ is infinite, then every element in the coset has infinite order. If the order of $a + B$ is $n < \infty$, then $na \in B$, so purity implies that $nb = na$ for some $b \in B$. Then $a - b(\in a + B)$ is of order $\leq n$, so of order n.

Conversely, if the stated condition holds, and if $na = b \in B$ for some $a \in A$, then choose $b' \in B$ such that $o(a-b') = o(a+B)$. Then $n(a-b') = 0$, and $nb' = na = b$ establishes the purity of B in A. □

We arrive at another important result:

Theorem 2.9 (Kulikov [1]). *If B is a pure subgroup of A such that A/B is a direct sum of cyclic groups, then B is a summand of A.*

Proof. In view of Lemma 2.4 in Chapter 2, it is enough to deal with the case of cyclic A/B. By Lemma 2.8, we can select a representative $a \in A$ in a generating coset $a + B$ such that $o(a) = o(a + B)$. Then $\langle B, a\rangle = A$ and $B \cap \langle a\rangle = 0$, so that $A = B \oplus \langle a\rangle$. □

In particular, *a pure subgroup of finite index is a summand.*

Theorem 2.10. *For a subgroup B of a group A the following conditions are equivalent:*

(i) *B is pure in A;*
(ii) *B/nB is a direct summand of A/nB for every $n \in \mathbb{N}$;*
(iii) *B is a direct summand of $n^{-1}B$.*

Proof. (i) ⇔ (ii) If (i) holds, then from Theorem 1.3(iii) we infer that B/nB is pure in A/nB. Hence (ii) follows at once from Theorem 2.5. Conversely, suppose (ii), and let $na = b \in B$ with $a \in A, n \in \mathbb{N}$. There is a subgroup $C \leq A$ such that $A/nB = B/nB \oplus C/nB$, so $a = b' + c$ with $b' \in B, c \in C$. Then $b = na = nb' + nc$ implies $nc = b - nb' \in C \cap B = nB$, whence $b \in nB$ follows.

(i) ⇔(iii) If B is pure in A, then the boundedness of $n^{-1}B/B$ implies (iii) because of Theorem 5.2 in Chapter 3 and Theorem 2.9. Conversely, assume again $na = b \in B$ with $a \in A, n \in \mathbb{N}$. By hypothesis, $n^{-1}B = B \oplus C$ for some $C \leq A$, where obviously $nC \leq B \cap C = 0$. Since $a \in n^{-1}B$, we have $a = b' + c$ with $b' \in B, c \in C$. But then $nb' = nb' + nc = na = b$, establishing purity. □

By making use of Theorem 2.9, we can verify the analog of Theorem 7.5 in Chapter 3 for Σ-cyclic torsion groups.

Theorem 2.11. *Suppose $0 = A_0 < A_1 < \dots < A_\sigma < \dots (\sigma < \kappa)$ with an uncountable regular cardinal κ is a smooth chain of pure subgroups of a p-group A such that*

(i) *the A_σ are Σ-cyclic of cardinality $< \kappa$, and*
(ii) *$A = \bigcup_{\sigma<\kappa} A_\sigma$.*

Then A is Σ-cyclic if and only if the set

$$E = \{\sigma < \kappa \mid \exists\, \rho > \sigma \;\; \text{such that } A_\rho/A_\sigma \text{ is not } \Sigma\text{-cyclic}\}$$

is not stationary in κ.

Proof. See the proof of Theorem 7.5 in Chapter 3. □

Solvability of Finite Systems Pure subgroups B of A were defined in terms of the solvability of single equations $nx = b \in B$ with one unknown. Proposition 4.3 in Chapter 4 shows that the same may fail for systems of equations with infinitely many unknowns. However, if we restrict ourselves to equations with a finite number of unknowns only, then we get the following result.

Theorem 2.12 (Prüfer [1]). *Let*

$$n_{i1}x_1 + \cdots + n_{im}x_m = b_i \qquad (b_i \in B, n_{ij} \in \mathbb{Z}, i \in I)$$

be a system of equations with a finite number of unknowns, $x_1, \ldots, x_m$, over a pure subgroup B of A. If it has a solution in A, then it is solvable also in B.

Proof. Let $x_j = a_j$ $(j = 1, \ldots, m)$ be a solution in A. $\langle B, a_1, \ldots, a_m\rangle/B$ is Σ-cyclic, so owing to Theorem 2.10, B is a summand of $C = \langle B, a_1, \ldots, a_m\rangle$, say, $C = B \oplus C'$. The B-coordinates of the a_j provide a solution in B. □

★ **Notes.** Theorems 2.5 and 2.9 are most essential results on pure subgroups, they are indispensable and will be used throughout without reference to them. Generalization of purity to modules over arbitrary (associative) rings was given by P. Cohn, using property in Theorem 2.12 with a finite number of equations. In the theory of modules over integral domains, a weaker version, the so-called 'relative divisibility' turned out to be most useful; this concept is due to Warfield [Pac. J. Math. **28**, 699–719 (1969)]. This adheres to the above definition of purity with ring elements replacing integers.

Exercises

(1) If a reduced p-group contains elements of arbitrarily large orders, then it also has cyclic summands of arbitrarily large orders.
(2) Let A be a bounded p-group, and B a subgroup of $A[p]$. A has a direct summand C such that $C[p] = B$.
(3) If G is a pure subgroup of $A = B \oplus C$ such that $G \cap C$ is essential both in G and in C, then $A = B \oplus G$.
(4) Call a group A **pure-simple** if it contains no pure subgroups other than 0 and A. Prove that A is pure-simple if and only if it isomorphic to a subgroup of $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ for some prime p.
(5) A group satisfies the maximum [minimum] condition on pure subgroups if and only if it is of finite rank.

(6) (Cutler, E. Walker) If the groups A and G satisfy $nA \cong nG$ for some $n \in \mathbb{N}$, then there exist groups A' and G' such that $A \oplus A' \cong G \oplus G'$ with $nA' = 0 = nG'$. [Hint: find maximal n-bounded subgroups.]
(7) If $nA = \oplus_{i \in I} C_i$ is a direct decomposition, then there exist subgroups B_i such that $A = \oplus_{i \in I} B_i$ with $nB_i = C_i$.
(8) If B is a pure subgroup of A, then $(A/B)[n] \cong A[n]/B[n]$ for every $n \in \mathbb{N}$.
(9) (Kulikov) Suppose $a \in A$ is an element of smallest finite order in the coset $a + pA$. Then $\langle a \rangle$ is a summand of A.
(10) (Mader) Let A be an infinite p-group such that $|A/p^nA| < |A|$ for some $n \in \mathbb{N}$. Then A contains a p^n-bounded summand of cardinality $|A|$.
(11) The closure C^- of a pure subgroup C of A (in the $\mathbb{Z}$-adic topology) is pure if and only if $(A/C)^1$ is a divisible group.
(12) Let $B \oplus C$ be a pure subgroup in a reduced torsion-free group A. Then the closures B^-, C^- in the $\mathbb{Z}$-adic topology of A are still disjoint, and $B^- \oplus C^-$ is pure in A.
(13) (Göbel–Goldsmith) In every group $\neq 0$, the set of all proper pure subgroups contains a maximal member. [Hint: argue separately for torsion-free groups.]
(14) A group is called **absolutely pure** if it is a pure subgroup in any group in which it is contained as a subgroup. Show that D is absolutely pure if and only if it is divisible.
(15) A subgroup B is κ-pure in A if and only if every system of equations over B with less than κ unknowns is solvable in B whenever it admits a solution in A. [Hint: Theorem 2.12.]
(16) A subgroup G of a group A can be embedded in an $\aleph_\sigma$-pure subgroup of cardinality $\leq |G|^{\aleph_\rho}$, where $\rho = \sigma - 1$ or σ according as σ is a successor or a limit ordinal. [Hint: argue as in Theorem 1.5 and preceding exercise.]

3 Pure-Exact Sequences

Pure-Exactness A short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{5.3}$$

is said to be **pure-exact** if $\operatorname{Im}\alpha$ is a pure subgroup of B. It is p-**pure-exact** if $\operatorname{Im}\alpha$ is p-pure in B.

To simplify notation, in the next theorem we shall use the same letter for homomorphisms and for maps that they induce.

Theorem 3.1. *An exact sequence* (5.3) *is pure-exact if and only if it satisfies one* (*and hence all*) *of the following equivalent conditions:*

(a) $0 \to nA \xrightarrow{\alpha} nB \xrightarrow{\beta} nC \to 0$ *is exact for every* $n \in \mathbb{N}$;
(b) $0 \to A/nA \xrightarrow{\alpha} B/nB \xrightarrow{\beta} C/nC \to 0$ *is exact for every* n;

(c) $0 \to A[n] \xrightarrow{\alpha} B[n] \xrightarrow{\beta} C[n] \to 0$ *is exact for every n;*

(d) $0 \to A/A[n] \xrightarrow{\alpha} B/B[n] \xrightarrow{\beta} C/C[n] \to 0$ *is exact for every n.*

Moreover, the sequences (b) *and* (c) *are splitting exact.*

Proof. We prove only (a) and (c), because then (b) and (d) will follow from the 3×3-lemma. That the compositions of α and β are 0 throughout is evident.

(a) It is clear that, for all n, α is monic if and only if it is monic in (5.3). $\operatorname{Ker} \beta$ is $\alpha A \cap nB$ which is $\alpha(nA)$ if and only if (5.3) is exact at B. Finally, for all n, β is epic exactly it is epic in (5.3).

(c) Again, for all n, α is monic if and only if it is monic in (5.3), and β is epic for every n exactly if every element of order n is the image of an element of order n in B. $\operatorname{Ker} \beta$ is equal to $\alpha(nA)$ if and only if (5.3) is exact at B.

The last claim follows straightforwardly by showing that (b) and (c) are pure-exact and the groups are bounded. □

The next theorem characterizes pure-exact sequences in terms of their injective and projective properties.

Theorem 3.2. *An exact sequence* (5.3) *is pure-exact if and only if the finite cyclic groups have the injective property relative to it if and only if the finite cyclic groups have the projective property relative to it.*

Proof. We prove the claim only for injectivity, a dual proof applies to projectivity. Let $\phi : A \to H$ be a homomorphism into a cyclic group H; without loss of generality we may assume ϕ is epic. The existence of a $\psi : B \to H$ with $\psi\alpha = \phi$ is equivalent to the extensibility of the isomorphism $A/\operatorname{Ker}\phi \cong H$ to a map $B \to H$, i.e. to the fact that $\alpha(A/\operatorname{Ker}\phi)$ is a summand of $B/\alpha \operatorname{Ker}\phi$. An appeal to Theorem 3.1 completes the proof. □

Direct Limits and Purity We turn our attention to the behavior of pure-exact sequences towards direct limits. Interestingly, direct limits of pure-exact sequences are again pure-exact.

Theorem 3.3. *Let* $\mathfrak{A} = \{A_i \ (i \in I); \pi_i^j\}$, $\mathfrak{B} = \{B_i \ (i \in I); \rho_i^j\}$ *and* $\mathfrak{C} = \{C_i \ (i \in I); \sigma_i^j\}$ *be direct systems of groups, and let* $\Phi : \mathfrak{A} \to \mathfrak{B}, \Psi : \mathfrak{B} \to \mathfrak{C}$ *be homomorphisms between them such that, for every* $i \in I$*, the sequence* $0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$ *is pure-exact. Then the sequence*

$$0 \to A_* \xrightarrow{\Phi_*} B_* \xrightarrow{\Psi_*} C_* \to 0 \tag{5.4}$$

of direct limits is likewise pure-exact.

Proof. In view of Theorem 4.6 in Chapter 2, we need only check the purity of $\Phi_*(A_*)$ in B_*. Let $nb = \Phi_* a$ for $a \in A_*, b \in B_*$, and for some $n \in \mathbb{N}$. Then there are $a_i \in A_i, b_i \in B_i$ for some $i \in I$ such that $\pi_i a_i = a, \rho_i b_i = b$ for the canonical maps $\pi_i : A_i \to A_*, \rho_i : B_i \to B_*$. Because of the commutativity of the diagram in Theorem 4.6 in Chapter 2, we have $\rho_i nb_i = \Phi_* \pi_i a_i = \rho_i \phi_i a_i$, whence we get $\rho_i(nb_i - \phi_i a_i) = 0$, and so $\rho_i^j(nb_i - \phi_i a_i) = 0$ for some index $j \geq i$. Therefore, $n\rho_i^j b_i = \rho_i^j \phi_i a_i = \phi_j \pi_i^j a_i$, and since $\rho_i^j b_i \in B_j$, $\pi_i^j a_i \in A_j$, the pure-exactness for index j implies that $\phi_j n a_j' = \phi_j \pi_i^j a_i$ for some $a_j' \in A_j$. Apply ρ_j, and observe that $\rho_j \phi_j = \Phi_* \pi_j$ to obtain $n\Phi_* a' = \Phi_* a$ with $a' = \pi_j a_j' \in A_*$. □

Next we point out a most interesting connection of purity with splitting exact sequences; this is another convincing evidence that purity is a very natural concept.

Theorem 3.4. *A sequence is pure-exact if and only if it is the direct limit of splitting exact sequences.*

Proof. That the direct limit of splitting exact sequences is pure-exact follows at once from the preceding theorem. To verify the converse, assume $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a pure-exact sequence, and $\{C_i \ (i \in I)\}$ is the family of finitely generated subgroups of C; here, I is a poset where $i \leq j$ means $C_i \leq C_j$. If $\sigma_i^j : C_i \to C_j$ is the injection map for $i \leq j$, then we may view C as the direct limit of the direct system $\mathfrak{C} = \{C_i \ (i \in I); \sigma_i^j\}$. It follows that B is the direct limit of the direct system $\mathfrak{B} = \{B_i = \beta^{-1} C_i \ (i \in I); \rho_i^j\}$ where ρ_i^j stands for the injection map $B_i \to B_j$. Finally, we let $\mathfrak{A} = \{A_i = A \ (i \in I); \pi_i^j = \mathbf{1}_A\}$. Then α and β induce homomorphisms $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{B} \to \mathfrak{C}$ such that all the sequences $0 \to A_i \xrightarrow{\alpha} B_i \xrightarrow{\beta} C_i \to 0$ are pure-exact; they are actually splitting because of Theorem 2.9. It is obvious that the direct limit of these splitting exact sequences is the exact sequence we started with. □

The last theorem can be applied to derive a useful corollary.

Corollary 3.5 (Yahya [1]). *Suppose F is a covariant additive functor $\mathcal{A}b \to \mathcal{A}b$ that commutes with direct limits. Then F carries a short pure-exact sequence into a pure-exact sequence.*

Proof. In view of Theorem 3.4, a pure-exact sequence $0 \to A \to B \to C \to 0$ can be represented as the direct limit of splitting exact sequences. Applying F to these sequences, we get a direct system of splitting exact sequences whose limit will be the pure-exact sequence $0 \to F(A) \to F(B) \to F(C) \to 0$. □

Tensor Products and Purity We hesitate to include in this section results on tensor product (and on Hom), because we have not as yet defined these functors. But these results are essential for purity, and we feel it is reasonable to discuss them here. First, we rephrase part of Theorem 3.1 in terms of functors.

Corollary 3.6. *The exact sequence* (5.3) *is pure-exact if and only if one* (*and hence both*) *of the following sequences is exact for every n:*

(a) $0 \to \mathbb{Z}(n) \otimes A \xrightarrow{\alpha} \mathbb{Z}(n) \otimes B \xrightarrow{\beta} \mathbb{Z}(n) \otimes C \to 0;$
(b) $0 \to \mathrm{Hom}(\mathbb{Z}(n), A) \xrightarrow{\alpha} \mathrm{Hom}(\mathbb{Z}(n), B) \xrightarrow{\beta} \mathrm{Hom}(\mathbb{Z}(n), C) \to 0.$

Proof. The proof is immediate, all that we have to observe is that both $\mathbb{Z}(n) \otimes G \cong G/nG$ and $\mathrm{Hom}(\mathbb{Z}(n), G) \cong G[n]$ are natural isomorphisms for every group G; see Sect. 1(B) in Chapter 8 and Example 1.2 in Chapter 7. $\square$

The next corollary is an important result on purity, it is intimately related to the foregoing discussions.

Corollary 3.7. *An exact sequence (5.3) is pure-exact if and only if, for every group G, the induced sequence*

$$0 \to G \otimes A \xrightarrow{\mathbf{1}_G \otimes \alpha} G \otimes B \xrightarrow{\mathbf{1}_G \otimes \beta} G \otimes C \to 0$$

is exact. It is even pure-exact, if so is (5.3).

Proof. Suppose (5.3) is an exact sequence. The exactness of the tensored sequence holds for finitely generated groups G, as is shown in Corollary 3.6(a). By Theorem 3.3, exactness is preserved under taking direct limits. Since tensor product commutes with direct limits, the claim also holds for direct limits of finitely generated groups, so also for any G. The converse is a consequence of Corollary 3.6 if we choose G as finite cyclic groups. Purity follows from Corollary 3.5 at once. $\square$

★ **Notes.** The results above show that purity has several remarkable characterizations. Several mathematicians noticed almost simultaneously that pure-exact sequences are exactly the direct limits of splitting exact sequences, this being true also in the module-theoretic version. Interestingly, for the generalization of purity to modules, the results of this section continue to hold *mutatis mutandis*. For integral domains, see, e.g., Fuchs–Salce, *Modules over non-Noetherian Domains* (2001).

In general, pure submodules of injective left R-modules need not be injective; they are always injective exactly if R is left noetherian. The von Neumann regular rings can be characterized as rings over which all exact sequences are pure-exact.

Exercises

(1) Show that (5.3) is pure-exact if and only if the induced sequence

$$0 \to \mathrm{Hom}(A, \mathbb{Z}(n)) \to \mathrm{Hom}(B, \mathbb{Z}(n)) \to \mathrm{Hom}(C, \mathbb{Z}(n)) \to 0$$

is exact for every $n \in \mathbb{N}$. [Hint: $\mathrm{Hom}(G, \mathbb{Z}(n))$.]

(2) If C is a pure subgroup of the p-group A, then $A[p]/C[p] \cong (A/C)[p]$.

(3) If (5.3) is a pure-exact sequence, then the sequence $0 \to tA \to tB \to tC \to 0$ of torsion subgroups is exact.

(4) Suppose (5.3) is pure-exact. The sequences $0 \to A^1 \to B^1 \to C^1 \to 0$ of first Ulm subgroups and $0 \to A/A^1 \to B/B^1 \to C/C^1 \to 0$ of first Ulm factors need not be exact; but if one is exact, then so is the other.

(5) Let $\mathcal{E}$ denote a class of exact sequences.

(a) The class of groups that enjoy the injective property with respect to every member of $\mathcal{E}$ is closed under taking direct products and summands.
(b) The same for 'projective' in place of 'injective,' but change 'direct product' to 'direct sum.'

4 Pure-Projectivity and Pure-Injectivity

In Theorem 3.2 the pure-exact sequences were characterized by properties that the finite cyclic groups have the projective as well as the injective property relative to them. Our next goal is to find all groups that have the projective, resp. the injective property relative to all pure-exact sequences.

Pure-Projective Groups A group P is called **pure-projective** if it enjoys the projective property relative to the class of pure-exact sequences; i.e., if every diagram

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

with pure-exact row can be completed by a map $\psi : P \to B$ such that $\beta\psi = \phi$.

Example 4.1. All cyclic groups are pure-projective: $\mathbb{Z}$ is because it is projective, and all finite cyclic groups because of Theorem 3.2. Hence all Σ-cyclic groups are pure-projective.

In order to find all pure-projective groups, we prove a lemma (which can be interpreted as the existence theorem on pure-projective resolutions; we say: *there are enough pure-projectives*).

Lemma 4.2. *Every group A can be embedded in a pure-exact sequence*

$$0 \to B \to P \xrightarrow{\alpha} A \to 0 \tag{5.5}$$

where P is Σ-cyclic (hence B as well).

Proof. For every $a \in A$, let $\langle c_a \rangle \cong \langle a \rangle$ be a cyclic group, and define $P = \oplus_{a \in A} \langle c_a \rangle$. Let $\alpha : P \to A$ act *via* $\alpha : c_a \mapsto a$ for all $a \in A$. This is a well-defined epimorphism, and if we set $B = \operatorname{Ker} \alpha$, then we get the exact sequence (5.5). B is pure in P, since by construction, every coset mod B is represented by an element of the same order (cf. Lemma 2.8). $\operatorname{Ker} \alpha = B$ is also Σ-cyclic, since it is a subgroup in a Σ-cyclic group (Theorem 5.7 in Chapter 3). $\square$

Theorem 4.3 (Maranda [1]). *A group is pure-projective if and only if it is Σ-cyclic.*

Proof. Theorem 3.2 implies that Σ-cyclic groups are pure-projective.

Conversely, assume that A is a pure-projective group. By Lemma 4.2, there exists a pure-exact sequence (5.5) with Σ-cyclic P. By pure-projectivity, there is a map $\alpha : A \to P$ such that $\eta\alpha = \mathbf{1}_A$. This means, A is isomorphic to a summand of P, and hence it is Σ-cyclic. □

Pure-Injective Groups Turning to the dual concept, a group H is defined to be **pure-injective** if it has the injective property relative to all pure-exact sequences. In other words, any diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & B \arrow[r, "\beta"] \arrow[dl, dashed, "\psi"] & C \arrow[r] & 0 \\ & H & & & \end{tikzcd}$$

with pure-exact row can be completed with a map $\psi : B \to H$ such that $\psi\alpha = \phi$. As we shall see, the theory of pure-injective groups can be incorporated in the theory of algebraically compact groups, so here we restrict ourselves to a few elementary results. For more information, we refer to Sect. 4 in Chapter 6.

Example 4.4. Injective groups are trivially pure-injective, and so are all the cocyclic groups (cf. Theorem 3.2).

Example 4.5. The group J_p is pure-injective. This will follow from Theorem 1.2 in Chapter 6 as J_p is a compact group.

The next lemma provides pure-injective resolutions (so *there are enough pure-injectives*).

Lemma 4.6 (Łoś [1]). *Every group can be embedded as a pure subgroup in a direct product of cocyclic groups.*

Proof. Let $\{H_i \ (i \in I)\}$ be the set of all cocyclic factor groups of the group A, and set $H = \prod_{i \in I} H_i$. The canonical maps $\eta_i : A \to H_i$ induce a homomorphism $\eta : A \to H$ which must be an embedding, since every non-zero $a \in A$ is excluded from the kernel of some η_i (see Proposition 5.5 in Chapter 4). To verify the purity of $\operatorname{Im}\eta$ in H, we show that if $a \in A$ is such that $a \notin p^n A$, then also $\eta a \notin p^n H$. Let C be a subgroup of A maximal with respect to the properties $p^n A \leq C$ and $a \notin C$. Then by Proposition 5.5 in Chapter 4 A/C is cocyclic, and since it is bounded, it must be cyclic of order p^k for some $k \leq n$. Thus $A/C = H_i$ for some i, where $\eta_i a$ is of height $k-1$, so that $\eta a \in p^n H$ is impossible. □

Theorem 4.7.

(a) *A group is pure-injective if and only if it is a summand of a direct product of cocyclic groups.*
(b) *Every group embeds as a pure subgroup in a pure-injective group.*

Proof.

(a) A straightforward argument shows that any injective property is preserved by taking direct products or passing to summand, so for the 'if' part it is enough to show that cocyclic groups are pure-injective. This is obvious for quasi-cyclic groups (they are injective), and this also holds for finite cyclic groups as is clear from Theorem 3.2.
On the other hand, if A is pure-injective, then by Lemma 4.6 A embeds as a pure subgroup in a direct product H of cocyclic groups. By pure-injectivity, the identity map $\mathbf{1}_A$ factors through $H \to A$, which shows that A is a summand of H.
(b) Combine Lemma 4.6 and (a). □

Pure-essential extensions and pure-injective hulls will be discussed in Sect. 4 in Chapter 6.

★ **Notes.** Pure-projectivity and pure-injectivity were studied by Maranda [1]. Over any ring (commutative or not), the pure-projective left modules are the summands of direct sums of finitely presented left modules. Hence it is easy to conclude that all cyclic left modules are pure-projective if and only if the ring is left noetherian. There is an extensive literature on pure-injective modules, they play an important role also in model theory.

Quasi-pure-projectivity and -injectivity have also been discussed in several publications. (In the definition of the latter imitate pure-injectivity with $H = B$ above.) See, e.g., Reid [4], and the literature cited there. Cf. also Chekhlov [2], and the survey Chekhlov–Krylov [1].

Exercises

(1) Give an example for a pure-injective group that is not a product of cocyclic groups (only a summand of such a group). [Hint: torsion-free.]
(2) Define p-pure-projectivity and p-pure-injectivity, and show that there are enough p-pure-projective and p-pure-injective groups.
(3) There are enough neat-projective and neat-injective groups.
(4) If G and H are pure-injective groups such that each of them is isomorphic to a pure subgroup of the other, then $G \cong H$. [Hint: $G \cong H \oplus A, H \cong G \oplus B$ implies that G contains a pure subgroup $\cong A^{(\aleph_0)} \oplus B^{(\aleph_0)}$.]
(5) (C. Walker) Let κ denote a non-limit cardinal. A group is κ-pure-projective if and only if it is (a summand of) a direct sum of groups of cardinalities $< \kappa$.
(6) A homomorphism $\alpha : A \to B$ is called **pure** if both $\operatorname{Ker}\alpha$ is pure in A and $\operatorname{Im}\alpha$ is pure in B. Groups with pure homomorphisms as morphisms form a subcategory of $\mathcal{A}b$ which has enough projectives and injectives.

5 Basic Subgroups

***p*-Independence** Let A be an arbitrary group, and p any prime which will be kept fixed in this section. A set $\{a_i\}_{i\in I}$ of non-zero elements of A is called ***p*-independent** if, for every finite subset $\{a_1, \dots, a_k\}$ and for every positive integer r,

$$n_1a_1 + \cdots + n_ka_k \in p^rA \quad (n_ia_i \neq 0, n_i \in \mathbb{Z}) \quad \text{implies} \quad p^r | n_i \ (i = 1, \dots, k).$$

Thus, by definition, p-independence is of finite character, thus every p-independent system can be extended to a maximal one. These maximal systems are of special interest.

(A) *A p-independent system is an independent set.* For, if $\{a_1, \dots, a_k\}$ is p-independent, and if $n_1a_1 + \cdots + n_ka_k = 0$ with $n_ia_i \neq 0$, then definition guarantees that $p^r|n_i$ holds for all $r > 0$, which is impossible. Thus the subgroup generated by a p-independent set $\{a_i\}_{i\in I}$ is $\oplus_{i\in I} \langle a_i\rangle$.

(B) *A subgroup generated by a p-independent system is p-pure.* Let C be the subgroup generated by the p-independent system $\{a_i\}_{i\in I}$. Assume $c \in C \cap p^rA$, i.e. $c = n_1a_1 + \cdots + n_ka_k \in p^rA$ with $n_ia_i \neq 0, n_i \in \mathbb{Z}$. By p-independence, we have $n_i = p^rm_i$ for suitable $m_i \in \mathbb{Z}$. This shows that $c = p^r(m_1a_1 + \cdots + m_ka_k) \in p^rC$.

(C) *If $a \in A$ belongs to a p-independent system, then $o(a)$ is either ∞ or p^ℓ for some $\ell \geq 1$.* For, if $o(a) = m$ and p^ℓ is the highest power of p that divides m, then $p^\ell a \in p^rA$ for every $r > 0$. Therefore, $p^r|p^\ell$ for all r, unless $p^\ell a = 0$.

(D) *If an independent set containing only elements of infinite and p-power orders generates a p-pure subgroup, then it is p-independent.* Let $\{a_i\}_{i\in I}$ be such an independent set in A, and $C = \langle \dots, a_i, \dots\rangle = \oplus_{i\in I} \langle a_i\rangle$. If $n_1a_1 + \cdots + n_ka_k \in p^rA$ with $n_ia_i \neq 0$, then by p-purity we have $n_1a_1 + \cdots + n_ka_k = p^r(m_1a_1 + \cdots + m_ka_k)$ for some $m_i \in \mathbb{Z}$. On account of independence, we infer that $n_ia_i = p^rm_ia_i$ for $i = 1, \dots, k$. Hence the hypothesis on the orders of the elements a_i implies $p^r|n_i$.

***p*-Basic Subgroups** By a ***p*-basic subgroup** B of A we mean a subgroup of A that satisfies the following three conditions:

(i) B is a direct sum of cyclic p-groups and infinite cyclic groups;
(ii) B is p-pure in A;
(iii) A/B is p-divisible.

Thus B has a basis which we will call a ***p*-basis** of A.

Evidently, A is a p-basic subgroup of itself if and only if (i) holds for A, and 0 is a p-basic subgroup of A if and only if A is p-divisible. If A is equipped with the p-adic topology, then conditions (i)–(iii) imply that B is Hausdorff in its p-adic topology that is the same as the p-adic topology inherited from A, and furthermore, B is dense in A.

Lemma 5.1. *A set of group elements is a p-basis if and only if it is a maximal p-independent system.*

Proof. Suppose $\{a_i\}_{i \in I}$ is a p-basis in the group A. Its p-independence follows from (i)–(ii) and (D). (iii) implies that for $0 \neq g \in A$ there is a relation of the form $g + n_1 a_1 + \cdots + n_k a_k \in pA$ with $n_i \in \mathbb{Z}$. This shows that adjoining g, the given set would loose its p-independence.

Conversely, let $\{a_i\}_{i \in I}$ denote a maximal p-independent system in A, and B the subgroup it generates. By (A), this system is independent, so (i) follows at once from (C). Since (B) implies (ii), only (iii) remains to be verified.

Let $g \in A \setminus B$. By maximality, there is a dependence relation $n_0 g + n_1 a_1 + \cdots + n_k a_k \in p^r A$ with $n_i a_i \neq 0$ ($n_i \in \mathbb{Z}$) such that p^r does not divide n_0. If $o(g) = p$, then necessarily $\gcd\{n_0, p\} = 1$, and g is divisible by p mod B. We induct on t to show that if $o(g) = p^t$, then g is divisible by p mod B. Let p^s denote the highest power of p that divides n_0. Evidently, $s < r$ and $n_i = p^s m_i$ for some $m_i \in \mathbb{Z}$ and for all i. Thus $p^s(m_0 g + m_1 a_1 + \cdots + m_k a_k) = p^r a$ for some $a \in A$. The element $g' = m_0 g - p^{r-s} a + m_1 a_1 + \cdots + m_k a_k$ has order dividing $p^s (< p^t)$, so the induction hypothesis implies that g' is divisible by p mod B. As $r - s \geq 1$, the same must be true for $m_0 g$, and so for g. For a g of infinite order, it suffices to point out that $o(g')$ is a power of p, so g', and hence g, is divisible by p mod B. □

Theorem 5.2 (Kulikov [2], Fuchs [11]). *Every group contains p-basic subgroups, for every prime p.*

Proof. Maximal p-independent systems exist in every group. By Lemma 5.1, each of them generates a p-basic subgroup. □

Example 5.3. In a divisible group, 0 is the only p-basic subgroup, for every p.

Example 5.4. The cyclic subgroup generated by 1 in J_p is a p-basic subgroup of J_p. For primes $q \neq p$, the q-basic subgroups are 0.

Example 5.5. Let $A = \oplus_{n=1}^{\infty} \langle a_n \rangle$ where $\langle a_n \rangle \cong \mathbb{Z}(p^n)$. Then A is a p-basic subgroup of itself. But A is not its only p-basic subgroup; e.g., $B = \oplus_{n=1}^{\infty} \langle a_n - p a_{n+1} \rangle$ is another p-basic subgroup, properly contained in A.

Example 5.6 (Kulikov [2]). Let $B = \oplus_{n=1}^{\infty} B_n$ where B_n is a direct sum of cyclic groups $\cong \mathbb{Z}(p^n)$. Then B is a p-basic subgroup of the torsion part $\overline{B}$ of the direct product $A = \prod_{n=1}^{\infty} B_n$. Observe that $\overline{B}$ consists of all vectors $c = (b_1, \dots, b_n, \dots)$ with $b_n \in B_n$ such that there is an integer k with $p^k b_n = 0$ for every n, while B contains only the vectors with almost all $b_n = 0$. Conditions (i) and (ii) in the definition of basic subgroups are clearly satisfied. The same is true for (iii), as $c - (b_1 + \cdots + b_k) = (0, \dots, 0, b_{k+1}, b_{k+2}, \dots)$ must be divisible by p in view of $p^k b_n = 0$ and $o(b_n) = p^n$.

Example 5.7. Consider the Prüfer group $H_{\omega+1}$. This group is generated by the elements a_n ($n < \omega$) subject to the defining relations $p a_0 = 0$, $p^n a_n = a_0$ ($n \geq 1$). $B = \oplus_{n \in \mathbb{N}} \langle a_n - p a_{n+1} \rangle$ is a basic subgroup, and $H_{\omega+1}/B \cong \mathbb{Z}(p^{\infty})$.

Example 5.8. In the Baer–Specker group $A = \mathbb{Z}^{\aleph_0}$, the subgroup B of bounded vectors is a free pure subgroup (Theorem 10.4 in Chapter 3). Moreover, A/B is divisible, because for every $a \in A$, for each prime p there is a $b \in B$ such that $p | a - b$. In fact, such a b can be chosen to have all coordinates from the set $\{0, 1, \dots, p-1\}$. Consequently, B is a p-basic subgroup of A for every prime p.

Let B be a p-basic subgroup of A. We collect the cyclic summands of the same order in a direct decomposition of B, and form their direct sums to obtain

$$B = B_0 \oplus B_1 \oplus \cdots \oplus B_n \oplus \dots \tag{5.6}$$

where B_0 denotes the direct sum of the infinite cyclic summands, while B_n for $n \geq 1$ is the direct sum of the cyclic summands of order p^n. Clearly, $B_1 \oplus \cdots \oplus B_n$ is a summand in B, and hence it is pure in A, so it is a bounded summand of A, for every n:

$$A = B_1 \oplus \cdots \oplus B_n \oplus A_n.$$

We are going to show that A_n can be chosen as

$$A_n = B_n^* + p^n A \quad \text{where } B_n^* = B_0 \oplus B_{n+1} \oplus B_{n+2} \oplus \dots \tag{5.7}$$

so that $A_n = B_{n+1} \oplus A_{n+1}$ for all $n \in \mathbb{N}$. In fact, (iii) above guarantees that, for any given n, each $a \in A$ is of the form $a = b + p^n c$ $(b \in B, c \in A)$, thus $B_1, \dots, B_n, B_n^*$ and $p^n A$ together generate A. If $a \in B_n^* + p^n A$ is also contained in $B_1 \oplus \cdots \oplus B_n$, then from $a = b + p^n c$ with $b \in B_n^*, c \in A$ we conclude that $p^n c \in B$. But $B_1 \oplus \cdots \oplus B_n$ is p^n-bounded, thus $p^n c \in B_n^*$. Consequently, $a \in B_n^* \cap (B_1 \oplus \cdots \oplus B_n) = 0$, and A_n can be chosen as stated. Thus $A_n = B_{n+1} \oplus A_{n+1}$, in fact.

Theorem 5.9 (Baer [1], Boyer [1]). *For a subgroup B in (5.6) of A to be a p-basic subgroup it is necessary and sufficient that*

(a) *B_0 is pure in A; and*
(b) *for every $n < \omega$, $A = B_1 \oplus \cdots \oplus B_n \oplus (B_n^* + p^n A)$ with B_n^* in (5.7).*

Proof. It remains to verify sufficiency. Let $B = \oplus_{n<\omega} B_n$ satisfy (a)-(b). Then B is obviously Σ-cyclic and p-pure in A. To prove that A/B is p-divisible, write $a \in A$ in the form $a = b_1 + \cdots + b_n + c + p^k g$ with $b_i \in B_i$, $c \in B_k^*$, $g \in A$. Then $a + B = p^k g + B = p^k(g + B)$, whence $p^k | a + B$. Thus A/B is p-divisible. □

***p*-Basic in Torsion-Free Groups** Let us have a closer look at two important special cases: when A is torsion-free and when A is a p-group.

If A is torsion-free, then its p-basic subgroups are free groups. Even if A is not free, it may very well happen that a free subgroup B is p-basic for every prime p. This is the case when A/B is torsion-free divisible. An example is the Baer–Specker group; see Example 5.8.

By the way, it is rather easy to construct a p-basic subgroup B in a torsion-free group A. Select a basis $a_i + pA$ $(a_i \in A, i \in I)$ in the vector space A/pA. We claim that $B = \oplus_{i \in I} \langle a_i \rangle$ is a p-basic subgroup of A. To see that the a_i are p-pure-independent, suppose that $n_1 a_1 + \cdots + n_k a_k \in p^r A$ for some $r > 0$ where $n_i \in \mathbb{Z}$. Since divisibility by p is unique in A, we may assume that not all of the n_i are divisible by p. Then $n_1 a_1 + \cdots + n_k a_k \in pA$, and omitting the terms $n_i a_i$ with $p | n_i$ we would get a non-trivial dependence relation for the $a_i + pA$. The maximality of the p-pure-independent set $\{a_i\}_{i\in I}$ follows from the isomorphism $B/pB \cong A/pA$ induced by the embedding $B \to A$ (cp. Sect. 6(B)).

Basic Subgroups in *p*-Groups p-basic subgroups play a significantly more important role in torsion groups than in torsion-free or mixed groups. Of course, in the torsion case, the focus is on p-groups. If A is a p-group, then its only non-trivial q-basic subgroup for a prime q is when $q = p$. In this case, we simply say 'basic subgroup' to mean 'p-basic subgroup.'

In order to better understand this concept, we mention here two different ways of obtaining basic subgroups in p-groups. One is based on using maximal bounded summands, and the other relies on the socle.

Proposition 5.10 (Szele [5]). *A Σ-cyclic subgroup $B = \oplus_{n=1}^{\infty} B_n$ of a p-group A, where B_n is a direct sum of cyclic groups of order p^n, is a basic subgroup if and only if, for every $n > 0$, $B_1 \oplus \dots \oplus B_n$ is a maximal p^n-bounded summand of A, i.e. a p^nA-high subgroup of A.*

Proof. If B is basic in A, then (using the above notation) $A_n = B_n^* + p^nA$ contains no elements of order p and of height $< n$, hence A_n has no cyclic summands of order $\le p^n$, i.e. $B_1 \oplus \dots \oplus B_n$ is a maximal p^n-bounded summand of A. Conversely, if B satisfies the stated condition, then (i) and (ii) in the definition of p-basic subgroups are obvious. If A/B were not divisible, then by Corollary 2.3 it would have a cyclic summand $C/B = \langle c + B\rangle \cong \mathbb{Z}(p^m)$ for some $m > 0$. By Theorem 2.9, $C = B \oplus \langle c\rangle$, and clearly, C is pure in A. But then $B_1 \oplus \dots \oplus B_m \oplus \langle c\rangle$ would be a larger p^m-bounded summand of A. The last claim is evident in view of Corollary 2.6. □

Proposition 5.11 (Charles [2]). *Let $B = \oplus_{n=1}^{\infty} B_n$ be a subgroup of a p-group A, where B_n denotes a direct sum of cyclic groups of order p^n. A necessary and sufficient condition for B to be a basic subgroup of A is that we have*

$$p^nA[p] = B_{n+1}[p] \oplus p^{n+1}A[p] \qquad \text{for every } n \in \mathbb{N}. \tag{5.8}$$

Proof. Necessity is a consequence of the equality $A_n = B_{n+1} \oplus A_{n+1}$ (notation as above) once we observe that $A_n[p] = p^nA[p]$; this equality follows by examining the socles of the B_i and B_n^*. To prove the converse, suppose (5.8) for all $n \in \mathbb{N}$. Then $A[p] = B_1[p] \oplus \dots \oplus B_{n+1}[p] \oplus p^{n+1}A[p]$, and it is clear that a p^{n+1}-bounded pure subgroup larger than $B_1 \oplus \dots \oplus B_{n+1}$ would intersect $p^{n+1}A[p]$, so the claim follows from Proposition 5.10. □

A basic subgroup isomorphic to $B \cong \oplus_{n=1}^{\infty} \mathbb{Z}(p^n)$ is often called **the standard basic subgroup**.

Subgroups of Basic Subgroups The next theorem characterizes the subgroups of basic subgroups. The analogy with Kulikov's Theorem 5.1 in Chapter 3 is apparent.

Theorem 5.12 (Kovács [1]). *A subgroup C of a p-group A is contained in a basic subgroup of A if and only if C is the union of an ascending chain*

$$0 = C_0 \leq C_1 \leq \cdots \leq C_n \leq \dots$$

of subgroups such that the heights of elements in C_n (computed in A) are bounded for every $n > 0$.

Proof. If the subgroup C is contained in a basic subgroup $B = \oplus_{n=1}^{\infty} B_n$, where $B_n \cong \oplus \mathbb{Z}(p^n)$, then the subgroups $C_n = C \cap (B_1 \oplus \cdots \oplus B_n)$ satisfy the stated condition.

To prove sufficiency, we may assume without loss of generality that $C_n \cap p^n A = 0$ (we may adjoin or delete members in the chain). Consider ascending chains $0 = G_0 \leq G_1 \leq \cdots \leq G_n \leq \dots$ of subgroups of A subject to the following conditions:

$$C_n \leq G_n \quad \text{and} \quad G_n \cap p^n A = 0 \ \ \forall\, n \in \mathbb{N}.$$

We introduce a partial order in the set of all such chains in the obvious manner: a chain of the G_n is $\leq$ than the chain of the G'_n if $G_n \leq G'_n$ for each $n \in \mathbb{N}$. Then Zorn's lemma ensures the existence of a maximal chain which we will denote (without danger of confusion) again by G_n.

We claim that the union $B = \cup_{n=1}^{\infty} G_n$ for a maximal chain is a basic subgroup of A. The main step in the proof is to show that G_n is $p^n A$-high in A. If we show this, then we can finish the proof quickly by referring to Khabbaz's theorem (Corollary 2.6) that G_n is a summand of A, clearly a maximal p^n-bounded one, so Szele's theorem (Proposition 5.10) applies.

By way of contradiction, suppose that there is an $a \in A$ such that, for some $n > 0$, $a \notin G_n$, $pa \in G_n$ satisfies $\langle G_n, a\rangle \cap p^n A = 0$. By the maximal choice of the chain, there is $m \geq n + 1$ with $\langle G_m, a\rangle \cap p^m A \neq 0$, and it is safe to assume that $m = n + 1$. Thus, $g_{n+1} + a = p^{n+1}c \neq 0$ for some $g_{n+1} \in G_{n+1}, c \in A$. Hence $pg_{n+1} + pa = p^{n+2}c \in G_{n+1} \cap p^{n+1}A = 0$ implies $pg_{n+1} = -pa \in G_n$. Now $g_{n+1} \notin G_n$, for otherwise $g_{n+1} + a = p^{n+1}c \in \langle G_n, a\rangle \cap p^n A = 0$, a contradiction. Then G_n being maximal in G_{n+1} with respect to being disjoint from $p^n A$, we argue that there is a $g_n \in G_n$ such that $g_{n+1} + g_n = p^n d \neq 0$ for some $d \in A$. But then $a - g_n = p^{n+1}c - p^n d \in \langle G_n, a\rangle \cap p^n A = 0$ implies $a = g_n \in G_n$, a contradiction. □

Corollary 5.13. *Let A be the torsion part of the direct product $\prod_{i\in I} A_i$ of p-groups A_i. Suppose $B_i = \oplus_{n=1}^{\infty} B_{in}$ is a basic subgroup of A_i where $B_{in} \cong \oplus \mathbb{Z}(p^n)$ for each $n \in \mathbb{N}$, and for each $i \in I$. Then*

$$B = \bigoplus\nolimits_{n=1}^{\infty} B_n \quad \textit{with} \quad B_n = \prod\nolimits_{i\in I} B_{in}$$

is a basic subgroup of A.

Proof. For every $n > 0$, we have $A_i = B_{i1} \oplus \cdots \oplus B_{in} \oplus A_{in}$ where the last summand has no cyclic summand of order $\leq p^n$. Hence we obtain $A = B_1 \oplus \cdots \oplus B_n \oplus A_n$ with

A_n as the torsion part of $\prod_{i\in I} A_{in}$. It is clear that A_n cannot have any cyclic summand of order $\leq p^n$. An appeal to Proposition 5.10 concludes the proof. ☐

Quasi-Basis The existence of basic subgroups makes it possible to find a simple generating set in p-groups (that will be used later on).

Let B denote a basic subgroup in a p-group A. We write

$$B = \oplus_{i\in I} \langle a_i \rangle \quad \text{and} \quad A/B = \oplus_{j\in J} C_j^* \ \text{ where } C_j^* \cong \mathbb{Z}(p^\infty).$$

If C_j^* is generated by the cosets c_{jn}^* $(n \in \mathbb{N})$ such that $pc_{j1}^* = 0$, $pc_{jn+1}^* = c_{jn}^*$ $(n \in \mathbb{N})$, then by the purity of B in A, we can pick a representative $c_{jn} \in A$ of each c_{jn}^* such that $o(c_{jn}) = o(c_{jn}^*)$. The c_{jn} satisfy the relations

$$pc_{j1} = 0, \quad pc_{jn+1} = c_{jn} - b_{jn} \quad (n \in \mathbb{N})$$

where $b_{jn} \in B$ is necessarily of order $\leq p^n$. We now consider the set $\{a_i\ (i \in I),\ c_{jn}\ (j \in J, n \in \mathbb{N})\}$, called a **quasi-basis** of A. It satisfies:

Proposition 5.14 (Fuchs [2]). *If $\{a_i, c_{jn}\}$ is a quasi-basis of the p-group A, then every $x \in A$ can be written in the form*

$$x = s_1 a_{i_1} + \cdots + s_k a_{i_k} + t_1 c_{j_1 n_1} + \cdots + t_m c_{j_m n_m}$$

with distinct indices, where $s_i, t_j \in \mathbb{Z}$ and every t_j is prime to p. This form is unique for the given quasi-basis in the sense that the terms sa_i and tc_{jn} are determined by x.

Proof. Given $x \in A$, we first write the coset $x^* = x + B$ as

$$x^* = t_1 c_{j_1 n_1}^* + \cdots + t_m c_{j_m n_m}^* \qquad \text{with } \gcd\{t_j, p\} = 1.$$

Then $x-(t_1 c_{j_1 n_1}+\cdots+t_m c_{j_m n_m}) \in B$ is equal to a linear combination $s_1 a_{i_1}+\cdots+s_k a_{i_k}$. That all this is done with uniquely determined terms is obvious. ☐

A frequently quoted corollary is the following (cp. also Theorem 3.2 in Chapter 10).

Corollary 5.15 (Kulikov [3]). *A basic subgroup B of a reduced p-group A satisfies*

$$|A| \leq |B|^{\aleph_0}.$$

Proof. Using the notation we developed for quasi-basis, we observe that if for indices $j \neq k$, the equality $b_{jn} = b_{kn}$ were true for all $n \in \mathbb{N}$, then the differences $c_{jn}-c_{kn}$ $(n \in \mathbb{N})$ would generate a divisible subgroup in A. Therefore, the cardinality of a reduced A cannot exceed the cardinality of the set of sequences $\{b_{jn}\}_{n\in\mathbb{N}}$ which is $|B|^{\aleph_0}$. ☐

★ **Notes.** Prüfer [1] proves that every countable p-group contains a subgroup satisfying conditions (i)–(iii) listed for basic subgroups. Also, Baer used a basic subgroup in a very special situation, but failed to recognize its potential. It was Kulikov [2] who developed the theory of basic subgroups for p-groups of arbitrary cardinalities. Szele should be given the credit for recognizing their utmost importance and popularizing them. It is hard to think of the theory of p-groups without basic subgroups.

The material of this section is based on two papers: the pioneering article by Kulikov [2] where basic subgroups of arbitrary p-groups were introduced, and a wealth of details about them was published, and a paper by the author Fuchs [11] where the p-basic subgroups were defined for every group. p-basic subgroups turn out to enjoy several properties of basic subgroups in p-groups without any need for localizing the groups. A major difference is that Szele's Theorem 6.10 fails in general. Kulikov also developed a theory of basic submodules over the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at a prime p and over J_p (which actually covers all discrete valuation domains). Further generalization (to arbitrary valuation domains) we refer to Fuchs–Salce, *Modules over non-Noetherian Domains* (Amer. Math. Soc., 2001).

Global basic subgroups (without reference to any prime) in torsion-free groups were investigated in several papers by Blass–Irwin. B is **basic** in the torsion-free group A if it is a free pure subgroup, and A/B is divisible. Of course, it is too much to demand from a torsion-free group to contain such a subgroup, but if it does contain one, then it may contain many, all of them isomorphic. Dugas–Irwin [2] prove that a strongly $\aleph_1$-free group of cardinality $\aleph_1$ has such a basic subgroup.

Mutzbauer–Toubassi [1] construct a quasi-basis that has additional properties.

Exercises

(1) If B_i is a p-basic subgroup in A_i for $i \in I$, then $\oplus_i B_i$ is p-basic in $\oplus_i A_i$.
(2) If B is p-basic in A, then for every integer m, mB is p-basic in mA.
(3) A subgroup C of A is p-pure in A if and only if a p-basic subgroup of C is p-pure in A.
(4) Let C_n $(n < \omega)$ be pure cyclic subgroups of different orders in the p-group A. The subgroup they generate is pure in A and is their direct sum.
(5) Let A be a reduced p-group.
   (a) $p^nA[p]$ is an essential subgroup of $A_n = B_n^* + p^nA$ in (5.6).
   (b) The subgroup A_n is an absolute direct summand in A.
(6) (a) Let A be a p-group without elements of infinite height. A countable subgroup of A can be embedded in a basic subgroup of A.
   (b) This fails in general for uncountable subgroups.
(7) (E. Walker) Generalize Corollary 5.15 as follows: If A is a reduced group, and if B is dense in A, then $|A| \leq |B|^{\aleph_0}$.
(8) (Irwin) If A is a p-group with elements of infinite height, then every A^1-high subgroup contains a basic subgroup of A.
(9) (Kaplansky, Bourbaki) Call $\{a_1, \dots, a_k\}$ **pure-independent** if $mx = n_1a_1 + \dots + n_ka_k$ with $x \in A,\ m, n_i \in \mathbb{Z}$ implies $m | n_ia_i$ for $i = 1, \dots, k$. An infinite set is pure-independent if every finite subset has this property.

(a) A set is pure-independent if and only if it is independent and generates a pure subgroup.
(b) A pure-independent set is contained in a maximal one.
(c) A pure-independent set S is maximal if and only if $A/\langle S\rangle$ is divisible.
(d) In a p-group A, a maximal pure-independent set is a p-basis.

(10) Define neat-independence, and analyze its properties.

6 Theorems on p-Basic Subgroups

In this section we prove a number of useful results on p-basic subgroups. We start with elementary properties most of which have been established by Kulikov for p-groups; they immediately extend to the general case.

Throughout the first part of this section, A will denote an arbitrary group, and B a p-basic subgroup of A, p any prime.

Results on Basic Subgroups

(A) *$A = B + p^nA$ for every $n \in \mathbb{N}$.* This is an immediate consequence of the p-divisibility of A/B.

(B) *For every integer $n > 0$, there is a natural isomorphism $B/p^nB \cong A/p^nA$.* By (A) and the first isomorphism theorem, we have $A/p^nA = (B + p^nA)/p^nA \cong B/(B \cap p^nA) = B/p^nB$. We have used the equality $B \cap p^nA = p^nB$ that holds because of the p-purity of B in A.

(C) *For every integer $n \geq 0$, $p^nA/p^nB \cong A/B$ is a natural isomorphism.* The proof is similar to (B).

(D) *Let C be a p-pure subgroup of A. A p-basic subgroup B of C is a summand of a suitable p-basic subgroup of A.* All that we have to do is to extend a p-basis of C to one of A.

(E) *Transitivity: a p-basic subgroup C of a p-basic subgroup B of A is p-basic in A.* The only non-trivial part is that A/C is p-divisible, but this follows at once from the p-divisibility of B/C and A/B.

(F) *Let $A_0 = A/A^1$ be the 0th Ulm factor of the p-group A. Then the image B' of a basic subgroup B of A under the canonical homomorphism $\phi : A \to A_0$ is a basic subgroup of A_0. Moreover, $\phi \restriction B$ is an isomorphism $B \to B'$.* Since $B \cap A^1 = 0$ is obvious, $\phi \restriction B$ is an isomorphism, so B' is Σ-cyclic. B pure in A implies ϕB pure in ϕA (Theorem 1.3). Finally, $\phi A/\phi B$ is p-divisible, since $\phi A/\phi B = (A/A^1)/[(B + A^1)/A^1] \cong A/(B + A^1)$ is an epic image of the p-divisible group A/B.

(G) *Let B be a p-basic subgroup of a group A that has no p-divisible subgroup $\neq 0$. If χ, η are endomorphisms of A such that $\chi \restriction B = \eta \restriction B$, then $\chi = \eta$.* The kernel of $\chi - \eta$ contains B, hence $\mathrm{Im}(\chi - \eta)$ is an epic image of A/B, and as such it is p-divisible, so 0.

(H) *If $C_0 < \cdots < C_n < \ldots$ is an ascending chain of basic subgroups of a p-group, then the union $C = \cup_{n<\omega} C_n$ is also a basic subgroup.* Everything is easy to check, except that C is also Σ-cyclic. But this follows at once from Theorem 5.5 in Chapter 3.

(J) Our results carry over to $\mathbb{Z}_{(p)}$- and J_p-modules. The only difference is that in the definition of basic subgroups instead of $\mathbb{Z}$, $\mathbb{Z}_{(p)}$, resp. J_p is to be used.

Basic Subgroups are Isomorphic A most relevant question is to what extent p-basic subgroups are unique. We will show that, though they are, in general, not unique as subgroups, they are still all isomorphic.

First of all we point out that in general a group contains infinitely many different p-basic subgroups.

Lemma 6.1. *Let $A = \oplus_{n=1}^{\infty} \langle a_n \rangle$, where either $o(a_n) = p^{k_n}$ with $k_1 < \cdots < k_n < \ldots$ or $o(a_n) = \infty$ for every n. Then A contains a proper p-basic subgroup.*

Proof. Define $b_n = a_n - p^{k_{n+1}-k_n} a_{n+1}$ for $n = 1, 2, \ldots$, where we can use any increasing sequence for the k_n in case the a_n are of infinite order. A straightforward calculation shows that $\{b_n\}_{n\in\mathbb{N}}$ is a p-independent system, and the subgroup $B = \oplus_{n=1}^{\infty} \langle b_n \rangle$ they generate does not contain a_1. Since A/B is p-divisible (generated by the cosets $a_n + B$ for all n), B is p-basic in A. □

Thus, if A is a p-group with an unbounded basic subgroup B, then Lemma 6.1 allows us to get a proper basic subgroup B' of B, which will be also basic in A; see (E). In this way, we can form an infinite descending chain $B > B' > B'' > \ldots$ of basic subgroups. However, we have:

Theorem 6.2 (Kulikov [2], Fuchs [11]). *For every prime p, all p-basic subgroups of a group are isomorphic.*

Proof. Suppose $B = \oplus_{n=0}^{\infty} B_n$ (where $B_0 \cong \oplus \mathbb{Z}, B_n \cong \oplus \mathbb{Z}(p^n)$) is p-basic in A. The number of cyclic summands in B_n $(n > 0)$ is equal to the number of cyclic summands of order p^n in $B/p^k B$ for every $k > n$. Since $B/p^n B \cong A/p^n A$ by (B), this number is determined by A, i.e. is independent of the choice of the p-basic subgroup B.

It remains to prove that also B_0 is unique up to isomorphism. First we show that $pB_0 = B_0 \cap (T + pA)$ where $T = tA$. The inclusion $\leq$ being obvious, let $b_0 \in B_0 \cap (T + pA)$, thus $b_0 = c + pa$ with $c \in T, a \in A$. By changing a, we may assume $o(c) = p^r$ for some p^r; then $p^r b_0 = p^{r+1} a \in B_0$, so $p^r b_0 = p^{r+1} b$ for some $b \in B_0$. By the torsion-freeness of B_0, we obtain $b_0 = pb \in B_0$, proving the reverse inclusion $\geq$. We now infer $B_0/pB_0 = B_0/[B_0 \cap (T + pA)] \cong (B_0 + T + pA)/(T + pA) = A/(T + pA)$ where we have used the inclusion relation $B_0 + T + pA \geq B + pA = A$. Consequently, the rank of B_0 which is obviously equal to the rank of B_0/pB_0 does not depend on the choice of B. □

A consequence of this theorem is that the p-basic subgroups provide most valuable invariants for the group. More precisely, if $B = \oplus_{n=0}^{\infty} B_n$ with $B_0 \cong \oplus_{\kappa_0} \mathbb{Z}, B_n \cong \oplus_{\kappa_n} \mathbb{Z}(p^n)$ is a p-basic subgroup of A, then the sequence

$\kappa_0, \kappa_1, \dots, \kappa_n, \dots$ of cardinal numbers yields a system of invariants for A. For every prime p, A has such a collection. However, in general, the set of these invariants (even if taken for all p) is far from being a complete system of invariants (but—as we shall see—they completely determine the pure-injective hull of A).

If A happens to be a p-group, then $\kappa_0 = 0$, and only the invariants for the same prime p are of interest. In this case, the invariant κ_n $(n \geq 1)$ is the $n-1$st UK-invariant $f_{n-1}(A) = p^{n-1}A[p]/p^nA[p]$ of A. To substantiate this claim, note that $\kappa_n = \dim B_n[p]$ and (in the notation of Sect. 5) $p^{n-1}A[p] = A_{n-1}[p] = B_n[p] \oplus A_n[p] = B_n[p] \oplus p^nA[p]$.

In which groups is a p-basic subgroup unique? We answer this question for p-groups, and delegate the general problem to the exercises.

Lemma 6.3 (Kulikov [2]). *A p-group has a unique basic subgroup if and only if it either bounded or divisible.*

Proof. If a p-group A is bounded, then the only basic subgroup of A is itself, while if A is divisible, then its only basic subgroup is 0. On the other hand, if A has an unbounded basic subgroup, then—as was pointed out above—it has an infinite descending chain of basic submodules. The only remaining case is when A is a direct sum of a bounded group B and a divisible group D, both non-zero. In this case, a cyclic summand $\langle b \rangle$ of B can be replaced by $\langle b + d \rangle$ with $d \in D[p]$ to obtain a basic subgroup different from B. $\square$

Basic Subgroups in p-Adic Modules We wish to state without repetition of proofs that *the results in this section above carry over to p-adic modules.* (As a matter of fact, they hold for modules over discrete valuation domains.) In the definition of basic subgroup, $\mathbb{Z}$ is usually replaced by $\mathbb{Z}_{(p)}$ or J_p.

Basic Subgroups in p-Groups In the rest of this section, we confine our attention to p-groups.

Let A denote an unbounded reduced p-group. For any basic subgroup B of A, the factor group A/B is divisible, so a direct sum of quasi-cyclic groups $\mathbb{Z}(p^\infty)$, the number $\mathrm{rk}(A/B)$ of such summands is an invariant of A/B. However, this number is not an invariant for A.

The set of cardinal numbers $\mathrm{rk}(A/B)$, taken for all basic subgroups B of A, contains a minimal cardinal, and we may select a basic subgroup B_u for which $\mathrm{rk}(A/B_u)$ is minimal. Such a B_u will be called an **upper basic subgroup** of A. If $\mathrm{rk}(A/B_u)$ is infinite, then B_u contains infinitely many different upper basic subgroups.

The following is a key example (already mentioned in Example 5.7). It is the Prüfer group of length $\omega + 1$, the simplest reduced p-group with elements of infinite height. Here we use it to show that $\mathrm{rk}(A/B_u) = 1$ (and hence $\mathrm{rk}(A/B_u) = k$ for any $k \in \mathbb{N}$) is a possibility for reduced p-groups A.

Example 6.4 (Prüfer [1]). Let $A = H_{\omega+1} = \langle a_0, a_1, \dots, a_n, \dots \rangle$ where the generators are subject to the defining relations

$$pa_0 = 0, \;\; pa_1 = a_0 \;\; p^2a_2 = a_0, \;\; \dots, \;\; p^na_n = a_0, \;\; \dots.$$

Manifestly, $o(a_n) \leq p^{n+1}$ for each n. If $C \cong \mathbb{Z}(p^\infty)$ is defined as in Sect. 3 in Chapter 1, then the correspondence $a_n \mapsto c_{n+1}$ $(n \leq \omega)$ gives rise to an epimorphism $A \to C$ (relations in A hold for the images in C), showing that $o(a_n) \geq p^{n+1}$. Since $a_0 \neq 0$ is of infinite height, A is not Σ-cyclic (but $A/\langle a_0\rangle$ is Σ-cyclic: the cosets of a_n $(n \geq 1)$ form a basis). $a \in A$ may be written in the form $a = t_0a_0 + t_1a_1 + \cdots + t_na_n$ with $0 \leq t_0 < p$ and $0 \leq t_i < p^i$ $(i = 1, \ldots, n)$ for some n. In case $t_na_n \neq 0$, the correspondence $a_0, a_1, \ldots, a_{n-1} \mapsto 0$ and $a_n \mapsto c_1$ extends to a homomorphism $\phi\colon A \to C$ such that $\phi a \neq 0$; this shows that $a = 0$ if and only if $t_0a_0 = t_1a_1 = \cdots = t_na_n = 0$. It is now routine to show that $A^1 = \langle a_0\rangle$ and $\{a_1 - pa_2, \ldots, a_n - pa_{n+1}, \ldots\}$ is a p-basis of A. The subgroup B generated by this p-basis is an upper basic subgroup, since clearly $A/B \cong \mathbb{Z}(p^\infty)$. Thus $\mathrm{rk}(A/B) = 1$.

The direct sum of k copies of $H_{\omega+1}$ yields an example where $\mathrm{rk}(A/B_u) = k$.

Final Rank Szele [5] defines the **final rank** of a p-group A as the infimum of the cardinals $\mathrm{rk}(p^nA)$ with $n < \omega$:

$$\text{fin rk}(A) = \inf_{n<\omega} \mathrm{rk}(p^nA).$$

Often it is inconvenient to work with p-groups whose ranks are larger than their final ranks. There is a remedy:

Lemma 6.5 (Szele [5]). *A reduced p-group A decomposes as $A = C \oplus A'$ where C is bounded and* $\mathrm{rk}\,A' = \text{fin rk}\,A' = \text{fin rk}\,A$.

Proof. We use the same notations as above. If $\mathrm{rk}(p^mA)$ is equal to the final rank of A, then decompose $A = B_1 \oplus \cdots \oplus B_m \oplus A_m$ where $C = B_1 \oplus \cdots \oplus B_m$ is a maximal p^m-bounded summand, and $p^mC = 0$. Then $p^mA = p^mA_m$, and $A = C \oplus A_m$ is a desired decomposition. □

Lower Basic Subgroups Resuming the question of factor groups A/B modulo basic subgroups B, it is clear that there is an upper bound for the ranks $\mathrm{rk}(A/B)$: they cannot exceed $\mathrm{rk}\,A$. Hence there is a least upper bound for the ranks $\mathrm{rk}(A/B)$; this is easy to describe. Property (C) above guarantees that $\mathrm{rk}(A/B) = \mathrm{rk}(p^nA/p^nB) \leq \mathrm{rk}(p^nA)$, so that always $\mathrm{rk}(A/B) \leq \text{fin rk}(A)$. Call a basic subgroup B_ℓ **lower** if it satisfies $\mathrm{rk}(A/B_\ell) = \text{fin rk}(A)$. Its existence is established in:

Proposition 6.6 (Fuchs [2]). *Every basic subgroup of a p-group contains a lower basic subgroup.*

Proof. Let A be a reduced p-group. If $\text{fin rk}(A)$ is finite, then $\text{fin rk}(B)$ is also finite, in which case it must be 0 as B is Σ-cyclic. Then B is bounded, it is the unique basic subgroup, so it is a lower basic.

Next assume $\kappa = \text{fin rk}(A)$ is infinite. If a basic subgroup B is not lower, then $\mathrm{rk}(A/B) = \mathrm{rk}(p^nA/p^nB) < \kappa \leq \mathrm{rk}(p^nA)$ implies $\mathrm{rk}(p^nB) = \mathrm{rk}(p^nA)$. This equality means that the cardinality of the cyclic summands $\langle a_i\rangle$ of order $> p^n$ in B is at least κ, for every integer n. If this is the case, then we may decompose B into a direct sum of groups C_j, where each C_j is an unbounded Σ-cyclic group, and j runs over an index set J of cardinality κ. We quote Lemma 6.1 to argue that each C_j contains properly a basic subgroup $B_j < C_j$. The direct sum $B' = \oplus_{j \in J} B_j$ is evidently a basic subgroup of B, and hence of A. This B' is lower, because by construction $\mathrm{rk}(A/B') \geq \mathrm{rk}(B/B') = \sum_{j \in J} \mathrm{rk}(C_j/B_j) \geq \kappa$. □

Another property of basic subgroups is featured in the next lemma.

Lemma 6.7. *Let A be a reduced p-group, and B a basic subgroup of a subgroup G of A. If $B \leq p^\sigma A$ for some ordinal σ, then also $G \leq p^\sigma A$.*

Proof. A/B is viewed as an extension of $p^\sigma A/B$ by $A/p^\sigma A$. The latter group is reduced, so every divisible subgroup of A/B, in particular G/B, ought to be contained in $p^\sigma A/B$. □

Szele's Theorem The next result provides a good illustration of how isomorphism properties between a p-group and its basic subgroup are applied in practice.

Lemma 6.8 (Fuchs [3]). *If the group G is an epimorphic image of the p-group A, then every basic subgroup of G is an epimorphic image of every basic subgroup of A.*

Proof. The restriction of an epimorphism $\phi : A \to G$ to $p^n A$ is an epimorphism $\phi_n : p^n A \to p^n G$ which induces the epic map $\bar{\phi}_n : p^n A/p^{n+1}A \to p^n G/p^{n+1}G$. Owing to (B), $p^n A/p^{n+1}A \cong p^n B/p^{n+1}B$ and $p^n G/p^{n+1}G \cong p^n C/p^{n+1}C$ are natural isomorphisms (B, C denote the basic subgroups). Take into account that $p^n B/p^{n+1}B$ is a direct sum of groups of order p, where the number of summands equals the number of summands in B that are of order $\geq p^{n+1}$, and the same holds for C. The existence of an epimorphism $p^n B/p^{n+1}B \to p^n C/p^{n+1}C$ implies an inequality between the cardinalities of sets of components of order $\geq p^{n+1}$ in B and in C, for every n. Considering that both B and C are Σ-cyclic groups, this suffices to guarantee the existence of a desired epimorphism (argument with cardinalities is needed). □

Incidentally, this theorem leads us to a very interesting corollary about Σ-cyclic groups that are epic images of a p-group.

Corollary 6.9 (Szele [5]). *If a Σ-cyclic group is an epimorphic image of a p-group A, then it is an epic image of every basic subgroup of A.*

Proof. Apply Lemma 6.8 with $G = C$. □

While we are still on this theme, let us reveal another piece of information on basic subgroups of p-groups: a most surprising feature. (We reproduce Szele's proof, a shorter proof can be given by using large subgroups (Sect. 2 in Chapter 10.)

Theorem 6.10 (Szele [5]). *In a p-group, basic subgroups are endomorphic images.*

Proof. Lemma 6.5 above allows us to reduce the proof to the case when $\text{fin rk} A = \text{rk } A = \kappa$. Let $\mathfrak{D}$ denote the set of cyclic summands $\langle b_i \rangle$ of a basic subgroup B in an arbitrary, but fixed decomposition. We define a function $f : \mathfrak{D} \to \mathfrak{D}$ subject to the following conditions:

(i) f is one-to-one; and
(ii) if $f : \langle b_i \rangle \mapsto \langle b_j \rangle$ and $o(b_i) = p^{k_i}$, $o(b_j) = p^{k_j}$, then $k_j \geq 2k_i$.

By the choice of κ, for every $k \in \mathbb{N}$, $\mathfrak{D}$ contains κ summands of order $\geq p^k$, so such a function f does exist. Once we have such an f, define a surjective endomorphism χ of B as follows:

(a) $\chi b_j = b_i$ whenever $f : \langle b_i \rangle \mapsto \langle b_j \rangle$;
(b) $\chi b_j = 0$ if $\langle b_j \rangle \notin \text{Im} f$.

We now extend χ to a homomorphism $\eta : A \to B$. If $a \in A$ is of order p^k, then decompose $A = B_1 \oplus \cdots \oplus B_n \oplus A_n$ with $n \geq 2k$. Write $a = b + c$ with $b \in B_1 \oplus \cdots \oplus B_n, c \in A_n$, and define $\eta a = \chi b$. This definition is independent of the choice of n (provided $n \geq 2k$), for the B_ℓ-coordinate of a in the decomposition $A = B_1 \oplus \cdots \oplus B_\ell \oplus A_\ell$ is divisible by $p^{\ell - k}$ and therefore it is carried into 0 by χ whenever $\ell \geq 2k$. That η preserves sums is evident, so it is an endomorphism of A with image B. □

Example 6.11. Let $B_n = \langle b_n \rangle \cong \mathbb{Z}(p^n)$ and $B = \oplus_{n=1}^{\infty} B_n$. Define A as the torsion subgroup of the direct product $\prod_{n=1}^{\infty} B_n$. With the notation of the preceding proof, we let $f : \langle b_n \rangle \mapsto \langle b_{2n} \rangle$, so the map $\chi : B \to B$ is defined by setting $\chi(b_{2n}) = b_n$ and $\chi(b_{2n-1}) = 0$ for all n. If $a = (a_1, \ldots, a_n, \ldots)$ $(a_n \in B_n)$ is of order p^r, then $\eta a = \sum_{n=1}^{\infty} \chi(a_n)$ which is actually a finite sum, because $\chi(a_n) = 0$ for all $n > 2r$.

Σ-Cyclic Summands Contained in Basic Subgroups We continue this section with an interesting feature of basic subgroups. First, a preliminary lemma.

Lemma 6.12 (Cutler–Irwin [1]). *Let G be a pure subgroup of final rank κ contained in the Σ-cyclic p-group F. Then G contains a summand of F that has final rank κ.*

Proof. Decompose the socle $F[p] = \oplus_{n<\omega} V_n$ where all the non-zero elements of V_n are of height n, and set $V_n = \oplus_{i \in I_n} \langle v_{ni} \rangle$. For each $n < \omega$, we select a maximal set of disjoint finite subsets $S_{nj} \subset \{v_{ni} \mid i \in I_n\}$ such that $(\langle S_{nj} \rangle + p^{n+1}F) \cap G \neq 0$. For each S_{nj}, pick $s_{nj} \in G$ which is a linear combination of elements in S_{nj} plus an element of greater height. If $g_{nj} \in G$ satisfies $p^n g_{nj} = s_{nj}$, then it is clear that the set of all such g_{nj} for a fixed n generates a summand G_n of F contained in G. These G_n are clearly independent, so $H = \oplus_{n<\omega} G_n \leq G$ is a summand of F.

In order to prove that $\text{fin rk } H = \kappa$, observe that $F[p] = S \oplus T$ where S is generated by the set union of all selected S_{nj}, and T is generated by all v_{ni} not in S. Evidently, $T \cap G = 0$ by the maximal choice of the set of the S_{nj}. Hence the final rank of H cannot be less than κ. □

Theorem 6.13 (Cutler–Irwin [1]). *If a p-group A contains a Σ-cyclic summand of final rank κ, then every basic subgroup of A contains such a summand of A.*

Proof. Assume $A = F \oplus G$ where F is a Σ-cyclic p-group of final rank κ, and $B = \oplus_{n \in \mathbb{N}} B_n$ is a basic subgroup of A; here, $B_n = \oplus \mathbb{Z}(p^n)$. Observe that the subgroup $(B_1 \oplus \cdots \oplus B_n) + G$ is a summand of A for each n, and therefore their union $B + G$ is pure in A. Hence $(B + G)/G$ is a pure subgroup of the Σ-cyclic group A/G; moreover, it must be a basic subgroup, since the density of B in A is preserved mod G. This means that $(B + G)/G \cong A/G$ by the uniqueness of basic subgroups up to isomorphism. By Lemma 6.12, there is a summand $(H + G)/G$ of A/G contained in $(B+G)/G$ of final rank κ. The generators of the cyclic summands in H may be chosen to belong to B. □

Starred *p*-Groups Not much, but still something can be said about reduced p-groups whose basic subgroups have the same cardinality as the group; these groups are called **starred** (Khabbaz [1]). It is pretty clear that this property is preserved by arbitrary direct sums, but not necessarily by summands.

A kind of converse to Theorem 6.13 holds for starred p-groups.

Theorem 6.14 (Khabbaz [1]). *Suppose that A is a starred p-group. Then A has a Σ-cyclic summand of cardinality $|A|$.*

Proof. First assume $A^1 = 0$. The countable case being trivial (since then A itself is Σ-cyclic), let $\kappa = |A| = |B| > \aleph_0$ where $B = \oplus_n B_n$ with $B_n = \oplus_{\kappa_n} \mathbb{Z}(p^n)$. There is no loss of generality in assuming that A is a subgroup of its torsion-completion $\overline{B}$ (Sect. 3 in Chapter 10), so every $a \in A \setminus B$ may be viewed as a vector $(b_1, \dots, b_n, \dots)$ with infinitely many non-zero coordinates. If $\kappa_n = \kappa$ for some n, there is nothing to prove: B_n is a sought-after summand. Thus, for the rest of the proof we may assume that $\kappa_n < \kappa$ for each n and $\sum_n \kappa_n = \kappa$, i.e. κ is a limit cardinal.

A moment reflection shows that there exists a sequence $n_0 < \cdots < n_i < \dots$ satisfying $\kappa_{n_0} < \cdots < \kappa_{n_i} < \dots$ with $\sum_i \kappa_{n_i} = \kappa$. Let X be a set of representatives of the cosets mod B; evidently $|X| \le \kappa$. Let $X = \cup_{i<\omega} X_i$ be a partition of X such that $|X_i| \le \kappa_{n_i}$. Each $x \in X$ is represented by an infinite vector with nth coordinate in B_n, and it is clear that no harm is done if we change a finite number of coordinates. Taking advantage of this freedom, we may assume that all the jth coordinates of every $x \in X_i$ are 0 whenever $j \le n_i$. Since the κ_i's are increasing, it follows that each $B_{n_{i+1}}$ has a summand B'_{i+1} of cardinality κ_i that contains all the $B_{n_{i+1}}$-coordinates of the $x \in X$, so its complement C_i in $B_{n_{i+1}}$ has cardinality κ_{i+1}. Then $C = \oplus_i C_i$ is a Σ-cyclic summand of A of cardinality κ, its complement is generated by the B'_i along with X and the B_n with $n \neq n_i$.

If $A^1 \neq 0$, then let $\phi : A \to A/A^1$ be the canonical map. What has been proved can be applied to $\phi(A)$ with basic subgroup $\phi(B) \cong B$ to find a decomposition $\phi(A) = C' \oplus G/A^1$ where C' is Σ-cyclic of cardinality $|A|$. Then G is pure in A and $A/G \cong C'$ is Σ-cyclic, so G is a summand of A whose complement is a summand of the desired kind. □

★ **Notes.** The role played by basic subgroups in p-groups is absolutely fundamental; their various characterizations enable us to view them from different angles. The existence of lower and upper basic subgroups shows that it is impossible to find a standard location for basic subgroups. Basic subgroups are crucial in the study of torsion-complete and algebraically compact groups.

So far no relevant application has been found for Szele's theorem which is a rather unexpected result on basic subgroups. But it was instrumental in leading Pierce to the definition of small homomorphisms. Keef [3] has an interesting, more general version of Theorem 6.10: for any sequence of p-groups G_n $(n < \omega)$, there is an epimorphism $\phi : t(\prod_{n<\omega} G_n) \to \oplus_{n<\omega} G_n$.

Exercises

(1) Let B denote a p-basic subgroup of A. For every $n \geq 0$, we have

  (a) $(A/B)[p] \cong A[p]/B[p]$;
  (b) $(p^nA/p^nB)[p] \cong p^nA[p]/p^nB[p]$;
  (c) $p^nA[p] = p^nB[p] + p^{n+1}A[p]$;
  (d) $p^{n+1}B[p] = p^nB[p] \cap p^{n+1}A[p]$;
  (e) $p^nB[p]/p^{n+1}B[p] \cong p^nA[p]/p^{n+1}A[p]$.

(2) If B is a p-basic subgroup of A, then for all integers $n, k > 0$, we have $p^nA = p^nB + p^{n+k}A$ and $p^nB/p^{n+k}B \cong p^nA/p^{n+k}A$.
(3) Let C be a p-pure subgroup of A, and $\{a_i\}_{i\in I}$ a p-basis of C. Furthermore, let $\{b_j + C\}_{j\in J}$ be a p-basis of A/C such that $o(b_j) = o(b_j + C)$ for each $j \in J$. Then $\{a_i, b_j\}_{i\in I, j\in J}$ is a p-basis of A.
(4) Let $0 \to A \to G \to C \to 0$ be a p-pure-exact sequence, and B_A, B_C p-basic subgroups of A, C. There is a p-basic subgroup B_G of G such that with the induced mappings the sequence $0 \to B_A \to B_G \to B_C \to 0$ is splitting exact. [Hint: Exercise 3.]
(5) In a p-group A, the intersection of all basic subgroups is either A or 0.
(6) (Khabbaz–Walker, Hill) If the p-group A has infinitely many basic subgroups, then it has $|A|^{|B|}$ different basic subgroups.
(7) Let κ be a cardinal satisfying $\mathrm{rk}(A/B_u) \leq \kappa \leq \mathrm{rk}(A/B_\ell)$ for an upper and a lower basic subgroup B_u, B_ℓ of a p-group A. There exists a basic subgroup B in A with $\mathrm{rk}(A/B) = \kappa$.
(8) Suppose A is the torsion part of $\prod_{n=1}^{\infty} \langle a_n \rangle$ where $o(a_n) = p^n$. Then every basic subgroup of A is both upper and lower.
(9) A p-group A with $A^1 = 0$ has a decomposition $A = C \oplus A'$ where C is Σ-cyclic, and every basic subgroup of A' is both upper and lower.
(10) Let B be a basic subgroup of the p-group A, and G fully invariant in A. Then $G \cap B$ is basic in G.
(11) (Irwin–Keef) If the p-group $A = G \oplus H$ has a Σ-cyclic summand of final rank $\aleph_0$, then either G or H has such a summand. [Hint: the direct sum of their basic subgroups has such a summand; use isomorphic refinement.]
(12) (Nunke) Let A be a reduced p-group. A basic subgroup B of A satisfies $|B| = |A|$ if and only if $|C| = |A|$ for every dense subgroup C of A.
(13) Let A be a reduced starred p-group. Every subgroup of A is an endomorphic image of A. [Hint: Theorem 6.10.]

(14) (Szele) Every p-group of cardinality $\leq \kappa$ (infinite cardinal) is an epic image of the p-group A if and only if $\kappa \leq \operatorname{fin}\operatorname{rk}(B)$ holds for its basic subgroup B.

Problems to Chapter 5

PROBLEM 5.1. For which cardinals κ is there a universal p-group for purity? We mean a p-group U_κ of cardinality of κ such that every p-group of cardinality $\leq \kappa$ embeds in U_κ as a pure subgroup. The same question for torsion-free groups.

PROBLEM 5.2. For which ordinals σ are there $\aleph_\sigma$-pure subgroups that fail to be $\aleph_{\sigma+1}$-pure?

The answer is affirmative for isolated ordinals σ up to the first inaccessible cardinal.

PROBLEM 5.3. What are the special features of the subgroups of A that are pure $\operatorname{End} A$-modules? (Purity in the module sense à la P. Cohn.)

Cf. Turmanov [1].

PROBLEM 5.4. Describe the pure-injective hull of a group over its own endomorphism ring.

See Vinsonhaler–Wickless [2] for injectivity.

PROBLEM 5.5. Study the pure-projective and pure-injective dimensions over the endomorphism ring.

Chapter 6
Algebraically Compact Groups

Abstract In the preceding chapter we have encountered groups that were summands in every group containing them as pure subgroups: the pure-injective groups. In this chapter, we collect a large amount of additional information about these groups. Interestingly, these are precisely the summands of groups admitting a compact group topology, and the reduced ones are nothing else than the groups complete in the $\mathbb{Z}$-adic topology. From Sect. 4 in Chapter 5 we know that every group can be embedded as a pure subgroup in a pure-injective (i.e., in an algebraically compact) group, and here we show that the significance of this embedding is enhanced by the fact that minimal embeddings exist and are unique up to isomorphism. Thus the theory of algebraically compact groups runs, in many respects, parallel to the theory of injective groups, a fact that was first pointed out by Maranda [1].

The theory of algebraically compact groups is quite satisfactory: these groups admit complete characterization by cardinal invariants. We shall often meet algebraically compact groups in subsequent discussions.

We close this chapter with the discussion of the exchange property. This is a remarkable, but rather rare phenomenon. Groups with this property show the best behavior as summands.

1 Algebraic Compactness

A group is called **algebraically compact** if it is (algebraically) a summand of a group that admits a compact group topology. Evidently, this property is inherited by summands and direct products.

Example 1.1. As the circle group $\mathbb{T}$ is compact, its summand $\mathbb{Q}/\mathbb{Z}$ is algebraically compact. Direct products of copies of a finite group are compact groups, thus their summands: the bounded groups are also algebraically compact.

Characterizations of Algebraic Compactness It is convenient to start the discussion with the main characterizations of algebraic compactness. The equivalence of (b) with (e) has already been observed in Theorem 4.7 in Chapter 5.

Theorem 1.2 (Balcerzyk [1], Łoś [1], Maranda [1]). *The following conditions on a group A are equivalent:*

(a) *A is a summand in every group which contains it as a pure subgroup;*
(b) *A is a summand of a direct product of cocyclic groups;*
(c) *A is algebraically compact;*

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(d) *a system of equations over A is solvable in A provided that each of its finite subsystems has a solution in A;*
(e) *A is pure-injective.*

Proof. (a) ⇒ (b) By Lemma 4.6 in Chapter 5, A can be embedded as a pure subgroup in a direct product of cocyclic groups. Hence it is clear that (a) implies (b).

(b) ⇒ (c) Since the direct product of compact groups is compact in the product topology, and since the property of being a summand is transitive, (c) will be established as soon as we can show that cocyclic groups share property (c). But this is evident, since the cyclic groups $\mathbb{Z}(p^k)$ $(k < \infty)$ are compact in the discrete topology, while the quasi-cyclic groups $\mathbb{Z}(p^\infty)$ are summands of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the reals mod 1) which is a well-known compact group.

(c) ⇒ (d) Consider the system

$$\sum\nolimits_{j\in J} n_{ij}x_j = a_i \in A \qquad (i \in I) \tag{6.1}$$

of linear equations where $n_{ij} \in \mathbb{Z}$ such that, for a fixed i, almost all the coefficients n_{ij} are 0, and suppose that every finite subsystem is solvable in A. By hypothesis (c), for some group B, the group $C = A \oplus B$ admits a compact group topology. We may equally well view our system (6.1) to be over C, and finitely solvable in C. A solution of the ith equation can be regarded as an element $(\dots, c_j, \dots)$ in the cartesian power C^J such that $x_j = c_j \in C$ $(j \in J)$ satisfies the ith equation. The set of all solutions to the ith equation is thus a subset X_i of the compact space C^J; moreover, it is a closed subset, since it is defined in terms of an equation. The hypothesis that every finite subsystem of (6.1) is solvable in C amounts to the condition that every finite set of the X_i $(i \in I)$ has a non-empty intersection. By the compactness of C^J, the intersection $\cap_{i\in I} X_i$ is not vacuous. This means that the entire system (6.1) admits a solution in C. The A-coordinates of a solution yield a solution of (6.1) in A.

(d) ⇒ (e) Let B denote a pure subgroup of C, and $\eta : B \to A$. Let $\{c_j\}_{j\in J}$ be a generating system of C mod B, and

$$\sum_{j\in J} n_{ij}c_j = b_i \in B \qquad (n_{ij} \in \mathbb{Z},\ i \in I)$$

the list of all the relations between these c_j and elements of B. We pass to the corresponding system

$$\sum_{j\in J} n_{ij}x_j = \eta b_i \in A \qquad (i \in I) \tag{6.2}$$

of equations. A finite subsystem of (6.2) contains but a finite number of unknowns $x_{j_1}, \dots, x_{j_k}$ explicitly. By purity, B is a direct summand of $B' = \langle B, c_{j_1}, \dots, c_{j_k}\rangle$, $B' = B \oplus C'$ (cf. Theorem 2.9 in Chapter 5), and the images of the B-coordinates of $c_{j_1}, \dots, c_{j_k}$ under η yield a solution in A. Consequently, (6.2) satisfies the hypotheses of (d), so we can infer that there is a solution $x_j = a_j \in A$ $(j \in J)$ of the entire system (6.2). The correspondence $c_j \mapsto a_j$ gives rise to an extension of η to a morphism $C \to A$, establishing (e).

(e) $\Rightarrow$ (a) Let A be a pure-injective group contained as a pure subgroup in a group G. Owing to pure-injectivity, the identity map $\mathbf{1}_A : A \to A$ can be factored as $A \to G \to A$. Thus A is a direct summand of G. □

The reduction of conditions (a), (d), and (e) to the countable case will be useful in the sequel.

Corollary 1.3. *We have* (a) $\Leftrightarrow$ (a′), (d) $\Leftrightarrow$ (d′)*, and* (e) $\Leftrightarrow$ (e′)*, where*

(a′) *A is a summand in every group G in which it is a pure subgroup with G/A countable;*

(d′) *every countable system of equations over A is solvable in A provided that each of its finite subsystems has a solution in A;*

(e′) *A is injective with respect to pure-exact sequences* $0 \to B \to C \to C/B \to 0$ *with countable C/B.*

Proof. We give a proof for (e′) $\Rightarrow$ (e), and leave the rest to the reader (Exercise 4). In order to extend a map $\eta: B \to A$ to $C \to A$ if C/B is uncountable, we take a pure $\aleph_0$-filtration $B/B < C_1/B < \dots < C_\sigma/B < \dots$ of C/B (with subgroups $C_\sigma \le C$). Since C_σ is pure in $C_{\sigma+1}$ of countable index, once we have a map $\eta_\sigma : C_\sigma \to A$, this can be extended to the next level. As always, at limit ordinals we take the union of maps already defined. □

For a reduced group, condition Theorem 1.2 (b) can be refined:

Corollary 1.4. *A reduced algebraically compact group is a summand of a direct product of cyclic p-groups. Its first Ulm subgroup is* 0.

Proof. If A is reduced algebraically compact, then for some group B, $A \oplus B = D \oplus E$ where D is the direct product of quasi-cyclic groups, while E is a direct product of finite cyclic groups. D is the divisible part, so fully invariant; thus, $D = (A \cap D) \oplus (B \cap D)$ where the first summand must vanish. Thus $D \le B$ and $A \oplus (B/D) \cong E$. □

From Theorem 1.2(b) we derive

Corollary 1.5. *Reduced algebraically compact groups are $\tilde{\mathbb{Z}}$-modules.*

Proof. Take into account that group summands of reduced $\tilde{\mathbb{Z}}$-modules are also $\tilde{\mathbb{Z}}$-module summands. □

Example 1.6. The group J_p of the p-adic integers is compact in the p-adic topology (so it is algebraically compact). Since J_p is the inverse limit of the cyclic groups $\mathbb{Z}(p^k)$ for $k = 1, 2, \dots$, it is contained in the group $\prod_{k<\omega} \mathbb{Z}(p^k)$ which is compact in the product topology. The inverse limit is a closed subgroup in the product, so the compactness of J_p is immediate.

Example 1.7 (Fuchs [IAG]). The additive group of an injective module M over any ring R is algebraically compact. Indeed, if D is the divisible hull of M as an abelian group, then M

is a submodule and hence a summand of the injective R-module $\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}, D)$ which is by Theorem 2.11 in Chapter 7 algebraically compact.

Example 1.8. The group $\mathbb{Z}$ is not algebraically compact. Perhaps the easiest way to see it is to point out that $\mathbb{Z}$ is not a $\tilde{\mathbb{Z}}$-module.

For the sake of easy reference we formulate the following immediate consequence of (b) or (c) in Theorem 1.2.

Corollary 1.9. *A direct product of groups is algebraically compact if and only if every component is algebraically compact.* □

Combining Lemma 4.6 in Chapter 5 with the theorem above, we conclude:

Corollary 1.10. *Every group can be embedded as a pure subgroup in an algebraically compact group.* □

We will see in Theorem 4.2 below that there is a minimal such algebraically compact group.

Algebraically Compact Factor Groups Next, we will show that algebraically compact groups are abundant among quotient groups of direct products.

Let G_i $(i \in I)$ be a family of groups with an arbitrary index set I, and let K denote an ideal in the Boolean lattice $\mathbf{2}^I$ of all subsets of I. K^* will stand for the σ-ideal generated by K, i.e. it consists of countable unions of subsets in K. (It is easy to see that these are exactly the countable unions of pairwise disjoint subsets from K.) Our concern is the relation between K- and K^*-products (for definition, see Sect. 1 in Chapter 2).

Theorem 1.11 (Fuchs [13]). *If K and K^* are defined as above, then the factor group*

$$A = \prod_{\mathsf{K}^*} G_i / \prod_{\mathsf{K}} G_i \tag{6.3}$$

is algebraically compact for any collection of groups G_i $(i \in I)$.

Proof. Ignoring a trivial case, assume $\mathsf{K} \neq \mathsf{K}^*$. In view of Theorem 1.2 and Corollary 1.3, it suffices to establish a solution in A for the system

$$\sum_{k=1}^{\infty} n_{jk} x_k = \bar{a}_j \in A \qquad (n_{jk} \in \mathbb{Z},\ j < \omega) \tag{6.4}$$

with a countable number of unknowns and equations, under the hypothesis that every finite subsystem of (6.4) admits a solution in A. Here $a_j \in \prod_{\mathsf{K}^*} G_i$, and bars denote cosets mod $\prod_{\mathsf{K}} G_i$.

Let $x_k = \bar{c}_k^m \in A$ be a solution of the system consisting of the first m equations, with the understanding that the value 0 is assigned to the unknowns not occurring explicitly in these equations. The representatives $a_j, c_k^m \in \prod_{\mathsf{K}^*} G_i$ in the cosets

$\bar{a}_j, \bar{c}_k^m$, respectively, are such that 0 is always chosen as a representative of the coset $\bar{0}$. All the supports $s(a_j)$ belong to $\mathsf{K}^\star$, and so does their union $Y = \bigcup_{j<\omega} s(a_j)$.

We now move to the system

$$\sum\nolimits_{k=1}^{\infty} n_{jk} x_k = a_j \in \prod_{\mathsf{K}^\star} G_i \qquad (j < \omega). \tag{6.5}$$

Plugging $x_k = c_k^m$ in the equations with indices $j \leq m$, the set Y_m of indices $i \in I$ for which the ith coordinates of $\sum_k n_{jk} c_k^m$ and a_j differ for $j = 0, 1, \dots, m$ certainly belongs to K. There is no difficulty in constructing subsets X_m $(m < \omega)$ so as to satisfy

$$X_0 = Y_0,\ X_1 \supseteq Y_1 \setminus Y_0,\ \dots,\ X_m \supseteq Y_m \setminus (Y_0 \cup \dots \cup Y_{m-1}), \dots,$$

subject to the condition $\bigcup_{m<\omega} X_m = Y \cup Y_0 \cup \dots \cup Y_m \cup \dots$.

Now the proof can easily be completed. Set $b_k = (\dots, c_{ki}, \dots)$, where c_{ki} is the ith coordinate of c_k^m if $i \in X_m$, and 0 otherwise. Clearly, $b_k \in \mathsf{K}^\star$. As the ith coordinate of $\sum_k n_{jk} b_k$ agrees with the ith coordinate of a_j for all $i \in X_m$ $(j \leq m)$, it follows that $x_k = \bar{b}_k \in A$ $(k \in \mathbb{N})$ is a solution of the system (6.4). □

There is a special case that is particularly worth mentioning. This is when the index set I is countable, and K is the ideal consisting of all finite subsets of I. Then $\mathsf{K}^* = I$, and we get a most interesting corollary.

Corollary 1.12 (Balcerzyk [2], Hulanicki [3]). *For any countable family of groups G_n $(n = 1, 2, \dots)$ the factor group*

$$\prod_{n=1}^{\infty} G_n / \bigoplus_{n=1}^{\infty} G_n$$

is algebraically compact. □

On the basis of this result it would be tempting to conjecture that direct products modulo direct sums are always algebraically compact. It turns out that Corollary 1.12 is true only for countable index sets (see Notes).

A proof similar to the one in Theorem 1.11 yields:

Theorem 1.13. *If $\{A_i \mid i \in I\}$ is a set of groups, and $\mathcal{F}$ is an ω_1-complete ultrafilter on the index set I, then the ultraproduct $\prod_{i\in I} A_i/\mathcal{F}$ is algebraically compact.* □

Large Products Mod Direct Sum For uncountable index sets I, the factor groups $\prod_{i\in I} A_i / \oplus_{i\in I} A_i$ are only exceptionally algebraically compact. As observed by Göbel–Rychkov–Wald [1], a weaker conclusion may be drawn which we prove in the next theorem (in an equivalent form). We start with the observation that all torsion epic images of an algebraically compact group are of the form $B \oplus D$, with bounded B and divisible D. The easiest way to prove this is to refer to Corollary 3.5.

Theorem 1.14. *For a non-measurable index set I and any groups A_i, every reduced torsion epic image of the factor group $\prod_{i\in I} A_i/\oplus_{i\in I} A_i$ is bounded.*

Proof. Let $\phi : A^* = \prod_{i\in I} A_i/\oplus_{i\in I} A_i \to T$, where T is a reduced torsion group. $\operatorname{Im}\phi = T$ may be assumed. We consider ϕ as a homomorphism of the product that vanishes on $\oplus A_i$. If $\operatorname{Im}\phi$ is unbounded, then the image of ϕ followed by an epimorphism ψ of T onto its basic subgroup B Theorem 6.10 in Chapter 5 is still unbounded. In this situation, in the application of Theorem 6.5 in Chapter 2 we can ignore I_0 and J_0, and conclude that there is an $m \in \mathbb{N}$ such that $m\operatorname{Im}(\psi\phi)$ is contained in the Ulm subgroup $B^1 = 0$. Hence $\operatorname{Im}(\psi\phi)$ is bounded, and so is $\operatorname{Im}\phi$. □

$\aleph_1$-Algebraic Compactness It seems reasonable to consider generalizations of algebraic compactness for higher cardinalities. $\aleph_1$-algebraic compactness was studied by Megibben [6]. He proved that this concept does not yield anything new. However, for higher cardinalities the situation is different, though it does not seem to be too interesting. Here we wish to discuss only the $\aleph_1$ case. In the proof, we need a couple of results to be proved later, in particular, the characterization of algebraic compactness from Proposition 5.8 in Chapter 9.

A group M is said to be **$\aleph_1$-algebraically compact** if it has the injective property relative to all $\aleph_1$-pure-exact sequences. Equivalently, it is a summand in every group in which it is contained as an $\aleph_1$-pure subgroup. All algebraically compact groups are evidently $\aleph_1$-algebraically compact.

In the proof of the next theorem, the following simple fact is needed from set theory. For an infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa < \kappa^{\aleph_0} (= 2^\kappa)$. (For instance, $\kappa = \sum_{i<\omega} \kappa_i$ is such a cardinal if $\kappa_0 = \lambda$ and $\kappa_{i+1} = 2^{\kappa_i}$ for all i; see note after Proposition 7.10 in Chapter 10.)

Theorem 1.15 (Megibben [6]). *A group is $\aleph_1$-algebraically compact if and only if it is algebraically compact.*

Proof. By Proposition 5.8 in Chapter 9, for the algebraic compactness of a group G, it suffices to show that $\operatorname{Ext}(\mathbb{Q}, G) = 0$ and $\operatorname{Pext}(\mathbb{Q}/\mathbb{Z}, G) = 0$ hold for G. We start the proof by showing that an $\aleph_1$-algebraically compact group G satisfies

(i) $\operatorname{Ext}(F, G) = 0$ for all $\aleph_1$-free groups F; and
(ii) $\operatorname{Pext}(T, G) = 0$ for all separable torsion groups T.

To prove (i), observe that if G is $\aleph_1$-algebraically compact, and if $0 \to G \to A \to F \to 0$ is an exact sequence with $\aleph_1$-free F, then the sequence has to split, being $\aleph_1$-pure-exact. The argument for (ii) is entirely similar with a pure-exact sequence.

Now suppose G is $\aleph_1$-algebraically compact. Choose a cardinal $\kappa > |G|$ such that $\kappa < \kappa^{\aleph_0}$. Let F be the $\mathbb{Z}$-adic closure of the free group $F_0 = \mathbb{Z}^{(\kappa)}$ in the direct product $\mathbb{Z}^\kappa$. F is $\aleph_1$-free and has cardinality $\kappa^{\aleph_0}$. Thus $F/F_0 \cong \oplus_{\kappa^{\aleph_0}} \mathbb{Q}$. We obtain the induced exact sequence

$$G^\kappa \cong \operatorname{Hom}(F_0, G) \to \operatorname{Ext}(F/F_0, G) \to \operatorname{Ext}(F, G) = 0.$$

Here $|G^\kappa| = 2^\kappa$, while $|\operatorname{Ext}(F/F_0, G)| = \prod_{\kappa^{\aleph_0}} |\operatorname{Ext}(\mathbb{Q}, G)|$ is either 0 or is at least $2^{\kappa^{\aleph_0}}$. The latter alternative is impossible, proving $\operatorname{Ext}(\mathbb{Q}, G) = 0$.

The proof for $\operatorname{Pext}(\mathbb{Q}/\mathbb{Z}, G) = 0$ is exactly the same, using embedding in torsion completion. □

★ **Notes.** The problem of describing the algebraic structure of compact abelian groups led Kaplansky [K] to the discovery of algebraically compact groups as summands (in the algebraic sense) of compact groups. A different line of investigation started with Łoś [1, 2], who considered groups that were summands in every group containing them as pure subgroups. Balcerzyk [1] noticed that the classes discussed by Kaplansky and Łoś were identical. The study of pure-injectivity as an analogue of injectivity was initiated by Maranda [1].

Łoś [1] considers abstract 'limits' (not in the topological sense), called ω_σ*-limits*, and shows that a group admits such a limit if and only if it is algebraically compact.

The theory of pure-injective modules over general rings was developed by Warfield [Pac. J. Math. **28**, 699–720 (1969)] based on P. Cohn's definition of purity. Lot of information is available about the structure of pure-injective modules, especially over particular rings.

Numerous papers deal with factor groups of direct products modulo direct sums. Gerstner [1] shows that $\mathbb{Z}^\kappa/\mathbb{Z}^{(\kappa)}$ is never algebraically compact if κ is uncountable. Rychkov [1] proves that the factor group $\prod_{i\in I} G_i/\oplus_{i\in I} G_i$ is algebraically compact if and only if, with the exception of at most countably many indices i, all the G_i are algebraically compact. Dugas–Göbel [1] studies such factor groups more generally, using filters of the index set. Franzen [1] discusses when the factor group of a filter-product modulo another filter-product is algebraically compact. Göbel–Rychkov–Wald [1] show that if I has non-measurable cardinality, then $A = \prod_{i\in I} A_i/\oplus_{i\in I} A_i$ is always a Fuchs-44-group (see Notes to Sect. 6 in Chapter 2). Wald [1] studies the quotients $\prod^{(\mu)} A_i/\prod^{(\kappa)} A_i$ for cardinals $\mu > \kappa$ (only vectors with supports $< \mu$ (resp. $< \kappa$) are taken).

Exercises

(1) (a) A^1 is the divisible part of an algebraically compact group A.
  (b) A group is algebraically compact if and only if it is the direct sum of an injective group and a reduced algebraically compact group.

(2) (Sąsiada) A group A is algebraically compact if it is a summand in every group G in which it is a pure subgroup such that $G/A \cong \mathbb{Q}$ or $\mathbb{Z}(p^\infty)$.

(3) (a) A pure subgroup C of an algebraically compact group A is algebraically compact provided A/C is reduced.
  (b) A group A need not be algebraically compact even if both the subgroup C and the factor group A/C are algebraically compact.

(4) Complete the proof of Corollary 1.3.

(5) The obvious map $A \to \prod_{n\in\mathbb{N}} A/nA \oplus E$ is an embedding of A as a pure subgroup in an algebraically compact group; here, E stands for the injective hull of A.

(6) (a) Show that A is isomorphic to a pure subgroup of the algebraically compact group $A^{\aleph_0}/A^{(\aleph_0)}$. [Hint: consider the diagonal in $A^{\aleph_0}$.]
  (b) Every group A can be embedded as a pure subgroup in an algebraically compact group of cardinality $\leq |A|^{\aleph_0}$.

(7) Prove that $\mathbb{Z}^{\aleph_0}/\mathbb{Z}^{(\aleph_0)} \cong (\mathbb{Q} \oplus \tilde{\mathbb{Z}})^{\aleph_0}$.
(8) Let G_n $(n = 1, 2, \dots)$ be reduced groups such that there is no integer $m > 0$ satisfying $mG_n = 0$ for almost all n. The factor group $\prod_n G_n / \oplus_n G_n$ is not reduced.
(9) (Baumslag–Blackburn) Let G_n $(n = 1, 2, \dots)$ be reduced groups. $\oplus_n G_n$ is a summand of $\prod_n G_n$ if and only if $mG_n = 0$ holds for some $m > 0$ and for almost all n. [Hint: for necessity see preceding exercise.]
(10) Let H denote the subgroup of $\prod_{i \in I} G_i$ consisting of elements with countable support. Prove that $H / \oplus_i G_i$ is a summand of $\prod_i G_i / \oplus_i G_i$.
(11) (a) If A is pure-injective, then for every cardinal κ, the codiagonal homomorphism $\nabla \colon A^{(\kappa)} \to A$ extends to a homomorphism: $A^\kappa \to A$.
(b) If A has this extension property for some infinite cardinal $\kappa > |A|$, then it is pure-injective.
(12) A group G is $\aleph_1$-algebraically compact if and only if every system of equations over G is solvable in G whenever every countable subsystem has a solution in G.

2 Complete Groups

In this section we examine more closely the groups that are complete in their $\mathbb{Z}$-adic or p-adic topologies. This will not be a new class: we shall see that these groups coincide with the reduced algebraically compact groups. (Recall that according to our agreement, only Hausdorff groups will be called complete.)

Example 2.1. The groups that are discrete in their $\mathbb{Z}$-adic topologies, i.e. the bounded groups, are evidently complete.

Example 2.2. The group J_p of p-adic integers is (reduced and compact, and hence) complete in its p-adic topology.

To simplify terminology, we agree that in the absence of firm indication to the contrary, **complete group** means a group that is complete in its $\mathbb{Z}$-adic topology (in p-groups, this is identical to the p-adic topology).

Properties of Complete Groups We start with a simple observation.

Lemma 2.3. *A group A that is complete in its p-adic topology is a J_p-module, and is complete also in the $\mathbb{Z}$-adic topology. It satisfies $qA = A$ for all primes $q \neq p$.*

Proof. Recall that such an A is in a natural way a J_p-module: if $a \in A$ and $\pi = s_0 + s_1 p + \cdots + s_n p^n + \cdots \in J_p$ $(0 \leq s_n < p)$, then

$$s_0 a,\ (s_0 + s_1 p)a,\ \dots,\ (s_0 + s_1 p + \cdots + s_n p^n)a,\ \dots$$

is a Cauchy sequence in A, hence it converges to a limit in A which we define as πa. $qJ_p = J_p$ implies $qA = A$ for all primes $q \neq p$. As a consequence, the p-adic and the $\mathbb{Z}$-adic topologies are identical on A. □

In a similar fashion, we can prove that every group that is complete in the $\mathbb{Z}$-adic topology is a $\tilde{\mathbb{Z}}$-module.

Recall that groups that are J_p-modules were called p-adic groups. p-adic groups have basic subgroups, unique up to isomorphism.

Theorem 2.4. (i) *A complete p-adic group is the completion of any of its basic subgroups.*
(ii) *A Σ-cyclic p-group is basic in its completion.*
(iii) *Two complete p-adic groups are isomorphic if and only if their basic subgroups are isomorphic.*

Proof. (i) This follows from the fact that a basic subgroup is dense and the induced topology on it is the same as its p-adic topology.
(ii) is clear from (i).
(iii) One way the claim is trivial. Conversely, a complete p-adic group is determined by a basic subgroup. □

Completeness and Algebraic Compactness The principal result on complete groups is the following theorem, which is essentially due to Kaplansky [K].

Theorem 2.5. *A group is complete in its $\mathbb{Z}$-adic topology if and only if it is a reduced algebraically compact group.*

Proof. Assume A is reduced and algebraically compact. Owing to Corollary 1.4, A is a summand of a direct product of cyclic groups $\mathbb{Z}(p^k)$. Each component is complete in its $\mathbb{Z}$-adic topology, so the same holds for the summands of their direct product (see Lemma 7.10 in Chapter 2).

Conversely, suppose that A is complete in its $\mathbb{Z}$-adic topology and is pure in the group G. If $G^1 = 0$, then we expand G to its completion $\tilde{G}$; this contains G, and hence also A as a pure subgroup. A basic subgroup B' of A is a summand of a basic subgroup $B = B' \oplus B''$ of $\tilde{G}$, whence $\tilde{G} = \tilde{B}' \oplus \tilde{B}'' = A \oplus \tilde{B}''$. Thus A is a summand of G, too. If $G^1 \neq 0$, then factoring out G^1, the image of A remains pure in G/G^1, so $(A + G^1)/G^1 \cong A$ is a summand of G/G^1. As $A \cap G^1 = 0$, A is a summand of G. Consequently, A is algebraically compact. □

Since the $\mathbb{Z}$-adic topology on A induces a topology on a subgroup B of A that is coarser than the $\mathbb{Z}$-adic topology of B, we combine the preceding theorem with Lemma 7.2 in Chapter 2 to derive at once:

Corollary 2.6. *Let A be a reduced algebraically compact group, and B a subgroup such that $(A/B)^1 = 0$. Then both B and A/B are algebraically compact.* □

Another noteworthy observation is the following.

Corollary 2.7. *If θ is an endomorphism of a complete group A, then both* $\operatorname{Ker}\theta$ *and* $\operatorname{Im}\theta$ *are complete groups.*

Proof. Apply the preceding corollary to $B = \operatorname{Ker}\theta$, and note that $A/\operatorname{Ker}\theta$ is complete by virtue of Lemma 7.2 in Chapter 2. ☐

Lemma 2.8 (Kaplansky [K]). *Let G be a pure subgroup of a complete group C. Then the $\mathbb{Z}$-adic closure of G in C is a summand of C. In particular, a pure closed subgroup is a summand.*

Proof. A basic subgroup B' of G extends to a basic subgroup $B = B' \oplus B''$ of C. Then $C = \tilde{B}' \oplus \tilde{B}''$ by Theorem 2.4, where $G \le \tilde{B}'$. Thus $\tilde{B}'$ is the closure of G in C. ☐

Completions We turn our attention to the completion process. In the next result we expand Theorem 7.7(i) in Chapter 2.

Theorem 2.9. *For any group A, the inverse limit*

$$\tilde{A} = \varprojlim (A/nA)$$

with the connecting maps $a + knA \mapsto a + nA$ ($a \in A$; $n, k \in \mathbb{N}$) is a complete group. The canonical map

$$\mu_A \colon\ a \mapsto (\dots, a + nA, \dots) \in \tilde{A}$$

has A^1 for its kernel, and an isomorphic copy of A/A^1 for its image. $\mu_A(A)$ is pure in $\tilde{A}$, and the factor group $\tilde{A}/\mu_A(A)$ is divisible.

Proof. Since the factor groups A/nA are bounded, and hence complete, $\prod(A/nA)$ is complete, and its pure closed subgroup (the inverse limit) is complete by Lemma 2.8. Clearly, $\mu a = 0$ amounts to $a \in nA$ for every n, whence $\operatorname{Ker}\mu = A^1$ and $\operatorname{Im}\mu \cong A/A^1$. If $\tilde{b} = (\dots, b_n + nA, \dots)$ ($b_n \in A$) satisfies $m\tilde{b} = \mu a$ for some $a \in A$ and $m \in \mathbb{N}$, then $mb_n - a \in nA$ for every n, in particular, for $n = m$, whence $a \in mA$, and the purity of μA in $\tilde{A}$ follows. To prove that $\tilde{A}/\mu A$ is divisible, we show that every $\tilde{b} = (\dots, b_n + nA, \dots)$ ($b_n \in A$) is divisible by every $m \in \mathbb{N}$ mod μA. Since $m | b_m - b_{km}$ for each $k \in \mathbb{N}$, we have $m | \tilde{b} - \mu b_m$, in fact. ☐

An immediate consequence of the preceding theorem is the inequality

$$|\tilde{A}| \le |A|^{\aleph_0}, \tag{6.6}$$

which is evident in view of $\tilde{A} \le \prod_n A/nA$.

The group $\tilde{A}$ is called the **$\mathbb{Z}$-adic completion** of A. The map $\mu_A : A \to \tilde{A}$ is natural in the categorical sense, so the correspondence $A \mapsto \tilde{A}$ is functorial. Indeed, we have

Proposition 2.10. *Every homomorphism $\alpha : A \to B$ induces a unique $\tilde{\mathbb{Z}}$-homomorphism $\tilde{\alpha} : \tilde{A} \to \tilde{B}$ making the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ \widetilde{A} & \xrightarrow{\widetilde{\alpha}} & \widetilde{B} \end{array}$$

commute.

Proof. The proof is given in the comments after Lemma 7.6 in Chapter 2. □

We hasten to point out that the completion functor: $A \mapsto \tilde{A}$ is a **pure-exact functor** in the sense that it carries short pure-exact sequences into pure-exact sequences. Moreover, a stronger result holds:

Theorem 2.11. *If* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is a pure-exact sequence, then the induced sequence*

$$0 \to \tilde{A} \xrightarrow{\tilde{\alpha}} \tilde{B} \xrightarrow{\tilde{\beta}} \tilde{C} \to 0 \tag{6.7}$$

of completions is splitting exact.

Proof. If the given sequence is pure-exact, then the induced sequence $0 \to A/nA \to B/nB \to C/nC \to 0$ is exact for every n; see Theorem 3.1 in Chapter 5. The completion functor (as inverse limit) is left-exact (Theorem 5.6 in Chapter 2), so for the exactness of (2) it suffices to show that $\tilde{\beta}$ is a surjective map. By Lemma 7.2 in Chapter 2, $\operatorname{Im}\tilde{\beta}$ is complete and contains βB as a dense subgroup, so necessarily $\operatorname{Im}\tilde{\beta} = \tilde{C}$. What remains to be proved is only the purity of $\operatorname{Im}\tilde{\alpha}$ in $\tilde{B}$. The map $a + A^1 \mapsto \alpha a + B$ carries A/A^1 onto a pure subgroup of B/B^1, which along with the purity of $\mu_B(B)$ in $\tilde{B}$ shows that $\mu_B(\alpha A)$ is pure in $\tilde{B}$. In view of $\mu_B\alpha = \tilde{\alpha}\mu_A$ and the divisibility of $\tilde{\alpha}\tilde{A}/\tilde{\alpha}\mu_A(A)$, we infer that $\tilde{\alpha}\tilde{A}$ must be pure in $\tilde{B}$. The algebraic compactness of $\tilde{A}$ implies the splitting. □

Corollary 2.12. *Under the canonical map* $\mu_A \colon A \to \tilde{A}$*, a* p*-basic subgroup of* A *maps upon a* p*-basic subgroup of* $\tilde{A}$*.*

Proof. Since $\operatorname{Ker}\mu = A^1$, Sect. 6(F) in Chapter 5 implies that μ maps p-basic subgroups of A isomorphically upon p-basic subgroups of μA. Owing to the divisibility of $\tilde{A}/\mu A$ and the purity of μA in $\tilde{A}$, the claim follows. □

Example 2.13. The p-adic (as well as the $\mathbb{Z}$-adic) completion of $\mathbb{Z}_{(p)}$ is J_p, and the $\mathbb{Z}$-adic completion of $\mathbb{Z}$ is $\prod_p J_p$ with p running over all primes.

Example 2.14. The $\mathbb{Z}$-adic completion of $\oplus_p \mathbb{Z}(p)$ is $\prod_p \mathbb{Z}(p)$.

Example 2.15. Let $B = \oplus_n B_n$ where $B_n \cong \oplus \mathbb{Z}(p^n)$ with an arbitrary number of components. The p-adic ($\mathbb{Z}$-adic) completion of B is C, where C/B denotes the divisible part of the factor group $(\prod_n B_n)/B$.

Direct Decompositions of Complete Groups We focus our attention on infinite direct decompositions of complete groups. We start with a lemma that has its own interest.

Lemma 2.16. *Assume that the complete group C is contained in the direct sum $A = \oplus_{i\in I} A_i$ of groups with $A_i^1 = 0$ for every i. Then there is an integer $m > 0$ such that mC is contained in the direct sum of a finite number of the A_i.*

Proof. If the conclusion fails, then there exist an increasing sequence of integers, $m_1 < \cdots < m_j < \ldots$ with $m_j | m_{j+1}$, and groups H_j, each being a direct sum of finitely many A_i, such that the H_j generate their direct sum in A and

$$m_j C \cap \oplus_{k=1}^{j-1} H_k < m_j C \cap \oplus_{k=1}^{j} H_k \qquad (j = 1, 2, \ldots).$$

Pick an element c_j in the right, which is not contained in the left side. Evidently, c_{j-1} has 0, while c_j has a non-zero coordinate in H_j. Thus the Cauchy sequence $c_1, \ldots, c_j, \cdots \in C$ cannot have a limit in A. □

We can now state:

Theorem 2.17. *If $C = \oplus_{i\in I} C_i$ is a direct decomposition of a complete group, then all the groups C_i are complete, and there exists an integer $m > 0$ such that $mC_i = 0$ for almost all i.*

Proof. The first claim is evident, while the second one is an immediate consequence of Lemma 2.16. □

★ **Notes.** The theory of complete groups is due to Kaplansky [K]; he gives credit to I. Fleischer for the torsion-free case. That complete groups can be characterized by their basic subgroups is a consequence of Theorem 2.4, and will also follow from the next section where this question is settled for algebraically compact groups.

Groups complete in the Prüfer topology are extremely important: they are the linearly compact groups. They will be discussed in the next section.

Exercises

(1) Let $m > 0$ be an integer. A is complete if and only if mA is complete.
(2) The direct product of groups C_i is complete if and only if each C_i is complete.
(3) (a) If $A = A_1 \oplus \cdots \oplus A_n$, then $\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_n$.
  (b) This fails in general for infinite direct sums.
(4) $\prod_p \mathbb{Z}(p)$ cannot be written as an infinite direct sum of non-zero groups.
(5) A complete torsion-free group may have decompositions into the direct sum of any finite number of summands, but never into the direct sum of infinitely many non-zero groups.

(6) Let B be a p-basic subgroup of a group A that is complete in its p-adic topology. Every homomorphism $\phi : B \to C$ extends uniquely to a homomorphism $\phi^* : A \to C$ provided that C is complete in the p-adic topology.

(7) (a) The inverse limit of reduced algebraically compact groups is again algebraically compact.

(b) The same is not true in general for non-reduced algebraically compact groups.

3 The Structure of Algebraically Compact Groups

Having got acquainted with several remarkable properties of algebraically compact groups, most of which may even serve as a characterization, it is time to have a closer look at their structure. The main theorem will tell us that they have a satisfactory structure theorem: they admit a complete and independent set of cardinal invariants.

***p*-adic Algebraically Compact Groups** Since an algebraically compact group is a direct sum of an injective group (which can be characterized by invariants (Theorem 3.1 in Chapter 4)) and a reduced algebraically compact group, we may restrict our considerations to the reduced case. We state a lemma that is essentially the same as Theorem 2.4.

Lemma 3.1. *Let A_p be complete in its p-adic topology, and let B_p denote a p-basic subgroup of A_p. Then $A_p = \tilde{B}_p$, and the correspondence $A_p \leftrightarrow B_p$ is a bijection between groups that are complete in the p-adic topology and groups that can be p-basic subgroups.* □

The Structure Theorem We are prepared for the proof of the structure theorem on algebraically compact groups.

Theorem 3.2 (Kaplansky [K]). *A reduced algebraically compact group A is of the form*

$$A = \prod_p A_p, \tag{6.8}$$

where for each prime p, A_p is a uniquely determined p-adic algebraically compact group. The invariants of the p-basic subgroups of A serve as a complete and independent set of invariants of A.

Proof. Let A be reduced and algebraically compact. Then, for a suitable B, $A \oplus B = C$ is a direct product of cyclic p-groups (see Corollary 1.4). Collect the components $\mathbb{Z}(p^k)$ belonging to the same prime p, and form their direct product C_p. The C_p are fully invariant in C, hence $C_p = A_p \oplus B_p$ with $A_p = A \cap C_p,\ B_p = B \cap C_p$, thus $C = \prod_p C_p = \prod_p A_p \oplus \prod_p B_p$. The closure of $A' = \oplus_p A_p$ in C must contain $\prod_p A_p$ because of the divisibility of $(\prod_p A_p)/A'$. A is closed in C, whence the inclusion

$\prod_p A_p \leq A$ is immediate. Analogously, we have $\prod_p B_p \leq B$, and consequently, $C = \prod_p A_p \oplus \prod_p B_p$ implies $\prod_p A_p = A$, $\prod_p B_p = B$. As a direct summand of a complete group, A_p must be complete in its $\mathbb{Z}$-adic topology, which is now identical with its p-adic topology.

Finally, the uniqueness of the components A_p in (6.8) follows at once from the equality $A_p = \cap_{q \neq p} q^k A$ $(k = 1, 2, \dots)$, where q denotes primes. This is a consequence of the relations $qA_p = A_p$ and $\cap_k q^k A_q = 0$. □

The group A_p in the preceding theorem is called the ***p*-adic component** of the algebraically compact group A. As is shown in Lemma 2.3, it is a J_p-module.

Let us emphasize that owing to the theorem, groups A_p that are complete in their p-adic topologies can be totally characterized by the invariants of their p-basic subgroups B_p, i.e. by the cardinal numbers κ_0 and κ_n $(n = 1, 2, \dots)$ of the sets of components $\cong \mathbb{Z}$, resp. of $\mathbb{Z}(p^n)$ in a direct decomposition of B_p. This countable set of invariants is complete: they determine A_p up to isomorphism, and, in addition, this set is independent: it can be chosen arbitrarily. More explicitly,

$$A_p \cong \tilde{B}_p \cong p\text{-adic completion of } \oplus_{\kappa_0} \mathbb{Z} \oplus (\oplus_{n=1}^{\infty} \oplus_{\kappa_n} \mathbb{Z}(p^n)).$$

Example 3.3. (a) We prove that $\tilde{\mathbb{Z}} \cong \prod_p J_p$. It is a reduced torsion-free algebraically compact group, so by the structure theorem it is the product of copies of J_p. Since $\tilde{\mathbb{Z}}/p\tilde{\mathbb{Z}} \cong \mathbb{Z}/p\mathbb{Z}$ (inverse limit), each J_p occurs exactly once in the product.

(b) Let κ denote an infinite cardinal. Then J_p^{κ} is isomorphic to the p-adic completion of a direct sum of 2^{κ} copies of J_p. For, it is a torsion-free p-adic algebraically compact group whose p-basic subgroups have the same rank as $J_p^{\kappa}/pJ_p^{\kappa} \cong \mathbb{Z}(p)^{\kappa} = \oplus_{2^{\kappa}} \mathbb{Z}(p)$.

Example 3.4. Let κ_n $(n = 1, 2, \dots)$ be arbitrary cardinals. Then the p-adic completion of $A = \oplus_{n=1}^{\infty} \oplus_{\kappa_n} \mathbb{Z}(p^n)$ is the subgroup of $C = \prod_{n=1}^{\infty} [\oplus_{\kappa_n} \mathbb{Z}(p^n)]$ which contains A and corresponds to the divisible part of C/A.

Corollaries to the Structure Theorem The structure theorem allows us to derive the following useful corollaries.

Corollary 3.5. *A reduced torsion group is algebraically compact if and only if it is bounded.*

Proof. Bounded groups are algebraically compact. It is evident that, in a decomposition (6.8) of a reduced torsion group A, only a finite number of $A_p \neq 0$ may occur. If the basic subgroup B_p is unbounded, then $\tilde{B}_p$ would contain elements of infinite order. Thus the p-basic subgroups of our A must be bounded. Then $A_p = B_p$, and the claim follows. □

Corollary 3.6 (Kaplansky [K]).

(i) *Every reduced algebraically compact group $\neq 0$ contains a summand isomorphic to $\mathbb{Z}(p^k)$ $(k = 1, 2, \dots)$ or to J_p, for some prime p.*

(ii) *The indecomposable algebraically compact groups are the following: $\mathbb{Z}(p^k)$ $(k \leq \infty)$ and J_p for all primes p, and $\mathbb{Q}$.*

Proof. (i) If A is algebraically compact and reduced, and if p is a prime with $A_p \neq 0$, then a p-basic subgroup B_p of A_p contains either a finite cyclic summand $\mathbb{Z}(p^k)$ or a summand $\cong \mathbb{Z}$. It follows that $\tilde{B}_p$ contains a pure subgroup $\cong \mathbb{Z}(p^k)$ or $\cong J_p$. These are summands, as they are algebraically compact.

(ii) This is immediate from (i). □

Example 3.7. The group $P = \prod_{n\in\mathbb{N}} \langle a_n \rangle$ (where $o(a_n) = p^n$) is algebraically compact, being a product of algebraically compact groups. The completion $\tilde{B}$ of the pure subgroup $B = \oplus_{n\in\mathbb{N}} \langle a_n \rangle$ is a summand of P such that $P/\tilde{B}$ is torsion-free, reduced, and algebraically compact. As $P/pP \cong \oplus_{2^{\aleph_0}} \mathbb{Z}(p)$, while $B/pB \cong \tilde{B}/p\tilde{B}$ is countable, it follows that $P/\tilde{B}$ is the p-adic completion of a free group of rank $2^{\aleph_0}$ (or of $\oplus_{2^{\aleph_0}} J_p$).

By the way, it is easy to find a direct summand $\cong J_p$ in P. E.g., the vector $x = (a_1, \dots, a_n, \dots)$ generates a J_p-submodule $J_px \cong J_p$ whose purity (as a group or as a module) is obvious. Hence J_px is a summand of P.

Linear Compactness We now consider briefly linearly compact groups. A **linearly compact group** is a group A with a linear topology such that if a collection $a_j + A_j$ ($j \in J$) of cosets modulo closed subgroups A_j has the finite intersection property (i.e., any finite number of them have a non-void intersection), then the intersection of all of them is not empty.

We make the following observations that are derived from the theory of topological groups (some follow from results proved here).

(a) *A subgroup of a linearly compact group is linearly compact in the induced topology if and only if it is closed.*
(b) *The image of a linearly compact group under a continuous homomorphism is again linearly compact.*
(c) *Direct products of linearly compact groups are linearly compact.*
(d) *Inverse limits of linearly compact groups are linearly compact.*
(e) *A reduced linearly compact group is complete in its topology.*
(f) *A group which is compact in a linear topology is linearly compact.*

Groups that are linearly compact in the discrete topology admit a complete classification.

Lemma 3.8 (Leptin, Schöneborn). *A group is linearly compact in the discrete topology if and only if it satisfies the minimum condition on subgroups.*

Proof. Assume A is linearly compact in the discrete topology; then so is every subgroup of A. A does not contain elements of infinite order, for $\mathbb{Z}$ is not linearly compact: if $\pi = s_0 + s_1p + s_2p^2 + \dots$ is a non-rational p-adic integer, then the closed cosets $(s_0 + s_1p + \dots + s_{n-1}p^{n-1}) + p^n\mathbb{Z}$ ($n \in \mathbb{N}$) have no elements in common. Thus A is torsion. Choose a basis $\{b_j\}_{j\in J}$ for its socle S with each b_j of prime order, and let B_k be the subgroup generated by all b_j with $j \neq k$. The cosets $b_k + B_k$ ($k \in J$) have the finite intersection property, but the intersection of all of them is empty unless J is a finite set. Hence S is finite which is equivalent to the minimum condition on subgroups (see Theorem 5.3 in Chapter 4).

Conversely, if A satisfies the minimum condition, and if the cosets $a_i + A_i$ ($i \in I$) have the finite intersection property, then there is a minimal finite intersection

$A_{i_1} \cap \cdots \cap A_{i_n}$. It follows that the intersection of all $a_i + A_i$ is equal to the intersection of the cosets corresponding to this minimal intersection, and so it is not empty. □

It follows that if A is any linearly compact group, and if U is an open subgroup of A, then A/U satisfies the minimum condition on subgroups. In fact, A/U is then linearly compact in the discrete topology.

Theorem 3.9. *A group is linearly compact if and only if it is an inverse limit of groups with minimum condition on subgroups.*

Proof. The 'if' part is an immediate consequence of Lemma 3.8 and (d). To prove the 'only if' part, assume A linearly compact, and let the subgroups U_i $(i \in I)$ form a fundamental system of neighborhoods about 0. Without loss of generality, the system of the U_i may be assumed to be closed under finite intersections. A/U_i is discrete and linearly compact, so it satisfies the minimum condition, for every $i \in I$. The groups A/U_i with the natural maps $\rho_i^j : a + U_i \mapsto a + U_j$ (for $U_i \leq U_j$) form an inverse system whose inverse limit must be A, since the image of A under the natural map is a dense subgroup of the inverse limit, and by (a), the image must be the whole inverse limit. □

Finally we verify:

Corollary 3.10. *Linearly compact groups are algebraically compact.*

Proof. We show that a linearly compact group A satisfies Theorem 1.2(d). A solution of the ith equation in the system $\sum_{j \in J} n_{ij} x_j = a_i \in A$ $(i \in I)$ is a coset of a closed subgroup C_i of A^J. The system is finitely solvable if these cosets have the finite intersection property. Linear compactness implies that the system is solvable. □

Example 3.11. J_p is linearly compact in the p-adic topology. J_p^κ is linearly compact in the finite index topology, for every cardinal κ.

Example 3.12. The additive group $\mathbb{Q}_p^*$ of the p-adic numbers is the inverse limit of groups $\mathbb{Z}(p^\infty)$; it is linearly compact in the Prüfer topology.

★ **Notes.** Kaplansky [K] gives a full description of algebraically compact groups. The theory of algebraically compact modules contains several remarkable generalizations of the group case and much more. See, e.g., Zimmermann-Huisgen–Zimmermann [1], Fuchs–Salce's book *Modules over non-Noetherian Domains*, 2001.

Eklof–Mekler [1] investigates the structure of ultraproducts of abelian groups and proves *inter alia* that over an ω-incomplete ultrafilter they are algebraically compact.

Exercises

(1) A countable algebraically compact group is the direct sum of a divisible group and a bounded group.

(2) The group $\mathbb{R}$ of reals admits infinitely many different linearly compact topologies.
(3) In an algebraically compact group, a finite set of elements is contained in a summand that is a finite direct sum of $\mathbb{Q}$, cocyclic groups, and copies of J_p for various primes p.
(4) Determine the invariants of the algebraically compact group $[\prod_{n=1}^{\infty} \mathbb{Z}(p^n)]^{\kappa}$ where κ is an infinite cardinal.
(5) (Leptin) A group with a linear topology is linearly compact if and only if it is complete in this topology, and the factor groups modulo open subgroups satisfy the minimum condition. [Hint: (e).]
(6) If A, B are algebraically compact groups such that $A \oplus A \cong B \oplus B$, then $A \cong B$.
(7) If A and B are algebraically compact groups, each isomorphic to a pure subgroup of the other, then $A \cong B$. [Hint: Theorem 3.2.]
(8) J_p has the cancellation property: if A, B are groups such that $A \oplus J_p \cong B \oplus J_p$, then $A \cong B$. [Hint: reduce the proof to reduced torsion-free groups.]
(9) (Balcerzyk) Prove that $\mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})} \cong D \oplus \prod_p A_p$, where D is a $\mathbb{Q}$-vector space of dimension $2^{\aleph_0}$, and A_p is the p-adic completion of $\oplus_{2^{\aleph_0}} J_p$. [Hint: the group $\mathbb{Z}^{\mathbb{N}}/(p\mathbb{Z}^{\mathbb{N}} + \mathbb{Z}^{(\mathbb{N})})$ has cardinality $2^{\aleph_0}$.]
(10) Are quasi-injective groups algebraically compact?
(11) The direct sum $\oplus_{\aleph_0} \mathbb{Z}(p^n)$ for any fixed $n \in \mathbb{N}$ is algebraically compact, but it cannot be linearly compact under any linear topology.

4 Pure-Injective Hulls

The striking analogy between injective and pure-injective groups can be pushed further by pointing out the analogue of the injective hull.

Pure-Essential Extensions We first introduce the following notation. For a pure subgroup G of A, $K(G, A)$ will denote the set of all subgroups $H \le A$ such that

(i) $G \cap H = 0$; and
(ii) $(G + H)/H$ is pure in A/H.

Since (i) implies $G + H = G \oplus H$, condition (ii) amounts to the fact that if $nx = g + h$ $(n \in \mathbb{N})$ with $g \in G, h \in H$ is solvable for x in A, then g is divisible by n in G. Hence the set $K(G, A)$ is closed under taking subgroups. The purity of G in A assures that $0 \in K(G, A)$, so $K(G, A)$ is never empty.

Moreover, the set $K(G, A)$ is inductive. To prove this, let H_i $(i \in I)$ be a chain of subgroups in $K(G, A)$, and H their union. H obviously satisfies (i). Suppose $nx = g + h$ $(g \in G, h \in H)$ is solvable in A. For some $i \in I$, $h \in H_i \in K(G, A)$ whence $g \in nG$, and so $H \in K(G, A)$.

Following Maranda [1], we call a group A a **pure-essential extension** of its pure subgroup G, and G a **pure-essential subgroup** of A, if $K(G, A) = \{0\}$. A is a **maximal pure-essential extension** of G if it is a pure-essential extension, but no group properly containing A is a pure-essential extension of G. It is easily checked that if A_i $(i \in I)$ is a chain of pure-essential extensions of G, then $A = \cup_{i \in I} A_i$ is also a pure-essential extension of G.

Lemma 4.1. *Suppose C is a pure-essential extension of G, and A an algebraically compact group containing G as a pure subgroup. Then the identity map $\mathbf{1}_G$ extends to an embedding $\phi : C \to A$.*

Proof. Due to the pure-injectivity of A, the existence of an extension $\phi : C \to A$ of $\mathbf{1}_G$ is evident. Since G is pure in ϕC, we have $\operatorname{Ker} \phi \in K(G, C)$. By pure-essentiality, $K(G, C) = \{0\}$, establishing our claim. □

Pure-Injective Hulls A is a **pure-injective hull** of G if it is a minimal pure-injective group containing G as a pure subgroup. To prove the existence of such a hull, we need a few important facts on pure-essential extensions.

Theorem 4.2 (Maranda [1]).

(i) *Every pure-essential extension of a group G is contained in a maximal pure-essential extension of G.*
(ii) *A is a maximal pure-essential extension of G if and only if it is a pure-injective hull of G.*
(iii) *Every group G has a pure-injective hull, unique up to isomorphism over G. Every pure-injective group containing G as a pure subgroup contains a pure-injective hull of G.*

Proof. (i) There exists a pure-injective group H containing G as a pure subgroup Theorem 4.7 in Chapter 5. From Lemma 4.1 we derive that every pure-essential extension of G must have cardinality $\leq |H|$. Therefore, the non-isomorphic pure-essential extensions of G form a set, so it only remains to appeal to Zorn's lemma to get a maximal member containing any given one.

(ii) Let A be a maximal pure-essential extension of G, and C a pure-injective group containing G as a pure subgroup. Select a maximal member M in the set $K(G, C)$. Passing mod M, we obtain $G \cong (G+M)/M \leq (A+M)/M \leq C/M$. By the maximal choice of M, $(G+M)/M$ is pure-essential in C/M. As G is pure-essential in A, we must have $A \cap M = 0$, so $A \cong (A+M)/M$. Therefore, $(A+M)/M = C/M$ by the maximality of A, whence $C = A \oplus M$, so A is pure-injective. To see that it is minimal pure-injective, assume that A' is a pure-injective group with $G \leq A' \leq A$. By Lemma 4.1, the identity $\mathbf{1}_G$ extends to a monomorphism $\phi : A \to A'$. Then ϕA is also a maximal pure-essential extension of G, and since $\phi A \leq A$, $\phi A = A$ is the only possibility. Hence $A' = A$ is indeed minimal.

With the same notations, there is a map $\phi : A \to C$ which is the identity on G; it has to be monic on A (see Lemma 4.1). By the preceding paragraph, A, and hence ϕA, is pure-injective, so $\phi A = C$ provided C is minimal. This means that C is a maximal pure-essential extension of G.

(iii) The statements (i) and (ii) guarantee the existence of a pure-injective hull A for every group G. If A' is any pure-injective group containing G as a pure subgroup, then the identity map $\mathbf{1}_G$ can be extended to a monomorphism $\phi : A \to A'$. If A' happens to be also minimal, then necessarily $\phi A = A'$. Thus ϕ is then an isomorphism. □

We next prove a useful criterion for the pure-injective hull.

Lemma 4.3. *A pure-injective group A containing G as a pure subgroup is the pure-injective hull of G exactly if*

(a) *the maximal divisible subgroup of A is the injective hull of G^1; and*
(b) *the factor group A/G is divisible.*

Proof. First, assume A is a pure-injective hull of G. Write $A = D \oplus C$ with D divisible and C complete. In view of the purity of G in A,

$$G^1 = \cap_n nG = \cap_n (G \cap nA) = G \cap A^1 = G \cap D.$$

Since every non-zero summand of D must intersect G, D has to be an injective hull of G^1. Define $E \leq A$ such that E/G is the first Ulm subgroup of A/G. Then $E = D \oplus (C \cap E)$, and $C/(C \cap E) \cong (C + E)/E = A/E \cong (A/G)/(E/G)$ has trivial Ulm subgroup. Thus $C \cap E$ is closed in C, so it is complete. Consequently, E is pure-injective containing G, so $E = A$ by minimality. Hence A/G coincides with its own Ulm subgroup, which means that it is divisible.

To prove sufficiency, assume (a) and (b) hold for the pure-injective A containing G as a pure subgroup. There is a pure-injective hull E of G contained in A. From the first part of the proof it follows that E/G is divisible, so E is pure in A, and hence it is a summand. It cannot be a proper summand, so $A = E$. □

It is now easy to describe how to obtain the pure-injective hull of a group.

Theorem 4.4. *The pure-injective hull of a group G is isomorphic to the direct sum of the injective hull of G^1 and the completion $\tilde{G}$ of G.*

Proof. Let D denote the injective hull of G^1, and let $\phi : G \to D$ be an extension of the inclusion map $G^1 \to D$. If $\mu : G \to \tilde{G}$ stands for the canonical map, then consider the map $\psi : G \to D \oplus \tilde{G}$ which is the composite of the diagonal map $G \to G \oplus G$ followed by $\phi \oplus \mu$. The purity of μG in $\tilde{G}$ guarantees that $\operatorname{Im} \psi$ is pure in $D \oplus \tilde{G}$. From Lemma 4.3 the assertion follows at once. □

The last theorem makes it possible to determine the complete system of invariants for the pure-injective hull of a group G in terms of certain invariants of G; cf. Theorem 3.1 in Chapter 4 and Theorem 3.2.

Pure-Essential is Not Transitive The analogy of 'essential' with 'pure-essential' subgroups breaks down when transitivity is considered.

Lemma 4.5 (Fuchs–Salce–Zanardo [1]).

(i) *The property of being a "pure-essential extension" is not transitive.*
(ii) *However, it is transitive for p-local groups.*

Proof. (i) Let $B = \oplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)$, a Σ-cyclic group. Since $\tilde{B}/B$ contains copies of $\mathbb{Q}$, it is easy to find a subgroup C of $\tilde{B}$ such that $C/B \cong \mathbb{Z}_{(q)}$ where $q \neq p$ is a prime. Manifestly, B is pure-essential in C, since B is not a summand in C, and C/B is torsion-free of rank 1. By Theorem 4.4, C is pure-essential in its $\mathbb{Z}$-adic completion which is $\tilde{C} = \tilde{B} \oplus J_q$ (see Theorem 2.11). However, B is not pure-essential in $\tilde{C}$, since it is contained in its first summand.

(ii) The arguments in the proof of Lemma 4.3 can be applied to show that a pure subgroup G of a p-local group A is pure-essential if and only if (a) G^1 is essential in A^1, and (b) A/G is p-divisible. Since properties (a) and (b) are transitive, the claim follows. □

★ **Notes.** Every module over any ring admits a pure-injective hull, unique up to isomorphism. However, pure-injective hulls cannot be obtained in general *via* completion. There is an extensive theory of pure-injectivity that started to develop following the pattern of groups. Pure-injective modules admit indecomposable summands, so that their structure is considerably simpler to study.

Pure-injective hulls may compete in significance with injective hulls. They play important roles in the structure theory of modules over arbitrary rings, not to mention their relevance in model theory. As illustration, let us refer to the interesting result by G. Sabbagh [C.R. Acad. Sci. Paris, **271**, A909–A912 (1970)] which states that for any ring R, an R-module is elementarily equivalent to its pure-injective hull. Thus the elementary theory of modules is reduced at once to the case of pure-injective modules.

Exercises

(1) If A is a pure-essential extension of G, and if B is a pure subgroup of A containing G, then B is a pure-essential extension of G.
(2) Prove that if G is pure in A, $H \leq A^1$, and $H \cap G = 0$, then $H \in K(G, A)$.
(3) Let A be a p-group and B its basic subgroup.
   (a) $K(B, A)$ is the set of all subgroups of A^1.
   (b) B is pure-essential in A if and only if $A^1 = 0$.
(4) If C is a pure-essential extension of G, and C/G is not divisible, then there is a pure-essential extension C' of G such that $C < C'$ and C'/A is divisible.
(5) Show that if A_i ($i \in I$) is a chain of pure-essential extensions of G, then $A = \cup_{i\in I} A_i$ is also a pure-essential extension of G.
(6) Let A be a pure-injective group containing G as a pure subgroup. A maximal pure-essential extension of G in A is a summand of A.
(7) Give an example where A_n ($n < \omega$) are the pure-injective hulls of the groups G_n, but $\prod_n A_n$ is not the pure-injective hull of $\prod_n G_n$.
(8) Using invariants of a group, determine a complete system of invariants for its pure-injective hull. [Hint: ranks.]
(9) Find conditions on the invariants of a pure-injective group A such that A has no proper summands isomorphic to A.

5 Locally Compact Groups

Having discussed compact and linearly compact groups, it is worth while making a few comments on the locally compact case. Besides the Pontryagin duality theory we rely heavily on homological machinery, so a reader not familiar with the material of Chapters 7–9 is advised to read this section after studying these chapters.

Locally Compact Extensions It is well known that a locally compact (abelian) group is the direct sum of $\mathbb{R}^n$ (for some integer $n \geq 0$) and a group G that contains an open compact subgroup C. Thus G fits into an exact sequence $0 \to C \to G \to A \to 0$ where C is compact and A is discrete. Conversely, any extension of a compact C by a discrete A yields a well-defined locally compact group G, as the only topology compatible with the given topologies of C and A is the one in which a base of neighborhoods of 0 in C is also a base of neighborhoods of 0 in G. Consequently, if C is compact and A is discrete, then the elements of $\mathrm{Ext}(A, C)$ may be viewed as the locally compact extensions of C by A (the equivalence of extensions is the same whether C is discrete or compact).

Recall that the **Pontryagin dual** (or **character group**) Char G of a locally compact group G, i.e., the group of all continuous homomorphisms of G into the compact group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, equipped with the **compact-open topology**, is again locally compact. We write

$$\hat{G} = \mathrm{Char}\, G = \mathrm{Hom}(G, \mathbb{T}),$$

where Hom now stands for continuous homomorphisms. Importantly, if G is compact, then $\hat{G}$ is discrete, and *vice versa*. The Pontryagin Duality asserts that the map

$$\eta_G\colon\ G \to \hat{\hat{G}}, \quad \text{acting as } \eta_G(x)(\chi) = \chi(x) \quad (x \in G, \chi \in \hat{G})$$

implements a canonical algebraic and topological isomorphism between G and its second dual $\hat{\hat{G}}$.

Hom, Ext with Compact Second Argument In the sequel, the letters M, N will stand for compact groups, so their duals $\hat{M}, \hat{N}$ are discrete groups. The group $\mathrm{Hom}(\hat{M}, N)$ carries a compact topology: the compact-open topology makes it into a compact group.

Lemma 5.1. *For compact groups M and N, there are natural* (*topological*) *isomorphisms of compact groups*

$$\mu_{MN}\colon\ \mathrm{Hom}(\hat{M}, N) \to \mathrm{Hom}(\hat{N}, M),$$

$$\nu_{MN}\colon\ \mathrm{Hom}(\hat{M}, N) \to \mathrm{Hom}(\hat{M} \otimes \hat{N}, \mathbb{T}) = (\hat{M} \otimes \hat{N})\hat{}.$$

Proof. It is enough to consider a sequence of natural isomorphisms:

$$\begin{aligned}\mathrm{Hom}(\hat{M}, N) &\cong \mathrm{Hom}(\hat{M}, \mathrm{Hom}(\hat{N}, \mathbb{T})) \cong \mathrm{Hom}(\hat{M} \otimes \hat{N}, \mathbb{T}) \\ &\cong \mathrm{Hom}(\hat{N} \otimes \hat{M}, \mathbb{T}) \cong \mathrm{Hom}(\hat{N}, \mathrm{Hom}_{\mathbb{Z}}(\hat{M}, \mathbb{T})) \cong \mathrm{Hom}(\hat{N}, M),\end{aligned}$$

where we have made repeated use of Theorem 1.13 in Chapter 8. □

If the isomorphism Theorem 3.8 in Chapter 9 is used with the following cast of characters: $A = \hat{M}, B = \hat{N}$ are discrete groups, and $C = \mathbb{T}$, then we get

$$\mathrm{Ext}(\hat{M}, \mathrm{Hom}(\hat{N}, \mathbb{T})) \;\cong\; \mathrm{Hom}(\mathrm{Tor}(\hat{M}, \hat{N}), \mathbb{T}).$$

Consequently,

Lemma 5.2. *For compact M and N, there is a natural isomorphism*

$$\mathrm{Ext}(\hat{M}, N) \;\cong\; (\mathrm{Tor}(\hat{M}, \hat{N}))\hat{}$$

that makes $\mathrm{Ext}(\hat{M}, N)$ *into a compact group.* □

Since Tor is symmetric in its arguments, Lemma 5.2 shows that the switch involution in $\mathrm{Tor}(\hat{N}, \hat{M})$ induces a natural isomorphism between the compact groups $\mathrm{Ext}(\hat{M}, N)$ and $\mathrm{Ext}(\hat{N}, M)$. This isomorphism can easily be described explicitly:

Theorem 5.3. *For compact groups M, N, there is a natural topological isomorphism*

$$\mathrm{Ext}(\hat{M}, N) \;\cong\; \mathrm{Ext}(\hat{N}, M)$$

that carries the extension $0 \to N \to G \to \hat{M} \to 0$ *to its Pontryagin dual* $0 \to M \to \hat{G} \to \hat{N} \to 0$.

Proof. As G is locally compact, Pontryagin duality yields a natural bijection θ_{MN} between the two Exts whose inverse is the map θ_{NM}. This bijection respects the group operations, since the Baer sum is defined by using diagonal and codiagonal maps (which are dual to each other). □

Duality of Long Exact Sequences The purpose of including this section in this volume was also to point out that the classical induced long exact sequences Theorem 2.4 in Chapter 8 and Theorem 2.3 in Chapter 9 become intertwined once topology gets involved. In fact, we can verify a surprising duality between these long sequences.

Theorem 5.4. *Let M be compact, A, B, C discrete, and the sequence $0 \to A \to B \to C \to 0$ exact. The Pontryagin dual of the exact sequence*

$$0 \to \mathrm{Tor}(A, \hat{M}) \to \mathrm{Tor}(B, \hat{M}) \to \mathrm{Tor}(C, \hat{M}) \to$$

$$\to A \otimes \hat{M} \to B \otimes \hat{M} \to C \otimes \hat{M} \to 0$$

of discrete groups is the exact sequence

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to$$

$$\to \mathrm{Ext}(C, M) \to \mathrm{Ext}(B, M) \to \mathrm{Ext}(A, M) \to 0$$

of compact groups (*where the maps are continuous*).

Proof. Application of the exact functor $\mathrm{Hom}(\bullet, \mathbb{T})$ to the first long exact sequence yields the second one, after making use of Lemma 5.1 and 5.2. The continuity of the maps is straightforward. □

Autodual Extensions As an application, consider the special case $M = N$. Then θ_{MM} is an involution of the compact group $\mathrm{Ext}(\hat{M}, M)$. Of particular interest are those extensions which are **autodual** in the sense that θ_{MM} carries them into equivalent extensions, and those which **antidual**, i.e. θ_{MM} acts on them as multiplication by -1. It is readily checked that both the autodual and the antidual extensions in $\mathrm{Ext}(\hat{M}, M)$ form a subgroup $\mathrm{Ext}(\hat{M}, M)_+$ and $\mathrm{Ext}(\hat{M}, M)_-$, respectively.

Proposition 5.5. *Suppose M is compact such that multiplication by 2 is an automorphism of* $\mathrm{Ext}(\hat{M}, M)$. *Then there is a direct decomposition*

$$\mathrm{Ext}(\hat{M}, M) = \mathrm{Ext}(\hat{M}, M)_+ \oplus \mathrm{Ext}(\hat{M}, M)_-$$

where the summands are closed subgroups.

Proof. All that we have to point out is that, for every element $\mathfrak{e}$ of $\mathrm{Ext}(\hat{M}, M)$, $\mathfrak{e}$ is the Baer sum of $\frac{1}{2}(\mathfrak{e} + \hat{\mathfrak{e}})$ and $\frac{1}{2}(\mathfrak{e} - \hat{\mathfrak{e}})$, which are evidently auto- and antidual extensions, respectively. □

Corollary 5.6 (Fuchs–Hofmann [1]). *All the extensions of a compact group M by its dual $\hat{M}$ are autodual if and only if the torsion part of $\hat{M}$ is locally cyclic.*

Proof. By Lemma 5.2, the condition can be rephrased as saying that the involution of $\mathrm{Tor}(\hat{M}, \hat{M})$ corresponding to the switch of the arguments is the identity. Since if $\hat{M} = A \oplus B$, then

$$\mathrm{Tor}(\hat{M}, \hat{M}) \cong \mathrm{Tor}(A, A) \oplus \mathrm{Tor}(B, A) \oplus \mathrm{Tor}(A, B) \oplus \mathrm{Tor}(B, B),$$

and the involution in question would switch the two middle summands, the condition $\mathrm{Tor}(A, B) = 0$ is necessary. This means that, for each prime p, one of the

p-components A_p, B_p must be 0, i.e. the torsion part of $\hat{M}$ is locally cyclic, i.e. $t\hat{M} \leq \mathbb{Q}/\mathbb{Z}$. As Tor vanishes for torsion-free groups, it is pretty clear that this condition is sufficient. □

Example 5.7 (Samelson). Consider the extension $0 \to J_p \to \mathbb{Q}_p^* \to \mathbb{Z}(p^\infty) \to 0$ where J_p carries the obvious compact topology. This extension is self-dual, and so are all extensions of J_p by $\mathbb{Z}(p^\infty)$.

★ **Notes.** More on the homological aspects of topological groups may be found in K.H. Hofmann–S.A. Morris, *The Structure of Compact Groups* (de Gruyter, 1998), and on autoduality in Fuchs–Hofmann [1].

Loth [3] investigates how group properties are reflected in the Pontryagin duals.

Exercises

(1) If $0 \to L \to M \to N \to 0$ is an exact sequence of compact groups, then for any discrete A, the sequence

$$0 \to \mathrm{Hom}(A, L) \to \mathrm{Hom}(A, M) \to \mathrm{Hom}(A, N) \to$$

$$\to \mathrm{Ext}(A, L) \to \mathrm{Ext}(A, M) \to \mathrm{Ext}(A, N) \to 0$$

of compact groups is exact with continuous maps.

(2) Find the Pontryagin dual of the long exact sequence in the preceding exercise.

(3) An extension equivalent to an autodual extension is autodual as well.

(4) Describe the auto- and antidual extensions in $\mathrm{Ext}(\mathbb{Z}(p) \oplus \mathbb{Z}(p^\infty), \mathbb{Z}(p) \oplus J_p)$ if $p \neq 2$.

6 The Exchange Property

We end this chapter with a brief study of the Exchange Property, a property that has come to prominence in the literature. It is closely connected with the problem of uniqueness of direct decompositions, with the Krull-Schmidt theorem, as we shall see below.

Exchange Property A group G is said to have the **(finite) exchange property** if, for any direct decomposition

$$A = G \oplus H = \oplus_{i \in I} A_i$$

of a group A, where A_i, H are arbitrary groups, and I is a (finite) index set, there always exist subgroups B_i of A_i for $i \in I$ such that

$$A = G \oplus \left(\oplus_{i \in I} B_i\right).$$

Evidently, B_i must be a summand of A_i: it is a summand of A contained in A_i. We shall see that some special groups, like injective, quasi-injective, and pure-injective groups, enjoy the exchange property.

The next lemma will be needed in the following proofs (it holds also for modules).

Lemma 6.1 (Crawley–Jónsson [1]). *It suffices to check the exchange property for G in case the A_i are isomorphic to subgroups of G.*

Proof. Let $A = G \oplus H = \oplus A_i$ with arbitrary A_i. Set $H_i = H \cap A_i$, and let overbars denote images under the canonical map $\phi : A \to A/H'$ where $H' = \oplus H_i$. Then $\bar{A} = \bar{G} \oplus \bar{H} = \oplus \bar{A}_i$, where

$$\bar{A}_i \cong A_i/(A_i \cap H') = A_i/(A_i \cap H) \cong (A_i + H)/H \leq A/H \cong G.$$

By hypothesis, there exist $\bar{B}_i \leq \bar{A}_i$ satisfying $\bar{A} = \bar{G} \oplus (\oplus \bar{B}_i)$. Setting $B_i = \phi^{-1}\bar{B}_i \cap A_i$, we clearly have $A = G \oplus (\oplus B_i)$. $\square$

Groups with the Exchange Property It seems sensible to start the study with indecomposable groups. (It is easily checked that, for indecomposable groups, the finite and the general exchange properties are equivalent.) Observe that a group with local endomorphism ring is necessarily indecomposable.

Theorem 6.2 (Warfield [2]). *An indecomposable group G has the finite exchange property if and only if its endomorphism ring* End G *is local.*

Proof. Suppose End G is local, and $A = G \oplus H = B \oplus C$. Let ϕ, β, γ denote the projections of A onto its summands G, B and C, respectively. $\beta + \gamma = \mathbf{1}_A$ implies $\phi\beta\phi + \phi\gamma\phi = \phi$ where the terms will be viewed as endomorphisms of G; in particular, ϕ acts as the identity on G. End G local implies that either $\phi\beta\phi$ or $\phi\gamma\phi$ is an automorphism of G; say, the first alternative holds. Setting $K = \operatorname{Im} \beta\phi$, it is clear that both $\phi\restriction_K : K \to G$ and $\beta\restriction_G : G \to K$ are isomorphisms. As $\phi K = \phi\beta G = G$, we have $A = K \oplus H$. Now $K \leq B$ implies $B = K \oplus B'$ for some $B' \leq B$, thus $A = G \oplus H = K \oplus B' \oplus C$. Taking into account that β induces an isomorphism of G with K, we conclude that $A = G \oplus B' \oplus C$, i.e. G enjoys the exchange property.

To prove the converse, assume End G is not local. Thus there are endomorphisms α and β of G, which are not automorphisms, such that $\alpha - \beta = \mathbf{1}_G$. Then $A = G_1 \oplus G_2$ ($G_i \cong G$) also decomposes as $A = H_1 \oplus H_2$ where $H_1 = \{(g, g) \mid g \in G\}$ and $H_2 = \{(\beta g, \alpha g) \mid g \in G\}$ are indecomposable. However, neither $A = G_1 \oplus H_1$ nor $A = G_1 \oplus H_2$, so G does not have the exchange property. $\square$

Several important groups have the exchange property. For instance,

Theorem 6.3 (Warfield, Fuchs). *Injective groups, and more generally, quasi-injective groups share the exchange property.*

Proof. We prove the claim for a quasi-injective G. Suppose $A = G \oplus H = \oplus_{i \in I} A_i$ where each A_i is isomorphic to a subgroup of G. Pick a subgroup B of A maximal with respect to the properties: (i) $B = \oplus_{i \in I} B_i$ with $B_i \leq A_i$, and (ii) $G \cap B = 0$. We claim $A = G \oplus B$.

The natural homomorphism $\phi : A \to A/B$ maps G isomorphically into A, so ϕG is a quasi-injective subgroup in $A/B = \oplus_i (A_i + B)/B$. Owing to the maximal choice of B, $\phi G \cap (A_i + B)/B$ is an essential subgroup in $(A_i + B)/B$ for each index i, whence we conclude that ϕG is essential in A/B. As $\phi A_i \leq A/B$, where A_i is isomorphic to a subgroup of ϕG, and ϕG is fully invariant in its injective hull (that contains A/B), we can argue that necessarily $\phi A_i \leq \phi G$. Therefore, $\phi A \leq \phi G$, whence $\phi G = A/B$, and $A = G \oplus B$ follows. $\square$

Our next task is to establish a sufficient condition on a group to have the exchange property, this will be needed in the applications. The theorems above suggest strong dependence on the endomorphism ring. This is confirmed by the next theorem.

An **exchange ring** is a (not necessarily commutative) ring E such that the left module ${}_{\mathsf{E}}\mathsf{E}$ has the finite exchange property.

Theorem 6.4 (Warfield). *A group G has the finite exchange property if and only if $\mathsf{E} = \operatorname{End} G$ is an exchange ring.*

Proof. In the proof we use the *ad hoc* notations $X^\vee = \operatorname{Hom}(X, G)$ for a group X, and $Y^\wedge = \operatorname{Hom}_{\mathsf{E}}(Y, G)$ for a left E-module Y (where G is viewed as a left E-module). Thus $X^\vee$ is a left E-module, and $Y^\wedge$ is a group. In this notation, $G^\vee = {}_{\mathsf{E}}\mathsf{E}$ and $\mathsf{E}^\wedge = G$. Consequently, the canonical maps $G \to G^{\vee\wedge}$ and $\mathsf{E} \to \mathsf{E}^{\wedge\vee}$ are isomorphisms.

Assuming G has the exchange property, let ${}_{\mathsf{E}}M = \mathsf{E} \oplus N = \mathsf{L}_1 \oplus \mathsf{L}_2$ with a left E-module N and left ideals L_i of E. Hence $M^\wedge = G \oplus N^\wedge = \mathsf{L}_1^\wedge \oplus \mathsf{L}_2^\wedge$ are group decompositions. Hypothesis implies the existence of summands $C_i \leq \mathsf{L}_i^\wedge$ such that $M^\wedge = G \oplus C_1 \oplus C_2$. Thus, $M^{\wedge\vee} = \mathsf{E} \oplus C_1^\vee \oplus C_2^\vee$. In view of the isomorphism $\mathsf{E} \to \mathsf{E}^{\wedge\wedge}$, the canonical maps $\mathsf{L}_i \to \mathsf{L}_i{}^{\wedge\wedge}$ are monic, so the map $M \to M^{\wedge\wedge}$ is likewise monic. It remains to set $B_i = \mathsf{L}_i \cap C_i^\wedge$ to obtain $M = \mathsf{E} \oplus B_1 \oplus B_2$, establishing the exchange property of E.

Since the group G and the ring E play symmetric roles in the last argument, the proof is complete. $\square$

Exchange Rings We now investigate properties of endomorphism rings that imply the exchange property. We need a definition: for an ideal H of the ring R, we say that **idempotents lift** modulo H if every idempotent coset in R/H contains an idempotent element of R.

Theorem 6.5. *A ring E is an exchange ring if it is von Neumann regular modulo its Jacobson radical J, and idempotents lift* mod J.

Proof. Assume E is a ring as stated. First we prove that if $a + b = 1$ $(a, b \in \mathsf{E})$, then there exists an idempotent $e \in \mathsf{E}a$ such that $1 - e \in \mathsf{E}b$. We start with the special case when $\mathsf{J} = 0$. By regularity, if given $a \in \mathsf{E}$, there exists a $c \in \mathsf{E}$ such that $aca = a$. Then $f = ca$ is an idempotent, and so is $e = f + (1 - f)a \in \mathsf{E}a$. A simple calculation leads to $1 - e = (1 - f)(1 - a) = (1 - f)b \in \mathsf{E}b$.

In the general case, let overbars denote cosets mod J. By the preceding paragraph, for $a \in \mathsf{E}$, there exists an $x \in \mathsf{E}$ such that $\bar{x}^2 = \bar{x} \in \bar{\mathsf{E}}\bar{a}$ and $\bar{1} - \bar{x} \in \bar{\mathsf{E}}\bar{b}$. We may assume, without loss of generality, that $x \in \mathsf{E}a$. Hypothesis guarantees that we can find an idempotent $f \in \bar{x}$. Then $u = 1 - f + x$ is a unit in E (as $x - f \in \mathsf{J}$), so $u^{-1} \in \mathsf{E}$ exists. Setting $g = u^{-1}fu = u^{-1}fx \in \mathsf{E}x \leq \mathsf{E}a$, and observing that $\bar{g} = \bar{f} = \bar{a}$, we have $g^2 = g$ and $(1-g) - y(1-a) \in \mathsf{J}$ for some $y \in \mathsf{E}$. Consequently, $g + y(1-a)$ is a unit in E, so it has an inverse $v \in \mathsf{E}$. It remains to define $e = g - (1-g)vg \in \mathsf{E}g \leq \mathsf{E}a$, and to check that $e^2 = e$ and $1 - e = (1-g)(1-vg) = (1-g)vy(1-a) \in \mathsf{E}b$.

Suppose E is the endomorphism ring of the group A. Let $A = G \oplus H = A_1 \oplus A_2$, and denote the projections onto G, A_1, A_2 by π, α_1, α_2, respectively. Evidently, $\pi\mathsf{E}\pi$ is the endomorphism ring of G, and $\pi\alpha_1\pi + \pi\alpha_2\pi = \pi$. By what has been proved above, there exist $\beta_i \in \pi\mathsf{E}\pi$ $(i = 1, 2)$ such that $\gamma_i = \beta_i\alpha_i\pi$ are orthogonal idempotent endomorphisms of G with $\gamma_1 + \gamma_2 = \pi$. Since γ_i is an idempotent, β_i may be chosen so as to satisfy $\gamma_i\beta_i = \beta_i$. Note that

$$\pi\beta_i = \beta_i = \beta_i\pi, \quad \pi\gamma_i = \gamma_i = \gamma_i\pi, \quad \beta_i\alpha_i\beta_i = \beta_i.$$

The maps $\eta_i = \alpha_i\beta_i\alpha_i$ $(i = 1, 2)$ are orthogonal idempotents satisfying $\eta_i\gamma_i = \alpha_i\beta_i\alpha_i\beta_i\alpha_i\pi = \alpha_i\beta_i\alpha_i\pi = \eta_i\pi$. Setting $B_i = A_i \cap \operatorname{Ker}\eta_i$, it is easily seen that $B_i = \operatorname{Im}[(1-\eta_i)\alpha_i] = \operatorname{Im}(\alpha_i - \eta_i)$. We now claim that $A = G \oplus B_1 \oplus B_2$.

First, if $a \in G \cap (B_1 \oplus B_2)$, then from $\eta_1 a = 0 = \eta_2 a$ we obtain that $a = \gamma_1 a + \gamma_2 a = \beta_1\alpha_1 a + \beta_2\alpha_2 a = (\beta_1\alpha_1\beta_1)\alpha_1 a + (\beta_2\alpha_2\beta_2)\alpha_2 a = \beta_1\eta_1 a + \beta_2\eta_2 a = 0$. Evidently, $A = \operatorname{Im}\eta_1 + \operatorname{Im}\eta_2 + \operatorname{Im}(\alpha_1 - \eta_1) + \operatorname{Im}(\alpha_2 - \eta_2)$, hence it only remains to prove that $\operatorname{Im}\eta_i \leq G + B_1 + B_2$. We have $\alpha_i\beta_i\eta_i = \alpha_i\beta_i\alpha_i\beta_i\alpha_i = \alpha_i\beta_i\alpha_i = \alpha_i\eta_i$, and for $i \neq j$, also $\eta_j\beta_i = \eta_j(\pi\beta_i) = (\eta_j\pi)(\gamma_i\beta_i) = (\eta_j\gamma_j)(\gamma_i\beta_i)$. It follows that $\operatorname{Im}(\eta_i - \beta_i\eta_i)$ is annihilated by both η_i and α_i, so it is contained in B_j $(j \neq i)$. Consequently, $\operatorname{Im}\eta_i \leq \operatorname{Im}\beta_i\eta_i + \operatorname{Im}(\eta_i - \beta_i\eta_i) \leq G + B_j$. □

For the proof of Theorem 6.7 below, we require the following lemma that is a special case of a result by Zimmermann-Huisgen–Zimmermann [1].

Lemma 6.6. *Let the ring* E *be algebraically compact as a left module over itself. Then* $\bar{\mathsf{E}} = \mathsf{E}/\mathsf{J}$ *is von Neumann regular, and its idempotents lift to* E.

Proof. Suppose $\bar{\mathsf{L}}$ is a finitely generated left ideal of $\bar{\mathsf{E}}$. Clearly, there is a finitely generated left ideal L of E that maps upon $\bar{\mathsf{L}}$ *via* the canonical homomorphism $\phi : \mathsf{E} \to \bar{\mathsf{E}}$. First we verify the existence of a finitely generated left ideal U minimal with respect to the property $\mathsf{L} + \mathsf{U} = \mathsf{E}$.

Let W be a left ideal of E satisfying $\mathsf{L}+\mathsf{W}=\mathsf{E}$, and $\{\mathsf{U}_i \mid i \in I\}$ a descending chain of finitely generated left ideals of E, contained in W, such that $\mathsf{L} + \mathsf{U}_i = \mathsf{E}$ for every $i \in I$. As $\mathsf{U} = \bigcap_{i \in I} \mathsf{U}_i$ is again a left ideal, it remains only to show that $\mathsf{L} + \mathsf{U} = \mathsf{E}$ (by minimality, U is then even singly generated, since $1 \in \mathsf{E}$). Let $\{a_1, \ldots, a_n\}$ and $\{u_{i1}, \ldots, u_{im_i}\}$ be sets of generators of L and U_i, respectively. The condition $\mathsf{L} + \mathsf{U}_i = \mathsf{E}$ translates into the solvability of the system of equations

$$x_1 a_1 + \cdots + x_n a_n + y_{i1} u_{i1} + \cdots + y_{im_i} u_{im_i} = 1 \quad (i \in I)$$

in E for the unknowns x_j, y_{ik}. The chain property of the set of the U_i guarantees that for every finite subset J of I, the system consisting of equations with indices in J admits a simultaneous solution. Algebraic compactness implies the existence of a global solution $x_j = e_j \in \mathsf{E}$ and $y_{ik} = c_{ik} \in \mathsf{E}$. Then U is the principal left ideal generated by $u = 1 - (e_1 a_1 + \cdots + e_n a_n)$.

We claim that for this U, the intersection $\mathsf{L} \cap \mathsf{U}$ is superfluous in E (i.e., it can be omitted from every generating system of E). First we show that it is superfluous in U. Suppose that $\mathsf{U} = (\mathsf{L} \cap \mathsf{U}) + \mathsf{V}$ for some left ideal $\mathsf{V} \leq \mathsf{U}$. Then $\mathsf{E} = \mathsf{L} + (\mathsf{L} \cap \mathsf{U}) + \mathsf{V} = \mathsf{L} + \mathsf{V}$ implies $\mathsf{V} = \mathsf{U}$ by the minimality of U. Next, let $\mathsf{E} = (\mathsf{L} \cap \mathsf{U}) + \mathsf{A}$ for a left ideal $\mathsf{A} \leq \mathsf{E}$. By the modular law, $\mathsf{U} = (\mathsf{L} \cap \mathsf{U}) + (\mathsf{A} \cap \mathsf{U})$, whence $\mathsf{U} = \mathsf{A} \cap \mathsf{U}$ follows as $\mathsf{L} \cap \mathsf{U}$ is superfluous in U. But then $\mathsf{U} \leq \mathsf{A}$, thus $\mathsf{E} = (\mathsf{L} \cap \mathsf{U}) + \mathsf{U} + \mathsf{A} = \mathsf{U} + \mathsf{A} = \mathsf{A}$, and $\mathsf{L} \cap \mathsf{U}$ is superfluous in E. Hence $\mathsf{L} \cap \mathsf{U} \leq \mathsf{J}$, and passing mod J, we obtain that the sum $\bar{\mathsf{L}} + \bar{\mathsf{U}} = \bar{\mathsf{E}}$ has to be direct. A ring in which every finitely generated left ideal is a summand is known to be von Neumann regular, so $\bar{\mathsf{E}}$ is von Neumann regular.

It remains to verify that idempotents lift modulo J. For an idempotent coset $x + \mathsf{J}$, consider the set $\mathcal{S}$ of all cosets $f + \mathsf{A}$ of ideals A of E such that $f + \mathsf{A} \leq x + \mathsf{J}$ (so $\mathsf{A} \leq \mathsf{J}$) and $f^2 \in f + \mathsf{A}$. The set $\mathcal{S}$ is not empty, and a proof like the one above (for the existence of a minimal U) will guarantee that it contains a minimal member, say, $e + \mathsf{B}$ ($\mathsf{B} \leq \mathsf{J}$). To complete the proof, we show that e is an idempotent.

Since $(1 - 2e)^2 = 1 + 4(e^2 - e) \in 1 + \mathsf{J}$ is invertible, so is $1 - 2e$. Hence $y = (e^2 - e)(1 - 2e)^{-1} \in \mathsf{B}$. Set $f = e + y \in x + \mathsf{J}$ and $\mathsf{C} = y^2\mathsf{E} \leq \mathsf{B} \leq \mathsf{J}$. Then $(e+y)^2 - (e+y) = e^2 - e + (2e-1)y + y^2 = y^2$ shows that we have $f + \mathsf{C} \in \mathcal{S}$. As $f + \mathsf{C} \leq e + \mathsf{B}$, necessarily $\mathsf{B} = \mathsf{C}$. Hence $y \in \mathsf{C}$ implies $y = y^2 r$ for some $r \in \mathsf{E}$, and from $1 - yr \in 1 + \mathsf{J}$ we obtain $y = 0$. Consequently, $e = f$ is idempotent. $\square$

We are now prepared for the proof of the following important theorem.

Theorem 6.7 (Warfield [3]). *Algebraically compact groups have the finite exchange property.*

Proof. Combining Theorem 6.5, Lemma 6.6 with Theorem 6.4, it follows that a group enjoys the exchange property whenever its endomorphism ring is a left algebraically compact ring. As left and right exchange rings are identical (this follows from Theorem 6.4, since the opposite ring $\mathsf{E}^{\mathrm{opp}}$ is the endomorphism ring of ${}_{\mathsf{E}}\mathsf{E}$). An appeal to Theorem 4.2 in Chapter 16 completes the proof. $\square$

The next theorem is the main result on direct sums of groups with local endomorphism rings. We state it here without proof.

Theorem 6.8 (Azumaya, Crawley–Jónsson [1]). *Assume*

$$A = \bigoplus_{i \in I} A_i$$

where each A_i is a countably generated group with local endomorphism ring. Then every direct decomposition of A can be refined to a direct decomposition isomorphic *to the given one. In particular, every summand C of A satisfies $C \cong \oplus_{i\in J} A_i$ for a suitable subset J of I.* □

A class $\mathcal{C}$ of groups is said to have the **Krull-Schmidt property** if every group in $\mathcal{C}$ is a direct sum of indecomposable members of $\mathcal{C}$, and such a direct decomposition is unique up to isomorphism. Hence Theorem 6.8 can be rephrased by stating that direct sums of countably generated groups with local endomorphism rings enjoy the Krull-Schmidt property.

★ **Notes.** The exchange property for algebraic systems was introduced by Crawley–Jónsson [1], it became a well-researched subject. It is still an open problem whether or not the finite exchange property implies the unrestricted exchange property (it is true for indecomposable modules). The proofs in the text are modeled after arguments by W.K. Nicholson [Trans. Amer. Math. Soc. **229**, 269–278 (1977)]. The exchange property for injective modules was proved by Warfield [Pacific J. Math. **31**, 263–276 (1969)] and for quasi-injectives by Fuchs [Annali Scuola Norm. Pisa **23**, 541–546 (1969)]. Monk [Proc. Amer. Math. Soc. **35**, 349–353 (1972)] characterizes exchange rings as rings R such that for all $a \in \mathsf{R}$ there exist $b, c \in \mathsf{R}$ with $bab = b$, $c(1-a)(1-ba) = 1 - ba$. For Theorems 6.4 and 6.7, see Warfield [Math. Ann. **199**, 31–36 (1972)], and for Theorem 6.8, Azumaya [Nagoya Math. J. **1**, 117–124 (1950)].

There are several results facilitating the proofs that certain groups (or modules) enjoy the exchange property. For instance, for the finite exchange property it suffices to check for an index set of cardinality 2 (Exercise 1). Furthermore, in Lemma 6.1 it is enough to assume that all the groups A_i are isomorphic to G (Zimmermann-Huisgen–Zimmermann [1]).

Ivanov [4] discusses an 'almost exchange property' of groups G. His definition reads as follows: if $A = G \oplus H = \oplus_{i\in I} A_i$, then there are a partition of I into finite subsets I_j and subgroups $C_j \leq \oplus_{i\in I_j} A_i$ such that $A = G \oplus (\oplus_j C_j)$. Closure under finite sums is established.

Exercises

(1) For the finite exchange property it suffices to check for 2 summands.

(2) (a) A finite direct sum has the finite exchange property exactly if all summands enjoy this property.

(b) In general, the exchange property is not preserved when taking infinite direct sums. [Hint: a direct sum of unbounded cyclic p-groups.]

(3) (a) Finite groups have the exchange property.

(b) $\mathbb{Z}$ does not have the exchange property.

(4) If $A = B \oplus C = G \oplus H$ and B is an unbounded Σ-cyclic p-group, then either G or H has an unbounded Σ-cyclic summand. [Hint: keep using Exercise 3.]

(5) Suppose that $G = A_1 \oplus \cdots \oplus A_n$ where all $\operatorname{End} A_i$ are local. Use the exchange property to show:

(a) a decomposition of G into indecomposable summands is unique up to isomorphism;

(b) a summand of G is isomorphic to a partial direct sum of the given decomposition.

Problems to Chapter 6

PROBLEM 6.1. Which reduced algebraically compact groups carry an injective module structure over some ring with 1?

PROBLEM 6.2. Which algebraically compact rings can be endomorphism rings of groups?

Such rings are not cotorsion-free, so the results in Sect. 7 in Chapter 16 are not applicable.

PROBLEM 6.3. Does the finite exchange property imply the general exchange property?

This is a well-known difficult open problem.

Chapter 7
Homomorphism Groups

Abstract The fact that the homomorphisms of a group into another group form an abelian group has proved extraordinarily profound not only in abelian group theory, but also in Homological Algebra where the functor Hom is one of the cornerstones of the theory. Our first aim is to find relevant properties of Hom both as a bifunctor and as a group.

It is rather surprising that in some significant cases $\mathrm{Hom}(A, C)$ is algebraically compact; for instance, when A is a torsion group, or when C is algebraically compact. In the special situation when C is the additive group $\mathbb{T}$ of the reals mod 1, in which case $\mathrm{Hom}(A, \mathbb{T})$, furnished with the compact-open topology, will be the character group of A, our description leads to a complete characterization of compact abelian groups by cardinal invariants. An analogous result deals with the linearly compact abelian groups.

The final section discusses special types of homomorphisms that play important roles in the theory of torsion groups.

1 Groups of Homomorphisms

Homomorphism Groups We have already noticed earlier that, if α and β are homomorphisms of A into C, then their sum $\alpha + \beta$, defined as

$$(\alpha + \beta)a = \alpha a + \beta a \qquad (a \in A),$$

is again a homomorphism $A \to C$. It is now routine to check that the homomorphisms of A into C form an abelian group under addition. This group is called the **homomorphism group** of A into C and is denoted by $\mathrm{Hom}(A, C)$. The zero in this group is the trivial homomorphism mapping A to $0 \in C$, and the inverse $-\alpha$ of $\alpha : A \to C$ maps $a \in A$ upon $-(\alpha a) \in C$.

If $A = C$, then the elements of $\mathrm{Hom}(A, A)$ are the endomorphisms of A, and the group $\mathrm{Hom}(A, A) = \mathrm{End}\, A$ is called the **endomorphism group** of A. This group carries a ring structure where the product $\alpha\beta$ of $\alpha, \beta \in \mathrm{End}\, A$ is defined by $(\alpha\beta)a = \alpha(\beta a)$ for $a \in A$ (observe the order of maps). The ring identity is the identity automorphism of A.

Next we list some simple facts on homomorphism groups.

(A) There are two important necessary conditions to satisfy when we are looking for homomorphisms $\alpha : A \to C$: one is that if $a \in A$ is annihilated by $n \in \mathbb{N}$, then also $n(\alpha a) = 0 \in C$, and the other is that we must have $h_p(\alpha a) \geq h_p(a)$.

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-319-19422-6_7

(B) $\mathrm{Hom}(A, C) = 0$ *in the following cases:* (i) *A is torsion and C is torsion-free;* (ii) *A is a p-group and C is a q-group, for primes $p \neq q$;* (iii) *A is divisible and C is reduced.*
(C) *If $C[n] = 0$ for some $n \in \mathbb{N}$, then* $\mathrm{Hom}(A, C)[n] = 0$ *for every group A.* Indeed, if $\alpha : A \to C$ and $n\alpha = 0$, then for $a \in A$ we have $n(\alpha a) = (n\alpha)a = 0$ whence $C[n] = 0$ implies $\alpha a = 0$, i.e. $\alpha = 0$.
(D) $\mathrm{Hom}(A, C)$ *is torsion-free whenever C is torsion-free.*
(E) *If C is torsion-free and divisible, then so is* $\mathrm{Hom}(A, C)$ *for every A.* In order to show that $\mathrm{Hom}(A, C)$ is now also divisible, pick an $\alpha \in \mathrm{Hom}(A, C)$ and an $n \in \mathbb{N}$. For $a \in A$, there exists a unique $c \in C$ with $nc = \alpha a$, and thus we may define a map $\beta : A \to C$ via $\beta a = c$. It follows readily that β is a homomorphism $A \to C$ satisfying $n\beta = \alpha$.
(F) *If $nA = A$ for some $n \in \mathbb{N}$, then* $\mathrm{Hom}(A, C)[n] = 0$. Indeed, let $\alpha \in \mathrm{Hom}(A, C)$ with $n\alpha = 0$. Write $a \in A$ as $a = nb$ for some $b \in A$. Then $\alpha a = \alpha(nb) = (n\alpha)b = 0$ shows that $\alpha = 0$.
(G) *If A is divisible, then* $\mathrm{Hom}(A, C)$ *is torsion-free.*
(H) *If A is torsion-free and divisible, then the same holds for* $\mathrm{Hom}(A, C)$*, for any C.* The proof is similar to the one in (E).

Example 1.1. If $A = \mathbb{Z}$, then every $\alpha : \mathbb{Z} \to C$ is completely determined by $\alpha(1) = c \in C$. Moreover, evidently, for every $c \in C$ there is a homomorphism $\gamma : \mathbb{Z} \to C$ such that $\gamma(1) = c$. Since $\alpha(1) = c_1$ and $\beta(1) = c_2$ imply $(\alpha + \beta)(1) = c_1 + c_2$, the correspondence $\gamma \mapsto c$ given by $\gamma(1) = c$ is a natural isomorphism between $\mathrm{Hom}(\mathbb{Z}, C)$ and C,

$$\mathrm{Hom}(\mathbb{Z}, C) \cong C \qquad \text{for all groups } C.$$

Example 1.2. If $A = \mathbb{Z}(m)$ with $m \in \mathbb{N}$, then again, every homomorphism $\alpha : \mathbb{Z}(m) \to C$ is determined by the image $\alpha(\bar{1}) = c$ of the coset $\bar{1} = 1 + m\mathbb{Z}$, but here $mc = 0$ must hold, i.e. $c \in C[m]$. Conversely, each such c gives rise to a homomorphism $\gamma : \bar{1} \mapsto c$, and as in the preceding example, the correspondence $\gamma \mapsto c$ given by $\gamma(\bar{1}) = c$ is a natural isomorphism

$$\mathrm{Hom}(\mathbb{Z}(m), C) \cong C[m] \qquad \text{for all groups } C.$$

Example 1.3. From the preceding example we obtain

$$\mathrm{Hom}(\mathbb{Z}(p^k), \mathbb{Z}(p^n)) \cong \mathbb{Z}(p^\ell) \quad \text{where } \ell = \min\{k, n\}.$$

Example 1.4. Next, let C be quasi-cyclic, say, $C = \langle c_1, \dots, c_n, \dots \rangle$ with the defining relations $pc_1 = 0, pc_{n+1} = c_n (n \geq 1)$. If η is an endomorphism of C, then write $\eta c_n = k_n c_n$ with an integer k_n $(0 \leq k_n < p^n)$ for every n. Now $k_n c_n = \eta c_n = \eta(pc_{n+1}) = p\eta c_{n+1} = pk_{n+1}c_{n+1} = k_{n+1}c_n$ implies $k_n \equiv k_{n+1} \bmod p^n$. This means that the sequence of the k_n is a Cauchy sequence in J_p, so it has a limit, say $\pi \in J_p$ is the limit. The correspondence $\eta \mapsto \pi$ between the endomorphisms η of C and the p-adic integers π is evidently additive. If the endomorphisms η_1 and η_2 define the same π, then $\eta_1 - \eta_2$ maps every c_n to 0, i.e. $\eta_1 = \eta_2$. On the other hand, if $\pi = s_0 + s_1 p + \dots + s_n p^n + \dots$ is any p-adic integer, then the correspondence $c_n \mapsto (s_0 + s_1 p + \dots + s_n p^n)c_n$ for all n (which we may write simply as πc_n) extends uniquely to an endomorphism η of C such that $\eta \mapsto \pi$. We conclude:

$$\mathrm{End}\, \mathbb{Z}(p^\infty) \cong J_p.$$

Example 1.5. Consider $\mathbb{Q}^{(p)}$, the group of rational numbers with powers of p as denominators, and $C = \mathbb{Z}(p^\infty)$. Suppose $\mathbb{Q}^{(p)} = \langle 1, p^{-1}, \dots, p^{-n}, \dots \rangle$ and C as in the preceding example (and $c_n = 0$ for $n \leq 0$). A p-adic number $\rho = p^k\pi$ (with a p-adic unit π and $k \in \mathbb{Z}$) induces a homomorphism $\eta\colon \mathbb{Q}^{(p)} \to \mathbb{Z}(p^\infty)$ such that $p^{-n} \mapsto \pi c_{n-k}$ for all n. As in the preceding example, we can convince ourselves that different p-adic numbers ρ give rise to different homomorphisms, and every homomorphism $\eta\colon \mathbb{Q}^{(p)} \to \mathbb{Z}(p^\infty)$ arises in this way. Consequently, $\operatorname{Hom}(\mathbb{Q}^{(p)}, \mathbb{Z}(p^\infty))$ is isomorphic to the additive group of all p-adic numbers, i.e., we have

$$\operatorname{Hom}(\mathbb{Q}^{(p)}, \mathbb{Z}(p^\infty)) \cong \oplus_\kappa \mathbb{Q} \qquad \text{with } \kappa = 2^{\aleph_0}.$$

Example 1.6. If $A = C = J_p$, then it is evident that multiplication by a fixed p-adic integer π is an endomorphism of J_p (which we denote by $\dot\pi$), and different p-adic integers yield different endomorphisms of J_p, since they map $1 \in J_p$ differently. Let $\xi \in \operatorname{End} J_p$ such that $\xi(1) = \pi$. Then ξ and $\dot\pi$ are identical on $\mathbb{Z} \leq J_p$, so $\mathbb{Z} \leq \operatorname{Ker}(\xi - \dot\pi)$. But $J_p/\mathbb{Z}$ is divisible, while J_p is reduced, so $J_p/\operatorname{Ker}(\xi - \dot\pi) = 0$. It follows that $\xi = \dot\pi$, and we have

$$\operatorname{End} J_p \cong J_p.$$

Hom and Direct Sums and Products Our next concern is the behavior of Hom towards direct sums and direct products. The following theorem is fundamental.

Theorem 1.7. *For an arbitrary index set I, there are natural isomorphisms*

$$\operatorname{Hom}(\oplus_{i\in I} A_i, C) \cong \prod_{i\in I} \operatorname{Hom}(A_i, C) \tag{7.1}$$

and

$$\operatorname{Hom}(A, \prod_{i\in I} C_i) \cong \prod_{i\in I} \operatorname{Hom}(A, C_i). \tag{7.2}$$

Proof. In order to prove (7.1), let $\rho_i : A_i \to \oplus A_i$ and $\pi_i : \oplus A_i \to A_i$ denote the injection and the projection maps, respectively. We map the left side of (7.1) to the right side by sending $\alpha : \oplus A_i \to C$ to $(\dots, \alpha\rho_i, \dots)$ where $\alpha\rho_i : A_i \to C$. This is evidently a homomorphism ϕ from the left to the right side. It is clear that ϕ maps α to 0 only if $\alpha = 0$. Since every $(\dots, \alpha_i, \dots) \in \prod \operatorname{Hom}(A_i, C)$ defines an $\alpha \in \operatorname{Hom}(\oplus A_i, C)$ *via* $\alpha = \oplus(\alpha_i\pi_i)$, ϕ is epic as well.

For the proof of (7.2), let $\sigma_i : C_i \to \prod C_i$ and $\tau_i : \prod C_i \to C_i$ denote the injection and the projection maps, respectively. Every $\beta \in \operatorname{Hom}(A, \prod C_i)$ defines a homomorphism $\tau_i\beta \in \operatorname{Hom}(A, C_i)$ for each i. As in the preceding paragraph, we conclude that the correspondence $\beta \mapsto (\dots, \tau_i\beta, \dots)$ is an isomorphism of the left-hand side of (7.2) with its right-hand side. □

We can now derive the following corollary.

Corollary 1.8. *Assume A is a torsion group with p-components A_p, and C is a group with p-components C_p. Then*

$$\operatorname{Hom}(A, C) \cong \prod_p \operatorname{Hom}(A_p, C_p).$$

Proof. Apply (7.1) and observe that $\mathrm{Hom}(A_p, C) = \mathrm{Hom}(A_p, C_p)$. $\square$

Example 1.9. For any group A,

$$\mathrm{Hom}(A, \mathbb{Q}) \cong \prod_{\mathrm{rk}_0(A)} \mathbb{Q}.$$

Because of (E), the description of $\mathrm{Hom}(A, \mathbb{Q})$ becomes a simple calculation in cardinal arithmetics. If F is a free subgroup of A generated by a maximal independent system of elements of infinite order only, then every $\phi : F \to \mathbb{Q}$ extends uniquely to a map $\alpha : A \to \mathbb{Q}$. This amounts to saying that $\mathrm{Hom}(F, \mathbb{Q}) \cong \mathrm{Hom}(A, \mathbb{Q})$ naturally. The former Hom is evaluated by using (7.1).

Hom As Bifunctor The correct way of viewing Hom is as a functor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$ associating the group $\mathrm{Hom}(A, C)$ with the ordered pair $(A, C) \in \mathcal{A}b \times \mathcal{A}b$. In the balance of this section we investigate the functorial behavior of Hom.

Let $\alpha : A' \to A$ and $\gamma : C \to C'$ be fixed homomorphisms. An $\eta \in \mathrm{Hom}(A, C)$ defines a homomorphism $A' \to C'$ as the composite $A' \xrightarrow{\alpha} A \xrightarrow{\eta} C \xrightarrow{\gamma} C'$. The correspondence $\eta \mapsto \gamma\eta\alpha$ is a homomorphism

$$\mathrm{Hom}(\alpha, \gamma) \colon \mathrm{Hom}(A, C) \to \mathrm{Hom}(A', C'),$$

called the **homomorphism induced by** α **and** γ. Clearly, $\mathrm{Hom}(\mathbf{1}_A, \mathbf{1}_C) = \mathbf{1}_{\mathrm{Hom}(A,C)}$. Furthermore, if $A'' \xrightarrow{\alpha'} A' \xrightarrow{\alpha} A$ and $C \xrightarrow{\gamma} C' \xrightarrow{\gamma'} C''$, then

$$\mathrm{Hom}(\alpha\alpha', \gamma'\gamma) = \mathrm{Hom}(\alpha', \gamma') \, \mathrm{Hom}(\alpha, \gamma).$$

Evidently, $\mathrm{Hom}(\alpha, \gamma)$ is additive in both arguments. Therefore, we can conclude:

Theorem 1.10. Hom *is an additive bifunctor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$, contravariant in the first and covariant in the second argument.* $\square$

It is often convenient to use abbreviated notations (provided there is no danger of confusion):

$$\alpha^* = \mathrm{Hom}(\alpha, \mathbf{1}_C) \qquad \text{and} \qquad \gamma_* = \mathrm{Hom}(\mathbf{1}_A, \gamma).$$

The following result describes the behavior of Hom towards direct and inverse limits.

Theorem 1.11 (Cartan–Eilenberg [CE]). *Assume*

$$\mathfrak{A} = \{A_i \ (i \in I); \pi_i^k\} \qquad \textit{and} \qquad \mathfrak{C} = \{C_j \ (j \in J); \rho_j^\ell\}$$

are a direct and an inverse system of groups, respectively, and let $A = \varinjlim A_i$, $C = \varprojlim C_j$ with canonical maps $\pi_i : A_i \to A$ and $\rho_j : C \to C_j$. Then

$$\mathfrak{H} = \{\mathrm{Hom}(A_i, C_j) \ ((i,j) \in I \times J; \mathrm{Hom}(\pi_i^k, \rho_j^\ell)\}$$

is an inverse system of groups whose inverse limit is $\mathrm{Hom}(A, C)$ *with* $\mathrm{Hom}(\pi_i, \rho_j)$ *as canonical maps.*

Proof. It is straightforward to check that $\mathfrak{H}$ is an inverse system; let H denote its inverse limit. From the required commutativity of the triangles we can conclude that there exists a unique map ξ rendering all triangles

$$\begin{tikzcd} \mathrm{Hom}(A, C) \arrow[rr, dashed, "\xi"] \arrow[dr, "\mathrm{Hom}(\pi_i, \rho_j)"'] & & H \arrow[dl, "\xi_{ij}"] \\ & \mathrm{Hom}(A_i, C_j) & \end{tikzcd}$$

commutative, where the ξ_{ij} are the canonical maps. To show that ξ is monic, let $\eta \in \operatorname{Ker} \xi$. Then $\xi_{ij}\xi\eta = 0$, that is, $\rho_j \eta \pi_i = \mathrm{Hom}(\pi_i, \rho_j)\eta = 0$ for all i, j. Thus the map $\eta\pi_i \colon A_i \to C$ is 0, because all of its jth coordinates are 0, and since $\cup_i \pi_i A_i = A$, we have $\eta = 0$.

Any $\chi \in H$ is of the form $\chi = (\dots, \chi_{ij}, \dots) \in \prod \mathrm{Hom}(A_i, C_j)$ where the coordinates χ_{ij} satisfy the requisite postulates. Define $\eta : A \to C$ as follows: if $a = \pi_i a_i$, then for this i set $\eta a = (\dots, \chi_{ij} a_i, \dots) \in \prod C_j$. It is straightforward to verify the independence of ηa of the choice of i as well as the homomorphism property of η. Considering that $\xi_{ij}\chi = \chi_{ij}$ and $\xi_{ij}\xi\eta = \rho_j\eta\pi_i = \chi_{ij}$, we must have $\xi\eta = \chi$, showing that ξ is epic. Thus ξ is an isomorphism. $\square$

Hom and Exact Sequences Next we prove a most frequently used application of the Hom functor.

Theorem 1.12. *If* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is an exact sequence, then so are the induced sequences*

$$0 \to \mathrm{Hom}(G, A) \xrightarrow{\alpha_*} \mathrm{Hom}(G, B) \xrightarrow{\beta_*} \mathrm{Hom}(G, C) \tag{7.3}$$

and

$$0 \to \mathrm{Hom}(C, G) \xrightarrow{\beta^*} \mathrm{Hom}(B, G) \xrightarrow{\alpha^*} \mathrm{Hom}(A, G) \tag{7.4}$$

for every group G. Equation (7.3) *can always be completed to an exact sequence with* $\to 0$ *if G is a free group, and* (7.4) *if G is a divisible group.*

Proof. Let $\eta : G \to A$. If $\alpha\eta = 0$, then α monic implies $\eta = 0$, so α_* is also monic. Furthermore, $\beta\alpha\eta = 0$ shows that $\beta_*\alpha_* = 0$ as well. If $\xi : G \to B$ is such that $\beta\xi = 0$, then $\operatorname{Im} \xi \leq \operatorname{Ker} \beta = \operatorname{Im} \alpha$, so there is a $\phi : G \to A$ with $\xi = \alpha\phi$. Thus (7.3) is exact. If we continue with $\to 0$, then exactness at $\mathrm{Hom}(G, C)$ would mean that for every $\zeta : G \to C$ there is a $\xi : G \to B$ such that $\beta\xi = \zeta$, this holds for free groups G, due to their projective property.

Next let $\eta : C \to G$. If $\eta\beta = 0$, then $\eta = 0$, since β is epic; thus β^* is monic. From $\eta\beta\alpha = 0$ we obtain $\alpha^*\beta^* = 0$. Assume $\xi : B \to G$ satisfies $\xi\alpha = 0$. This means that $\operatorname{Ker}\beta = \operatorname{Im}\alpha \leq \operatorname{Ker}\xi$, so there is a $\phi\colon C \to G$ with $\xi = \phi\beta$. Thus (7.4) is exact. If G is injective, then for every $\zeta : A \to G$ there is a $\xi : B \to G$ such that $\zeta = \xi\alpha$, so $\to 0$ can be added to (7.4). □

More can be said if we start with a pure-exact sequence.

Proposition 1.13 (Fuchs [11]). *If the sequence given in the preceding theorem is pure-exact* (*p-pure-exact*)*, then so are* (7.3) *and* (7.4).

Proof. First, let $\eta : G \to B$, $\xi : G \to A$ satisfy $n\eta = \alpha\xi$ $(n \in \mathbb{N})$. Thus $n\eta$ maps G into αA, and so $\operatorname{Im}\eta \leq n^{-1}\alpha A$. By Theorem 2.10 in Chapter 5, $n^{-1}\alpha A = \alpha A \oplus X$ where $nX = 0$. If π denotes the projection onto the first summand, then $\phi = \alpha^{-1}\pi\eta\colon G \to A$ satisfies $n\phi = \xi$, establishing the first claim.

Next, assume $n\eta = \xi\beta$ holds for $\eta : B \to G$, $\xi : C \to G$ and $n \in \mathbb{N}$. Then $\eta(n\alpha) = \xi\beta\alpha = 0$ shows that $n\alpha A \leq \operatorname{Ker}\eta$. Owing to Theorem 2.10 in Chapter 5, there is a direct decomposition $B/(n\alpha A) = \alpha A/(n\alpha A) \oplus B'/(n\alpha A)$ for some $B' \leq B$. Define ϕ as the composite map $B \to B/(n\alpha A) \to B'/(n\alpha A) \to G$, where the second map is the canonical projection, while the third is induced by η. Clearly, $n\phi = n\eta$ and $\alpha A \leq \operatorname{Ker}\phi$. Because of this inclusion, there is a homomorphism $\theta : C \to G$ such that $\phi = \theta\beta$. Hence $n(\theta\beta) = n\phi = n\eta = \xi\beta$, completing the proof. □

Small Groups A group G is said to be **small** if there is a natural isomorphism

$$\operatorname{Hom}(G, \oplus_{i\in I} C_i) \cong \oplus_{i\in I} \operatorname{Hom}(G, C_i)$$

for every set of groups C_i. Equivalently, the image of every homomorphism of G into an infinite direct sum is already contained in the direct sum of a finite number of summands. If the groups C_i in the definition are restricted to a class $\mathcal{C}$ of groups (e.g., to torsion-free groups), then G is called **$\mathcal{C}$-small**. In particular, if $\mathcal{C}$ is the direct sum of copies of G itself, then G is **self-small**.

Example 1.14. (a) Finitely generated groups are small, while finite rank torsion-free groups are $\mathcal{F}$-small, where $\mathcal{F}$ denotes the class of torsion-free groups.

(b) The quasi-cyclic group $\mathbb{Z}(p^\infty)$ is not small: it has a homomorphic image in $\oplus_{\aleph_0}\mathbb{Z}(p^\infty)$ that has non-zero projection in every summand (elements of order p^n have non-zero coordinates in the first n summands).

(a) Epic images of small groups are small.
(b) A finite direct sum of groups is small if and only if each component is small.
(c) *A group is small if and only if it is finitely generated.* If G is small, then its copy in an injective group must be contained in the direct sum of finitely many summands, whence we infer that G is of finite rank. The torsion subgroup of G cannot have infinitely many non-zero p-components, nor a quasi-cyclic summand (see Example 1.14b), so it must be finite. G/tG is a small torsion-free group, hence all of its torsion homomorphic images have to be small, so finite. Hence G/tG is a finite extension of a finitely generated free subgroup, so itself finitely generated.
(d) *A torsion-free group is small in the category of torsion-free groups if and only if it is of finite rank.*

★ **Notes.** The group structure of Hom has been the main topic of numerous investigations. It is impossible to survey them without the extensive knowledge of the material in later chapters. Perhaps the most important results are due to Pierce [1] that give a very precise description of Hom in case of p-groups, making use of Theorem 2.1. No comparable study is expected for torsion-free groups.

One question which we would like to point out here is concerned with the problem as to what extent the functor $\mathrm{Hom}(A, *)$ determines the group A. That A is by no means determined by this functor was proved by Hill [12] for p-groups and by Sebel'din [2] for torsion-free groups. The counterexamples are: 1) $A = \oplus_{2^{\aleph_0}} B$ and $A' = \oplus_{2^{\aleph_0}} \overline{B}$ where $B = \oplus_{\aleph_0} \oplus_{n<\omega} \mathbb{Z}(p^n)$ ($\overline{B}$ is the torsion-completion of B, see Sect. 3 in Chapter 10); and 2) $A = \oplus_{\aleph_0}\mathbb{Z} \oplus \mathbb{Q}$ and $A' = A \oplus \mathbb{Q}$. Then $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(A', G)$ holds for all G. Albrecht [7] deals with this question for p-groups and cotorsion groups.

An important generalization of homomorphism groups is concerned with groups with distinguished subgroups. The objects of the category $\mathcal{A}b_n$ are $\mathbf{A} = \{A; A_i (i < n)\}$ where $A \in \mathcal{A}b$, and $A_0, \dots, A_{n-1}$ are fixed subgroups of A. If $\mathbf{C} = \{C; C_i (i < n)\}$ is another object in this category, then $\phi : \mathbf{A} \to \mathbf{C}$ is a morphism if $\phi \in \mathrm{Hom}(A, C)$ such that $\phi(A_i) \leq C_i$ for all $i < n$. Results on homomorphism groups in such categories are instrumental in several questions concerning ordinary homomorphism groups.

Exercises

(1) Show that $\mathrm{Hom}(A, C)$ is isomorphic to a subgroup of C^A.

(2) We have $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(C, A)$ and $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong A$ if both A and C are finite groups.

(3) If A is torsion-free and C is divisible, then $\mathrm{Hom}(A, C)$ is divisible.

(4) Prove that $\mathrm{Hom}(A, \mathbb{Z}(m)) \cong \mathrm{Hom}(A/mA, \mathbb{Z}(m))$ for all $m \in \mathbb{N}$.

(5) If C is torsion-free, then $\mathrm{Hom}(\mathbb{Q}, C)$ is isomorphic to the maximal divisible subgroup of C.

(6) If the sequence $0 \to A \to B \to C \to 0$ is pure-exact, then (7.3) can be completed with $\to 0$ if G is Σ-cyclic, and (7.4) can so be completed if G is pure-injective.

(7) (a) If A is a torsion group, then the set union $\cup \, \mathrm{Im}\, \alpha$, taken for all $\alpha \in \mathrm{Hom}(A, C)$, is a subgroup of C.
 (b) The same is not necessarily true if A is torsion-free. [Hint: A of rank 2 with $\mathrm{End}\, A \cong \mathbb{Z}$, and $C = A \oplus A$.]

(8) Prove $\mathrm{End}\, J_p \cong J_p$ *via* the isomorphism $\mathrm{End}\, J_p \cong \varprojlim_n \mathrm{Hom}(J_p, \mathbb{Z}(p^n))$.

(9) Describe the structures of $\mathrm{End}(\oplus_\kappa \mathbb{Q})$ and $\mathrm{End}(\oplus_\kappa J_p)$ for a cardinal κ.

(10) If either A or C is a p-group, then $\mathrm{Hom}(A, C)$ is a J_p-module.

(11) If $\alpha \in \mathrm{Aut}\, A$, $\gamma \in \mathrm{Aut}\, C$, then $\mathrm{Hom}(\alpha, \gamma)$ is an automorphism of $\mathrm{Hom}(A, C)$.

(12) (Gerdt) G is small if and only if $G \leq \oplus_{i \in I} C_i$ implies $G \leq \oplus_{i \in J} C_i$ for some finite subset $J \subset I$. (Thus it suffices to consider monomorphisms.)

(13) (Gerdt) If $\mathcal{D}$ is the class of divisible groups, then $\mathcal{D}$-small groups are small.

2 Algebraically Compact Homomorphism Groups

Having considered elementary properties of Hom as well as the exact sequences involving Homs, we turn our attention to special situations when $\mathrm{Hom}(A, C)$ is of great interest. We concentrate on cases in which Hom is algebraically compact.

Hom for Torsion Groups We start with the remarkable fact that if A is a torsion group, then $\mathrm{Hom}(A, C)$ has to be algebraically compact, and hence it can be characterized by invariants describable in terms of the invariants of A and C.

Theorem 2.1 (Harrison [2], Fuchs [11]). *If A is a torsion group, then* $\mathrm{Hom}(A, C)$ *is a reduced algebraically compact group, for any C.*

Proof. It suffices to prove that if A is a p-group, then $H = \mathrm{Hom}(A, C)$ is complete in its p-adic topology. To show that H is Hausdorff, suppose $\eta \in H$ is divisible by every power of p. If $a \in A$ is of order p^k, and if $\chi \in H$ satisfies $p^k\chi = \eta$, then $\eta a = p^k\chi a = \chi p^k a = 0$ shows that $\eta = 0$. Next, let $\eta_1, \dots, \eta_n, \dots$ be a Cauchy sequence in the p-adic topology of H; dropping to a subsequence if necessary, we may assume it is neat: $\eta_{n+1} - \eta_n \in p^n H$ for each n, i.e. $\eta_{n+1} - \eta_n = p^n\chi_n$ for some $\chi_n \in H$. Let

$$\eta = \eta_1 + (\eta_2 - \eta_1) + \dots + (\eta_{n+1} - \eta_n) + \dots .$$

This is a well-defined map $A \to C$, since for $a \in A$ of order p^k, we have $(\eta_{n+1} - \eta_n)a = 0$ for all $n \geq k$, so that the image $\eta a = \eta_1 a + (\eta_2 - \eta_1)a + \dots + (\eta_k - \eta_{k-1})a$ is well defined. Furthermore,

$$\eta - \eta_n = (\eta_{n+1} - \eta_n) + (\eta_{n+2} - \eta_{n+1}) + \dots = p^n(\chi_n + p\chi_{n+1} + \dots),$$

where the infinite sum in the parentheses belongs to H. Thus $\eta - \eta_n \in p^n H$, and η is the limit of the given Cauchy sequence. Consequently, H is complete. $\square$

We give a second, shorter proof based on Theorem 1.11. As a torsion group, A is the direct limit of its finite subgroups A_i. By Theorem 1.11, $\mathrm{Hom}(A, C)$ is then the inverse limit of the groups $\mathrm{Hom}(A_i, C)$ which are bounded in view of Sect. 1(F). Hence $\mathrm{Hom}(A, C)$ is the inverse limit of complete groups, and the assertion follows from Sect. 2, Exercise 7 in Chapter 6.

The invariants of $\mathrm{Hom}(A, C)$ for p-groups A can be computed, but the computation is very technical and lengthy, so we just refer the interested reader to Pierce [1]; see also Fuchs [IAG].

Hom for Compact Groups A most interesting case is when the Hom is a compact group. Next, we take a look at this situation.

We want to give an algebraic characterization of those groups that can carry a compact group topology. In Sect. 5 in Chapter 6 we have introduced the group $\mathrm{Char}\, G = \hat{G}$ as the group of all continuous homomorphisms of the topological group G into the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and observed that $\hat{G}$ is compact if and only if G is discrete. Actually, this is all that we need from the Pontryagin duality to describe the structure of compact groups:

Proposition 2.2. *A group A can carry a compact group topology if and only if it is of the form*

$$A = \operatorname{Hom}(G, \mathbb{T})$$

for some group G. □

Here Hom can be viewed in the algebraic or in the topological sense.

Character Groups Accordingly, our problem became purely algebraic: to classify the groups of the form $\operatorname{Hom}(G, \mathbb{T})$. Algebraically, $\mathbb{T}$ is nothing else than the direct product of quasi-cyclic groups, one for each prime p. Hence

$$\operatorname{Char}\ G \cong \prod_p \operatorname{Hom}(G, \mathbb{Z}(p^\infty)).$$

Consequently, it suffices to deal with $\operatorname{Hom}(G, \mathbb{Z}(p^\infty))$ only.

In describing the structure of this Hom, crucial role is played by the p-basic subgroups of G. So let us fix a p-basic subgroup B of G, and write

$$B = \oplus_{n=0}^{\infty} B_n \quad \text{where} \quad B_0 = \oplus_{\kappa_0} \mathbb{Z},\ B_n = \oplus_{\kappa_n} \mathbb{Z}(p^n) \ \text{ for } \ n \geq 1.$$

Here κ_n ($n \geq 0$) are cardinal numbers, uniquely determined by G. The p-component of G/B is of the form $\oplus_\kappa \mathbb{Z}(p^\infty)$; the fact that the cardinal κ depends on the choice of B is not relevant (as we shall see below), but it can be made unique by choosing e.g. a lower basic subgroup in the p-component of G. Finally, we let $\lambda = \operatorname{rk}_0(G/B)$. A full characterization of $\operatorname{Hom}(G, \mathbb{Z}(p^\infty))$ may be given with the aid of these cardinal numbers.

Theorem 2.3 (Fuchs [10]). *Using the above notation, for any group G we have*

$$\operatorname{Hom}(G, \mathbb{Z}(p^\infty)) \cong \prod_{\kappa_0} \mathbb{Z}(p^\infty) \oplus \prod_{n=1}^{\infty} \prod_{\kappa_n} \mathbb{Z}(p^n) \oplus \prod_{\kappa} J_p \oplus \prod_{\lambda \aleph_0} \mathbb{Q}. \tag{7.5}$$

Proof. The p-pure exact sequence $0 \to B \to G \to G/B \to 0$ induces the p-pure-exact sequence

$$0 \to \operatorname{Hom}(G/B, \mathbb{Z}(p^\infty)) \to \operatorname{Hom}(G, \mathbb{Z}(p^\infty)) \to \operatorname{Hom}(B, \mathbb{Z}(p^\infty)) \to 0.$$

Now Theorem 1.7 shows that

$$\operatorname{Hom}(B, \mathbb{Z}(p^\infty)) = \prod_{n=0}^{\infty} \operatorname{Hom}(B_n, \mathbb{Z}(p^\infty)) \cong \prod_{\kappa_0} \mathbb{Z}(p^\infty) \oplus \prod_{n=1}^{\infty} \prod_{\kappa_n} \mathbb{Z}(p^n).$$

If we write $G/B = \oplus_\kappa \mathbb{Z}(p^\infty) \oplus H$ with zero p-component for H, then because of $\mathrm{Hom}(\oplus_\kappa \mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong \prod_\kappa \mathrm{End}(\mathbb{Z}(p^\infty)) \cong \prod_\kappa J_p$, it remains to evaluate $\mathrm{Hom}(H, \mathbb{Z}(p^\infty))$. The p-pure subgroup L in H generated by a maximal independent set of elements of infinite order is torsion-free and p-divisible, and H/L is a torsion group with zero p-component, so $L = \oplus_\lambda \mathbb{Q}^{(p)}$. (This group L is not unique, not even its cardinality is well defined, but this does not influence the outcome.) The exactness of $0 \to L \to H \to H/L \to 0$ implies that of $0 = \mathrm{Hom}(H/L, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(H, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(L, \mathbb{Z}(p^\infty)) \to 0$, thus we obtain

$$\mathrm{Hom}(H, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(L, \mathbb{Z}(p^\infty)) \cong \prod_\lambda \mathrm{Hom}(\mathbb{Q}^{(p)}, \mathbb{Z}(p^\infty)) = \prod_\lambda (\prod_{\aleph_0} \mathbb{Q}),$$

where we have used Example 1.5. We observe that $\mathrm{Hom}(G/B, \mathbb{Z}(p^\infty))$ is algebraically compact, so its purity in $\mathrm{Hom}(G, \mathbb{Z}(p^\infty))$ implies that it is a summand. This completes the proof. $\square$

If we determine the cardinal numbers $\kappa_0, \kappa_n, \kappa, \lambda$ for all primes, then Char G will be the direct product of groups (7.5) with p ranging over all primes. The group on the right side of (7.5) does not depend on the choice of κ, since the second summand always has a summand that is the product of κ' copies of J_p where $\kappa' = \text{fin rk}\, tB$, so (7.5) has always the product of fin rk tG copies of J_p. A similar comment applies to the choice of λ (see also Theorem 2.6 below).

Observe that the first and the fourth summands in (7.5) come from elements of infinite order, while the two middle summands from the torsion subgroup of G. Hence:

Corollary 2.4. Char *G is reduced if and only if G is a torsion group, and is divisible if and only if G is torsion-free.* $\square$

Since groups G can be found with arbitrarily chosen cardinals κ_n and κ, for every prime p, we can conclude:

Corollary 2.5 (Hulanicki [1], Harrison [1]). *A reduced group is the character group of some* (*torsion*) *group exactly if it is the direct product of finite cyclic groups and groups J_p for* (*distinct or equal*) *primes p.* $\square$

For divisible groups, a simple inequality must be satisfied.

Theorem 2.6 (Hulanicki [1], Harrison [1]). *A divisible group $\neq 0$ is the character group of some* (*torsion-free*) *group if and only if it is of the form*

$$\prod_p \prod_{\mu_p} \mathbb{Z}(p^\infty) \oplus \prod_\mu \mathbb{Q} \qquad \textit{where} \;\; \mu \geq \aleph_0.$$

Proof. If G is a torsion-free group, then its rank is, in the above notation, $\kappa_0(p) + \lambda(p)$ (the dependence on p must be indicated, but the sum is the same for every prime p). This shows that Char G will have the stated form with $\mu_p = \kappa_0(p)$, unless $\kappa_0(p) = 0$ for every prime p. In this case, the direct sum with $\prod_{\aleph_0} \mathbb{Q}$ does not change the isomorphy class of the first direct product.

Conversely, given a divisible group of the stated form, it is an easy exercise to check that μ may be replaced by $\mu + \sum_p \mu_p$. This says, in short, that $\mu \geq \mu_p$ may be assumed. Define G as a direct sum of rational groups G_i such that, for every prime p, μ_p of them satisfy $pG_i \neq G_i$ and μ of them satisfy $pG_i = G_i$. Then Char G will be as desired. □

Example 2.7. (a) For discrete groups $\mathbb{Z}(p^\infty), \mathbb{Q}$ we have Char $\mathbb{Z}(p^\infty) \cong J_p$ and Char $\mathbb{Q} \cong \mathbb{R}$.
(b) For discrete J_p, Char $J_p \cong \mathbb{Z}(p^\infty) \oplus \oplus_{2^{\aleph_0}} \mathbb{Q}$.

Corollary 2.8 (Kakutani). *The character group of a group of infinite cardinality κ is of the power 2^κ.*

Proof. The cardinality of an infinite group G is the sum of the cardinalities used in Theorem 2.3, taken for all primes p. (7.5) implies that then the group Char G must have cardinality 2^κ. □

The theorems above are convincing evidence that the algebraic structure of compact groups is extremely special. The cardinality of the set of all non-isomorphic groups of infinite cardinality $\leq 2^\kappa$ is 2^{2^κ}, but the number of those that can carry a compact group structure is minuscule. For instance, if $\kappa = \aleph_\alpha$ with $|\alpha| \leq \aleph_0$, then there are only countably many, pairwise (algebraically) non-isomorphic groups of cardinality κ that can be compact topological groups, provided we assume GCH. Indeed, then the cardinal invariants in Theorems 2.3 and 2.6 can be chosen not more than countably many ways, and they are unique due to GCH.

Example 2.9. This is an example of a group that can carry one and only one compact group topology: J_p^κ for any cardinal κ (it is compact in the finite index topology). In fact, Theorem 2.3 shows that the only discrete group whose character group is $\cong J_p^\kappa$ is the group $\oplus_\kappa \mathbb{Z}(p^\infty)$ where κ is unique if GCH holds (note that $|J_p^\kappa| = 2^\kappa$).

In contrast, some groups may be furnished with as many distinct compact topologies as possible, as is shown by the following theorem:

Theorem 2.10 (Fuchs [10]). *For any infinite cardinal κ, there exist 2^κ non-isomorphic compact topological groups of power 2^κ that are algebraically all isomorphic.*

Proof. In the proof we refer to Corollary 3.8 in Chapter 11 that asserts the existence of 2^κ non-isomorphic p-groups of cardinality κ: they can be chosen with isomorphic basic subgroups $\oplus_{n=1}^\infty \oplus_\kappa \mathbb{Z}(p^n)$, and they have the same final rank κ. By virtue of Theorem 2.3, their character groups are algebraically isomorphic to $\prod_{n=1}^\infty \prod_\kappa \mathbb{Z}(p^n) \oplus \prod_\kappa J_p$; however, by the Pontryagin duality theory, they are not isomorphic as topological groups. □

While we are still on the subject of compactness, it is worthwhile pointing out that Hom preserves (algebraic) compactness in the second argument.

Theorem 2.11. *If A is (algebraically) compact, then* $\mathrm{Hom}(G,A)$ *is (algebraically) compact for every group G.*

Proof. $\mathrm{Hom}(G,A)$ is isomorphic to a subgroup of the group A^G of all functions from G to A. If A is compact, then A^G is a compact group in which Hom is a closed subset. Hence $\mathrm{Hom}(G,A)$ is a compact group. (For the proof of algebraic compactness, use the summand property.) □

Linearly Compact Groups The structure of linearly compact groups (see Sect. 3 in Chapter 6) is similar to the compact case, though there are some notable differences. First and foremost is that the Kaplansky duality replaces the Pontryagin duality.

In Kaplansky's theory, the duality is established between the category of linearly compact and the category of discrete p-adic modules, for a fixed prime p. Thus only those abelian groups are participating in the duality that are also J_p-modules. The characters are continuous homomorphisms into the discrete p-adic module $\mathbb{Z}(p^\infty)$. If M is a discrete p-adic module, then its character module $\mathrm{Hom}_{J_p}(M,\mathbb{Z}(p^\infty))$ is a linearly compact J_p-module, furnished with the compact-open topology. On the other hand, if M is a linearly compact J_p-module, then the continuous homomorphisms of M into $\mathbb{Z}(p^\infty)$ yield a discrete J_p-module.

Example 2.12. The Kaplansky dual of the discrete group $\mathbb{Z}(p^\infty)$ is the linearly compact group J_p, and *vice versa*.

Consequently, the linearly compact p-adic modules are, from the pure algebraic point of view, nothing else than the groups $\mathrm{Hom}_{J_p}(M,\mathbb{Z}(p^\infty))$ where M ranges over the class of discrete p-adic modules. Hence, from the proof above on the character groups one can derive:

Theorem 2.13 (Fuchs [15]). *A group admits a linearly compact topology if and only if it is the direct product of groups of the following types:*

(a) *Cocyclic groups:* $\mathbb{Z}(p^n), \mathbb{Z}(p^\infty)$ *for any prime p and $n \in \mathbb{N}$;*
(b) *The additive group J_p of the p-adic integers and the additive group of the field $\mathbb{Q}_p^*$ of the p-adic numbers, for each prime p.* □

★ **Notes.** The character group of a discrete left module (over any ring) is a right module that is compact in the compact-open topology. Theorem 2.11 also extends to modules. It was S. Lefschetz who introduced linearly compact vector spaces, and later the theory was extended to modules. Linearly compact modules have an extensive theory.

Exercises

(1) Prove that $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}^{\aleph_0}$.
(2) (a) If A is algebraically compact, and if H is a pure subgroup of a group G, then $\mathrm{Hom}(G,A) \cong \mathrm{Hom}(H,A) \oplus \mathrm{Hom}(G/H,A)$.
 (b) Show that then $\mathrm{Char}\ G \cong \mathrm{Char}\ H \oplus \mathrm{Char}\ G/H$ (isomorphism in the algebraic sense only).

(3) The additive group $\mathbb{R}$ of the reals can be furnished with infinitely many distinct topologies, each yielding non-isomorphic compact groups. [Hint: Char $(\oplus_n \mathbb{Q})$.]
(4) The group $A = \mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ is the character group of $\oplus_{\aleph_0}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$. [Hint: Exercise 9 in Sect. 3 in Chapter 6.]
(5) If C is a complete group, then $\mathrm{Hom}(A, C)$ is the inverse limit of bounded groups.
(6) (a) Assume A is equipped with a non-discrete compact topology. Then it has a subgroup (algebraically) isomorphic either to 1) J_p for some p, or to 2) an infinite direct product of cyclic groups of prime orders. [Hint: Theorem 2.3.]
(b) A group admits a non-discrete locally compact topology if and only if it has a subgroup of kind 1) or 2).
(7) (Faltings) Let A be a p-group. Then $t(\mathrm{Hom}(A, \mathbb{Z}(p^\infty)) \cong A$ if and only if A is torsion-complete with finite UK-invariants.
(8) Calculate the invariants of the algebraically compact group $\mathrm{Hom}(A, C)$ in case A is Σ-cyclic and $C = \oplus_\kappa \mathbb{Z}(p^\infty)$.
(9) Find the invariants of $\mathrm{Hom}(A, C)$ if A is torsion-free and $C = \oplus_\kappa \mathbb{Z}(p^\infty)$. [Hint: take p-basic in A.]
(10) Give a detailed proof of Theorem 2.13 for linearly compact groups.

3 Small Homomorphisms

This section should be read after getting familiar with the basic material from Chapter 10; in particular, with large subgroups to be discussed in Sect. 2 there.

Small Homomorphisms Let A, C be p-groups. Following Pierce [1], we call a homomorphism $\phi : A \to C$ **small** if $\mathrm{Ker}\,\phi$ contains a large subgroup of A. In other words, the Pierce condition (Sect. 2 in Chapter 10) must be satisfied: given $k > 0$, there exists $n > 0$ such that

$$p^n A[p^k] \leq \mathrm{Ker}\,\phi.$$

Example 3.1. The map η in the proof of Szele's theorem 6.10 in Chapter 6 is a small endomorphism of the p-group A; its image is a basic subgroup.

(A) *Elements of infinite height belong to the kernel of every small homomorphism,* since the first Ulm subgroup is contained in each large subgroup (see Sect. 2(D) in Chapter 10).
(B) *The small homomorphisms $\phi : A \to C$ form a subgroup of the group* $\mathrm{Hom}(A, C)$. Observe that if $p^{n_1}A[p^k] \leq \mathrm{Ker}\,\phi_1$ and $p^{n_2}A[p^k] \leq \mathrm{Ker}\,\phi_2$, then $p^n A[p^k] \leq \mathrm{Ker}(\phi_1 + \phi_2)$ holds with $n = \max\{n_1, n_2\}$. The group of small homomorphisms will be denoted by $\mathrm{Hom}_s(A, C)$.
(C) $\mathrm{Hom}_s(A, C) = \mathrm{Hom}(A, C)$ *whenever either A or C is bounded.* The latter holds as $p^m A$ is a large subgroup of A for each $m > 0$.

(D) *The factor group* $\mathrm{Hom}(A, C)/\mathrm{Hom}_s(A, C)$ *is torsion-free.* This follows from the fact that if $p^m\phi$ is a small homomorphism for some $\phi: A \to C$ and $m > 0$, then ϕ must be small, as well.

(E) *If* $\phi : A \to C$ *is a small homomorphism, then* $B + \mathrm{Ker}\,\phi = A$ *for any basic subgroup* B *of* A. This is a consequence of the definition of large subgroups.

Lemma 3.2 (Pierce [1]). *Let B be a basic subgroup of the p-group A. There is a natural isomorphism*

$$\mathrm{Hom}_s(A, C) \to \mathrm{Hom}_s(B, C)$$

given by the restriction map: $\phi \mapsto \phi \restriction B$ *where* $\phi: A \to C$.

Proof. If $\phi: A \to C$ is a small homomorphism, then (E) shows that, for every $a \in A$, the image ϕa is the same as ϕb if $a \equiv b \bmod \mathrm{Ker}\,\phi$ $(b \in B)$. Hence it is clear that $\phi \restriction B$ is different for distinct ϕ's.

Conversely, we show that if we are given a small homomorphism $\psi : B \to C$, then we can extend it to a small $\phi: A \to C$. By definition, there is a large subgroup $B(\underline{u})$ of B contained in $\mathrm{Ker}\,\psi$. Then $L = A(\underline{u})$ is a large subgroup of A such that $L \cap B = B(\underline{u})$ by the purity of B in A. Since $A/L = (L + B)/L \cong B/(L \cap B)$ and there is a natural homomorphism $B/(L \cap B) \to B/\mathrm{Ker}\,\psi$, we have the composite map $\phi : A \to A/L \to B/\mathrm{Ker}\,\psi$, which is evidently small and coincides with ψ on B. □

Hom$_s$ As a Summand Perhaps more interesting is that $\mathrm{Hom}_s(A, C)$ is a summand of $\mathrm{Hom}(A, C)$. This is demonstrated by the next theorem.

Theorem 3.3 (Pierce [1]). *For p-groups A, C,* $\mathrm{Hom}_s(A, C)$ *is a direct summand of* $\mathrm{Hom}(A, C)$*, complete in its p-adic topology. We have*

$$\mathrm{Hom}(A, C) \cong \tilde{F} \oplus \mathrm{Hom}_s(A, C)$$

where $\tilde{F}$ is the p-adic completion of a free group F.

Proof. To show that $\mathrm{Hom}_s(A, C)$ is complete, let $\phi_1, \dots, \phi_i, \dots$ be a neat Cauchy sequence in $\mathrm{Hom}_s(A, C)$. It is Cauchy also in $\mathrm{Hom}(A, C)$, thus, by the completeness of this Hom, it has a limit in $\mathrm{Hom}(A, C)$, which must be $\psi = \phi_1 + (\phi_2 - \phi_1) + \dots + (\phi_{i+1} - \phi_i) + \dots$. It remains to show that ψ is small. By the Cauchy property, $\phi_{i+1} - \phi_i = p^i\psi_i$ for some $\psi_i \in \mathrm{Hom}_s(A, C)$. Pick a $k \in \mathbb{N}$ and let $p^{n_i}A[p^k] \leq \mathrm{Ker}\,\psi_i$ for suitable $n_i \in \mathbb{N}$, for each i. Since $A[p^k] \leq \mathrm{Ker}\,p^k\psi_i \leq \mathrm{Ker}\,p^i\psi_i$ whenever $i \geq k$, if we choose $n = \max\{n_0, n_1, \dots, n_k, k\}$, then $p^nA[p^k] \leq \mathrm{Ker}(\phi_{i+1} - \phi_i)$ for all i, showing that ψ is a small homomorphism. This proves that $\mathrm{Hom}_s(A, C)$ is a complete group.

Since by (D) $\mathrm{Hom}(A, C)/\mathrm{Hom}_s(A, C)$ is torsion-free, $\mathrm{Hom}_s(A, C)$ is by algebraic compactness a summand of $\mathrm{Hom}(A, C)$. A complementary summand is p-adically complete (as a summand of $\mathrm{Hom}(A, C)$) and torsion-free, so it must be the p-adic completion of a free group. (The rank of F can be computed as λ^κ where κ (λ) denotes the final rank of the basic subgroup of A (resp. C) if these are infinite.) $\square$

The special case $A = C$ leads to the subgroup $\mathrm{End}_s(A)$ of $\mathrm{End}\,A$ consisting of the small endomorphisms of A. It is a two-sided ideal: that $\eta\phi$ is small for all $\eta \in \mathrm{End}\,A$ whenever $\phi \in \mathrm{End}_s A$ is pretty obvious. That the same holds for $\phi\eta$ too follows easily, see, e.g., Exercise 2. Thus $\mathrm{End}(A)/\mathrm{End}_s(A)$ is a torsion-free ring on an algebraically compact group.

Exact Sequence for Hom_s We now prove an analogue of Theorem 1.12 for Hom_s.

Proposition 3.4 (Pierce [1]). *Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be an exact sequence of p-groups. Then the induced sequence*

$$0 \to \mathrm{Hom}_s(G, A) \xrightarrow{\alpha_*} \mathrm{Hom}_s(G, B) \xrightarrow{\beta_*} \mathrm{Hom}_s(G, C)$$

is likewise exact for every group G. If the first sequence is pure-exact, then so is the induced sequence even if we append $\to 0$ *to the end.*

Proof. For the first part, the only non-obvious claim is $\mathrm{Im}\,\alpha_* \geq \mathrm{Ker}\,\beta_*$. If $\eta \in \mathrm{Hom}_s(G, B)$ satisfies $\beta\eta = 0$, then $\beta(\eta G) = 0$, so $\eta G \leq \mathrm{Im}\,\alpha$. Hence there is a $\xi : G \to A$ such that $\alpha\xi = \eta$. Since evidently $\mathrm{Ker}\,\xi = \mathrm{Ker}\,\eta$ (α being monic), we have $\xi \in \mathrm{Hom}_s(G, A)$, as desired.

Assume the given sequence is pure-exact, and $\theta \in \mathrm{Hom}_s(G, C)$. If H denotes a basic subgroup of G, then by Theorem 4.3 in Chapter 5 there is a homomorphism $\gamma : H \to B$ such that $\beta\gamma = \theta \restriction H$. It is readily checked that the map $\phi : G = H + \mathrm{Ker}\,\theta \to B$ is well defined if we apply γ to H and send $\mathrm{Ker}\,\theta$ to 0. Furthermore, ϕ is small, and $\beta_*(\phi) = \theta$, so β_* is surjective. To verify purity, let $\phi \in \mathrm{Hom}_s(G, B)$ satisfy $p^k\phi \in \alpha(\mathrm{Hom}_s(G, A))$. Then by Proposition 1.13 there is a $\psi : G \to A$ satisfying $\alpha(p^k\psi) = p^k\phi$. From the equality of the kernels we derive that $p^k\psi$ is a small homomorphism. Hence ψ is small as well, and the proof is complete. $\square$

Note that we do not claim that the sequence for $\mathrm{Hom}_s(*, G)$ is exact. As Pierce [1] points out, in contrast to Proposition 3.4, this is not true in general.

★ **Notes.** Megibben [2] shows that an unbounded torsion-complete p-group has a non-small homomorphism into a separable p-group C if and only if C has an unbounded torsion-complete subgroup. A result by Monk [2] states that the finite direct decompositions of $\mathrm{End}\,A/\,\mathrm{End}_s\,A$ are induced by those of $\mathrm{End}\,A$, so that they correspond to certain direct decompositions of A.

The concept of small homomorphism has been extended to the torsion-free and mixed cases by Corner–Göbel [1]. The general version, called **inessential homomorphism**, is based on the ideal $\mathrm{Ines}\,A$ of $\mathrm{End}\,A$; this is the set of all kinds of endomorphisms that are always present in groups (like those with finite rank images). Interested readers are advised to consult this interesting paper.

Exercises

(1) (Pierce) Every small homomorphism $\phi: p^k A \to p^k C$ can be extended to a small homomorphism $A \to C$.
(2) (Pierce) (a) $\phi: A \to C$ is small if and only if for every $k > 0$ there is an $n > 0$ such that $o(a) \leq p^k$ and $h(a) \geq n$ imply $\phi(a) = 0$.
(b) ϕ is small if and only if, for every $k > 0$, there is an $n > 0$ such that $p^n a \neq 0$ $(a \in A)$ implies $o(\phi a) \leq o(p^k a)$.
(3) (Pierce) Prove that $\mathrm{Hom}_s(A_1 \oplus A_2, C) \cong \mathrm{Hom}_s(A_1, C) \oplus \mathrm{Hom}_s(A_2, C)$ and $\mathrm{Hom}_s(A, C_1 \oplus C_2) \cong \mathrm{Hom}_s(A, C_1) \oplus \mathrm{Hom}_s(A, C_2)$.
(4) (Pierce) If G is a pure subgroup of the p-group A, then every small homomorphism $\phi: G \to C$ extends to a small homomorphism $A \to C$.
(5) (Pierce) Let A, C be arbitrary p-groups. $\mathrm{Hom}_s(A, C) = \mathrm{Hom}(A, C)$ if either (a) A has bounded basic subgroups and C is reduced, or (b) C is bounded.
(6) The composite of two small homomorphisms is small.
(7) Suppose A, C are p-groups. $\phi: A \to C$ is small exactly if $\phi(A^1) = 0$ and the induced map $A/A^1 \to C$ is small.
(8) Let A be a p-group with unbounded basic subgroup B. Prove Szele's theorem 6.10 in Chapter 5 by first mapping B onto itself by a small endomorphism, and then applying Lemma 3.2.
(9) Let A be a p-group and B a basic subgroup of A. Then, for every p-group C, the torsion subgroups of $\mathrm{Hom}(A, C)$ and $\mathrm{Hom}(B, C)$ are isomorphic. [Hint: the elements in these Homs are small homomorphisms.]

Problems to Chapter 7

PROBLEM 7.1. Can the groups $\mathrm{Hom}(M, \mathbb{Z})$ be characterized for monotone subgroups M of $\mathbb{Z}^{\aleph_0}$?

See Sect. 2 in Chapter 13 for monotone subgroups.

PROBLEM 7.2. Call $\phi \in \mathrm{Hom}(A, B)$ a *right universal homomorphism* for $A \to B$ if every $\psi \in \mathrm{Hom}(A, B)$ factors uniquely as $\psi = \eta\phi$ with $\eta \in \mathrm{End}\,B$. It is *left universal* if $\psi = \phi\chi$ with unique $\chi \in \mathrm{End}\,A$. Study the cases when uniqueness is not required, so Hom is singly generated (on the right or on the left) over End.

Right universal homomorphisms, called *localizations,* were discussed by Dugas [3] for torsion-free groups. Left universal homomorphisms were completely described by Chachólski–Farjoun–Göbel–Segev [1] for divisible groups B under the name of *cellular cover*, and for arbitrary abelian groups by Fuchs–Göbel [2]. See also Dugas [4].

Chapter 8
Tensor and Torsion Products

Abstract The tensor product of groups is one of the most important concepts and indispensable tools in the theory of abelian groups. They compete in importance with homomorphism groups, but their features are totally different.

Tensor products can be introduced in various ways. We define them *via* generators and defining relations, and then we show that they have the universal property for bilinear maps. Tensoring is a bifunctor that is right exact in both arguments. The exact sequence of tensor products is a most useful asset, both as a tool in proofs and as a device in discovering new facts. Exactness on the left can be restored by introducing the functor Tor, the torsion product, that is of independent interest.

If one of the groups is a torsion group, then the tensor product can be completely described by invariants. In particular, the tensor product of two torsion groups is always a direct sum of cyclic groups. The torsion product behaves differently, it raises more challenging problems. The tensor product of torsion-free groups is a difficult subject.

Various facts concerning groups that were proved originally in an *ad hoc* fashion may be verified more clearly, and perhaps more elegantly, by using homological methods, in particular, the long exact sequences connecting the tensor and torsion products (as well as Hom and Ext).

1 The Tensor Product

Bilinear Functions and the Tensor Product Suppose A and C are arbitrary groups, and g is a function defined on the set $A \times C$ with values in a group G, $g : A \times C \to G$. We say that g is a **bilinear function** if it satisfies

$$g(a_1 + a_2, c) = g(a_1, c) + g(a_2, c) \tag{8.1}$$

and

$$g(a, c_1 + c_2) = g(a, c_1) + g(a, c_2) \tag{8.2}$$

for all $a, a_1, a_2 \in A, c, c_1, c_2 \in C$. It follows at once that a bilinear function obeys the following simple rules: $g(a, 0) = 0 = g(0, c), g(na, c) = ng(a, c) = g(a, nc)$ for all $a \in A, c \in C$, and $n \in \mathbb{Z}$.

We are going to define a group (that will be denoted by $A \otimes C$) as well as a bilinear function $e: A \times C \to A \otimes C$, such that e is the most general bilinear function in the sense that any bilinear function $g : A \times C \to G$ for any group G factors

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uniquely through it: there is a unique group homomorphism $\phi : A \otimes C \to G$ such that $g = \phi e$. This is the **universal property** of the tensor product.

Define X as the free group on the set $A \times C$; i.e. the free generators of X are the pairs (a, c) with $a \in A, c \in C$. Let Y denote the subgroup of X generated by the elements of the form

$$(a_1 + a_2, c) - (a_1, c) - (a_2, c) \quad \text{and} \quad (a, c_1 + c_2) - (a, c_1) - (a, c_2) \tag{8.3}$$

for all $a, a_1, a_2 \in A, c, c_1, c_2 \in C$. The **tensor product** of A and C is defined as

$$A \otimes C = X/Y.$$

If we write the coset $(a, c) + Y$ as $a \otimes c$, then a typical element of $A \otimes C$ will be written as a finite sum $u = \sum_i k_i(a_i \otimes c_i)$ with $a_i \in A, c_i \in C, k_i \in \mathbb{Z}$, where the elements are subject to the rules

$$(a_1 + a_2) \otimes c = a_1 \otimes c + a_2 \otimes c \quad \text{and} \quad a \otimes (c_1 + c_2) = a \otimes c_1 + a \otimes c_2. \tag{8.4}$$

We also observe that $na \otimes c = n(a \otimes c) = a \otimes nc$ for all $a \in A, c \in C, n \in \mathbb{Z}$. As a consequence, the elements of $A \otimes C$ can be written in a simplified form as $u = \sum_i (a_i \otimes c_i)$ where $a_i \in A, c_i \in C$. However, it should be emphasized that ordinarily u has many expressions of this form.

The notation $a \otimes c$ is in principle ambiguous: it has a different meaning when it is considered as an element of $A \otimes C$ or as an element of $A' \otimes C'$ for subgroups $A' \leq A, C' \leq C$ containing a and c, respectively (e.g., it can be 0 in $A \otimes C$, but not in $A' \otimes C'$; see (H)). Therefore, one always has to specify to which tensor product the element $a \otimes c$ belongs, unless its membership is clear from the context.

From the definition it is clear that $e \colon (a, c) \mapsto a \otimes c$ is a bilinear map of $A \times C$ into $A \otimes C$. If $g \colon A \times C \to G$ is an arbitrary bilinear map, then the correspondence $a \otimes c \mapsto g(a, c)$ extends to a homomorphism $\phi \colon A \otimes C \to G$, since the generators of $A \otimes C$ are subject only to the relations (8.3) (and to their consequences) that hold for the respective images in G. Furthermore, it is obvious that no other map ϕ can satisfy $g = \phi e$, i.e. given g, there is a unique ϕ making the triangle

$$\begin{tikzcd}
 & A \times C \arrow[dl, "e"'] \arrow[dr, "g"] & \\
A \otimes C \arrow[rr, dashed, "\phi"] & & G
\end{tikzcd}$$

commute. We can now state:

Theorem 1.1. *Given a pair A, C of groups, there exist a group $A \otimes C$ and a bilinear function*

$$e \colon A \times C \to A \otimes C$$

with the following property: if $g\colon A \times C \to G$ is a bilinear function into any group G, then there is a unique homomorphism $\phi\colon A \otimes C \to G$ making the above triangle commute. This property determines $A \otimes C$ up to isomorphism.

Proof. It remains to prove the last statement. Assume there are a group B, and a bilinear function $f : A \times C \to B$ with the same properties as $A \otimes C$ and e. By the first part of the theorem, there is a unique homomorphism $\phi : A \otimes C \to B$ with $e = \phi f$, and by the hypothesis on B, there is a unique $\psi\colon B \to A \otimes C$ with $f = \psi e$. Hence $e = \psi\phi e, f = \phi\psi f$, and in view of the uniqueness property, we conclude that $\psi\phi = \mathbf{1}_{A\otimes C}$ and $\phi\psi = \mathbf{1}_B$, so $A \otimes C \cong B$, as claimed. □

As mentioned above, $A \otimes C$ is called the tensor product of A and C, and we will refer to the map $e : A \times C \to A \otimes C$ as the **tensor map**. From the symmetry of the roles of A and C in the definition it follows right away that the correspondence $a \otimes c \mapsto c \otimes a$ induces a natural isomorphism

$$A \otimes C \cong C \otimes A.$$

It is easy to check, for arbitrary groups A, B, C, the associative law

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

either by the universal property or by the equality of generators.

Basic Properties Before stating some relevant properties of tensor products, we record a simple lemma that is instrumental in exploring tensor products.

Lemma 1.2. *Let $a \in A, c \in C$.*

(a) *If $m, n \in \mathbb{Z}$ such that $ma = 0 = nc$, then $d(a \otimes c) = 0$ where $d = \gcd\{m, n\}$;*
(b) *if $m|a$ and $mc = 0$, then $a \otimes c = 0$;*
(c) *if $m|a$ and $n|c$, then $mn|(a \otimes c)$.*

Proof. (a) If $s, t \in \mathbb{Z}$ such that $sm + tn = d$, then $d(a \otimes c) = sma \otimes c + tna \otimes c = sma \otimes c + a \otimes tnc = 0$.
(b) If $a = ma'$ $(a' \in A)$, then $a \otimes c = ma' \otimes c = a' \otimes mc = 0$.
(c) If also $c = nc'$ $(c' \in C)$, then $a \otimes c = mn(a' \otimes c')$. □

Here is a short list of the most useful properties of tensor products. (Note how important an ingredient the universal property is in the proofs of (A)-(B).) The proofs of (C)-(H) are straightforward, and will be left to the reader.

(A) *There is a natural isomorphism $\mathbb{Z} \otimes C \cong C$ for every group C.*
The elements of $\mathbb{Z} \otimes C$ may be brought to the simpler form $\sum_i (n_i \otimes c_i) = \sum_i (1 \otimes n_i c_i) = 1 \otimes c$ for $c = \sum_i n_i c_i \in C$. Thus the map $\phi\colon c \mapsto 1 \otimes c$ is an epimorphism $C \to \mathbb{Z} \otimes C$. Clearly, $(m, c) \mapsto mc$ is a bilinear map $\mathbb{Z} \times C \to C$, so the universal property implies that there is a unique $\psi : \mathbb{Z} \otimes C \to C$ acting as $\psi(1 \otimes c) = c$. Consequently, ϕ and ψ are inverse to each other.
(B) *There is a natural isomorphism $\mathbb{Z}(m) \otimes C \cong C/mC$ for every integer $m > 0$ and group C.*

Here again, $\phi : c \mapsto \bar{1} \otimes c$ is an epimorphism $C \to \mathbb{Z}(m) \otimes C$ where $\bar{1} = 1 + m\mathbb{Z}$. We have $mC \leq \operatorname{Ker} \phi$, since $1 \otimes mc = m \otimes c = 0 \otimes c = 0$. Now $(n, c) \mapsto nc + mC$ is a bilinear map $\mathbb{Z}(m) \times C \to C/mC$, so by the universal property, there is a homomorphism $\psi : \mathbb{Z}(m) \otimes C \to C/mC$ such that $\psi\phi$ is the canonical map $C \to C/mC$. Thus $\operatorname{Ker} \phi = mC$.

(C) *The heights satisfy* $h_p(a \otimes c) \geq h_p(a) + h_p(c)$ $(a \in A, c \in C)$. Thus if either A or C is p-divisible (resp. divisible), then so is $A \otimes C$.

(D) *If* $h_p(a) \geq \omega$ *for* $a \in A$ *and* C *is a* p*-group, then* $a \otimes c = 0$ *for every* $c \in C$. Thus if A is p-divisible and C is a p-group, then $A \otimes C = 0$.

(E) *If* $a \in mA$ *and* $c \in C[m]$ *for some* $m \in \mathbb{Z}$, *then* $a \otimes c = 0$.

(F) *If either* A *or* C *is a* p*-group* (*torsion group*), *then so is* $A \otimes C$.

(G) *If* A *is a* p*-group and* C *is a* q*-group for different primes* p, q, *then* $A \otimes C = 0$.

Example 1.3. We have the following natural isomorphisms: $\mathbb{Z} \otimes \mathbb{Z}(n) \cong \mathbb{Z}(n)$, $\mathbb{Z}(p^n) \otimes \mathbb{Z}(p^m) \cong \mathbb{Z}(p^k)$ with $k = \min\{m, n\}$, and $\mathbb{Z}(n) \otimes \mathbb{Z}(m) \cong \mathbb{Z}(d)$ where $d = \gcd\{m, n\}$.

Example 1.4. Suppose A is a rational group. Then every $x \in A \otimes C$ can be written in the form $x = a \otimes c$ with some $a \in A, c \in C$. In fact, as always, $x = \sum_{i=1}^{n} (a_i \otimes c_i)$ holds with $a_i \in A, c_i \in C$, but in the present case A is locally cyclic, i.e. there exists an $a \in A$ such that each a_i is an integral multiple of a, say $a_i = m_i a$ $(m_i \in \mathbb{Z})$. Thus

$$x = \sum (m_i a \otimes c_i) = \sum (a \otimes m_i c_i) = a \otimes (\sum m_i c_i)$$

where $\sum m_i c_i = c \in C$. This form is not unique: if $a = ma'$, then also $x = a' \otimes mc$.

(H) It is very important to keep in mind that *if* B *is a subgroup of* A, *then* $B \otimes C$ *need not be a subgroup of* $A \otimes C$. For instance, $\mathbb{Z} < \mathbb{Q}$, but $\mathbb{Z} \otimes \mathbb{Z}(p) \cong \mathbb{Z}(p)$ is not a subgroup of $\mathbb{Q} \otimes \mathbb{Z}(p) = 0$; or, $\mathbb{Z}(p) < \mathbb{Z}(p^\infty)$, but $\mathbb{Z}(p) \otimes \mathbb{Z}(p) \cong \mathbb{Z}(p)$ is not a subgroup of $\mathbb{Z}(p^\infty) \otimes \mathbb{Z}(p) = 0$. (We will see later that this cannot happen if the containment is pure.)

Tensor Product As Bifunctor Let $\alpha : A \to A'$, $\gamma : C \to C'$ be homomorphisms. Clearly, $(a, c) \mapsto \alpha a \otimes \gamma c$ is a bilinear map $A \times C \to A' \otimes C'$, hence there is a unique homomorphism $\phi : A \otimes C \to A' \otimes C'$ such that $\phi(a \otimes c) = \alpha a \otimes \gamma c$. This map will be denoted as $\phi = \alpha \otimes \gamma$, thus

$$(\alpha \otimes \gamma)(a \otimes c) = \alpha a \otimes \gamma c \qquad (a \otimes c \in A \otimes C, \ \alpha a \otimes \gamma c \in A' \otimes C').$$

To avoid complicated notations, we will often write simply

$$\alpha_* = \alpha \otimes \mathbf{1}_C, \qquad \gamma_* = \mathbf{1}_A \otimes \gamma$$

whenever there is no danger of confusion. We also observe:

$$(\alpha \otimes \gamma)(\alpha' \otimes \gamma') = \alpha\alpha' \otimes \gamma\gamma',$$

$$(\alpha_1 + \alpha_2) \otimes \gamma = (\alpha_1 \otimes \gamma) + (\alpha_2 \otimes \gamma), \ \alpha \otimes (\gamma_1 + \gamma_2) = (\alpha \otimes \gamma_1) + (\alpha \otimes \gamma_2)$$

for matching homomorphisms $\alpha, \alpha_1, \alpha_2, \gamma, \gamma_1, \gamma_2$. Hence we can state:

Theorem 1.5 (Cartan–Eilenberg [CE]). *The tensor product is an additive bifunctor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$, covariant in both variables.* $\square$

In the next lemma we prove an important property: tensor products commute with direct limits, and, in particular, with direct sums.

Lemma 1.6. *The tensor product commutes with direct limits.*

Proof. Let $\mathfrak{A} = \{A_i \ (i \in I); \pi_i^j\}$ be a direct system of groups with direct limit A and canonical maps $\pi_i : A_i \to A$. Then

$$\mathfrak{C} = \{A_i \otimes C \ (i \in I);\ \pi_i^j \otimes \mathbf{1}_C\}$$

is also a direct system; let H denote its direct limit, and $\rho_i : A_i \otimes C \to H$ the canonical maps. The homomorphisms $\pi_i \otimes \mathbf{1}_C : A_i \otimes C \to A \otimes C$ satisfy $\pi_i \otimes \mathbf{1}_C = (\pi_j \otimes \mathbf{1}_C)(\pi_i^j \otimes \mathbf{1}_C)$ for $i \leq j$. By Theorem 1.1, there is a unique $\sigma : H \to A \otimes C$ such that $\sigma\rho_i = \pi_i \otimes \mathbf{1}_C$ for each i. Our claim is that σ is an isomorphism. Given $a \in A$, write $a = \pi_i a_i$ for some $a_i \in A_i$, and observe that $g : (a, c) \to \rho_i(a_i \otimes c)$ is a well defined bilinear map $A \times C \to H$ (i.e., it does not depend on the choice of i and a_i). Therefore, there is a homomorphism $\phi : A \otimes C \to H$ satisfying $\phi(a \otimes c) = g(a, c)$. Now $a \otimes c = (\pi_i \otimes \mathbf{1}_C)(a_i \otimes c) = \sigma\rho_i(a_i \otimes c) = \sigma g(a, c)$ shows that σ and ϕ are inverse on the generators of $A \otimes C$ and H. $\square$

Corollary 1.7. (i) *For all groups A_i, C_j we have*

$$(\oplus_{i \in I} A_i) \otimes (\oplus_{j \in J} C_j) \cong \oplus_{i \in I} \oplus_{j \in J} (A_i \otimes C_j)$$

(ii) *In case $A = \oplus_p A_p$ is a torsion group with p-components A_p, then $A \otimes C = \oplus_p (A_p \otimes C)$ for every C. If C is also a torsion group, $C = \oplus_p C_p$, then $A \otimes C = \oplus_p (A_p \otimes C_p)$.* $\square$

Tensor Product and Exact Sequences We have come to study the behavior of the tensor product towards short exact sequences. In view of the symmetry, we may confine ourselves to tensoring from one side only. The next theorem shows that $\otimes$ is a right exact functor.

Theorem 1.8 (Cartan–Eilenberg [CE]). *If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence of groups, then so is the sequence*

$$A \otimes G \xrightarrow{\alpha_*} B \otimes G \xrightarrow{\beta_*} C \otimes G \to 0$$

for every group G.

Proof. Since $\beta\alpha = 0$, we have $\beta_*\alpha_* = (\beta\alpha)_* = 0$. Thus we need only prove that the homomorphism $\phi : H = (B \otimes G)/\operatorname{Im}\alpha_* \to C \otimes G$ induced by β_* is an isomorphism. Given $c \in C$, choose a $b \in B$ such that $\beta b = c$. The mapping

$(c, g) \mapsto (b \otimes g) + \operatorname{Im} \alpha_* \in H$ is clearly well-defined and bilinear, so there is a homomorphism $\psi : C \otimes G \to H$ such that $\psi(c \otimes g) = (b \otimes g) + \operatorname{Im} \alpha_*$. It is now clear that ϕ and ψ are inverse to each other on the generators. □

Example 1.9. To show that in Theorem 1.8 the tensored sequence cannot be extended with $0 \to$ on the left, consider the inclusion map $\mathbb{Z}(p) \to \mathbb{Z}(p^\infty)$. If it is tensored with $\mathbb{Z}(p)$, then the induced map $\mathbb{Z}(p) \otimes \mathbb{Z}(p) = \mathbb{Z}(p) \to \mathbb{Z}(p^\infty) \otimes \mathbb{Z}(p) = 0$ is no longer monic. (See also (H).)

Example 1.10. We show that $A = J_p \otimes J_p$ contains an uncountable divisible torsion-free subgroup. The exact sequence $0 \to \mathbb{Z}_{(p)} \to J_p \to \mathbb{Q}^{(2^{\aleph_0})} \to 0$ induces the exact sequence $0 \to \mathbb{Z}_{(p)} \otimes J_p \to J_p \otimes J_p \to \mathbb{Q}^{(2^{\aleph_0})} \otimes J_p \to 0$ (Theorem 3.5 will show that tensoring with a torsion-free group preserves exactness). From the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}_{(p)} \to \oplus_{q \neq p} \mathbb{Z}(q^\infty) \to 0$ it follows that $\mathbb{Z}_{(p)} \otimes J_p \cong J_p$, so algebraically compact, while $\mathbb{Q}^{(2^{\aleph_0})} \otimes J_p$ is a large torsion-free divisible group. Hence the tensored sequence splits, and the claim follows: $J_p \otimes J_p \cong J_p \oplus \mathbb{Q}^{(2^{\aleph_0})}$.

A repeated applications of the last theorem shows that if $\beta : B \to C$ and $\beta' : B' \to C'$ are epimorphisms, then so is $\beta \otimes \beta' : B \otimes B' \to C \otimes C'$. With a somewhat greater effort one can prove a little bit more:

Corollary 1.11. *Let* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *and* $A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C' \to 0$ *be exact sequences. Then the sequence*

$$(A \otimes B') \oplus (B \otimes A') \xrightarrow{\xi} B \otimes B' \xrightarrow{\beta \otimes \beta'} C \otimes C' \to 0$$

is exact where $\xi = \nabla[(\alpha \otimes \mathbf{1}_{B'}) \oplus (\mathbf{1}_B \otimes \alpha')]$.

Proof. Consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & B \otimes A' \arrow[r] \arrow[d, "\mathbf{1}_B \otimes \alpha'"'] & C \otimes A' \arrow[d, "(\mathbf{1}_C \otimes \alpha')"] & \\
A \otimes B' \arrow[r, "\alpha \otimes \mathbf{1}_{B'}"] \arrow[d, "\mathbf{1}_A \otimes \beta'"'] & B \otimes B' \arrow[r, "\beta \otimes \mathbf{1}_{B'}"] \arrow[d, "\mathbf{1}_B \otimes \beta'"'] & C \otimes B' \arrow[r] \arrow[d, "\mathbf{1}_C \otimes \beta'"] & 0 \\
A \otimes C' \arrow[r, "\alpha \otimes \mathbf{1}_{C'}"] \arrow[d] & B \otimes C' \arrow[r, "\beta \otimes \mathbf{1}_{C'}"] \arrow[d] & C \otimes C' \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Hence it is clear that the map $\beta \otimes \beta' = (\mathbf{1}_C \otimes \beta')(\beta \otimes \mathbf{1}_{B'})$ is surjective, and it is easy to verify by simple diagram chasing that its kernel is the union of $\operatorname{Ker}(\beta \otimes \mathbf{1}_{B'})$ and $\operatorname{Ker}(\mathbf{1}_B \otimes \beta')$. □

Vanishing Elements in Tensor Products An immediate corollary to Lemma 1.6 is that *if* $\sum_{i=1}^n (a_i \otimes c_i)$ *vanishes in the tensor product* $A \otimes C$*, then there are* *finitely generated subgroups $A' \leq A, C' \leq C$ with $a_i \in A', c_i \in C'$ such that $\sum_{i=1}^{n}(a_i \otimes c_i) = 0$ as an element of $A' \otimes C'$.* Actually, we can obtain more detailed information about what forces an element of a tensor product to become 0.

Lemma 1.12. *In the tensor product $A \otimes C$ of the groups A, C, a relation*

$$\sum_{i=1}^{n}(a_i \otimes c_i) = 0 \qquad (a_i \in A,\ c_i \in C)$$

holds if and only if there exist elements $b_j \in C$ $(j = 1, \dots, m)$ as well as integers $r_{ij} \in \mathbb{Z}$ $(i = 1, \dots, n;\ j = 1, \dots, m)$ such that

$$c_i = \sum_{j=1}^{m} r_{ij} b_j \quad \textit{for all } i \qquad \textit{and} \qquad \sum_{i=1}^{n} r_{ij} a_i = 0 \quad \textit{for all } j.$$

Proof. Sufficiency follows from a simple computation:

$$\sum_{i=1}^{n}(a_i \otimes c_i) = \sum_{i=1}^{n}[a_i \otimes (\sum_{j=1}^{m} r_{ij} b_j)] = \sum_{j=1}^{m}[(\sum_{i=1}^{n} r_{ij} a_i) \otimes b_j] = 0.$$

For the converse, assume $\sum_{i=1}^{n}(a_i \otimes c_i) = 0$. Let $F = \oplus_{k \in I} \langle x_k \rangle$ be a free group, and $\phi \colon F \to A$ an epimorphism such that $x_i \mapsto a_i$ for $i = 1, \dots, n$ (we assume that these indices are in I). Hypothesis implies that the element $\sum_{i=1}^{n}(x_i \otimes c_i)$ in the second term of the exact sequence $\operatorname{Ker}\phi \otimes C \to F \otimes C \to A \otimes C \to 0$ belongs to the image of the first map. Hence there exist elements $y_j \in \operatorname{Ker}\phi$ and $b_j \in C$ such that $\sum_{i=1}^{n}(x_i \otimes c_i) = \sum_{j=1}^{m}(y_j \otimes b_j)$. Writing $y_j = \sum_k r_{kj} x_k$ $(r_{kj} \in \mathbb{Z})$, we get

$$\sum_{i=1}^{n}(x_i \otimes c_i) = \sum_{k}\sum_{j=1}^{m}(x_k r_{kj} \otimes b_j) = \sum_{k}\sum_{j=1}^{m}(x_k \otimes r_{kj} b_j).$$

Since the x_i are free generators, hence we obtain $c_i = \sum_{j=1}^{m} r_{ij} b_j$ and $\sum_{i=1}^{n} r_{ij} a_i = \phi y_j = 0$. □

For later application we point out that the elements b_j in the proof may be chosen in any preassigned pure subgroup C' of C that contains the c_i. In fact, it will follow from Theorem 3.1 that $A \otimes C' \to A \otimes C$ is a monic map, so if $\sum(a_i \otimes c_i)$ vanishes in $A \otimes C$, then it must be 0 in $A \otimes C'$.

A Natural Isomorphism A most important natural isomorphism connecting Hom with the tensor product is the content of the following theorem.

Theorem 1.13. *For any three groups A, B, C, there is a natural isomorphism*

$$\phi : \operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(A, \operatorname{Hom}(B, C)),$$

acting as follows: if $\eta\colon A\otimes B\to C$, *then*

$$[(\phi\eta)a](b)=\eta(a\otimes b)\qquad \textit{for all } a\in A, b\in B.$$

Proof. Clearly, $\phi(\eta)$ assigns to $a\in A$ a map $(\phi\eta)a: B\to C$ that sends $b\in B$ to $\eta(a\otimes b)\in C$. The homomorphism properties of $(\phi\eta)a$ are rather obvious. It is also clear that the correspondence $\phi:\eta\mapsto(\phi\eta)a$ is a homomorphism, so it remains to find an inverse to ϕ. Pick a $\chi\colon A\to \mathrm{Hom}(B,C)$, and observe that $(a,b)\mapsto(\chi a)(b)$ is a bilinear function $g: A\times B\to C$. Hence there exists a unique $\eta: A\otimes B\to C$ such that $\eta(a\otimes b)=(\chi a)(b)$. The mapping $\psi:\chi\mapsto\eta$ is readily seen to be inverse to ϕ. □

★ **Notes.** Tensor products for groups were introduced by H. Whitney. It is one of the fundamental functors in Homological Algebra and in Category Theory. Tensor products $A\otimes_{\mathsf{R}} C$ for modules over a ring R are defined when A is a right and C is a left R-module, and in forming Y, the above generators are complemented with those of the form $(ar,c)-(a,rc)$ for all $a\in A, c\in C, r\in\mathsf{R}$. Theorem 1.13 is an important result: it claims that the tensor product and the Hom are adjoint functors.

Exercises

(1) The bilinear functions $A\times C\to G$ form a group under addition. This group is $\cong \mathrm{Hom}(A\otimes C,G)$.

(2) There is a (non-natural) isomorphism $A\otimes C\cong\mathrm{Hom}(A,C)$ for finite groups A,C.

(3) Characterize the groups C that satisfy in turn the following conditions:

  (a) $A\otimes C\cong A$ for every group A;
  (b) $A\otimes C\cong A/mA$ with a fixed integer m, for every A;
  (c) $A\otimes C$ is a divisible group for every A;
  (d) $A\otimes C=0$ for every torsion group A;
  (e) $A\otimes C=0$ for every p-group A;
  (f) $A\otimes C$ is a torsion group for every A.

(4) Describe the structure of the following tensor products: (a) $J_p\otimes\mathbb{Q}$; (b) $J_p\otimes J_q$ for primes $p\neq q$; (c) $J_p\otimes C$ for a torsion group C.

(5) (a) If $a\in A, c\in C$ are elements of infinite order, then the same holds for $a\otimes c$ in $A\otimes C$.
  (b) Let $\{a_i\}_{i\in I}$ and $\{c_j\}_{j\in J}$ be maximal independent sets in the torsion-free groups A and C, respectively. Show that $\{a_i\otimes c_j\}_{i\in I, j\in J}$ is a maximal independent set in $A\otimes C$.
  (c) We have $\mathrm{rk}_0(A\otimes C)=\mathrm{rk}_0(A)\cdot\mathrm{rk}_0(C)$ for any groups A,C.

(6) (a) If A is torsion-free, then $\mathbb{Q} \otimes A$ is naturally isomorphic to the divisible hull of A; a natural embedding being given by $a \mapsto 1 \otimes a$.
(b) Claim (a) is false if A has elements $\neq 0$ of finite order.

(7) Assume that the element $a \in A$ is such that $a \otimes c = 0$ in $A \otimes C$ for all torsion groups C and for all $c \in C$. Prove that $a \in A^1$.

(8) Define multilinear functions $g\colon A_1 \times \cdots \times A_n \to G$, the corresponding tensor product $A_1 \otimes \cdots \otimes A_n$, and the tensor map e for multilinear functions. Prove the universal property and the associativity relations

$$(A_1 \otimes A_2) \otimes A_3 \cong A_1 \otimes A_2 \otimes A_3 \cong A_1 \otimes (A_2 \otimes A_3).$$

(9) There is a natural homomorphism

$$A \otimes \prod_i C_i \to \prod_i (A \otimes C_i)$$

which in general fails to be an isomorphism. [Hint: $A = \mathbb{Q}, C_i$ unbounded torsion groups. See Mittag-Leffler groups in Sect. 8 in Chapter 3.]

(10) If $\alpha \in \operatorname{End} A$, $\gamma \in \operatorname{End} C$, then $\alpha \otimes \mathbf{1}_C$ and $\mathbf{1}_A \otimes \gamma$ are commuting endomorphisms of $A \otimes C$.

(11) (a) Assume A and C are torsion-free groups, and $p^t | a \otimes c$ holds for some $a \in A, c \in C$, prime p, and integer $t > 0$. Show that there are integers $r, s \geq 0$ such that $p^r | a, p^s | c$ with $r + s = t$.
(b) Using the notion of type of torsion-free groups to be introduced in Sect. 1 in Chapter 12, find the type of $A \otimes C$ if A, C are rank one torsion-free groups.

(12) (Corner) (a) If $A \otimes C$ is p-divisible, then one of A, C is p-divisible. [Hint: argue with p-basic subgroups.]
(b) Let A be a torsion group. $A \otimes C = 0$ if and only if (i) $pA \neq A$ implies $p(tC) = tC$, and (ii) $A_p \neq 0$ implies $p(C/tC) = C/tC$.

(13) (Head) Let B be a subgroup of a p-group A, and C a reduced group. The canonical map $B \otimes C \to A \otimes C$ is monic exactly if $B \cap p^n A = p^n B$ holds whenever C has a cyclic summand of order p^n.

2 The Torsion Product

In Theorem 1.8 it was proved that the tensor product is right exact, and Example 1.9 shows that it is not left exact. Now the question arises whether or not it is possible to salvage some sort of exactness by introducing another functor. It is our next goal to define a bifunctor that leads to a long exact sequence continuing the tensor sequence to the left.

Torsion Products The **torsion product** $\operatorname{Tor}(A, C)$ of the groups A, C is defined as the group generated by the triples (a, m, c) with $a \in A, c \in C, m \in \mathbb{N}$ such that $ma = 0 = mc$, subject to the following relations:

$$(a_1 + a_2, m, c) = (a_1, m, c) + (a_2, m, c) \quad \text{where} \quad ma_1 = ma_2 = mc = 0,$$
$$(a, m, c_1 + c_2) = (a, m, c_1) + (a, m, c_2) \quad \text{where} \quad ma = mc_1 = mc_2 = 0,$$
$$(a, mn, c) = (na, m, c) \quad \text{where} \quad mna = mc = 0,$$
$$(a, mn, c) = (a, m, nc) \quad \text{where} \quad ma = mnc = 0$$

for all $a, a_1, a_2 \in A,\ c, c_1, c_2 \in C,\ m, n \in \mathbb{N}$. (Thus the relations are assumed to hold whenever the right-hand sides are defined.) In view of the apparent symmetry, we have

$$\mathrm{Tor}(A, C) \cong \mathrm{Tor}(C, A) \qquad \text{for all } A, C,$$

this being a natural isomorphism *via* $(a, m, c) \leftrightarrow (c, m, a)$.

The first two relations above imply $(0, m, c) = 0 = (a, m, 0)$; furthermore, $(na, m, c) = n(a, m, c) = (a, m, nc)$ for each $n \in \mathbb{Z}$ whenever the symbols are meaningful. Thus every element x of $\mathrm{Tor}(A, C)$ is a finite sum $x = \sum_i (a_i, m_i, c_i)$ with $m_i a_i = 0 = m_i c_i$ ($a_i \in A, c_i \in C, m_i \in \mathbb{N}$), so there exist finite subgroups $A' \le A$, $C' \le C$ such that $x \in \mathrm{Tor}(A', C') \le \mathrm{Tor}(A, C)$. Note that $m(a, m, c) = 0$, so Tor is always a torsion group (as its name indicates).

We list a number of elementary properties of Tor.

(A) *There is a natural isomorphism* $\mathrm{Tor}(A, C) \cong \mathrm{Tor}(t(A), t(C))$. This follows from the fact that in the definition of Tor, only elements of finite order are involved. In particular, $\mathrm{Tor}(A, C) = 0$ if either A or C is torsion-free.
(B) *If $nA = 0$ for some $n \in \mathbb{Z}$, then $n\,$*$\mathrm{Tor}(A, C) = 0$ *for every C.* Clearly, all the generators of Tor are now annihilated by n.
(C) $\mathrm{Tor}(A, C) = 0$ *if A is p-group and C is a q-group for different primes p, q.*
(D) *Multiplication by an integer n on A or on C induces multiplication by the same n on* $\mathrm{Tor}(A, C)$. This is an immediate consequence of the additivity of Tor.
(E) *If $C = \langle c \rangle$ is a cyclic group of order n, then there is a natural isomorphism* $\mathrm{Tor}(A, C) \cong A[n]$. Now, every $x \in \mathrm{Tor}(A, C)$ is annihilated by n, so x can be written in the form $x = (a, n, c)$ with $na = 0$. In this special case, the elements of Tor have unique forms, since $(a, n, c) = 0$ follows from the relations only if $a = 0$. Hence $\mathrm{Tor}(A, C) \cong A[n]$.
(F) *We have* $\mathrm{Tor}(\mathbb{Z}(p^k), \mathbb{Z}(p^\ell)) \cong \mathbb{Z}(p^{\min\{k,\ell\}})$ *and* $\mathrm{Tor}(\mathbb{Z}(p^k), \mathbb{Z}(q^\ell)) = 0$ *if p, q are distinct primes.* This follows from (E) *via* a straightforward calculation.

Tor As Bifunctor In order to examine the behavior of Tor as a bifunctor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$, choose homomorphisms $\alpha : A \to A', \gamma : C \to C'$. Evidently, if (a, m, c) is a generator of $\mathrm{Tor}(A, C)$, then $(\alpha a, m, \gamma c)$ will be a generator of $\mathrm{Tor}(A', C')$. Moreover, the function $(a, m, c) \mapsto (\alpha a, m, \gamma c)$ between the generators extends uniquely to a group homomorphism

$$\mathrm{Tor}(\alpha, \gamma)\colon\ \mathrm{Tor}(A, C) \to \mathrm{Tor}(A', C').$$

It is pretty obvious that $\mathrm{Tor}(\alpha,\gamma)\,\mathrm{Tor}(\alpha',\gamma') = \mathrm{Tor}(\alpha\alpha',\gamma\gamma')$ for matching homomorphisms, as well as $\mathrm{Tor}(\mathbf{1}_A,\mathbf{1}_C) = \mathbf{1}_{\mathrm{Tor}(A,C)}$. Hence we conclude:

Theorem 2.1 (Cartan–Eilenberg [CE]). *The torsion product is an additive bifunctor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$, covariant in both variables.* □

Next we prove the important result:

Theorem 2.2. Tor *commutes with direct limits* (*and so also with direct sums*)*:*

$$\mathrm{Tor}(\varinjlim_i A_i, C) = \varinjlim_i \mathrm{Tor}(A_i, C).$$

Proof. Let $\mathfrak{A} = \{A_i\ (i \in I); \pi_i^j\}$ be a direct system of groups with direct limit A, and canonical maps $\pi_i : A_i \to A$. We get the direct system $\mathfrak{C} = \{\mathrm{Tor}(A_i, C)\ (i \in I); \mathrm{Tor}(\pi_i^j, \mathbf{1}_C)\}$; let T denote its direct limit, and ρ_i the canonical maps $\mathrm{Tor}(A_i, C) \to T$. The homomorphisms $\mathrm{Tor}(\pi_i, \mathbf{1}_C) : \mathrm{Tor}(A_i, C) \to \mathrm{Tor}(A, C)$ satisfy $\mathrm{Tor}(\pi_i, \mathbf{1}_C) = \mathrm{Tor}(\pi_j, \mathbf{1}_C)\,\mathrm{Tor}(\pi_i^j, \mathbf{1}_C)$ for $i \leq j$. T being a direct limit, there is a unique $\sigma : T \to \mathrm{Tor}(A, C)$ such that $\sigma\rho_i = \mathrm{Tor}(\pi_i, \mathbf{1}_C)$. To show that σ is an isomorphism, let $(a, m, c) \in \mathrm{Tor}(A, C)$ where of course $ma = 0 = mc$. We can write $a = \pi_i a_i$ for some $a_i \in A_i$ with $ma_i = 0$, whence $\mathrm{Tor}(\pi_i, \mathbf{1}_C) : (a_i, m, c) \mapsto (a, m, c)$ shows that σ is a surjective map. If $x \in \mathrm{Ker}\,\sigma$, then there is an index $i \in I$ with $\rho_i y = x$ for some $y \in \mathrm{Tor}(A_i, C)$. Now $\mathrm{Tor}(\pi_i, \mathbf{1}_C)y = \sigma\rho_i y = \sigma x = 0$ implies the existence of a $j \geq i$ such that $\mathrm{Tor}(\pi_i^j, \mathbf{1}_C)y = 0$. Apply ρ_j, and notice that $\rho_j\,\mathrm{Tor}(\pi_i^j, \mathbf{1}_C) = \rho_i$ to conclude that $x = \rho_i y = 0$. This proves that σ is an isomorphism. □

An immediate consequence of the preceding theorem, combined with (A), is that if A, C are any groups, then $\mathrm{Tor}(A, C) \cong \oplus_p \mathrm{Tor}(A_p, C_p)$ holds for the p-components A_p and C_p.

(G) *For any three groups A, B, C, there is a natural isomorphism*

$$\mathrm{Tor}(A, \mathrm{Tor}(B, C)) \cong \mathrm{Tor}(\mathrm{Tor}(A, B), C).$$

We can represent both sides as direct limits of the respective finitely generated subgroups, since Tor commutes with direct limits. Consequently, it suffices to verify the isomorphism for finitely generated groups. These are Σ-cyclic groups, Tor commutes with direct sums, so a further reduction to cyclic groups is immediate. If one of the groups is infinite cyclic or if not all of A, B, C belong to the same prime, then both sides are 0. If A, B, C are cyclic of orders p^k, p^ℓ, p^m, respectively, then a quick calculation (using (E)) shows that both sides are isomorphic to $\mathbb{Z}(p^n)$ with $n = \min\{k, \ell, m\}$. Hence the stated isomorphism is evident. It is easy to check that the isomorphism is natural for cyclic groups, and direct limits preserve the naturality of maps.

The following natural isomorphism makes it possible to identify the torsion part of a group *via* Tor.

Theorem 2.3. *For every group C, we have*

$$\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, C) \cong t(C).$$

Proof. We start observing that $\mathbb{Q}/\mathbb{Z}$ is the direct limit of the direct system $\mathfrak{A} = \{\mathbb{Z}(m); \pi_m^n\}$ where $\pi_m^n : \mathbb{Z}(m) \to \mathbb{Z}(n)$ is the natural injection for $m|n$, acting as $1 + m\mathbb{Z} \mapsto nm^{-1} + n\mathbb{Z}$. From Theorem 2.2 we conclude that $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, C)$ is the direct limit of the direct system $\mathfrak{C} = \{\mathrm{Tor}(\mathbb{Z}(m), C); \mathrm{Tor}(\pi_m^n, \mathbf{1}_C)\}$. In view of (E), we can replace $\mathrm{Tor}(\mathbb{Z}(m), C)$ by $C[m]$, and $\mathrm{Tor}(\pi_m^n, \mathbf{1}_C)$ becomes the inclusion map $C[m] \to C[n]$, as is easily verified. Thus the direct limit will be the union of the $C[m]$ for all $m > 0$, which is exactly $t(C)$. $\square$

Tor and the Tensor Product Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{8.5}$$

be an exact sequence. Define the **connecting homomorphism**

$$\delta \colon \mathrm{Tor}(G, C) \to G \otimes A$$

as follows: if $(g, m, c) \in \mathrm{Tor}(G, C)$ and if $\beta b = c$ $(b \in B)$, then let $\delta \colon (g, m, c) \mapsto g \otimes a$ where $\alpha a = mb$ $(a \in A)$. Thus δ is defined on the set of generators, and in view of the relations in Tor and in $\otimes$, it extends to all of $\mathrm{Tor}(G, C)$. Importantly, the definition is independent of the choice of b, a, because if b', a' are other choices, then $b' = b + \alpha x$ $(x \in A)$, $a' = a + mx$, and so $g \otimes a' = g \otimes a + g \otimes mx$ where $g \otimes mx = mg \otimes x = 0$.

The exactness of the sequence in the following theorem is one of the most important results.

Theorem 2.4 (Cartan–Eilenberg [CE]). *If* (8.5) *is exact, then so is for any group G the sequence*

$$0 \to \mathrm{Tor}(G, A) \xrightarrow{\alpha_*} \mathrm{Tor}(G, B) \xrightarrow{\beta_*} \mathrm{Tor}(G, C) \xrightarrow{\delta}$$
$$\to G \otimes A \xrightarrow{\mathbf{1}_G \otimes \alpha} G \otimes B \xrightarrow{\mathbf{1}_G \otimes \beta} G \otimes C \to 0.$$

Proof. Since both tensor and torsion products commute with direct limits, and since direct limits of exact sequences are exact, it will be enough to verify exactness for $G \cong \mathbb{Z}$ and $G \cong \mathbb{Z}(m)$ $(m \in \mathbb{N})$ only. The first case is trivial, since it just leaves the given exact sequence (8.5) unchanged (all Tors are 0). If $G = \langle g \rangle \cong \mathbb{Z}(m)$, then the elements of the Tors are of the form (g, m, x) with $mx = 0$, while those in the tensor products are of the form $g \otimes x$ with x taken mod mX (here $X = A, B, C$). That the composite of any two maps is 0 is clear except at places where δ is involved. Then $\delta\beta_*(g, m, b) = \delta(g, m, \beta b) = g \otimes a = 0$, because $\alpha a = mb = 0$, and $(\mathbf{1}_G \otimes \alpha)\delta(g, m, c) = (\mathbf{1}_G \otimes \alpha)(g \otimes a) = (\mathbf{1}_G \otimes \alpha a) = 0$ as $\alpha a \in mB$.

It remains to prove that the kernels are included in previous images for the first three maps. If $\alpha_*(g, m, a) = (g, m, \alpha a) = 0$, then $\alpha a = 0$ and $a = 0$, thus α_* is monic. If $\beta(g, m, b) = 0$, then similarly $\beta b = 0$, thus some $a \in A$ satisfies $\alpha a = b$ with $ma = 0$. Hence $\alpha_*(g, m, a) = (g, m, b)$, and the exactness at the second Tor follows. Next, suppose $\delta(g, m, c) = 0$ where $mc = 0$ ($c \in C$). Then with the above notation $g \otimes a = 0$, so $a \in mA$, i.e. $a = ma'$ for some $a' \in A$. Thus $\alpha ma' = \alpha a = mb$, i.e. $m(b - \alpha a') = 0$. Hence $(a, m, \beta(b - \alpha a')) \in \mathrm{Tor}(G, B)$ maps upon (g, m, c). Finally, if $(\mathbf{1}_A \otimes \alpha)(g \otimes a) = 0$, then $\alpha a = mb$ for some $b \in B$, and for this b we have $\delta(g, m, \beta b) = g \otimes a$. This implies exactness at $G \otimes A$, completing the proof. $\square$

By the way, if (8.5) is chosen as a free resolution of C, say, $0 \to H \to F \to C \to 0$ with free group F, then $\mathrm{Tor}(G, C)$ can be defined as the kernel of the induced map $G \otimes H \to G \otimes F$. This interpretation leads to an alternative set of generators and defining relations for $\mathrm{Tor}(G, C)$.

★ **Notes.** The torsion product $\mathrm{Tor}(A, C)$ is defined over any ring R if A is a right and C is a left module. Theorem 2.4 carries over, but the left 0 has to be removed and the sequence extended to the left by terms of higher functors $\mathrm{Tor}_n^{\mathsf{R}}(*, *)$.

S.E. Dickson [Trans. Amer. Math. Soc. **121**, 223–235 (1966)] defines **torsion theory** for abelian categories $\mathcal{A}$ as a pair $(\mathcal{T}, \mathcal{F})$ of classes in $\mathcal{A}$ such that $\mathcal{T}, \mathcal{F}$ are maximal with respect to the property $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$. $\mathcal{T}$ is closed under homomorphic images, direct sums and extensions, while $\mathcal{F}$ is closed under taking kernels, direct products and extensions. An example for a torsion theory in $\mathcal{A}b$ is (besides the obvious one) when $\mathcal{T}$ is the class of divisible, and $\mathcal{F}$ is the class of reduced groups.

A more general concept is the radical. A functor $R : \mathcal{A}b \to \mathcal{A}b$ defines a **radical** in $\mathcal{A}b$ if $R(A) \leq A$ and $R(A/R(A)) = 0$ for all $A \in \mathcal{A}b$. E.g. p^σ (for any ordinal σ) is a radical functor in the category of torsion groups. More generally, if $\mathcal{X}$ is any class of groups, then $R_{\mathcal{X}}(A) = \cap_\phi \mathrm{Ker}\,\phi$ with $\phi \in \mathrm{Hom}(A, X)$ ($\forall X \in \mathcal{X}$) is a radical functor. An idempotent radical defines a torsion theory.

Exercises

(1) Prove that $(a, m_1 + m_2, c) = (a, m_1, c) + (a, m_2, c)$ holds for the generators of Tor whenever the right hand side makes sense.
(2) For finite groups A, C, $\mathrm{Tor}(A, C) \cong A \otimes C$ is a (non-natural) isomorphism.
(3) Assume $(a, p^n, c_1) = (a, p^n, c_2)$ in $\mathrm{Tor}(A, C)$ where $A = \langle a \rangle \cong \mathbb{Z}(p^n)$ and $o(c_1) = o(c_2) = p^n$. Prove that $c_1 = c_2$.
(4) (a) A is torsion-free if and only if $\mathrm{Tor}(A, C) = 0$ for every group C.
(b) $\mathrm{Tor}(A, A) = 0$ implies that A is torsion-free.
(5) If A is torsion-free, then $A \otimes \mathrm{Tor}(B, C) \cong \mathrm{Tor}(A \otimes B, C)$. [Hint: direct limit.]
(6) For all A, C, and $m \in \mathbb{Z}$, one has $\mathrm{Tor}(A \otimes \mathbb{Z}(m), C) \cong A \otimes \mathrm{Tor}(\mathbb{Z}(m), C)$ and $\mathrm{Tor}(\mathbb{Z}(m), A) \otimes \mathrm{Tor}(\mathbb{Z}(m), C) \cong \mathrm{Tor}(\mathrm{Tor}(\mathbb{Z}(m), A), C)$.
(7) If A satisfies $\mathrm{Tor}(A, C) \cong t(C)$ for every group C, then $A \cong \mathbb{Q}/\mathbb{Z} \oplus G$ with G torsion-free.

(8) Formulate and prove the dual of Corollary 1.11 for Tor.
(9) Give counterexamples to show that Tor does not commute with (a) direct products; (b) inverse limits. [Hint: $A = \prod_p \mathbb{Z}(p^\infty)$.]

3 Theorems on Tensor Products

We wonder under what conditions the exactness of a short exact sequence is preserved when tensoring it with a group G or when forming the torsion product with G. We offer two independent conditions, one relates the sequence itself (pure-exactness), while the other specifies the groups G (torsion-freeness).

Tensor Product and Pure-Exact Sequences We verify a frequently used property of pure-exact sequences.

Theorem 3.1 (Harrison [1], Fuchs [11]). *If $0 \to A \to B \to C \to 0$ is a pure-exact sequence, then so are the sequences*

$$0 \to A \otimes G \to B \otimes G \to C \otimes G \to 0 \tag{8.6}$$

and

$$0 \to \mathrm{Tor}(A, G) \to \mathrm{Tor}(B, G) \to \mathrm{Tor}(C, G) \to 0 \tag{8.7}$$

for every group G.

Proof. Pure-exactness for both sequences follows from Theorem 3.1 or Corollary 3.6 in Chapter 5, since the claim is true for cyclic groups, and both tensor and torsion products commute with direct limits. □

An important consequence of this theorem is that if A', C' are pure subgroups in A, C, respectively, then $A' \otimes C'$ may be regarded as a (pure) subgroup of $A \otimes C$ under the natural map. In other words, $a' \otimes c'$ ($a' \in A'$, $c' \in C'$) is the same if viewed as an element of $A' \otimes C'$ or as one of $A \otimes C$. Needless to say, the same conclusion can be stated for p-purity.

Lemma 3.2 (Fuchs [11]). *If B is a p-basic subgroup of A, and C is a p-group, then there is a natural isomorphism*

$$A \otimes C \cong B \otimes C.$$

Proof. We start with the p-pure-exact sequence $0 \to B \to A \to A/B \to 0$, and tensor it with C to obtain the exact sequence $0 \to B \otimes C \to A \otimes C \to (A/B) \otimes C \to 0$. Since A/B is p-divisible and C is a p-group, their tensor product vanishes. The exactness of $0 \to B \otimes C \to A \otimes C \to 0$ amounts to the claim in the lemma. □

Tensor Products of Torsion Groups The last result enables us to determine explicitly the tensor product of two groups if one of them is a torsion group. Let B_p denote a p-basic subgroup of A, and C_p the p-component of the torsion group C, then

$$A \otimes C \cong \oplus_p (A \otimes C_p) \cong \oplus_p (B_p \otimes C_p). \tag{8.8}$$

The last tensor products are easily computed as the groups B_p are Σ-cyclic. Note that these isomorphisms also show that, for a torsion group C, we have $A \otimes C \cong (t(A) \otimes C) \oplus (A/t(A) \otimes C)$. More generally,

Lemma 3.3 (Harrison [2]). *If B is a pure subgroup of A, and if C is a torsion group, then*

$$A \otimes C \cong [B \otimes C] \oplus [(A/B) \otimes C].$$

Proof. Referring again to the direct limit property, the proof can be simplified to the special case $C \cong \mathbb{Z}(p^k)$ of cocylic groups. If k is an integer, then we know that the pure-exact sequence $0 \to B \to A \to A/B \to 0$ splits when tensored with a finite cyclic group. If $k = \infty$, then (8.6) along with the injectivity of $B \otimes \mathbb{Z}(p^\infty)$ implies the claim. $\square$

If both A and C are torsion groups, then their tensor product is very simple to describe:

Theorem 3.4 (Harrison [1], Fuchs [6]). *The tensor product of torsion groups is a direct sum of finite cyclic groups.*

Proof. If both A and C are torsion groups, then Lemma 3.2 permits us to replace C_p in (8.8) by its basic subgroup, so $B_p \otimes C_p$ will be a direct sum of finite cyclic groups. $\square$

Tensor Products and Torsion-Free Groups The other case when exactness is preserved under tensor products is recorded in the following theorem.

Theorem 3.5 (Dieudonné). *Suppose* $0 \to A \to B \to C \to 0$ *is an exact sequence. If G is a torsion-free group, then the induced sequence* (8.6) *is also exact.*

Hence the tensor product of torsion-free groups is again torsion-free.

Proof. As $\mathrm{Tor}(G, *) = 0$ for torsion-free G, the claim follows at once from Theorem 2.4. $\square$

Very little is known about the structure of the tensor product of torsion-free groups, except when the groups are completely decomposable. In this case, complete information is available about the tensor product, based on the tensor product of two rank one torsion-free groups; see Chapter 12. A very modest result is related to p-basic subgroups.

Lemma 3.6. *Let A and C be torsion-free groups with p-basic subgroups B and D, respectively. Then $A \otimes C$ is torsion-free, and its p-basic subgroups are isomorphic to $B \otimes D$.*

Proof. From the p-pure-exact sequence $0 \to B \to A \to A/B \to 0$ we deduce the p-pure-exact sequence $0 \to B \otimes C \to A \otimes C \to (A/B) \otimes C \to 0$. As $(A/B) \otimes C$ is p-divisible and torsion-free, the p-basic subgroups of $B \otimes C$ are also p-basic in $A \otimes C$. A repetition of this argument leads to the conclusion that $B \otimes D$ is p-basic in $A \otimes C$. $\square$

As far as mixed groups are concerned, we are able to get some information about their torsion and torsion-free parts.

Theorem 3.7. *For any groups A, C, there are isomorphisms*

$$t(A \otimes C) \cong [t(A) \otimes t(C)] \oplus [t(A) \otimes C/t(C)] \oplus [A/t(A) \otimes t(C)],$$
$$(A \otimes C)/t(A \otimes C) \cong A/t(A) \otimes C/t(C).$$

Proof. It is easily checked that the kernel of the natural epimorphism $A \otimes C \to A/t(A) \otimes C/t(C)$ is generated by $A \otimes t(C)$ and $t(A) \otimes C$ (which are subgroups of $A \otimes C$; cf. Theorem 3.3). Evidently, $A \otimes t(C) \cong [t(A) \otimes t(C)] \oplus [A/t(A) \otimes t(C)]$, and $t(A) \otimes C \cong [t(A) \otimes t(C)] \oplus [t(A) \otimes C/t(C)]$ are torsion groups. $\square$

★ **Notes.** Theorem 3.5 sounds trivial, but one should not forget that it was published before the powerful homological machinery existed. It is a special case of a general theorem on modules that states that exactness is preserved by tensoring a short exact sequence with a flat module. By the way, flat modules are defined by the property that Tor vanishes identically for them. (Flat abelian group is the same as torsion-free group.)

Tensor products need not be reduced even if the components are reduced as is shown by Example 1.10. A. Fomin [1] studied the tensor powers of torsion-free groups A of rank n and p-corank k. He found that $A \otimes \cdots \otimes A$ ($k+1$ factors) contains a p-divisible subgroup of rank $\binom{n}{k+1}$.

Exercises

(1) The following are equivalent:

(a) The sequence $0 \to A \to B \to C \to 0$ is pure-exact;
(a) $0 \to A \otimes G \to B \otimes G \to C \otimes G \to 0$ is exact for every G;
(b) $0 \to \operatorname{Tor}(A, G) \to \operatorname{Tor}(B, G) \to \operatorname{Tor}(C, G) \to 0$ is exact for each G.

(2) (a) (Harrison) If $0 \to A \to B \to C \to 0$ is a pure-exact sequence, then $0 \to A \otimes G \to B \otimes G \to C \otimes G \to 0$ is splitting exact for every torsion G.
(b) If a group G has property (a) for every pure-exact sequence, then it must be torsion. [Hint: B free and $C = \mathbb{Q}$.]

(3) (a) If $A \otimes C$ contains a copy of $\mathbb{Z}(p^\infty)$, then either A or C has a subgroup $\cong \mathbb{Z}(p^\infty)$.
(b) Examine when $A \otimes C \cong \mathbb{Z}(p^\infty)$.
(4) (a) Prove that $A \otimes C \cong \mathbb{Z}$ implies $A \cong C \cong \mathbb{Z}$.
(b) If $A \otimes C$ is a non-trivial free group, then both A and C are free.
(5) If A is torsion-free, then $\tilde{A} \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A \otimes \mathbb{Q}/\mathbb{Z})$ ($\mathbb{Z}$-adic completion).
(6) If the sequence $0 \to A \to B \to C \to 0$ is pure-exact, then the same holds for the sequence $0 \to tA \to tB \to tC \to 0$ of torsion subgroups. [Hint: $\mathrm{Tor}(\mathbb{Z}(p^\infty), *)$.]

4 Theorems on Torsion Products

While tensoring with a p-group implements drastic structural simplification, this is not the case for the torsion product, though we may notice some smoothing effect. In this section, we explore some interesting features of Tor.

In order to get more information about Tor, we want to take advantage of its left exactness. The first and foremost fact to be observed is that if $A' \le A$ and $C' \le C$, then $\mathrm{Tor}(A', C') \le \mathrm{Tor}(A, C)$. This is a powerful property that will be used throughout without mentioning it explicitly.

Elementary Facts Our first concern is how Tor behaves towards multiplication by integers.

Lemma 4.1 (Nunke [4]). *For every integer n, we have*

$$n\,\mathrm{Tor}(A, C) = \mathrm{Tor}(nA, nC).$$

Proof. Starting with the exact sequence $0 \to nA \to A \to A/nA \to 0$ and the same for C, we obtain the exact sequence

$$0 \to \mathrm{Tor}(nA, nC) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A/nA, C) \oplus \mathrm{Tor}(A, C/nC),$$

using the commutative diagram dual to Corollary 1.11. Hence we conclude that $n\,\mathrm{Tor}(A, C) \le \mathrm{Tor}(nA, nC)$, since the direct sum in the displayed formula is annihilated by n. To prove the converse inclusion, pick a generator $x = (na, m, nc) \in \mathrm{Tor}(nA, nC)$ where $mna = 0 = mnc$ ($a \in A, c \in C$). Here $x = (a, nm, nc) = n(a, nm, c) \in n\mathrm{Tor}(A, C)$. ☐

Interestingly, in the last lemma the integer n can be replaced by p^σ for any ordinal σ.

Lemma 4.2 (Nunke [4]). *For all ordinals σ and p-groups A, C, we have*

$$p^\sigma\,\mathrm{Tor}(A, C) = \mathrm{Tor}(p^\sigma A, p^\sigma C).$$

Thus the length of the p-group $\operatorname{Tor}(A, C)$ *can exceed neither the length of A nor the length of C.*

Proof. First we observe that, by the left exactness of Tor, $\operatorname{Tor}(p^\sigma A, p^\sigma C)$ may be regarded as a subgroup of $\operatorname{Tor}(A, C)$. To prove the claim by induction, note that the case $\sigma = 0$ is trivial, and the step from σ to $\sigma + 1$ follows from the preceding lemma along with the induction hypothesis:

$$p \cdot p^\sigma \operatorname{Tor}(A, C) = p \operatorname{Tor}(p^\sigma A, p^\sigma C) = \operatorname{Tor}(p \cdot p^\sigma A, p \cdot p^\sigma C).$$

If σ is a limit ordinal, then

$$p^\sigma \operatorname{Tor}(A, C) = \cap_{\rho<\sigma} p^\rho \operatorname{Tor}(A, C) = \cap_{\rho<\sigma} \operatorname{Tor}(p^\rho A, p^\rho C) \geq \operatorname{Tor}(p^\sigma A, p^\sigma C).$$

The reverse inclusion follows in the same way as in the proof of Lemma 4.1. □

Lemma 4.3 (Nunke [4]).

(i) *For all groups* A, C, *and integers* $n > 0$,

$$\operatorname{Tor}(A, C)[n] \cong \operatorname{Tor}(A[n], C[n]).$$

(ii) *The ranks satisfy* $\operatorname{rk}_p(\operatorname{Tor}(A, C)) = \operatorname{rk}_p(A) \cdot \operatorname{rk}_p(C)$.

Proof. (i) Multiplication by n yields the exact sequence $0 \to A[n] \to A \xrightarrow{\dot{n}} nA \to 0$, whence we derive the exact sequence $0 \to \operatorname{Tor}(A[n], C) \to \operatorname{Tor}(A, C) \xrightarrow{\dot{n}} \operatorname{Tor}(nA, C)$. Thus $\operatorname{Tor}(A, C)[n] = \operatorname{Ker} \dot{n} = \operatorname{Tor}(A[n], C)$. Applying the same argument to C, we obtain the desired isomorphism.

(ii) Since the rank of a p-group is the dimension of its socle as a $\mathbb{Z}/p\mathbb{Z}$-vector space, and since Tor commutes with direct sums, the equality follows from the proved isomorphism, applied for $n = p$. □

A consequence of this lemma is that *if* A, C *are non-zero p-groups, then* $\operatorname{Tor}(A, C)$ *is never* 0.

UK-Invariants of Tor The following result will enable us to determine the UK-invariants of Tor for p-groups in terms of their invariants.

Theorem 4.4 (Nunke [4]). *If* A, C *are p-groups, then the* σth *UK-invariant of their torsion product is*

$$f_\sigma(\operatorname{Tor}(A, C)) = f_\sigma(A) f_\sigma(C) + f_\sigma(A) \operatorname{rk}(p^{\sigma+1} C) + \operatorname{rk}(p^{\sigma+1} A) f_\sigma(C).$$

Proof. Write $p^\sigma A[p] = p^{\sigma+1}A[p] \oplus G$ and $p^\sigma C[p] = p^{\sigma+1}C[p] \oplus H$ where $\mathrm{rk}(G) = f_\sigma(A)$ and $\mathrm{rk}(H) = f_\sigma(C)$. Combining Lemmas 4.2 and 4.3 we obtain $(p^\sigma \operatorname{Tor}(A, C))[p] = \operatorname{Tor}(p^\sigma A[p], p^\sigma C[p])$ whence

$$(p^\sigma \operatorname{Tor}(A,C))[p] = \operatorname{Tor}(p^{\sigma+1}A[p], p^{\sigma+1}C[p]) \oplus \operatorname{Tor}(G, p^{\sigma+1}C[p]) \\ \oplus \operatorname{Tor}(p^{\sigma+1}A[p], H) \oplus \operatorname{Tor}(G,H)[p].$$

The first summand on the right is $(p^{\sigma+1} \operatorname{Tor}(A, C))[p]$, so the σth UK-invariant of Tor will be the sum of the ranks of the remaining three summands. The claim follows from the observation that the rank of the last summand is $\mathrm{rk}(G)\,\mathrm{rk}(H) = f_\sigma(A) f_\sigma(C)$. □

Anticipating results from Chapter 11 that direct sums of countable p-groups are completely determined by their UK-invariants, we can state:

Corollary 4.5. *If the p-groups A, C are direct sums of countable groups, then so is their torsion product. In this case,* $\operatorname{Tor}(A, C)$ *can be completely characterized by the UK-invariants of A and C.*

Proof. As Tor commutes with direct sums, the first claim is evident. The rest follows from our discussion above. □

Tor and Intersection of Subgroups Additional relevant properties are listed in the following two lemmas.

Lemma 4.6 (Nunke [4]). *Suppose A', A'' are subgroups of A and C', C'' are subgroups of C. Then*

(i) $\operatorname{Tor}(A' \cap A'', C) = \operatorname{Tor}(A', C) \cap \operatorname{Tor}(A'', C)$;
(ii) $\operatorname{Tor}(A', C') = \operatorname{Tor}(A, C') \cap \operatorname{Tor}(A', C)$;
(iii) $\operatorname{Tor}(A' \cap A'', C' \cap C'') = \operatorname{Tor}(A', C') \cap \operatorname{Tor}(A'', C'')$.

Proof. (i) From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' \cap A'' & \longrightarrow & A' & \longrightarrow & A'/(A' \cap A'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'' & \longrightarrow & A & \longrightarrow & A/A'' & \longrightarrow & 0 \end{array}$$

with exact rows and monic vertical maps we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Tor}(A' \cap A'', C) & \longrightarrow & \operatorname{Tor}(A', C) & \longrightarrow & \operatorname{Tor}(A'/(A' \cap A''), C) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Tor}(A'', C) & \longrightarrow & \operatorname{Tor}(A, C) & \longrightarrow & \operatorname{Tor}(A/A'', C) \end{array}$$

whose rows are exact and the vertical maps are monic. Diagram chasing shows that $\mathrm{Tor}(A' \cap A'', C)$ must be the stated intersection, thus (i) holds.

(ii) follows in the same way from the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tor}(A', C') & \longrightarrow & \mathrm{Tor}(A, C') & \longrightarrow & \mathrm{Tor}(A/A', C') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Tor}(A', C) & \longrightarrow & \mathrm{Tor}(A, C) & \longrightarrow & \mathrm{Tor}(A/A', C) \end{array}$$

with exact rows and monic vertical maps.

(iii) Using (i) and (ii), we calculate

$$\begin{aligned} \mathrm{Tor}(A', C') \cap \mathrm{Tor}(A'', C'') &= \mathrm{Tor}(A, C') \cap \mathrm{Tor}(A', C) \ \cap \mathrm{Tor}(A, C'') \cap \mathrm{Tor}(A'', C) \\ &= \mathrm{Tor}(A' \cap A'', C) \ \cap \mathrm{Tor}(A, C' \cap C'') \\ &= \mathrm{Tor}(A' \cap A'', C' \cap C''), \end{aligned}$$

as desired.

□

Lemma 4.7 (Nunke [4]).

(i) *For every* $x \in \mathrm{Tor}(A, C)$*, there exist unique finite subgroups* $A' \leq A$, $C' \leq C$ *with* $x \in \mathrm{Tor}(A', C')$ *such that if* $x \in \mathrm{Tor}(A'', C'')$ $(A'' \leq A, C'' \leq C)$*, then* $A' \leq A''$ *and* $C' \leq C''$.

(ii) *For every infinite subgroup* $G < \mathrm{Tor}(A, C)$*, there exist subgroups* $A' \leq A$, $C' \leq C$ *such that* $G \leq \mathrm{Tor}(A', C')$ *and* $|A'| + |C'| \leq |G|$.

Proof. (i) Consider pairs A'', C'' of finite subgroups with $A'' \leq A$, $C'' \leq C$ satisfying $x \in \mathrm{Tor}(A'', C'')$. From Lemma 4.6(iii) we conclude that the sets of such A'' and C'' are closed under intersection, so there is a unique minimal pair A', C'. (ii) follows from (i). □

We can say something definitive about the basic subgroups of Tor, also for those of the subgroups of Tor.

Lemma 4.8 (Nunke [4]). *Assume* A, C *are reduced* p*-groups. Every subgroup* G *of* $\mathrm{Tor}(A, C)$ *is starred, i.e. its basic subgroups are of the same cardinality as* G *itself.*

Proof. (Keef) Let B denote a basic subgroup of G. There is nothing to prove if B is finite, so assume it is infinite. By Lemma 4.7, there are subgroups $A' \leq A, C' \leq C$ of cardinality $\leq |B|$ such that $B \leq \mathrm{Tor}(A', C')$. There are maps $\alpha : \mathrm{Tor}(A, C) \to \mathrm{Tor}(A/A', C)$ and $\gamma : \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, C/C')$ with kernels $\mathrm{Tor}(A', C)$ and $\mathrm{Tor}(A, C')$, respectively, so that the kernel of the map $\alpha \oplus \gamma : \mathrm{Tor}(A, C) \to \mathrm{Tor}(A/A', C) \oplus \mathrm{Tor}(A, C/C')$ is equal to $\mathrm{Tor}(A', C) \cap \mathrm{Tor}(A, C') = \mathrm{Tor}(A', C')$. Since B is mapped by $\alpha \oplus \gamma$ onto 0 in a reduced group, the image of G under $\alpha \oplus \gamma$

must also be 0 (see Lemma 6.7 in Chapter 5). This means that $G \leq \mathrm{Tor}(A', C')$, thus $|G| \leq |\mathrm{Tor}(A', C')| = |B|$. $\square$

Dependence on the Underlying Set We now show that, interestingly, the structure of Tor might depend on the set-theoretical model in which it is considered. (For the torsion-completion $\overline{B}$, see Sect. 3 in Chapter 10.)

Theorem 4.9 (Keef [2]). *Let B be an unbounded countable Σ-cyclic p-group. Then* $\mathrm{Tor}(\overline{B}, \overline{B})$ *is Σ-cyclic if and only if* CH *holds.*

Proof. Assume CH, so that we can create a filtration of the torsion-complete p-group $\overline{B}$ (whose cardinality is $2^{\aleph_0}$) with *countable* pure (Σ-cyclic) subgroups C_σ ($\sigma < \omega_1$). Consider the pure-exact sequences

$$0 \to \mathrm{Tor}(C_\sigma, C_\sigma) \to \mathrm{Tor}(C_\sigma, C_{\sigma+1}) \to \mathrm{Tor}(C_\sigma, C_{\sigma+1}/C_\sigma) \to 0,$$

$$0 \to \mathrm{Tor}(C_\sigma, C_{\sigma+1}) \to \mathrm{Tor}(C_{\sigma+1}, C_{\sigma+1}) \to \mathrm{Tor}(C_{\sigma+1}/C_\sigma, C_{\sigma+1}) \to 0.$$

These sequences split, as the last Tors are Σ-cyclic, so $\mathrm{Tor}(C_{\sigma+1}, C_{\sigma+1}) \cong \mathrm{Tor}(C_\sigma, C_\sigma) \oplus \mathrm{Tor}(C_\sigma, C_{\sigma+1}/C_\sigma) \oplus \mathrm{Tor}(C_{\sigma+1}/C_\sigma, C_{\sigma+1})$. Therefore, in the smooth chain of the $\mathrm{Tor}(C_\sigma, C_\sigma)$, every group is a summand of the next one with Σ-cyclic complementary summand, and since evidently $\mathrm{Tor}(\overline{B}, \overline{B}) = \cup_{\sigma<\omega_1} \mathrm{Tor}(C_\sigma, C_\sigma)$, that $\mathrm{Tor}(\overline{B}, \overline{B})$ is Σ-cyclic follows at once from Theorem 2.11 in Chapter 5.

Conversely, by way of contradiction, suppose $T = \mathrm{Tor}(\overline{B}, \overline{B})$ is Σ-cyclic, and $\aleph_1 < 2^{\aleph_0}$. Using a fixed direct decomposition of T into cyclic summands, a standard back-and-forth argument leads us to a pure subgroup C such that $B < C < \overline{B}$, $|C| = \aleph_1$, and $\mathrm{Tor}(C, C)$ is a summand of T. $\mathrm{Tor}(C, C)$ is obviously a summand of $\mathrm{Tor}(\overline{B}, C)$ as well, hence the exact sequence $0 \to \mathrm{Tor}(C, C) \to \mathrm{Tor}(\overline{B}, C) \to \mathrm{Tor}(\overline{B}/C, C) \to 0$ is splitting. Consequently, the last Tor is Σ-cyclic. But $\overline{B}/C$ is divisible of cardinality $2^{\aleph_0}$, hence $\mathrm{Tor}(\overline{B}/C, C) \cong \oplus C \neq 0$, and C is Σ-cyclic. But C cannot be Σ-cyclic, since it is uncountable and its basic subgroups are countable. $\square$

When Tor is κ-Cyclic Call a p-group κ**-cyclic** if every subgroup which has cardinality $< \kappa$ is Σ-cyclic. The main interest lies of course in groups of cardinality $\geq \kappa$ that are κ-cyclic.

Example 4.10. It is clear that for every uncountable cardinal κ, κ-cyclic p-groups are separable (in the sense of Sect. 1 in Chapter 10), and in turn, every separable p-group is $\aleph_1$-cyclic by Prüfer's theorem.

Lemma 4.11 (Nunke [4]). *If the p-groups A and C are κ-cyclic, then* $\mathrm{Tor}(A, C)$ *is κ^+-cyclic.*

Proof. It suffices to prove that $T = \mathrm{Tor}(A, C)$ is Σ-cyclic provided $|A| = |C| = \kappa$. Choose smooth chains $0 = A_0 < \cdots < A_\sigma < \ldots$ and $0 = C_0 < \cdots < C_\sigma < \ldots$ ($\sigma < \kappa$) of pure subgroups with unions A and C, respectively, such that the links have cardinality $< \kappa$, so they are Σ-cyclic by hypothesis. Then T is the union of the smooth chain $T_\sigma = \operatorname{Tor}(A_\sigma, C_\sigma)$ $(\sigma < \kappa)$ of pure subgroups. For each σ, there is a pure-exact sequence

$$0 \to T_\sigma \to T_{\sigma+1} \to \operatorname{Tor}(A_{\sigma+1}/A_\sigma, C_{\sigma+1}) \oplus \operatorname{Tor}(A_{\sigma+1}, C_{\sigma+1}/C_\sigma)$$

(the dual of Corollary 1.11). The last direct sum of the Tors is Σ-cyclic, since $C_{\sigma+1}$ and $A_{\sigma+1}$ are both Σ-cyclic, thus $T_{\sigma+1} = T_\sigma \oplus H_\sigma$ for some Σ-cyclic H_σ. Hence we conclude that $T = \oplus_{\sigma<\kappa} H_\sigma$ is Σ-cyclic. □

In particular, if A, C are separable p-groups (thus $\aleph_1$-cyclic), then $\operatorname{Tor}(A, C)$ is $\aleph_2$-cyclic. A simple induction shows that if $A_1, \dots, A_{2^n}$ are separable p-groups, then their torsion product $\operatorname{Tor}(A_1, \dots, A_{2^n})$ is $\aleph_{n+1}$-cyclic.

★ **Notes.** More on Tor can be found in Nunke [4]. For iterated torsion products, see Keef [2]. Keef [9] gives a complete answer to Nunke's question as to when $\operatorname{Tor}(A, C)$ is Σ-cyclic. He introduces new invariants whose values are collections of finite sets of uncountable regular cardinals.

The problem has been raised about the relation of reduced p-groups A, C if they satisfy $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(C, G)$ for all reduced groups G. Hill [12] has several positive results, and Cutler [1] provides examples showing that A, C need not be isomorphic. Keef [7] shows that reduced p-groups A, C satisfy $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(C, G)$ for all reduced p-groups G if and only if they have the same UK-invariants and $A \oplus X \cong C \oplus Y$ for some Σ-cyclic groups X, Y. For an excellent survey of Tor, see Keef [8].

Exercises

(1) For all A, C, we have $|\operatorname{Tor}(A, C)| \leq |A| \cdot |C|$.
(2) If A, C are p-groups of length λ_A, λ_C, then the length of $\operatorname{Tor}(A, C)$ is exactly $\min\{\lambda_A, \lambda_C\}$.
(3) The divisible part of $\operatorname{Tor}(A, C)$ is the torsion product of the maximal divisible subgroups of A and C.
(4) (Richman) If A, C are p-groups, and $u = (a, p^k, c) \in \operatorname{Tor}(A, C)$ is a generator, then $h_p(u) = \min\{h_p(a), h_p(c)\}$.
(5) (Nunke) Let $a \in A, c \in C$ be of the same order p^m, and $A' \leq A, C' \leq C$ such that the generator (a, p^m, c) of $\operatorname{Tor}(A, C)$ belongs to $\operatorname{Tor}(A', C')$. Then also $a \in A', c \in C'$.
(6) The torsion product $T = \operatorname{Tor}(A, C)$ of two reduced p-groups has a Σ-cyclic summand of the same cardinality as T. (We could be more specific and claim a summand isomorphic to a basic subgroup of T.) [Hint: Lemma 4.8 and Lemma 6.12 in Chapter 5.]
(7) (Nunke) If A, C are p-groups such that $\operatorname{Tor}(A, C)$ is, but C is not Σ-cyclic, then A must be separable.

(8) (Keef) If $A' \leq A, C' \leq C$, then $\mathrm{Tor}(A', C')$ is isotype (Sect. 5 in Chapter 11) in the direct sum $\mathrm{Tor}(A, C') \oplus \mathrm{Tor}(A', C)$. [Hint: Lemmas 4.2 and 4.6.]

5 Localization

Localization at Primes Let S denote a set of primes. The **localization of $\mathbb{Z}$ at** S is defined as the set of rational numbers $\frac{r}{s}$ such that all the prime divisors of the denominator s belong to S. It is easy to check that these rational numbers form a subring of $\mathbb{Q}$; it is denoted by $S^{-1}\mathbb{Z}$. By far the most important special case is when S consists of all primes with the exception of a single prime p, in which case the localization is denoted as $\mathbb{Z}_{(p)}$, and is called the **localization at** p.

We are particularly interested in the localization of groups at a prime p. The **localization of group** A **at** p is defined by the formula

$$A_{(p)} = \mathbb{Z}_{(p)} \otimes A.$$

Thus $A_{(p)}$ is a $\mathbb{Z}_{(p)}$-module. The **localization map** $\theta_p(A) \colon A \to A_{(p)}$ acting as $\theta_p(A) \colon a \mapsto \mathbf{1} \otimes a$ is a natural homomorphism. (We often simplify notation and write θ_p if there is no danger of confusion.) θ_p has the universal property that if $\psi : A \to M$ is a group homomorphism into a $\mathbb{Z}_{(p)}$-module M, then there exists a unique $\mathbb{Z}_{(p)}$-homomorphism $\psi' : A_{(p)} \to M$ such that $\psi = \psi'\theta_p$. It is routine to check that $\mathrm{Ker}\,\theta_p$ is the direct sum of the q-components of the torsion subgroup of A for all primes $q \neq p$.

If $\theta_p(A)$ is an isomorphism, we say that A is p**-local**. A p-local group is nothing else than a group with unique q-divisibility for every prime $q \neq p$. Group homomorphisms between p-local groups are automatically $\mathbb{Z}_{(p)}$-homomorphisms.

(A) *The localization at p is functorial*: if $\phi : A \to C$ is a homomorphism, then there exists a unique $\mathbb{Z}_{(p)}$-homomorphism $\phi_p \colon A_{(p)} \to C_{(p)}$ such that $\theta_p(C)\phi = \phi_p\theta_p(A)$. The map ϕ_p carries $\mathbf{1} \otimes a$ $(a \in A)$ to $\mathbf{1} \otimes \phi a$.

(B) *Localization is an exact functor $\mathcal{A}b \to \mathcal{M}od\text{-}\mathbb{Z}_{(p)}$ from the category of abelian groups into the category of $\mathbb{Z}_{(p)}$-modules.* More explicitly, this means that if $0 \to B \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$ is an exact sequence, then so is the sequence $0 \to B_{(p)} \xrightarrow{\alpha_p} A_{(p)} \xrightarrow{\beta_p} C_{(p)} \to 0$. In fact, tensoring by the torsion-free $\mathbb{Z}_{(p)}$ preserves exactness. In particular, we have

$$A_{(p)}/B_{(p)} \cong (A/B)_{(p)}.$$

(C) *If A is a torsion group, then its localization at p is just its p-component A_p*. In our notation, $A_{(p)} = A_p$. Thus localization at p leaves a p-group unchanged.

(D) *Localization commutes with arbitrary direct sums:*

$$(\oplus_{i\in I} A_i)_{(p)} = \oplus_{\imath\in I}\,(A_i)_{(p)}.$$

(The same fails for direct products.)

(E) *For every group A, there is a canonical embedding*

$$\theta\colon A \to \prod_p A_{(p)},$$

acting as $\theta(x) = (\dots, \theta_p(x), \dots)$. This is clear, since the intersection of the kernels of the θ_p for all p is 0. Consequently, $A = 0$ if and only if $A_{(p)} = 0$ for all primes p.

(F) *A homomorphism* $\phi : A \to C$ *is injective* (*respectively, surjective*) *if and only if* $\phi_p : A_{(p)} \to C_{(p)}$ *is injective* (*respectively, surjective*) *for all primes p.* If ϕ is monic, then ϕ_p is also monic by (B). Conversely, the sequence $0 \to K \to A \xrightarrow{\phi} C$ is exact ($K = \operatorname{Ker}\phi$), so the same is true for the localization sequence $0 \to K_{(p)} \to A_{(p)} \xrightarrow{\phi_p} C_{(p)}$. By hypothesis, $K_{(p)} = 0$ for all p, so $K = 0$ by (E). A similar proof applies to the epic case.

Thus $\phi : A \to C$ is an isomorphism if and only if $\phi_p : A_{(p)} \to C_{(p)}$ is an isomorphism for every prime p. However, it should be kept in mind that $A \cong C$ *need not be true even if* $A_{(p)} \cong C_{(p)}$ *for all p* (Exercise 4).

Localization of Torsion-Free Groups If G is torsion-free, then the canonical map $\theta_p(G)$ is monic for each p. Thus $\theta_p(G)$ embeds G as a subgroup in $G_{(p)}$ which in turn is embedded in the divisible hull of G. These embeddings are natural which makes it possible to form intersections of distinct localizations.

Lemma 5.1. *For any torsion-free group G,*

$$G = \cap_p G_{(p)}$$

where the intersection is taken in the injective hull of G.

Proof. This is a special case of a general theorem on localization over commutative rings, but it is easy to give a quick proof for groups. Since $G \leq G_{(p)}$ for all primes p, it is enough to prove that every $x \in \cap_p G_{(p)}$ belongs to G. For a p, there is $n \in \mathbb{N}$, coprime to p, such that $nx \in G$. Also, for every prime divisor p_j of n we can find an $n_j \in \mathbb{N}$ coprime to p_j with $n_j x \in G$. If $t, t_j \in \mathbb{Z}$ satisfy $tn + t_1 n_1 + \cdots + t_\ell n_\ell = 1$, then $x = tnx + t_1 n_1 x + \cdots + t_\ell n_\ell x \in G$. □

★ **Notes.** There is an increasing interest in localizations. They can be defined as idempotent functors $L\colon \mathcal{A}b \to \mathcal{A}b$ along with the natural transformation $\mathbf{1}_{\mathcal{A}b} \to L$.

Exercises

(1) If F is a free group, then $F_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module.
(2) Localization commutes with finite intersections: $(G \cap H)_{(p)} = G_{(p)} \cap H_{(p)}$.
(3) Find a counterexample to show that in general direct products do not commute with localization.
(4) A and C need not be isomorphic even if $A_{(p)} \cong C_{(p)}$ for every prime p. [Hint: $\mathbb{Z}$ and the group of rational numbers with square-free denominators.]
(5) $\operatorname{Hom}(A, C)_{(p)} \cong \operatorname{Hom}(A_{(p)}, C_{(p)})$ need not be true, not even for torsion-free A, C. [Hint: $A = \mathbb{Z}_{(p)}, C = \mathbb{Z}$.]
(6) (*Hasse principle* for abelian groups) Let B be a subgroup of A with A/B torsion, and let $\phi : B \to C$ be a homomorphism. Suppose that, for each prime p, the localization map $\phi_p : B_{(p)} \to C_{(p)}$ has an extension $\psi_p : A_{(p)} \to C_{(p)}$. Then there exists a homomorphism $\phi^* : A \to C$ that extends ϕ and satisfies $\phi^*_p = \psi_p$ for each prime p.

Problems to Chapter 8

PROBLEM 8.1. Find the groups A with $\operatorname{Tor}(A, A) \cong A$.

PROBLEM 8.2. Determine the Ulm factors of $\operatorname{Tor}(A, C)$ for p-groups A, C.

PROBLEM 8.3. When can a reduced p-group A be written as $\operatorname{Tor}(B, C)$ for reduced p-groups B, C?

PROBLEM 8.4. When is a rank 4 torsion-free group the tensor product of two rank 2 torsion-free groups? More generally, when is a torsion-free group a non-trivial tensor product?

Lausch [1] has results on the rank 4 case.

Chapter 9
Groups of Extensions and Cotorsion Groups

Abstract The extension problem for abelian groups (as a special case of the general group-theoretical question formulated by O. Schreier) consists in constructing a group from a normal subgroup and the corresponding factor group. The classical way of discussing extensions is *via* factor sets which we follow in our presentation (simplified for the abelian case). Then we introduce Baer's group Ext, an extremely important device, and discuss its fundamental properties. The intimate relationship between Hom and Ext has been pointed out by Eilenberg–MacLane [1]; this led to the interpretation of Ext as a derived functor of Hom and has been exploited extensively in Homological Algebra. Another important functor is Pext, the group of pure extensions, which appears unexpectedly as the first Ulm subgroup of Ext.

The investigation of the group structure of Ext leads to the concept of cotorsion group, a generalization of algebraic compactness. We give special prominence to cotorsion groups that occur not only as Ext, but also in several other forms.

1 Group Extensions

The Extension Problem Given two groups, A and C, the *extension problem* consists in finding all groups B such that B contains a subgroup A' isomorphic to A and $B/A' \cong C$. This situation can be expressed in terms of the exact sequence

$$\mathfrak{e}:\ 0 \to A \xrightarrow{\mu} B \xrightarrow{\nu} C \to 0, \tag{9.1}$$

where μ stands for the inclusion map, and ν is an epimorphism with kernel μA. In this case, we say that B is an **extension of** A **by** C.

It is our next aim to survey all extensions for fixed A and C. This can be done in different ways. We first describe extensions *via* factor sets (in a pedestrian way), and then we discuss them by using short exact sequences (which is a more attractive and more powerful method).

Let $a, b, \ldots$ denote elements of A, and $u, v, w, \ldots$ those of C. Assuming that B is an extension of A by C, we pick a **transversal**; this is a function $g : C \to B$ that assigns an element of B in the coset corresponding to u, i.e. $g(u) \in \nu^{-1}u$. For convenience, it is always assumed that $g(0) = 0$. Once the function g is selected, every $b \in B$ has a unique form $b = g(u) + \mu a$ with $u \in C,\ a \in A$. Since $g(u) + g(v)$ and $g(u + v)$ belong to the same coset mod μA, there is an $f(u, v) \in A$ such that

$$g(u) + g(v) = g(u + v) + \mu f(u, v).$$

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
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Thus we have a function $f: C \times C \to A$, uniquely determined by the extension B and by the choice of representatives $g(u)$. The commutative and associative laws imply that f satisfies

$$f(u, v) = f(v, u), \quad f(u, v) + f(u + v, w) = f(u, v + w) + f(v, w) \tag{9.2}$$

for all $u, v, w \in C$. A function f satisfying these two equations is called a **factor set on** C **to** A. Evidently, a splitting map for ν (if it exists) is also a transversal for which the factor set is identically 0.

Assume, conversely, that we are given two groups, A and C, along with a factor set $\{f(u, v)\}_{u,v \in C}$ on C to A. We will construct a group B as the set of all pairs $(u, a) \in C \times A$ with the operation

$$(u, a) + (v, b) = (u + v, a + b + f(u, v)).$$

The commutative and associative laws in B are consequences of conditions (9.2), while $(0, 0)$ is the zero in B, and $(-u, -a - f(-u, a))$ is the inverse to (u, a). Manifestly, the mappings $\mu : a \mapsto (0, a), \nu : (u, a) \mapsto u$ will make the sequence (9.1) exact. Thus B is an extension of A by C, where the choice $g(u) = (u, 0)$ corresponds to the factor set f.

The relation between extensions and factor sets being clarified, it is now obvious that the extension problem amounts to finding all factor sets. The direct sum $C \oplus A$ is always a solution to the extension problem; it is called the **splitting extension**. If $g(u) = (u, 0) \in C \oplus A$ is chosen, then $f(u, v)$ is identically 0.

Equivalence of Extensions While a factor set determines the extension uniquely, the converse is not true: a factor set depends also on the representatives selected. In order to remedy the problem with this ambiguity, an equivalence relation is introduced for factor sets. We shall call the factor sets f_1 and f_2 **equivalent** if they are coming from the same extension by using another set of representatives. This means that the new representatives will be $g(u) + \mu h(u)$ with some $h(u) \in A$, and accordingly, the new factor set will look like $f(u, v) + h(u) + h(v) - h(u + v)$. A factor set $f'(u, v)$ of the form $f'(u, v) = h(u) + h(v) - h(u + v)$ with any function $h : C \to A$ is called a **transformation set** or **coboundary**; we always choose $h(0) = 0$. Consequently, by definition, *two factor sets are equivalent if and only if their difference is a coboundary.*

One should be aware of the fact that two equivalent, but different, factor sets do not necessarily define identical extensions. However, the two extensions, B and B', are essentially the same in the sense that the correspondence $\beta : (u, a) \mapsto (u, a + h(u))'$ is not only an isomorphism $\beta : B \to B'$, but it is a special one that induces the identity maps both on A and on C. This is tantamount to saying that the diagram

$$\begin{array}{ccccccccc}
\mathfrak{e}\colon\ 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \| & & \\
\mathfrak{e}'\colon\ 0 & \longrightarrow & A & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C & \longrightarrow & 0
\end{array}$$

is commutative. In this case, the extensions $\mathfrak{e}$ and $\mathfrak{e}'$ themselves are called **equivalent**. Thus, *there is a bijective correspondence between the equivalence classes of extensions of A by C and the equivalence classes of factor sets $f : C \times C \to A$.* In this correspondence, the splitting extensions form an equivalence class, and the corresponding equivalence class of factor sets consists of factor sets of the form $h(u) + h(v) - h(u + v)$ for arbitrary functions $h\colon C \to A$ (with $h(0) = 0$ as agreed).

The Group Ext We have come to a leading idea: instead of being satisfied with a survey of the collection of all extensions, one tries to furnish this set with a proper algebraic structure which would provide a more powerful tool in the exploration. This was done by R. Baer who introduced a group structure, creating a fascinating theory.

If $f, f' : C \times C \to A$ are factor sets, then their sum $f + f'$ defined as

$$(f + f')(u, v) = f(u, v) + f'(u, v)$$

is again a factor set. Under this composition, the factor sets form a group, denoted $Z(C, A)$. The coboundaries form a subgroup $B(C, A)$, and what has been concluded above can be rephrased by saying that there is a bijective correspondence between the equivalence classes of extensions of A by C and the elements of the factor group $Z(C, A)/B(C, A)$. This factor group is generally called the **group of extensions of** A **by** C:

$$\operatorname{Ext}(C, A) = Z(C, A)/B(C, A).$$

Having defined Ext in terms of factor sets, we now proceed to another interpretation: *via* short exact sequences (9.1). If we think of extensions of A by C as such sequences, then it seems reasonable to create and to study first a category whose objects are short exact sequences. In doing so, the first order of business is to define the morphisms between two exact sequences, $\mathfrak{e}$ and $\mathfrak{e}'$. The right definition is pretty clear: it is a triple (α, β, γ) of group homomorphisms rendering the diagram

$$\begin{array}{ccccccccc}
\mathfrak{e}\colon\ 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
\mathfrak{e}'\colon\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C' & \longrightarrow & 0
\end{array}$$

commutative. There is no difficulty in showing that in this way we have indeed defined a category which will be denoted $\mathcal{E}$.

In accordance with our definition of equivalent extensions above, we will say that the exact sequences $\mathfrak{e}$ and $\mathfrak{e}'$ (*qua* extensions) are **equivalent** (notation: $\mathfrak{e} \equiv \mathfrak{e}'$) if $A = A', C = C'$ and there is a morphism $(\mathbf{1}_A, \beta, \mathbf{1}_C)$ from $\mathfrak{e}$ to $\mathfrak{e}'$. That β is then an isomorphism follows at once from Lemma 2.6 in Chapter 1.

The beauty of treating extensions as short exact sequences lies in the fact that we can work with commutative diagrams, making most proofs more transparent. But first we must learn the basic facts about the category $\mathcal{E}$.

To start with, we concentrate on extensions of a fixed group A. If $\gamma : C' \to C$ is a homomorphism, then there is a pull-back square

$$\begin{tikzcd} & & B' \arrow[r, "\nu'"] \arrow[d, "\beta"'] & C' \arrow[d, "\gamma"'] & \\ 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

From Sect. 3(a) in Chapter 2 we know that ν' is an epimorphism (since so is ν), and the pull-back property shows that $\operatorname{Ker} \nu' \cong \operatorname{Ker} \nu \cong A$. Hence there exists a monic map $\mu' : A \to B'$ acting as $\mu' : a \mapsto (\mu a, 0) \in B' (\leq B \oplus C')$, so that the diagram

$$\begin{tikzcd} \mathfrak{e}\gamma\colon 0 \arrow[r] & A \arrow[r, "\mu'"] \arrow[d, equal] & B' \arrow[r, "\nu'"] \arrow[d, "\beta"'] & C' \arrow[r] \arrow[d, "\gamma"'] & 0 \\ \mathfrak{e}\colon 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

has exact rows and commutative squares. The top row is an extension of A by C' which we have denoted by $\mathfrak{e}\gamma$ to indicate its origin from $\mathfrak{e}$ *via* γ. Note that $\gamma^* = (\mathbf{1}_A, \beta, \gamma)$ is a morphism $\mathfrak{e} \to \mathfrak{e}\gamma$ in the category $\mathcal{E}$.

Now suppose we have a similar commutative diagram

$$\begin{tikzcd} \mathfrak{e}''\colon 0 \arrow[r] & A \arrow[r, "\mu''"] \arrow[d, equal] & B'' \arrow[r, "\nu''"] \arrow[d, "\beta'"'] & C' \arrow[r] \arrow[d, "\gamma"'] & 0 \\ \mathfrak{e}\colon 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows, for another group B''. By the pull-back property there is a unique $\phi : B'' \to B'$ such that $\nu'\phi = \nu''$ and $\beta\phi = \beta'$. Since the maps $\phi\mu'', \mu' : A \to B'$ are such that $\beta(\phi\mu'') = \beta'\mu'' = \mu = \beta\mu'$ and $\nu'(\phi\mu'') = \nu''\mu'' = 0 = \nu'\mu'$, the uniqueness assertion on pull-backs implies $\phi\mu'' = \mu'$. Hence $(\mathbf{1}_A, \phi, \mathbf{1}_{C'})$ is an

$\mathcal{E}$-morphism $\mathfrak{e}'' \to \mathfrak{e}\gamma$, and so $\mathfrak{e}'' \equiv \mathfrak{e}\gamma$. This shows that $\mathfrak{e}\gamma$ is unique up to equivalence. Once this has been established, we can then assert that $\mathfrak{e}\mathbf{1}_C \equiv \mathfrak{e}$ and $\mathfrak{e}(\gamma\gamma') \equiv (\mathfrak{e}\gamma)\gamma'$ for $C'' \xrightarrow{\gamma'} C' \xrightarrow{\gamma} C$. If we view $\operatorname{Ext}(*, A)$ for a fixed A as a functor $\mathcal{A}b \to \mathcal{A}b$, then its contravariant character is evident.

Next we switch the roles: we keep C fixed and let A vary. Given a map $\alpha : A \to A'$, let B' be defined by the push-out square (on the left)

$$\begin{array}{rccccccccc}
\mathfrak{e}\colon & 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\
& & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \| & & \\
\alpha\mathfrak{e}\colon & 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C & \longrightarrow & 0
\end{array}$$

where we have already completed the diagram to have an exact sequence at the bottom row. This could be done, since μ' is a monomorphism, and if B' is defined as a factor group of $A' \oplus B$, then $\nu'((a', b) + H) = \nu b$ makes the diagram commute. The bottom row is an extension of A' by C which was denoted as $\alpha\mathfrak{e}$. As before, we can show (using dual arguments) that $\alpha\mathfrak{e}$ is unique up to equivalence, furthermore, $\mathbf{1}_A\mathfrak{e} \equiv \mathfrak{e}$ and $(\alpha'\alpha)\mathfrak{e} \equiv \alpha'(\alpha\mathfrak{e})$ for $A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A''$. Thus, for a fixed C, $\operatorname{Ext}(C, *)$ is a covariant functor $\mathcal{A}b \to \mathcal{A}b$.

Combining the maps $\alpha : A \to A'$ and $\gamma : C' \to C$, we obtain the important associative law: $\alpha(\mathfrak{e}\gamma) \equiv (\alpha\mathfrak{e})\gamma$. Indeed, by making use of the pull-back property of $(\alpha\mathfrak{e})\gamma$, it is easy to verify the existence of a morphism $(\alpha, \beta', \mathbf{1}_C) \colon \mathfrak{e}\gamma \to (\alpha\mathfrak{e})\gamma$ and the commutativity of the square

$$\begin{array}{ccc}
\mathfrak{e}\gamma & \xrightarrow{(\mathbf{1},\beta',\gamma)} & \mathfrak{e} \\
{\scriptstyle(\alpha,\beta',\mathbf{1})}\downarrow & & \downarrow{\scriptstyle(\alpha,\beta,\mathbf{1})} \\
(\alpha\mathfrak{e})\gamma & \xrightarrow{(\mathbf{1},\beta,\gamma)} & \alpha\mathfrak{e}
\end{array}$$

The Baer Sum of Extensions It was shown above that the extensions (more precisely, the equivalence classes of extensions) of A by C form a group. We wonder how to describe the group operation in the language of short exact sequences. We will use the diagonal map $\Delta_G : G \to G \oplus G$ (where $g \mapsto (g, g)$), as well as the codiagonal map $\nabla_G \colon G \oplus G \to G$ (where $(g_1, g_2) \mapsto g_1 + g_2$). If we understand by the **direct sum of extensions** $\mathfrak{e}_i \colon 0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$ $(i = 1, 2)$ the extension

$$\mathfrak{e}_1 \oplus \mathfrak{e}_2 \colon\ 0 \to A_1 \oplus A_2 \xrightarrow{\alpha_1 \oplus \alpha_2} B_1 \oplus B_2 \xrightarrow{\beta_1 \oplus \beta_2} C_1 \oplus C_2 \to 0,$$

then we can state:

Proposition 1.1 (Baer). *The sum of the extensions* $\mathfrak{e}_1$ *and* $\mathfrak{e}_2$ *of* A *by* C *is the extension*

$$\mathfrak{e}_1 + \mathfrak{e}_2 = \nabla_A(\mathfrak{e}_1 \oplus \mathfrak{e}_2)\Delta_C.$$

Proof. What we have to establish is that if $f_i : C \times C \to A$ is a factor set belonging to $\mathfrak{e}_i$ ($i = 1, 2$), then $f_1 + f_2$ belongs to $\nabla_A(\mathfrak{e}_1 \oplus \mathfrak{e}_2)\Delta_C$. It is clear that $(f_1(c_1, c_2), f_2(c_1', c_2'))$ with $c_i, c_i' \in C$ is a factor set belonging to the direct sum $\mathfrak{e}_1 \oplus \mathfrak{e}_2$, and $(f_1(c_1, c_2), f_2(c_1, c_2))$ is one corresponding to $(\mathfrak{e}_1 \oplus \mathfrak{e}_2)\Delta_C$. An application of ∇_A yields the factor set $f_1(c_1, c_2) + f_2(c_1, c_2)$. □

It is now easy to conclude that for the homomorphisms $\alpha_i : A \to A'$ and $\gamma_i : C' \to C$ the following equivalences hold for the extensions $\mathfrak{e}, \mathfrak{e}_1, \mathfrak{e}_2$ of A by C:

$$\alpha(\mathfrak{e}_1 + \mathfrak{e}_2) \equiv \alpha\mathfrak{e}_1 + \alpha\mathfrak{e}_2, \quad (\mathfrak{e}_1 + \mathfrak{e}_2)\gamma \equiv \mathfrak{e}_1\gamma + \mathfrak{e}_2\gamma, \tag{9.3}$$

$$(\alpha_1 + \alpha_2)\mathfrak{e} \equiv \alpha_1\mathfrak{e} + \alpha_2\mathfrak{e}, \quad \mathfrak{e}(\gamma_1 + \gamma_2) \equiv \mathfrak{e}\gamma_1 + \mathfrak{e}\gamma_2. \tag{9.4}$$

The equivalences (9.3) express the fact that $\alpha_* : \mathfrak{e} \mapsto \alpha\mathfrak{e}$ and $\gamma^* : \mathfrak{e} \mapsto \mathfrak{e}\gamma$ are group homomorphisms

$$\alpha_* : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, A'), \quad \gamma^* : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C', A),$$

while (9.4) asserts that $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$, i.e. the correspondence $\mathrm{Ext} : C \times A \to \mathrm{Ext}(C, A)$ with $\gamma \times \alpha \mapsto \gamma^*\alpha_* = \alpha_*\gamma^*$ is an additive bifunctor $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$. This fact is important enough to be recorded as a theorem:

Theorem 1.2 (Eilenberg–MacLane [1]). Ext *is an additive bifunctor* $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$*, contravariant in the first, and covariant in the second variable.* □

In order to be consistent with the functorial notation for homomorphisms, we shall also use the notation $\mathrm{Ext}(\gamma, \alpha)$ for $\gamma^*\alpha_* = \alpha_*\gamma^*$, thus $\mathrm{Ext}(\gamma, \alpha) : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C', A')$ acting as $\mathfrak{e} \mapsto \alpha\mathfrak{e}\gamma$. In the sequel, we will often deal with diagrams of extensions, emphasizing that in all of these diagrams extensions can be replaced by equivalent extensions without causing any harm to commutativity or exactness.

A useful observation: if C is a p-group, then there are natural isomorphisms

$$\mathrm{Ext}(C, A) \cong \mathrm{Ext}(C, \mathbb{Z}_{(p)} \otimes A) \quad \text{and} \quad \mathrm{Ext}(A, C) \cong \mathrm{Ext}(\mathbb{Z}_{(p)} \otimes A, C).$$

In both cases the right-hand sides can also be interpreted as $\mathrm{Ext}_{\mathbb{Z}_{(p)}}$.

★ **Notes.** If we wish to develop extensions solely *qua* short exact sequences, then the Baer sum would serve as the definition of the sum of two extensions, and then we have to verify: (1) $\mathfrak{e}_1 + \mathfrak{e}_2$ is indeed an extension of A by C which stays in the same equivalence class if $\mathfrak{e}_1, \mathfrak{e}_2$ are replaced by equivalent extensions, and (2) the equivalence classes form a group under this operation. For details of this approach, we refer to MacLane [M]. A third method of introducing Ext is as the derived functor of Hom, this is a popular way of defining Ext in Homological Algebra, where also the higher Ext^n functors are dealt with (for $n > 1$, these are trivial for abelian groups).

It was O. Schreier (1926) who started a more systematic investigation of the extensions of non-commutative groups (in a multiplicative notation): given the groups A, C, new groups G have to be constructed such that G contains a normal subgroup N isomorphic to A satisfying $G/N \cong C$. Besides factor sets, he used also automorphisms of A (these are not needed in the commutative case) to describe extensions. Extensions without factor sets were discussed by Baer (1935). Topological applications were discovered by S. Eilenberg and S. Mac Lane (1942).

A note on the terminology is in order. A few authors apply a reverse terminology: if $B/A \cong C$, then they say that B is an extension of C by A. It seems more natural to extend on the top than in the bottom.

Exercises

(1) A factor set on any group C to a divisible group D is a transformation set. [Hint: set up a system of equations.]
(2) Let B be a basic subgroup of the reduced p-group A. For any group C, any factor set $f: C \times C \to A$ is equivalent to a factor set $g: C \times C \to B$.
(3) The exact sequences $0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}(3) \to 0$ and $0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}(3) \to 0$ represent different elements in $\mathrm{Ext}(\mathbb{Z}(3), \mathbb{Z})$ if $\beta : 1 \mapsto 1 + 3\mathbb{Z}$, $\gamma : 1 \mapsto 2 + 3\mathbb{Z}$.
(4) There are p inequivalent extensions of $\mathbb{Z}(p)$ by $\mathbb{Z}(p)$, but only two non-isomorphic ones.
(5) In terms of factor sets f, the extension $\mathfrak{e}\gamma$ corresponds to the composite map $C' \times C' \xrightarrow{\gamma \times \gamma} C \times C \xrightarrow{f} A$, and $\alpha\mathfrak{e}$ to $C \times C \xrightarrow{f} A \xrightarrow{\alpha} A'$ (notation as in the text).
(6) If $(\alpha, \beta, \gamma): \mathfrak{e} \to \mathfrak{e}'$ is a morphism in the category $\mathcal{E}$, then $\alpha\mathfrak{e} \equiv \mathfrak{e}'\gamma$.
(7) (a) Let $\alpha : A \to A'$ be an epimorphism. Then $\alpha\mathfrak{e}$ with $\mathfrak{e}$ as in (9.1) is equivalent to the extension $0 \to A/\operatorname{Ker}\alpha \to B/\mu(\operatorname{Ker}\alpha) \to C \to 0$ (obvious maps).
    (b) Let $\gamma : C' \to C$ be a monic map. Then $\mathfrak{e}\gamma$ is equivalent to $0 \to A \to \nu^{-1}(\operatorname{Im}\gamma) \to \operatorname{Im}\gamma \to 0$.
(8) (a) If an extension is represented by a pure-exact sequence, then the same holds for all equivalent extensions.
    (b) Show that the Baer sum of two extensions represented by pure-exact sequences is again pure-exact.
(9) (a) Let α be an automorphism of A. When is $\mathfrak{e}$ equivalent to $\alpha\mathfrak{e}$?
    (b) When does $\mathfrak{e}\gamma \equiv \mathfrak{e}$ hold for some $\gamma \in \operatorname{Aut} C$?

2 Exact Sequences for Hom and Ext

The functorial aspects of Ext having been settled, we move on to gather more information about its formal properties, in particular, its behavior towards exact sequences. What might come as a surprise is that it is intimately connected with the functor Hom: Ext emerges as an "error term" for the failure of exactness of the Hom sequence in Sect. 1 in Chapter 7.

Preparatory Lemmas We need to do a little technical work before we prove our main result in this section. We start with an easy, but very useful observation. (Compare with Lemmas 2.1 and 2.2 in Chapter 1.)

Lemma 2.1. *Suppose we are given the diagram*

$$\begin{tikzcd} & & & G \arrow[dl, dashed, "\xi"'] \arrow[d, "\eta"] & \\ \mathfrak{e}:\ 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\chi"'] & B \arrow[r, "\beta"] \arrow[dl, dashed, "\zeta"] & C \arrow[r] & 0 \\ & H & & & \end{tikzcd}$$

with exact row. Then

(i) *there exists a map $\xi : G \to B$ such that $\beta\xi = \eta$ if and only if $\mathfrak{e}\eta$ splits;*
(ii) *there is a map $\zeta : B \to H$ such that $\zeta\alpha = \chi$ if and only if $\chi\mathfrak{e}$ splits.*

Proof. If there is such a ξ, then the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus G \arrow[r] \arrow[d, "\nabla(\alpha\oplus\xi)"] & G \arrow[r] \arrow[d, "\eta"] & 0 \\ \mathfrak{e}:\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

(with the obvious maps in the top row) commutes, hence the top row is $\equiv \mathfrak{e}\eta$. Conversely, if $\mathfrak{e}\eta : 0 \to A \to B' \to G \to 0$ splits, then a splitting map followed by $B' \to B$ yields a map $\xi : G \to B$ with the desired property. For ζ the proof is dual. $\square$

The following lemma unveils the basic relationship between Hom and Ext. Actually, it is the jumping board to Theorem 2.3; it deals with a special case needed in the proof of Theorem 2.3.

Lemma 2.2. *Let $0 \to H \to F \to C \to 0$ be a free resolution of C and $0 \to A \to D \to N \to 0$ an injective resolution of A.*

(i) *There exists a natural map $\delta^* : \mathrm{Hom}(H,A) \to \mathrm{Ext}(C,A)$ such that following the sequence is exact:*

$$0 \to \mathrm{Hom}(C,A) \to \mathrm{Hom}(F,A) \to \mathrm{Hom}(H,A) \xrightarrow{\delta^*} \mathrm{Ext}(C,A) \to 0. \qquad (9.5)$$

(ii) *There is a natural map* $\delta_* : \text{Hom}(C, N) \to \text{Ext}(C, A)$ *that makes the following sequence exact:*

$$0 \to \text{Hom}(C, A) \to \text{Hom}(C, D) \to \text{Hom}(C, N) \xrightarrow{\delta_*} \text{Ext}(C, A) \to 0. \qquad (9.6)$$

Proof. Starting from the middle row, consider the following diagram with exact rows and commutative squares:

$$\begin{tikzcd}
\mathfrak{e}_1\colon & 0 \arrow[r] & H \arrow[r, "\mu"] \arrow[d, "\chi"'] & F \arrow[r] \arrow[d, "\phi"] \arrow[dl, dashed, "\zeta"'] & C \arrow[r] \arrow[d, equal] & 0 \\
\mathfrak{e}\colon & 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "\beta"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "\eta"] \arrow[dl, dashed, "\xi"'] & 0 \\
\mathfrak{e}_2\colon & 0 \arrow[r] & A \arrow[r] & D \arrow[r, "\nu"] & N \arrow[r] & 0,
\end{tikzcd}$$

where ϕ exists, because F is free, and ψ exists, because D is divisible; then χ and η are automatic. Thus $\mathfrak{e} \equiv \chi\mathfrak{e}_1$ and $\mathfrak{e} \equiv \mathfrak{e}_2\eta$, showing that the maps in (9.5) and (9.6) between the Hom and Ext are surjective. To prove exactness at the last Homs, we refer to the preceding lemma to argue that $\mathfrak{e} \equiv \chi\mathfrak{e}_1$ is splitting if and only if there is a map $\zeta : F \to A$ such that $\zeta\mu = \chi$, and similarly, $\mathfrak{e} \equiv \mathfrak{e}_2\eta$ is splitting exactly if there is a map $\xi : C \to D$ such that $\nu\xi = \eta$. This amounts to saying that the kernels are the images of the maps $\text{Hom}(F, A) \to \text{Hom}(H, A)$ and $\text{Hom}(C, D) \to \text{Hom}(C, N)$, respectively. We leave it as an exercise to the reader to show that the maps δ^*, δ_* (called **connecting maps**) are natural. □

The Hom-Ext Exact Sequences We now have all the ingredients to verify the extremely important long exact sequences.

Theorem 2.3 (Cartan–Eilenberg [CE]). *If* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is an exact sequence, then so are the sequences*

$$0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to$$
$$\xrightarrow{\delta^*} \text{Ext}(C, G) \xrightarrow{\beta^*} \text{Ext}(B, G) \xrightarrow{\alpha^*} \text{Ext}(A, G) \to 0 \qquad (9.7)$$

and

$$0 \to \text{Hom}(G, A) \to \text{Hom}(G, B) \to \text{Hom}(G, C) \to$$
$$\xrightarrow{\delta_*} \text{Ext}(G, A) \xrightarrow{\alpha_*} \text{Ext}(G, B) \xrightarrow{\beta_*} \text{Ext}(G, C) \to 0 \qquad (9.8)$$

for every group G. All the maps are natural.

Proof. The proof relies on the Snake Lemma 2.5 in Chapter 1. We first choose free resolutions $0 \to H_1 \to F_1 \to A \to 0$ and $0 \to H_2 \to F_2 \to C \to 0$, and note that there is a free resolution $0 \to H_1 \oplus H_2 \to F_1 \oplus F_2 \to B \to 0$ (see Sect. 1, Exercise 7 in Chapter 3). These are used in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(C,G) \arrow[r, "\rho_1"] \arrow[d] & \mathrm{Hom}(B,G) \arrow[r, "\sigma_1"] \arrow[d] & \mathrm{Hom}(A,G) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(F_2,G) \arrow[r, "\mu"] \arrow[d, "\alpha"] & \mathrm{Hom}(F_1 \oplus F_2,G) \arrow[r, "\nu"] \arrow[d, "\beta"'] & \mathrm{Hom}(F_1,G) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(H_2,G) \arrow[r, "\mu'"] \arrow[d] & \mathrm{Hom}(H_1 \oplus H_2,G) \arrow[r, "\nu'"] \arrow[d] & \mathrm{Hom}(H_1,A) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Ext}(C,G) \arrow[r, "\rho_2"] & \mathrm{Ext}(B,G) \arrow[r, "\sigma_2"] & \mathrm{Ext}(A,G) \arrow[r] & 0
\end{tikzcd}$$

where the middle rows are exact, because F_i, H_i are free groups, while the columns (bordered with 0's, not shown) are exact, thanks to Lemma 2.2. By Lemma 2.5 in Chapter 1, (9.7) follows from the diagram. The proof of (9.8) is similar. □

Purity in the Exact Sequence on Ext We complement Theorem 2.3 with the following result where the starting short exact sequence is pure-exact.

Lemma 2.4. *Suppose* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is a pure-exact sequence. Then, for every group G, the induced homomorphisms*

$$\beta^* : \mathrm{Ext}(C,G) \to \mathrm{Ext}(B,G) \quad \textit{and} \quad \alpha_* : \mathrm{Ext}(G,A) \to \mathrm{Ext}(G,B)$$

map onto pure subgroups.

Proof. We give a detailed proof for the first part (using Theorem 5.2 below), and leave the rest to the reader as an exercise. In the following commutative diagram, we start from the bottom row and then form the preceding rows by using the maps in the right column as indicated:

$$\begin{tikzcd}
\mathfrak{e}\beta\sigma\rho : 0 \arrow[r] & G \arrow[r] \arrow[d, equal] & K' \arrow[r] \arrow[d] & C[n] \arrow[r] \arrow[d, "\rho"] & 0 & \\
\mathfrak{e}\beta\sigma : 0 \arrow[r] & G \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & B[n] \arrow[r] \arrow[d, "\sigma"] & 0 & \\
\mathfrak{e}\beta : 0 \arrow[r] & G \arrow[r] \arrow[d, equal] & H' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\beta"] & 0 & (\in \mathrm{Im}\,\beta^*) \\
\mathfrak{e} : 0 \arrow[r] & G \arrow[r] & H \arrow[r] & C \arrow[r] & 0 &
\end{tikzcd}$$

Here the rows are exact, σ denotes the injection map, and ρ exists, due to the purity of the given exact sequence. The composite map $\beta\sigma\rho\colon C[n] \to C$ will be the natural injection. If the last but one row belongs to $n\,\mathrm{Ext}(B,G)$, then Theorem 5.2 below shows that the second row splits, and hence so does the top row. This means that the bottom exact sequence belongs to $n\,\mathrm{Ext}(C,G)$, so the last but one row is in $n\,\mathrm{Im}\,\beta^*$. This holds for every $n \in \mathbb{N}$, thus $\mathrm{Im}\,\beta^*$ is pure in $\mathrm{Ext}(B,G)$. □

Ext and Direct Sums, Products We have to find out how Ext behaves towards direct sums and products. In view of Theorem 2.3, it should not come as a surprise that Ext imitates Hom in this respect.

Theorem 2.5. *For all groups A, A_i, C, C_i, there exist natural isomorphisms*

$$\mathrm{Ext}(\oplus_{i\in I} C_i, A) \cong \prod_{i\in I} \mathrm{Ext}(C_i, A), \tag{9.9}$$

$$\mathrm{Ext}\left(C, \prod_{i\in I} A_i\right) \cong \prod_{i\in I} \mathrm{Ext}(C, A_i). \tag{9.10}$$

Proof. We prove (9.9), the proof of (9.10) runs dually. We start with free resolutions of the C_i, $0 \to H_i \to F_i \to C_i \to 0$ with F_i free, to obtain the exact sequences $\mathrm{Hom}(F_i, A) \to \mathrm{Hom}(H_i, A) \to \mathrm{Ext}(C_i, A) \to \mathrm{Ext}(F_i, A) = 0$. The exact sequence $0 \to \oplus_i H_i \to \oplus_i F_i \to \oplus_i C_i \to 0$ induces the top exact sequence in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\oplus_i F_i, A) \arrow[r] \arrow[d] & \mathrm{Hom}(\oplus_i H_i, A) \arrow[r] \arrow[d] & \mathrm{Ext}(\oplus_i C_i, A) \arrow[r] \arrow[d] & 0 \\
\prod_i \mathrm{Hom}(F_i, A) \arrow[r] & \prod_i \mathrm{Hom}(H_i, A) \arrow[r] & \prod_i \mathrm{Ext}(C_i, A) \arrow[r] & 0
\end{tikzcd}$$

We know from Theorem 1.7 in Chapter 7 that the first two vertical maps represent natural isomorphisms, whence we can conclude that there is a natural isomorphism between the two Exts in the diagram. □

★ **Notes.** The long exact sequence for Hom-Ext generalizes for modules over arbitrary rings. However, the 0 at the right end is then replaced by sequences of higher Exts.

The functor Ext does not behave in the same way towards direct and inverse limits as Hom. If $\{C_i (i \in I) | \pi_i^j\}$ is a direct system of groups with direct limit C, then $\{\mathrm{Ext}(C_i, A)(i \in I) | \mathrm{Ext}(\pi_i^j, \mathbf{1}_A)\}$ is an inverse system. The most we can say in general is that there is a natural homomorphism $\phi : \mathrm{Ext}(C, A) \to \varprojlim \mathrm{Ext}(C_i, A)$. For the special case when the direct system is indexed by the natural numbers, see Lemma 5.9. In general, the so-called Mittag-Leffler condition guarantees the surjectivity of ϕ.

Göbel–Prelle [1] ask for groups G with the properties like $\mathrm{Ext}(\prod A_i, G) \cong \prod_i \mathrm{Ext}(A_i, G)$ for all choices of the A_i, and show that "only $G \cong \mathbb{Q}$" is the answer.

Exercises

(1) The group G is divisible if and only if, for every monomorphism $\alpha: A \to B$, the induced map $\alpha^*: \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G)$ is epic.
(2) G is free if and only if, for every epimorphism $\beta: B \to C$, the induced map $\beta_*: \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C)$ is epic.
(3) Let $\alpha: A \to B$ be a monomorphism such that $\alpha_*: \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, B)$ is monic for every G. Prove that $\mathrm{Im}\, \alpha$ is a summand of B.
(4) Let $\beta: B \to C$ be an epimorphism such that $\beta^*: \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G)$ is monic for every G. Then $\mathrm{Ker}\, \beta$ is a summand of B.
(5) Suppose A satisfies $\mathrm{Hom}(A, \mathbb{Z}) = 0$. Then $\mathrm{rk}_p \mathrm{Ext}(A, \mathbb{Z}) = \mathrm{rk}_p \mathrm{Ext}(A/pA, \mathbb{Z})$.
(6) Prove that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{Q}^{\aleph_0}$. [Hint: $0 \to \mathbb{Z} \to \mathbb{Q} \to \oplus_p \mathbb{Z}(p^\infty) \to 0$.]
(7) Verify the isomorphism (for any p) $\mathrm{Ext}(\mathbb{Z}(p^\infty), J_p) \cong J_p$. [Hint: $0 \to J_p \to Q_p^* \to \mathbb{Z}(p^\infty) \to 0$.]
(8) Assume B is a basic subgroup of the p-group A. For any group G, the groups $\mathrm{Ext}(G, A)$ and $\mathrm{Ext}(G, B)$ are epimorphic images of each other. [Hint: Theorem 6.10 in Chapter 5.]

3 Basic Properties of Ext

In this section our objective is to record a number of elementary and some advanced properties of the group of extensions. We shall make extensive use of the long exact sequences stated in Theorem 2.3 without referring to them explicitly.

Induced Endomorphisms If $\mathfrak{e}: 0 \to A \to B \to C \to 0$ is an exact sequence, and $\alpha \in \mathrm{End}\, A$, $\gamma \in \mathrm{End}\, C$, then both $\alpha\mathfrak{e}$ and $\mathfrak{e}\gamma$ are again extensions of A by C. It is a routine calculation to verify that the correspondences $\alpha_*: \mathfrak{e} \mapsto \alpha\mathfrak{e}$ and $\gamma^*: \mathfrak{e} \mapsto \mathfrak{e}\gamma$ are endomorphisms of $\mathrm{Ext}(C, A)$; they will be called the **induced endomorphisms** of Ext. Their actions commute, since $\alpha_*\gamma^* = \gamma^*\alpha_*$. Consequently, $\mathrm{Ext}(C, A)$ is a bimodule over the endomorphism rings $\mathrm{End}\, A$ and $\mathrm{End}\, C$.

Our next lemma points out a most useful fact.

Lemma 3.1. *Multiplication by an integer n on A or on C induces multiplication by the same n on* $\mathrm{Ext}(C, A)$. *The same holds for multiplication by a p-adic integer (provided it is defined on A or on C).*

Proof. If $\alpha_i \in \mathrm{End}\, A$, then $(\alpha_1 + \cdots + \alpha_n)_* = (\alpha_1)_* + \cdots + (\alpha_n)_*$. Since $\mathbf{1}_A$ obviously acts as the identity on Ext, it is manifest that $\dot{n} \in \mathrm{End} A$ induces multiplication by n on $\mathrm{Ext}(C, A)$. The proof for C is analogous.

A p-adic integer π is the limit of a sequence $n_i \in \mathbb{Z}$ $(i < \omega)$ in the p-adic topology, say, $p^i | \pi - n_i$ for $i < \omega$. By what has been proved, multiplication by π on $\mathrm{Ext}(C, A)$ is the limit of multiplications by n_i, and hence it can be identified with the multiplication by π. $\square$

Elementary Facts on Ext Next, we collect several elementary results on Ext; they are useful in computing Ext. We start with two rather trivial remarks which reveal important features of Ext.

(A) *A group C satisfies* $\mathrm{Ext}(C,A) = 0$ *for all A exactly if it is free.* Thus, we claim that every extension by C splits if and only if C is free. The "if" part is the same as Theorem 1.5 in Chapter 3, while the "only" if part follows from Theorem 1.7 in Chapter 3.

(B) *A group A satisfies* $\mathrm{Ext}(C,A) = 0$ *for all C if and only if it is divisible.* This is a restatement of the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 2.6 in Chapter 4.

(C) *If, for some $m \in \mathbb{N}$, either $mA = 0$ or $mC = 0$, then also* $m\,\mathrm{Ext}(C,A) = 0$. This is an obvious consequence of Lemma 3.1.

(D) *If $mA = A$ for some integer $m \in \mathbb{N}$, then also* $m\,\mathrm{Ext}(C,A) = \mathrm{Ext}(C,A)$. This follows from the exactness of $\mathrm{Ext}(C,A) \xrightarrow{\dot{m}} \mathrm{Ext}(C,A) \to 0$ which is a consequence of the exactness of $A \xrightarrow{\dot{m}} A \to 0$.

(E) $\mathrm{Ext}(\mathbb{Q},A)$ *is torsion-free divisible for each group A.* This is obvious if we combine (C) and (D).

(F) $C[m] = 0$ *implies* $m\,\mathrm{Ext}(C,A) = \mathrm{Ext}(C,A)$ *for every $m \in \mathbb{N}$. In particular,* $\mathrm{Ext}(C,A)$ *is divisible whenever C is torsion-free.* Hypothesis guarantees that the sequence $0 \to C \xrightarrow{\dot{m}} C$ is exact, whence the exactness of the induced $\mathrm{Ext}(C,A) \xrightarrow{\dot{m}} \mathrm{Ext}(C,A) \to 0$ follows.

(G) *For all groups A, and for every integer $m > 0$,*

$$\mathrm{Ext}(\mathbb{Z}(m),A) \cong A/mA.$$

From the exact sequence $0 \to \mathbb{Z} \xrightarrow{\dot{m}} \mathbb{Z} \to \mathbb{Z}(m) \to 0$ we obtain the exact sequence

$$\mathrm{Hom}(\mathbb{Z},A) \cong A \xrightarrow{\dot{m}} \mathrm{Hom}(\mathbb{Z},A) \cong A \to \mathrm{Ext}(\mathbb{Z}(m),A) \to \mathrm{Ext}(\mathbb{Z},A) = 0.$$

A moment's reflection shows that the image of the first Hom in the second one is mA, whence the stated isomorphism is evident.

Example 3.2. If m,n are relatively prime integers satisfying $mA = 0$ and $nC = 0$, then $\mathrm{Ext}(C,A) = 0$. This is an immediate consequence of (C).

Example 3.3. Using (G), a simple calculation with the integers m,n shows that

$$\mathrm{Ext}(\mathbb{Z}(m),\mathbb{Z}(n)) \cong \mathbb{Z}(d),$$

where $d = \gcd\{m,n\}$. Thus $\mathrm{Ext}(\mathbb{Z}(m),\mathbb{Z}(n)) = 0$ whenever m,n are relatively prime.

(H) *For any group C and integer m,*

$$\mathrm{Ext}(C,\mathbb{Z}(m)) \cong \mathrm{Ext}(C[m],\mathbb{Z}(m)).$$

The exact sequence $0 \to C[m] \to C \xrightarrow{\dot{m}} mC \to 0$ induces the exact sequence $\mathrm{Ext}(mC, \mathbb{Z}(m)) \xrightarrow{\dot{m}} \mathrm{Ext}(C, \mathbb{Z}(m)) \to \mathrm{Ext}(C[m], \mathbb{Z}(m)) \to 0$. The image of the first map must be 0 owing to Lemma 3.1 and (F), whence the claim is immediate.

(I) *If A is p-divisible and C is a p-group, then* $\mathrm{Ext}(C, A) = 0$. If D is the divisible hull of A, then D/A is torsion divisible with zero p-component. This means that $\mathrm{Hom}(C, D/A) = 0$. The exactness of the sequence $\mathrm{Hom}(C, D/A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, D) = 0$ establishes the assertion.

(J) *An automorphism $\alpha \in \mathrm{Aut}\, A$ (or $\gamma \in \mathrm{Aut}\, C$) induces an automorphism of* $\mathrm{Ext}(C, A)$. Using the inverse $\bar{\alpha}$ of α, observe that the induced endomorphisms of $\mathrm{Ext}(C, A)$ will satisfy $\alpha_* \bar{\alpha}_* = \bar{\alpha}_* \alpha_* = \mathbf{1}$. The same argument applies to γ^*.

Example 3.4. Let R denote the group of the rational numbers with square-free denominators, and T the direct sum $\oplus_p \mathbb{Z}(p)$ with p running over the prime numbers. The exact sequence $0 \to \mathbb{Z} \to R \to T \to 0$ induces the exact sequence

$$0 \to \mathrm{Hom}(T, T) \to \mathrm{Hom}(R, T) \to \mathrm{Hom}(\mathbb{Z}, T) \cong T$$
$$\to \mathrm{Ext}(T, T) \to \mathrm{Ext}(R, T) \to 0.$$

It is easily checked that the map between the first Homs is an isomorphism, and $\mathrm{Ext}(T, T) \cong \prod_p \mathrm{Ext}(\mathbb{Z}(p), \mathbb{Z}(p)) \cong \prod_p \mathbb{Z}(p)$. Thus $\mathrm{Ext}(R, T) \cong (\prod_p \mathbb{Z}(p))/T \cong \mathbb{Q}^{\aleph_0}$.

Isomorphisms Involving Ext For later reference, it is necessary to point out:

Theorem 3.5 (Eilenberg–MacLane [1]). *Let A be a torsion-free group, and D its injective hull. If C is torsion, then*

$$\mathrm{Ext}(C, A) \cong \mathrm{Hom}(C, D/A).$$

Thus $\mathrm{Ext}(C, A)$ *is reduced algebraically compact whenever C is torsion and A is torsion-free.*

Proof. Our starting point is the exact sequence $0 \to A \to D \to D/A \to 0$. In the induced exact sequence $0 = \mathrm{Hom}(C, D) \to \mathrm{Hom}(C, D/A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, D) = 0$, the first Hom vanishes because C is torsion and D is torsion-free. Hence the stated isomorphism follows at once. The second claim is a consequence of Theorem 2.1 in Chapter 7. □

The choice $A = \mathbb{Z}$ leads to an interesting corollary on character groups:

Corollary 3.6. *For a torsion group C, we have the natural isomorphism*

$$\mathrm{Ext}(C, \mathbb{Z}) \cong \mathrm{Char}\, C.$$

Proof. Just note that, for C torsion, $\mathrm{Hom}(C, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(C, \mathbb{Q}/\mathbb{Z})$. □

Example 3.7.
(a) $\operatorname{Ext}(\mathbb{Z}(p^\infty), \mathbb{Z}) = \operatorname{Char} \mathbb{Z}(p^\infty) = J_p$ for every prime p.
(b) $\operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) = \operatorname{Char} \mathbb{Q}/\mathbb{Z} = \tilde{\mathbb{Z}} = \prod_p J_p$.

Finally, we state without proof two important isomorphisms (that are derived from the Künneth relations).

Theorem 3.8 (Cartan–Eilenberg [CE]). *For all groups A, B, C we have*

$$\operatorname{Ext}(A, \operatorname{Ext}(B, C)) \cong \operatorname{Ext}(\operatorname{Tor}(A, B), C). \tag{9.11}$$

In case C is injective, also

$$\operatorname{Ext}(A, \operatorname{Hom}(B, C)) \cong \operatorname{Hom}(\operatorname{Tor}(A, B), C). \tag{9.12}$$

$\square$

★ **Notes.** There is an extensive literature on the structure of Ext; for instance, what its torsion-free or p-rank is if it is divisible, or when it can be isomorphic to $\mathbb{Q}$. Interestingly, some questions turn out undecidable in ZFC. But, tempting as it is, the discussion of these results would take us too far afield from our principal aim here. However, let us point out that there is a full characterization of the divisible group $\operatorname{Ext}(G, \mathbb{Z})$ for torsion-free groups G, assuming that V = L and there is no weakly compact cardinal. See Shelah–Strüngmann [2], and the literature quoted there.

Eklof–Huber [1] investigate when Ext vanishes, by using a new invariant, called Γ-invariant, which is an equivalence class in filtrations.

It is natural to ask to what extent the functor $\operatorname{Ext}(C, *)$ or $\operatorname{Ext}(*, C)$ determines the group C. For countable p-groups A, B of finite Ulm length, Moskalenko [2] shows (under CH) that they are isomorphic whenever $\operatorname{Ext}(A, C) \cong \operatorname{Ext}(B, C)$ holds for all groups C. For finite rank torsion-free groups, see Notes to Sect. 1 in Chapter 12.

Exercises

(1) For finite groups A, C, there is a (non-natural) isomorphism $\operatorname{Ext}(C, A) \cong \operatorname{Hom}(C, A)$.
(2) $\operatorname{Ext}(C, A) = 0$ if A is a p-group and C is a q-group for different primes p, q.
(3) For any prime p, $\operatorname{Ext}(\mathbb{Z}(p^\infty), A) = 0$ is equivalent to $pA = A$.
(4) (a) $\operatorname{Ext}(C, A)/m \operatorname{Ext}(C, A) \cong \operatorname{Ext}(C, A/mA)$ for all groups A, C ($m \in \mathbb{N}$).
 (b) Verify the following converse of (D): If, for some group A and for some $m \in \mathbb{N}$, the equality $m \operatorname{Ext}(C, A) = \operatorname{Ext}(C, A)$ holds for every group C, then $mA = A$.
 (c) Suppose the group A and the integer $m \in \mathbb{N}$ have the property that $m \operatorname{Ext}(C, A) = 0$ for every group C. Then mA is divisible.
(5) Assume that $mA = A$ and $A[m] = 0$; then also $m \operatorname{Ext}(C, A) = \operatorname{Ext}(C, A)$ and $\operatorname{Ext}(C, A)[m] = 0$.
(6) (a) If $mC = C$ and $C[m] = 0$ for some $m \in \mathbb{N}$, then $m \operatorname{Ext}(C, A) = \operatorname{Ext}(C, A)$ and $\operatorname{Ext}(C, A)[m] = 0$ as well. Conclude that $\operatorname{Ext}(C, A)$ is torsion-free divisible if so is C.

(b) Prove the following converse of (F): if C and $m \in \mathbb{N}$ satisfy $m\,\mathrm{Ext}(C,A) = \mathrm{Ext}(C,A)$ for all groups A, then $C[m] = 0$.
(c) If C is such that $\mathrm{Ext}(C,A)$ is torsion-free divisible for every group A, then C is also torsion-free divisible.

(7) (Baer) If the groups A, C satisfy $p\,\mathrm{Ext}(C,A) = \mathrm{Ext}(C,A)$ for some prime p, then either $C[p] = 0$ or $pA = A$.
(8) If A is torsion-free, then $\mathrm{Ext}(C,A) \cong \mathrm{Ext}(t(C),A) \oplus \mathrm{Ext}(C/t(C),A)$.
(9) Prove the isomorphism $\mathrm{Ext}(\mathbb{Z}_{(p)},\mathbb{Z}) \cong \mathbb{Z}(p^\infty) \oplus \mathbb{Q}^{\aleph_0}$. [Hint: $0 \to \mathbb{Z} \to \mathbb{Z}_{(p)} \to \oplus_{q \neq p}\mathbb{Z}(q^\infty) \to 0$, apply $\mathrm{Hom}(*,\mathbb{Z})$.]
(10) Verify $\mathrm{Ext}(\mathbb{Q}^{(p)},\mathbb{Z}) \cong \mathbb{Z}(p^\infty) \oplus \mathbb{Q}^{\aleph_0}$. [Hint: $0 \to \mathbb{Z} \to \mathbb{Q}^{(p)} \to \mathbb{Z}(p^\infty) \to 0$.]
(11) Prove $\mathrm{Ext}(J_p,\mathbb{Z}) \cong \mathbb{Z}(p^\infty) \oplus \mathbb{Q}^{2^{\aleph_0}}$. [Hint: $0 \to J_p \to \mathbb{Q}_p^* \to \mathbb{Z}(p^\infty) \to 0$.]
(12) Using the fact that $\prod_p \mathbb{Z}(p)$ is a reduced group with $\oplus_p\mathbb{Z}(p)$ as torsion subgroup, show that $\mathrm{Ext}(\mathbb{Q}, \oplus_p\mathbb{Z}(p)) \neq 0$.
(13) (Nunke) Prove that $\mathrm{Hom}(C,\mathbb{Z}) = 0 = \mathrm{Ext}(C,\mathbb{Z})$ implies $C = 0$. [Hint: $\mathrm{Ext}(t(C),\mathbb{Z}) = 0$ and $\mathrm{Ext}(C/mC,\mathbb{Z}) = 0$, then Exercise 6.]
(14) (Nunke) Every torsion-free divisible group D is representable as an Ext. [Hint: try $\mathrm{Ext}(\mathbb{Q},T)$ with torsion T.]
(15) For a torsion-free A, $\mathrm{Ext}(A,\mathbb{Z})$ is torsion-free if and only if A has the projective property with respect to the exact sequences $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ for all $n \in \mathbb{N}$.
(16) (Hill–Megibben) Let A be a p-group. An exact sequence $0 \to A \to G \to \mathbb{Z}(p^\infty) \to 0$ represents an extension of finite order in $\mathrm{Ext}(\mathbb{Z}(p^\infty),A)$ if and only if G is not reduced.

4 Lemmas on Ext

We will need several important lemmas involving Ext that are most useful in the theory. They relate Ext to ascending chains.

Eklof's Lemmas We start with two lemmas due to P. Eklof. The first provides a sufficient condition for an Ext to vanish (with no restriction on the sizes), while the second sharpens this to a necessary and sufficient condition in the Constructible Universe.

Lemma 4.1 (Eklof's Lemma). *Let A be any group, and*

$$0 = C_0 < C_1 < \cdots < C_\nu < \cdots \quad (\nu < \kappa) \tag{9.13}$$

(*where κ is any infinite cardinal*) *a smooth chain of subgroups of a group C such that*

(a) $\bigcup_{\nu<\kappa} C_\nu = C$;
(b) *for each* $\nu < \kappa$, $\mathrm{Ext}(C_{\nu+1}/C_\nu, A) = 0$.

Then $\mathrm{Ext}(C, A) = 0$.

Proof. Let $0 \to A \to G \xrightarrow{\phi} C \to 0$ be an extension of A by C, and let G_ν denote the subgroup of G that is the complete preimage of C_ν under ϕ. Then G is the union of the smooth chain $A = G_0 < G_1 < \cdots < G_\nu < \cdots$. We are going to define by transfinite induction a smooth chain $0 = B_0 < B_1 < \cdots < B_\nu < \cdots (\nu < \kappa)$ of subgroups of G such that $G_\nu = A \oplus B_\nu$ where $B_\nu \cong C_\nu$.

The starting point $\nu = 0$ is obvious, and so is the case when ν is a limit ordinal, since then we have no choice other than taking unions. So suppose that, for some ordinal $\mu < \kappa$, G_ν and B_ν have been defined in the desired way for all $\nu \le \mu$. By induction, we may assume that $\mathrm{Ext}(C_\mu, A) = 0$. Now $G_{\mu+1}/B_\mu$ can be viewed as an extension of A by $G_{\mu+1}/G_\mu \cong C_{\mu+1}/C_\mu$, so condition (b) guarantees that $(A + B_\mu)/B_\mu \cong A$ is a summand, i.e. $G_{\mu+1}/B_\mu = (A + B_\mu)/B_\mu \oplus B_{\mu+1}/B_\mu$ for some subgroup $B_{\mu+1} \le G_{\mu+1}$. Clearly, $G_{\mu+1} = A \oplus B_{\mu+1}$. Once we have the complete chain of the B_ν for $\nu < \kappa$, we can argue that the union $B = \cup_{\nu<\kappa} B_\nu$ satisfies $G = A \oplus B$, i.e. the given exact sequence splits. □

Much greater effort is needed if we wish to prove something similar with a necessary and sufficient condition. This can be done in Gödel's Constructible Universe, i.e. under the hypothesis V = L. We will need an easy observation which we display as a lemma for easy reference.

In the proofs of the next lemmas, it will be convenient to consider an extension of A by C (or by a subgroup $C' < C$) to be a group built on the set $C \times A$ (on $C' \times A$). This can be done without loss of generality.

Lemma 4.2. *Suppose $\phi : C' \to C$ is an inclusion map, and A is a group such that*

$$\mathrm{Ext}(C, A) = 0 \quad \textit{and} \quad \mathrm{Ext}(C/C', A) \neq 0.$$

Given splitting exact sequences with maps ρ, ρ' in (9.14)*, there exists a homomorphism $\chi : G' \to G$ making the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & G' \arrow[r, "\rho'"] \arrow[d, "\chi"'] & C' \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & A \arrow[r] & G \arrow[r, "\rho"] & C \arrow[r] & 0
\end{tikzcd} \tag{9.14}$$

commute such that there is no splitting map $\sigma : C \to G$ with $\chi\sigma' = \sigma\phi$, for any splitting map $\sigma' : C' \to G'$ of ρ'.

Proof. By hypothesis, in the exact sequence $\mathrm{Hom}(C, A) \to \mathrm{Hom}(C', A) \to \mathrm{Ext}(C/C', A) \to \mathrm{Ext}(C, A) = 0$ the map between the two Homs is not epic. If $\eta : C' \to A$ is a map that does not extend to any $C \to A$, then define χ to map $(a, c') \in G' = A \times C'$ onto $(a + \eta c', \phi c') \in G = A \times C$. It is readily checked that, by the choice of η, no splitting map $\sigma : C \to A \oplus C$ with $\chi\sigma' = \sigma\phi$ may exist. □

Keep in mind that the main point is that, under the stated hypotheses, the map χ can be chosen such that the two splitting sequences have no matching splitting maps.

Lemma 4.3 (Eklof). *Assuming* $\mathrm{V} = \mathrm{L}$*, let* κ *be an uncountable regular cardinal,* A *any group of cardinality* $\leq \kappa$*, and* (9.13) *a* κ*-filtration of a group* C *of cardinality* κ *such that*

$$\mathrm{Ext}(C_\nu, A) = 0 \quad \textit{for all } \nu < \kappa.$$

Then $\mathrm{Ext}(C, A) = 0$ *if and only if the set*

$$E = \{\nu < \kappa \mid \exists\, \mu > \nu \ \textit{ such that } \mathrm{Ext}(C_\mu/C_\nu, A) \neq 0\}$$

is not stationary in κ*.*

Proof. If E is not stationary in κ, then choose a cub X that does not intersect E. Keeping only the subgroups C_ν with $\nu \in X$, an appeal to Lemma 4.1 shows that the hypotheses are satisfied for the modified chain, and therefore $\mathrm{Ext}(C, A) = 0$.

Conversely, suppose E is stationary in κ. We can clearly drop those indices that are not needed, and still have a stationary subset of κ, so there is no loss of generality in assuming $\mu = \nu + 1$ in the definition of E. Let $\{A_\nu\}_{\nu<\kappa}$ be a filtration of A (repetitions are allowed). In view of the Diamond Principle $\diamondsuit$ (it holds because $\mathrm{V} = \mathrm{L}$), there exists a family $\{g_\nu\}_{\nu \in E}$ of functions $g_\nu \colon C_\nu \to C_\nu \times A_\nu$ ($\nu \in E$) such that, for every function $g \colon C \to C \times A$, the set $\{\nu \in E \mid g \restriction C_\nu = g_\nu\}$ is stationary in κ.

With the aid these g_ν, we are going to construct a non-split exact sequence $\mathfrak{e}$: $0 \to A \to G \xrightarrow{\rho} C \to 0$ as the direct limit of splitting exact sequences $\mathfrak{e}_\nu$: $0 \to A \to G_\nu \xrightarrow{\rho_\nu} C_\nu \to 0$ ($\nu < \kappa$) such that whenever $\mu < \nu$, there is a commutative diagram

$$\begin{array}{ccccccccccc}
\mathfrak{e}_\mu : & 0 & \longrightarrow & A & \longrightarrow & G_\mu & \xrightarrow{\rho_\mu} & C_\mu & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \gamma_\mu^\nu} & & \downarrow{\scriptstyle \mathrm{incl}} & & \\
\mathfrak{e}_\nu : & 0 & \longrightarrow & A & \longrightarrow & G_\nu & \xrightarrow{\rho_\nu} & C_\nu & \longrightarrow & 0
\end{array} \tag{9.15}$$

where the right vertical map is the inclusion map. Let $\nu < \kappa$, and assume that $\mathfrak{e}_\mu$ has been defined for every $\mu < \nu$ such that required diagrams are commutative. We distinguish three cases. (Now G_ν is assumed to be built on the set $C_\nu \times A$, so the γ_μ^ν are also inclusion maps.)

Case 1. If ν is a limit ordinal, then we let $\mathfrak{e}_\nu$ be the direct limit of the exact sequences $\mathfrak{e}_\mu$ for $\mu < \nu$ with connecting maps $(\mathbf{1}_A, \gamma_\mu^\nu, \text{incl})$. This is a splitting sequence, since by hypothesis C_ν satisfies $\operatorname{Ext}(C_\nu, A) = 0$.

Case 2. Let $\nu = \delta + 1$. If $\delta \notin E$ or if g_δ is not a splitting map for ρ_δ, then let $\mathfrak{e}_\nu : 0 \to A \to G_\nu \xrightarrow{\rho_\nu} C_\nu \to 0$ be a splitting extension with any choice of $\gamma_\delta^\nu : G_\delta \to G_\nu$ such that (9.15) commutes (with $\mu = \delta$).

Case 3. The remaining case is when $\nu = \delta + 1$, $\delta \in E$, and the selected g_δ: $C_\delta \to C_\delta \times A_\delta \subset C \times A$ is a splitting map for ρ_δ. From Lemma 4.2 we conclude that there are a map $\gamma_\delta^\nu : G_\delta \to G_\nu$ and a splitting extension $\mathfrak{e}_\nu : 0 \to A \to G_\nu \xrightarrow{\rho_\nu} C_\nu \to 0$ with a commutative diagram (9.15) such that there exists no splitting map $\sigma : C_\nu \to G_\nu$ for ρ_ν extending g_δ.

We now define $\mathfrak{e} : 0 \to A \to G \xrightarrow{\rho} C \to 0$ as the direct limit of the splitting exact sequences $\mathfrak{e}_\nu : 0 \to A \to G_\nu \xrightarrow{\rho_\nu} C_\nu \to 0$ for $\nu < \kappa$ where $G = C \times A = \cup_{\nu<\kappa}(C_\nu \times A)$ as sets. By way of contradiction, assume there is a splitting homomorphism g: $C \to G$ for ρ. Note that we must then have $g(C_\nu) \le G_\nu$ for every $\nu < \kappa$. By the choice of the functions g_ν, there is a $\delta \in E$ (actually, stationarily many of them) such that $g \restriction C_\delta = g_\delta$. This means that $\mathfrak{e}_{\delta+1}$ has been constructed according to Case 3 above. Since $g \restriction C_{\delta+1}$ is both a splitting map for $\mathfrak{e}_{\delta+1}$ and an extension of $g \restriction C_\delta$, we have reached a contradiction to the existence of a splitting g. Thus $\operatorname{Ext}(C, A) \neq 0$, indeed. $\square$

Ext When Both Arguments Are Chains In the following lemma we turn to the case when both arguments in Ext are unions of chains. This is a version of a lemma by Eklof–Fuchs; it requires a different approach.

Lemma 4.4. *Let κ be an uncountable regular cardinal, and B_ν $(\nu < \kappa)$ arbitrary groups. We set $A = \oplus_{\nu<\kappa} B_\nu$ and $A_\mu = \oplus_{\nu<\mu} B_\nu$ $(\mu < \kappa)$. Furthermore, let again* (9.13) *be a κ-filtration of a group C of cardinality κ. Suppose that*

$$\operatorname{Ext}(C_\nu, A_\nu) = 0 \quad \textit{for all } \nu < \kappa.$$

If the set

$$E = \{\nu < \kappa \mid \exists\, \mu > \nu \ \textit{ such that } \operatorname{Ext}(C_\mu/C_\nu, A_\mu/A_\nu) \neq 0\}$$

is stationary in κ, then $\operatorname{Ext}(C, A) \neq 0$.

Proof. As noted above, there is no loss of generality in assuming $\mu = \nu + 1$ in the application of E. Observing that $\operatorname{Ext}(C_\mu, B_\nu) = 0$ if $\mu \ge \nu$, the exact sequence $0 \to C_\nu \to C_{\nu+1} \to C_{\nu+1}/C_\nu \to 0$ induces the exact sequence

$$\operatorname{Hom}(C_{\nu+1}, B_\nu) \to \operatorname{Hom}(C_\nu, B_\nu) \to \operatorname{Ext}(C_{\nu+1}/C_\nu, B_\nu) \to 0.$$

We now define homomorphisms $\chi_\nu : C_\nu \to B_\nu$ $(\nu < \kappa)$. If $\nu \in E$, then by hypothesis the last Ext $\neq 0$, and we can choose a χ_ν that has no extension to

$C_{\nu+1} \to B_\nu$. If $\nu \notin E$, then select $\chi_\nu = 0$. Define a direct system of splitting short exact sequences $\mathfrak{e}_\nu$ $(\nu < \kappa)$ with commutative diagrams

$$\begin{array}{lccccccccc}
\mathfrak{e}_\nu\colon 0 & \longrightarrow & A_\nu & \longrightarrow & A_\nu \oplus C_\nu & \xrightarrow{\phi_\nu} & C_\nu & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle\gamma_\nu}\downarrow & & \downarrow & & \\
\mathfrak{e}_{\nu+1}\colon 0 & \longrightarrow & A_{\nu+1} & \longrightarrow & A_{\nu+1} \oplus C_{\nu+1} & \xrightarrow{\phi_{\nu+1}} & C_{\nu+1} & \longrightarrow & 0
\end{array}$$

where the extremal vertical arrows are the injection maps, $\phi_\nu, \phi_{\nu+1}$ are the obvious projections, while γ_ν is defined as follows:

$$\gamma_\nu\colon\ (a_\nu, c_\nu) \to (a_\nu + \chi_\nu c_\nu, c_\nu),$$

where $a_\nu \in A_\nu$, $c_\nu \in C_\nu$. At limit ordinals μ, we form the direct limit of the system $\{\mathfrak{e}_\nu \mid \nu < \mu\}$ which will clearly be an extension of A_μ by C_μ. Therefore, in view of the hypothesis $\operatorname{Ext}(C_\mu, A_\mu) = 0$, it will be equivalent to, and thus identifiable with, the splitting sequence $\mathfrak{e}_\mu$.

Suppose $\mathfrak{e}\colon 0 \to A \to G \xrightarrow{\phi} C \to 0$ is the limit of the direct system $\{\mathfrak{e}_\nu \mid \nu < \kappa\}$ with the indicated maps, and let $\rho_\nu : A_\nu \oplus C_\nu \to G$ denote the map induced by the canonical morphisms $\mathfrak{e}_\nu \to \mathfrak{e}$. Thus $\rho_{\nu+1}\gamma_\nu = \rho_\nu$ for all $\nu < \kappa$. By way of contradiction, assume that $\mathfrak{e}$ splits, and let $\psi : C \to G$ be a splitting map for ϕ; thus, we can write $G = A \oplus \psi C$. $|C_\nu| < \kappa$ implies that ψ maps C_ν into the direct sum of C_ν and a set of less than κ of the B_μ's. Routine arguments lead us to the conclusion that the set

$$S = \{\nu < \kappa \mid \psi C_\nu \leq \rho_\nu(A_\nu \oplus C_\nu)\}$$

is a cub in κ. Evidently, for each $\nu \in S$, ψ induces a splitting map $\psi_\nu : C_\nu \to A_\nu \oplus C_\nu$ for ϕ_ν such that $\rho_\nu \psi_\nu = \psi \restriction C_\nu$.

We will denote by ζ_ν the composition of ψ_ν with the projection onto A_ν, and by $-\xi_\nu$ the map ρ_ν followed by the projection onto A. It is clear that for all $c_\nu \in C_\nu$, $\rho_\nu\psi_\nu c_\nu = \rho_\nu(\zeta_\nu c_\nu, c_\nu) = (\zeta_\nu c_\nu - \xi_\nu c_\nu, c_\nu)$ is equal to $\psi c_\nu = (0, c_\nu)$, whence $\zeta_\nu = \xi_\nu$ follows. Furthermore, $\rho_\nu\psi_\nu c_\nu$ also equals

$$\begin{aligned}
\rho_{\nu+1}\gamma_\nu\psi_\nu c_\nu =& \rho_{\nu+1}\gamma_\nu(\zeta_\nu c_\nu, c_\nu) = \rho_{\nu+1}(\zeta_\nu c_\nu + \chi_\nu c_\nu, c_\nu) \\
=& (\zeta_\nu c_\nu + \chi_\nu c_\nu - \zeta_{\nu+1} c_\nu, c_\nu),
\end{aligned}$$

thus $\zeta_{\nu+1} c_\nu = \zeta_\nu c_\nu + \chi_\nu c_\nu \in A_\nu \oplus B_\nu = A_{\nu+1}$. Consequently, $\zeta_{\nu+1}$ followed by the projection of $A_{\nu+1}$ onto B_ν carries $c_\nu \in C_\nu$ onto $\chi_\nu c_\nu$, therefore it yields a homomorphism $C_{\nu+1} \to B_\nu$ extending χ_ν. This shows that $\nu \notin E$, i.e., E is not stationary in κ. $\square$

★ **Notes.** Lemma 4.4 was published by Eklof–Fuchs [Annali Mat. Pura Appl. **90**, 363–374 (1988)] for the special case (on valuation domain) needed there.

Let us call attention to the basic difference between Lemmas 4.3 and 4.4. In the former lemma, the second argument of Ext was kept fixed, while in the latter lemma the second argument could grow as large as necessary to match the size of the group in the first argument. This explains why we needed the Diamond Principle in one case, but not in the other, this difference will be more apparent in Sect. 7 in Chapter 13.

Exercises

(1) Suppose that A satisfies $\mathrm{Ext}(X, A) = 0$ for all rank 1 torsion-free groups X. Then $\mathrm{Ext}(C, A) = 0$ for all torsion-free groups C.
(2) If $\mathrm{Ext}(\mathbb{Z}(p), A) = 0$ holds for A, then $\mathrm{Ext}(C, A) = 0$ for all p-groups C.
(3) If A satisfies $\mathrm{Ext}(\mathbb{Z}(p), A) = 0$ for all primes p, then A is divisible.
(4) Derive Pontryagin's Theorem 7.1 in Chapter 3 from Lemma 4.1.
(5) (Eklof–Huber) If C is a torsion-free group of countable rank such that $\mathrm{Ext}(B, A) = 0$ holds for all finite rank subgroups $B < C$, then also $\mathrm{Ext}(C, A) = 0$ for any A.

5 The Functor Pext

That the extensions of A by C in which A is a pure subgroup (we will call them **pure-extensions**) play a distinguished role does not seem to be obvious at the outset. The truly significant thing here is that these extensions form a subgroup of Ext which can be identified as the first Ulm subgroup of Ext.

Preliminary Lemma We recall that if $\alpha : A \to A$ and $\gamma : C \to C$ are endomorphisms, then there are induced endomorphisms α_* and γ^* of $\mathrm{Ext}(C, A)$. We begin with investigating the actions of α_* and γ^* in $\mathrm{Ext}(C, A)$. In the proof of the next lemma, conveniently we may view A as a subgroup of B.

Lemma 5.1 (Baer [4]). *Given the exact sequence*

$$\mathfrak{e}\colon\ 0 \to A \to B \xrightarrow{\beta} C \to 0 \tag{9.16}$$

and the endomorphisms $\alpha : A \to A$ *and* $\gamma : C \to C$*, we have:*

(i) $\mathfrak{e} \in \mathrm{Im}\,\alpha_*$ *if and only if* $A/\alpha A$ *is a summand of* $B/\alpha A$*;*
(ii) $\mathfrak{e} \in \mathrm{Im}\,\gamma^*$ *if and only if* A *is a summand of* $\beta^{-1}\,\mathrm{Ker}\,\gamma$*;*
(iii) *if* $\mathrm{Ker}\,\gamma = 0$*, then* $\mathfrak{e} \in \mathrm{Ker}\,\gamma^*$ *if and only if* $\beta^{-1}\,\mathrm{Im}\,\gamma \cong A \oplus \mathrm{Im}\,\gamma$.

Proof.

(i) (Tellman [1]) The sequence $\mathrm{Ext}(C, \alpha A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, A/\alpha A) \to 0$ is obviously exact. Since the induced map $\mathrm{Ext}(C, A) \xrightarrow{\alpha_*} \mathrm{Ext}(C, \alpha A)$ is surjective, the sequence

$$\mathrm{Ext}(C, A) \xrightarrow{\alpha_*} \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, A/\alpha A) \to 0$$

is also exact. It shows that $\mathfrak{e} \in \mathrm{Im}\, \alpha_*$ exactly if $\mathfrak{e}$ is mapped upon the splitting extension of $A/\alpha A$ by C. This is tantamount to saying that $A/\alpha A$ is a summand of $B/\alpha A$.

(ii) An argument similar to the one in (i) yields the exact sequence

$$\mathrm{Ext}(C, A) \xrightarrow{\gamma^*} \mathrm{Ext}(C, A) \to \mathrm{Ext}(\mathrm{Ker}\, \gamma, A) \to 0.$$

This shows that the extension $\mathfrak{e}$ satisfies $\mathfrak{e} \in \mathrm{Im}\, \gamma^*$ exactly if it is mapped upon the splitting extension of A by $\mathrm{Ker}\, \gamma$.

(iii) It is clear from Lemma 2.1 that $\mathfrak{e} \in \mathrm{Ker}\, \gamma^*$ if and only if the top row in the pull-back diagram

$$\begin{array}{rccccccccc}
\mathfrak{e}\gamma : & 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\
\mathfrak{e} : & 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & C & \longrightarrow & 0
\end{array}$$

is splitting. As γ is monic, this means that the sequence $\mathfrak{e}\gamma : 0 \to A \to \beta^{-1}\, \mathrm{Im}\, \gamma \to \mathrm{Im}\, \gamma \to 0$ is splitting. □

Pext as First Ulm Subgroup of Ext Our main goal in proving the preceding lemma was to apply it to the case when the endomorphisms are multiplications by some non-zero integer n. From Lemma 3.1 we know that then the induced endomorphisms on Ext are likewise multiplications by the same n. Consequently, Lemma 5.1(i) tells us that (9.16) represents an element of $n\, \mathrm{Ext}(C, A)$ if and only if A/nA is a summand of B/nA. Observing that $\mathfrak{e} \in n\, \mathrm{Ext}(C, A)$ implies $\mathfrak{e} \in m\, \mathrm{Ext}(C, A)$ for all $m|n$, we are now in a position to derive:

Theorem 5.2 (Nunke [1], Fuchs [AG]). *The exact sequence* (9.16) *represents*

(i) *an element of* $n\, \mathrm{Ext}(C, A)$ *if and only if* $mA = A \cap mB$ *for all* $m|n$;

(ii) *an element of the first Ulm subgroup* $\mathrm{Ext}(C, A)^1$ *if and only if it is pure-exact. In other words,*

$$\mathrm{Pext}(C, A) = \mathrm{Ext}(C, A)^1,$$

where $\mathrm{Pext}(C, A)$ *denotes the set of pure-extensions of* A *by* C. □

Thus $\mathrm{Pext}(C,A)$ is not only a subset, it is also a subgroup of $\mathrm{Ext}(C,A)$, called the **group of pure-extensions**.

If $\mathfrak{e} : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a pure-extension of A by C, and if $\mu : A \to A'$, $\nu : C' \to C$ are homomorphisms, then the extensions $\mu\mathfrak{e}$ and $\mathfrak{e}\nu$ are again pure-extensions, this is easy to check directly, but also follows immediately from Theorem 5.2 by observing that the map $\mathrm{Ext}(\nu,\mu) : \mathrm{Ext}(C,A) \to \mathrm{Ext}(C',A')$ carries Ulm subgroups into Ulm subgroups. We can thus claim that Pext *is an additive bifunctor* $\mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$, *contravariant in the first, and covariant in the second variable.*

From Theorem 4.3 in Chapter 5 we derive at once:

Corollary 5.3. *A group C satisfies* $\mathrm{Pext}(C,A) = 0$ *for all groups A if and only if it is Σ-cyclic.* □

Exact Sequences for Pext Having had a first glimpse of Pext, it looks interesting enough to pursue its properties. To reveal its behavior towards exact sequences, we prove a couple of results. The next two lemmas are more general than we need them right now, but we will find their generality useful later on.

Lemma 5.4. *Suppose* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is an exact sequence such that αA is contained in the σth Ulm subgroup B^σ of B. Then β maps B^σ onto C^σ.*

Similarly, if $\alpha A \le p^\sigma B$, then $\beta(p^\sigma B) = p^\sigma C$.

Proof. It is pretty clear that $\beta(B^\sigma) \le C^\sigma$, so only the surjectivity requires a proof. Since $\mathrm{Ker}\,\beta \le B^\sigma$, the transfinite heights of the elements of B not in B^σ are the same in every coset mod $\mathrm{Ker}\,\beta$. As the height of an element in an epic image is the supremum of the heights of its preimages, no element can belong to C^σ without being contained in $\beta(B^\sigma)$.

The proof for $p^\sigma B$ is the same. □

Lemma 5.5. *If* $\mathfrak{e} : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *represents an element in the σth Ulm subgroup* $\mathrm{Ext}(C,A)^\sigma$ *of* $\mathrm{Ext}(C,A)$, *then for every group G, the connecting homomorphisms δ^* and δ_* act as*

$$\delta^* : \mathrm{Hom}(A,G) \to \mathrm{Ext}(C,G)^\sigma, \quad \delta_* : \mathrm{Hom}(G,C) \to \mathrm{Ext}(G,A)^\sigma.$$

In particular, if $\mathfrak{e}$ is pure-exact, then $\mathrm{Im}\,\delta^*$ *and* $\mathrm{Im}\,\delta_*$ *are contained in* $\mathrm{Pext}(C,G)$ *and* $\mathrm{Pext}(G,A)$, *respectively.*

Similarly, if $\mathfrak{e} \in p^\sigma\,\mathrm{Ext}(C,A)$, *then* $\mathrm{Im}\,\delta^* \in p^\sigma\,\mathrm{Ext}(C,G)$ *and* $\mathrm{Im}\,\delta_* \in p^\sigma\,\mathrm{Ext}(G,A)$.

Proof. An $\eta \in \mathrm{Hom}(A,G)$ induces a map $\eta_* : \mathrm{Ext}(C,A) \to \mathrm{Ext}(C,G)$ that carries the top row into the bottom row in the commutative diagram

$$\begin{tikzcd}
\mathfrak{e}\colon 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\eta"'] & B \arrow[r, "\beta"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
\eta_*\mathfrak{e}\colon 0 \arrow[r] & G \arrow[r] & B' \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

Homomorphisms map Ulm subgroups into Ulm subgroups, so necessarily $\eta_*\mathfrak{e} \in \operatorname{Ext}(C, G)^\sigma$. But the bottom row is exactly the image of η under δ^*, whence the claim is immediate. The dual proof establishes the statement involving δ_*, while the last claim is a simple corollary, as Lemma 5.4 is available. ☐

Now we can fit together the pieces of information we obtained in these lemmas to prove that Pext admits the same kind of long exact sequences as Ext does.

Theorem 5.6 (Harrison [1]). *Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be a pure-exact sequence. Then, for every group G, the following induced sequences are exact:*

$$0 \to \operatorname{Hom}(C, G) \to \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G) \to$$
$$\xrightarrow{\delta^*} \operatorname{Pext}(C, G) \xrightarrow{\beta^*} \operatorname{Pext}(B, G) \xrightarrow{\alpha^*} \operatorname{Pext}(A, G) \to 0 \tag{9.17}$$

and

$$0 \to \operatorname{Hom}(G, A) \to \operatorname{Hom}(G, B) \to \operatorname{Hom}(G, C) \to$$
$$\xrightarrow{\delta_*} \operatorname{Pext}(G, A) \xrightarrow{\alpha_*} \operatorname{Pext}(G, B) \xrightarrow{\beta_*} \operatorname{Pext}(G, C) \to 0. \tag{9.18}$$

All the maps are natural.

Proof. In view of Lemma 5.5 and the functorial behavior of Pext, it is evident that (9.17) and (9.18) make sense (i.e., the images of the indicated maps are correct), only their exactness has to be established. We do not have to worry about exactness at the Homs (see Theorem 2.3), so let us concentrate on the Pext part. Theorem 2.3 also assures the exactness of the sequence

$$\operatorname{Hom}(A, G) \xrightarrow{\delta^*} \operatorname{Ext}(C, G) \xrightarrow{\beta^*} \operatorname{Ext}(B, G) \xrightarrow{\alpha^*} \operatorname{Ext}(A, G) \to 0.$$

By Lemma 5.5, $\operatorname{Im}\delta^* \leq \operatorname{Ext}(C, G)^1$, so from Theorem 5.2 we infer that β^* maps $\operatorname{Ext}(C, G)^1 = \operatorname{Pext}(C, G)$ upon $(\operatorname{Im}\beta^*)^1 \leq \operatorname{Ext}(B, G)^1$, thus α^* for Pext is surjective. But then $(\operatorname{Im}\beta^*)^1 \leq \operatorname{Ext}(B, G)^1$ cannot be a proper inclusion. This establishes the exactness of (9.17). The proof of (9.18) is similar. ☐

We observe that Theorem 5.6 *also holds if we start with a p-pure-exact sequence, and assume that G is a p-adic group.*

Pext and Algebraic Compactness Pext turns out to be a most versatile device to test algebraic compactness. First of all, it is clear that the definition of algebraic compactness can be rephrased by saying that A is algebraically compact if and only if $\mathrm{Pext}(C, A) = 0$ for all groups C. A reduction to certain C yields an especially convenient criterion (Proposition 5.8). But first a comment that will be used in the next proof.

Assume A is reduced and algebraically compact. The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces the exact sequence $0 \to \mathrm{Hom}(\mathbb{Z}, A) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, A) \to \mathrm{Ext}(\mathbb{Q}, A) = 0$, where the last Ext vanishes by the algebraic compactness of A. Hence we obtain the natural isomorphism $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, A) \cong A$ for a reduced algebraically compact (= complete) A.

Proposition 5.7. *Let A be a group such that $A^1 = 0$. Then*

$$0 \to \mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, A) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, A) \to \tilde{A} \to 0$$

is an exact sequence where $\tilde{A}$ denotes the $\mathbb{Z}$-adic completion of A. There is a natural isomorphism

$$\mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, A) \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \tilde{A}/A).$$

Proof. If $A^1 = 0$, then A is a pure subgroup of its completion, and we can use the pure-exact sequence $0 \to A \to \tilde{A} \to \tilde{A}/A \to 0$ to derive two exact sequences

$$0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \tilde{A}/A) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, A) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, \tilde{A}) \cong \tilde{A} \to 0$$

and

$$0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \tilde{A}/A) \to \mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, A) \to \mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, \tilde{A}) = 0,$$

where we have used the divisibility of $\tilde{A}/A$ to get 0 at the right end of the first sequence. Hence the claims are evident. □

Observe that Proposition 5.7 also tells us the interesting fact that *the completion of a group A with trivial Ulm subgroup can also be obtained by forming the initial Ulm factor of* $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, A)$.

Proposition 5.8. *A group A is algebraically compact if and only if it satisfies*

$$\mathrm{Ext}(\mathbb{Q}, A) = 0 \quad \textit{and} \quad \mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, A) = 0.$$

Proof. For the proof of sufficiency, we assume that A is reduced and the indicated Ext and Pext vanish. Let D be the divisible hull of A and D' the divisible hull of A^1 in D. Then $A + D'$ contains A as a pure subgroup with $(A + D')/A$ torsion divisible. The assumption $\mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, A) = 0$ implies $\mathrm{Pext}(E, A) = 0$ for all torsion divisible E, so A is a summand in $A + D'$. This can happen only if $D' = 0$, and $A^1 = 0$. Thus A

is a pure subgroup in $\tilde{A}$ with divisible $\tilde{A}/A$. Now $\tilde{A}/A$ cannot contain any copy of $\mathbb{Q}$, since $\mathrm{Ext}(E, A) = 0$ for all torsion-free divisible E, and neither can it contain any torsion divisible subgroup because of $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \tilde{A}/A) \cong \mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, A) = 0$. Consequently, $\tilde{A}/A = 0$, establishing the claim. □

More on Pext In general, neither Ext nor Pext needs to convert a direct limit to an inverse limit of Exts or Pexts, but in the countable case something definite can be stated.

Lemma 5.9 (Nunke [1]). *Let C_n $(n < \omega)$ be a direct system with connecting maps $\phi_n \colon C_n \to C_{n+1}$, and limit C. Then the natural map*

$$\psi\colon \mathrm{Ext}(C, A) \to \varprojlim_{n<\omega} \mathrm{Ext}(C_n, A) \tag{9.19}$$

is surjective.

Proof. The direct system of the C_n gives rise to an inverse system with connecting maps $\phi_n^* \colon \mathrm{Ext}(C_{n+1}, A) \to \mathrm{Ext}(C_n, A)$. Since the natural maps $\gamma_n \colon C_n \to C$ induce $\gamma_n^* \colon \mathrm{Ext}(C, A) \to \mathrm{Ext}(C_n, A)$ which satisfy $\gamma_n^* \phi_n^* = \gamma_{n+1}^*$, ψ is well defined. To prove its surjectivity, let $\mathfrak{e}$ be an element in the inverse limit, say, represented by a sequence of extensions $\mathfrak{e}_n \in \mathrm{Ext}(C_n, A)$ $(n < \omega)$ fitting in the commutative diagrams

$$\begin{array}{rccccccccc}
\mathfrak{e}_n\colon & 0 & \longrightarrow & A & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\
& & & \| & & \downarrow & & \downarrow{\scriptstyle \phi_n} & & \\
\mathfrak{e}_{n+1}\colon & 0 & \longrightarrow & A & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0
\end{array}$$

It is clear that then the $\mathfrak{e}_n$ form a direct system of exact sequences whose limit $\mathfrak{e}^*$ belongs to $\mathrm{Ext}(C, A)$. A straightforward check convinces us that $\psi(\mathfrak{e}^*)$ must be equal to $\mathfrak{e}$. □

In the following statement we keep the notations of the preceding lemma.

Proposition 5.10. *Let C be a countable group which is the union of the chain C_n $(n < \omega)$ of finitely generated subgroups. Then*

(i) $\mathrm{Ker}\,\psi = \mathrm{Pext}(C, A)$.
(ii) (Jensen [Je]) $\mathrm{Pext}(C, A) \cong \varprojlim^1 \mathrm{Hom}(C_n, A)$.

Proof.

(i) The inclusions $\gamma_n \colon C_n \to C$ induce epimorphisms $\gamma_n^* \colon \mathrm{Ext}(C, A)$ $\to \mathrm{Ext}(C_n, A)$, and evidently, $\mathrm{Ker}\,\psi$ is contained in the intersection of the $\mathrm{Ker}\,\gamma_n^*$. As an extension $0 \to A \to B \to C \to 0$ belongs to $\mathrm{Pext}(C, A)$ if and only if it is carried to 0 by all γ_n^*, the claim is evident.

(ii) (Schochet [1]) We observe that C fits in the pure-exact sequence $0 \to \oplus_{n<\omega} C_n \xrightarrow{\gamma} \oplus_{n<\omega} C_n \to C \to 0$ where γ is the Eilenberg map. By making use of Theorem 5.6, hence we deduce the exact sequence

$$\mathrm{Hom}(\oplus_{n<\omega} C_n, A) \to \mathrm{Hom}(\oplus_{n<\omega} C_n, A) \to \mathrm{Pext}(C, A) \to 0.$$

These Homs can be rewritten as direct products, so that the cokernel of the Eilenberg map between them is $\varprojlim^1 \mathrm{Hom}(C_n, A)$ (see, e.g., Sect. 5 in Chapter 2). $\square$

★ **Notes.** It is remarkable that the pure-extensions form the first Ulm subgroup of Ext. This fact is the source of several generalizations of purity for groups, see e.g. Sect. 8 in Chapter 11. For modules over integral domains, in general, the first Ulm submodule of Ext fails to be the collection of pure-extensions.

Mekler [1] proves that it is consistent with GCH that there exist non-Σ-cyclic p-groups C such that $\mathrm{Ext}(C, A) = 0$ for all countable groups A. Compare this with Corollary 5.3. For more on $\mathrm{Ext}(C, A)$, see Harrison [3].

Exercises

(1) Prove that an extension equivalent to a pure-extension is also a pure-extension.
(2) If C is torsion-free, then $\mathrm{Pext}(C, A) = \mathrm{Ext}(C, A)$ for every A. Conclude that $\mathrm{Ext}(C, A)$ is divisible provided C is torsion-free.
(3) (Irwin–Walker–Walker) An extension $\mathfrak{e} : 0 \to A \to B \to C \to 0$ belongs to the divisible subgroup of $\mathrm{Ext}(C, A)$ if and only if the induced sequence $0 \to tA \to tB \to tC \to 0$ is splitting exact.
(4) $\mathfrak{e} : 0 \to A \to B \to C \to 0$ is in the Frattini subgroup of $\mathrm{Ext}(C, A)$ exactly if it is neat-exact.
(5) Prove $\mathrm{Ext}(C, A/pA) \cong \mathrm{Ext}(C[p], A/pA) \cong \mathrm{Ext}(C[p], A)$ for all A, C.
(6) Verify the natural isomorphisms

$$\mathrm{Pext}(\oplus_{i\in I} C_i, A) \cong \prod_{i\in I} \mathrm{Pext}(C_i, A), \ \mathrm{Pext}\left(C, \prod_{i\in I} A_i\right) \cong \prod_{i\in I} \mathrm{Pext}(C, A_i).$$

(7) If C is torsion, then $\mathrm{Pext}(C, A) \cong \mathrm{Pext}(C, tA)$. [Hint: Lemma 1.4 in Chapter 5).]
(8) If A is a Σ-cyclic p-group and C is a divisible p-group, then those extensions of A by C in which A is a basic subgroup form a subgroup in $\mathrm{Ext}(C, A)$.
(9) If $A^1 = 0$, then $\mathrm{Ext}(\mathbb{Q}, A) \cong \mathrm{Hom}(\mathbb{Q}, \tilde{A}/A)$.
(10) If $0 \to A \to B \to C \to 0$ is a pure-exact sequence and F is Σ-cyclic, then the sequence $0 \to \mathrm{Ext}(F, A) \to \mathrm{Ext}(F, B) \to \mathrm{Ext}(F, C) \to 0$ is splitting exact.
(11) (Schoeman) A group C has the property that $\mathrm{Ext}(C, A)$ is algebraically compact for all groups A if and only if tC is Σ-cyclic.

6 Cotorsion Groups

In this section we get acquainted with a remarkable class of groups which fits perfectly into the structure theory of the groups of extensions. They were introduced independently and almost simultaneously by Harrison [1], Nunke [1], and Fuchs [11]. The name was coined by Harrison who pointed out their dual behavior to torsion groups (Theorem 7.4).

Cotorsion Groups Here is the definition: a group G is called **cotorsion** if it satisfies

$$\mathrm{Ext}(A, G) = 0 \quad \text{for all torsion-free groups } A.$$

Visibly, cotorsion generalizes the concept of algebraic compactness.

Every torsion-free group A can be embedded in a direct sum of copies of $\mathbb{Q}$. The inclusion map $A \to \oplus\, \mathbb{Q}$ implies that the sequence $\mathrm{Ext}(\oplus\, \mathbb{Q}, G) = \prod \mathrm{Ext}(\mathbb{Q}, G) \to \mathrm{Ext}(A, G) \to 0$ is exact. Consequently, *the single equality*

$$\mathrm{Ext}(\mathbb{Q}, G) = 0$$

suffices to guarantee that the group G is cotorsion. This is a handy criterion for cotorsionness.

Example 6.1. From Theorem 3.8 we can deduce that every Ext is cotorsion. Indeed, we have $\mathrm{Ext}(\mathbb{Q}, \mathrm{Ext}(C, A)) \cong \mathrm{Ext}(\mathrm{Tor}(\mathbb{Q}, C), A) = 0$, since the Tor vanishes. (We will give another proof in Theorem 6.5 that is independent of Theorem 3.8 whose proof has been omitted.)

As usual, we start with a list of elementary consequences of the definition. The immediate goal is to show that the class of cotorsion groups is closed under direct products, extensions and epic images.

(A) *A direct product $\prod_{i \in I} G_i$ is cotorsion if and only if every summand G_i is cotorsion.* This is a straightforward consequence of the natural isomorphism $\mathrm{Ext}(\mathbb{Q}, \prod_i G_i) \cong \prod_i \mathrm{Ext}(\mathbb{Q}, G_i)$.

(B) *A group G is cotorsion if a subgroup H and the factor group G/H are cotorsion.* The exact sequence $0 \to H \to G \to G/H \to 0$ implies the exactness of $\mathrm{Ext}(\mathbb{Q}, H) \to \mathrm{Ext}(\mathbb{Q}, G) \to \mathrm{Ext}(\mathbb{Q}, G/H)$.

(C) *Epimorphic images of cotorsion groups are cotorsion.* If G is cotorsion, and H is an epic image of G, then the sequence $\mathrm{Ext}(\mathbb{Q}, G) \to \mathrm{Ext}(\mathbb{Q}, H) \to 0$ is exact, whence the claim is evident.

(D) *Assume G is reduced and cotorsion. A subgroup H of G is cotorsion exactly if the factor group G/H is reduced.* The exact sequence $0 \to H \to G \to G/H \to 0$ leads to the exact sequence

$$0 = \mathrm{Hom}(\mathbb{Q}, G) \to \mathrm{Hom}(\mathbb{Q}, G/H) \to \mathrm{Ext}(\mathbb{Q}, H) \to \mathrm{Ext}(\mathbb{Q}, G) = 0.$$

Thus the middle terms are isomorphic; this shows that $\mathrm{Ext}(\mathbb{Q}, H) = 0$ if and only if $\mathrm{Hom}(\mathbb{Q}, G/H) = 0$.

(E) *For every endomorphism θ of a reduced cotorsion group G, both* $\mathrm{Ker}\,\theta$ *and* $\mathrm{Im}\,\theta$ *are cotorsion.* The claim follows at once from (D) and (C), respectively.

(F) *Inverse limits of reduced cotorsion groups are again reduced cotorsion.* The inverse limit G^* of reduced groups G_i is a subgroup in the direct product $\prod_i G_i$ which is, by (A), cotorsion (and reduced) if so are the G_i. Because of (D), it remains to show that the factor group $\prod_i G_i/G^*$ is reduced. Recall that G^* was the intersection of kernels of certain endomorphisms θ_j of $\prod_i G_i$ (see Sect. 5(C) in Chapter 2), and consequently, $\prod_i G_i/G^*$ is a subdirect product of the groups $\mathrm{Im}\,\theta_j$. These groups are reduced, and hence the claim is immediate.

Characterization of Cotorsion Groups The first evidence of the intimate relation between Ext and cotorsion groups is our next theorem.

Theorem 6.2. *A reduced group G is cotorsion if and only if there is an isomorphism*

$$\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \cong G.$$

A reduced cotorsion group is a $\tilde{\mathbb{Z}}$-module.

Proof. If this isomorphism holds, G is cotorsion by Example 6.1 (it will follow from Theorem 6.5 too). For the converse, we start off with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ which induces the exact sequence

$$0 = \mathrm{Hom}(\mathbb{Q}, G) \to \mathrm{Hom}(\mathbb{Z}, G) \cong G \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}, G) \to 0.$$

The connecting homomorphism between Hom and Ext is an isomorphism if and only if the last Ext vanishes. This is the case if and only if G is cotorsion. As $\mathbb{Q}/\mathbb{Z}$ is a $\tilde{\mathbb{Z}}$-module, so is $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ by the induced endomorphisms. □

Note that the proof shows that the isomorphism in the last theorem may be regarded to be natural for reduced cotorsion groups G.

Corollary 6.3. *A reduced cotorsion group is algebraically compact exactly if it is Hausdorff in the $\mathbb{Z}$-adic topology.*

Proof. A reduced cotorsion group G satisfies $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \cong G$, therefore, $\mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, G) \cong G^1$. G is Hausdorff if and only if $G^1 = 0$, and the claim follows from Proposition 5.8. □

Structure of Cotorsion Groups The last theorem has an important consequence: the structure theory of cotorsion groups can be reduced to the p-adic case. Indeed, cotorsion groups enjoy the following canonical decomposition.

Theorem 6.4. *A reduced cotorsion group G is a direct product,*

$$G = \prod_p G_p,$$

where, for each prime p, G_p is a reduced cotorsion group which is a module over the ring J_p of p-adic integers. The G_p are uniquely determined fully invariant subgroups.

Proof. In view of the isomorphism $\mathbb{Q}/\mathbb{Z} \cong \oplus_p \mathbb{Z}(p^\infty)$, the preceding theorem implies $G \cong \operatorname{Ext}(\oplus_p \mathbb{Z}(p^\infty), G) = \prod_p G_p$ where $G_p = \operatorname{Ext}(\mathbb{Z}(p^\infty), G)$ is cotorsion. G_p is the intersection of all $q^n G$ for primes $q \neq p$ and all $n < \omega$, so it is fully invariant in G. □

More on Ext The best way to explain the significance of cotorsion groups is to present a few illustrative applications. The first and foremost one is (i) in the following theorem that was the source for inspiration of the concept "cotorsion" (the name is a different story).

Theorem 6.5.

(i) *The group* $\operatorname{Ext}(C, A)$ *is cotorsion for all groups A and C.*
(ii) *If C is torsion, then* $\operatorname{Ext}(C, A)$ *is reduced cotorsion for all A.*

Proof. From the injective resolution $0 \to A \to E \to D \to 0$ of A (where E, D are injective) we derive the exact sequence $0 \to \operatorname{Hom}(C, A) \to \operatorname{Hom}(C, E) \to \operatorname{Hom}(C, D) \to \operatorname{Ext}(C, A) \to 0$.

(i) As D is injective, $\operatorname{Hom}(C, D)$ is algebraically compact (Theorem 2.11 in Chapter 7), so by (C), its epic image $\operatorname{Ext}(C, A)$ is cotorsion.
(ii) If C is a p-group, then the Homs are reduced and algebraically compact (Theorem 2.1 in Chapter 7), so the image X of the monic map between the two first Homs is cotorsion. In view of the exact sequence $0 \to X \to \operatorname{Hom}(C, D) \to \operatorname{Ext}(C, A) \to 0$, property (D) implies that $\operatorname{Ext}(C, A)$ is reduced. If C is torsion, then $\operatorname{Ext}(C, A)$ is a product of reduced cotorsion groups, so is itself of the same kind. □

We offer one more result on Ext showing that there are cases in which Ext is not only cotorsion, but even algebraically compact. Recall that by Corollary 3.6, $\operatorname{Ext}(C, \mathbb{Z})$ is compact if C is torsion.

Proposition 6.6.

(i) *For every C,* $\operatorname{Ext}(C, \mathbb{Z})$ *is algebraically compact.*
(ii) $\operatorname{Ext}(C, A)$ *is algebraically compact if so is A.*

Proof.

(i) The exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ induces the exact sequence $\operatorname{Hom}(C, \mathbb{R}) \to \operatorname{Hom}(C, \mathbb{R}/\mathbb{Z}) \to \operatorname{Ext}(C, \mathbb{Z}) \to \operatorname{Ext}(C, \mathbb{R}) = 0$. Here $\operatorname{Hom}(C, \mathbb{R})$ is torsion-free divisible, so its image in $\operatorname{Hom}(C, \mathbb{R}/\mathbb{Z})$ is a summand. Therefore, $\operatorname{Ext}(C, \mathbb{Z})$ is isomorphic to a summand of the compact group $\operatorname{Hom}(C, \mathbb{R}/\mathbb{Z})$.
(ii) By Theorem 1.2 in Chapter 6, an algebraically compact A is a summand of a direct product of cocylic groups. Hence $\operatorname{Ext}(C, A)$ is a summand of a product of groups $\operatorname{Ext}(C, \mathbb{Z}(p^k))$ for various primes p and various $k \in \mathbb{N} \cup \{\infty\}$. These

groups are bounded and 0, respectively. Thus $\mathrm{Ext}(C, A)$ is a summand of a product of bounded groups, so algebraically compact. □

Let us stop for a moment to record a useful consequence of the proof of Theorem 6.5.

Corollary 6.7. *Every reduced group G can be embedded in the cotorsion group*

$$G^{\bullet} = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$$

such that $G^{\bullet}/G$ is torsion-free divisible.

Proof. Consider the exact sequence $0 \to G \cong \mathrm{Hom}(\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}, G) \to 0$. We know from Theorem 6.5 that $G^{\bullet} = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ is cotorsion, and from Sect. 3(E) that $\mathrm{Ext}(\mathbb{Q}, *)$ is torsion-free and divisible. □

Occurrence of Cotorsion Groups In addition to Theorem 6.5, the following two propositions afford classes of natural examples of cotorsion groups.

Proposition 6.8 (De Marco–Orsatti [1]). *Let G be a topological group in a metrizable linear topology, and $\check{G}$ its completion in this topology. Then $\check{G}/G$ is a cotorsion group.*

Proof. Assuming G non-discrete, let $G = U_1 > U_2 > \cdots > U_n > \cdots$ be a base of neighborhoods for 0 consisting of subgroups with $\cap_n U_n = 0$. Define the map $\psi : \prod U_n \to \check{G}$ by sending the infinite vector $(u_1, \dots, u_n, \dots)$ $(u_n \in U_n)$ to $\sum u_n \in \check{G}$ (note that the infinite series $\sum u_n$ converges in $\check{G}$, since the sequence of the u_n converges to 0). Manifestly, ψ maps the finite vectors in $\prod U_n$ into G. Furthermore, $\mathrm{Im}\,\psi = \check{G}$, since every element of $\check{G}$ is the limit of a Cauchy sequence in G. Hence $\check{G}/G$ is an epic image of the factor group $\prod U_n / \oplus U_n$ which is by Corollary 1.12 in Chapter 6 an algebraically compact group. Therefore, $\check{G}/G$ is cotorsion. □

As a final illustration we mention the first derived functor $\varprojlim^1$ of the inverse limit functor (see Sect. 5 in Chapter 2).

Proposition 6.9 (Huber–Warfield [1]). *If $\{A_n\ (n < \omega)\}$ is a countable inverse system of groups, indexed by the natural numbers, then the derived functor yields a cotorsion group $\varprojlim^1 A_n$.*

Proof. Let $\alpha_n : A_{n+1} \to A_n$ denote the connecting homomorphisms in the inverse system. The derived functor $\varprojlim^1 A_n$ is isomorphic to the cokernel of the Eilenberg map $\delta \colon \prod A_n \to \prod A_n$ given by

$$\delta \colon (a_0, a_1, \dots, a_n, \dots) \mapsto (a_0 - \alpha_0(a_1), a_1 - \alpha_1(a_2), \dots, a_n - \alpha_n(a_{n+1}), \dots).$$

It is clear that $\mathrm{Im}\,\delta$ contains the direct sum of the A_n, and therefore $\varprojlim^1 A_n$ is an epimorphic image of the factor group $\prod A_n / \oplus A_n$. As above, it follows that $\varprojlim^1 A_n$ is cotorsion. □

When Cotorsion Is a Direct Sum We conclude this section with an interesting property of cotorsion groups that is related to their direct decompositions.

Proposition 6.10. *Let $G = \oplus_{i \in I} C_i$ be a direct decomposition of a reduced cotorsion group. Then there is an integer $m > 0$ such that*

$$mC_i = 0 \quad \textit{for almost all} \ \ i \in I.$$

Proof. We anticipate Proposition 8.3 which states that the Ulm factors of cotorsion groups are algebraically compact. If we factor out the first Ulm subgroups throughout, then we get a direct decomposition $G/G^1 = \oplus_{i \in I} C_i/C_i^1$ of the reduced algebraically compact group G/G^1. By Theorem 2.17 in Chapter 6, there is an integer $m > 0$ such that almost all summands C_i/C_i^1 are m-bounded. If C_i/C_i^1 is bounded, then C_i^1 ought to be 0. It follows that almost all of the summands C_i are bounded by m. □

★ **Notes.** Huber–Warfield [1] also proved that every cotorsion group can be obtained in the way described in Proposition 6.9, and if we allow systems of length ω_1 as well, then every group is obtainable as $\varprojlim^1$.

Cotorsion modules over integral domains were studied extensively by E. Matlis [Memoirs Amer. Math. Soc. **49** (1964)]. Other generalizations were given by E. Enochs and R. Warfield, Jr. The theory of cotorsion modules is flourishing, since Salce [2] initiated the theory of cotorsion pairs. Cf. the monograph *Approximations and Endomorphism Algebras of Modules* (2006) by R. Göbel and J. Trlifaj. The cotorsion theories for abelian groups were described by Göbel–Shelah–Wallutis [1].

Exercises

(1) A countable cotorsion group is a direct sum of a divisible and a bounded group. [Hint: consider G/G^1.]
(2) Let G be a cotorsion group, and D the divisible hull of G^1. Then $G + D$ is the pure-injective hull of G.
(3) A is torsion-free if $\mathrm{Ext}(A, G) = 0$ for all (reduced) cotorsion groups G.
(4) For a cotorsion G, there is a natural isomorphism $\mathrm{Ext}(C, G) \cong \mathrm{Ext}(tC, G)$ for every C.
(5) Suppose $0 \to A \to B \to C \to 0$ is pure-exact, and G is algebraically compact. Argue that the induced sequence $0 \to \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G) \to 0$ is splitting exact. [Hint: Lemma 5.5, Proposition 6.6(ii).]
(6) If $G^1 = 0$, then $(G^\bullet)^1 \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \tilde{G}/G)$ and $G^\bullet/(G^\bullet)^1 \cong \tilde{G}$.
(7) For all A, C, $\mathrm{Ext}(C, A)$ admits a decomposition $\mathrm{Ext}(C, A) \cong D \oplus \mathrm{Ext}(tC, A)$ where the first summand is an epimorphic image of the divisible group $\mathrm{Ext}(C/tC, A)$, and the second summand is a reduced group. [Hint: Theorem 6.5(ii).]

(8) (Wald) If G_i ($i \in I$) are cotorsion groups, then so is the subgroup of $\prod_{i\in I} G_i$ that consists of the vectors whose supports are countable. [Hint: consider $\mathrm{Ext}(\mathbb{Z}(p^\infty), *)$.]
(9) (Wald) G is cotorsion if every countable subgroup of G is contained in a cotorsion subgroup of G. [Hint: $\mathrm{Ext}(\mathbb{Q}, G) = 0$.]

7 Cotorsion vs. Torsion

The material of this section constitutes an important part of our study of cotorsion groups. The highlight is a decomposition of reduced cotorsion groups into two summands, mimicking the decompositions of torsion groups into reduced and divisible parts. The relationship to torsion groups is much deeper than one would expect, torsion and cotorsion are related to an amazing extent: the two summands belong to categories which are equivalent to the subcategories of reduced torsion and divisible torsion groups, respectively. These category equivalences were discovered by Harrison [1].

Adjusted Cotorsion Groups To begin with, we state a relevant definition. A cotorsion group that is reduced and admits no non-zero torsion-free summands is called **adjusted**. (This simply says that it is reduced and no J_p is a summand.)

Lemma 7.1 (Harrison [1]). *Let T be a reduced torsion group. Then* $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ *is an adjusted cotorsion group whose torsion subgroup is isomorphic to T, and whose factor group modulo T is* (*torsion-free*) *divisible.*

Proof. From the standard exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ we derive the exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}, T) \cong T \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \to \mathrm{Ext}(\mathbb{Q}, T) \to \mathrm{Ext}(\mathbb{Z}, T) = 0.$$

Thus the claim on the factor group will follow as soon as we can show that $\mathrm{Ext}(\mathbb{Q}, T)$ is torsion-free and divisible. But this is obvious from Sect. 3(E). What we have proved so far implies that any torsion-free summand of $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ must be divisible. Because of Theorem 6.5(ii), it has to be 0. $\square$

Lemma 7.2. *If T denotes the torsion subgroup of a group G, then*

$$\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \oplus \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G/T).$$

Proof. From the exact sequence $0 \to T \to G \to G/T \to 0$ we obtain the exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G/T) \to 0.$$

In view of Theorem 3.5 we know that the last Ext is $\cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D)$ where D is the cokernel of G/T in its divisible hull. Thus this Ext is the direct product of groups $\mathrm{Hom}(\mathbb{Z}(p^\infty), D)$ with p ranging over various primes, and Sect. 1(G) in Chapter 7 shows that these Homs are torsion-free. Since every Ext is cotorsion, the displayed exact sequence must split. □

Canonical Decomposition of Cotorsion Groups We do not need any more preparation for the proof of the basic decomposition theorem of reduced cotorsion groups mentioned above.

Theorem 7.3 (Harrison [1]). *Let G denote a reduced cotorsion group, and T its torsion subgroup. There is a decomposition*

$$G \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \oplus \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G/T),$$

where the first summand is an adjusted cotorsion group, and the second summand is a torsion-free algebraically compact group. This direct decomposition of G into an adjusted cotorsion and a torsion-free algebraically compact group is unique up to isomorphism.

Proof. Combining Theorem 6.2 and Lemma 7.2, we obtain the stated decomposition. The first summand is, by Lemma 7.1, an adjusted cotorsion group (the cotorsion hull of T, Sect. 9), while the second summand is torsion-free and algebraically compact as a consequence of Theorem 3.5. The uniqueness of the decomposition follows from the fact that if $G = C \oplus A$ is a decomposition with C adjusted, and A torsion-free algebraically compact, then $T \leq C$ and $T^\bullet/T$ divisible imply $T^\bullet \leq C$. Hence $T^\bullet = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ is a summand of C, but it cannot be a proper one, because a complement would be torsion-free divisible. □

Accordingly, every cotorsion group G decomposes, uniquely up to isomorphism, as $G = D \oplus C \oplus A$ where D is divisible, C is adjusted cotorsion, and A is reduced torsion-free algebraically compact. The summands D and A can fully be characterized by cardinal invariants, while the classification of adjusted cotorsion groups remains open, since it is equivalent to the unsolved classification problem of torsion groups; this equivalence will be more transparent in the light of Theorem 7.4 below.

Two Category Equivalences We are now entering the discussion of two category equivalences between certain subcategories of cotorsion and torsion groups.

Consider the category $\mathcal{T}$ of reduced torsion groups and the category $\mathcal{C}$ of adjusted cotorsion groups; we regard them as full subcategories of $\mathcal{A}b$. Observe that if $\phi : T \to T'$ is a map between two reduced torsion groups, then there is an induced map $\phi^\bullet : \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T')$ which is canonical, so that it is an isomorphism provided so is ϕ.

We have all the tools in our arsenal to verify the following remarkable dual roles of the functors Ext and Tor.

Theorem 7.4 (Harrison [1]). *The category $\mathcal{T}$ of reduced torsion groups and the category $\mathcal{C}$ of adjusted cotorsion groups are equivalent categories. The equivalence is provided by the covariant functors:*

$$\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, *)\colon\ \mathcal{T} \to \mathcal{C} \quad \textit{and} \quad \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, *)\colon\ \mathcal{C} \to \mathcal{T}.$$

Proof. If $T \in \mathcal{T}$, then $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \in \mathcal{C}$ as stated in Lemma 7.1. The same lemma implies $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)) \cong T$, remembering that the functor $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, *)$ picks up the torsion subgroup (Theorem 2.3 in Chapter 8). On the other hand, if $C \in \mathcal{C}$ and $T = tC$, then $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, C) \cong T$, and $C' = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ is a reduced adjusted cotorsion group with torsion subgroup $\cong T$. But it follows from Lemma 7.1 straightforwardly that if both C and C' are reduced adjusted cotorsion groups with isomorphic torsion subgroups, then they are themselves isomorphic. Therefore, $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, C)) \cong C$. □

Example 7.5. Let T be torsion and Hausdorff in the $\mathbb{Z}$-adic topology (i.e., $T^1 = 0$). Then $T^\bullet = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ is reduced, an extension of T by the divisible torsion-free group $\mathrm{Ext}(\mathbb{Q}, T)$. Its first Ulm subgroup is the algebraically compact $\mathrm{Pext}(\mathbb{Q}/\mathbb{Z}, T) \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \tilde{T}/T)$ modulo which $T^\bullet$ is isomorphic to the completion $\tilde{T}$.

The other category equivalence is between the category $\mathcal{D}$ of divisible torsion groups and the category $\mathcal{F}$ of reduced torsion-free algebraically compact groups. Both are viewed as full subcategories of $\mathcal{A}b$. Since the reduced algebraically compact groups are exactly the complete groups, we will switch to the shorter name in formulating the following theorem.

Theorem 7.6 (Harrison [1]). *The category $\mathcal{D}$ of divisible torsion groups is equivalent to the category $\mathcal{F}$ of complete torsion-free groups. The equivalence is given by the covariant functors*

$$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, *)\colon\ \mathcal{D} \to \mathcal{F} \quad \textit{and} \quad \mathbb{Q}/\mathbb{Z} \otimes *\colon\ \mathcal{F} \to \mathcal{D}.$$

Proof. Let C denote a complete torsion-free group, and E its divisible hull. In the arising exact sequence $0 \to C \to E \to D \to 0$, the group D is divisible torsion. The long exact sequences for Tor-tensor and for Hom-Ext yield

$$0 \to \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, D) \cong D \to \mathbb{Q}/\mathbb{Z} \otimes C \to 0$$

and

$$0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, C) \cong C \to 0,$$

respectively. We have taken into account that $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, *)$ selects the torsion subgroup, and $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, *)$ ignores torsion-free groups. The last isomorphism is supported by Theorem 6.2. Thus there are natural isomorphisms $D \cong \mathbb{Q}/\mathbb{Z} \otimes C$ and $C \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D)$, which proves that the given functors induce an equivalence on the objects. They act in the same way on the morphisms, as witnessed by the commutative diagrams

$$\begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},*)"'] & D' \arrow[d, "\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},*)"] \\ C \arrow[r, "\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},\delta)"] & C' \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "\mathrm{Tor}(\mathbb{Q}/\mathbb{Z},*)"'] & C' \arrow[d, "\mathrm{Tor}(\mathbb{Q}/\mathbb{Z},*)"] \\ D \arrow[r, "\mathrm{Tor}(\mathbb{Q}/\mathbb{Z},\gamma)"] & D' \end{tikzcd}$$

□

An especially convenient and handy way to capture the correspondence $D \leftrightarrow C$ is *via* the exact sequence $0 \to C \to E \to D \to 0$, where, we repeat, E is the divisible hull of C, a direct sum of copies of $\mathbb{Q}$.

Example 7.7. If $D = \mathbb{Z}(p^\infty)$, then $C = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}(p^\infty)) \cong J_p$. There is an exact sequence $0 \to J_p \to \oplus\mathbb{Q} \to \mathbb{Z}(p^\infty) \to 0$. If $D = \mathbb{Q}/\mathbb{Z}$, then $C = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \tilde{\mathbb{Z}}$, and the sequence $0 \to \tilde{\mathbb{Z}} \to \oplus\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact.

★ **Notes.** The dual behavior of Ext-Tor as well as Hom-tensor is most interesting and most important; there are other similar, less relevant, examples of duality in Homological Algebra. While the generalization of Theorem 7.6 has widespread applications in the theory of modules over integral domains (this is the Matlis category equivalence), so far Theorem 7.4 remains an isolated result, though it easily generalizes to modules over integral domains.

Exercises

(1) A non-zero cotorsion group has a summand isomorphic to one of the following groups: $\mathbb{Q}$, $\mathbb{Z}(p^k)$ $(k \le \infty)$, J_p, for some prime p.
(2) Show that $\mathcal{T} \cap \mathcal{C}$ is the class of bounded groups.
(3) Let $D = \mathbb{Z}(p^\infty)^{(\aleph_0)}$. Find $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D)$.
(4) If G is adjusted cotorsion, then $|G| \le |tG|^{\aleph_0}$.
(5) In the category equivalence of Theorem 7.6, what subcategory of $\mathcal{F}$ corresponds to the category of divisible p-groups?
(6) If G, H are adjusted cotorsion groups, then $\mathrm{Hom}(G, H) \cong \mathrm{Hom}(tG, tH)$ naturally.
(7) $\mathrm{End}\, G \cong \mathrm{End}(tG)$ for an adjusted cotorsion G.

8 More on Ext

Cotorsion Groups and Algebraic Compactness We continue our analysis of cotorsion groups. First, let us mention a few more results on the relationship between cotorsion and algebraically compact groups. As we shall see later on, they can be quite useful.

Lemma 8.1.

(i) *A reduced cotorsion group is algebraically compact if and only if its first Ulm subgroup vanishes.*
(ii) *A torsion-free group is cotorsion if and only if it is algebraically compact.*

Proof.

(i) This is basically equivalent to Corollary 6.3.
(ii) It is enough to prove this for a reduced group G. If G is torsion-free and reduced, then $G^1 = 0$. The claim follows from (i). □

Proposition 8.2. *Every reduced cotorsion group is a quotient of a torsion-free algebraically compact group modulo an algebraically compact subgroup.*

Proof. Let $0 \to G \to E \to D \to 0$ be an injective resolution of the reduced cotorsion group G. We obtain the induced exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, E) \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, D) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, E) = 0.$$

The Homs are torsion-free and algebraically compact, while the first Ext is $\cong G$, as stated in Theorem 6.2. □

The Ulm Factors of Cotorsion Groups The last proposition confirms that cotorsion groups can be derived from algebraically compact groups as epic images. Miraculously, algebraically compact groups are implanted in a natural fashion in cotorsion groups:

Proposition 8.3 (Fuchs [11]). *Ulm subgroups of cotorsion groups are cotorsion, and Ulm factors of cotorsion groups are algebraically compact.*

Proof. If G^σ is the σth Ulm subgroup of the cotorsion group G, then G/G^σ is a reduced group. Hence the first statement follows from Sect. 6(D). Then also the σth Ulm factor $G^\sigma/G^{\sigma+1}$ is cotorsion, as it is guaranteed by Sect. 6(C). Its first Ulm subgroup vanishes, so we can finish the proof by invoking Lemma 8.1(i). □

We note, in passing, that this result can be complemented by claiming that, in case G has finite Ulm length, a converse is also true: if the Ulm factors of G are algebraically compact, then G is cotorsion. (This is a consequence of Sect. 6(B).)

Corollary 8.4 (Harrison [1]). *A torsion group A is cotorsion if and only if it is of the form $A = B \oplus D$, where B is bounded and D is divisible.*

Proof. Sufficiency being obvious, assume A is torsion and cotorsion. By Proposition 8.3 its initial Ulm factor is algebraically compact. By Corollary 3.5 in Chapter 6, a torsion group that is reduced and algebraically compact is bounded. Hence the reduced part of A is bounded. □

Our next result relates the Ulm length of the torsion group T to the corresponding adjusted cotorsion group $T^\bullet = G$. Specifically, we prove:

Proposition 8.5. *Let T be a reduced torsion group of Ulm length σ. Then the cotorsion group $G = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ has Ulm length σ or $\sigma + 1$. The subgroup G^σ is torsion-free, and there is a natural isomorphism*

$$G^\sigma \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, G/(G^\sigma + T)).$$

Proof. Since $T < G$, it is clear that the Ulm length of G cannot be less than σ. We also have $G^\sigma \cap T = T^\sigma = 0$, so G^σ is torsion-free cotorsion. By Lemma 8.1(ii), G^σ is algebraically compact, so its first Ulm subgroup vanishes, $G^{\sigma+1} = 0$.

The sequence $0 \to T \to G/G^\sigma \to G/(G^\sigma + T) \to 0$ is exact, and the last factor group is divisible as an epic image of G/T. Hence Theorem 2.3 leads us to the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, G/(G^\sigma + T)) &\to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \cong G \\ &\to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G/G^\sigma) \cong G/G^\sigma \to 0, \end{aligned}$$

where the last isomorphism holds because G/G^σ is reduced cotorsion. The map between the two Exts is natural, so its kernel must be G^σ. □

Groups That Can be Ext The groups that appear in the form $\mathrm{Ext}(C, A)$ when the arguments belong to certain subclasses have been under close scrutiny by abelian group theorists. There are numerous publications about this subject, not only structural results in ZFC, but also under additional set-theoretical hypotheses, not to mention several consistency theorems. These are deeper results whose proofs demand more substantial machinery, so we cannot discuss them here. We only include a result with a relatively easy proof.

Theorem 8.6 (Jensen [Je]). *If A is torsion-free of countable rank, then* $\mathrm{Ext}(A, \mathbb{Z})$ *is either* 0 *or is a direct sum of the following summands:*

(1) *a direct sum of continuously many copies of* $\mathbb{Q}$, *and,*
(2) *for each prime p, a direct sum of finitely or continuously many copies of* $\mathbb{Z}(p^\infty)$.

Proof. By Corollary 8.3 in Chapter 3, A is a direct sum of a free group and a torsion-free group which has no non-trivial homomorphism into $\mathbb{Z}$. Evidently, it suffices to consider the second summand only, i.e. we may assume $\mathrm{Hom}(A, \mathbb{Z}) = 0$. In this case, for a prime p, the exact sequence $0 \to A \xrightarrow{\dot{p}} A \to A/pA \to 0$ induces the exact sequence $0 \to \mathrm{Ext}(A/pA, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \xrightarrow{\dot{p}} \mathrm{Ext}(A, \mathbb{Z}) \to 0$, where in view of Sect. 3(F), $\mathrm{Ext}(A, \mathbb{Z})$ is a divisible group. The image of the first Ext in the second

Ext is exactly the p-socle of $\mathrm{Ext}(A, \mathbb{Z})$, thus the number of summands $\mathbb{Z}(p^\infty)$ in $\mathrm{Ext}(A, \mathbb{Z})$ is given by the dimension of the elementary p-group $\mathrm{Ext}(A/pA, \mathbb{Z})$ as a $\mathbb{Z}/p\mathbb{Z}$-vector space. This is equal to the dimension of A/pA if this is finite, otherwise it is of the power of the continuum if A/pA is countably infinite.

If $A \neq 0$ and $\mathrm{Hom}(A, \mathbb{Z}) = 0$, then A contains a finite rank pure subgroup B which is not free (Theorem 7.1 in Chapter 3), i.e. B contains a finitely generated free subgroup F such that B/F is infinite. From the exact sequence $0 \to F \to B \to B/F \to 0$ we derive the exact sequence $\mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Ext}(B/F, \mathbb{Z}) \to \mathrm{Ext}(B, \mathbb{Z}) \to 0$. Here Hom is finitely generated free and the torsion-free rank of the first Ext is of the power of the continuum (see, e.g., Sect. 7(e) in Chapter 13), so the last Ext must have the same torsion-free rank. As $\mathrm{Ext}(B, \mathbb{Z})$ is an epic image of $\mathrm{Ext}(A, \mathbb{Z})$, the proof is complete. ☐

★ **Notes.** For more detailed information about the Ulm subgroups and Ulm factors of the groups $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$, we refer to Harrison [3].

The groups $\mathrm{Ext}(G, \mathbb{Z})$ have been studied by several authors. The study splits into the cases according as G is torsion or torsion-free. $\mathrm{Ext}(G, \mathbb{Z})$ is isomorphic to the character group of G if G is torsion (so it is a compact group). For countable torsion-free G, see Theorem 8.6. For larger torsion-free G, see the Notes to Sect. 3. Additional information: Hiller–Huber–Shelah [1] show in L that $\mathrm{Hom}(G, \mathbb{Z}) = 0$ implies that $\mathrm{Ext}(G, \mathbb{Z})$ admits a compact topology.

Schultz [5] calls G a **splitter** if $\mathrm{Ext}(G, G) = 0$ (e.g., torsion-free algebraically compact groups). His study includes groups whose infinite direct sums or products are also splitters. See Göbel–Shelah [3] for more on splitters, using new ideas and methods.

Exercises

(1) For a reduced p-group T, these conditions are equivalent:

   (a) $\mathrm{Ext}(\mathbb{Z}(p^\infty), T)$ is algebraically compact;
   (b) $\mathrm{Ext}(\mathbb{Z}(p^\infty), T) \cong \tilde{T}$;
   (c) T is torsion-complete.

(2) Find non-isomorphic reduced cotorsion groups with 2 Ulm factors such that the corresponding Ulm factors are isomorphic. [Hint: B the standard basic, T pure in torsion-completion $\overline{B}$ such that $|\overline{B} : T| = |T : B| = 2^{\aleph_0}$; compare the groups $\mathrm{Ext}(\mathbb{Z}(p^\infty), B)$ and $\mathrm{Ext}(\mathbb{Z}(p^\infty), T)$.]

(3) (a) Let G denote a reduced cotorsion group. There is a cotorsion group A such that $A^1 \cong G$.
   (b) (Kulikov) Every cotorsion group G can be realized as $G \cong \mathrm{Pext}(C, A)$ for suitable A, C.

(4) Let A be a reduced torsion-free algebraically compact group, and C an algebraically compact subgroup. Show that there is a transfinite well-ordered descending chain of algebraically compact subgroups A_σ from A down to B such that all the factors $A_\sigma/A_{\sigma+1}$ are algebraically compact. (At limit ordinals, intersections are taken.) [Hint: A/C is cotorsion.]

(5) (Hiller–Huber–Shelah) Let A be torsion-free such that $\mathrm{Hom}(A, \mathbb{Z}) = 0$. Then the rank of the p-component of $\mathrm{Ext}(A, \mathbb{Z})$ is either finite or of the form 2^κ for an infinite cardinal κ.

(6) For every totally disconnected compact abelian group A, there is a torsion group T such that $\mathrm{Ext}(T, \mathbb{Z}) \cong A$. [Hint: Corollary 3.6.]

9 Cotorsion Hull and Torsion-Free Cover

Algebraically compact groups can be characterized as pure-injective groups. This brings up the obvious question: *Do cotorsion groups display any injective behavior?* This is the first question which we wish to address in this section.

The Relative Injectivity To begin with, we introduce a new kind of exact sequence. Call an exact sequence $\mathfrak{e} : 0 \to A \to B \to C \to 0$ **torsion-splitting** if the sequence $\mathfrak{e}\tau : 0 \to A \to B' \to tC \to 0$ (with $B' \leq B$) splits for the injection map $\tau : tC \to C$.

(A) If C is torsion-free, or if the sequence is already splitting, then it is trivially torsion-splitting.
(B) $\mathfrak{e}$ is torsion-splitting if and only if torsion groups have the projective property with respect to $\mathfrak{e}$.
(C) $\mathfrak{e}$ is torsion-splitting exactly if the induced sequence $0 \to tA \to tB \to tC \to 0$ of torsion subgroups is splitting exact.

The following proposition gives only a vague idea what torsion-splitting exact sequences are like, but it helps identifying them within Ext.

Proposition 9.1. *An exact sequence* $\mathfrak{e} : 0 \to A \to B \to C \to 0$ *is torsion-splitting if and only if it represents an element in the divisible subgroup of* $\mathrm{Ext}(C, A)$.

Proof. We use the exact sequence $0 \to T \to C \to C/T \to 0$ (where $T = tC$) to derive the exact sequence $\mathrm{Ext}(C/T, A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(T, A) \to 0$. The extension $\mathfrak{e} \in \mathrm{Ext}(C, A)$ is torsion-splitting if and only if it maps upon $0 \in \mathrm{Ext}(T, A)$, i.e. it comes from the first Ext which is divisible (C/T is torsion-free). As $\mathrm{Ext}(T, A)$ is reduced, $\mathfrak{e}$ is torsion-splitting exactly if it belongs to the divisible subgroup of $\mathrm{Ext}(C, A)$. □

It follows, in particular, that the torsion-splitting extensions of A by C form a subgroup in $\mathrm{Ext}(C, A)$.

We can now answer in the affirmative the question posed above:

Theorem 9.2. *A group is cotorsion if and only if it has the injective property relative to torsion-splitting exact sequences; in particular, relative to exact sequences* $0 \to A \to B \to C \to 0$ *with torsion-free* C.

Proof. The most relevant part of this claim is a weaker form of the "only if" part: if $\phi : A \to G$ is a homomorphism of a group A into a cotorsion group G, and if A is a subgroup of H with torsion-free quotient H/A, then ϕ extends to a map $\psi : H \to G$. This is immediately seen from the exact sequence $\mathrm{Hom}(H, G) \to \mathrm{Hom}(A, G) \to \mathrm{Ext}(H/A, G) = 0$.

We continue assuming G cotorsion. If $0 \to A \to B \to C \to 0$ is a torsion-splitting exact sequence and $T = tC$, then evidently, every map $\phi : A \to G$ extends to a map $\phi' : B' \to G$ where $B' \leq B, B'/A \cong T$. As B/B' is torsion-free, by what has been proved in the preceding paragraph, ϕ' extends to some $\psi : B \to G$.

Conversely, suppose G has the injective property relative to all torsion-splitting exact sequences. Without loss of generality, G may be assumed reduced. By Corollary 6.7, there is an exact sequence $0 \to G \to G^\bullet \to D \to 0$ with $G^\bullet$ cotorsion and D torsion-free divisible. Since G has the injective property with respect to this exact sequence, the sequence splits and G is a summand of the cotorsion group $G^\bullet$ (hence $G = G^\bullet$). □

Thus *torsion groups are projective, and cotorsion groups are injective objects for the torsion-splitting exact sequences.*

Cotorsion Hull The embedding $A \to A^\bullet$ of a reduced group A in a reduced cotorsion group deserves special attention. $A^\bullet$ is actually the **cotorsion hull** of A in the following sense: $A^\bullet$ is the minimal cotorsion group containing A with torsion-free quotient. In fact, we have

Proposition 9.3.

(i) *Let A be a reduced group and $A^\bullet = \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, A)$. Any homomorphism $\phi : A \to G$ into a reduced cotorsion group G extends uniquely to $\phi^\bullet : A^\bullet \to G$.*

(ii) *The correspondence $A \mapsto A^\bullet$ is functorial: every homomorphism $\alpha : A \to B$ has a unique extension $\alpha^\bullet : A^\bullet \to B^\bullet$ making the following square commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow & & \downarrow \\ A^\bullet & \xrightarrow{\alpha^\bullet} & B^\bullet \end{array}$$

Proof.

(i) The exact sequence $0 \to A \xrightarrow{\mu} A^\bullet \to A^\bullet/A \to 0$ leads to the exact sequence

$$0 = \operatorname{Hom}(A^\bullet/A, G) \to \operatorname{Hom}(A^\bullet, G) \xrightarrow{\mu^\bullet} \operatorname{Hom}(A, G) \to \operatorname{Ext}(A^\bullet/A, G) = 0.$$

We infer that $\mu^\bullet$ is an isomorphism, proving (i).

(ii) follows in the same way, using $B^\bullet$ in place of G. □

Proposition 9.4. *A torsion-splitting exact sequence $\mathfrak{e} : 0 \to A \to B \to C \to 0$ of reduced groups induces a splitting exact sequence*

$$\mathfrak{e}^\bullet : 0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0.$$

Proof. Applying the Hom-Ext exact sequence with $\mathbb{Q}/\mathbb{Z}$ to $\mathfrak{e}$, the exactness of $\mathfrak{e}^\bullet$ is immediate. Its splitting will follow once we can establish that it is torsion-splitting. Since $tC^\bullet = tC$, from the commutative diagram

$$\begin{tikzcd}
\mathfrak{e}\colon 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
\mathfrak{e}^\bullet\colon 0 \arrow[r] & A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & 0
\end{tikzcd}$$

it is routine to derive the projective property of torsion groups relative to the exact sequence $\mathfrak{e}^\bullet$. $\square$

The cotorsion hull of a mixed group can be computed easily from those of the torsion and torsion-free parts. In fact, from Lemma 7.2 it follows that for any reduced group A, we have

$$A^\bullet \cong (tA)^\bullet \oplus (A/tA)^\bullet.$$

Example 9.5. Let $B = \oplus_n B_n$ with $B_n = \oplus \mathbb{Z}(p^n)$ for each $n \in \mathbb{N}$. Then $B^\bullet$ is equal to the subgroup $G \leq \prod_n B_n$ such that G/B is the largest torsion-free divisible subgroup in $(\prod_n B_n)/B$.

Example 9.6. For a reduced torsion-free group A, $A^\bullet \cong \tilde{A}$.

Torsion-Free Cover The class of cotorsion groups (and especially their generalizations) has attracted much attention in recent years, because they form a so-called **cotorsion pair** along with the class of torsion-free groups. This means that G is cotorsion if and only if $\text{Ext}(A, G) = 0$ for every torsion-free A, and *vice versa*, A is torsion-free if and only if $\text{Ext}(A, G) = 0$ holds for every cotorsion G. There is a very rich, fast growing literature available on this subject for modules. One of the objects of research is the existence of covers and envelopes (or hulls) in the respective classes; in our case: torsion-free covers and cotorsion hulls. The latter has been settled above, so let us move to the question of torsion-free covers.

Needless to say, every group is an epic image of a torsion-free group, for instance, of a free group. We wonder if there is a minimal one among such torsion-free groups. Better yet, if there is a unique minimal one. Enochs [2] succeeded in showing that there is always a minimal one that is even unique up to isomorphism. To make this fact precise, we have to clarify what "minimality" should mean.

Let $\phi\colon F \to A$ be a homomorphism of the torsion-free group F into A. ϕ is called a **torsion-free cover** of A if

(i) for every torsion-free group G and homomorphism $\gamma : G \to A$ there is a map $\eta\colon G \to F$ such that $\phi\eta = \gamma$, and
(ii) $\text{Ker}\,\phi$ contains no non-zero pure subgroup of F.

Condition (i) says, in more prosaic terms, that maps from torsion-free groups into A must factor through ϕ. Hence it follows that the map ϕ has to be surjective (since there is a free group mapping onto A). Trivially, a torsion-free group is a torsion-free cover of itself.

We do not wish to enter into a full-fledged study of torsion-free covers, since this would take us too far afield. But not too much effort is needed in establishing their existence.

Lemma 9.7 (Enochs [2]). *Every group A admits a torsion-free cover $\phi : F \to A$, and* $\operatorname{Ker}\phi$ *is a reduced algebraically compact group.*

Proof. Let D denote the injective hull of the torsion subgroup $T = tA$. By the Harrison category equivalence, there is an exact sequence $0 \to C \to E \to D \to 0$ where C is a complete torsion-free group, and E is an injective torsion-free group (a direct sum of copies of $\mathbb{Q}$). Cutting down from D to T, we get an exact sequence $0 \to C \to B \to T \to 0$. Here C cannot contain any pure subgroup of B, because every rank 1 pure subgroup of B maps upon a non-zero subgroup of T (T is essential in D).

We claim that there exists an exact sequence in the middle row, unique up to equivalence, making the diagram

$$\begin{tikzcd}
 & C \arrow[r, equal] \arrow[d] & C = \operatorname{Ker}\phi \arrow[d] & & \\
0 \arrow[r] & B \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "\phi"'] & H \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T \arrow[r] & A \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

with exact rows commute where $H = A/T$. Indeed, there is a natural bijection between the extensions of T by H and those of B by H, as it is clear from the exact sequence $\operatorname{Ext}(H, C) \to \operatorname{Ext}(H, B) \to \operatorname{Ext}(H, T) \to 0$, where the first Ext vanishes (H is torsion-free and C is algebraically compact). We claim that $\phi : F \to A$ is a torsion-free cover of A. If $\gamma : G \to A$ with a torsion-free group G, then the exactness of $\operatorname{Hom}(G, F) \to \operatorname{Hom}(G, A) \to \operatorname{Ext}(G, C) = 0$ implies the existence of an $\eta : G \to F$ as required by (i). Furthermore, (ii) is also satisfied, since C does not contain any pure subgroup of B, and B is pure in F. $\square$

Example 9.8. The group J_p is the torsion-free cover of $\mathbb{Z}(p^k)$ for every integer $k > 0$, where $\phi : J_p \to \mathbb{Z}(p^k)$ with $\operatorname{Ker}\phi = p^k J_p$. Condition (ii) in the definition of torsion-free cover is obviously satisfied. To check (i), let $\gamma : G \to \mathbb{Z}(p^k)$ with torsion-free G, and pick $g \in G$ such that $\langle \gamma g \rangle = \operatorname{Im}\gamma$. Then there is $u \in J_p$ such that $\phi u = \gamma g$. Since $\langle u \rangle$ is p-pure in J_p, the correspondence $g \mapsto u$ extends to a desired map $\eta : G \to J_p$ in view of the algebraic compactness of J_p.

Example 9.9. The torsion-free cover of $\mathbb{Z}(p^\infty)$ is the additive group $\mathbb{Q}_p^*$ of the field of p-adic numbers (i.e., the quotient field of J_p). This follows from the isomorphism $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}(p^\infty)) \cong \mathbb{Q}_p^*$.

★ **Notes.** Cotorsion hulls of groups have been investigated by several authors. For the cotorsion hulls of separable p-groups, see Moskalenko [1], and Kemoklidze [1], where also their full transitivity has been studied.

Torsion-free covers over arbitrary integral domains are unique up to isomorphism. For a proof, we refer to Enochs–Jenda, *Relative Homological Algebra* (2000), Theorem 4.2.1. A torsion-free F is called a **torsion-free precover** of A if it satisfies condition (i) in the definition of torsion-free cover. It is an important fact that every precover has a summand isomorphic to the cover.

Exercises

(1) If C is a cotorsion group, then $\operatorname{Hom}(A, C)$ is cotorsion for every A. [Hint: start with $tA, A/tA$.]
(2) Verify the inequality $|A^{\bullet}| \leq |A|^{\aleph_0}$.
(3) A group has the projective property relative to all torsion-splitting-exact sequences if and only if it is the direct sum of a free group and a torsion group.
(4) If $A \leq G$ and G is a reduced cotorsion group with G/A torsion-free divisible, then $G \cong A^{\bullet}$.
(5) (Rangaswamy) (a) For a group G, $\operatorname{Ext}(C, G)$ is reduced for all groups C if and only if G is cotorsion.
(b) For a fixed C, $\operatorname{Ext}(C, G)$ is reduced for all groups G if and only if C is a direct sum of a torsion and a free group.
(6) Show that $\mathbb{Z} \to \mathbb{Z}(p)$ cannot be a torsion-free cover. [Hint: rational groups $\to \mathbb{Z}(p)$.]
(7) The torsion-free cover of a divisible group D is given by $\operatorname{Hom}(\mathbb{Q}, D) \to \operatorname{Hom}(\mathbb{Z}, D) = D$.

Problems to Chapter 9

PROBLEM 9.1. Which cotorsion groups cannot be represented as $\operatorname{Ext}(C, A)$ for any A, C?

PROBLEM 9.2. Is it possible to give an upper bound for the Ulm length of $\operatorname{Ext}(C, A)$ in terms of the Ulm lengths of A and C?

PROBLEM 9.3. Characterize p-groups A such that $p^{\sigma} \operatorname{Ext}(A, \mathbb{Z}) = 0$.

PROBLEM 9.4. Study the cotorsion hulls of groups over their endomorphism rings.

PROBLEM 9.5. Relate a torsion group to its torsion-free cover.

Chapter 10
Torsion Groups

Abstract We are now prepared to plunge into an in-depth study of the major classes of abelian groups. Divisible groups have been fully characterized, so we can concentrate on reduced groups. Our discussion begins with the theory of torsion groups. Since a torsion group is a direct sum of uniquely determined p-groups, it is clear that the study of torsion groups reduces immediately to p-groups. This chapter is primarily concerned with p-groups without elements of infinite heights (called separable p-groups), while the next chapter will concentrate on p-groups containing elements of infinite heights.

Separable p-groups are distinguished by the property that every finite set of elements is contained in a finite summand. This proves to be a very powerful property. However, as it turns out, it does not simplify the group structure to the extent one hopes for: though the full potential of this condition has not been realized, it looks probable (if not certain) that a reasonable classification in terms of the available invariants is impossible. Every separable p-group is a pure subgroup between its basic subgroup B and the largest separable p-group $\overline{B}$ with the same basic subgroup. Much of the interest in these torsion-complete p-groups $\overline{B}$ comes from their numerous remarkable algebraic and topological features, one of which is that they admit complete systems of invariants.

There is a large body of work available on separable p-groups, and a fair amount of material will be covered on such groups that are distinguished by interesting properties, mostly those shared by torsion-complete groups.

1 Preliminaries on p-Groups

Before embarking on an in-depth study of p-groups, we have to get familiar with the basic tools.

Transfinite Heights and Indicators We start with transfinite heights that provide a lot of information about how an element sits inside the group. Let A be a p-group. Recall that for an ordinal σ, the subgroup $p^\sigma A$ was defined recursively: $pA = \{pa \mid a \in A\}$, $p^{\sigma+1}A = p(p^\sigma A)$, and $p^\sigma A = \cap_{\rho<\sigma} p^\rho A$ if σ is a limit ordinal. For cardinality reason, there is a smallest ordinal τ with $p^{\tau+1}A = p^\tau A$; this subgroup must be the divisible part of A which we may denote as $p^\infty A$. We form the descending chain

$$A > pA > \cdots > p^n A > \cdots > p^\sigma A > \cdots > p^\tau A = p^\infty A.$$

The ordinal τ is called the **length** of A.

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(A) *If C is a subgroup of the p-group A such that $p^nA \leq C \leq A$ for some $n \in \mathbb{N}$, and if A has infinite length, then C has the same length.*
(B) *If A has length σ and $S \leq A[p^n]$, then A/S has length $\leq \sigma + n$.* In fact, this is clear from $p^\sigma(A/S) \leq A[p^n]/S$.

The **transfinite height** $h(a)$ (henceforth called briefly *the height*) of an element $0 \neq a \in A$ is defined to be the ordinal σ if $a \in p^\sigma A \setminus p^{\sigma+1}A$, and we set $h(a) = \infty$ for elements $a \in p^\infty A$. We write $h_p(a)$ if the prime p is not obvious from the context.

More information is contained in the indicator. Let $a \in A$ be an element of order p^n. With a we associate the strictly increasing sequence

$$\underline{\mathrm{u}}(a) = (h(a), h(pa), \ldots, h(p^{n-1}a), h(p^n a) = \infty) \tag{10.1}$$

of ordinals with the symbol ∞ at the end. $\underline{\mathrm{u}}(a)$ is called the **indicator** of a in A. It is sometimes reasonable to continue the indicator with symbols ∞; if we do so, then it is easier to define the partial order for the characteristics: $\underline{\mathrm{u}}(a) \leq \underline{\mathrm{u}}(b)$ to mean that $h(p^n a) \leq h(p^n b)$ for all $n \in \mathbb{N}$.

If $h(p^i a)+1 < h(p^{i+1}a)$, then we say there is a **gap between** $h(p^i a)$ **and** $h(p^{i+1}a)$, or simply a **gap at** $h(p^i a)$. If $o(a) = p^n$ in a reduced p-group, there is certainly a gap between $h(p^{n-1}a)$ and $h(p^n a) = \infty$.

Lemma 1.1 (Kaplansky [K]). *Let A be a reduced p-group of length τ, and $\sigma_0 < \sigma_1 < \cdots < \sigma_{n-1} < \tau$ a strictly increasing sequence of ordinals. Then $\underline{\mathrm{u}} = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n = \infty)$ is the indicator of some $a \in A$ if and only if it satisfies the* **gap condition***:*

(*) *If there is gap between σ_i and σ_{i+1}, then the σ_ith UK-invariant $f_{\sigma_i}(A)$ of A is $\neq 0$.*

Proof. Assume that in $\underline{\mathrm{u}}$ a gap occurs at $h(p^i a)$ for some $a \in A$. This means that there is a $b \in A$ such that $p^{i+1}a = pb$ with $h(b) > h(p^i a) = \sigma_i$. Now $c = p^i a - b$ is of order p and of height $\min\{h(p^i a), h(b)\} = \sigma_i$. Hence the group $p^{\sigma_i}A[p]/p^{\sigma_i+1}A[p] \neq 0$, so its dimension $f_{\sigma_i}(A) \neq 0$.

Conversely, let $\underline{\mathrm{u}} = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n = \infty)$ $(\sigma_{n-1} \neq \infty)$ satisfy (*). Then there is an $a_{n-1} \in A[p]$ of height σ_{n-1}. We distinguish two cases. If there is no gap between σ_{n-2} and σ_{n-1}, then pick an a_{n-2} of height σ_{n-2} such that $pa_{n-2} = a_{n-1}$. If there is a gap between them, then (*) ensures the existence of a $b \in A[p]$ of height σ_{n-2}, and there is a $c \in A$ of height $> \sigma_{n-2}$ with $pc = a_{n-1}$. Now $a_{n-2} = b + c$ is of height σ_{n-2} and satisfies $pa_{n-2} = a_{n-1}$. Proceeding in the same way, we get successively elements $a_{n-1}, a_{n-2}, \ldots, a_0$ such that $pa_i = a_{i+1}$ and $h(a_i) = \sigma_i$ $(i = 0, \ldots, n-1)$. By construction, $\underline{\mathrm{u}}(a_0) = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n = \infty)$. ☐

Subsocles A subgroup S of the socle $A[p]$ of A is called a **subsocle** of A. A subsocle is said **to support** the subgroup C of A if $C[p] = S$. A subsocle S carries the topology inherited from the p-adic topology of A, so it will make sense to talk of a **closed** and of a **dense subsocle**, meaning closed resp. dense in $A[p]$. S is **discrete** in case $S \cap p^m A = 0$ for some $m \in \mathbb{N}$, i.e. the heights of the non-zero elements of S are bounded by $m - 1$.

We make an important observation about pure subgroups whose socles support summands.

Lemma 1.2 (Irwin–Walker [1], Enochs [1]). *A pure subgroup supported by the socle of a summand is itself a summand.*

Proof. Suppose $A = B \oplus C$ is a p-group, and G is a pure subgroup of A such that $G[p] = B[p]$. Clearly, $G \cap C = 0$ and $G + C$ contains $A[p]$. Given $a \in A[p]$, write $a = g + c$ with $g \in G[p], c \in C[p]$. Then the height of a in $G + C$ is $\geq \min\{h(g), h(c)\} = h_A(a)$. Therefore, $G + C$ is pure in A. For a pure, essential subgroup, we have $G \oplus C = A$ (see Sect. 1(F) in Chapter 5). □

Pure subgroups supported by dense subsocles are of special interest.

Theorem 1.3 (Hill–Megibben [2]). *Let S be a dense subsocle of the p-group A. If C is a subgroup of A maximal with respect to the property $C[p] = S$, then C is pure and dense in A.*

Proof. The existence of subgroups C of the stated kind follows at once from Zorn's lemma. We use induction on n to prove $C \cap p^nA \leq p^nC$. For $n = 1$, let $pa = c \in C$ with $a \in A$. If $a \notin C$, then by the maximal choice of C, there is $b \in \langle C, a\rangle \cap A[p]$ such that $b \notin S$. We can write $b = -c' + ka$ for some $c' \in C$ and integer k $(1 \leq k < p)$ which may be assumed to be 1 without loss of generality. Then $pc' = p(a - b) = pa = c$, and the case $n = 1$ is done.

Now assume that $C \cap p^nA \leq p^nC$ holds for some $n \geq 1$, and let $a \in A$ satisfy $p^{n+1}a \in C$. By what has been shown, $p^{n+1}a = pc$ for some $c \in C$. Since $p^na - c \in A[p]$, the density of S in $A[p]$ guarantees the existence of a $d \in S$ such that $p^na - c - d \in p^nA$. Then $c + d \in C \cap p^nA$, so by induction hypothesis, some $c_1 \in C$ satisfies $p^nc_1 = c + d$. Hence $p^{n+1}c_1 = pc = p^{n+1}a$, and the purity of C follows.

We refer to Lemma 2.8 in Chapter 5 to argue that the cosets mod C that are of order p can be represented by elements of $A[p]$. Owing to the density of S, the elements of order p in A/C are of infinite height in A/C. Hence A/C is divisible (cf. Sect. 1(C) in Chapter 4), and C is dense in A. □

Indicators with Integers In case the indicator contains only finite ordinals, an important conclusion may be drawn. Of course, no infinite ordinal may occur if the p-group is **separable**, i.e. every non-zero element is of finite height. An important technical lemma:

Lemma 1.4 (Baer [5]). *Let A be a p-group, and*

$$\underline{u}(a) = (r_0, r_1, \ldots, r_{n-1}, r_n = \infty)$$

the indicator of $a \in A$, *where* r_j $(j < n)$ *are integers. Let* $r_{n_1}, \dots, r_{n_s} = r_n$ *denote the places before which gaps occur. Define the integers* $k_1, \dots, k_s$ *by the rules:*

$$k_1 = r_0,\ k_2 = r_{n_1} - n_1,\ \dots,\ k_s = r_{n_{s-1}} - n_{s-1}.$$

Then $0 < n_1 < \cdots < n_s$ *and* $0 \le k_1 < \cdots < k_s$, *and there exist elements* $c_1, \dots, c_s \in A$ *such that*

(a) $o(c_i) = p^{n_i+k_i}$ *for* $i = 1, \dots, s$;
(b) $C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_s \rangle$ *is a summand of* A;
(c) $a = p^{k_1}c_1 + \cdots + p^{k_s}c_s$.

Proof. We induct on $o(a)$. If this is p, then $\underline{u}(a) = (r_0, \infty)$ and everything follows from Corollary 2.2 in Chapter 5. Thus let $o(a) = p^n$ $(n > 1)$, and assume the claim true for elements of orders $\le p^{n-1}$. The inequalities for the n_i and k_i are obvious. Consider the element $p^{n_s-1}a$ which is of height r_{n-1}, and choose a $c_s \in A$ with $p^{r_{n-1}}c_s = p^{n_s-1}a$. Then $\langle c_s \rangle$ is pure in A and is of order $p^{n_s+k_s}$. The element $a' = a - p^{k_s}c_s$ is of order $p^{n_1} \le p^{n-1}$ and—as is readily verified—its indicator is $\underline{u}(a') = (r_0, r_1, \dots, r_{n_{s-1}-1}, \infty)$. Invoking induction hypothesis, we argue that for a' there are elements $c_1, \dots, c_{s-1} \in A$ of the desired kind. Note that $\langle c_s \rangle$ intersects $C' = \langle c_1 \rangle \oplus \cdots \oplus \langle c_{s-1} \rangle$ trivially, since $p^{n_s+k_s-1}c_s$ is of order p and of height $n_s + k_s - 1$, while C' does not contain any such element. We now set $C = C' \oplus \langle c_s \rangle$. C must be pure in A, since elements in its socle have the same height in C as they have in A (Sect. 1(E) in Chapter 5). A finite pure subgroup is a summand. □

We can now derive an important corollary to the preceding lemma.

Corollary 1.5. *Let* A *be a separable* p*-group. There is an endomorphism of* A *mapping* $a \in A$ *to* $b \in A$ *if and only if* $\underline{u}(a) \le \underline{u}(b)$.

Proof. Since endomorphisms never decrease heights, the necessity is obvious. To prove sufficiency, assume $\underline{u}(a) \le \underline{u}(b)$ for $a, b \in A$. By the preceding lemma, we can embed a and b in direct summands $C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_s \rangle$ and $D = \langle d_1 \rangle \oplus \cdots \oplus \langle d_t \rangle$ of A, respectively, subject to the stated conditions. It suffices to exhibit a map $\eta \colon C \to D$ with $\eta a = b$ to complete the proof. Write $a = p^{k_1}c_1 + \cdots + p^{k_s}c_s$ as in condition (c) and similarly, $b = p^{\ell_1}d_1 + \cdots + p^{\ell_t}d_t$ where $o(d_j) = p^{m_j+\ell_j}$, $0 < m_1 < \cdots < m_t$, $0 \le \ell_1 < \cdots < \ell_t$. Now $\infty = h(p^{n_s}a) \le h(p^{n_s}b)$ shows that $p^{n_s}b = 0$, thus $n_s \ge m_t$, and therefore, we may map c_s upon $p^{\ell_j-k_s}d_j + \cdots + p^{\ell_t-k_s}d_t$ where j is the smallest index with $n_s \ge m_j$ (so that the image will be of order $\le o(c_s) = p^{n_s+k_s}$). The elements $a' = a - p^{k_s}c_s$ and $b' = b - p^{\ell_j}d_j - \cdots - p^{\ell_t}d_t$ satisfy $\underline{u}(a') \le \underline{u}(b')$, and by induction hypothesis $C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_{s-1} \rangle$ has a homomorphism into $D = \langle d_1 \rangle \oplus \cdots \oplus \langle d_{j-1} \rangle$ mapping a' upon b'. This extends to a homomorphism $\eta \colon C \to D$ with η acting on c_s as indicated. □

Kaplansky [K] calls a reduced p-group A **fully transitive** if it has the property stated in Corollary 1.5. A is **transitive** if the same holds for automorphisms with inequality replaced by equality. Corollary 1.5 asserts the full transitivity of p-groups with no non-zero elements of infinite height.

Arbitrarily Large Lengths An obvious question concerning p-groups is whether or not there exist p-groups of arbitrarily large length. The answer is yes, but it is not obviously so. The first existence proof was given independently and almost simultaneously by Kulikov [3] and the author [2]; the result follows immediately from Theorem 1.9 below. With the advent of the theory of totally projective p-groups, it is easy to construct examples of p-groups of arbitrarily large length.

Theorem 1.6. *There exist reduced p-groups of arbitrarily large length.*

Proof. Starting with the group $H_0 = 0$, we shall construct a p-group H_σ, for every ordinal σ, such that

(a) H_σ is of length σ;
(b) $p^\sigma H_{\sigma+1}$ is cyclic of order p and $H_{\sigma+1}/p^\sigma H_{\sigma+1} \cong H_\sigma$;
(c) $H_\sigma \cong \oplus_{\rho<\sigma} H_\rho$ if σ is a limit ordinal;
(d) all the UK-invariants of H_σ are $\leq |\sigma|$.

The construction is straightforward, the stated conditions indicate how to proceed. Since $H_0 = 0$, from (b) we conclude by induction that, for an integer $n > 0$, H_n is cyclic of order p^n. Then (c) implies that $H_\omega = \oplus_{n<\omega} \mathbb{Z}(p^n)$.

If σ is a limit ordinal, (c) tells us how to form H_σ. For a non-limit ordinal, write $\sigma = \mu + n$ with limit ordinal μ and integer $n > 0$. Two cases will be distinguished according as $n = 1$ or $n \geq 2$.

In case $n \geq 2$, we may assume that $H_{\mu+n-1}/p^\mu H_{\mu+n-1}$ is a cyclic group C of order p^{n-1}. Let B be a cyclic group of order p^n and $\gamma : B \to C$ an epimorphism. As the induced map $\gamma_* : \mathrm{Ext}(H_\mu, B) \to \mathrm{Ext}(H_\mu, C)$ is surjective, there exists a group $H_{\mu+n}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\gamma"'] & H_{\mu+n} \arrow[r] \arrow[d, "\delta"] & H_\mu \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] & H_{\mu+n-1} \arrow[r] & H_\mu \arrow[r] & 0
\end{tikzcd}$$

commute and its first row exact. It is readily seen that $H_{\mu+n} = H_\sigma$ satisfies conditions (a), (b), and (d).

If $n = 1$, then a different kind of construction is needed. The trick is to view H_μ as the direct sum of $H_{\rho+1}/p^\rho H_{\rho+1}$ for $\rho < \mu$ where all the subgroups $p^\rho H_{\rho+1}$ are cyclic of order p. Using the codiagonal map $\nabla : \oplus_{\rho<\mu} p^\rho H_{\rho+1} \to C \cong \mathbb{Z}(p)$, we define H_σ *via* the push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{\rho<\mu}\, p^\rho H_{\rho+1} & \longrightarrow & \oplus_{\rho<\mu}\, H_{\rho+1} & \longrightarrow & H_\mu & \longrightarrow & 0 \\
 & & \big\downarrow \nabla & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & C & \longrightarrow & H_{\mu+1} & \longrightarrow & H_\mu & \longrightarrow & 0
\end{array}$$

where $C \leq p^\mu H_{\mu+1}$, since ∇ is epic on each summand. The reverse inclusion is a consequence of $p^\mu H_\mu = 0$, so $H_{\mu+1} = H_\sigma$ satisfies (a)–(b). That (d) also holds in all cases follows rather easily from the definition by transfinite induction. □

We recognize $H_{\omega+1}$ as an old acquaintance of ours. In fact, it is identical to the Prüfer group constructed in Example 5.7 in Chapter 5. This group is a simplest example of a reduced p-group *with* non-zero elements of infinite height. Because of this resemblance, the group H_σ defined above was named by Nunke [5] a **generalized Prüfer group of length** σ.

The groups H_σ not only provide us with p-groups of arbitrary length, but, more importantly, they will serve as building blocks of the significant class of totally projective p-groups, to be discussed in Sects. 3–6 in Chapter 11.

As an additional information on these H_σ, let us point out:

Corollary 1.7. *The Ulm factors of H_σ are Σ-cyclic groups.*

Proof. We argue *via* transfinite induction. For $\sigma \leq \omega$, H_σ is just a direct sum of cyclic p-groups. If $\sigma = \mu + n$ with limit ordinal μ and $n \in \mathbb{N}$, then $H_\sigma/p^\mu H_\sigma$ has the stated property by induction hypothesis, while the last Ulm subgroup of H_σ is cyclic of order p^n. On the other hand, if σ is a limit ordinal, then the construction above shows that the Ulm factors of H_σ will be the direct sums of the Ulm factors of the H_ρ $(\rho < \sigma)$, whence the claim is evident. □

Every Group Can Be $p^\sigma A$ Another obvious question related to the existence of p-groups is answered by the following result that is proved in a more general setting (for arbitrary G).

Proposition 1.8. *Given a prime p, an ordinal σ, and a group G, there exists a group A such that $p^\sigma A \cong G$.*

Proof. First we construct a mixed group M_σ to fit into the exact sequence $0 \to \langle x_\sigma \rangle \cong \mathbb{Z} \to M_\sigma \to H_\sigma \to 0$ such that $p^\sigma M_\sigma = \langle x_\sigma \rangle$ (thus $h_p(x_\sigma) = \sigma$). This can be done in the same way as in the proof of Theorem 1.6, using the existing H_σ's, with an infinite cyclic group at the very end (in place of a cyclic group of order p). (Cf. Nunke groups in Sect. 1 in Chapter 15.)

Let $B = \oplus_{i\in I} \langle b_i \rangle$ be a p-basic subgroup of G. For each $i \in I$, select a copy C_i of M_σ, with c_i a generator of the cyclic group $p^\sigma C_i$. Define A as the push-out in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus_{i\in I}\langle c_i\rangle & \longrightarrow & \oplus_{i\in I} C_i & \longrightarrow & \oplus_{i\in I} H_\sigma & \longrightarrow & 0 \\ & & \gamma\downarrow & & \downarrow\alpha & & \| & & \\ 0 & \longrightarrow & G & \longrightarrow & A & \longrightarrow & \oplus_{i\in I} H_\sigma & \longrightarrow & 0 \end{array}$$

where γ is defined by the correspondence $c_i \mapsto b_i$. It remains to prove that $p^\sigma A = G$. The inclusion $p^\sigma A \leq G$ is a consequence of $p^\sigma H_\sigma = 0$. Clearly, the generators b_i of the basic subgroup B of G will have height σ. The rest follows from Lemma 6.7 in Chapter 5. □

Conditions on the Ulm Sequence For completeness' sake, we state a general necessary and sufficient condition on the Ulm sequence of p-groups. The theorem answers a question posed by A.G. Kurosh. The proofs are too long for inclusion, and since we are not going to need the result, we just show the necessity part, and for the rest we refer to the original articles.

We recall that, for a p-group A, the αth Ulm subgroup A^α is defined inductively as $A^1 = \cap_{n<\omega} p^n A$, $A^{\alpha+1} = (A^\alpha)^1$, and $A^\gamma = \cap_{\alpha<\gamma} A^\alpha$ if γ is a limit ordinal. The factor group $A_\alpha = A^\alpha/A^{\alpha+1}$ is the αth Ulm factor, and the sequence

$$A_0, A_1, \dots, A_\alpha, \dots \quad (\alpha < \lambda) \tag{10.2}$$

is the **Ulm sequence** of A. B_α will denote a basic subgroup of A_α.

Theorem 1.9 (Kulikov [3], Fuchs [2]). *Let (10.2) be a well-ordered sequence of p-groups, and κ an infinite ordinal. There exists a reduced p-group A of cardinality κ with Ulm sequence (10.2) if and only if the following conditions are satisfied:*

(a) A_α *is separable for each* $\alpha < \lambda$;
(b) $\sum_{0\leq\alpha<\lambda} |A_\alpha| \leq \kappa \leq \prod_{0\leq n<\min\{\omega,\lambda\}} |A_n|$;
(c) $\operatorname{rk} B_{\alpha+1} \leq \operatorname{fin}\operatorname{rk} A_\alpha$ *for every* $\alpha + 1 < \lambda$;
(d) $\sum_{\beta<\alpha<\lambda} |A_\alpha| \leq |A_\beta|^{\aleph_0}$ *for* $0 \leq \beta < \lambda$.

Proof. We establish necessity only. (a) is obvious, and so is (b) for finite ordinals λ. For infinite ordinals, the first inequality in (b) follows from $A = \cup_{\alpha<\lambda}(A^\alpha \setminus A^{\alpha+1})$ and $|A_\alpha| \leq |A^\alpha \setminus A^{\alpha+1}|$. If $|A_m| = \min_{n<\omega} |A_n|$, then (cf. Corollary 5.15 in Chapter 5)

$$|A| = |A_0||A_1|\cdots|A_{m-1}||A^m| \leq |A_0||A_1|\cdots|A_{m-1}||A^m|^{\aleph_0} \leq \prod_{0\leq n<\omega} |A_n|.$$

(d) follows from (b) applied to A^β and from Corollary 5.15 in Chapter 5: $\sum_{\beta<\alpha<\lambda} |A_\alpha| \leq |A^\beta| \leq |A_\beta|^{\aleph_0}$. To verify (c), note that for each $n \in \mathbb{N}$, there is an $x_i \in A^\alpha$ such that $p^n x_i = b_i$ for every b_i in a basis of $B_{\alpha+1}$. These x_i are independent mod $A^{\alpha+1}$, for otherwise we have $r_1x_1 + \cdots + r_kx_k = c \in B_{\alpha+1}$, whence $r_1b_1 + \cdots + r_kb_k = p^n c \in B_{\alpha+1}$. By the purity of $B_{\alpha+1}$, $p^n | r_i$ for $i = 1, \dots, k$, which is tantamount to the independence of the x_i in A_α. It follows that $A_\alpha = A^\alpha/A^{\alpha+1}$ contains an independent set of elements of p^n whose cardinality is not less than $\operatorname{rk} B_{\alpha+1}$. □

Notation In order to simplify notation, we shall write $p^\sigma A[p]$ for the accurate $(p^\sigma A)[p]$ to denote the socle of $p^\sigma A$.

★ **Notes.** Transitivity and full transitivity have been investigated by several papers. Corner [7] gives examples establishing the independence of these notions. Files–Goldsmith [1] prove that A is fully transitive if and only if $A \oplus A$ is transitive. Separable p-groups are both transitive and fully transitive, and so are the totally projective p-groups (Sect. 6 in Chapter 11).

It is not an easy task to find a p-group that fails to contain proper subgroups isomorphic to the entire group. Stringall [1] exhibits such an example with the standard basic subgroup.

Exercises

(1) Prove that $p^\sigma(p^\rho A) = p^{\rho+\sigma} A$ for any ordinals ρ, σ and any p-group A. [Hint: induction.]
(2) Let A be a finite group of type $(p^{k_1}, \dots, p^{k_n})$ with $k_1 < \dots < k_n$. Describe all possible indicators of elements in A.
(3) The indicators of elements in a separable p-group form a distributive lattice under the partial order defined above.
(4) A pure subgroup is dense in A exactly if its socle is dense in $A[p]$.
(5) (Hill–Megibben) A neat subgroup supported by a dense subsocle is a pure subgroup.
(6) In a Σ-cyclic p-group, (a) every subsocle supports a pure subgroup, and (b) pure subgroups with the same support are isomorphic.
(7) (Hill) Let $B' = \oplus_n \langle a_{2n-1}\rangle$ and $B'' = \oplus_n \langle a_{2n}\rangle$ with $o(a_k) = p^{2k}$. Define $C = \oplus_n \langle a_{2n-1} + pa_{2n}\rangle$. In the torsion-complete group $\overline{B}' \oplus \overline{B}''$ (Sect. 3), the pure subgroups $G = B' \oplus \overline{B}''$ and $H = \overline{C} + B''$ have the same socle, but they are not isomorphic. [Hint: under an isomorphism $G \to H$, G must have elements carried outside H.]
(8) (Charles) Let A be a p-group, and $a \in A$ with $\langle a\rangle \cap A^1 = 0$. Then a can be embedded in a minimal pure subgroup of A, and any two minimal pure subgroups containing a are isomorphic over $\langle a\rangle$.
(9) Prove Corollary 1.5 for automorphisms, replacing inequality by equality.
(10) (Megibben) Check that the following group is not fully transitive: $A = G \oplus H$ where $G^1 \cong H^1 \cong \mathbb{Z}(p)$, G/G^1 is torsion-complete and H/H^1 is Σ-cyclic. [Hint: G^1 is fully invariant in A, because every homomorphism $G/G^1 \to H/H^1$ is small.]
(11) (Kulikov) The set of non-isomorphic p-groups of cardinality $\leq \kappa$ has cardinality 2^κ. [Hint: Theorem 1.9; choose each Ulm factor A_σ such that its invariants are non-zero only for p^k where k is either only odd or only even.]

2 Fully Invariant and Large Subgroups

Fully Invariant Subgroups Recall that a subgroup G of a group A is called **fully invariant** (resp. **characteristic**) if every endomorphism (automorphism) of A carries G into itself. Trivially, fully invariant subgroups are characteristic, but the converse is not true in general, as is demonstrated by examples of suitable 2-groups and torsion-free groups.

Example 2.1 (Kaplansky [K]). Let $A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle a_3 \rangle$ where $o(a_i) = 2^i$. Consider the subgroup G generated by all $g \in A$ such that $o(g) = 4, h(g) = 0, h(2g) = 2$; these elements are: $a_1 \pm 2a_3$ and $a_1 + 2a_2 \pm 2a_3$. Every automorphism carries generators into generators, so G is characteristic. However, it is not fully invariant, since $a_1 \notin G$, but the projection $A \to \langle a_1 \rangle$ maps G onto $\langle a_1 \rangle$. (For torsion-free groups, see Sect. 1, Exercise 9 in Chapter 17.)

No complete description of fully invariant subgroups of p-groups is known so far, a satisfactory characterization is available only in special cases; see Baer [5], Shiffman [1], Kaplansky [K]. These particular cases include the most important subclasses of p-groups, like the separable and the totally projective p-groups.

In order to describe the fully invariant subgroups of fully transitive p-groups A, let

$$\underline{u} = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$$

be a strictly increasing sequence of ordinals, followed possibly by a sequence of symbols ∞. The meaning of $\underline{u} \leq \underline{v}$ should be clear. We associate with $\underline{u}$ the subgroup

$$A(\underline{u}) = A(\sigma_0, \sigma_1, \dots, \sigma_n, \dots) = \{a \in A \mid h_p(p^n a) \geq \sigma_n \ (n < \omega)\}. \tag{10.3}$$

This is evidently a fully invariant subgroup. We note that

(a) $A(\underline{u}) = \cap_{n \in \mathbb{N}} p^{-n}(p^{\sigma_n} A)$;
(b) if $A = \oplus_{i \in I} A_i$, then $A(\underline{u}) = \oplus_{i \in I} A_i(\underline{u})$;
(c) *every homomorphism* $\phi\colon A \to C$ *maps* $A(\underline{u})$ *into* $C(\underline{u})$.

As before, we say that $\underline{u}$ has a **gap at** σ_n if $\sigma_n + 1 < \sigma_{n+1}$, and that $\underline{u}$ satisfies the **gap condition** for A if u having a gap at σ_n is equivalent to A having an element of order p and of height σ_n (in other words, the σ_nth UK-invariant $f_{\sigma_n}(A) \neq 0$).

(d) *If both* $\underline{u}$ *and* $\underline{v}$ *satisfy the gap condition for A, then does* $\underline{u} \wedge \underline{v}$ *too.* For, if $\underline{u} \wedge \underline{v} = (\rho_0, \dots, \rho_n, \dots)$ (point-wise infimum) has a gap at ρ_n, then either $\underline{u}$ or $\underline{v}$ has ρ_n at the nth place, and the same must have a gap at this place. Thus the ρ_nth UK-invariant $f_{\rho_n}(A) \neq 0$.

Theorem 2.2 (Kaplansky [K]). *Suppose A is a fully transitive p-group. A subgroup G of A is fully invariant if and only if it can be written in the form* $G = A(\underline{u})$ *where* $\underline{u}$ *satisfies the gap condition.* $\underline{u}$ *is uniquely determined by G.*

Proof. We have noticed above that every subgroup of the form (10.3) is fully invariant. Assume, conversely, that G is a fully invariant subgroup, and define σ_n as the minimum of the heights $h(p^n g)$ with g running over G. The sequence $\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$ is obviously strictly increasing (except when it reaches ∞). To verify the gap condition, suppose $\sigma_k + 1 < \sigma_{k+1}$ for some k. Surely, there exists an $x \in G$ with $h(p^k x) = \sigma_k$, and by definition $h(p^{k+1}x) \geq \sigma_{k+1}$; thus, this x has a gap at σ_k in its indicator. By Lemma 1.1, A must contain an element of order p and of height σ_k.

The inclusion $G \leq A(\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$ is obvious. We now verify the existence of a $g \in G$ such that $h(p^i g) = \sigma_i$ for $i = 0, 1, \ldots, n-1$. If there is no gap in the sequence $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$, and if $g \in G$ satisfies $h(p^{n-1}g) = \sigma_{n-1}$, then this g is already as desired. If there is a gap in this sequence, and if the first gap appears between σ_{j-1} and σ_j, then there is a $g_j \in G$ such that $h(p^i g_j) = \sigma_i$ for $i = 0, 1, \ldots, j-1$. If the second gap lies between σ_{k-1} and σ_k $(j < k)$, then some $g' \in G$ exists with $h(p^i g') = \sigma_i$ for $i = j, \ldots, k-1$. By Lemma 1.1, A contains a g_k such that $h(p^i g_k) \geq \max\{h(p^i g'), \sigma_i + 1\}$ for $i = 0, 1, \ldots, j-1$ and $h(p^i g_k) = h(p^i g')$ for $i \geq j$. Because of full invariance, and hence full transitivity, as well as Corollary 1.5, $g_k \in G$. Thus proceeding, we construct elements $g_j, g_k, \ldots, g_\ell \in G$ for the gaps in $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$, and at the end, $g = g_j + g_k + \cdots + g_\ell$ will satisfy $h(p^i g) = \sigma_i$ for $i = 0, 1, \ldots, n-1$. Thus $h(g) \leq h(a)$ for every $a \in A(\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$ of order $\leq o(g)$. Full transitivity shows that $a \in G$, i.e. G is of the form (10.3).

If $(\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$ and $(\sigma'_0, \sigma'_1, \ldots, \sigma'_n, \ldots)$ are different sequences, both satisfying the gap condition, then let n be the first index with $\sigma_n \neq \sigma'_n$, say, $\sigma_n < \sigma'_n$. There exists an $a \in A$ with $h(p^i a) = \sigma_i$ for $i = 0, 1, \ldots, n$. This a belongs to $A(\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$, but not to $A(\sigma'_0, \sigma'_1, \ldots, \sigma'_n, \ldots)$. $\square$

Large Subgroups There is a kind of fully invariant subgroup in p-groups which is of particular interest. It was called by Pierce [1] a **large** subgroup. It is defined as a fully invariant subgroup G satisfying $G + B = A$ for every basic subgroup B of A.

We list some relevant properties.

(A) 0 *is a large subgroup if and only if A is bounded.*
(B) *In a bounded group, all fully invariant subgroups are large.*
(C) *If G is a large subgroup of A, then so is $p^n G$ for every n.* Indeed, $p^n G$ is fully invariant and satisfies $p^n G + B = p^n G + p^n B + B = p^n A + B = A$ provided $G + B = A$.
(D) *If G is a large subgroup of A, then A/G is Σ-cyclic.* In order to prove this, observe that $G + B = A$ implies $A/G \cong B/(G \cap B)$. Write $B = \oplus_{i \in I} \langle b_i \rangle$, and notice that the $\langle b_i \rangle$ are summands of A, so the projection of G on $\langle b_i \rangle$ is $G \cap \langle b_i \rangle$. Hence $G \cap B = \oplus_{i \in I} (G \cap \langle b_i \rangle)$, and $B/(G \cap B)$ is the direct sum of the cyclic groups $\langle b_i \rangle / (G \cap \langle b_i \rangle)$.
(E) A^1 *is contained in every large subgroup of A.* This follows from (D), since if A/G is Σ-cyclic, then necessarily $A^1 \leq G$.

Our next purpose is to single out the large subgroups from among the fully invariant subgroups. We will say that a subgroup G of a p-group A satisfies the

Pierce condition if for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every $a \in A$ with $o(a) \leq p^k$ and $h(a) \geq n$ is contained in G, i.e.

$$p^n A[p^k] \leq G.$$

Since (B) completely characterizes large subgroups in bounded groups, our focus is on unbounded groups.

Theorem 2.3 (Pierce [1]). *Let A be an unbounded, reduced p-group. For a fully invariant subgroup G of A, these are equivalent:*

(i) *G is a large subgroup of A*;
(ii) $G = A(r_0, r_1, \dots, r_n, \dots)$ *with a strictly increasing sequence of non-negative integers r_n and symbols ∞* (*satisfying the gap condition*);
(iii) *the Pierce condition holds for G.*

Proof.

(i) $\Rightarrow$ (ii) By Theorem 2.2, $G = A(\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ for suitable σ_n, and in view of (E), we have $\sigma_n < \omega$ for all $n \in \mathbb{N}$ for which $\sigma_n \neq \infty$.
(ii) $\Rightarrow$ (iii) Given $k \in \mathbb{N}$, let $n = r_{k-1} - k + 1$. Then none of $r_{k-2} - k + 2$, $r_{k-3} - k + 3, \dots, r_0$ exceeds n. Therefore, if $a \in A$ is such that $o(a) \leq p^k$ and $h(a) \geq n$, then $h(p^i a) \geq n + i \geq r_i$ for $i = 0, 1, \dots, k-1$, and $h(p^i a) = \infty$ for $i \geq k$. Thus $\underline{u}(a) \geq (r_0, r_1, \dots, r_n, \dots)$, and $a \in G$.
(iii) $\Rightarrow$ (i) We have to prove $G + B = A$ for any basic subgroup B of A. Pick $a \in A$ with $o(a) = p^k$, and choose $n \in \mathbb{N}$ according to the Pierce condition. By the divisibility of A/B, we have $a = b + p^n c$ for some $b \in B$, $c \in A$. We may assume $o(b) \leq p^k$, because $0 = p^k a = p^k b + p^{n+k} c$ implies $p^k b = p^{n+k} b'$ for some $b' \in B$, so also $a = (b - p^n b') + p^n(b' + c)$ where $o(b - p^n b') \leq p^k$. But then $o(p^n c) \leq p^k$ as well, and therefore (iii) implies $p^n c \in G$. This proves $G + B = A$. □

★ **Notes.** It was Kaplansky [K] who characterized the fully invariant subgroups in fully transitive p-groups. The theory of large subgroups is due to Pierce [1]. Numerous papers deal with the relation of the structure of the group to the structure of its large subgroups; e.g. large subgroups are totally projective if and only if the group itself is totally projective.

Anyone in search for more cardinal invariants of p-groups might try to test the vector spaces $A(\underline{u})/A(\underline{v})$ in case $\underline{u} = (\rho_0, \rho_1, \dots, \rho_n, \dots)$ and $\underline{v} = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ satisfy $\rho_n \leq \sigma_n < \rho_{n+1}$ for every $n < \omega$. However, in this way no new invariants are gained, since the dimension of the vector space in question turns out to be just the sum of dimensions of $p^{\rho_n} A[p]/p^{\sigma_n} A[p]$ for $n < \omega$.

Exercises

(1) Find the maximal length of chains of fully invariant subgroups in a bounded p-group.
(2) If L_i is a large subgroup of the p-group A_i for $i \in I$, then $\oplus_{i \in I} L_i$ is large in $\oplus_{i \in I} A_i$.

(3) Let A be a separable p-group, and $a \in A$. What is the fully invariant subgroup of A generated by a?
(4) If G is a large subgroup in the p-group A, then there is an $n < \omega$ such that $G[p] = p^n A[p]$.
(5) If A is a reduced p-group and has a large subgroup that is Σ-cyclic, then A itself must be Σ-cyclic. [Hint: preceding exercise.]
(6) For a pure subgroup C and for a large subgroup $A(\underline{u})$ of a p-group A, we always have $C(\underline{u}) = C \cap A(\underline{u})$.
(7) (a) Give an example when $A(\underline{u}) = A(\underline{v})$, but $\underline{u} \neq \underline{v}$ (no gap condition is required).
(b) If $A(\underline{u}) = A(\underline{v})$ and if $\underline{u}$ satisfies the gap condition, then $\underline{u} \leq \underline{v}$.
(8) Let A be a fully transitive p-group.

(a) For a family $G_j = A(\underline{u}_j)$ $(j \in J)$ of fully invariant subgroups write their union and intersection in the form (10.3).
(b) The fully invariant subgroups of A form a complete distributive sublattice $\mathbf{K}$ in $\mathbf{L}(A)$.
(c) The large subgroups form a filter in $\mathbf{K}$.

(9) What is the intersection of all large subgroups of a p-group?
(10) The lattices of large subgroups of the p-groups A and C are isomorphic if the two groups have isomorphic basic subgroups. [Hint: gaps.]
(11) (Pierce) Any single basic subgroup $B \neq A$ is a test group for a fully invariant subgroup G to be large: if $A = G + B$ for the selected B, then $A = G + B'$ holds for all basic subgroups B'.
(12) (Pierce) Let G be a large subgroup in a separable p-group A. If B is a basic subgroup of A, then $G \cap B$ is a basic subgroup of G. [Hint: for purity show $B \cap pG = p(B \cap G)$ and use (C).]
(13) (Pierce) A large subgroup of a large subgroup is a large subgroup. [Hint: preceding exercise.]
(14) For p-groups A, C, a homomorphism $\phi : A \to C$ is small if and only if $\operatorname{Ker}\phi$ satisfies the Pierce condition if and only if $\operatorname{Ker}\phi$ contains a large subgroup of A.
(15) Is it true that in a reduced p-group, for every element $a \neq 0$ there exists a fully invariant subgroup G maximal with respect to the property $a \notin G$?

3 Torsion-Complete Groups

Our study of p-groups continues with p-groups without elements $\neq 0$ of infinite height. This is the case when the first Ulm subgroup $A^1 = 0$, i.e. A is Hausdorff in the p-adic (equivalently, in the $\mathbb{Z}$-adic) topology. The closure X^- of a subgroup X of A in the p-adic topology is obtained by taking the first Ulm subgroup $X^-/X = (A/X)^1$ in A/X.

Separability We may refer to a separable p-group as a group in which every finite subset $\{a_1, \dots, a_n\}$ can be embedded in a finite direct summand. In fact, it is straightforward to check that this definition is equivalent to the one used in Sect. 1: the non-trivial part follows from Lemma 1.4 by using induction.

Torsion-Complete Groups Undoubtedly, the most significant class of separable p-groups, besides the class of direct sums of cyclic p-groups, is the class of torsion-complete p-groups. Their study began with the work of Kulikov [2] (under the name of *closed p-groups*), and was brought to maturity by several authors in the 1960s. These groups have profound implications in the theory of p-groups. Fortunately, they admit a satisfactory classification by cardinal invariants.

In this section and in the next one, we assemble various results on torsion-complete p-groups. We will adhere to the following notation: B_n will denote a direct sum of cyclic groups of order p^n, $B_n = \oplus_{\kappa_n} \mathbb{Z}(p^n)$ for an unspecified cardinal κ_n, and B will denote their direct sum $\oplus_{n \in \mathbb{N}} B_n$.

By a **torsion-complete p-group** is meant the torsion subgroup of the p-adic completion $\tilde{B}$ of a direct sum B of cyclic p-groups. $t\tilde{B}$ is uniquely determined by B, so we may denote it by $\overline{B}$—this notation will be standard in this volume. We will see in due course that a torsion-complete p-group can be written in the form $\overline{B}$ for any of its basic subgroups B.

We also note that in the category of torsion groups, the categorical product is the torsion subgroup of the direct product. Therefore, torsion-complete p-groups are nothing else than categorical products of bounded p-groups. It is no wonder that they share several properties with bounded groups.

Since $\tilde{B}$ is a subgroup of $\prod_n B_n$, the same holds for $\overline{B}$. Hence an element $g \in \overline{B}$ can be written uniquely in the form

$$g = (b_1, b_2, \dots, b_n, \dots) \qquad \text{with } \ b_n \in B_n. \tag{10.4}$$

A vector g in (10.4) will belong to $\overline{B}$ if and only if the orders of its coordinates b_n are bounded. The elements of B are represented by vectors with finite supports. In view of the structures of the B_n, for any sequence $\{b_n\}_{n \in \mathbb{N}}$ with $b_n \in B_n$ of orders bounded by p^m, we have $h(b_n) \geq n - m$ for $n \geq m$; thus, $h(g) = \min\{h(b_1), \dots, h(b_{m-1})\}$ whenever $b_i \neq 0$ for at least one of $i = 1, \dots, m-1$.

Example 3.1. All bounded p-groups are torsion-complete. The simplest unbounded example for a torsion-complete group is when B is the standard basic subgroup, i.e. $B_n = \langle a_n \rangle \cong \mathbb{Z}(p^n)$ for each n. The elements of this $\overline{B}$ are the vectors $b = (k_1 a_1, \dots, k_n a_n, \dots)$, where $0 \leq k_n < p^n$, and there is an integer m such that $p^m k_n a_n = 0$ for all n.

We record a few important facts on torsion-complete p-groups.

(A) $\overline{B} = B$ *if and only if B is bounded.* We observe that if B is unbounded, then $\overline{B}$ will contain vectors (10.4) with infinite support.

(B) *For two Σ-cyclic p-groups, B and B', we have* $\overline{B \oplus B'} = \overline{B} \oplus \overline{B'}$.

(C) *B is a basic subgroup of $\overline{B}$.* In fact, as we saw in Theorem 2.4 in Chapter 6, B is a basic subgroup in $\tilde{B}$, so it is also a basic subgroup in its torsion subgroup $\overline{B}$.

Theorem 3.2. *Let B be a Σ-cyclic p-group of infinite cardinality κ. Then $|\overline{B}| \le \kappa^{\aleph_0}$. Equality holds if also* $\operatorname{fin}\operatorname{rk} B = \kappa$.

Proof. The stated inequality is an immediate consequence of the representation of elements of $\overline{B}$ in the form (10.4): $|\overline{B}| = \prod_{n\in\mathbb{N}} \kappa_n \le (\sum_{n\in\mathbb{N}} \kappa_n)^{\aleph_0}$. If also $\operatorname{fin}\operatorname{rk} B = \kappa$, then we distinguish two cases according as the set of cardinalities $\kappa_n = |B_n|$ ($n \in \mathbb{N}$) contains only finitely many or infinitely many κ_n equal to κ. In the second alternative the claim is obvious. In the first case, for the proof we may ignore any B_n of cardinality κ, and assume that $\kappa_n < \kappa$ for all n. Necessarily, infinitely many of the κ_n are different, and we must have $\sum_n \kappa_n = \kappa = \sup_n \kappa_n$. Jech [J] tells us that in these circumstances $\prod_n \kappa_n = \kappa^{\aleph_0}$. □

We are now in a position to derive the following theorem that solves at once the structure problem for torsion-complete p-groups. It shows that they are completely characterized by their UK-invariants.

Theorem 3.3 (Kulikov [2]). *Two torsion-complete p-groups are isomorphic if and only if their basic subgroups are isomorphic. Thus the sequence $\kappa_1, \dots, \kappa_n, \dots$ of cardinal invariants of $B = \oplus_{n<\omega} B_n$ (where $B_n = \oplus_{\kappa_n} \mathbb{Z}(p^n)$) is a complete and independent set of invariants for $\overline{B}$.*

Proof. The 'only if' part is a trivial consequence of (C) and the uniqueness (up to isomorphism) of basic subgroups. □

Torsion-Completion We wish to clarify the relation between arbitrary p-groups A and torsion-complete p-groups. Let $B = \oplus_n B_n$ (with the adopted notation) be a basic subgroup of A. As in Sect. 5 in Chapter 5, we have a sequence $A = B_1 \oplus \dots \oplus B_n \oplus A_n$ ($n \in \mathbb{N}$) of direct decompositions such that the $(n+1)$st is obtained from its predecessor by separating the summand B_{n+1} from A_n. In this way, we can associate with each $a \in A$ a sequence $b_1, \dots, b_n, \dots$ with $b_n \in B_n$ such that $a = b_1 + \dots + b_n + a_n$ for some $a_n \in A_n$. This gives rise to a correspondence

$$\eta\colon\ a \mapsto (b_1, \dots, b_n, \dots) \qquad (b_n \in B_n) \tag{10.5}$$

which is obviously a homomorphism of A into $\prod_n B_n$. As the order of a is a bound for the orders of the b_n, η maps A into $\overline{B}$. It is clear that η acts isomorphically on B, mapping its elements to vectors with finite supports. As $\overline{B}$ is a separable group, we have $A^1 \le \operatorname{Ker}\eta$. On the other hand, if $\eta a = 0$ for some $a \in A$ of order p^k, then $a = a_n \in A_n$ for every n, thus $h(a) = h(a_n) \ge n + 1 - k$ for every n; this means that $h(a) \ge \omega$ and $a \in A^1$. Consequently, $\operatorname{Ker}\eta = A^1$.

Theorem 3.4. *Let A be a p-group, and B a basic subgroup of A. Then η in (10.5) is a homomorphism of A onto a pure subgroup of $\overline{B}$. The kernel of η is the first Ulm subgroup A^1, and $\overline{B}/\eta A$ is divisible.*

In particular, a separable p-group A with basic subgroup B is isomorphic to a pure (dense) subgroup of $\overline{B}$ containing B.

Proof. It remains to check the purity of Im η in $\overline{B}$. By Sect. 6(F) in Chapter 5, ηB is a basic subgroup in ηA, so $\eta A/\eta B$ is a divisible group. Manifestly, ηB is basic in $\overline{B}$, whence the purity of ηA in $\overline{B}$ is immediate. □

It is appropriate to regard the torsion-complete group $\overline{B}$ obtained from the p-group A in the preceding theorem as the **torsion-completion** of A. It is unique up to isomorphism.

As a consequence, a separable p-group may be thought of (and treated as) a pure and dense subgroup of a torsion-complete group. This point of view gives us considerably more leverage than regarding it as a subgroup of a divisible group. To illustrate this, we interrupt momentarily the discussion of our main topic, and prove the following slight generalization of Prüfer's Theorem 5.3 in Chapter 3.

Theorem 3.5. *Let A be a separable p-group, and B a pure Σ-cyclic subgroup of A. If A/B is countable, then A is Σ-cyclic.*

Proof. As B can be expanded to be a basic subgroup, we may assume B is a basic subgroup of A. A is treated as a pure subgroup of $\overline{B}$ containing B. Let $a_1, \dots, a_m, \dots$ be a countable set that, together with B, generates A. We can write $a_m = (b_{m1}, \dots, b_{mn}, \dots) \in \overline{B}$ with $b_{mn} \in B_n$. Each B_n is a direct sum of cyclic groups of order p^n, so there is a decomposition $B_n = B'_n \oplus B''_n$ such that $b_{mn} \in B'_n$ for all m, and B'_n is countable. Setting $B' = \oplus_n B'_n$ and $B'' = \oplus_n B''_n$, we obtain $B = B' \oplus B''$ and $A = A' \oplus B''$ where $A' = \langle B', a_1, \dots, a_m, \dots \rangle$. Here A' is countable and separable, so by Theorem 5.3 in Chapter 3 it is Σ-cyclic. □

Theorems on Torsion-Complete Groups We return to torsion-complete groups, and continue with various algebraic characterizations. The main algebraic features are encapsulated in the following theorem.

Theorem 3.6. *For a reduced p-group A, the following conditions are equivalent.*

(i) *A is torsion-complete;*
(ii) *A is the torsion subgroup of a reduced algebraically compact group;*
(iii) *A is pure-injective in the category of p-groups (i.e., it has the injective property relative to pure-exact sequences of p-groups);*
(iv) *A is a direct summand in every p-group in which it is contained as a pure subgroup.*

Proof.

(i) ⇒ (ii) This implication is obvious, since $\overline{B}$ is by definition the torsion subgroup in the algebraically compact (= complete) group $\tilde{B}$.

(ii) ⇒ (iii) Next, suppose (ii), i.e. A is the torsion part of an algebraically compact group C. Let $0 \to G \xrightarrow{\alpha} H \xrightarrow{\beta} K \to 0$ be a pure-exact sequence of p-groups, and $\eta : G \to A$ a homomorphism. If η is viewed as a map $G \to C$, then by pure-injectivity there is a $\chi\colon H \to C$ such that $\eta = \chi\alpha$. As H is torsion, χ must land in A, so (iii) follows.

(iii) $\Rightarrow$ (iv) If (iii) holds for A, and if A is a pure subgroup in a p-group G, then the identity map $\mathbf{1}_A$ extends to a map $G \to A$, establishing the summand property of A. Thus (iv) is immediate.

(iv) $\Rightarrow$ (i) Finally, assume (iv). If $A^1 \neq 0$, then A would be pure, but not a summand, in the torsion part G of the pure-injective hull of A (recall: G/A is divisible and every divisible subgroup $\neq 0$ of G intersects A non-trivially). Hence $A^1 = 0$, and so by Theorem 3.4 A may be thought of as a pure subgroup of $\overline{B}$ where B denotes a basic subgroup of A. By hypothesis (iv), we have $\overline{B} \cong A \oplus \overline{B}/A$, where the second summand must vanish, since it is divisible, and $\overline{B}$ is reduced. Hence (i) follows. □

We go on to point out that condition (iv) can be rephrased by stating that $\operatorname{Pext}(T, A) = 0$ for all torsion groups T. As a matter of fact, this criterion can considerably be improved: $\mathbb{Z}(p^\infty)$ is a kind of 'test' group for torsion-completeness.

Corollary 3.7. *A p-group A is torsion-complete if and only if*

$$\operatorname{Pext}(\mathbb{Z}(p^\infty), A) = 0;$$

equivalently, $\operatorname{Ext}(\mathbb{Z}(p^\infty), A)$ *is algebraically compact.*

Proof. Start with a pure-exact sequence $0 \to C \to T \to \oplus\, \mathbb{Z}(p^\infty) \to 0$ where C is a basic subgroup of an arbitrarily chosen p-group T. From Theorem 5.6 in Chapter 9 we derive the exact sequence

$$\operatorname{Pext}(\oplus\, \mathbb{Z}(p^\infty), A) = \prod \operatorname{Pext}(\mathbb{Z}(p^\infty), A) \to \operatorname{Pext}(T, A) \to \operatorname{Pext}(C, A) = 0$$

whence the implication $\operatorname{Pext}(\mathbb{Z}(p^\infty), A) = 0 \Rightarrow \operatorname{Pext}(T, A) = 0$ is evident.

Pext is the first Ulm subgroup of Ext, so the stated condition is equivalent to saying that the first Ulm subgroup of $\operatorname{Ext}(\mathbb{Z}(p^\infty), A)$ vanishes which means the algebraic compactness of this Ext (Corollary 6.3 in Chapter 9). □

(D) We note that *if $A = \overline{B}$ for a basic subgroup B of A, then also $A = \overline{B'}$ for any basic subgroup B' of A*. In fact, if $B'(\leq A)$ is a basic subgroup, then A is pure in $\overline{B'}$, thus the torsion-complete A is a summand of $\overline{B'}$. The complement is divisible, so it must be 0.

(E) It should be pointed out that, *for a Σ-cyclic p-group B, $\overline{B}$ is the largest separable p-group whose basic subgroup is B*. In fact, every separable p-group with B as basic subgroup is embeddable in $\overline{B}$.

(F) *The natural map* $\operatorname{Hom}(\overline{B}, \overline{B}) \to \operatorname{Hom}(B, \overline{B})$ *is an isomorphism.* This follows from the pure-exact sequence $0 \to B \to \overline{B} \to \oplus\, \mathbb{Z}(p^\infty) \to 0$, applying Theorem 5.6 in Chapter 9 and taking Corollary 3.7 into account.

Subgroups of Torsion-Complete Groups Next, we want to find out when subgroups of torsion-complete p-groups are again torsion-complete.

Corollary 3.8 (Irwin–O'Neill [1]). *A subgroup G of a torsion-complete p-group A is torsion-complete if the factor group A/G is reduced.*

Proof. If A/G is reduced, then the exact sequence $0 \to G \to A \to A/G \to 0$ induces the exact sequence $0 = \mathrm{Hom}(\mathbb{Z}(p^\infty), A/G) \to \mathrm{Ext}(\mathbb{Z}(p^\infty), G) \to \mathrm{Ext}(\mathbb{Z}(p^\infty), A)$. Thus $\mathrm{Ext}(\mathbb{Z}(p^\infty), G)$ can be regarded as a subgroup of $\mathrm{Ext}(\mathbb{Z}(p^\infty), A)$. If the first Ulm subgroup (i.e., Pext) of the latter group vanishes, then the same holds for the former group, i.e. $\mathrm{Pext}(\mathbb{Z}(p^\infty), G) = 0$. By Corollary 3.7, the proof is complete. □

More can be said if the subgroup is pure.

Corollary 3.9. *For any pure subgroup G of a torsion-complete p-group A, the factor group A/G is a direct sum of a divisible group and a torsion-complete group. Thus the closure of a pure subgroup is a summand of A.*

Proof. Because of Theorem 5.6 in Chapter 9, the pure-exactness of $0 \to G \to A \to A/G \to 0$ implies that the induced map $\mathrm{Pext}(\mathbb{Z}(p^\infty), A) \to \mathrm{Pext}(\mathbb{Z}(p^\infty), A/G)$ is onto. By Corollary 3.7, the first Pext vanishes, and therefore so does the second. It follows that the reduced part of A/G is torsion-complete. Consequently, the factor group G^-/G is divisible for the closure G^- of G in A, so G^- is also pure in A. Thus A/G^- is torsion-complete, hence G^- is torsion-complete by Corollary 3.8. It is pure in A, so a summand. □

We draw particular attention to the next two theorems. The first result provides a relevant information about the isomorphy of pure dense subgroups in torsion-complete groups.

Theorem 3.10 (Leptin [1]). *Two pure and dense subgroups, G and H, of a torsion-complete p-group A are isomorphic if and only if A has an automorphism carrying G onto H. Moreover, every isomorphism between G and H can be extended uniquely to an automorphism of A.*

Proof. We show that any isomorphism $\phi : G \to H$ is induced by a unique automorphism α of A. Viewing ϕ as a map $G \to A$, from the pure-exact sequence $0 \to G \to A \to A/G \to 0$ we conclude that there is an $\alpha : A \to A$ extending $G \to A$; cf. Theorem 3.6(iii). Changing the roles of G and H, we obtain a map $\beta : A \to A$ which extends $\phi^{-1} : H \to A$. Now both $\beta\alpha$ and $\alpha\beta$ are endomorphisms of A: the former is the identity on G, the latter is the identity on H, thus both are identities on the respective basic subgroups. From our hypothesis it follows that both G and H contain basic subgroups of A, thus $\beta\alpha$ and $\alpha\beta$ are identities on these basic subgroups. Section 6(G) in Chapter 5 implies that $\beta\alpha = \mathbf{1}_A = \alpha\beta$; thus $\alpha \in \mathrm{Aut}\, A$. The uniqueness of α is a consequence of the same (G), since any two extensions of ϕ induce the same map on a basic subgroup. □

An immediate consequence of Theorem 3.10 is that an isomorphism between basic subgroups of a torsion-complete group A extends uniquely to an automorphism of A. Actually, this property characterizes the torsion-complete groups, as is shown by

Theorem 3.11 (Leptin [1], Enochs [3]). *A reduced p-group A is torsion-complete if and only if every isomorphism between basic subgroups extends to an automorphism of A.*

Proof. In order to verify the 'if' part, suppose A has the stated property. If A is bounded, then it has only one basic subgroup, viz. itself, and there is nothing to prove. So assume A is unbounded, and let B be a basic subgroup of A. As B is unbounded, $B < \overline{B}$. Let B' be a subgroup of $\overline{B}$ such that $B < B' < \overline{B}$ and B'/B is countable divisible. Because of Theorem 3.5, B' is again a basic subgroup of $\overline{B}$, so by Theorem 3.10 there is an automorphism β of $\overline{B}$ such that $\beta B' = B$. Then $\beta B(< B)$ is also basic in $\overline{B}$, and in A as well, so by hypothesis, there is an $\alpha \in \operatorname{Aut} A$ such that $\alpha \restriction B = \beta \restriction B$.

Define a map $\phi\colon \overline{B} \to A$ as follows. Put $\phi b = b$ for every $b \in B$. If $x \in \overline{B} \setminus B$, then it is included in some B', and we set $\phi x = \alpha^{-1}\beta x$. This definition is unambiguous: if B'' is another basic subgroup of $\overline{B}$ containing B', and if $\beta' \in \operatorname{Aut} \overline{B}$ with $\beta' B'' = B$ and $\alpha' \in \operatorname{Aut} A$ with $\alpha' \restriction B = \beta' \restriction B$, then $\beta'\beta^{-1}B = \beta' B' \leq \beta' B'' = B$ and $\alpha'\alpha^{-1}$ agrees with $\beta'\beta^{-1}$ on $\beta B = \alpha B$, and hence on B. We conclude that $\alpha'\alpha^{-1} = \beta'\beta^{-1}$, showing that $\alpha^{-1}\beta = \alpha'^{-1}\beta'$, so that ϕ is well defined. If $x \in \operatorname{Ker} \phi$, then $\alpha^{-1}\beta x = 0$, $\beta x = 0$, and $x = 0$, that is, ϕ is monic.

Therefore, $\phi\overline{B}$ is a subgroup of A such that $\phi B = B$. This means that $\phi\overline{B}$ is pure in A, and so Theorem 3.6(iv) implies that it is a direct summand of A. But A is reduced and $A/\phi\overline{B}$ is divisible, thus necessarily $\phi\overline{B} = A$. This proves that ϕ is an isomorphism, and A is torsion-complete. □

★ **Notes.** Only a few classes of separable p-groups are well explored. In the countable case, by Prüfer's theorem, they are just Σ-cyclic groups, so the focus should be on the uncountable case. No general structure theorem is available, and none is expected, but much is known about a few classes with special properties (torsion-complete, quasi-complete, summable, etc. p-groups).

The theory of torsion-complete p-groups was developed by Kulikov [2], proving many of the essential results. He also proved that any two direct decompositions of a torsion-complete group have isomorphic refinements. Observe that the cardinality of any torsion-complete p-group A is 'essentially' of the form $\mu^{\aleph_0}$ for some μ. What we mean is that if $A = G \oplus H$ with bounded H, and $\lambda = |G|$ is minimal, then $\lambda = \mu^{\aleph_0}$.

There is a large body of work on torsion-complete groups, and numerous characterizations exist for torsion-completeness. Waller [1] considers generalized torsion-completeness to p-groups of countable limit length λ. Torsion-completion in the p^{λ}-topology retains some of the pleasant features of torsion-complete groups.

In retrospect, it is no surprise that torsion-completeness attracted so much attention, even before the theory of algebraically compact groups was developed. The similarity of the two theories is understandable, but, as far as importance is concerned, algebraic compactness is overwhelming.

It is instructive to study generalizations of separability involving higher cardinals. A p-group is called κ**-separable**, if every subgroup of cardinality $< \kappa$ embeds in a Σ-cyclic summand of size $< \kappa$. Megibben [8] investigates $\aleph_1$-separable p-groups, and shows that their summands inherit this property; he also proves several results that require additional set-theoretical hypotheses.

Exercises

(1) The torsion part of a direct product of p-groups is torsion-complete if and only if each summand is torsion-complete.

(2) Using the notation of the text, prove that $(\prod_n B_n)/\overline{B}$ is a torsion-free algebraically compact group which is not divisible unless 0.

(3) Large subgroups in torsion-complete groups are likewise torsion-complete.

(4) Using the notation in the text, an element $g = (b_1, \dots, b_n, \dots) \in \overline{B}$ of order p^n generates a cyclic summand of $\overline{B}$ if and only if $o(b_n) = p^n$.

(5) (a) Let A be a separable p-group, and B an upper basic subgroup of A. If $B \neq A$, then $|A/B| \geq \aleph_1$, and there is a decomposition $A = A' \oplus A''$ such that A' is Σ-cyclic and $|A''| = |A/B|$. [Hint: argue as in Theorem 3.5.]

(b) Every separable p-group A can be written as $A = A' \oplus A''$ such that A' is Σ-cyclic, and every basic subgroup of A'' is both upper and lower.

(6) In a torsion-complete p-group, all basic subgroups are lower as well as upper. [Hint: Theorem 3.11.]

(7) (Kemoklidze) If B is a basic subgroup of the p-group A, and if every endomorphism of B extends to A, then either $A = B$ or $A = \overline{B}$.

(8) Let $\overline{B}$ be a torsion-complete p-group with fin rk $B = \kappa$. There is a direct decomposition $\overline{B} = G \oplus H$, where H is bounded and G is torsion-complete of cardinality $\kappa^{\aleph_0}$.

(9) Kernels (but not all images) of endomorphisms of torsion-complete groups are again torsion-complete.

(10) (Faltings) A p-group A is isomorphic to the torsion part of $\mathrm{Hom}(A, \mathbb{Z}(p^\infty))$ if and only if it is torsion-complete with finite UK-invariants.

(11) Let A be a separable p-group with basic subgroup B.

(a) If $\overline{B}/A \cong (\mathbb{Z}(p^\infty))^{(\kappa)}$, then $\mathrm{Pext}(\mathbb{Z}(p^\infty), A) \cong \widetilde{(J_p)^{(\kappa)}}$.

(b) The Ulm factors of $\mathrm{Ext}(\mathbb{Z}(p^\infty), A)$ are: $\tilde{B}$ and $\widetilde{(J_p)^{(\kappa)}}$.

(12) (Richman) Let A be a separable p-group, and G a bounded subgroup. A is torsion-complete if and only if so is A/G. [Hint: Corollary 3.8.]

(13) Every height-preserving automorphism of the socle of $\overline{B}$ can be extended to an automorphism of $\overline{B}$.

(14) (Megibben) Let A be an unbounded torsion-complete p-group, and C an arbitrary separable p-group. There exists a homomorphism of A into C which is not small exactly if C contains an unbounded torsion-complete subgroup. [Hint: exhibit an unbounded torsion-complete group in the image of a non-small homomorphism.]

(15) Let T be a pure dense subgroup in the torsion-complete p-group $\overline{B}$ containing the basic subgroup B such that $\mathrm{rk}(\overline{B}/T) = n \in \mathbb{N}$. Then

(a) $\mathrm{Pext}(\mathbb{Z}(p^\infty), T) \cong \oplus_n J_p$; and

(b) if S is a separable p-group containing T as a pure dense subgroup with $\mathrm{rk}(S/T) = n$, then $S \cong \overline{B}$.

4 More on Torsion-Complete Groups

With regard to the question of direct decompositions of torsion-complete groups, we have definitive results.

Finite Direct Decompositions of Torsion-Complete Groups An unbounded torsion-complete p-group has numerous direct decompositions into a finite number of unbounded summands.

Proposition 4.1. *Let $\overline{B}$ be a torsion-complete p-group with basic subgroup B. For every direct decomposition $B = C_1 \oplus \cdots \oplus C_k$ with a finite number of summands, there is a direct decomposition*

$$\overline{B} = C_1^- \oplus \cdots \oplus C_k^-$$

where bars denote p-adic closures. Also, $C_i^- \cong \overline{C}_i$ for $i = 1, \ldots, k$.

Proof. We represent the elements of $\overline{B}$ as in (10.4) of Sect. 3. Every $a \in \overline{B}$ can be written uniquely as $a = a_1 + \cdots + a_k$ where $a_i = (c_{i1}, \ldots, c_{in}, \ldots)$ and $c_{in} \in C_i$ for all n. Evidently, $a_i \in C_i^-$ for $i = 1, \ldots, k$. The stated equality follows from this representation. □

Infinite Direct Decompositions of Torsion-Complete Groups The behavior of torsion-complete groups in infinite direct decompositions is quite unusual. We begin with a preliminary lemma.

Lemma 4.2 (Enochs [1]). *If a torsion-complete p-group A is contained in a direct sum $C = \oplus_{i \in I} C_i$ of separable p-groups, then there exist an integer m and a finite number of summands such that*

$$p^m A[p] \leq C_{i_1} \oplus \cdots \oplus C_{i_k}.$$

Proof. The proof is similar to the one in Lemma 2.16 in Chapter 6, with the understanding that m_j should mean p^j, and the a_j are selected in the socles. □

Further specialization $A = C$ yields the following result that provides a pretty complete information about the kind of direct decompositions torsion-complete p-groups might have.

Theorem 4.3 (Kulikov [2]). *If a torsion-complete p-group A is a direct sum $A = \oplus_{i \in I} C_i$, then the C_i are torsion-complete, and for a sufficiently large integer m, $p^m C_i = 0$ for almost all i.*

Proof. The torsion-completeness of the summands is evident, while the existence of m as stated follows from the preceding lemma. □

Exchange Property for Torsion-Completeness The p-groups that are torsion-complete belong to the exclusive club of groups that enjoy the exchange property. This is proved like Theorem 6.7 in Chapter 6, the reference to Theorem 4.2 in Chapter 16 includes torsion-complete groups as well.

Theorem 4.4 (Crawley–Jónsson [1]). *Torsion-complete p-groups have the exchange property.* □

Large Subgroup Topology The balance of this section is devoted to the topological aspects of torsion-completeness, in particular, to the **large subgroup topology**.

Let A be a separable p-group, and $\{U_j\}_{j\in J}$ the family of its large subgroups which we now declare as a subbase of open neighborhoods of 0. This is a linear topology which we will denote by $\mathbf{w}$. Thus the open subgroups in $\mathbf{w}$ are those that contain a large subgroup, i.e. which satisfy the Pierce condition.

(a) *Separable p-groups are Hausdorff in the large subgroup topology* $\mathbf{w}$. This is obvious, since already the intersection of the large subgroups p^nA $(n \in \mathbb{N})$ is the first Ulm subgroup A^1.
(b) *The topology* $\mathbf{w}$ *is finer than the p-adic topology.* This follows from the fact that p^nA is a large subgroup for each $n < \omega$.
(c) *For a pure subgroup C of a separable p-group A, the large subgroup topology is the same as the topology inherited from the large subgroup topology of A.* In fact, purity guarantees that $C(\underline{u}) = A(\underline{u}) \cap C$.
(d) *Every homomorphism* $\phi : A \to C$ *between separable p-groups is continuous in the large subgroup topology.* Let $c \in C$ be the image of $a \in A$ under ϕ, and $c+V$ an open neighborhood of c. We may assume that V is a large subgroup of C, say, defined by the sequence $\{r_n\}_{n\in\mathbb{N}}$ (cp. Theorem 2.3). If U is the large subgroup of A defined by the very same sequence of integers, then clearly $\phi(a+U) \subseteq c+V$.
(e) *In a separable p-group, a bounded sequence is Cauchy in the p-adic topology if and only if it is Cauchy in the large subgroup topology.* The 'if' part being obvious, let $\{a_i\}_{i<\omega}$ be a p^k-bounded Cauchy sequence in the p-adic topology, and $U = A(r_0, \dots, r_n, \dots)$ a neighborhood of 0 in the large subgroup topology. There is an index m such that $a_i - a_j \in p^{r_k-k}A$ for all $i, j \geq m$. Then the indicator of $a_i - a_j$ is $\geq (r_k - k, r_k - k + 1, \dots, r_k, \infty \dots)$. As $r_k - k + \ell \geq r_\ell$ $(\ell \leq k)$, we have $a_i - a_j \in U$ for $i, j \geq m$.
(f) *A Cauchy net in the large subgroup topology is bounded.* By way of contradiction, assume that there is an unbounded Cauchy net in $\mathbf{w}$. This is also Cauchy in the p-adic topology, so it contains an unbounded cofinal subsequence $a_0, \dots, a_i, \dots$. We may assume that the orders $o(a_i) = p^{n_i}$ increase with i, and so do the heights h_i of the differences $c_i = a_i - a_{i-1}$. Define $\underline{u} = (r_0, r_1, \dots, r_n, \dots)$ such that the r_i form an increasing sequence of positive integers satisfying the inequalities: $r_{n_i-1} > h_i + n_i$ for each i. Then the indicator of c_i is $(h_i, h_i + 1, \dots, h_i + n_i - 1, \infty, \dots)$, thus $c_i \notin A(\underline{u})$ for all i, contradicting the Cauchy property of the subsequence.

Charles [3] defines the **inductive topology** on a p-group A by declaring those subgroups G of A to form a subbase of neighborhoods about 0 which satisfy: for each $k \in \mathbb{N}$,

$$G[p^k] = p^nA \cap A[p^k] \qquad \text{for some } n \in \mathbb{N}.$$

(g) *The large subgroup topology on p-groups coincides with the inductive topology.* One way it is obvious: G satisfies the Pierce condition, and so it contains a large subgroup of A. For the converse, observe that a group G satisfying the above conditions is fully invariant in A: it is the union of the fully invariant subgroups $p^n A[p^k]$. In addition it satisfies the Pierce condition, so it is a large subgroup.

The principal result on the relation of torsion-completeness and the large subgroup topology is as follows.

Theorem 4.5 (Kulikov [2], Charles [3], Cutler–Stringall [1]). *For a separable p-group A, the following are equivalent:*

(i) *A is torsion-complete;*
(ii) *every bounded Cauchy sequence in the p-adic topology of A is convergent;*
(iii) *A is complete in the large subgroup topology.*

Proof. To fix the notation, as usual let $B = \oplus_{n\in\mathbb{N}} B_n$ with $B_n = \oplus \mathbb{Z}(p^n)$ be a basic subgroup of A. Then $a \in A$ can be identified with a vector $(b_1, \dots, b_n, \dots) \in \overline{B}$ $(b_n \in B_n)$, where $p^m b_n = 0$ for every n if $o(a) = p^m$.

(i) ⇒ (ii) Assuming $A = \overline{B}$, let $a_k = (b_{k1}, \dots, b_{kn}, \dots) \in \overline{B}$ $(k = 1, 2, \dots)$ be a bounded Cauchy sequence in the p-adic topology where $b_{kn} \in B_n$. For convenience, suppose the sequence is neat, so we have

$$a_{k+\ell} - a_k = (b_{k+\ell,1} - b_{k1}, \dots, b_{k+\ell,n} - b_{kn}, \dots) \in p^k\overline{B}$$

for all $k, \ell \in \mathbb{N}$. This implies that $b_{k+\ell,1} = b_{k1}, \dots, b_{k+\ell,k} = b_{kk}$, showing that the first k coordinates of $a_k, a_{k+1}, \dots$ are the same, and in addition, $b_{k+\ell,n} - b_{k,n} \in p^k B_n$ for each $n \in \mathbb{N}$. Define $b = (b_{11}, b_{22}, \dots, b_{nn}, \dots)$ which belongs to $\overline{B}$ in view of the boundedness of the sequence $\{a_k\}_{k<\omega}$. This b is the limit of the a_k, because obviously

$$b - a_k = (0, \dots, 0, b_{k+1,k+1} - b_{k,k+1}, b_{k+2,k+2} - b_{k,k+2}, \dots) \in p^k\overline{B}.$$

(ii) ⇒ (iii) Just observe that (e) implies that a Cauchy net in the large subgroup topology is a bounded Cauchy net in the p-adic topology.

(iii) ⇒ (i) Let $b = (b_1, \dots, b_n, \dots)$ $(b_n \in B_n)$ be an element, say, of order p^m, in the torsion-complete group $\overline{B} \geq A$. Consider the sequence $a_n = b_1 + \cdots + b_n \in A$ for $n \in \mathbb{N}$. This is a Cauchy sequence in the large subgroup topology, since it is bounded and $a_{n+k} - a_n = b_{n+1} + \cdots + b_{n+k} \in p^{n-m}A$ for all $k \geq 1$. It ought to have a limit in A, which cannot be anything else than b. Thus $\overline{B} \leq A$, and (i) follows. □

The preceding theorem says in effect that torsion-complete groups are exact analogues of complete groups: the large subgroup topology takes over the role of the p-adic topology.

We wind up this section with the following interesting corollary.

Corollary 4.6. *The completion of a separable p-group in its large subgroup topology is its torsion-completion.*

Proof. Let A be a pure subgroup of the torsion-complete p-group $\overline{B}$, and G the completion of A in its large subgroup topology. The identity map of A extends to a continuous map $\delta : G \to \overline{B}$ which has to be surjective, because A is dense in $\overline{B}$. There is a similar map $\gamma : \overline{B} \to G$, and the composite $\delta\gamma : \overline{B} \to \overline{B}$ must be the identity. □

★ **Notes.** Leptin [1] gave additional characterizations for torsion-completeness. In addition to Theorem 3.10, he proved that if a basic subgroup B of a separable p-group A has the property that every automorphism of B extends to an automorphism of A, then either $A = B$ or $A = \overline{B}$.

Crawley [1] solves one of the Kaplansky test problems for p-groups in the negative, by pointing out that it is possible to have p-groups A, C satisfying $A \cong C \oplus C \oplus A$, but $A \ncong C \oplus A$.

In the proceedings of the 1967 Montpellier conference, both Charles [3] and Cutler–Stringall [1] pointed out the completeness of torsion-complete groups, using different approaches: the former used the inductive topology arising from the topologies of the subgroups $A[p^k]$, induced by the p-adic topology, while the latter used the large subgroup topology.

Exercises

(1) Every subgroup C of a torsion-complete p-group A can be embedded in a summand G of A such that $|G| \leq |C|^{\aleph_0}$. [Hint: embed in pure subgroup.]
(2) A torsion-complete p-group that is contained in a Σ-cyclic p-group must be bounded.
(3) A torsion-complete p-group A fails to have a proper pure subgroup $\cong A$ if and only if its UK-invariants are finite.
(4) If C is a torsion-complete subgroup of a separable p-group A such that A/C is bounded, then A is also torsion-complete. [Hint: apply $\mathrm{Ext}(\mathbb{Z}(p^\infty), ^*)$.]
(5) (Koyama–Irwin) A separable p-group A is torsion-complete if and only if every decomposition $B = C_1 \oplus C_2$ of its basic subgroup B induces a direct decomposition $A = C_1^- \oplus C_2^-$.
(6) (Irwin–O'Neill) Suppose $A = B \oplus C$ is a separable p-group that contains an unbounded torsion-complete subgroup. Then either B or C has an unbounded torsion-complete subgroup.

5 Pure-Complete and Quasi-Complete p-Groups

It does not seem unreasonable to anticipate that torsion-complete p-groups have many more properties of interest. Some of these are shared by wider classes of groups as well. In this section, we discuss briefly two such special properties. Both are related to the existence of certain pure subgroups, motivated by the observations that in torsion-complete p-groups every subsocle supports a pure subgroup, and closures of pure subgroups in the p-adic topology preserve purity.

Pure-Completeness A p-group A is called **pure-complete** if every subsocle of A supports a pure subgroup of A. Because of Theorem 1.3, this property is shared by all dense subsocles.

Since in a pure-complete p-group, also the subsocle consisting of all the elements of infinite heights must support a pure subgroup (which cannot be anything else than a divisible subgroup), we conclude that every such group decomposes into the direct sum of a separable and a divisible group.

Example 5.1. Let B be the direct sum of cyclic groups of fixed order p^k. Then every subsocle of B supports a direct summand of B, so B is pure-complete. More generally, a torsion group with bounded p-components is pure-complete.

Example 5.2. Divisible p-groups are pure-complete.

It is rather obvious that summands of pure-complete groups are again pure-complete. However, the direct sum of two pure-complete p-groups need not be pure-complete as is shown by the following example.

Example 5.3 (Hill–Megibben [2]). Suppose A and C are non-isomorphic p-groups, and there is a height-preserving isomorphism $\phi\colon A[p] \to C[p]$ between their socles (such p-groups do exist, see e.g. Theorem 9.2). Assuming A is quasi-complete (see below), it follows from the characterization to be stated in the Notes below that so is C, and hence by Corollary 5.6 *infra* they are pure-complete. However, the subsocle $S = \{(x, \phi x) \mid x \in A[p]\}$ of the direct sum $A \oplus C$ does not support any pure subgroup. In fact, a pure subgroup supported by S ought to be a subdirect sum of A and C with 0 kernels. But such a subdirect sum could exist only if A and C were isomorphic.

Lemma 5.4 (Hill–Megibben [3]).

(i) *Σ-cyclic p-groups are pure-complete.*
(ii) *The direct sum of a countable number of torsion-complete p-groups is pure-complete.*

Proof. (i) This will be a simple consequence of (ii), since we can write a Σ-cyclic p-group as $A = \oplus_{n\in\mathbb{N}} B_n$ where $B_n = \oplus\, \mathbb{Z}(p^n)$ is torsion-complete.

(ii) First we show that a torsion-complete p-group A is pure-complete. Actually, we prove somewhat more: if S is a subsocle of A, and H is a pure subgroup of A with $H[p] \leq S$, then A contains a pure subgroup G such that $H \leq G$ and $G[p] = S$. Let B' be a basic subgroup of H, and B'' a maximal Σ-cyclic pure subgroup of A that contains B' and satisfies $B''[p] \leq S$. Finally, let B be a basic subgroup of A that contains B'' as a summand, say, $B = B'' \oplus C$. We then have $A = \overline{B}'' \oplus \overline{C}$. Since $B''[p]$ must be dense in S, we have $S \leq \overline{B}''[p]$. An appeal to Theorem 1.3 establishes the existence of a G with the desired properties.

Now suppose $A = \oplus_{n=1}^{\infty} A_n$, where the A_n are torsion-complete p-groups. Let S denote a subsocle of A. Then $S_n = S \cap (A_1 \oplus \cdots \oplus A_n)$ is a subsocle of the torsion-complete p-group $A_1 \oplus\cdots\oplus A_n$, so it supports a pure subgroup G_n. What we have proved in the preceding paragraph shows that the pure subgroups G_n can be selected so as to form a chain $G_1 \leq \cdots \leq G_n \leq \dots$. It is then clear that $G = \cup_n G_n$ is pure in A and is supported by S. ☐

Quasi-Complete Groups Let A be a reduced p-group. It is called **quasi-complete** (Head[1]) if the closure G^- (in the p-adic topology) of every pure subgroup G is again pure in A. Since G^-/G can be identified as the first Ulm subgroup of A/G, the quasi-completeness of A means that, for any pure subgroup G of A, the first Ulm subgroup $(A/G)^1$ of A/G is pure in A/G, i.e. divisible.

We list a few trivialities for quasi-complete groups.

(A) *Reduced quasi-complete p-groups are separable.* This follows from $0^- = A^1$.
(B) *Torsion-complete groups are quasi-complete.* This is a simple consequence of Corollary 3.9.
(C) *Closed pure subgroups in a quasi-complete group are quasi-complete.*
(D) *If G is a closed pure subgroup of a quasi-complete p-group A, then A/G is likewise quasi-complete.* For, if H/G is a pure subgroup of A/G, then H is pure in A, so $[(A/G)/(H/G)]^1 \cong (A/H)^1$ must be divisible.

Next we show that the property that we verified for torsion-complete groups in the proof of Lemma 5.4 actually characterizes quasi-completeness.

Proposition 5.5 (Irwin–Richman–Walker [1], Koyama [1]). *A reduced p-group A is quasi-complete if and only if, for every pure subgroup H of A, and for every subsocle S with $H[p] \leq S$, there exists a pure subgroup G of A such that $H \leq G$ and $G[p] = S$.*

Proof. Let A be quasi-complete, H a pure subgroup of A such $H[p] \leq S \leq A[p]$, and G a pure subgroup maximal with respect to the properties $H \leq G$ and $G[p] \leq S$. To prove $G[p] = S$, by way of contradiction assume that there is an $x \in S \setminus G$. If the coset $x + G$ has finite height k in A/G, then write $x + G = p^k y + G$ with $y \in A$. Now $\langle y + G \rangle$ is a summand of A/G, and $G \oplus \langle y \rangle$ is a pure subgroup supported by $G[p] \oplus \langle x \rangle \leq S$, a contradiction. If $x + G$ has infinite height in A/G, then by quasi-completeness it is contained in a subgroup $G'/G \cong \mathbb{Z}(p^\infty)$. Clearly, G' is pure in A and is supported by $G[p] \oplus \langle x \rangle \leq S$, again a contradiction.

Conversely, let A have the stated property. First choosing $S = A^1[p]$, we obtain A^1 divisible, and hence 0. Next, for any pure subgroup H, choose G pure containing H with socle $S = H^-[p]$. Then G/H is pure in A/H with socle $(A/H)^1[p]$. Consequently, G/H is divisible and equal to H^-/H, so A is quasi-complete. □

Hence it follows at once:

Corollary 5.6. *Quasi-complete p-groups are pure-complete.* □

We isolate a preparatory result in the next lemma that will be needed in later proofs.

Lemma 5.7 (Hill–Megibben [3]). *Suppose A is a quasi-complete p-group, and B' is an unbounded summand of a basic subgroup B of A. Then $\overline{B} = A + \overline{B'}$.*

Proof. Let $B = B' \oplus B''$, and $\pi : \overline{B} \to \overline{B''}$ the projection map. Then $\pi(A)$ is pure in $\overline{B''}$, since $B'' \leq \pi(A)$ and $\mathrm{Coker}(\pi \restriction A)$ is divisible. We want to prove that $\mathrm{Coker}(\pi \restriction A) = 0$ by showing that every $b \in \overline{B''}[p]$ is in $\pi(A)$. Let C denote a basic subgroup of B' with $B'/C \cong \mathbb{Z}(p^\infty)$, and $u \in B' \setminus C$ an element of order p^2. One can easily construct a pure Σ-cyclic subgroup $X < A$ with $X[p] = C[p]$ and $b + u \in X^-$. By quasi-completeness, the closure $A \cap X^-$ of X in A is pure in A, so $pu \in A \cap X^-$ implies that $pa = pu$ for some $a \in A \cap X^-$. Hence $b = (b + u - a) - (u - a) \in X^-[p] + A[p] \leq \overline{B'} + A$, completing the proof. □

Another way of looking at quasi-completeness relies on the property stated in Corollary 3.9 for torsion-complete groups.

Theorem 5.8 (Hill–Megibben [2]). *A separable p-group A is quasi-complete if and only if A/G is a direct sum of a divisible and a torsion-complete group, whenever G is any unbounded pure subgroup of A.*

Proof. For sufficiency, note that the stated condition implies that the closure of G will also be pure in A. For the proof of necessity, assume A is quasi-complete and contained in $\overline{B}$, where B is a basic subgroup of A. The p-adic closure H (in $\overline{B}$) of an unbounded pure subgroup G is by Corollary 3.9 a summand of $\overline{B}$, hence torsion-complete. Since A contains a basic subgroup of G, and hence of H, from Lemma 5.7 we conclude that $A + H = \overline{B}$. In view of quasi-completeness, $(A \cap H)/G \cong (A/G)^1$ is divisible. Therefore, A/G is isomorphic to a direct sum of a divisible group and $A/(A \cap H) \cong (A + H)/H = \overline{B}/H$, a summand of $\overline{B}$. □

It follows from Lemma 5.7 that if $B = B' \oplus B''$ is a decomposition of a basic subgroup of a quasi-complete p-group A, then

$$(A \cap \overline{B'}) \oplus (A \cap \overline{B''}) \leq A \leq \overline{B'} \oplus \overline{B''}$$

and A is a subdirect sum of $\overline{B'}$ and $\overline{B''}$ with kernels $A \cap \overline{B'}$ and $A \cap \overline{B''}$. This remark is now used to prove that quasi-complete p-groups that are not torsion-complete are few and far between.

Theorem 5.9 (Hill–Megibben [2]). *A quasi-complete p-group of final rank $> 2^{\aleph_0}$ is torsion-complete.*

Proof. Let A be quasi-complete of final rank $> 2^{\aleph_0}$, and $B = B' \oplus B''$ a decomposition of its basic subgroup such that B' is countable and unbounded. In view of the mentioned subdirect sum representation, we have $\overline{B''}/(A \cap \overline{B''}) \cong \overline{B'}/(A \cap \overline{B'})$ where the last quotient has cardinality $\leq |\overline{B'}| \leq 2^{\aleph_0}$. Let S denote a complete set of representatives of $\overline{B''}$ modulo $A \cap \overline{B''}$. Clearly, S must be contained in the completion of a summand C of B'' which has cardinality not exceeding $2^{\aleph_0}$. Thus $\overline{B''} = \overline{C} \oplus \overline{C'}$ for some $C' < B''$ of final rank $> 2^{\aleph_0}$ where $\overline{C'} \leq A$. By Lemma 5.7, $\overline{B} = A + \overline{C'} \leq A$, and hence A is torsion-complete. □

Thus, if we are asking for an example of a quasi-complete p-group that is not torsion-complete, then we should not look beyond the continuum. Such an example (of cardinality exactly $2^{\aleph_0}$) was supplied by Hill–Megibben [2].

★ **Notes.** Lady [1] proves that the torsion subgroup of a direct product of Σ-cyclic p-groups is pure-complete.

It was Head [1] who first studied quasi-completeness. Let us mention the following useful characterization of quasi-completeness. Let A be a separable p-group embedded in the torsion-completion $\overline{B}$ of its basic subgroup B. Hill–Megibben [2] prove that A is quasi-complete if and only if $\overline{B}[p] = A[p] + S^-$ holds for every subsocle S of A where the heights of elements of S are unbounded (S^- is calculated in $\overline{B}$)—actually, this follows from the proof of Lemma 5.7. Another interesting result is concerned with unbounded torsion-complete p-groups T of cardinality $2^{\aleph_0}$. Given any countable subgroup A of T, there exists a proper quasi-complete pure subgroup $G < T$ such that $G \cap A = 0$ and T/G is divisible.

Exercises

(1) Define a pure-complete torsion group in the same way as it was done for p-groups, and show that a torsion group is pure-complete if and only if all of its p-components are pure-complete.
(2) (Hill–Megibben) If A is pure-complete, and C is Σ-cyclic, then $A \oplus C$ is also pure-complete.
(3) (Hill–Megibben) If A is a pure-complete, and C is a torsion-complete p-group with finite UK-invariants, then $A \oplus C$ is again pure-complete.
(4) The direct sum of two quasi-complete p-groups need not be quasi-complete. [Hint: quasi-complete plus large torsion-complete.]
(5) (Hill–Megibben) Show that in any direct decomposition of a quasi-complete p-group that is not torsion-complete, only one of the summands can be unbounded (i.e., the group is essentially indecomposable). [Hint: Theorem 5.8.]
(6) (Benabdallah–Irwin) Call a subgroup G of a p-group A **almost dense** if, for every pure subgroup C of A containing G, the factor group A/C is divisible. Show that G is almost dense in A exactly if $p^nA[p] \leq G + p^{n+1}A$ holds for all $n < \omega$.

6 Thin and Thick Groups

Small homomorphisms have been discussed in Sect. 3 in Chapter 7. Several features make them especially interesting for study. Here we discuss two different types of p-groups defined in terms of small homomorphisms. Their relation to torsion-completeness will be evident.

Thin Groups A p-group C is said to be **thin** (Richman [1]) if every homomorphism $\phi: \overline{B} \to C$ from the standard torsion-complete p-group $\overline{B}$ into C is small (recall: $B \cong \oplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)$). (In the category of p-groups, thin groups play similar role as slender groups in the category of torsion-free groups; see Sect. 2 in Chapter 13.) It is easily checked:

(a) *Subgroups of thin groups are thin.*
(b) *Extension of a thin group by a thin group is again thin.*

(c) *Homomorphisms of any unbounded torsion-complete p-group into a thin group are small.* This will follow immediately from the more general Lemma 6.4 that is needed in the proof of the subsequent theorem.

Lemma 6.1 (Richman [1]). *Direct sums of thin groups are thin.*

Proof. The proof follows the arguments in Lemma 2.16 in Chapter 6 and in Lemma 4.2 with obvious modifications. □

Example 6.2 (Megibben [5]). Σ-cyclic groups are thin. This is an immediate consequence of Lemma 6.1, observing that cyclic p-groups are trivially thin.

Example 6.3. The group $\mathbb{Z}(p^\infty)$ is not thin. The standard torsion-complete p-group $\overline{B}$ evidently has a map ϕ onto $\mathbb{Z}(p^\infty)$ such that $B \leq \operatorname{Ker}\phi$. Such a $\operatorname{Ker}\phi$ cannot contain a large subgroup of $\overline{B}$, since $B + \operatorname{Ker}\phi \neq \overline{B}$.

Lemma 6.4 (Keef [6]). *Let A_i $(i < \omega)$ be p-groups. If*

$$\psi : t(\prod_{i<\omega} A_i) \to C$$

is a homomorphism into a thin group C, then for each $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that

$$\psi(\prod_{i>n} p^n A_i[p^k]) = 0.$$

Proof. If this were not true, then for some k we could find elements $a_n \in \prod_{i>n} p^n A_i[p^k]$ for all $n < \omega$ such that $\psi(a_n) \neq 0$. Let $\langle b_n \rangle$ be a cyclic group of order p^{k+n}, and $\eta : t(\prod_{n<\omega} \langle b_n \rangle) \to t(\prod_{i<\omega} A_i)$ a homomorphism mapping $p^n b_n$ upon a_n. Then the composite map $\psi\eta : t(\prod_{n<\omega} \langle b_n \rangle) \to C$ would not be small (its kernel fails the Pierce condition), contrary to our hypothesis on C. □

The condition of non-measurability appears in the next result: another evidence of its intimate relation to direct products.

Theorem 6.5 (Keef [6]). *Let $\rho_i : A_i \to A = t(\prod_{i\in I} A_i)$ denote the ith injection map, where the A_i are p-groups, and I is a non-measurable index set. A p-group C is thin if and only if it satisfies:*

(*) *a homomorphism $\phi : A \to C$ is small exactly if $\phi\rho_i$ is small for every $i \in I$.*

Proof. First, suppose C satisfies (*), and $\phi : \overline{B} \to C$ is a homomorphism, where $\overline{B}$ is the torsion-completion of the standard basic subgroup B. Since the induced maps $B_n \to C$ are small homomorphisms, hypothesis implies that so is ϕ. Consequently, C is thin.

Conversely, assume that C is thin, and let $\phi : A \to C$ be such that $\phi\rho_i$ is small for every $i \in I$, but ϕ is not small. Then in view of the definition of small homomorphisms, there is a $k \in \mathbb{N}$ such that $\phi(\prod_{i\in I} p^n A_i[p^k]) \neq 0$ for all $n \in \mathbb{N}$. Consider the set $\mathcal{S}$ of all subsets $J \subseteq I$ such that $\phi(\prod_{i\in J} p^n A_i[p^k]) \neq 0$ for all $n \in \mathbb{N}$. As the $\phi\rho_i$ are small, none of $J \in \mathcal{S}$ is finite. From Lemma 6.4 it follows that if

$J_1, \dots, J_j, \dots$ are pairwise disjoint subsets in $\mathcal{S}$, then $\cup_{j>m} J_j \notin \mathcal{S}$ must hold for some $m \in \mathbb{N}$. The proof of Lemma 6.4 in Chapter 2 applied to ϕ (in place of f) convinces us that I is measurable, a contradiction. □

Example 6.6. To show that the last theorem may fail for measurable cardinals, let I be a measurable index set, and A_i copies of the same countable, unbounded Σ-cyclic p-group X, and also $C = X$. In order to define $\phi : A = t(\prod_{i\in I} A_i) \to X$, choose a vector $a = (\dots, a_i, \dots) \in A$, and regard the coordinates as elements of X. There are only countably many different coordinates, so the supports of the equal ones give rise to a countable partition of I into disjoint subsets, exactly one of which has measure 1, and the rest has measure 0. If the support of $x \in X$ is the one of measure 1, then set $\phi(a) = x$. This is a well-defined homomorphism; it is not small, though $\phi\rho_i$ is small for each $i \in I$.

Thick Groups A p-group G is called **thick** (Megibben [2]) if every homomorphism $\phi : G \to C$ into a Σ-cyclic group C is small.

Example 6.7. Torsion-complete p-groups are thick. This is a consequence of Example 6.2.

Example 6.8 (Megibben [2]). A pure dense subgroup G of a torsion-complete p-group $\overline{B}$ is thick whenever $|\overline{B}/G| \leq \aleph_0$. Indeed, let $\phi : G \to C$ be a homomorphism, where C is Σ-cyclic. Then $A = \operatorname{Im}\phi$ is likewise Σ-cyclic. Now, ϕ extends uniquely to $\overline{\phi} : \overline{B} \to \overline{A}$, where $\overline{A}$ denotes the torsion-completion of A. As $\operatorname{Im}\overline{\phi}/A$ is countable, by Theorem 3.5 $\operatorname{Im}\overline{\phi}$ is also Σ-cyclic. Thus $\overline{\phi}$ is small, and so is $\phi = \overline{\phi} \restriction G$.

In the following theorem, which is a dual to Theorem 6.5, π_i denotes the ith coordinate projection.

Theorem 6.9 (Rychkov–Thomé [1], Keef [6]). *A p-group G is thick if and only if the following holds:*

(**) *for every set $\{A_i \mid i \in I\}$ of separable p-groups, a homomorphism $\phi : G \to \oplus_{i\in I} A_i$ is small exactly if $\pi_i\phi$ is small for every $i \in I$.*

Proof. If G has property (**) and the A_i are cyclic p-groups, then ϕ small implies G is thick. For the converse, assume $\phi : G \to \oplus_{i\in I} A_i$ is a non-small homomorphism, though all of $\pi_i\phi$ are small. Thus for each $k < \omega$ there is $n \in \mathbb{N}$ such that $\phi(p^n G[p^k]) \neq 0$. Because the homomorphisms $\pi_i\phi$ are small, we can find inductively integers n_j, elements $x_j \in p^{n_j} G[p^k]$, and different indices $i_j \in I$ such that $\pi_{i_j}\phi(x_j) \neq 0$. As A_{i_j} is separable, there is a map $\gamma_j : A_{i_j} \to B_i$, to a cyclic p-group, such that $\gamma_i(\pi_{i_j}\phi(x_j)) \neq 0$. The γ_j's yield the composite map $\gamma : \oplus_{i\in I} A_i \to \oplus_{j<\omega} A_{i_j} \to \oplus_{j<\omega} B_i$ such that $\gamma\phi(x_j) \neq 0$, showing that $\gamma\phi$ is not small, so G is not thick. □

Theorem 6.10 (Keef [6]). *If the p-groups A_i $(i \in I)$ are thick, then so is the torsion subgroup of their direct product.*

Proof. If I is finite, then the assertion follows from the fact that the direct sum of large subgroups of the summands is large in the direct sum. Suppose I is infinite, and $\phi : t(\prod_{i\in I} A_i) \to C$ where C is Σ-cyclic. By the analogue of Theorem 6.7 in Chapter 2, there are a finite subset $J \subset I$ and an integer m such that $p^m t(\prod_{i\in I\setminus J} A_i) \neq 0$ is mapped by ϕ into a finite summand of C. If this summand is annihilated by p^k, then $\operatorname{Ker}\phi$ contains, along with a direct sum of large subgroups of A_i ($i \in J$), also $p^{m+k}t(\prod_{i\in I} A_i)$. Therefore, ϕ is small, and $t(\prod_{i\in I} A_i)$ is thick. $\square$

★ **Notes.** Megibben [2] shows that a separable p-group is thin exactly if it does not contain an unbounded torsion-complete p-group. Generalizations of thin and thick groups have been discussed by Rychkov–Thomé [1].

Keef [3] proves an interesting result on the torsion subgroup of a countable product of p-groups (generalizing Theorem 6.10 in Chapter 5): for any collection $\{A_n\}_{n\in\mathbb{N}}$ of p-groups, there exists an epimorphism

$$\phi\colon t\left(\prod_{n\in\mathbb{N}} A_n\right) \to \oplus_{n\in\mathbb{N}} A_n.$$

Exercises

(1) Every homomorphism $\overline{B} \to B$ is small (B is a Σ-cyclic p-group).
(2) (Keef)

  (a) Given a p-group A, let $S(A)$ denote the class of p-groups C such that all homomorphisms $\phi : A \to C$ are small. Prove that $S(A)$ is closed under taking subgroups.
  (b) If A is unbounded and reduced, then every group in $S(A)$ is thin.

(3) In the definition of thick p-groups, can the Σ-cyclic C be restricted to the standard $B = \oplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)$?
(4) A p-group is thick if and only if its initial Ulm factor is thick.
(5) (Keef) Let A, C be reduced unbounded p-groups. There exists a non-small homomorphism $A \to C$ if either A is not thick or C is not thin.

7 Direct Decompositions of Separable p-Groups

Every non-trivial separable p-group has an indecomposable summand: a cyclic group, but the group itself need not be a direct sum of indecomposable groups. The obvious question is: what kind of decompositions a separable p-group might have? We would like to use the knowledge that we have gained on torsion groups so far to draw attention to some not so common direct decompositions. We will mention a few results of this sort. We point out right away that the presence of cyclic summands rules out the existence of superdecomposable p-groups.

Direct Decompositions We know that a separable p-group has an ample supply of finite summands. These are of course Σ-cyclic groups. How about unbounded Σ-cyclic summands? We refer to Khabbaz' Theorem 6.14 in Chapter 5 showing that in case the basic subgroups have the same cardinality as the p-group A, then A has Σ-cyclic summands as large as its cardinality permits.

$\aleph_1$-Separability Almost free groups have analogues in the torsion case: p-groups in which every subgroup of smaller cardinality is Σ-cyclic, see Corollary 8.11 in Chapter 3. A stronger version requires that every subgroup of smaller cardinality can be embedded in a Σ-cyclic summand.

Theorem 7.1 (Hill [8], Crawley–Megibben [1]). *There exists a separable p-group of cardinality $\aleph_1$, not a Σ-cyclic group, such that every countable subgroup embeds in a Σ-cyclic direct summand.*

Proof. Let $B = \oplus_n B_n$, $B_n \cong \oplus_{\aleph_0} \mathbb{Z}(p^n)$, and C a basic subgroup of B such that $B/C \cong \mathbb{Z}(p^\infty)$. We construct a p-group A of cardinality $\aleph_1$ as the union of an ascending chain of pure subgroups, all isomorphic to B.

We start with $A_0 = 0$. Suppose that for some ordinal $\beta < \omega_1$, the countable Σ-cyclic groups A_α have been defined for all $\alpha < \beta$ to form a smooth chain, each being pure in its successors. If β is a limit ordinal, then we have to set $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$. As a countable separable p-group, A_β will then be Σ-cyclic (Theorem 5.5 in Chapter 3). If $\beta = \gamma + 1$ and γ is a successor ordinal, then define $A_\beta = A_\gamma \oplus F_\gamma$ where $F_\gamma \cong B$. Finally, if $\beta = \gamma + 1$ and γ is a limit ordinal, then choose $A_\beta \cong B$ with A_γ embedded in A_β as C is embedded in B. Then all the groups A_α are Σ-cyclic and pure in their union $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Moreover, each A_α with non-limit α is a direct summand of every A_β with $\beta > \alpha$, where the complements form a chain, so it is a summand of A as well. Since A has cardinality $\aleph_1$, and every countable subset of A is contained in some A_α, it is clear that A has the required embedding property. However, A cannot be Σ-cyclic, since it violates the criterion of Theorem 7.5 in Chapter 3 (see the Remark after it): $A_{\alpha+1}/A_\alpha \cong \mathbb{Z}(p^\infty)$ is not Σ-cyclic for every limit ordinal $\alpha < \omega_1$, and the set of limit ordinals $< \omega_1$ is stationary in ω_1. $\square$

Summands of Various Properties Features that we want to consider next are concerned with the size of direct decompositions: when one of the summands ought to be small or large in size.

A p-group A is said to be **essentially indecomposable** if it is incapable of decomposition into two *unbounded* summands; i.e. if $A = B \oplus C$, then either B or C is bounded. Bounded p-groups share this property, but no unbounded torsion-complete group has it (see Proposition 4.1). The next example exhibits an essentially indecomposable p-group of the cardinality of the continuum (for another example see Sect. 5, Exercise 5).

Example 7.2. Pierce [1] shows that the torsion-complete group $\overline{B}$ (where B denotes the standard basic subgroup) contains a pure subgroup $G > B$ of cardinality $2^{\aleph_0}$ such that $\operatorname{End} G \cong J_p \oplus \operatorname{End}_s G$. Idempotents in $\operatorname{End}_s G$ yield bounded summands, therefore only idempotents mod $\operatorname{End}_s G$ could produce two unbounded summands. But J_p has only a single idempotent $\neq 0$, so such a G must be essentially indecomposable.

From the existence Theorem 7.3 in Chapter 16 one can derive the existence of arbitrarily large essentially indecomposable p-groups, even a large family of such groups with the property that all homomorphisms between different members are small.

A class more general than essentially indecomposable groups was studied by Cutler–Irwin [1]. They call a p-group A **essentially finitely indecomposable**, or **e.f.i.** for short, if for every direct decomposition of A there is an integer $n > 0$ such that almost all summands are p^n-bounded. An equivalent condition is: A has no unbounded Σ-cyclic subgroup as a summand.

Example 7.3. Thick groups are e.f.i. To verify this, assume A is a p-group that is not e.f.i., so it has an unbounded Σ-cyclic summand B such that $A = B \oplus C$ with unbounded C. The projection $A \to B$ is not a small homomorphism, A is not thick.

Lemma 7.4 (Cutler–Irwin [1]). *A reduced p-group is e.f.i. if and only if its initial Ulm factor is e.f.i.*

Proof. If A has an unbounded Σ-cyclic group as a summand, then this summand remains summand mod the first Ulm subgroup A^1. Conversely, if A/A^1 has an unbounded Σ-cyclic summand, then by Theorem 6.13 in Chapter 5 every basic subgroup of A/A^1 has such a summand. Basic subgroups of A/A^1 come from those of A, so also A has an unbounded Σ-cyclic summand. □

Quasi-Indecomposable Groups A group A is called **quasi-indecomposable** if, for every cardinal $\kappa < |A|$, in every direct decomposition of A (into any number of summands) there is always a summand of cardinality $> \kappa$. It is obvious that

(a) a reduced countable p-group is quasi-indecomposable exactly if it is unbounded;
(b) if $|A|$ is a successor cardinal, then quasi-indecomposability means that in every direct decomposition one of the summands has the same cardinality as A.

Example 7.5. The torsion-complete group $\overline{B}$ with the standard basic subgroup B is quasi-indecomposable of cardinality $2^{\aleph_0}$.

Theorem 7.6 (Kulikov [1, 2]). *For every infinite cardinal λ, there exist quasi-indecomposable separable p-groups of cardinality λ.*

Proof. Ignoring the countable case settled by (a), we distinguish three mutually exclusive cases.

Case 1: There is an infinite cardinal κ satisfying $\kappa < \lambda \leq \kappa^{\aleph_0}$. Define A as a pure dense subgroup of cardinality λ in the torsion-complete group $\overline{B}$ with basic subgroup $B = \oplus_{n=1}^{\infty} B_n$ where $B_n = \oplus_{\kappa} \mathbb{Z}(p^n)$. Suppose $A = \oplus_{i \in I} C_i$ with $C_i \neq 0$. Here necessarily $|I| \leq |B| = \kappa$, because every summand must have a non-zero basic subgroup. Therefore, if $\mu < \lambda$, then $|C_i| \leq \mu$ for all $i \in I$ is impossible, thus A is quasi-indecomposable.

Case 2: $\lambda = \aleph_{\sigma+1}$ is a successor cardinal, and $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Then $\aleph_{\sigma}^{\aleph_0} = \aleph_{\sigma}$, and by the Hausdorff formula (Jech [J]), $\aleph_{\sigma+1}^{\aleph_0} = \aleph_{\sigma+1}$ also holds, i.e. $\lambda^{\aleph_0} = \lambda$. Define $A = \overline{B}$ with $B = \oplus_n B_n$, $B_n = \oplus_{\lambda} \mathbb{Z}(p^n)$. Manifestly, A has cardinality and final rank λ. If $A = \oplus_{i \in I} C_i$, then by Theorem 4.3 there is an integer m such that almost all $p^m C_i = 0$. At least one of the unbounded C_i must have final rank λ.

Case 3: λ is a limit cardinal such that $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Write $\lambda = \sum_\sigma \kappa_\sigma$ with an increasing sequence of cardinals $\kappa_\sigma < \lambda$ ($\sigma < \operatorname{cf} \lambda$), and for each σ construct a torsion-complete group $A_\sigma = \overline{B}_\sigma$ (as in Case 1) with the cardinal κ_σ. Then $A = \oplus_{\sigma < \operatorname{cf} \lambda} A_\sigma$ will have cardinality and final rank λ. Suppose $A = \oplus_{i \in I} C_i$. By Theorem 4.3, for each σ, there is an integer m_σ such that $p^{m_\sigma} A_\sigma[p]$ is contained in the direct sum of a finite number of the C_i, and it is clear that one of these must have cardinality $\geq \kappa_\sigma$. Thus for every $\mu < \lambda$ there is a C_i of cardinality $> \mu$, so A is quasi-indecomposable. □

κ-Indecomposable Groups Another property we wish to look at relates to the number of summands. A group A is **κ-indecomposable** if it cannot be written as a direct sum of κ non-zero groups. Of course, in this case, only groups of cardinality $\geq \kappa$ are of interest.

Example 7.7. The group in Case 1 of Theorem 7.6 above is κ-indecomposable (of cardinality $> \kappa$). In fact, every direct decomposition of A induces a direct decomposition of a basic subgroup of A. Every basic subgroup is of cardinality κ, so the group cannot be a direct sum of more than κ non-zero summands.

In view of Theorem 5.3 in Chapter 3, no countably infinite, $\aleph_0$-indecomposable separable p-group may exist. However, for larger cardinalities we have a simple criterion for the existence:

Theorem 7.8 (Szele [7], Fuchs [4], Khabbaz [1]). *For an uncountable infinite cardinal λ, there exist λ-indecomposable separable p-groups of cardinality $\geq \lambda$ if and only if either*

(i) *there is a cardinal κ such that $\kappa < \lambda \leq \kappa^{\aleph_0}$; or*
(ii) *λ is a limit cardinal of cofinality ω.*

(i) *is a necessary and sufficient condition for the existence of groups of the stated kind to be of cardinality equal to λ.*

Proof. First assume there is a λ-indecomposable p-group A of cardinality λ. Let $B = \oplus_n B_n$ be a basic subgroup of A, where $B_n = \oplus_{\kappa_n} \mathbb{Z}(p^n)$. As B_n is a summand, we must have $\kappa_n < \lambda$ for each n. Then $\sum_n \kappa_n = |B| \leq |A| \leq |B|^{\aleph_0}$. If the first inequality is strict, we get case (i). In the remaining cases $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$, and either $\lambda = \kappa^+$ with $\kappa^{\aleph_0} = \kappa$, or λ is a limit cardinal with $\sum_n \kappa_n = \lambda$, in which case $\operatorname{cf} \lambda = \omega$ is a must.

If λ satisfies (i), then let A be defined as in Case 1 above in Theorem 7.6. Example 7.7 establishes the existence of such an A.

To rule out the case when $\lambda = \kappa^+$ is a successor cardinal and $\kappa^{\aleph_0} = \kappa$, note that for such a λ the basic subgroup B of any separable p-group A of cardinality $\geq \lambda$ must also be of cardinality $\geq \lambda$. If $|B| = \lambda$, then also $|A| = (\lambda)^{\aleph_0} = \lambda$ (by Hausdorff formula), and Theorem 6.14 in Chapter 5 implies that A has a Σ-cyclic summand of cardinality λ. If $|B| > \lambda$, then B, and hence also A, must have a bounded summand of size λ. Such an A cannot be λ-indecomposable.

If λ is like in (ii), then let $\kappa_0 < \cdots < \kappa_n < \dots$ be cardinals with $\sum_n \kappa_n = \lambda$; we may assume $(\kappa_n)^{\aleph_0} = \kappa_n$ for each n. Let $A = \overline{B}$ be the torsion-complete group with basic subgroup $B = \oplus_n B_n$, where $B_n = \oplus_{\kappa_n} \mathbb{Z}(p^n)$. Then $\operatorname{rk} B = \operatorname{fin} \operatorname{rk} B = \lambda$, thus $|A| = \lambda^{\aleph_0}$—as it is clear from Theorem 3.2; here, $\lambda^{\aleph_0} > \lambda$ by Jech [J]. Now if $A = \oplus_{i\in I} C_i$, then by Theorem 4.3 there is an integer m such that almost all of $p^m C_i$ vanish. Obviously, the cardinality of a p^m-bounded summand cannot exceed $\kappa_m < \lambda$, so the index set I has cardinality $< \lambda$, and A is λ-indecomposable.

To prove the second claim, it only remains to show that (ii) is not an option if we wish the group to have cardinality equal to λ, and (i) fails. If λ is a limit cardinal such that $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$, then in any separable p-group of cardinality λ, the basic subgroups ought to have cardinality λ. We invoke Theorem 6.14 in Chapter 5 to argue that then the group has a Σ-cyclic summand of cardinality λ. □

Direct Product vs. Direct Sum The following theorem records a remarkable feature of torsion groups. Interestingly, but not surprisingly, the natural boundary for the result is the first measurable cardinal. To simplify notation, we confine our discussion to p-groups.

Theorem 7.9 (Zimmermann-Huisgen [1]). *Let A_i $(i \in I)$ be reduced p-groups, and I a non-measurable index set. In any infinite direct decomposition of the direct product*

$$A = \prod_{i\in I} A_i = \oplus_{j\in J} C_j,$$

no more than a finite number of the C_j's are torsion-free.

Proof. There is a smallest ordinal τ such that $p^\tau A_i = 0$ for all $i \in I$. Then also $p^\tau C_j = 0$ for all $j \in J$, so there is a smallest ρ such that $p^\rho C_j$ is torsion for almost all $j \in J$. We will be done if we show that $\rho = 0$.

By way of contradiction, assume $\rho > 0$. Theorem 6.7 in Chapter 2 guarantees the existence of an ordinal $\sigma < \rho$ as well as finite subsets $I_0 \subseteq I, J_0 \subseteq J$ such that

$$p^\sigma B_{I_0} \le (\oplus_{j\in J_0} C_j) + (\oplus_{j\in J} p^\rho C_j),$$

where $B_{I_0} = \prod_{i\in I\setminus I_0} A_i$. No harm is done if we replace $\oplus_{j\in J_0} C_j$ by $\oplus_{j\in J_0} p^\sigma C_j$ in the last containment relation. Passing mod $p^\rho A$, we obtain $p^\sigma B_{I_0}/p^\rho B_{I_0} \to \oplus_{j\in J_0} (p^\sigma C_j/p^\rho C_j)$, a monomorphism. Both sides may be viewed as subgroups of $p^\sigma A/p^\rho A$, therefore there is an epimorphism between the cokernels: $\prod_{i\in I_0}(p^\sigma A_i/p^\rho A_i) \to \oplus_{j\in J\setminus J_0} (p^\sigma C_j/p^\rho C_j)$. Here the finite product is a p-group, so the same has to be true for the direct sum. This means that σ is also an ordinal such that almost all of $p^\sigma C_j$ are torsion. Thus ρ cannot be minimal unless $\rho = 0$. □

Number of Separable *p*-Groups The question as to the cardinality of the set of non-isomorphic separable p-groups of cardinality κ arises naturally. Shelah [2] proved that the cardinality is 2^κ for every infinite cardinal κ, using a more general result of his on the number of non-isomorphic models. Here we cannot give details of his proof; instead, we offer a weaker theorem showing only that there exist arbitrarily large cardinals with this property.

Proposition 7.10. *Let κ be an infinite cardinal such that $\kappa^{\aleph_0} = 2^\kappa$. Then there exist 2^{2^κ} non-isomorphic separable p-groups of cardinality 2^κ. Moreover, they can be chosen so as to have isomorphic basic subgroups.*

Proof. The claim is a bonus of Theorem 9.2 below. □

Regarding Proposition 7.10, we point out that there are arbitrarily large cardinals κ satisfying the hypothesis of this proposition (Griffith [6]). In fact, starting with any infinite cardinal μ_0, set $\mu_{n+1} = 2^{\mu_n}$ and $\kappa = \sum_{n<\omega} \mu_n$. Then

$$\kappa^{\aleph_0} \geq \prod_{n<\omega} \mu_n = \prod_{n<\omega} 2^{\mu_n} = 2^{\sum \mu_n} = 2^\kappa$$

along with the reverse inequality implies $\kappa^{\aleph_0} = 2^\kappa$.

★ **Notes.** Monk [2] proves that a p-group A is essentially indecomposable if and only if the factor ring $\operatorname{End} A/\operatorname{End}_s A$ has no idempotents other than 0 and 1. An interesting variant is Irwin's essentially finitely indecomposable p-groups defined above. The class of these groups is closed under finite direct sums and summands.

Rychkov [3] considers p-groups A of cardinality $\kappa > \aleph_0$ and of final rank κ that are not Σ-cyclic, but every subgroup of cardinality $< \kappa$ embeds in a Σ-cyclic summand of cardinality $< \kappa$. Assuming V = L, he establishes the existence of such an A for every uncountable, regular, not weakly compact cardinal κ. A has the additional property that it cannot be decomposed into the direct sum of two summands of the same final rank.

Shelah [2] establishes the existence of families of separable p-groups like in Proposition 7.10 such that every homomorphism between two different groups is small.

Exercises

(1) A torsion-complete p-group A with $\operatorname{fin\,rk} A = |A|$ is quasi-indecomposable.

(2) An unbounded quasi-complete p-group with a countable basic subgroup is quasi-indecomposable.

(3) In a separable p-group A, the summand C constructed for $a \in A$ in Lemma 1.4 is a minimal summand containing a.

(4) (Soifer, Göbel–Ziegler) A group is called **almost λ-decomposable** for a cardinal λ if, for every cardinal $\kappa < \lambda$, it admits a direct decomposition with κ non-zero summands, but has no decomposition into λ such summands. There exist almost λ-decomposable p-groups of cardinality λ for any infinite λ satisfying $\mu^+ = \lambda \leq \mu^{\aleph_0}$ for some cardinal μ. [Hint: suitable $\overline{B}$.]

(5) The direct sum of a finite number of e.f.i. groups is e.f.i.

(6) There exists no $\aleph_0$-indecomposable reduced p-group of cardinality $\aleph_0$.

8 Valuated Vector Spaces

Though in a p-group A, the socle $A[p]$ plays a discernible role, the information about the group itself provided by its structure is, however, next to nothing: it is a vector space over $\mathbb{Z}/p\mathbb{Z}$, and in vector spaces all subspaces of equal dimensions are alike. The situation changes drastically, if the elements of the socle are viewed along with their heights that they acquired from the group. We take this as our inspiration to embark into the study of vector spaces in which the vectors are equipped with a 'value.' In order to understand what such valuated vector spaces can offer, we have to have a closer look at them.

Valuation of Vector Spaces With applications to reduced p-groups in mind, we will restrict our discussions to vector spaces over the prime field $\mathbb{Z}/p\mathbb{Z}$ of characteristic p. Such vector spaces are easy to handle, since their one-dimensional subspaces are finite. To simplify discussion, the values will be taken from the class of ordinals with a symbol ∞ adjoined (which is to be viewed larger than any ordinal); this class will be denoted in this section by Γ.

To set the stage, let V denote a vector space over the prime field $\mathbb{Z}/p\mathbb{Z}$ of characteristic p. By a **valuation** of V is meant a function v of V into Γ such that, for all $x, y \in V$,

(i) $v(x) = \infty$ if and only if $x = 0$;
(ii) $v(kx) = v(x)$ if $\gcd\{k, p\} = 1$;
(iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

Note that (ii) implies that in a one-dimensional vector subspace all non-zero vectors have the same value. It is a good idea to keep in mind an immediate consequence of (iii):

$$v(x + y) = \min\{v(x), v(y)\} \quad \text{whenever } v(x) \neq v(y).$$

In fact, this follows from combining (iii) with $v(y) \geq \min\{v(-x), v(x+y)\}$ if $v(x) > v(y)$.

The pair $(V; v)$ consisting of a vector space V and its valuation v is called a **valuated vector space**. Its **support** is defined as

$$\operatorname{supp} V = \{v(a) \mid 0 \neq a \in V\} \subset \Gamma.$$

V is **homogeneous** if $\operatorname{supp} V$ is a singleton $\{\gamma\}$, and is **finite-valued** if $\operatorname{supp} V$ is a finite set.

A **morphism** between valuated vector spaces $(V; v)$ and $(W; w)$ is a linear transformation $\chi : V \to W$ (i.e., a group homomorphism) that does not decrease values, i.e. $w(\chi x) \geq v(x)$ for all $x \in V$.

Category $\mathcal{V}$ The valuated vector spaces $(V; v)$ with these morphisms form a category $\mathcal{V}$. Vector space isomorphisms that preserve values are the category isomorphisms; these will be referred to as **isometries**.

Example 8.1. A one-dimensional valuated vector space $(V; v)$ is isometric to $\mathbb{Z}(p)$ where every element $\neq 0$ has the same value γ. If $g \in V$ is a generator, then for a valuated vector space $(W; w)$ there is non-zero morphism $\phi : V \to W$ if and only if there is an $x \in W$ with $\gamma \leq w(x)$, in which case $\phi g = x$ gives rise to a morphism.

Example 8.2. For any reduced p-group A, the socle $A[p]$ of A becomes a valuated vector space if the values are the heights measured in A. If $\phi : A \to B$ is a homomorphism of A into a reduced p-group B, then $A[p]$ is mapped into $B[p]$ such that the map does not decrease heights.

A subspace U of V carries the induced valuation $u = v \restriction U$, and $(U; u)$ becomes a subobject of $(V; v)$ in the category $\mathcal{V}$. (Usually it is harmless to denote the induced valuation in U also by the same symbol v.) The quotient space V/U can be equipped with the valuation

$$v^\star(x + U) = \sup_{a \in U} v(x + a) \qquad (x \in V)$$

making it into a valuated vector space $(V/U; v^\star)$, and the natural map $V \to V/U$ into a $\mathcal{V}$-morphism. It follows that the category $\mathcal{V}$ admits kernels and cokernels, so that we can talk about exact sequences of valuated vector spaces:

$$0 \to (U; u) \to (V; v) \to (W; w) \to 0.$$

More explicitly, this means that $(U; u)$ is a subobject of $(V; v)$ such that the vector space V/U with the induced valuation is isometric to $(W; w)$. Moreover, we have:

Proposition 8.3. *The category of Γ-valued $\mathbb{Z}/p\mathbb{Z}$-vector spaces V is an additive category with kernels and cokernels. It admits products and coproducts.*

Proof. The first claim is obvious in view of what has been said above. The product of the valuated vector spaces $(V_i; v_i)$ with $i \in I$ is easily seen to be the cartesian product $\prod V_i$; i.e. it consists of all vectors $x^\star = (\dots, x_i, \dots)$ with $x_i \in V_i$ where the valuation is given *via* $v(x^\star) = \inf_{i \in I} \; v_i(x_i)$. We shall denote this categorical product by the usual symbol $\prod_i V_i$.

It is readily checked that the coproduct $\coprod V_i$ of the $(V_i; v_i)$ is the vector space direct sum $\oplus V_i$ with the valuation defined as for the product. $\square$

Example 8.4. For reduced p-groups, direct decompositions $A = \oplus_{i \in I} A_i$ are reflected in the socles as coproducts $A[p] = \coprod_{i \in I} A_i[p]$, and *vice versa*.

Example 8.5. Let $B = \oplus_{n=1}^{\infty} B_n$ where B_n is a direct sum of cyclic groups of the same order p^n. For each n, $B_n[p]$ is a homogeneous vector space with constant value $n-1$, and $B[p] = \coprod_{n=1}^{\infty} B_n[p]$. Of course, each $B_n[p]$ is a coproduct of one-dimensional vector spaces.

The following lemma is elementary, but most essential.

Lemma 8.6. *Let $(A; v)$ be a subspace of the valuated vector space $(V; v)$ such that there is a $\gamma \in \Gamma$ satisfying $v(a) \geq \gamma$ for all $a \in A$, and $v(x) \leq \gamma$ for all $x \notin A$. Then*

$V = A \coprod B$ for some subobject B of V which can be chosen so as to contain any given subspace C of V disjoint from A.

Proof. Let C be any subspace of V disjoint from A. In view of Zorn's lemma, there is a subspace B of V which is maximal with respect to the properties: $C \leq B$ and $A \cap B = 0$. Then evidently, $V = A \oplus B$ as vector spaces. From the stated conditions on the values it follows at once that $v(a + b) = \min\{v(a), v(b)\}$ for all $a \in A$ and $b \in B$. This means that $V = A \coprod B$, indeed. $\square$

This leads to most useful decompositions of $(V; v)$. For $\gamma \in \Gamma$ set

$$V^\gamma = \{a \in V \mid v(a) \geq \gamma\} \quad \textit{and} \quad V_\gamma = \{a \in V \mid v(a) > \gamma\}.$$

These are fully invariant subspaces in the sense that every endomorphism of $(V; v)$ (endomorphisms are $\mathcal{V}$-maps!) carries them into themselves. From the preceding lemma we conclude that there are subspaces $A(\gamma) \leq A'(\gamma)$ of V such that

$$V = A(\gamma) \coprod V^\gamma \quad \textit{and} \quad V = A'(\gamma) \coprod V_\gamma.$$

Hence there is a subspace $B(\gamma)$ of V such that $V^\gamma = B(\gamma) \coprod V_\gamma$. Obviously, $B(\gamma)$ is a homogeneous subspace isometric to the vector space V^γ/V_γ with constant value γ. Its dimension $i_\gamma(V) = \dim V^\gamma/V_\gamma$ is uniquely determined by V, it may be called the γ**th Ulm-Kaplansky invariant**, briefly the γ**th UK-invariant**, of V.

Lemma 8.7. *Let A be a homogeneous subspace of the valuated vector space V, and C a subspace of V such that $A + C = A \coprod C$. Then $V = A \coprod B$ for some $B \geq C$.*

Proof. If $\gamma \in \Gamma$ is the common value of the non-zero elements of A, then we pass mod V_γ to conclude that $A^* + C^* = A^* \coprod C^*$ where stars indicate cosets. By Lemma 8.6, we have $V^* = A^* \coprod B^*$ for some $B^* \geq C^*$. The complete inverse image B of B^* in V satisfies $V = A \coprod B$. $\square$

Free Valuated Vector Spaces A valuated vector space F is called **free** if it is the coproduct of one-dimensional valuated vector spaces. Thus F is of the form $F = \coprod_{i \in I} V_i$ where $(V_i; v_i)$ are one-dimensional valuated vector spaces, each with constant value γ_i. If $x_i \in V_i$ is a basis element, then every function $f: X \to W$ from the basis $X = \{x_i\}_{i \in I}$ extends (uniquely) to a category morphism of F into a valuated vector space $(W; w)$ if and only if $\gamma_i \leq w(f(x_i))$ for all $i \in I$. This property can be used as a definition of free valuated vector spaces, it is an exact analogue of what we have learned about free groups; of course, the stated inequality is a must. Note that:

(A) *Every homogeneous valuated vector space is free.* In fact, such a space is just a vector space with constant valuation.

(B) *Two free valuated vector spaces are isometric if and only if, for every ordinal γ, the cardinalities of the free generators of value γ are the same for both spaces.*

(C) *Every valuated vector space is a $\mathcal{V}$-homomorphic image of a free valuated vector space.*

(D) *A finite-valued subspace A of a valuated vector space V is a free summand of V.* For, if supp A consists of the values $\gamma_1 < \cdots < \gamma_k$, then by Lemma 8.7 we can write $V = A^{\gamma_k} \coprod V_1$ for some subspace V_1 such that $A = A^{\gamma_k} \coprod (A \cap V_1)$ (A^{γ_k} denotes the γ_k-homogeneous summand of A). The conclusion follows by a straightforward induction.

Proposition 8.8 (R. Brown). *Countable dimensional valuated vector spaces are free.*

Proof. In fact, if dim V is countable, then V is the union of an ascending chain $0 = V_0 < V_1 < \cdots < V_n < \ldots$ of subspaces where vectors in V_n have n different values. Therefore, by (D) there are decompositions $V_{n+1} = U_n \coprod V_n$ for suitable subspaces U_n ($n < \omega$). Hence $V = \coprod_{n<\omega} U_n$ follows at once. □

Example 8.9. The socle of a countable p-group is a free valuated vector space under the height valuation. This is clear from Proposition 8.8.

From Theorem 2.5 in Chapter 2 and Proposition 8.8 follows at once:

Theorem 8.10. *Summands of a free valuated vector space are free.* □

We do not intend to pursue the exploration of valuated vector spaces, since we do not wish to make extensive use of them, but we cannot leave the subject without mentioning a few interesting facts. The valuation defines a non-archimedean metric, called **ultra-metric** on the valuated vector space $(V; v)$. The **distance** of $a, b \in V$ is defined as $d(a, b) = v(a - b)$, so it is an ordinal. The basic properties of this distance function are not the same as for the real-valued distances:

(i) $d(a, b) = \infty$ is equivalent to $a = b$;
(ii) $d(b, a) = d(a, b)$; and
(iii) $d(a, c) \geq \min\{d(a, b), d(b, c)\}$ for all $a, b, c \in V$.

The last inequality is a strengthened version of the triangle inequality. A **ball** $B(a, \gamma)$ with center $a \in V$ and radius $\gamma \in \Gamma$ is defined as

$$B(a, \gamma) = \{x \in V \mid v(x - a) \geq \gamma\} = a + V^{\gamma}.$$

Note that $b \in B(a, \gamma)$ implies $B(a, \gamma) = B(b, \gamma)$, so every element in the ball can be viewed as a center. Moreover, if two balls intersect, then one is contained in the other. The ultra-metric defines a topology where convergence, density, etc. make good sense.

The most important case is when the values are $< \omega$. The reader should observe the close analogy of the dense free subspace in the next lemma to basic subgroups.

Lemma 8.11. *Let V be a valuated vector space with values $< \omega$. It contains dense free valuated subspaces; these are all isometric.*

Proof. Define $B(n)$ *via* $V^n = B(n) \coprod V_n$ for each $n < \omega$. Thus $B(n)$ is homogeneous, so free with values n. We claim that the free valuated vector space $B = \coprod_{n<\omega} B(n)$ is dense in V. Indeed, if $a \in V \setminus B$, then for each $n < \omega$ it has a decomposition $a = b_0 + \cdots + b_n + c_n$ with $b_i \in B(i)$ for all i, and $c_n \in V_n$; thus, $v(a - b_0 - \cdots - b_n) > n$. The UK-invariants of B are the same as those of V, whence the claim on isometry becomes evident. □

★ **Notes.** The idea of using the elements of the socle equipped with their heights can be traced back to Charles' proof of Kulikov's Theorem 5.1 in Chapter 3. Hill [3] used the metric defined by the height in the socle. A systematic study of valuated vector spaces was initiated in the author's paper [J. Algebra **35**, 23–38 (1975)], using a more general setting. Every valuated vector space embeds in a complete one. We point out that subspaces of free valuated vector spaces need not be free, but summands are always free. Generalization to valuated groups turns out more important, it has an extensive literature (for a brief discussion of valuated groups, we refer to Sect. 3 in Chapter 15).

Exercises

(1) If A and B are subspaces of the valuated vector space V such that $a \mapsto a + B$ is an isometry from A to V/B, then $V = A \coprod B$.

(2) Show that B is isometric to V/A if A and B are subspaces of the valuated vector space V such that $V = A \coprod B$.

(3) (a) A valuated vector space with countable support is free if and only if it is the union of a countable ascending chain of finite-valued subspaces.
  (b) A subspace of a free valuated vector space with countable support is free.
  (c) Give an example of a countably valued vector space that is not free. [Hint: rephrase the claims for socles of p-groups.]

(4) For every valuated vector space V with values $< \omega$, there exists a separable p-group A such that $A[p]$ with height valuation is isometric to V.

(5) If $V = A \coprod B$ and if C is a subspace of V containing A, then $C = A \coprod (B \cap C)$.

(6) Prove the claim that two balls in a valuated vector space can intersect only if one is contained in the other.

(7) Call $a \in V$ **orthogonal** to a subspace U of the valuated vector space (V, v) (in notation: $a \perp U$) if $v(a + u) \leq v(a)$ for all $u \in U$. We say a subspace W of V is orthogonal to U (and write $W \perp U$) if $a \perp U$ holds for all $a \in W$. Prove that

  (a) $W \perp U$ implies $U \perp W$.
  (b) $W \perp U$ if and only if $W + U = W \coprod U$.

9 Separable p-Groups That Are Determined by Their Socles

The results in this section will elucidate the importance of viewing the socle of a p-group as a vector space furnished with the height-valuation. In this section, the socles will be assumed to carry this valuation.

It is an immediate question as to what extent a p-group is determined by its socle as a valuated vector space. Further questions include characterization of the socles of p-groups in some well-defined class $\mathcal{C}$ or cases when the isometry of socles implies the isomorphy of the p-groups within the class $\mathcal{C}$. We deal briefly with these questions. We should probably stress at the outset that the valuated vector space point of view has many more applications than those treated here.

Subgroups Supported by Dense Subsocles If we are to talk about these questions, we should probably start with showing the opposite situation: it can very well happen that as many non-isomorphic separable p-groups as possible have isometric socles. Before stating the relevant theorem, we prove a lemma.

Lemma 9.1 (Hill–Megibben [3]). *Let A be a reduced p-group. A proper dense subsocle S of A supports 2^λ different pure and dense subgroups, where $\lambda = |pA[p] \cap S|$.*

Proof. Let $\{c_i\}_{i \in I}$ denote a basis of the vector space $pA[p] \cap S$, and choose $a_i \in A$ $(i \in I)$ such that $pa_i = c_i$. By hypothesis, there is a $u \in A[p] \setminus S$. For a subset J of I, let X_J be the set of the elements g_i where $g_i = a_i$ if $i \in J$ and $g_i = a_i + u$ otherwise. S is still the socle of the subgroup generated by X_J, so by Theorem 1.3 for each J, X_J is contained in a pure and dense subgroup C_J of A, supported by S. If K is a different subset of I, then $C_J \neq C_K$, since no C_J may contain both a_i and $a_i + u$. As $|I| = \lambda$, we have 2^λ different subgroups C_J. $\square$

Regarding the cardinal hypothesis in the following result, let us note that there exist arbitrarily large cardinals κ satisfying $\kappa^{\aleph_0} = 2^\kappa$. (See the comment following Proposition 7.10.)

Theorem 9.2 (Hill–Megibben [3]). *Let κ be a cardinal satisfying $\kappa^{\aleph_0} = 2^\kappa$, and A a separable p-group of cardinality 2^κ whose basic subgroups have rank and final rank equal to κ. Then a proper dense subsocle S of cardinality 2^κ supports 2^{2^κ} non-isomorphic pure (and dense) subgroups of A.*

Proof. Let $\overline{B}$ be the torsion-complete group whose basic subgroup B has rank and final rank κ. By Theorem 3.2, we have $|\overline{B}| = 2^\kappa$. Applying Lemma 9.1 to the case $\lambda = 2^\kappa$, we conclude that there are 2^{2^κ} different pure subgroups supported by a proper dense subsocle S. Owing to Theorem 3.10, isomorphic dense subgroups are carried into each other by an automorphism of $\overline{B}$. We complete the proof by showing that $\overline{B}$ has less than 2^{2^κ} automorphisms.

Each automorphism of $\overline{B}$ is entirely determined by its action on B. As any generator of B is mapped by an automorphism upon one of λ different elements, the cardinality of $\operatorname{Aut}\overline{B}$ is at most $\lambda^\kappa = (2^\kappa)^\kappa = 2^\kappa < 2^{2^\kappa}$. $\square$

When Socles Determine the Group We now turn to the discussion of positive results. Our intention is to show that the Σ-cyclic p-groups, the direct sums of countable p-groups, and the direct sums of torsion-complete p-groups are determined by their socles as valuated vector spaces, within the respective classes of groups.

Theorem 9.3.

(i) *A p-group is Σ-cyclic if and only if its socle is a free valuated vector space with values $< \omega$.*
(ii) *Groups with isometric free socles of values $< \omega$ are isomorphic.*
(iii) *Every free valuated vector space over $\mathbb{Z}/p\mathbb{Z}$ with values $< \omega$ is isometric to the socle of a Σ-cyclic p-group.*

Proof. (i) If A is Σ-cyclic, then $A[p]$ is the coproduct of the socles of the cyclic summands. These are one-dimensional spaces, so $A[p]$ is free as a valuated vector space. Conversely, a free valuated vector space is the coproduct of one-dimensional spaces each of which supports a cyclic group which is of order p^{n+1} if n is the common value of the non-zero vectors. Hence also (ii) is evident.

(iii) is straightforward. □

For the class of countable p-groups, the analogues of (i)–(ii) hold (see Theorem 1.6 in Chapter 11, while the analogue of (iii) holds true only under additional conditions on the invariants; see Exercise 2.

Theorem 9.4.

(i) *If A is a direct sum of countable p-groups, then its socle $A[p]$ is a free valuated vector space with values $< \omega_1$.*
(ii) *Direct sums of countable p-groups are isomorphic if and only if their socles are isometric.*

Proof. (i) It suffices to prove the claim for countable p-groups A. Then $\dim A[p]$ is countable, so Proposition 8.8 implies that $A[p]$ is free as a valuated vector space.
(ii) This is equivalent to Kolettis' theorem (Corollary 6.7 in Chapter 11). □

As expected, also the torsion-complete groups are determined by their socles. However, it is not so obvious that the same holds for their arbitrarily large direct sums.

Theorem 9.5 (Hill [5]). *Two direct sums of torsion-complete p-groups are isomorphic if and only if they have isometric socles.*

Proof. To begin with, we show that two torsion-complete p-groups with isometric socles are isomorphic. The 'basic' subspaces (cp. Lemma 8.11) in the socles are isometric, so the basic subgroups of the torsion-complete groups are isomorphic, and hence the groups themselves are isomorphic.

It suffices to prove the 'if' part of the statement. Our first step is to show that two separable p-groups A, C with isometric socles can be embedded isomorphically as pure subgroups in a torsion-complete p-group such that they share the same socle. An isometry $\phi: A[p] \to C[p]$ carries the socle $A'[p]$ of a basic subgroup A' of A onto the socle $C'[p]$ of a basic subgroup C' of C. As ϕ is height-preserving, it extends to an isomorphism $\phi': A' \to C'$ which in turn extends to an isomorphism $\bar{\phi}: \overline{A} \to \overline{C}$ between the torsion-completions of A and C. Then $\bar{\phi}(A)$ and C have the same socle in $\overline{C}$.

Thus it is harmless to assume that the p-groups $A = \oplus_{i \in I} A_i$ and $C = \oplus_{j \in J} C_j$ (where A_i, C_j are torsion-complete) are pure subgroups with the same socle in a torsion-complete group $\overline{B}$. Under this hypothesis, we prove that for any summand A_i, C admits a summand D_i such that $D_i[p] = A_i[p]$. From Lemma 4.2 we argue that there is an integer m such that $p^m A_i[p]$ is contained in a finite direct sum $C_{j_1} \oplus \cdots \oplus C_{j_k}$. This direct sum is torsion-complete, so by Lemma 5.4 it contains a pure subgroup U supported by $p^m A_i[p]$. If we write $A_i[p] = X \oplus p^m A_i[p]$, then it is clear that X (of bounded heights) supports a summand V of C, thus there is a pure subgroup $D_i = U \oplus V \leq C$ as stated. Now A_i, as a pure torsion-complete subgroup, is a summand of $\overline{B}$. A pure subgroup supported by the support of a summand is itself a summand (Lemma 1.2), so D_i is a summand of $\overline{B}$, and hence of C. It is evidently $\cong A_i$.

Clearly, the subgroups D_i generate their direct sum in C such that $D = \oplus_{i \in I} D_i$ has the same socle as $A[p] = C[p]$. The proof will be complete if we can show that D is pure in C. Let $x \in (D_{i_1} \oplus \cdots \oplus D_{i_k})[p]$ be of height h in C. It has the same height in A, so all of its coordinates in the A_i have height $\geq h$. x has exactly the same coordinates in the D_i with the same heights. Thus the elements in $D[p]$ cannot have smaller heights in D than in C, and so D is pure in C. $\square$

Other classes of p-groups in which only isomorphic groups can have isometric socles are the class of simply presented and the class of $p^{\omega+1}$-projective p-groups. These groups will be discussed later in Sects. 3 and 10 in Chapter 11, respectively.

★ **Notes.** By making use of the Diamond Principle, both Cutler [2], Dugas–Vergohsen [1] proved that the only separable p-groups of cardinality $\leq \aleph_1$ that are determined by their socles are the Σ-cyclic and the torsion-complete groups. Mekler–Shelah [5] established the independence of this question in ZFC. The **Keef class** K_p of p-groups is the smallest class that contains the cyclic p-groups, and is closed under taking summands, direct sums, and torsion subgroups of non-measurable direct products. Keef [4] shows that the groups in K_p are characterized by their socles within this class (generalizing Theorem 9.5).

As we have already mentioned above, subspaces of free valuated vector spaces need not be free. The valuated socles of totally projective p-groups are never free once the lengths of the groups exceed ω_1, but are always embeddable in free valuated vector spaces. Valuated vector spaces that support totally projective p-groups were characterized in the author's paper [Symposia Math. **23**, 47–62 (1979)].

There are interesting results on direct sums of torsion-complete groups. One of those is that summands are again such direct sums. Another theorem states that any two decompositions have isomorphic refinements. See Irwin–Richman–Walker [1], Hill [4], Enochs [1]. Ivanov [2] has an extensive study of countable direct sums of algebraically compact and torsion-complete groups, including cancellation properties, solutions to Kaplansky's test problems, etc.

Hill [1] gave the first example of non-isomorphic pure subgroups supported by the same socle. In his paper [18], he examines to what extent the socle can determine the group. Cutler [4] raises the question as to the existence of non-isomorphic pure and dense subgroups with the same p^n-socle in a separable p-group, and shows that it is undecidable in ZFC.

Exercises

(1) (Hill) Show that the decompositions $\oplus_i A_i = \oplus_j C_j$ where A_i, C_j are torsion-complete p-groups have common refinement. [Hint: argue with the exchange property.]
(2) Find necessary and sufficient conditions on the UK-invariants of a free valuated vector space V (with values $< \omega_1$) to exist a countable p-group with socle isometric to V. [Hint: Theorem 1.9 in Chapter 11.]

Problems to Chapter 10

PROBLEM 10.1. Let $\{C_\sigma \mid \sigma < \tau\}$ be a smooth increasing chain of p-groups such that C_σ is pure in $C_{\sigma+1}$, and $C_{\sigma+1}/C_\sigma$ is a thin group for all $\sigma < \tau$. Is the union $C = \cup_{\sigma<\tau} C_\sigma$ a thin group?

PROBLEM 10.2. Let $\{C_\sigma \mid \sigma < \tau\}$ be a smooth increasing chain of p-groups such that C_σ is pure in $C_{\sigma+1}$, and $C_{\sigma+1}/C_\sigma$ is a thick group for all $\sigma < \tau$. Is the union $C = \cup_{\sigma<\tau} C_\sigma$ a thick group?

PROBLEM 10.3. Call the p-groups A, C *almost disjoint* if only the Σ-cyclic groups can have isomorphic copies in both of them. Is there any bound for the cardinality of sets of non-isomorphic pairwise almost disjoint p-groups?

This problem for torsion-free groups was discussed by Eklof–Mekler–Shelah [1].

PROBLEM 10.4. Characterize the socle $A[p]$ as a valuated vector space.

Much has been done by Dugas–Vergohsen [1] under the hypothesis V = L.

PROBLEM 10.5. When do the "slices" $A[p^{n+1}]/A[p^n]$ as valuated vector spaces determine a separable p-group A?

Chapter 11
p-Groups with Elements of Infinite Height

Abstract We continue our study of torsion groups concentrating on p-groups (with unspecified prime p) in the general case when the groups contain elements of infinite height. Matters are more subtle here as one has to deal with transfinite heights that are the central concept both in the search for invariants and in the proofs.

The focus of the structure theory is on p-groups that can be described by their UK-invariants. Accordingly, this chapter is primarily devoted to countable p-groups and their generalizations: the totally projective p-groups. The theory is perhaps the most interesting and highly satisfactory classification of a fairly large class of p-groups in terms of well-ordered sequences of cardinal numbers (provided by their UK-invariants). The four main approaches to the theory of totally projective p-groups (simple presentation, total projectivity, nice systems, and balanced-projectivity) underline the extreme importance of these groups; this theory is unparalleled in beauty and richness in abelian group theory.

Once the equivalence of the four main characterizations is established, there remain still some intriguing questions to be answered. For instance, which well-ordered sequences of cardinals may be the UK-invariants of a totally projective p-group? or, which is the largest class of p-groups that includes the generalized Prüfer groups, is closed under direct sums and summands, and whose members are distinguishable *via* their UK-invariants?

Needless to say, there have been various attempts to extend the well-rounded theory of totally projective p-groups, and various generalizations have been considered in the literature. So far these theories have produced only less remarkable results. Though several innovative techniques have been discovered, it seems that so far they have fallen short of true significance.

The final sections of this chapter deal with questions that are spin-offs of the theory of totally projective p-groups, and offer a glimpse into classes depending on ordinal numbers.

1 The Ulm-Zippin Theory

The main objective of this section is to present the celebrated Ulm-Zippin theory of countable p-groups. As before, p denotes an arbitrary, but fixed prime number, and for an element $a \in A$, $h_A(a)$ (or $h(a)$) will mean its transfinite height at p.

Hill Invariants Let A be a p-group, and G a subgroup of A. An element $a \in A \setminus G$ is said to be **proper with respect to** G if the heights satisfy

$$h(a) \geq h(x) \qquad \text{for all } x \in a + G.$$

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-319-19422-6_11

In other words, a has the maximal height in its coset mod G. Thus an element $a \in A$ of height σ is proper with respect to G if and only if $a \notin p^{\sigma+1}A + G$. In this case,

$$h_A(a) = h_{A/G}(a + G).$$

Of course such an element a need not exist in a coset, though it can be shown easily that if the height of a coset $a + G$ in A/G is a non-limit ordinal, then the coset does contain an element of this height. Note that if a is proper with respect to G, then we have

$$h(a + g) = \min\{h(a), h(g)\} \qquad \text{for all } g \in G.$$

Evidently, if G is finite, then every coset mod G contains elements proper with respect to G. The same holds if $G = p^\sigma A$ for an ordinal σ.

Let now A be a reduced p-group, and G a subgroup of A. For an ordinal σ, we introduce the notation

$$G(\sigma) = (p^{\sigma+1}A + G) \cap p^\sigma A[p];$$

this is a subgroup between $p^\sigma A[p]$ and $p^{\sigma+1}A[p]$. From our remark above it follows that an element $a \in A$ of order p and of height σ belongs to $G(\sigma)$ if and only if it is *not* proper with respect to G. Consequently, representatives of the non-zero cosets in the factor group

$$f_\sigma(A, G) = p^\sigma A[p]/G(\sigma)$$

are exactly the elements of A that are of order p, of height σ, and proper with respect to G. This $\mathbb{Z}/p\mathbb{Z}$-vector space is called the σth **UK-invariant of** A **relative to** G or the σth **Hill invariant of** A **relative to** G. Sometimes we understand by this invariant the dimension of the vector space. We can allow this ambiguity, since from the context it will always be clear what we mean: vector space or its dimension. Evidently, $f_\sigma(A, G) \le f_\sigma(A)$ and $f_\sigma(A, 0) = f_\sigma(A)$. Here $f_\sigma(A) = p^\sigma A[p]/p^{\sigma+1}A[p]$ is the **Ulm-Kaplansky-** or **UK-invariant** of A.

Example 1.1. Let $A = \mathbb{Z}(p^{n+1})$ and $G = pA$. Then $G(k) = p^k A[p] = A[p]$ for $k = 0, \dots, n$, and $G(k) = 0$ for all $k > n$. Thus the Hill invariants $f_\sigma(A, G)$ vanish for all σ.

Example 1.2. Let A be a reduced p-group, and B its basic subgroup. Then $B(n) = (p^{n+1}A + B) \cap p^n A[p] = A \cap p^n A[p] = p^n A[p]$, thus $f_n(A, B) = 0$ for all $n < \omega$. If $\sigma \ge \omega$, then $B(\sigma) = (p^{\sigma+1}A + B) \cap p^\sigma A[p] = p^{\sigma+1}A[p]$, thus $f_\sigma(A, B) = (p^{\sigma+1}A[p])/(p^\sigma A[p]) = f_\sigma(A)$.

Example 1.3. Let A be a reduced p-group, and $G = p^\rho A$ for some ordinal ρ. Then $G(\sigma) = (p^{\sigma+1}A + p^\rho A) \cap p^\sigma A[p] = p^{\sigma+1}A[p]$ or $p^\sigma A[p]$ according as $\sigma < \rho$ or $\sigma \ge \rho$. Thus $f_\sigma(A, p^\rho A)$ equals $f_\sigma(A)$ if $\sigma < \rho$ and is 0 otherwise.

In order to illustrate the role of Hill invariants, we prove a simple lemma.

Lemma 1.4. *Assume A, C are p-groups, G is a subgroup of A, and H is a subgroup of C such that $f_\sigma(A,G) = f_\sigma(C,H)$ for some σ. There exists an $a \in A[p]$ of height σ and proper with respect to G if and only if there is a $c \in C[p]$ of height σ and proper with respect to H.*

Proof. Observe that $x \in G(\sigma)$ means that $x \in A[p]$ is of height σ, and not proper with respect to G, since—we repeat—$x \in p^{\sigma+1}A + G$ is another way of saying that the coset $x + G$ has a representative in $p^{\sigma+1}A$. □

Let A and C be reduced p-groups, and $G \leq A$, $H \leq C$ subgroups. An isomorphism $\phi\colon G \to H$ is called **height-preserving** if

$$h_C(\phi g) = h_A(g) \qquad \text{for every } g \in G.$$

It is important to keep in mind that heights are always computed in the large groups. Manifestly, the restriction of an isomorphism of groups $A \to C$ to subgroups $G \to H$ is always height-preserving.

Kaplansky–Mackey Lemma The next lemma is crucial in extending isomorphisms between subgroups to larger subgroups. It is the most essential ingredient in the proof of Ulm's theorem below. The groups A, C in Lemma 1.5 need not be p-groups, but the Hill invariants are computed for the chosen prime p.

Lemma 1.5 (Kaplansky–Mackey [1]). *Let A and C be reduced groups, G a subgroup of A, and H a subgroup of C such that A/G and C/H are p-groups. Suppose $\phi : G \to H$ is a p-height-preserving isomorphism; furthermore, for all σ, $f_\sigma(A,G) \leq f_\sigma(C,H)$ and*

$$\alpha_\sigma\colon\ p^\sigma A[p]/G(\sigma) \to p^\sigma C[p]/H(\sigma) \tag{11.1}$$

are monomorphisms. If $a \in A$ is proper with respect to G and $pa \in G$, then ϕ can be extended to a height-preserving isomorphism

$$\phi^*\colon\ \langle G, a\rangle \to \langle H, c\rangle$$

for a suitable $c \in C$ such that

$$\alpha_\sigma(\langle G, a\rangle(\sigma)/G(\sigma)) = \langle H, c\rangle(\sigma)/H(\sigma) \qquad \textit{for all } \sigma.$$

Proof. Assume that a with $h(a) = \sigma$ has been chosen in its coset mod G, in addition to being proper with respect to G, to satisfy also $h(pa) > \sigma + 1$ whenever possible. We distinguish two cases according as such a choice is possible or not.

Case I: $h(pa) > \sigma + 1$. In this case $pa = pb$ for some $b \in A$ of height $\geq \sigma + 1$. Then $h(a - b) = \sigma$, and $a - b$ is proper with respect to G, for otherwise there would exist a $g \in G$ with $h(a - b + g) > \sigma$, leading to the contradiction $h(a + g) > \sigma$. By an earlier remark, $a - b \notin G(\sigma)$, thus there is a $u \in p^\sigma C[p]$

such that α_σ maps $a - b + G(\sigma)$ to $u + H(\sigma)$. This u is evidently proper with respect to H. Since ϕ is height-preserving, there is a $d \in C$ of height $\geq \sigma + 1$ such that $pd = \phi(pa) \in H$. Now $c = d + u$ is proper with respect to H and satisfies $h(c) = \sigma$, $pc = \phi(pa) \in H$.

Case II: $h(pa) = \sigma + 1$. Pick a $c \in C$ of height σ such that $\phi(pa) = pc$. To see that $c \in H$ is contradictory, assume that $\phi(g) = c$ for some $g \in G$. Then $pg = pa$, and $p(a - g) = 0$. As a was proper with respect to G, we obtain $h(a - g) = \min\{h(a), h(g)\} = \sigma$, showing that $a' = a - g$ is likewise proper with respect to G. But $h(pa') = \infty > \sigma + 1$ contradicts the choice of a. Thus $c \notin H$, and we now show that it is proper with respect to H. If we had an $x \in H$ with $h(c + x) > \sigma$, then necessarily $h(x) = \sigma$, whence we conclude that $\sigma < h(c + x) < h(pc + px) = h(pa + p\phi^{-1}x)$ implies $a' = a + \phi^{-1}x$ satisfies $h(a') = \min\{h(a), h(x)\} = \sigma$ and $h(pa') > \sigma + 1$, contrary to the choice of a. Thus $c \notin H$ is proper with respect to H.

In both cases, with the chosen $c \in C$, we will extend ϕ to ϕ^* by letting c correspond to a. That we get an isomorphism $\phi^* : \langle G, a\rangle \to \langle H, c\rangle$ is clear from the selection of c. To show that it is height-preserving, observe that, for all $g \in G$,

$$h(a + g) = \min\{h(a), h(g)\} = \min\{h(c), h(\phi g)\} = h(c + \phi g).$$

To verify the claim on the action of ϕ^*, take into consideration that if $\rho < \sigma$, then $h(a) \geq \rho + 1$ implies $\langle G, a\rangle(\rho) > G(\rho)$, while if $\rho > \sigma$, then the same inequality is the consequence of a being proper with respect to G. Similar inequalities hold for H and c. If $\rho = \sigma$, then in Case I, $\langle G, a\rangle(\sigma) = \langle a - b\rangle \oplus G(\sigma)$ and $\langle H, c\rangle(\sigma) = \langle u\rangle \oplus H(\sigma)$, and the choice of u guarantees that α_σ induces an isomorphism as desired. In Case II, no $g \in G$ and $a' \in A$ with $h(a') \geq \sigma + 1$ may satisfy $p(a - g + a') = 0$, so that again $\langle G, a\rangle(\sigma) = G(\sigma)$, and similarly $\langle H, c\rangle(\sigma) = H(\sigma)$. □

Whenever the monomorphisms α_σ are not preassigned, in Case I any $u \in p^\sigma C[p]/H(\sigma)$ may be chosen.

A corollary to the preceding proof is that the ρth Hill invariants of A relative to G and relative to $\langle G, a\rangle$ are the same except possibly when $\rho = \sigma$ in which case $f_\sigma(A, \langle G, a\rangle) + 1 = f_\sigma(A, G)$ may hold. But certainly we have (in the notation above)

$$f_\sigma(A, \langle G, a\rangle) = f_\sigma(C, \langle H, c\rangle).$$

Ulm's Theorem Equipped with this lemma, we are prepared for the proof of the main result.

Theorem 1.6 (Ulm [1]). *For the countable reduced p-groups, A and C, the following conditions are equivalent:*

(i) *A and C have the same Ulm length τ, and for each $\sigma < \tau$, the Ulm factors A_σ and C_σ are isomorphic;*
(ii) *A and C have the same UK-invariants: $f_\sigma(A) = f_\sigma(C)$ for all σ;*
(iii) *A and C are isomorphic.*

Proof. The implication (i) $\Rightarrow$(ii) is a simple consequence of the fact that the number of cyclic summands of order p^n in the decomposition of A_σ is equal to the $(\omega\sigma + n - 1)$st UK-invariant of A, while (iii) $\Rightarrow$ (i) is trivial. It remains to show that (ii) $\Rightarrow$ (iii).

Assuming (ii), for each σ we fix an isomorphism $\alpha_\sigma : p^\sigma A[p]/p^{\sigma+1}A[p] \to p^\sigma C[p]/p^{\sigma+1}C[p]$. The elements of A, C can be arranged in countable sequences: $A = \{a_0, \ldots, a_n, \ldots\}$, $C = \{c_0, \ldots, c_n, \ldots\}$; we may even assume $pa_{n+1} \in \langle a_0, \ldots, a_n\rangle$, $pc_{n+1} \in \langle c_0, \ldots, c_n\rangle$ for all n. Suppose we have constructed chains of finite subgroups $0 = G_0 \leq G_1 \leq \cdots \leq G_n$ of A and $0 = H_0 \leq H_1 \leq \cdots \leq H_n$ of C, as well as height-preserving isomorphisms $\phi_i \colon G_i \to H_i$ such that $\phi_n \restriction G_i = \phi_i$ for $i = 0, 1, \ldots, n$. We may assume that α_σ induces isomorphisms $p^\sigma A[p]/G_i(\sigma) \to p^\sigma C[p]/H_i(\sigma)$ for $i \leq n$. If n is odd, select the first a_k in the sequence that does not belong to G_n. We wish to extend ϕ_n to a height-preserving isomorphism ϕ_{n+1} of $G_{n+1} = \langle G_n, a_k\rangle$ with a subgroup H_{n+1} of C, containing H_n. This can be done by using Lemma 1.5. If n is even, then we select the first $c_\ell \in C$ with $c_\ell \notin H_n$, and extend ϕ_n^{-1} to an isomorphism of $H_{n+1} = \langle H_n, c_\ell\rangle$ with a subgroup G_{n+1} in A. After each step, (11.1) will be satisfied, as is clear from Lemma 1.5. It is evident that $\phi = \cup_{n<\omega} \phi_n$ will establish the isomorphy of A and C. □

Before moving on to the existence question, we record an important corollary to the preceding theorem. Though it will follow from the more general Lemma 4.5 below, we give a short proof.

Corollary 1.7 (Zippin [1]). *If A and C are countable p-groups with the same UK-invariants, then every isomorphism between $p^\rho A$ and $p^\rho C$ (for any fixed ordinal ρ) extends to an isomorphism $A \to C$.*

Proof. The proof is essentially the same as for the preceding theorem, just we have to start with a prescribed isomorphism $\psi : p^\rho A \to p^\rho C$. The only UK-invariants needed in the proof are those of indices $\sigma < \rho$. As cosets of finite extensions of $p^\rho A$ always contain elements proper with respect to them, a repetition of the proof in Theorem 1.6 establishes our claim. □

Example 1.8. The Prüfer group $H_{\omega+1}$ defined in Sect. 1 in Chapter 10 has the UK-invariants $f_\sigma(H_{\omega+1}) = 1$ for $\sigma = 0, 1, \ldots, \omega$. All countable p-groups with the same sequence of invariants are isomorphic to $H_{\omega+1}$.

Zippin's Theorem Once we have a classification theorem, we also need a result saying what values of the invariants can occur. In order to find out what sequences $f_0(A), \ldots, f_\sigma(A), \ldots$ of cardinals may serve as UK-invariants in Ulm's theorem, we turn our attention to Zippin's main theorem. (For generalization, see Theorem 3.7.)

Theorem 1.9 (Zippin [1]). *There exists a countable reduced p-group A of length τ and with UK-invariants $n_0, \ldots, n_i, \ldots, n_\omega, \ldots, n_\sigma, \ldots$ $(\sigma < \tau)$ if and only if*

(a) *$\tau = \omega\alpha + k$ is a countable ordinal where $k \geq 0$ is an integer;*
(b) *each n_σ is a non-negative integer or $\aleph_0$ such that for each $\rho < \tau$, infinitely many of the cardinals $n_\rho, \ldots, n_{\rho+i}, \ldots$ $(i < \omega)$ are non-zero.*

Proof. The necessity of the conditions is immediate, the sole non-trivial argument relies on the fact that only the last Ulm factor can be bounded.

For sufficiency, we first develop a sequence A_ν ($\nu \leq \alpha$) of Σ-cyclic p-groups whose UK-invariants are $n_{\omega\nu+j}$ for $j < \omega$ (for $j \leq k$ when $\nu = \alpha$). By (b), the A_ν are all unbounded with the possible exception of the last one if such exists. The theorem will be proved if we succeed in defining a p-group whose Ulm factors are A_ν. We use transfinite induction on α. If $\alpha = 0$, or if $\alpha = 1, k = 0$, then the sequence has only one Ulm factor, and $A = A_0$ is as desired. Hence we assume that either $\alpha = 1$ and $k > 0$, or $\alpha \geq 2$, and the existence has been established for shorter sequences. The proof distinguishes several cases.

Case I. $\alpha - 1$ *exists and* A_α *is a cyclic group, say, of order* p^ℓ. Let C denote a countable p-group with Ulm sequence A_ν ($\nu < \alpha - 1$), $A'_{\alpha-1}$ where $A'_{\alpha-1} = \oplus_{i<\omega}\langle c_i\rangle$ with $o(c_i) = p^{\ell+\ell_i}$ provided $A_{\alpha-1} = \oplus_{i<\omega}\langle b_i\rangle$ with $o(b_i) = p^{\ell_i}$. Define A as the quotient of C modulo the subgroup generated by all $p^{\ell_i}c_i - p^{\ell_j}c_j$ ($i,j < \omega$). If we denote by a the coset of $p^{\ell_i}c_i$, then it is clear that $A/\langle a\rangle$ is a countable p-group with Ulm sequence A_ν ($\nu \leq \alpha - 1$). Since the exponents ℓ_i are unbounded, a must have a height not smaller than the heights of the c_i, so it belongs to the αth Ulm subgroup of A, which is $\langle a\rangle$. That $\langle a\rangle$ has order p^ℓ can be verified by taking a copy of $\mathbb{Z}(p^\infty)$, mapping each c_i upon an element of the same order, watching that all the $p^{\ell_i}c_i$ should have the same image, and then extending this map to all of C stepwise starting with $A_{\alpha-1}$. It only remains to point out that such a map factors through A.

Case II. $\alpha - 1$ *exists and* A_α *is a* Σ*-cyclic group.* We reduce the construction to Case I. We decompose each A_ν ($\nu < \alpha - 1$) into a direct sum of a countable number of unbounded summands, and for each cyclic summand of A_α we apply the above construction with one of the summands. Finally, we take the direct sum of the groups constructed.

Case III. α *is a limit ordinal and* $k = 0$. We decompose each A_ν ($\nu < \alpha$) into a countable direct sum of unbounded groups: $A_\nu = \oplus_{\nu\leq\mu<\alpha} G_{\nu\mu}$. We appeal to the induction hypothesis to conclude the existence of countable p-groups G_μ of length μ ($\mu < \alpha$) with Ulm sequence $G_{0\mu}, G_{1\mu}, \dots, G_{\mu\mu}$. Then $A = \oplus_{\mu<\alpha} G_\mu$ is a countable p-group with the prescribed Ulm sequence.

Case IV. α *is a limit ordinal and* A_α *is a cyclic group of order* p^ℓ. We decompose each $A_\nu = \langle b_\nu\rangle \oplus A'_\nu$ where the cyclic summands $\langle b_\nu\rangle$ are selected such that for each $\rho < \alpha$ and for each $m \in \mathbb{N}$, there is a $\nu < \alpha$ with $o(b_\nu) = p^{\ell_\nu} > p^m$. By induction hypothesis, there is a countable p-group C with Ulm factors $C = \langle c_\nu\rangle \oplus A'_\nu$ ($\nu < \alpha$) where $o(c_\nu) = p^{\ell+\ell_\nu}$. Define A as the factor group of C modulo the subgroup generated by all $p^{\ell_\nu}c_\nu - p^{\ell_\mu}c_m$ ($\nu, \mu < \alpha$). In the same way as in Case I, it follows that A will have the prescribed Ulm sequence.

Case V. α *is a limit ordinal and* $A_\alpha \neq 0$ *is a* Σ*-cyclic group.* This case can be reduced to Case IV by imitating the method used in Case II. □

Example 1.10. Let G_0, G_1 be unbounded countable Σ-cyclic p-groups, and write $G_0 = \oplus_{i<\omega}\langle a_i\rangle$, $G_1 = \oplus_{i<\omega}\langle g_i\rangle$. We define a group A whose Ulm factors are isomorphic to the given groups as follows. Let $B_0 = \oplus_{i<\omega}\langle b_i\rangle$ be a basic subgroup of G_0 such that G_0/B_0 is a countable direct sum: $G_0/B_0 = \oplus_{i<\omega} C_i$ with $C_i \cong \mathbb{Z}(p^\infty)$. Let $c_i \in G_0$ be a representative of a generator of the socle of C_i. The group A will be defined as the group generated by $\{a_i, g_i\}_{i<\omega}$ subject to the existing relations for the b_i, elements of C_i, and for the g_i (e.g., of the form $p^m g_i = 0$), to the relations between the c_i and the b_i, and in addition to $pc_i = g_i$ for all $i < \omega$. Then the elements g_i will have infinite heights in A. To complete the argument, we still need to ascertain that the new relations do not force any collapse in A. We need not worry about the elements generated by the c_i, because none of them becomes 0, not even modulo the subgroup generated by the g_i. If $a \in A$ contains generators g_i, then define a map $\phi\colon A \to \mathbb{Z}(p^\infty)$ by mapping one of these generators to a non-zero element, the rest of the g_i to 0, and extend this map to all of A. Then $\phi(a) \neq 0$, so $a \neq 0$.

Theorems 5.3 in Chapter 3, 1.6, 1.9 of Prüfer, Ulm, and Zippin yield a complete classification of countable reduced p-groups. We regard this theory as the greatest achievement in abelian group theory in the first half of the twentieth century.

A few comments may be inserted here about this theory. If A is a countable reduced p-group of length $\tau = \alpha\omega + k$, then we can arrange its UK-invariants in a convenient matrix form (to which we will refer as the **UK-matrix of** A):

$$\mathbb{U}(A) = \begin{pmatrix} f_0(A) & f_1(A) & \dots & f_n(A) & \dots \\ f_\omega(A) & f_{\omega+1}(A) & \dots & f_{\omega+n}(A) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ f_{\omega\rho}(A) & f_{\omega\rho+1}(A) & \dots & f_{\omega\rho+n}(A) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \|f_\sigma(A)\|$$

where the rows are indexed by ordinals $\rho < \alpha$ (or $\rho \leq \alpha$ if $k > 0$) and the columns by non-negative integers. The cardinal numbers in the ρth row are the invariants of the ρth Ulm factor of A. The matrix $\mathbb{U}(A)$ satisfies:

(i) it is an $\alpha \times \omega$-matrix with a countable ordinal α;
(ii) the entries are non-negative integers or $\aleph_0$;
(iii) every row (with the possible exception of the very last one) contains infinitely many non-zero entries.

Ulm's theorem can be interpreted as saying that two reduced countable p-groups are isomorphic, $A \cong C$, if and only if the corresponding matrices are equal: $\mathbb{U}(A) = \mathbb{U}(C)$, while Zippin's theorem says that every matrix with properties (i)–(iii) is the UK-matrix of some reduced countable p-group.

Corollaries We conclude this section with two corollaries.

Proposition 1.11. *A countably infinite reduced p-group decomposes into the direct sum of infinitely many non-trivial groups.*

Proof. Let A be a group as stated, and let $\mathbb{U}(A)$ be its UK-matrix. It is an easy exercise in cardinal arithmetics to decompose $\mathbb{U}(A)$ into the sum of countably many matrices satisfying (i)–(iii). By Zippin's theorem, each of these matrices defines a group A_n ($n < \omega$), and Ulm's theorem assures that $A \cong \oplus_{n<\omega} A_n$. $\square$

Proposition 1.12 (Baer [1]). *A countable reduced p-group A has the property that any two of its direct decompositions have isomorphic refinements if and only if* $A^1 = 0$, *i.e. A is* Σ*-cyclic.*

Proof. If $A^1 = 0$, i.e. if A is a direct sum of cyclic p-groups, then evidently, its direct decompositions have isomorphic refinements: we decompose each summand into the direct sum of cyclic p-groups.

Conversely, assume that $A^1 \neq 0$. We decompose the UK-matrix $\mathbb{U}(A)$ into the sum of two matrices $\mathbb{U}(A) = \mathbb{M}_1 + \mathbb{M}_2$ such that they satisfy conditions (i)–(iii) and have never non-zero entries at the same location. Next, define matrices $\mathbb{M}'_1, \mathbb{M}'_2$ by switching the first rows of $\mathbb{M}_1$ and $\mathbb{M}_2$, while keeping the rest of the rows intact. There are countable reduced p-groups A_1, A_2, C_1, C_2 with UK-matrices $\mathbb{M}_1, \mathbb{M}_1, \mathbb{M}'_1, \mathbb{M}'_2$, respectively, and by Ulm's theorem we have $A \cong A_1 \oplus A_2 \cong C_1 \oplus C_2$. However, no summand of A_i that contains elements $\neq 0$ of infinite height can be isomorphic to such a summand of C_i for $i = 1, 2$. $\square$

It is no longer true for p-groups of higher cardinalities that they are isomorphic whenever their corresponding Ulm factors are isomorphic.

Example 1.13 (Kulikov [2]). Let $G_0 = \overline{B}$ be the torsion-complete p-group with the standard basic subgroup $B = \oplus_{n=1}^{\infty} \langle b_n \rangle$ ($o(b_n) = p^n$), and G_1 the direct sum of $2^{\aleph_0}$ cyclic groups $\langle g_i \rangle$ ($i \in I$) of order p. Two groups, A and C, will be defined with G_0, G_1 as Ulm sequence such that $|A[p]/A^1| = \aleph_0$ and $|C[p]/C^1| = 2^{\aleph_0}$. Let $G_0/B = \oplus_{i \in I} D_i$ ($D_i \cong \mathbb{Z}(p^\infty)$), and select a canonical generating set $d_{in} \in D_i$ of order p^n ($n \in \mathbb{N}$) for each i. Now A is generated by all of b'_n, d'_{in}, g'_i subject to the same relations as the corresponding b_n, d_{in}, g_i except that $pd_{i1} = 0$ is replaced by $pd'_{i1} = g'_i$ for every $i \in I$. Furthermore, for C, we write $I = I_1 \cup I_2$ with disjoint subsets, both of cardinality $2^{\aleph_0}$, and keep the same generating set and relations including $pd_{i1} = 0$ but only for $i \in I_1$, and put $pd_{i1} = g'_i$ for $i \in I_2$. The groups A and C cannot be isomorphic, because $|A[p]/A^1| = |B[p]| = \aleph_0$, while $|C[p]/C^1| \geq |I_1| = 2^{\aleph_0}$.

Example 1.14 (Richman [3]). Let $\overline{B}$ and $\overline{C}$ be torsion-complete groups with basic subgroups $B = \oplus_{n \in \mathbb{N}} B_n$ ($B_n \cong \mathbb{Z}(p^n)$) and $C = \oplus_{n \in \mathbb{N}} C_n$ ($C_n \cong \mathbb{Z}(p^{2n})$), respectively. Then $G = \overline{B}/B[p]$ and $H = \overline{C} \oplus \overline{C}/C[p]$ are extensions of an elementary p-group by a torsion-complete group. They have the same UK-invariants: the nth invariant is 1 for all $n < \omega$, and the ωth invariant is $2^{\aleph_0}$. However, G and H are not isomorphic: $G[p]/G^1$ is countable, while $H[p]/H^1$ has the power of the continuum.

For larger cardinalities, we have the following theorem which can be derived from Theorem 1.6 with only set-theoretical argument. It will be restated as a Theorem 6.7, so we skip its proof now.

Theorem 1.15 (Kolettis [1]). *If the p-groups A and C are direct sums of countable groups, then they are isomorphic if and only if their corresponding UK-invariants are equal.* $\square$

★ **Notes.** Prüfer's group $H_{\omega+1}$ (defined in 1923) was the first example for a reduced p-group with non-zero elements of infinite height. A decade later, in 1933 Ulm [1] used ingenious matrix-theoretical arguments to prove that the Ulm factors of a countable reduced p-group determine the group up to isomorphism. Ulm translated the problem from p-groups into infinite-dimensional linear algebra, and applied (his thesis supervisor) Toeplitz's theory to obtain a kind of normal form. Subsequently in 1935, Zippin [1] published a group-theoretical proof, and at the same time established the important existence Theorem 1.9. Surprisingly, no fruit has been born out of this remarkable theory for a quarter of a century, and even applications were very scarce. In 1960 the situation changed with Kolettis [1] where Ulm's theory was extended to direct sums of countable p-groups. The elegant proof by Kaplansky–Mackey [1] opened the door for substantial generalizations of Ulm's theorem. Hill [2] and Richman–Walker [2] show that Kolettis' theorem can be derived from Ulm's just by pure set-theoretical argument.

There are numerous results in the literature involving countable p-groups, and their direct sums. We will see some later on. In these cases, luckily, the question of isomorphy boils down to the equality of the UK-invariants.

Rogers [1] notes that—ignoring addition—a p-group A becomes a tree under multiplication by p: the nodes are the elements of A, and two nodes $a, b \in A$ are connected by an oriented edge if $a = pb$. She defines UK-invariants, and derives Ulm's theorem. This idea goes back to Crawley–Hales [1, 2], and is explored more fully in the paper Hunter–Richman–Walker [3].

Crawley [3] points out that p-groups with a finite number of Ulm factors are determined by the UK-invariants even if they are uncountable, but still Σ-cyclic (this is a consequence of Theorem 12.2). However, Lemma 12.3 will show that this is no longer true if the Ulm factors are not Σ-cyclic. In another paper [2] he shows that a reduced countable torsion group has the cancellation (substitution) property if and only if all of its UK-invariants are finite for all primes. In the same vein, Göbel–May [1] describe when a direct sum of countable p-groups is cancellable.

Szmielew [1] initiated a model-theoretical study of abelian groups. Barwise–Eklof [1] considered the classification problem of p-groups in the infinitary language $L_{\infty\omega}$, and show that the (modified) UK-invariants determine an *arbitrary* p-group up to $L_{\infty\omega}$-equivalence. This is a genuine generalization of Ulm's theorem, since for countable groups, $L_{\infty\omega}$-equivalence implies isomorphism.

Exercises

(1) If $G \leq H$ are subgroups of a p-group A, then $f_\sigma(A, G) \geq f_\sigma(A, H)$ for all σ.

(2) Let $A = B \oplus C$ be a p-group. Then $f_\sigma(A, B) = f_\sigma(C)$ for each ordinal σ.

(3) If σ is a countable ordinal and $a \in p^\sigma A$, then a embeds in a countable subgroup C of A such that $a \in p^\sigma C$.

(4) (a) For each countable ordinal τ, the set of non-isomorphic countable p-groups of length τ has the power of the continuum.
(b) Is the same true for the set of *all* non-isomorphic countable p-groups?

(5) Let A, C be countable p-groups. State a necessary and sufficient condition on their UK-invariants for A having (a) a subgroup isomorphic to C; (b) a summand isomorphic to C.

(6) (a) For every countable ordinal τ, there exists a countable reduced p-group $M(\tau)$ of length τ such that every countable reduced p-group of length $\leq \tau$ is isomorphic to a summand of $M(\tau)$.
(b) There is a reduced p-group M of length ω_1 and of cardinality $\aleph_1$ such that every countable reduced p-group is isomorphic to a summand of M.

(7) (Kaplansky) (a) If A is a countable p-group, and if C satisfies $A \oplus A \cong C \oplus C$, then $A \cong C$.
(b) For a countable p-group A with finite UK-invariants, $A \oplus B \cong A \oplus C$ implies $B \cong C$, where B, C are countable p-groups.
(c) The claim in (b) may fail if A has infinite UK-invariants.

(8) Find precise conditions, in terms of the UK-invariants, for a countable reduced p-group A to satisfy $A \oplus A \cong A$. Then also $A^{(\aleph_0)} \cong A$.

(9) (Irwin–Walker) Let H be a $p^\sigma A$-high subgroup of the p-group A, for some ordinal σ. Show that $f_\rho(H) = f_\rho(A)$ for $\rho < \sigma$.

(10) Let A be a reduced countable p-group, and C a summand of A. Suppose that $p^\sigma A = p^\sigma C \oplus X$ holds for some ordinal σ and a subgroup X. Then $A = C \oplus B$ for some B with $p^\sigma B = X$.

(11) Let A be a p-group of length $\tau \geq \omega$, and $C \cong \mathbb{Z}(p^n)$ a cyclic group. There exists a group G such that $p^\tau G \cong C$ and $G/p^\tau G \cong A$. [Hint: distinguish according as the last Ulm factor of A is bounded or not.]

2 Nice Subgroups

The insight one obtains from the Ulm-Zippin theory guides the in-depth study of p-groups of higher cardinalities. It was Hill who isolated the concept of nice subgroup in an effort to find conditions under which Ulm's theorem extends to a larger class of p-groups. A close analysis of the Kaplansky–Mackey proof of Ulm's theorem (see Lemma 1.5) led him to the discovery of this significant type of subgroup which embodies the property of finite subgroups relevant to the proof. We shall see that the abundance of such subgroups in a p-group will make it possible to prove an Ulm type theorem for a class of uncountable p-groups.

Nice Subgroups A subgroup N of a p-group A is called **nice** if every coset of A mod N contains an element proper with respect to N. In other words, every coset $a + N$ ($a \in A \setminus N$) contains an $a + x$ ($x \in N$) such that $h_A(a + x) = h_{A/N}(a + N)$.

We point out right away that in order to check the niceness of a subgroup N, we need to look only at cosets of limit heights in A/N. In fact, first of all, every element in a coset of height 0 has 0 height. Applying transfinite induction, suppose that we know that cosets of heights $\leq \sigma$ contain elements proper with respect to N, and $h_{A/N}(a + N) = \sigma + 1$. Then $pb + N = a + N$ for some $b \in A$ with $h_{A/N}(b + N) = h_A(b) = \sigma$ (by hypothesis). Thus $h_A(pb) \geq \sigma + 1$, and since strict inequality is out of question, we see that pb is proper with respect to N. The case of limit ordinals is included in the hypothesis.

The next lemma offers a most important characterization of niceness.

Lemma 2.1 (Hill [11]). *A subgroup N of a p-group A is nice exactly if*

$$p^\sigma(A/N) = (p^\sigma A + N)/N \quad \textit{for every (limit) ordinal } \sigma \textit{ and } \infty. \tag{11.2}$$

Proof. Observe that $p^\sigma(A/N)$ is the set of cosets whose heights in A/N are $\geq \sigma$, while $(p^\sigma A + N)/N$ is the image of $p^\sigma A$ under the canonical map $A \to A/N$. Hence the inclusion $\geq$ holds for every subgroup N of A. Now N is nice in A if and only if every coset of height σ can be represented by an element of A of height σ. Thus the reverse inclusion holds for every σ exactly if N is nice in A. (The reduction to limit ordinals follows from the remarks before the lemma.) □

We can rephrase the last lemma by saying that in the exact sequence $0 \to N \to A \to C \to 0$ the subgroup N is nice if and only if the induced sequence $p^\sigma N \to p^\sigma A \to p^\sigma C \to 0$ is exact for every σ (the emphasis being on the exactness at $p^\sigma C$).

Example 2.2.

(a) Finite subgroups are trivially nice. More generally, finite extensions of a nice subgroup are nice. This follows from the simple observation that a finite extension cannot create new cosets of limit heights.
(b) If A/N is a separable p-group, then N is nice in A.

Example 2.3. For every ordinal σ, the subgroup $p^\sigma A$ is nice in the p-group A. Recall that heights $< \sigma$ are preserved passing mod $p^\sigma A$.

Properties of Nice Subgroups Let us acquaint ourselves with some elementary properties of niceness.

(A) *Direct summands are nice subgroups.*
(B) *Subgroups closed in the p-adic topology are nice.* (In this case, the cosets have finite heights only.) In particular, in a torsion-complete group, a pure subgroup is nice if and only if it is a summand.
(C) *The subgroup $\oplus_{i\in I} N_i$ is nice in $\oplus_{i\in I} A_i$ exactly if N_i is nice in A_i for each $i \in I$.*
(D) *A nice subgroup N of A need not be nice in a subgroup B that lies between N and A.* This can be seen in a divisible group A where all subgroups are nice.
(E) *The property of being nice is not transitive, in general* (see Exercise 7), but the following special case is true: *a subgroup N of $p^\sigma A$ is nice in A if and only if it is nice in $p^\sigma A$.* This is a consequence of the relation $h_A(a) = \sigma + h_{p^\sigma A}(a)$ $(a \in p^\sigma A)$. Moreover, if N is nice in A, then $N \cap p^\sigma A$ is nice in $p^\sigma A$ for every σ.

Visibly, (D)–(E) show that niceness does not behave like purity. However, the following two useful properties are reminiscent of purity.

Lemma 2.4 (Hill [11]). *Let A be a p-group, and M, N subgroups with $N \le M$. Then the following hold:*

(i) *if M is nice in A, then M/N is nice in A/N;*
(ii) *if N is nice in A and M/N is nice in A/N, then M is nice in A.*

Proof.

(i) By our remark above, it suffices to check niceness at limit ordinals. We use induction on the height. Let $h_{A/N}(a + M/N) = \sigma$ be a limit ordinal, and assume that all cosets mod M/N of heights $< \sigma$ contain cosets mod N proper with respect to M/N. This means that for every ordinal $\rho < \sigma$, there exists an $x_\rho \in M$ such that $h_{A/N}(a + x_\rho + N) > \rho$. Therefore, $h_{A/M}(a + M) \ge \sigma$, and since the reverse inequality is trivially true, $h_{A/M}(a + M) = \sigma$. By hypothesis, M is nice in A, so there is an $x \in M$ such that $h_A(a + x) = \sigma$. It follows that $h_{A/N}(a + x + N) = \sigma$, since strict inequality $>$ is impossible.
(ii) By the hypotheses of (ii), $h_{A/M}(a + M) \ge \sigma$ implies the existence of a $y \in M$ satisfying $h_{A/N}(a + y + N) = \sigma$, and of a $z \in N$ satisfying $h_A(a + y + z) = \sigma$. □

Consequently, if N is a nice subgroup of A, then under the natural correspondence between subgroups of A containing N and subgroups of A/N, nice subgroups correspond to nice subgroups.

★ **Notes.** Niceness is a crucial concept in the theory. We shall see that nice subgroups play crucial role also in mixed groups; see Chapter 15.

Exercises

(1) If M, N are subgroups of a p-group A, and if $a \in A$ is proper with respect to both of them, then a is proper with respect to $M \cap N$ as well.
(2) A subgroup N is nice in the p-group A if and only if the equality $N + p^\sigma A = \cap_{\rho<\sigma}(N + p^\rho A)$ holds for all limit ordinals σ.
(3) If N is nice in A, then so is $N + p^\sigma A$ for every σ.
(4) (Hill) Let A be a p-group. For a subgroup N of A to be nice it is necessary and sufficient that $N \cap p^\sigma A$ is nice in $p^\sigma A$ and $(N + p^\sigma A)/p^\sigma A)$ is nice in $A/p^\sigma A$ for all ordinals σ.
(5) A p-group has the property that all of its subgroups are nice if and only if it is a direct sum of a bounded group and a divisible group.
(6) The union of an ascending chain of nice subgroups need not be nice.
(7) Let B denote an unbounded Σ-cyclic p-group, and $\overline{B}$ its torsion-completion. Show that (a) $B[p]$ is nice in $\overline{B}[p]$ (it is a summand); (b) $\overline{B}[p]$ is nice in $\overline{B}$; but (c) $B[p]$ is not nice in $\overline{B}$.
(8) Give an example where $N < M < A$, M/N is nice in A/N, but M fails to be nice in A.
(9) The extensions of a p-group A by a p-group C in which A is a nice subgroup form a subgroup in $\mathrm{Ext}(C, A)$. [Hint: Baer sum.]

3 Simply Presented p-Groups

We now embark on an ambitious generalization of Ulm's theorem: this section and the three following ones are devoted to the theory of p-groups that can be characterized by their UK-invariants. This theory is undoubtedly one of the major achievements in the theory of p-groups of arbitrary cardinality; it provides the most penetrating results known today on p-groups. We develop the theory on parallel lines:

1. simply presented p-groups;
2. p-groups with nice systems (or nice composition series) of subgroups;
3. balanced-projective p-groups; and
4. totally projective p-groups.

Simple Presentation In this section, we deal with p-groups that have a special kind of presentation. As a motivation, let us recall that a Σ-cyclic p-group can be presented by a set of symbols x as the set of generators (corresponding to a basis), with defining relations of the form $p^n x = 0$ for certain integers $n > 0$; i.e., all the relations involve exactly *one* generator. In contrast, no particular property is needed for a group A to be presented with defining relations containing no more than *three* generators. Indeed, if we choose the set of elements of A as generating system, then the existing equalities like $a + b = c$ for $a, b, c \in A$ provide relations of the desired kind. Our present aim is to investigate those p-groups that are in between, i.e. which can be presented by a set of generators such that all the defining relations contain at most *two* generators.

Thus given a set X of generators, we now assume that every defining relation is of the form $nx = 0$ or $nx = my$ where $x, y \in X$ and $m, n \in \mathbb{N}$. Since we are dealing with p-groups, we can get rid of integers prime to p, so that we may assume that every defining relation is of the form

$$p^n x = 0 \qquad \text{or} \qquad p^n x = p^m y \tag{11.3}$$

where $x, y \in X$. We will call a p-group A defined in this way a **simply presented** p-group, and write $A = \langle X; \Xi \rangle$ where Ξ stands for the collection of defining relations. These groups, under the name of *T-groups*, were introduced by Crawley–Hales [1].

Faithful Simple Presentation We will need more control on the elements, and therefore we simplify the presentation (at the expense of enlarging the generating set). First, we replace $p^n x = 0$ by the relations

$$px = x_1, \ px_1 = x_2, \ \dots, \ px_{n-1} = 0$$

after adjoining $x_1, \dots, x_{n-1}$ to X as additional generators. The same can be done for the relations of the form $p^n x = p^m y$. As a result, we may assume without loss of generality that every relation in Ξ is of the form $px = 0$ or $px = y$ $(x, y \in X)$.

Next, in order to avoid the nuisance of working with generators that become 0 in A, and different generators that become equal in A, first we delete those $x \in X$ that are 0 in A, and replace them by 0 in the relations. By doing so, we give the same name in the relations to generators $x \neq y$ in X that become equal in A. To summarize, we have now a presentation $A = \langle X; \Xi \rangle$ such that

(i) each defining relation is of the form $px = 0$ or $px = y$ with $x, y \in X$;
(ii) if $x \in X$, then $x \neq 0$ in A;
(iii) if x, y are different elements in X, then $x \neq y$ in A.

A presentation satisfying (i)–(iii) will be called **faithful**. Thus if the simple presentation of a p-group is faithful, then every generator occurs in exactly one relation with coefficient p, but may occur in numerous relations with coefficient 1.

In this section we will assume (unless stated otherwise) that *all the presentations considered are faithful*.

Example 3.1.

(a) Cyclic groups of order p^n are simply presented: $\mathbb{Z}(p^n) = \langle x_1, x_2, \dots, x_n; px_1 = 0, px_2 = x_1, \dots, px_n = x_{n-1}\rangle$ (faithful presentation).
(b) Quasi-cyclic groups are simply presented: $\mathbb{Z}(p^\infty) = \langle x_1, \dots, x_n, \dots; px_1 = 0, px_2 = x_1, \dots, px_n = x_{n-1}, \dots\rangle$ (faithful presentation).
(c) The Prüfer group $H_{\omega+1}$ is simply presented: $H_{\omega+1} = \langle x_0, x_1, \dots, x_n, \dots; px_0 = 0, px_1 = x_0, \dots, p^n x_n = x_0, \dots\rangle$ (this is not a faithful presentation).

Example 3.2. A clever method of constructing simply presented groups was suggested by Walker [4]. For an ordinal β, we define a simply presented p-group P_β of length $\beta + 1$ as follows. The generators are finite "strings" $(\beta\beta_1 \cdots \beta_n)$ for $n \geq 0$ where $\beta > \beta_1 > \dots > \beta_n$ is a strictly decreasing sequence of ordinals. The defining relations are:

$$p(\beta\beta_1 \dots \beta_n) = (\beta\beta_1 \dots \beta_{n-1}) \quad \text{for } n \geq 1, \quad \text{and} \quad p(\beta) = 0.$$

It is obvious that, for every ordinal β, P_β is simply presented, and by induction it is not difficult to verify that its length is $\beta + 1$.

A faithful simple presentation gives rise to a partial order in the set X of generators. For $x, y \in X$ we define

$$y < x \qquad \text{provided} \quad p^n x = y \quad \text{for some } n > 0.$$

This is a partial order that satisfies the minimum condition. The minimal elements are precisely those $z \in X$ for which $pz = 0$ is a relation in Ξ. Note that the set $X_z = \{x \in X \mid x \geq z\}$ is a tree with nodes corresponding to the generators in X_z, and edges representing the relations between these generators in Ξ.

Properties of Simple Presented Groups Divisible groups are simply presented, but the main interest lies in the reduced case. To elicit properties of reduced simply presented p-groups, we now record a few basic facts on them. (All groups are *reduced*, and simple presentations are *faithful*.)

(A) *Direct sums of simply presented groups are simply presented.* This is obvious.
(B) *If M is the set of minimal elements of X in the partial ordering, then $X = \dot\bigcup_{z \in M} X_z$ (disjoint union) and*

$$A = \oplus_{z \in M} \langle X_z \rangle.$$

This is easy to see, since every $x \in X$ belongs to some X_z, and X_z, X_y are disjoint sets provided $y \neq z$ in M.
(C) *A simply presented p-group of limit length is the direct sum of simply presented groups of smaller lengths.* This can be derived at once from (B) where the subgroups $\langle X_z \rangle$ with $z \in M$ are of smaller lengths.
(D) *Every non-zero element a of a simply presented p-group can be written uniquely in the form*

$$a = r_1 x_1 + \dots + r_k x_k \quad (k \geq 1) \tag{11.4}$$

where $x_1, \dots, x_k$ are distinct elements of X and $0 < r_i < p$ for $i = 1, \dots, k$. In view of (i)–(iii), the existence of such a representation is evident. To verify uniqueness, suppose $a = r_1x_1 + \cdots + r_kx_k = s_1x_1 + \cdots + s_kx_k$ holds for distinct $x_1, \dots, x_k \in X$ and for integers $0 \le r_i, s_i < p$. Let x_1 be a maximal one among $x_1, \dots, x_k$ in the natural partial order of X. There is a homomorphism ϕ of $\langle x_1, \dots, x_k\rangle$, and hence of A, into $\mathbb{Z}(p^\infty)$ mapping x_1 upon an element of order p and $x_2, \dots, x_n$ to 0. Now $\phi a = r_1(\phi x_1) = s_1(\phi x_1)$ entails $r_1 = s_1$, and the rest follows by induction.

(E) *If* (11.4) *is the unique representation of $a \neq 0$ in a simply presented p-group A, then $a \in p^\sigma A$ if and only if $x_i \in p^\sigma A$ for $i = 1, \dots, k$.* Proof of necessity by induction on σ. The assertion being trivially true for $\sigma = 0$, suppose that $a = pb$ where $b = s_1y_1 + \cdots + s_\ell y_\ell \in p^\sigma A$ for some σ and distinct $y_1, \dots, y_\ell \in X$; here, $0 < s_j < p$ for $j = 1, \dots, \ell$. By induction hypothesis $y_j \in p^\sigma A$ for all j. Hence $a = s_1(py_1) + \cdots + s_\ell(py_\ell)$, and the set $\{x_1, \dots, x_k\}$ is coming from the set of non-vanishing py_j. Hence necessarily $x_i \in p^{\sigma+1}A$, and we are done.

In particular, $h(a) = \min\{h(x_1), \dots, h(x_k)\}$ if a is as in (11.4).

(F) *If A is simply presented, then so are $A/p^\sigma A$ and $p^\sigma A$ for every σ. Hence also the Ulm factors are simply presented.* This follows from (E).

(G) *The Ulm factors of a simply presented p-group are Σ-cyclic groups.* First assume A is separable. Then the minimal elements in a faithful generating system are of finite height, so from (E) it follows that the heights of the elements in each $\langle X_z\rangle$ in the decomposition (B) are bounded. Therefore, the groups $\langle X_z\rangle$ are Σ-cyclic. In the general case, by (F) the Ulm factors are simply presented, they are also separable, so also Σ-cyclic.

(H) *An infinite simply presented p-group A satisfies*

$$|A| = |A_0| = \sum_{n<\omega} f_n(A),$$

where $A_0 = A/A^1$ is the initial Ulm factor. As the Ulm factors are Σ-cyclic groups, the second equality is evident. To prove the first, we induct on the length λ of the Ulm sequence. If $A^\lambda \neq 0$, but $A^{\lambda+1} = 0$, then Theorem 1.9(c) in Chapter 10 implies $|A/A^\lambda| \ge |A^\lambda|$. As A/A^λ is also simply presented and has the same initial Ulm factor, by induction we obtain $|A| = |A/A^\lambda||A^\lambda| = |A/A^\lambda| = |A_0|$. If the first ordinal λ with $A^\lambda = 0$ is a limit ordinal, then $A = \oplus A_i$ with summands for which the induction hypothesis applies. Hence the claim is clear.

A consequence of (H) is that the ranks of the Ulm factors of simply presented groups never increase with the indices: $\mathrm{rk}\, A_\alpha \ge \mathrm{rk}\, A_\beta$ if $\alpha < \beta$.

The following lemma is crucial.

Lemma 3.3. *The generalized Prüfer groups H_σ are simply presented.*

Proof. From the definition of H_σ it is manifest that if H_σ is simply presented, then so is $H_{\sigma+1}$. Invoking (C) and (A) at limit ordinals, the claim is established. □

We will refer to the next result in the proof of the main Theorem 3.6.

Lemma 3.4. *Let $A = \langle X; \Xi \rangle$ be a simply presented p-group, and Y a subset of X. The subgroup $N = \langle Y \rangle$ is a nice subgroup of A.*

Proof. Let $a \in A \setminus N$, and write $a = r_1x_1 + \cdots + r_kx_k + s_1y_1 + \cdots + s_\ell y_\ell$ as in (11.4) where x_i and y_j are distinct elements in $X \setminus Y$ and in Y, respectively; here r_i, s_j are positive integers $< p$. We claim that $b = r_1x_1 + \cdots + r_kx_k \in a + N$ is proper with respect to N. Pick any $c = t_1y'_1 + \cdots + t_my'_m \in N$, and calculate the height of the element $b + c \in a + N$. Evidently, we have $h(b + c) \leq \min_{i,j}\{h(x_i), h(y'_j)\} \leq \min_i\{h(x_i)\} = h(b)$. $\square$

From the preceding proof it is clear that an $x \in X$ satisfying $px \in \langle Y \rangle$ is proper with respect to $\langle Y \rangle$.

Crawley–Hales Theorem We now have all the ingredients to establish the main structure theorem. This remarkable result does no less than characterizes a large class of p-groups in terms of cardinal invariants.

Theorem 3.5 (Crawley–Hales [1]). *Two simply presented p-groups are isomorphic if and only if their corresponding UK-invariants are identical.*

Some advantage is to be gained by proving this result in the following strengthened form, in order to be applicable to mixed groups as well.

Theorem 3.6 (Hill [11], Walker [3]). *For a prime p, let G, H be nice subgroups of A, C, respectively, such that $A/G = \langle X; \Sigma_X \rangle$ and $C/H = \langle Y; \Sigma_Y \rangle$ are simply presented p-groups of length $\leq \tau$. If*

(i) *there is a p-height-preserving isomorphism $\phi\colon G \to H$; and*
(ii) *the Hill invariants for the prime p are equal:*

$$f_\sigma(A, G) = f_\sigma(C, H) \qquad \textit{for every } \sigma \leq \tau,$$

then ϕ extends to an isomorphism $\phi^*: A \to C$.

If only an inequality $f_\sigma(A, G) \leq f_\sigma(C, H)$ holds for every σ, then there is an extension ϕ^* which is a homomorphism.

Proof. Since certain restraints are needed to control the step-wise extension of ϕ, we select, for each σ, an arbitrary, but fixed isomorphism

$$\alpha_\sigma\colon\ p^\sigma A[p]/G(\sigma) \to p^\sigma C[p]/H(\sigma).$$

Consider the set of all pairs (G_ν, ϕ_ν) subject to the following conditions:

(a) $G_\nu = \langle G, X_\nu \rangle$ (for some subset $X_\nu \subseteq X$) is a nice subgroup of A;
(b) ϕ_ν is a height-preserving isomorphism of G_ν with a nice subgroup $H_\nu = \langle H, Y_\nu \rangle$, where $Y_\nu \subseteq Y$;
(c) $\phi_\nu \restriction G = \phi$;
(d) for each σ, α_σ induces an isomorphism $G_\nu(\sigma)/G(\sigma) \to H_\nu(\sigma)/H(\sigma)$.

The set of pairs (G_ν, ϕ_ν) is not empty, and is partially ordered in the obvious way: $(G_\nu, \phi_\nu) \le (G_\mu, \phi_\mu)$ exactly if $G_\nu \le G_\mu$ and $\phi_\nu = \phi_\mu \restriction G_\nu$. Application of Zorn's lemma yields a maximal pair (G^*, ϕ^*), where ϕ^* is a height-preserving isomorphism of $G^* = \langle G, X^* \rangle$ with $H^* = \langle H, Y^* \rangle$ for some $X^* \subseteq X$ and $Y^* \subseteq Y$. Note that (d) guarantees that $f_\sigma(A, G^*) = f_\sigma(C, H^*)$ for every σ. We claim that $X^* = X$ and $Y^* = Y$.

Suppose the contrary, i.e. some $x \in X$ is missing from X^*. Then there is also such an x with $px \in G^*$. By making use of Lemma 1.5, we can extend ϕ^* to a height-preserving isomorphism $\phi_0^* : G_0^* = \langle G^*, x \rangle \to H_0^* \le C$ (where $H^* \le H_0^*$) such that (d) is preserved. There are a finite number of generators $y_1, \dots, y_m \in Y$ such that $H_0^* \le H_1^* = \langle H, y_1, \dots y_m \rangle$. Now we invert the procedure: by the niceness of H_0^*, Lemma 1.5 is applicable to H_0^*, ϕ_0^{*-1}, and to the generators $y_1, \dots, y_m$ to obtain a finite extension G_1^* of G_0^* with a height-preserving isomorphism $\phi_1^* : G_1^* \to H_1^*$ such that $\phi_1^* \restriction G_0^* = \phi_0^*$ and $\alpha_\sigma(G_1^*(\sigma)/G(\sigma)) = H_1^*(\sigma)/H(\sigma)$. If $G_1^* \ne A$, then we select $x_1, \dots, x_n \in X$ such that $G_1^* \le \langle G_0^*, x_1, \dots, x_n \rangle = G_2^*$. We repeat this process, alternately in C and A, to find a countable ascending chain $G^* < G_0^* \le G_1^* \le \dots$ of subgroups of A along with height-preserving isomorphisms $\phi_k : G_k^* \to H_k^* \le C$ satisfying the requisite conditions, in particular, $\phi_{k+1}^* \restriction G_k^* = \phi_k^*$ for $k < \omega$. Setting $G^{**} = \cup_{k<\omega} G_k^*$ and $\phi^{**} = \cup_{k<\omega} \phi_k^*$, we get a pair (G^{**}, ϕ^{**}) that also belongs to the set of pairs under consideration. This contradicts the maximal choice of G^*, so we must have $G^* = A$.

To show that the simply presented group H^* is all of C, suppose $y \in Y$ of height σ is missing from H^*. As $py = u \in Y \cap H^*$ can be assumed, there is $v \in Y \cap H^*$ such that $pv = u$ and $h(v) \ge \sigma$. By construction, $H^*(\sigma) = 0$, which amounts to $p^\sigma C[p] \le p^{\sigma+1}C + H^*$, so we can write $y - v = z + w$ with $z \in C$, $h(z) \ge \sigma + 1$ and $w \in H^*$. We must have $h(w) \ge \sigma$, thus if we write $z = y - v - w$ in the canonical form (11.4), using the generators Y, then y will have coefficient 1, so the right-hand side cannot equal z.

The final claim follows from our argument. The proof of Theorem 3.6 is complete, and Theorem 3.5 is the special case $G = H = 0$. ☐

UK-Invariants of Simply Presented p-Groups The structure Theorem 3.5 will be much more satisfactory if we can answer the question as to which well-ordered sequences

$$\kappa_0, \kappa_1, \dots, \kappa_\sigma, \dots \qquad (\sigma < \tau)$$

of cardinals can be realized as UK-invariants of simply presented p-groups. We note right away that in view of the existence of generalized Prüfer groups, there is no restriction on the length τ of the sequences. In order to state the precise condition we need a new definition.

We call a function ψ_τ from the set of ordinals $\sigma < \tau$ to the cardinals **τ-admissible** if it satisfies

(i) $\tau = \sup\{\sigma + 1 \mid \psi_\tau(\sigma) \ne 0\}$; and
(ii) $\sum_{\sigma+\omega\le\rho} \psi_\tau(\rho) \le \sum_{n<\omega} \psi_\tau(\sigma + n)$ for all σ with $\sigma + \omega < \tau$.

Theorem 3.7 (Crawley–Hales [1], Hill [11]). *Let ψ_τ be a function from the ordinals $\sigma < \tau$ to a set of cardinals. There exists a simply presented p-group A of length τ such that $f_\sigma(A) = \psi_\tau(\sigma)$ for all $\sigma < \tau$ if and only if the function ψ_τ is τ-admissible.*

Proof. For necessity, we show that ψ_τ satisfies (i)–(ii) if A is simply presented of length τ and $\psi_\tau(\sigma) = f_\sigma(A)$. (i) is obvious. (ii) is nothing else than Theorem 1.9(c) in Chapter 3 combined with (H), observing that the UK-invariants of the αth Ulm factor A_α are $f_\beta(A)$ where $\omega\alpha \leq \beta < \omega(\alpha+1)$, so $\operatorname{rk} A_\alpha$ is equal to the sum of these cardinals.

To prove sufficiency, let ψ_τ be a τ-admissible function; by adjoining 0's if needed we may assume that $\tau = \omega\lambda$ is a limit ordinal. First of all, note that the future Ulm factors A'_α can be constructed as Σ-cyclic groups with invariants $\psi_\tau(\beta)$ where $\omega\alpha \leq \beta < \omega(\alpha+1)$. Hence the case $\tau \leq \omega$ is settled. Write $\lambda = \lambda' + n$ where λ' is a limit ordinal and $n < \omega$. We distinguish three cases.

Case I: $n \geq 2$. The first step is to choose a basis $\{b_i \mid i \in I\}$ in A'_λ. By induction hypothesis, there is a simply presented G whose Ulm factors are $G_\alpha \cong A'_\alpha$ for $\alpha \leq \lambda - 2$, while $G_{\lambda-1}$ is $A'_{\lambda-1}$, but modified as follows. Decompose $A'_{\lambda-1} = \oplus_{i\in I} H_i$ into unbounded summands; (ii) ensures that this can be done with I as index set. If $x \in H_i$ is a basis element, then we change it to be x' of order $o(x)o(b_i)$, to get $G_{\lambda-1}$ as modified $A'_{\lambda-1}$. Finally, A is defined as the factor group of G by identifying all the $o(x)x'$ for the same i, for every i. It is easily seen that A is simply presented, and $A_\alpha \cong A'_\alpha$ for each $\alpha \leq \lambda$.

Case II: $n = 0$, *i.e.* λ *is a limit ordinal.* We will show below: ψ_τ can be written as a sum $\sum \psi_{\tau_i}$ of τ_i-admissible functions of lengths $\tau_i < \tau$. If A_i is a simply presented p-group with the ψ_i as the function for the sequence of UK-invariants, then $A = \oplus_i A_i$ will have the prescribed function ψ_τ.

The proof of the claim on τ-admissible functions for limit ordinals τ requires a delicate set-theoretical argument. In the set of cardinals $\mu_\rho = \sum_{\rho\leq\sigma<\tau} \psi_\tau(\sigma)$ for all $\rho < \tau$, let μ_γ denote the smallest with the smallest index. Set $J = \{\beta \mid \gamma \leq \beta < \tau\}$, and consider the cardinals

$$\nu_\rho = \min_{k<\omega} \sum_{k\leq n<\omega} \psi_\tau(\rho + n)$$

for all limit ordinals $\rho < \tau$. It is an elementary exercise in cardinal arithmetic to show that the restriction ξ_ρ of ψ_τ to the interval $[\rho, \rho+\omega)$ decomposes as $\xi_\rho = \sum_{\beta\in J} \xi_{\rho\beta}$ such that $\sum_{k\leq n<\omega} \xi_{\rho\beta}(\rho+n) \geq \nu_\rho$ for all $k < \omega$. (This is essentially the same argument as the one needed to decompose a Σ-cyclic p-group of final rank ν into a direct sum of ν summands, each of final rank ν.) We extend $\xi_{\rho\beta}$ by putting $\xi_{\rho\beta}(\sigma) = 0$ whenever $\sigma < \rho$ or $\sigma \geq \rho + \omega$. Next, define $\eta_\beta = \sum_\rho \xi_{\rho\beta}$ for $\beta \in J$. Then we have $\psi_\tau = \sum_{\beta\in J} \eta_\beta$ and $\sum_{n<\omega} \eta_\beta(\sigma+n) \geq \sum_{\rho\geq\sigma+\omega} \psi_\tau(\rho)$ for every σ and for every β. Finally, let

$$\psi_\beta(\sigma) = \begin{cases} \eta_\beta(\sigma) & \text{if } \sigma < \beta, \\ \sum_{\gamma \le \beta} \eta_\gamma(\sigma) & \text{if } \sigma = \beta, \\ 0 & \text{if } \sigma > \beta. \end{cases}$$

Then ψ_β has length $\beta + 1 < \tau$, and $\sum_{\beta \in J} \psi_\beta = \psi_\tau$. In view of the inequalities

$$\sum_{n<\omega} \psi_\beta(\sigma + n) = \sum_{n<\omega} \eta_\beta(\sigma + n) \ge \sum_{\rho \ge \sigma + n} \psi(\rho) \ge \sum_{\rho \ge \sigma + n} \psi_\beta(\rho)$$

for every σ with $\sigma + \omega < \beta + 1$, we infer that ψ_β is $(\beta + 1)$-admissible.

Case III: $n = 1$. Before entering into the discussion of this case, it should be pointed out that a corollary to the proof in Case II is that the constructed simply presented group (we call it now G) decomposes into the direct sum of groups G_i such that for every ordinal $\rho < \lambda$ as many as $|J|$ of them have length $> \omega\rho$. By construction, this G has the Ulm sequence A'_α $(\alpha < \lambda)$. Again choosing a basis $\{b_i \mid i \in I\}$ in A'_λ, we observe that (ii) implies $|I| \le |J|$. As a result, the proof reduces to the case when the last Ulm factor is a cyclic group $\langle b_i \rangle$. It is not difficult to see that there is no loss of generality in assuming that each G_i has a last Ulm factor that is a cyclic group. Then the method of Case I can be applied, and the direct sum of the constructed groups will be as desired. □

Finally, we estimate the cardinality of the set of non-isomorphic simply presented groups of a given cardinality.

Corollary 3.8. *For every infinite cardinal κ, there exist 2^κ non-isomorphic simply presented p-groups of cardinality κ.*

Proof. We apply Theorem 3.7 with $\tau = \kappa$, and refer to its proof where the Ulm factors were chosen as unbounded Σ-cyclic groups. If we let each $\psi_\kappa(\sigma)$ be κ or 0 (or either only even or only odd invariants), then we have quite a freedom to satisfy the required condition for admissibility, so that we will have 2^κ different choices. Distinct choices mean non-isomorphic simply presented groups. (All constructed groups will have isomorphic basic subgroups if we fix the initial Ulm factor.) □

★ **Notes.** Simple presentation is not the historical route to the theory of p-groups that can be classified by their UK-invariants, but—as Hill points out—it is considered by many as the most aesthetic approach. The study of simply presented p-groups was initiated by Crawley–Hales [1]; they proved most of the main results on them.

Exercises

(1) Give a faithful simple presentation of $\mathbb{Q}$.

(2) A separable p-group is simply presented if and only if it is Σ-cyclic.

(3) If A is a p-group such that both $p^\sigma A$ and $A/p^\sigma A$ are simply presented, then so is A.
(4) If A is simply presented and reduced, then fin rk A = fin rk B for any basic subgroup B of A.
(5) A reduced simply presented p-group is fully transitive. [Hint: Theorem 3.6.]
(6) If A is a simply presented p-group, and C satisfies $A \oplus A \cong C \oplus C$, then $A \cong C$.
(7) Two simply presented p-groups are isomorphic if each of them is isomorphic to an isotype subgroup of the other (for isotypeness, see Sect. 5). [Hint: the UK-invariants of an isotype subgroup cannot be larger than those of the group.]
(8) Let P_β denote the group defined in Example 3.2. Prove that if $\beta = \gamma + \delta$, then $p^\gamma P_\beta = P_\delta$.

4 p-Groups with Nice Systems

Our main concern here is to make good our claim that p-groups with an abundance of nice subgroups can be classified by their UK-invariants. In this section, we are making a big step towards this goal.

Nice Composition Chains Let A be a p-group and

$$0 = N_0 < N_1 < \cdots < N_\nu < \cdots < N_\mu = A \tag{11.5}$$

a smooth chain of subgroups in A such that

(a) each N_ν is a nice subgroup of A; and
(b) $N_{\nu+1}/N_\nu \cong \mathbb{Z}(p)$ for every $\nu < \mu$.

Such a chain will be called a **nice composition chain** for A.

Nice Systems There is a stronger version of nice composition chain. By a **nice system** is meant an $H(\aleph_0)$-family $\mathcal{N}$ of nice subgroups (Hill [11]). It is an easy exercise to extract a nice composition chain from a nice system: just select from a nice system a smooth chain with countable factor groups, and then refine the chain by making use of Example 2.2. Surprisingly, the converse is also true: every group with a nice composition chain admits a nice system—this will follow from Theorem 5.9 below.

Example 4.1.

(a) Every countable p-group admits a nice system: the set of all finite subgroups is such a system. Recall that finite subgroups are always nice.
(b) Divisible groups have nice systems, since all of their subgroups are nice.
(c) Simply presented p-groups have a nice system: the set of subgroups generated by various subsets of the generating set.

Lemma 4.2 (Hill [11]). *The class of groups that admit nice systems is closed under direct sums and summands.*

Proof. Let $A = \oplus_{i\in I} A_i$ where each A_i has a nice system $\mathcal{N}_i$. Define $\mathcal{N}$ to consist of all subgroups $N = \oplus_{i\in I} N_i$ with $N_i \in \mathcal{N}_i$ $(i \in I)$. Then each N is nice in A, and it is straightforward to see that $\mathcal{N}$ is closed under taking sums. Finally, a countable subset X of A has non-zero projection X_i only for i in a countable subset $J \subseteq I$. For each $i \in J$, there is an $M_i \in \mathcal{N}_i$ such that $N_i \cup X_i \subseteq M_i$ and $|M_i/N_i| \leq \aleph_0$. Then $M \in \mathcal{N}$ obtained from N by replacing the N_i by M_i for $i \in J$ is as desired.

For summands, we appeal to Lemma 5.4 in Chapter 1 which tells us that a summand B of a group A with an $H(\aleph_0)$-family $\mathcal{N}$ also admits an $H(\aleph_0)$-family $\mathcal{M}$ of subgroups that are summands of groups in $\mathcal{N}$. Summands are nice subgroups, and summands of groups in $\mathcal{N}$ remain nice in B, so groups in $\mathcal{M}$ ought to be nice in B. □

The following is an important observation, a restatement of Example 3.1(c).

Proposition 4.3 (Crawley–Hales [1], Hill [7]). *The generalized Prüfer groups H_σ, and more generally, simply presented p-groups admit nice systems.*

Proof. From the construction of the generalized Prüfer groups it is clear that if H_σ has a system $\mathcal{N}$ of nice subgroups, then the preimages of the subgroups in $\mathcal{N}$ under the natural epimorphism $H_{\sigma+1} \to H_\sigma$ yield a nice system in $H_{\sigma+1}$. The limit case is obvious. The rest is a trivial corollary to Lemma 4.2. □

The next two results provide more information about the groups under consideration.

Lemma 4.4. *If A is a p-group such that both its nice subgroup N and the factor group A/N admit nice composition chains, then so does A.*

Proof. It suffices to refer to the simple fact that a nice composition chain in A/N lifts to a nice composition chain between N and A. □

Groups with Nice Composition Chains In the next few results we explore the implications of the hypothesis of having a nice composition chain. The following lemma was proved by Hill [11] for nice systems, and later by Fuchs [IAG] for nice composition chains.

Lemma 4.5. *Suppose A, C are reduced p-groups, and ϕ is a height-preserving isomorphism between a nice subgroup G of A and a subgroup H of C. Suppose that*

(i) *the factor group A/G has a nice composition chain, and*
(ii) *the Hill invariants satisfy $f_\sigma(A, G) \leq f_\sigma(C, H)$ for each ordinal σ.*

Then ϕ extends to a height-preserving isomorphism ψ of A with a subgroup of C.

Proof. For each σ, we pick a monic map $\alpha_\sigma : p^\sigma A[p]/G(\sigma) \to p^\sigma C[p]/H(\sigma)$. We also select a nice composition chain $G = G_0 < G_1 < \cdots < G_\nu < \cdots < G_\lambda = A$, and consider the set of all pairs (G_ν, ϕ_ν) such that

(a) ϕ_ν is a height-preserving isomorphism of G_ν with a subgroup H_ν of C;
(b) $\phi_\nu \restriction G = \phi$, and ϕ_ν is the restriction of $\phi_{\nu'}$ to G_ν if $\nu < \nu'$;
(c) for each σ, α_σ induces an isomorphism $G_\nu(\sigma)/G(\sigma) \to H_\nu(\sigma)/H(\sigma)$.

Evidently, there is a maximal pair (G_μ, ϕ_μ) in this set. (c) ensures that $f_\sigma(A, G_\mu) \leq f_\sigma(C, H_\mu)$ for each σ. If $\mu < \lambda$, then owing to (a), we are in the situation of Lemma 1.5, so $\phi_\mu : G_\mu \to H_\mu$ can be extended to a height-preserving isomorphism of $G_{\mu+1}$ with a subgroup of C such that (c) is not violated. This is a contradiction to the choice of μ, so $\mu = \lambda$, and $N_\mu = A$, proving the assertion. □

We derive the following immediate corollary.

Corollary 4.6. *The preceding lemma continues to hold if condition* (ii) *is dropped, ϕ is assumed not to decrease heights, and the extension ψ is required to be a non-height-decreasing homomorphism only.*

Proof. We apply Lemma 4.5 to the groups A and $A \oplus C$ with the nice subgroups G and $G \oplus H$. The UK-invariants satisfy condition (ii), and the homomorphism $\phi : G \to G \oplus H$ is height-preserving. Hence Lemma 4.5 guarantees that there is a height-preserving monomorphism $A \to A \oplus C$. The projection to C is a map ψ as desired. □

★ **Notes.** Groups with nice systems were studied by Hill [7]; he used the terminology "Axiom-3 groups" (referring to their $H(\aleph_0)$-family property). He proved the generalized Ulm theorem for these groups—a major result. That it suffices to assume a priori the existence of nice composition chains was observed in [IAG]. It turns out that this is not a weaker hypothesis after all; actually, we do not need a deep result like Theorem 5.9 to show that a group with a nice composition chain has a nice system: Hill has an ingenious direct proof in [16], based on his Theorem 5.5 in Chapter 1.

Exercises

(1) Let A be a p-group, and $n > 0$ an integer. $p^n A$ has a nice system if and only if A has one.
(2) An unbounded torsion-complete p-group has no nice composition chain. [Hint: it is enough to prove if basic subgroup is countable; in a nice composition chain from a countable index on the cokernels are not nice subgroups.]
(3) The torsion subgroup of a direct product of infinitely many unbounded reduced p-groups with nice composition chains never admits such a chain. [Hint: hint in preceding exercise.]
(4) Let C be a p-group. A reduced p-group A with a nice composition chain is isomorphic to an isotype subgroup (see next section) of C exactly if $f_\sigma(A) \leq f_\sigma(C)$ for all σ.

5 Isotypeness, Balancedness, and Balanced-Projectivity

We shall now turn our attention to subgroups that are of immense help in developing the theory of totally projective p-groups. We start with an important device that refines the concept of purity in exactly the same way as transfinite heights improve on ordinary heights.

Isotype Subgroups A subgroup G of a p-group A is called **isotype** (Kulikov [3]) if

$$p^{\sigma}G = G \cap p^{\sigma}A \qquad \text{for every ordinal } \sigma;$$

in other words, transfinite heights of elements in G are the same whether computed in G or in A. Equivalently, the exact sequence $0 \to G \to A \to A/G \to 0$ implies the exactness of

$$0 \to p^{\sigma}G \to p^{\sigma}A \to p^{\sigma}(A/G).$$

Let us learn some basic facts about isotype subgroups.

(a) *An isotype subgroup is isotype in every isotype subgroup that contains it.*
(b) *The property of being isotype is transitive.*
(c) *Isotypeness is an inductive property.*
(d) *G is isotype in A if and only if $p^{\sigma}G[p] = G \cap p^{\sigma}A[p]$ for every σ; i.e. the elements in the socle of G have the same heights in G as in A.* This can be verified in the same way as it was done in Sect. 1(F) in Chapter 5 for finite heights.
(e) *If G is isotype in A, then for every ordinal σ, $G/p^{\sigma}G$ is isotype in $A/p^{\sigma}G$.* Moreover, if $G/p^{\sigma}G$ is viewed as a subgroup of $A/p^{\sigma}A$ in the natural way, then we can claim that $G/p^{\sigma}G$ is isotype in $A/p^{\sigma}A$. These follow at once from the preservation of heights $< \sigma$ mod $p^{\sigma}A$.

We do not intend to pursue the exploration of isotypeness much further, but we wish to point out an interesting application of this concept. (For a stronger result, see Theorem 8.3 below.)

Lemma 5.1 (Irwin–Walker [2]). *$p^{\sigma}A$-high subgroups of a p-group A are isotype, for any ordinal σ.*

Proof. To prove this claim, let G be p^{σ}-high in A. Transfinite induction is used to establish the inclusion relation $G \cap p^{\rho}A \leq p^{\rho}G$ for all $\rho \leq \sigma$. This inclusion holds trivially for $\rho = 0$, and is true for a limit ordinal provided that it holds for all smaller ordinals. Thus, it remains to verify that it holds for $\rho+1$, assuming it holds for ρ. Let $g \in G \cap p^{\rho+1}A$; thus $g = pa$ for some $a \in p^{\rho}A$. If $a \in G$, then $a \in G \cap p^{\rho}A = p^{\rho}G$, and $g \in p^{\rho+1}G$. If $a \notin G$, then $\langle a, G\rangle \cap p^{\sigma}A \neq 0$, say, $na + g_0 = a_0 \neq 0$ with $g_0 \in G, a_0 \in p^{\sigma}A, n \in \mathbb{Z}$. As $pa \in G$, n must be coprime to p, so $n = 1$ may be assumed. From $pa + pg_0 = pa_0 \in G \cap p^{\sigma}A = 0$ we obtain $pa_0 = 0$ and $-pg_0 = g$. Now $\rho < \sigma$ implies $g_0 = a_0 - a \in G \cap p^{\rho}A = p^{\rho}G$, whence $g = -pg_0 \in p^{\rho+1}G$ is immediate. ◻

Balanced Subgroups We now combine the concepts of nice and isotype subgroups to create a major concept, called balanced subgroup, which plays a more important role in the theory. Balancedness crystallizes an idea that is common to tractable classes of torsion, torsion-free and mixed groups. Its real significance can be better observed if we treat the torsion and torsion-free cases separately, before embarking into the general discussion needed for mixed groups (see Chapter 15).

A subgroup B of a p-group A is called **balanced** if it is both nice and isotype. Equivalently, an exact sequence $0 \to B \to A \to C \to 0$ is **balanced-exact** if and only if the induced sequence

$$0 \to p^\sigma B \to p^\sigma A \to p^\sigma C \to 0 \tag{11.6}$$

is exact for every ordinal σ. (An interesting observation that follows from what has been said above is that the exactness of (11.3) at a non-limit ordinal σ is responsible for the isotypeness of B in A, and at limit ordinals for the niceness of B in A.) Thus, for the last balanced-exact sequence, the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & p^\sigma B \arrow[r] \arrow[d] & p^\sigma A \arrow[r] \arrow[d] & p^\sigma C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/p^\sigma B \arrow[r] \arrow[d] & A/p^\sigma A \arrow[r] \arrow[d] & C/p^\sigma C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

has exact columns and exact first and second rows. By virtue of the 3×3-Lemma 2.7 in Chapter 1, we conclude that even the third row is exact:

$$0 \to B/p^\sigma B \to A/p^\sigma A \to C/p^\sigma C \to 0. \tag{11.7}$$

Conversely, the exactness of (11.7) for all σ is equivalent to the balancedness of $0 \to B \to A \to C \to 0$.

From what has been said it follows readily that if B is balanced in A, then $p^\sigma B$ is balanced in $p^\sigma A$, and $B/p^\sigma B$ is balanced in $A/p^\sigma A$ for all σ.

That there is an abundance of balanced subgroups which are not direct summands will be apparent from the existence of balanced-projective resolutions; see below. The next examples are of special interest.

Example 5.2. Let $\overline{B}$ be an unbounded torsion-complete p-group, and $0 \to G \to C \to \overline{B} \to 0$ a pure-projective resolution of $\overline{B}$; thus, C, G are Σ-cyclic p-groups. Then G is closed in the p-adic topology of C, so it is nice in C. As G is pure with no elements of infinite height, it is isotype in C. Thus G is balanced in C. Since C is, but $\overline{B}$ is not a Σ-cyclic group, it is evident that G is not a summand.

Example 5.3. Consider the group C in Example 1.13. For every $c \in C[p]$, select a homomorphism ϕ_c of a copy A_c of the Prüfer group $H_{\omega+1}$ into C such that some $a_c \in A_c$ of height $h(c)$ maps upon c. In this way, we get an epimorphism

$$\phi = \nabla(\oplus\phi_c)\colon A = \oplus_{c\in C[p]} A_c \to C$$

whose kernel will be balanced in A (this will follow e.g. from Proposition 5.5 below). However, $\operatorname{Ker}\phi$ cannot be a summand, since C is not a direct sum of countable groups.

Properties of Balancedness Some needed facts about the behavior of balancedness are presented in the next lemma. The resemblance to purity is apparent.

Lemma 5.4. *Let A be a p-group, and $C < B$ subgroups in A.*

(i) *If C is balanced in A, then it is balanced in B.*
(ii) *If C is balanced in B, and B is balanced in A, then C is balanced in A.*
(iii) *If B is balanced in A, then B/C is balanced in A/C.*
(iv) *If C is balanced in A, and B/C is balanced in A/C, then B is balanced in A.*

Proof.

(i) We only need to prove that C is nice in B. Let $h_{B/C}(b + C) = \sigma$ $(b \in B)$ for a limit ordinal σ, and assume, by induction, that cosets of B mod C of heights $< \sigma$ contain elements proper with respect to C. Thus, for every $\rho < \sigma$, there is a c_ρ with $b + c_\rho \in p^\rho B$. As C is balanced in A, $b + c \in p^\sigma A$ for some $c \in C$. Then $c - c_\rho \in p^\rho A \cap C = p^\rho C$ whence $b + c = (b + c_\rho) + (c - c_\rho) \in p^\rho B$ for all $\rho < \sigma$, and $h_B(b + c) = \sigma$.

(iii) In view of Lemma 2.4(i), it suffices to ascertain that B/C is isotype in A/C. If $b + C \in p^\sigma(A/C)$ for $b \in B$, then some $a \in p^\sigma A$ and $c \in C$ satisfy $b + c = a \in B \cap p^\sigma A = p^\sigma B$. Hence $b + C \in p^\sigma B + C \le p^\sigma(B/C)$, and (iii) follows.

(ii) Because of the transitivity of isotypeness, it is enough to show that C is nice in A. Let $h_{A/C}(a + C) = \sigma$ $(a \in A)$. If $a \in B$, then (iii) implies that $a + C$ has height σ in B/C, thus $h(a+c) = \sigma$ for some $c \in C$. If $a \notin B$, then $h(a+B) \ge \sigma$ and $h(a + b) = h(a + B)$ for some $b \in B$. Visibly, $h(-a - b + a + C) \ge \sigma$, whence $h(-b + c) \ge \sigma$ and $h(a + c) = h(a + b - b + c) \ge \sigma$ for some $c \in C$. Consequently, $h(a + c) = \sigma$, and (ii) is done.

(iv) In view of Lemma 2.4(ii), only isotypeness needs to be checked. Suppose $b \in B \cap p^\sigma A$. Then $b + C \in B/C \cap p^\sigma(A/C) = p^\sigma(B/C)$ implies that $b - b_1 \in C$ holds for some $b_1 \in p^\sigma B$. Therefore, $b - b_1 \in C \cap p^\sigma A = p^\sigma C$, and $b \in p^\sigma B$. □

We draw attention to the following proposition that records most essential characterizations of balancedness.

Proposition 5.5. *For an exact sequence $0 \to B \to A \xrightarrow{\phi} C \to 0$ of p-groups, the following are equivalent:*

(*i*) *B is balanced in A;*
(*ii*) $0 \to p^\sigma B \to p^\sigma A \to p^\sigma C \to 0$ *is exact for every ordinal σ;*

(*iii*) $0 \to B(\underline{u}) \to A(\underline{u}) \to C(\underline{u}) \to 0$ *is exact for all increasing sequences* $\underline{u} = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ *of ordinals and* ∞*;*
(*iv*) $\phi(p^\sigma A[p]) = p^\sigma C[p]$ *holds for each* σ.

Proof. The equivalence of (*i*) and (*ii*) has already been established above.

To prove (*ii*) $\Rightarrow$ (*iii*), note that the map $B(\underline{u}) \to A(\underline{u})$ is monic, since $h_B(p^n b) = h_A(p^n b)$ for all $b \in B$ and $n \geq 0$. It is also clear that $\phi A(\underline{u}) \leq C(\underline{u})$. In order to prove that this is not a proper inclusion, we use induction on the order of $c \in C(\underline{u})$, and show that $p^\sigma C[p^k] \leq \phi(p^\sigma A[p^k])$ for $k \geq 1$. For $k = 1$, pick $c \in p^\sigma C[p]$, and argue that there is an $a \in p^\sigma A$ with $\phi a = c$. Now $pa \in p^{\sigma+1}A \cap B = p^{\sigma+1}B$ implies that $pb = pa$ for some $b \in p^\sigma B$. Since α maps $a - b \in p^\sigma A[p]$ upon c, the reverse inclusion follows. Assume the assertion is true for $k - 1$ and for all σ. If $c \in p^\sigma C[p^k]$ $(k > 1)$, then there is $x \in A$ such that $\phi(x) = c$, and by induction hypothesis, some $y \in p^{\sigma+1}A[p^{k-1}]$ satisfies $\phi(y) = pc \in p^{\sigma+1}C[p^{k-1}]$. Choose an $a_0 \in p^\sigma A[p^k]$ with $pa_0 = y$, and note that $\phi(x - a_0) = c - \phi(a_0) \in p^\sigma C[p]$. Consequently, $c - \phi(a_0) = \phi(a_1)$ for some $a_1 \in p^\sigma A[p]$, and $c = \phi(a_0 + a_1)$ with $a_0 + a_1 \in p^\sigma A[p^k]$. Hence (*iii*) is exact at $C(\underline{u})$.

(*iv*) is a special case of (*iii*).

For the implication (*iv*) $\Rightarrow$ (*ii*), it remains to check that $B \cap p^\sigma A = p^\sigma B$ for all σ. This being trivially true for $\sigma = 0$, suppose it holds for σ. To verify it for $\sigma + 1$, pick a $b \in B \cap p^{\sigma+1}A$ and an $a_0 \in p^\sigma A$ with $pa_0 = b$. As $\phi(a_0) \in p^\sigma C[p]$, we can find an $a_1 \in p^\sigma A[p]$ such that $\phi(a_1) = \phi(a_0)$. Then $a = a_0 - a_1 \in B \cap p^\sigma A = p^\sigma B$, whence $b = pa \in p^{\sigma+1}B$ follows. The induction step for limit ordinals is automatic. □

The Extension Lemma We have come to a crucial lemma about extending maps that do not decrease heights. The idea is borrowed from Hill [11].

Lemma 5.6. *Given a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & N & \longrightarrow & G & & \\
 & & & & \psi\downarrow & & \downarrow\phi & & \\
0 & \longrightarrow & B & \longrightarrow & A & \xrightarrow{\alpha} & C & \longrightarrow & 0
\end{array}$$

with exact rows, suppose that

(i) *B is a balanced subgroup of A;*
(ii) *ψ does not decrease heights (heights in N are computed in G);*
(iii) *there is a $g \in G$ proper with respect to N satisfying $pg \in N$.*

Then ψ can be extended to a map

$$\psi^*\colon \langle N, g\rangle \to A$$

which does not decrease heights either, and satisfies $\alpha\psi^*(g) = \phi(g)$.

Proof. Using the notation $h(g) = \sigma$, (i) implies that some $a \in p^\sigma A$ satisfies $\alpha a = \phi g$. Visibly, $pa - \psi(pg) \in B \cap p^{\sigma+1}A = p^{\sigma+1}B$, whence the existence of a $b \in p^\sigma B$ with $pb = pa - \psi(pg)$ is immediate. To define ψ^*, it suffices to specify $\psi^* \restriction N = \psi$ and $\psi^* g = a - b \in A$. It is readily seen that this is a homomorphism $\langle N, g\rangle \to A$ such that $\alpha\psi^* g = \alpha a = \phi g$. Obviously, $h(a-b) \geq \sigma = h(g)$. In order to check that ψ^* does not decrease heights, it is enough to show that $h(g + x) \leq h(a - b + \psi x)$ for every $x \in N$. From (iii) we infer that, on one hand, $h(g + x) = \min\{h(g), h(x)\}$, and, on the other hand, $h(a-b+\psi x) \geq \min\{h(a-b), h(\psi x)\} \geq \min\{\sigma, h(x)\}$. This completes the proof. □

Balanced-Projective *p*-Groups Fundamental to our treatment is the concept of balanced-projectivity. A p-group G will be called **balanced-projective** if it has the projective property with respect to balanced-exact sequences of p-groups. Our next aim is to characterize these p-groups, and at the same time to obtain relevant information about groups with nice composition chains. The following two results are preludes to Theorem 5.9 below. (Recall the H_σ denotes the σth generalized Prüfer group.)

Lemma 5.7 (Nunke [5]). *Suppose A is a p-group, and $a \in A$ is of order $\leq p^n$ and of height $\geq \sigma$. Then there exists a homomorphism*

$$\phi\colon H_{\sigma+n} \to A \qquad \textit{such that} \quad \phi z = a$$

where $\langle z\rangle = p^\sigma H_{\sigma+n}$ is cyclic of order p^n.

Proof. The correspondence $z \mapsto a$ gives rise to a homomorphism $\eta\colon p^\sigma H_{\sigma+n} \to \langle a\rangle$ that does not decrease heights, computed in $H_{\sigma+n}$ and A, respectively. It remains to refer to Corollary 4.6 to complete the proof. □

Proposition 5.8. *Let A be a reduced p-group of length τ. There exist a group G that is a direct sum of generalized Prüfer groups of lengths $\leq \tau$, and a balanced-exact sequence*

$$0 \to K \to G \to A \to 0. \tag{11.8}$$

Proof. For $a \in A$ of order p^n and of height σ, we select a copy G_a of $H_{\sigma+n}$ and a map $\phi_a : G_a \to A$ as described in Lemma 5.7. Define the group $G = \oplus_{a\in A} G_a$, and the map $\phi = \nabla(\oplus\phi_a) : G \to A$. It is evident that $\phi(p^\sigma H[p]) = p^\sigma A[p]$ for every $\sigma \leq \tau$, so that by virtue of Proposition 5.5 we can claim with confidence that $\operatorname{Ker}\phi = K$ is balanced in G. □

The next theorem will show that the generalized Prüfer groups are balanced-projective. Consequently, (11.8) may be viewed as a **balanced-projective resolution** of A.

We now have all the tools for the core theorem that completely describes the balanced-projective p-groups. Main portions of this result are due to Hill [7].

Theorem 5.9. *For a reduced p-group A, the following conditions are equivalent:*

(*a*) *A has a nice system;*
(*b*) *A admits a nice composition chain;*
(*c*) *A is balanced-projective;*
(*d*) *A is a summand of a direct sum of generalized Prüfer groups.*

Proof. The implication (*a*) ⇒ (*b*) is trivial.

(*b*) ⇒ (*c*) We start with a balanced-exact sequence in the bottom row in the following diagram, and with a map $\phi : A \to C$:

$$\begin{tikzcd} & & & A \arrow[d, "\phi"] \arrow[dl, dashed, "\psi"'] & \\ 0 \arrow[r] & B \arrow[r] & G \arrow[r, "\alpha"] & C \arrow[r] & 0 \end{tikzcd}$$

We select a nice composition chain $0 = N_0 < N_1 < \cdots < N_\nu < \cdots < N_\mu = A$, and define the map $\psi : A \to G$ step-wise, by transfinite recursion for $\psi_\nu : N_\nu \to G$: at limit ordinals we take unions, while passing from N_ν to $N_{\nu+1}$ we simply refer to Lemma 5.6. Then $\alpha\psi = \phi$, proving (*c*).

(*c*) ⇒ (*d*) Owing to Proposition 5.8, there exists a balanced-exact sequence $0 \to K \to G \to A \to 0$ where G is a direct sum of generalized Prüfer groups. (*c*) implies that the sequence splits, whence (*d*) is evident.

(*d*) ⇒ (*a*) This implication is an immediate consequence of Proposition 4.3. □

★ **Notes.** Griffith [7] discusses a subfunctor of Ext that is actually Bext (subgroup of balanced extensions) for torsion groups. Balancedness was studied under different names separately for *p*-groups and torsion-free groups, until Hunter [1] developed a unified theory, including mixed groups. Balanced-projectivity was also studied by Warfield [7].

The problem of balanced-injective *p*-groups was settled by Griffith [7]: there is nothing new, they are just the torsion-complete *p*-groups. To prove this in one direction, observe that maps into a torsion-complete *p*-group kill elements of infinite heights, so balancedness reduces to purity.

Exercises

(1) G is isotype in A if and only if $p^\sigma G$ is pure in $p^\sigma A$ for every σ.
(2) A pure subgroup G of A is isotype whenever A/G is separable.
(3) (Kulikov) A subgroup C of A is isotype if and only if, for an ordinal ρ, $p^\rho C$ is isotype in $p^\rho A$, and $C/p^\rho C$ is isotype in $A/p^\rho A$.
(4) If G is isotype in A, then $f_\sigma(G) \leq f_\sigma(A)$ for every σ.
(5) (Irwin–Walker) In a p-group A, every subgroup with elements of heights $\leq \sigma$ can be embedded in an isotype subgroup of cardinality $\leq |\sigma|\aleph_0$ whose elements are again of heights $\leq \sigma$.
(6) Let A be generated by symbols $\{a_n, b_n\}_{n<\omega}$ subject to the relations $pa_0 = 0$, $pb_0 = a_0$, $p^n a_n = a_0$, $p^n b_n = b_0$ for all $n \geq 1$. Then $G = \langle a_0, \dots, a_n, \dots \rangle$ is pure, but not isotype in A.

(7) A balanced subgroup of a countable p-group is a summand. [Hint: balanced-projectivity.]
(8) Prove Lemma 5.7 by induction on σ. [Hint: it is trivial for finite σ, and easy for limit σ; for successor ordinals use $y \in p^{\rho}H_{\sigma+n-1}$ with $py = z$ and $b \in A$ with $pb = a, h(b) \geq \sigma - 1$, and extend η to $\langle y \rangle \to \langle b \rangle$.]
(9) (Warfield) A short exact sequence of p-groups relative to which the generalized Prüfer groups have the projective property is balanced-exact.
(10) (E. Walker) In the next section we show that the balanced-projective p-groups are exactly the simply presented ones. Using this, verify the following method of getting a balanced-projective resolution $0 \to K \to G \to A \to 0$ of a p-group A. G is generated by x_a, one generator for each $a \in A$, subject to the defining relations $x_0 = 0$, and $px_a = x_b$ if and only if $pa = b$ holds in A. The correspondence $x_a \mapsto a$ induces an epimorphism $\phi\colon G \to A$, and $K = \operatorname{Ker}\phi$ is balanced in G.
(11) If G is a balanced subgroup of A, then $\operatorname{Tor}(G, X)$ is balanced in $\operatorname{Tor}(A, X)$ for every group X. [Hint: Lemma 4.2 and Theorem 3.1 in Chapter 8.]

6 Totally Projective *p*-Groups

A most important type of p-group was discovered by Nunke [3] *via* homological considerations. Later, it turned out that the class of p-groups with nice systems introduced by Hill coincides with this class.

Total Projectivity Recall that Σ-cyclic groups can be characterized as groups A satisfying $\operatorname{Pext}(A, C) = 0$ for all groups C. If we restrict our attention to p-groups, and take into consideration that Pext is just the first Ulm subgroup of Ext, then we can claim that a p-group A is Σ-cyclic exactly if $p^{\omega}\operatorname{Ext}(A, C) = 0$ for all groups C. Nunke defined classes of p-groups A using this observation by replacing ω by an arbitrary ordinal.

Let σ stand for an ordinal. A p-group A is called p^{σ}**-projective** if

$$p^{\sigma}\operatorname{Ext}(A, C) = 0 \qquad \text{for all groups } C,$$

and **totally projective** if

$$p^{\sigma}\operatorname{Ext}(A/p^{\sigma}A, C) = 0 \qquad \text{for all ordinals } \sigma \text{ and all groups } C. \tag{11.9}$$

Accordingly, A is totally projective if and only if the factor group $A/p^{\sigma}A$ is p^{σ}-projective for each σ.

Observe that if $\rho < \sigma$, then p^{ρ}-projective implies p^{σ}-projective.

Example 6.1.

(a) A p^n-bounded group is p^n-projective, and an unbounded Σ-cyclic p-group is p^{ω}-projective.
(b) A p^n-bounded group is totally projective, and so is any Σ-cyclic p-group.

(c) The Prüfer group $H_{\omega+1}$ is both $p^{\omega+1}$-projective and totally projective. This will follow from Lemmas 6.3 and 6.4 *infra*.

We will need the following result on Ext.

Proposition 6.2 (Irwin–Walker–Walker [1], Nunke [5]). *If the subgroup B of A satisfies $B \leq p^\sigma A$ for some ordinal σ, then for every group G, the exact sequence $0 \to B \xrightarrow{\beta} A \xrightarrow{\alpha} C \to 0$ induces the exact sequence*

$$\mathrm{Hom}(G, C) \to \mathrm{Ext}(G, B) \xrightarrow{\beta^*} p^\sigma \mathrm{Ext}(G, A) \xrightarrow{\alpha^*} p^\sigma \mathrm{Ext}(G, C) \to 0.$$

Proof. We refer to Theorem 2.3 in Chapter 9, and show that dropping to subgroups is legitimate. Consider the Nunke group N_σ of length σ for the prime p (Sect. 1 in Chapter 15) with the exact sequence $0 \to \mathbb{Z} \to N_\sigma \to H_\sigma \to 0$ where $p^\sigma N_\sigma \cong \mathbb{Z}$. There is a natural homomorphism $\delta_A : A \cong \mathrm{Hom}(\mathbb{Z}, A) \to \mathrm{Ext}(H_\sigma, A)$ whose kernel is $\mathrm{Ker}\,\delta_A = \mathrm{Im}(\mathrm{Hom}(N_\alpha, A) \to A) = p^\sigma A$. δ_A induces the map $\delta_A^* : \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, \mathrm{Ext}(H_\sigma, A))$. We also have $\delta_{\mathrm{Ext}(G,A)} : \mathrm{Ext}(G, A) \to \mathrm{Ext}(H_\sigma, \mathrm{Ext}(G, A))$, where the last Ext is naturally isomorphic to $\mathrm{Ext}(G, \mathrm{Ext}(H_\sigma, A))$ (see Theorem 3.8 in Chapter 9). Owing to Lemma 5.5 in Chapter 9, $\mathrm{Im}\,\delta_A \leq p^\sigma \mathrm{Ext}(H_\sigma, A) = 0$, thus $\delta_A \beta = 0$, and so $\delta_A^* \beta^* = 0 = \delta_{\mathrm{Ext}(G,A)} \beta^*$. Hence $\mathrm{Im}\,\beta^* \leq \mathrm{Ker}\,\delta_{\mathrm{Ext}(G,A)} = p^\sigma \mathrm{Ext}(G, A)$. That the displayed sequence is exact at the last two Exts is straightforward. □

A few simple remarks are inserted here.

(A) *The class of p^σ-projective p-groups (for a fixed σ) as well as the class of totally projective p-groups is closed under direct sums and summands.*

(B) *A p^σ-projective group A has length $\leq \sigma$.* To prove this, consider for any C the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}(A/p^\sigma A, C/p^\sigma C) \arrow[r] \arrow[d] & \mathrm{Hom}(A, C/p^\sigma C) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \mathrm{Hom}(p^\sigma A, C/p^\sigma C) \arrow[d] & \\
\mathrm{Ext}(A/p^\sigma A, p^\sigma C) \arrow[r] & \mathrm{Ext}(A, p^\sigma C) \arrow[r, "\beta"] \arrow[d] & \mathrm{Ext}(p^\sigma A, p^\sigma C) \arrow[r] & 0 \\
& p^\sigma\ \mathrm{Ext}(A, C) = 0 & &
\end{tikzcd}$$

where γ is surjective (Proposition 6.2), and $\alpha = 0$ ($p^\sigma A$ is in the kernel of every map $A \to C/p^\sigma C$). We conclude that $\beta = 0$, whence $\mathrm{Ext}(p^\sigma A, p^\sigma C) = 0$. This implies $p^\sigma A = 0$, for otherwise we could choose C, e.g. such that $p^\sigma C \cong \mathbb{Z}(p)$ to have the last Ext $\neq 0$.

In particular, a p^n-projective group is p^n-bounded ($n \in \mathbb{N}$).

(C) It can very well happen that a $p^{\sigma+1}$-projective p-group has length σ even if it is not p^σ-projective. See Sect. 9.

Lemma 6.3 (Nunke [3], Irwin–Walker–Walker [1]). *If S is a subsocle of a p-group A such that A/S is p^σ-projective, then A is $p^{\sigma+1}$-projective.*

Proof. The claim is obvious if $\sigma < \omega$, so assume $\sigma \geq \omega$. We have an exact sequence $\operatorname{Hom}(S,*) \xrightarrow{\gamma} \operatorname{Ext}(A/S,*) \xrightarrow{\delta} \operatorname{Ext}(A,*) \to \operatorname{Ext}(S,*) \to 0$. Here $p\operatorname{Ext}(S,*) = 0$ implies $p\operatorname{Ext}(A,*) \leq \operatorname{Im}\delta$, so it suffices to verify $p^{\sigma+1}\operatorname{Im}\delta = 0$, because then $p^{\sigma+1}\operatorname{Ext}(A,*) = 0$ (observe that $1+\sigma = \sigma$ as $\sigma \geq \omega$).

More generally, we show that if S is a subsocle of a p-group E satisfying $p^\sigma E = 0$, then $p^{\sigma+1}(E/S) = 0$. But this is clear from $p^\sigma(E/S) \leq E[p]/S$. (See Sect. 1(B) in Chapter 10.) □

The same argument applies to establish the corresponding result on total projectivity.

Lemma 6.4 (Nunke [3]). *If A is a p-group satisfying $p^{\sigma+1}A = 0$ such that $A/p^\sigma A$ is totally projective, then A is likewise totally projective.* □

An immediate consequence of this lemma is that *for the total projectivity of a p-group A, it suffices to check that $p^\sigma\operatorname{Ext}(A/p^\sigma A, C) = 0$ holds for limit ordinals σ.* Indeed, it is evident from Lemma 6.4 that if (11.9) holds for σ, then it also holds for $\sigma + 1$.

A Characterization of Total Projectivity We now relate nice composition chains to total projectivity.

Theorem 6.5 (Hill [11]). *A p-group is totally projective if and only if it satisfies one (and hence all) of the equivalent conditions of Theorem 5.9.*

Proof. The first task is to show that a direct summand of a direct sum of generalized Prüfer groups is totally projective. Because of (A), it suffices to establish the total projectivity of H_σ for all σ. As expected, a straightforward induction on σ will do the job. Indeed, passage from σ to $\sigma + 1$ is trivial by virtue of Lemma 6.4, while for limit ordinals σ it is a consequence of being the direct sum of groups H_ρ with $\rho < \sigma$.

Conversely, suppose A is totally projective. We induct on the length σ of A. If $\sigma = n \in \mathbb{N}$, then A is a bounded group, so Σ-cyclic and balanced-projective. If A is totally projective of length $\sigma + 1$, then $A/p^\sigma A$ is totally projective of length σ, consequently, by induction hypothesis, it has a nice composition chain. So does the elementary p-group $p^\sigma A$ that is nice in A, thus also A has a nice composition chain.

Finally, assume that the length σ of A is a limit ordinal. We show that any balanced-exact sequence $\mathfrak{e} : 0 \to B \to G \to A \to 0$ is splitting. From Sect. 5 we know that the sequence $\mathfrak{e}_\rho : 0 \to B/p^\rho B \to G/p^\rho G \to A/p^\rho A \to 0$ is exact for all $\rho < \sigma$, and the large diagram commutes. Starting from the bottom row, we construct the following diagram:

$$\begin{array}{cccccccccc}
\mathfrak{e}: & 0 & \longrightarrow & B & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & A & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle\mu} & & {\scriptstyle\nu}\downarrow & & \| & & \\
\mathfrak{e}_\rho\phi: & 0 & \longrightarrow & B/p^\rho B & \xrightarrow{\gamma} & H & \xrightarrow{\delta} & A & \longrightarrow & 0 \\
& & & \| & & {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} & & \\
\mathfrak{e}_\rho: & 0 & \longrightarrow & B/p^\rho B & \xrightarrow{\epsilon} & G/p^\rho G & \xrightarrow{\xi} & A/p^\rho A & \longrightarrow & 0,
\end{array}$$

We get the middle row as a pull-back. The canonical map $\zeta : G \to G/p^\rho G$ followed by ξ equals $\phi\beta$, so the pull-back property ensures the existence of a unique $\nu : G \to H$ such that $\zeta = \psi\nu$ and $\delta\nu = \beta$. If $\mu : B \to B/p^\rho B$ denotes the canonical map, then for the commutativity of the diagram it only remains to check that $\gamma\mu = \nu\alpha$. Now $\psi\gamma\mu = \epsilon\mu = \zeta\alpha = \psi\nu\alpha$ and $\delta\gamma\mu = 0 = \beta\alpha = \delta\nu\alpha$ show that $\gamma\mu$ and $\nu\alpha$ are maps $B \to H$ which become equal if followed by ψ or δ. Since H was a pull-back, $\gamma\mu = \nu\alpha$ follows.

By a remark in Sect. 5, after (11.7), $B/p^\rho B$ is balanced in $G/p^\rho G$, and therefore $\mathfrak{e}_\rho$ must split (by induction hypothesis, $A/p^\rho A$ is balanced-projective). It follows that $\mathfrak{e}_\rho\phi$ also splits, and hence in view of the exact sequence $\mathrm{Ext}(A, p^\rho B) \xrightarrow{\eta_*} \mathrm{Ext}(A, B) \xrightarrow{\mu_*} \mathrm{Ext}(A, B/p^\rho B)$ we conclude that $\mathfrak{e} \in \mathrm{Im}\,\eta_*$ where $\eta : p^\rho B \to B$ is the inclusion map. By Proposition 6.2, $\mathrm{Im}\,\eta_* \le p^\rho \mathrm{Ext}(A, B)$, and therefore $\mathfrak{e} \in \cap_{\rho<\sigma} p^\rho \mathrm{Ext}(A, B) = p^\sigma \mathrm{Ext}(A, B)$. The last group is 0, if A is totally projective, i.e. $\mathfrak{e}$ is splitting. □

Before continuing, let us summarize briefly the various characterizations of the class of p-groups we are studying. The theorems we have proved so far already show that for a reduced p-group A the following are equivalent:

1. A is simply presented;
2. A has a nice system (or a nice composition chain);
3. A is a summand of a direct sum of generalized Prüfer groups;
4. A is balanced-projective;
5. A is totally projective.

And more importantly, these groups are characterized by their UK-invariants.

Though these groups can be referred to by any of the listed properties, we more often use the term 'totally projective p-groups,' because chronologically this property was first introduced. However, total projectivity is perhaps the least practical criterion to recognize groups in this class.

Consequences of the Main Result Capitalizing on Theorem 6.5, we can get valuable information about how totally projective p-groups are related to direct sums of countable groups.

Theorem 6.6 (Nunke [5]). *A reduced p-group is a direct sum of countable groups if and only if it is totally projective of length $\le \omega_1$.*

Proof. Combining Example 4.1, Theorems 5.9 and 6.5, it follows that a direct sum of countable p-groups is totally projective; evidently, its length cannot exceed ω_1. Conversely, a totally projective p-group A of length $\leq \omega_1$ is a summand of a direct sum of generalized Prüfer groups H_σ of lengths $\sigma < \omega_1$. These H_σ are countable, and an appeal to Kaplansky's Theorem 2.5 in Chapter 2 completes the proof. □

Corollary 6.7 (Kolettis [1]). *Direct sums of countable p-groups can be classified by their UK-invariants.* □

If we concentrate on the entire *class* of totally projective p-groups rather than on the individual groups, then we can establish two remarkable characterizations: one as the smallest, and the other as the largest class satisfying certain conditions. Accordingly, we now prove two theorems. In the next result, an **elongation** of a p-group B by a p-group C means an extension A of B by C such that $B = p^\sigma A$ for some ordinal σ (which is equal to the length of C).

Theorem 6.8 (Parker–Walker [1]). *The class of totally projective p-groups is the smallest class $\mathcal{C}$ of groups that is closed under isomorphism, and shares the following properties:*

(a) $\mathbb{Z}(p) \in \mathcal{C}$;
(b) *$\mathcal{C}$ is closed under the formations of arbitrary direct sums and direct summands;*
(c) *$\mathcal{C}$ is closed under elongations.*

Proof. From (a)–(c) it follows that cyclic and Σ-cyclic p-groups are in $\mathcal{C}$. Then by (c) the generalized Prüfer groups of length $\omega + n$ and $\omega 2$ also belong to this class. A straightforward induction convinces us that all generalized Prüfer groups H_σ belong to $\mathcal{C}$, and then (b) guarantees that all totally projective p-groups are members of $\mathcal{C}$. Since the class of totally projective p-groups shares the listed properties, the assertion follows. □

The second theorem we hinted at above asserts that the class of totally projective p-groups is the natural boundary for the extension of Ulm's theorem.

Theorem 6.9. *The class of totally projective p-groups is the largest class $\mathcal{B}$ of p-groups that*

(a) *contains the generalized Prüfer p-groups,*
(b) *is closed under direct sums and summands, and*
(c) *non-isomorphic groups in $\mathcal{B}$ are distinguishable by their UK-invariants.*

Proof. We need to show that only totally projective groups can belong to $\mathcal{B}$. Let A be a member of $\mathcal{B}$, say, of length τ. Consider the direct sum $H = \oplus_\kappa \oplus_{\sigma \leq \tau} H_\sigma$ of generalized Prüfer groups where $\kappa = |A|\aleph_0$. By (a)–(b), $H \in \mathcal{B}$, and it is clear that $f_\sigma(H) = \kappa$ for all $\sigma \leq \tau$. By (b), $H \oplus A \in \mathcal{B}$. Evidently, $f_\sigma(H \oplus A) = f_\sigma(H)$ for all $\sigma < \tau$, thus (c) implies $H \oplus A \cong H$. This amounts to saying that A is a summand of a direct sum of generalized Prüfer groups, so it is totally projective. □

Transitivity and Full Transitivity We plan to show that totally projective p-groups A are transitive (fully transitive) in the sense that if $a, b \in A$ satisfy $\underline{\mathrm{u}}(a) = \underline{\mathrm{u}}(b)$ $(\underline{\mathrm{u}}(a) \leq \underline{\mathrm{u}}(b))$, then there exists an $\alpha \in \operatorname{Aut} A$ $(\alpha \in \operatorname{End} A)$ such that $\alpha(a) = b$.

Theorem 6.10. *Totally projective p-groups are both transitive and fully transitive.*

Proof. Suppose A is totally projective. If $a, b \in A$ satisfy $\underline{\mathrm{u}}(a) = \underline{\mathrm{u}}(b)$, then there exists a height-preserving isomorphism $\phi : \langle a \rangle \to \langle b \rangle$; these subgroups are finite, and hence nice. Furthermore, we have $A/\langle a \rangle \cong A/\langle b \rangle$, both totally projective. Indeed, these factor groups have the same UK-invariants: this is obvious if the only gap in $\underline{\mathrm{u}}(a)$ is at ∞, and it follows by induction for a finite number of gaps. Theorem 3.6 implies the existence of $\alpha \in \operatorname{Aut} A$ such that $\alpha(a) = b$.

In case $\underline{\mathrm{u}}(a) \leq \underline{\mathrm{u}}(b)$, we apply what has been proved for the group $A \oplus A'$ (where $A' \cong A$) and to the elements $(a, 0)$ and (b, a'). After application of $\alpha \in \operatorname{Aut}(A \oplus A')$, we project to the first summand. □

★ **Notes.** Totally projective p-groups were introduced by R. Nunke in order to give a homological description to direct sums of countable p-groups. He proved that a totally projective p-group is a direct sum of countable groups provided its length is $\leq \omega_1$. In view of Kolettis' [1] remarkable result that direct sums of countable p-groups are classifiable by UK-invariants, E. Walker conjectured that such a classification extends to totally projective p-groups. A good progress was made toward the proof of this conjecture when it was confirmed for groups of lengths $\leq \omega_1$ by Parker–Walker [1]. The credit of extending Ulm's theorem to arbitrary p-groups with nice systems belongs to Hill. Walker [3] simplified the proof, and pointed out that the extension lemma may start from any group, so it can be applied to mixed groups as well.

There is a large literature on various aspects of total projectivity. For example, Hill [15] shows that the set union of a countable number of totally projective p-groups is again totally projective. Göbel–May [4] give conditions on a direct sum of countable groups when it is cancellable on both sides of a direct sum. The compact groups that are Pontryagin duals of totally projective p-groups were described by Kiefer [1].

It is interesting that in several cases, for a p-group A, the group $V(A)/A$ is totally projective. Here $V(A)$ denotes the (multiplicative) group of normalized units in the group algebra $\mathsf{F}(A)$ over a field F.

Since the classification of all p-groups seems to be an unrealistic goal at this time, one has to be content with aiming at classifying significant subclasses. There are a couple of successful attempts using additional types of invariants describing classes of p-groups larger than that of totally projective p-groups whose members admit complete sets of invariants. Space forbids our presenting any of these generalizations, but we will call attention to A- and S-groups in Notes to Sect. 7.

Exercises

(1) Show that in the definition of total projectivity, C may be restricted to p-groups. [Hint: check the proofs.]

(2) (Irwin–Walker–Walker) If A is p^σ-projective, then for every group C, $\operatorname{Ext}(A, C) \cong \operatorname{Ext}(A, C/p^\sigma C)$.

(3) If A is a totally projective p-group, and C is a subgroup satisfying $p^n A \leq C \leq A$ for some $n \in \mathbb{N}$, then C is also totally projective. [Hint: $p^\omega C = p^\omega A$ and $C/p^\omega C$ is Σ-cyclic.]
(4) (Nunke, Hill–Megibben) Let σ denote a countable ordinal. If A is a p-group such that both $p^\sigma A$ and $A/p^\sigma A$ are direct sums of countable groups, then A is also a direct sum of countable groups.
(5) (Nunke) Let A be a p-group whose nth Ulm subgroup vanishes for some integer n. If all the Ulm factors of A are Σ-cyclic groups, then A is a direct sum of countable groups.
(6) (Nunke) If A is a p^σ-projective p-group, then so is $\mathrm{Tor}(A, C)$ for any C. [Hint: trivial proof by $\mathrm{Ext}(A, \mathrm{Ext}(B, G)) \cong \mathrm{Ext}(\mathrm{Tor}(A, C), G)$.]
(7) A p-group is totally projective if and only if it is a summand of a direct sum of copies of groups P_β for various β (see Example 3.2).
(8) (Irwin–Walker–Walker [1]) A subgroup B of a p-group A is contained in the Ulm subgroup A^1 if and only if the sequence $0 \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, A/B) \to \mathrm{Ext}(G, B) \to \mathrm{Pext}(G, A) \to \mathrm{Pext}(G, A/B) \to 0$ is exact for all groups G.

7 Subgroups of Totally Projective p-Groups

When is a subgroup of a totally projective group again totally projective? We have a better chance for obtaining a useful answer to this question if we concentrate on special subgroups, like fully invariant or isotype. But first let us see an example demonstrating that even subgroups of direct sums of countable groups need not be such direct sums.

Example 7.1 (Nunke [5]). Let B be an unbounded countable Σ-cyclic p-group, and $H_{\omega+1}$ the Prüfer group of length $\omega + 1$. The group $A = \mathrm{Tor}(\overline{B}, H_{\omega+1})$ is contained in $\mathrm{Tor}(D, H_{\omega+1})$ (where $D = \oplus \mathbb{Z}(p^\infty)$ is the divisible hull of $\overline{B}$), so A is contained in a direct sum of countable groups $H_{\omega+1}$. By Lemma 4.2 in Chapter 8, A has length $\leq \omega$, therefore, if A is a direct sum of countable groups, then it must be Σ-cyclic. But A cannot be Σ-cyclic, because the pure-exact sequence $0 \to B \to \overline{B} \to \oplus \mathbb{Z}(p^\infty) \to 0$ induces the pure-exact sequence $0 \to \mathrm{Tor}(B, H_{\omega+1}) \to \mathrm{Tor}(\overline{B}, H_{\omega+1}) \to \mathrm{Tor}(\oplus \mathbb{Z}(p^\infty), H_{\omega+1}) \to 0$, where the first Ulm subgroup of the last Tor has cardinality $2^{\aleph_0}$, and such a group cannot be the image of a Σ-cyclic p-group A modulo a countable subgroup $\mathrm{Tor}(B, H_{\omega+1})$.

Fully Invariant in Totally Projective To begin with, we point out that every fully invariant subgroup of a totally projective p-group A is of the form $A(\underline{u})$ for some $\underline{u} = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$ with an increasing sequence of ordinals and symbols ∞. In fact, totally projective groups are fully transitive by Theorem 6.10, and the rest follows from Theorem 2.2 in Chapter 10.

Theorem 7.2 (Fuchs–E. Walker, Linton [1]). *Fully invariant subgroups of totally projective p-groups are likewise totally projective.*

Proof. Since full invariance commutes with arbitrary direct sums, it suffices to prove the claim for the generalized Prüfer groups of the form $H_{\sigma+1}$. Consider $H_{\sigma+1}(\underline{u})$ where $\underline{u} = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$. If $\sigma + 1 < \sigma_n$ for some n, then $H_{\sigma+1}(\underline{u})$ is bounded, so it is trivially totally projective. If $\sigma_n < \sigma + 1$ for all $n \in \mathbb{N}$, then we induct on σ. Evidently, we have

$$[p^{-n}(p^{\sigma_n}H_{\sigma+1})]/p^{\sigma}H_{\sigma+1} = p^{-n}[p^{\sigma_n}(H_{\sigma+1}/p^{\sigma}H_{\sigma+1})] \cong p^{-n}(p^{\sigma_n}H_{\sigma})$$

(naturally), and $H_{\sigma+1}(\underline{u}) = \cap_{n\in\mathbb{N}} p^{-n}(p^{\sigma_n}H_{\sigma+1})$. Consequently, we have $H_{\sigma+1}(\underline{u})/p^{\sigma}H_{\sigma+1} \cong H_{\sigma}(\underline{u})$. Thus $H_{\sigma+1}(\underline{u})$ is totally projective as an extension of the cyclic group $p^{\sigma}H_{\sigma+1}$ by the totally projective $H_{\sigma}(\underline{u})$. □

Separative Subgroups The following concept will be needed; it was introduced by Hill [17]. It generalizes the notion of nice subgroups.

A subgroup B of a p-group A is called **separative** (or, **separable in the sense of Hill**) if, for each $a \in A$, there is a countable subset $\{b_n \mid n < \omega\}$ in B such that

$$\sup\{h_A(a + x) \mid x \in B\} = \sup\{h_A(a + b_n) \mid n < \omega\}.$$

In other words, the set $\{h_A(a + x) \mid x \in B\}$ of ordinals either contains a largest element or is cofinal with ω. By a **separative chain** from A to G is meant a smooth chain $A = A_0 < \cdots < A_\alpha < \cdots < A_\tau = G$ (for some ordinal τ) of separative subgroups of G, such that, for each $\alpha < \tau$, the factor group $A_{\alpha+1}/A_\alpha$ is countable.

Let us point out that $\sup\{h_A(a+x) \mid x \in B\}$ is equal to $h_{A/B}(a+B)$, provided the latter is $\neq \infty$. An easy example for the ∞ case is a separable p-group A where this supremum for any non-zero coset modulo a basic subgroup B is ω, but the coset has height ∞ in A/B.

Example 7.3.

(i) Let C be a subgroup of a separable p-group A such that A/C has an element of height ω_1. Then C is not separative in A.
(ii) A separable p-group A admits a separative chain through any of its basic subgroups. All factor groups can be chosen to be cocyclic.

To start with, the following important properties of separative subgroups should be pointed out.

(A) It is obvious from the definition that *nice subgroups, countable subgroups, and more generally, subgroups with heights bounded by a countable ordinal are separative.*
(B) *Subgroups modulo which the factors groups have lengths $< \omega_1$ are separative.*
(C) *Countable extensions of separative subgroups are separative.* Suppose $C < B < A$, where C is separative in A, and B/C is countable. Let $\{b_n \in B,\ n < \omega\}$ be a complete set of representatives of B mod C. By hypothesis, for each $n < \omega$, there is a countable cofinal subset $\{h_A(a + b_n + c_{ni}) \mid c_{ni} \in C,\ i < \omega\}$ in $\{h_A(a + b_n + c) \mid c \in C\}$. Then the countable set $\{h_A(a + b_n + c_{ni}) \mid b_n \in B,\ c_{ni} \in C;\ n, i < \omega\}$ is cofinal in $\{h_A(a + b) \mid b \in B\}$, thus B is separative in A.

(D) *If* $B_0 < B_1 < \cdots < B_n < \ldots$ *is a countable chain of separative subgroups of* A*, then their union* $\bigcup_{n<\omega} B_n$ *is likewise separative in* A.

(E) *Separativity is transitive.* In fact, suppose $C < B < A$, and let C be separative in B, and B in A. Given $a \in A \setminus B$, we consider the set $\{h_A(a+c) \mid c \in C\}$. By hypothesis, the set $\{h_A(a+b) \mid b \in B\}$ contains a countable cofinal subset $\{h_A(a+b_n) \mid b_n \in B, n < \omega\}$. Similarly, for each n, the set $\{h_A(b_n+c) \mid c \in C\}$ contains a countable cofinal subset $\{h_A(b_n+c_{ni}) \mid c_{ni} \in C,\ i < \omega\}$. To show that $\{h_A(a-c_{ni}) \mid c_{ni} \in C;\ n,i < \omega\}$ is cofinal in $\{h_A(a+c) \mid c \in C\}$, note that there is an $n < \omega$ such that $h_A(a+c) \leq h_A(a+b_n)$. Then $h_A(a+c) \leq h_A(b_n-c)$, so there is an $i < \omega$ with $h_A(b_n-c) \leq h_A(b_n+c_{ni})$. Hence $h_A(a+c) \leq h_A(a+b_n-b_n-c_{ni}) = h_A(a-c_{ni})$.

(F) *Let* $C < B < A$ *where* C *is nice in* A*. Then* B/C *is separative in* A/C *if and only if* B *is separative in* A. To prove this, note that by the niceness of C, for any $a \in A$, $h_{A/C}(a+b+C)$ with $b \in B$ is equal to $h_A(a+b+c)$ for some $c \in C$ (depending on b). Therefore, $\{h_{A/C}(a+b+C) \mid b \in B\}$ contains a countable cofinal subset if and only if so does the set $\{h_A(a+b) \mid b \in B\}$.

Absolute Separativity The next theorem shows how well separativity fits when total projectivity is discussed.

A p-group is said to be **absolutely separative** if it is a separative subgroup in every p-group in which it is contained as an isotype subgroup.

Theorem 7.4 (Hill [17]). *Totally projective p-groups are absolutely separative.*

Proof. Let A be a totally projective p-group, and $\mathcal{N}$ an $H(\aleph_0)$-family of nice subgroups of A. Suppose A is an isotype subgroup in a p-group G, but for some $g \in G$, the set $\{h_G(g+a) \mid a \in A\}$ has cofinality $> \omega$. Manifestly, there is a chain of subgroups $A_n \in \mathcal{N}$ $(n < \omega)$ such that

$$\sup\{h_G(g+a) \mid a \in A_1\} < \cdots < \sup\{h_G(g+a) \mid a \in A_n\} < \cdots$$

The union C of the A_n also belongs to $\mathcal{N}$. By our hypothesis on cofinality, there is a $b \in A$ such that $h_G(g+b) > \sup\{h_G(g+x) \mid x \in C\}$. Hence

$$\sup\{h_G(b-x) \mid x \in C\} = \sup\{h_G(g+x) \mid x \in C\}.$$

C is nice in A, so $h_G(b-c) = \sup\{h_G(b-x) \mid x \in C\}$ for some $c \in C$. It follows that $h_G(b-c) = \sup\{h_G(g+x) \mid x \in C\}$. But $h_G(b-c) > h_G(g+c)$ implies $h_G(g+b) = h_G(g+c)$, a contradiction. $\square$

Compatibility For subgroups A and B of a p-group G, we say that A **is compatible with** B (in notation: $A \| B$, see Hill [17]) if for every pair $(a,b) \in A \times B$, we have

$$h_G(a+b) \leq h_G(a+x) \qquad \text{for some } x \in A \cap B. \tag{11.10}$$

(a) Note that the inequality (11.10) implies $h_G(a + b) \leq h_G(b - x)$; hence, *compatibility is a symmetric relation*.
(b) Trivially, $A \| B$ whenever $B \leq A$.

Example 7.5.

(i) Let G be an unbounded separable p-group, and A a pure subgroup of G. Then $A \| B$ holds, e.g. for a subgroup $B < G$ for which $A \cap B$ is a summand of G. Observe that every $a \in A$ is contained in a summand $A' \geq A \cap B$ of G, so the A'-coordinate $a + x$ of $a + b$ ($b \in B$) will satisfy $h(a + b) \leq h(a + x)$.
(ii) Let A be a finite subgroup, and B a basic subgroup in a separable p-group G such that $A \not\leq B$. Then B is not compatible with A. Indeed, for every $a \in A \setminus B$ there is a $b \in B$ such that $h(a + b) > h(x)$ for every $x \in A$, so also $h(a + x) < h(a + b)$.

(c) *Compatibility is an inductive property*: if $B_1 < B_2 < \cdots < B_\sigma < \cdots$ is any chain of subgroups of G, each compatible with A, then the union $\bigcup_\alpha B_\sigma$ is likewise compatible with A.
(d) *If A, B, C are subgroups of G, then $A \| (B + C)$ and $B \| C$ imply $(A + B) \| C$.* To verify this, let $a \in A, b \in B, c \in C$. Then there is an $x \in A \cap (B + C)$ such that $h(a + b + c) \leq h(b + c + x)$. Setting $x = b' + c'$ with $b' \in B, c' \in C$, we have $h(b+c+x) = h(b+b'+c+c') \leq h(y+c+c')$ for some $y \in B \cap C$, and therefore $h(a+b+c) \leq h(a+b-(y+c'))$. Here $y+c' \in C$ and $y+c' = y+x-b' \in A+B$, so $(A + B) \| C$, indeed.

Lemma 7.6. *Let A and B be subgroups of a p-group G such that B is nice in G and compatible with A. Then the following are true:*

(i) *$A \cap B$ is nice in A.*
(ii) *If A is isotype in G, then $(A + B)/B$ is isotype in G/B.*
(iii) *A subgroup $B' > B$ satisfies $(B'/B) \| (A + B)/B$ exactly if $B' \| A$.*

Proof.

(i) If $a \in A \setminus (A \cap B) = A \setminus B$, then there is $b' \in B$ such that $h(a + b) \leq h(a + b')$ for all $b \in A \cap B$ (by niceness). Compatibility implies $h(a + b') \leq h(a + x)$ for some $x \in A \cap B$.
(ii) Given $a \in A$, for an $x \in A \cap B$, we have inequalities $h_{G/B}(a+B) = h_G(a+b) \leq h_G(a + x) = h_A(a + x) \leq h_{(A+B)/B}(a + B)$.
(iii) Suppose $(B'/B) \| (A + B)/B$. Given $a \in A$ and $b' \in B'$, we have $h_G(a + b') \leq h_{G/B}(a + b' + B) \leq h_{G/B}(a + x + B)$ for some $x \in B' \cap (A + B) = (B' \cap A) + B$, which can clearly be chosen such that $x \in B' \cap A$. Now $h_{G/B}(a + x + B) = h_G(a + x + b)$ for some $b \in B$, and $A \| B$ implies that the last height is $\leq h_G(a + x + y)$ for some $y \in B \cap A$. Since $x + y \in B' \cap A$, the desired $B' \| A$ follows.

Conversely, assume $B' \| A$. Let again $a \in A$ and $b' \in B'$. Manifestly, $h_{G/B}(a + b' + B) = h_G(a + b' + b)$ for some $b \in B$, which is $\leq h_G(x + b' + b)$ for some $x \in B' \cap A$, which in turn is $\leq h_{G/B}(x + b' + B)$. Here $x + B \in (B' \cap A) + B = B' \cap (A + B)$. □

The next lemma is a convincing evidence that separative subgroups are compatible with many subgroups.

Lemma 7.7 (Hill [17]). *Given a separative subgroup A, and a subgroup X of G, there is a subgroup B of G such that*

(i) $X \leq B$,
(ii) $|B| \leq |X|\aleph_0$, *and*
(iii) $A\|B$.

In particular, every countable subset of G can be embedded in a countable subgroup B of G such that $A\|B$.

Proof. Set $B_0 = X$. For each $x \in B_0$ choose elements $a_n(x) \in A$ $(n < \omega)$ such that $\{h_G(x + a_n(x)) \mid n < \omega\}$ is cofinal in $\{h_G(x + a) \mid a \in A\}$. Define B_1 to be generated by B_0 and all the $a_n(x) \in A$ $(x \in B_0, n < \omega)$. Repeat this argument for B_1 in the role of B_0, to obtain a subgroup B_2, and keep repeating this procedure ω times. Let B be the union of the chain $B_0 < B_1 < \cdots < B_i < \cdots$ $(i < \omega)$. Then B obviously satisfies (i) and (ii). Given $a \in A$ and $b \in B$, there is an i such that $b \in B_i$. By construction, there is an $a_n(b) \in A$ such that $h_G(b + a) \leq h_G(b + a_n(b))$. Since $a_n(b) \in B_{i+1} \leq B$, $A\|B$ follows. □

When Isotype Implies Totally Projective Compatibility has been introduced to answer our question posed above. Here is the answer.

Theorem 7.8 (Hill [17]). *An isotype subgroup A of a totally projective p-group G is totally projective if and only if G admits a $G(\aleph_0)$-family of (nice) subgroups that are compatible with A.*

Proof. Suppose A is totally projective. Let $\mathcal{A}$ and $\mathcal{G}$ be $G(\aleph_0)$-families of nice subgroups of A and G, respectively, and define

$$\mathcal{H} = \{X \in \mathcal{G} \mid X\|A \text{ and } X \cap A \in \mathcal{A}\}.$$

Since compatibility is preserved under unions of chains, in order to show that $\mathcal{H}$ is likewise a $G(\aleph_0)$-family, it suffices to ascertain that if $X \in \mathcal{H}$ and if C is a countable subgroup of G, then there is a $Y \in \mathcal{H}$ containing both X and C such that Y/X is countable. By Lemma 7.6(ii), $(A+X)/X$ is isotype in G/X, and since it is isomorphic to the totally projective group $A/(A \cap X)$, by Theorem 7.4, it is separative in G/X. From Lemma 7.7 we infer that there is a countable subgroup Y/X of G/X that contains $(C + X)/X$ and is compatible with $(A + X)/X$. Using a familiar back-and-forth argument, it follows that Y may be chosen so as to satisfy both $Y \in \mathcal{G}$ and $Y \cap A \in \mathcal{A}$. Hence Lemma 7.6(iii) shows that $Y\|A$, i.e. $Y \in \mathcal{H}$.

Conversely, let $\mathcal{H} = \{X \mid X\|A\}$ be a $G(\aleph_0)$-family in G. Since the intersection of two $G(\aleph_0)$-families is again one, we may even assume that $\mathcal{H}$ is a subfamily of $\mathcal{G}$ (defined in the preceding paragraph), i.e. the subgroups in $\mathcal{H}$ are nice in G. Evidently, $\mathcal{A} = \{X \cap A \mid X \in \mathcal{H}\}$ is a $G(\aleph_0)$-family in A. By Lemma 7.6(i), $X \cap A$ is nice in A for each $X \in \mathcal{H}$. Since A contains a $G(\aleph_0)$-family of nice subgroups, it is totally projective. □

From this theorem we derive several corollaries. But first, a definition.

Let A be a subgroup of the p-group G. We will say that $\mathcal{S} = \{S_i \mid i \in I\}$ (where $A \leq S_i \leq G$) is a $G(\aleph_0)$**-family of separative subgroups over** A if each $S_i \in \mathcal{S}$ is separative in G and $\{S_i/A \mid i \in I\}$ is a $G(\aleph_0)$-family in G/A.

Corollary 7.9 (Hill [17]). *For an isotype subgroup A of a totally projective p-group G the following are equivalent:*

(i) *A is totally projective;*
(ii) *G admits a $G(\aleph_0)$-family $\mathcal{S}$ of separative subgroups over A;*
(iii) *there is a $G(\aleph_0)$-family $\mathcal{T}$ of totally projective groups over A.*

Proof. (i) $\Rightarrow$ (ii) If A is totally projective, then by Theorem 7.8, G admits a $G(\aleph_0)$-family $\mathcal{H}$ of nice subgroups compatible with A. Without loss of generality, we may moreover assume that each $X \in \mathcal{H}$ satisfies $X \cap A \in \mathcal{A}$ for a $G(\aleph_0)$-family $\mathcal{A}$ of nice subgroups of A. Manifestly, $(A+X)/X \cong A/(A \cap X)$ is totally projective, and hence separative in G/X. Then (F) implies $A+X$ is separative in G; consequently, $\mathcal{S} = \{A+X \mid X \in \mathcal{H}\}$ is a $G(\aleph_0)$-family of separative subgroups over A.

(ii) $\Rightarrow$ (iii) Let $\mathcal{S}$ be a $G(\aleph_0)$-family of separative subgroups over A, and $\mathcal{G}$ a $G(\aleph_0)$-family of nice subgroups of G. We now claim that the set $\mathcal{H} = \{X \in \mathcal{G} \mid X||A$ and $A+X \in \mathcal{S}\}$ is a $G(\aleph_0)$-family in G. (F) implies that the factor groups $(A+X)/X$ are separative in G/X, so Lemma 7.7 guarantees, for any given countable subgroup C of G, the existence of a countable subgroup Y/X of G/X that contains $(C+X)/X$ and is compatible with $(A+X)/X$. Using back-and-forth arguments, we may even assume that both $Y \in \mathcal{G}$ and $A+Y \in \mathcal{S}$. Hence Lemma 7.6(iii) implies $Y\|A$, thus $\mathcal{H}$ is in fact a $G(\aleph_0)$-family. Clearly, $\mathcal{T} = \{T \in \mathcal{S} \mid T = A+X$ with $X \in \mathcal{H}\}$ is still a $G(\aleph_0)$-family of separative subgroups over A. In view of Lemma 7.6(i), $X \cap A$ with $X \in \mathcal{H}$ is nice in A, thus $\mathcal{A} = \{X \cap A \mid X \in \mathcal{H}\}$ is a $G(\aleph_0)$-family of nice subgroups. Evidently, $T \in \mathcal{T}$ is the extension of X by $(A+X)/X \cong A/(A \cap X)$ where both groups are totally projective, the latter because of $X \cap A \in \mathcal{A}$. Since X is nice in G, T is likewise totally projective.

(iii) $\Rightarrow$ (i) is trivial, since $A \in \mathcal{T}$. □

An immediate corollary follows by using (A):

Corollary 7.10 (Hill [17]). *Isotype subgroups of a totally projective p-group of countable lengths are again totally projective* (*thus direct sums of countable groups*). □

We deduce one more consequence of our discussion above; this is concerned with countable chains of totally projective subgroups.

Corollary 7.11 (Hill [17]). *Let $G_0 < G_1 < \cdots < G_n < \cdots$ be a countable ascending chain of p-groups each of which is isotype in their union G. If every G_n is totally projective, then so is G.*

Proof. Let $\mathcal{G}_0$ be a $G(\aleph_0)$-family of nice subgroups of G_0. Since G_0 is totally projective and isotype in G_1, by Lemma 7.6, G_1 admits a $G(\aleph_0)$-family $\mathcal{G}_1$ of nice subgroups compatible with G_0. Arguing in the same way, we can find a $G(\aleph_0)$-family $\mathcal{G}_n$ of nice subgroups in G_n, compatible with the groups $G_0, \ldots, G_{n-1}$.

Observe that since G_i contains G_n if $i \geq n$, G_i is trivially compatible with every subgroup in G_n. Define

$$\mathcal{H} = \{H \leq G \mid H \cap G_n \in \mathcal{G}_n \text{ for each } n < \omega\}.$$

A routine back-and-forth argument convinces us that $\mathcal{H}$ is a $G(\aleph_0)$-family in G. We claim that each $H \in \mathcal{H}$ is nice in G. Given $g \in G$, $g \in G_j$ holds for some index j. As $H_j = H \cap G_j$ is nice in G_j, there is an $a \in H_j$ such that $h_G(g + x) \leq h_G(g + a)$ for all $x \in H_j$. If $b \in H_n$ $(n > j)$, then by compatibility $h_G(g + b) \leq h_G(g + c)$ for some $c \in H_j \cap H_n = H_j$. Thus $h_G(g + b) \leq h_G(g + a)$ for all $b \in H$, and therefore $g + a$ is proper with respect to H. □

More Total Projectivity For later reference, we will find useful to have the following lemma.

Lemma 7.12. *Let C be a subgroup of the reduced p-group A.*

(i) *If C is finite, then A is totally projective if and only if so is A/C.*
(ii) *If C is of finite index in A, then A is totally projective if and only if so is C.*
(iii) (Wallace [1]) *If $|A/C| \leq \aleph_0$ and C is totally projective, then A is totally projective.*

Proof.

(i) Finite subgroups being nice, the claim follows easily from Lemma 2.4.
(ii) Only the case $|A : C| = p$ needs to be considered. Then C is an extension of pA by C/pA. Here pA is nice both in A and C, so C is totally projective if and only if so is pA.
(iii) There is a countable subgroup $H < A$ such that $A = C + H$, and since C is totally projective, we may assume that $C \cap H$ belongs to a nice system in C. Then $\phi\colon c + (C \cap H) \mapsto c + H$ $(c \in C)$ yields an isomorphism $C/(C \cap H) \cong A/H$, showing that A/H is totally projective. It remains to show that H is nice in A. Let $h_p(c + H) = \sigma$ for some $c \in C$. Then ϕ implies $h_p(c + (C \cap H)) = \sigma$, so by niceness there is $x \in C \cap H$ such that $h_p(c + x) = \sigma$. Thus the element $c + x \in A$ has the same height as its coset mod H. □

Relative Balanced-Projective Resolution The following relative resolution of a p-group with respect to an isotype subgroup reveals another close relation between total projectivity and separative chains. (For an analogous idea, see Sect. 3 in Chapter 12.)

Lemma 7.13. *Let A be a subgroup of the p-group G. There exists a balanced-exact sequence*

$$0 \to B \xrightarrow{\alpha} A \oplus C \xrightarrow{\phi} G \to 0 \tag{11.11}$$

where $\phi \restriction A$ is the inclusion map, and C is a totally projective p-group. Then $\operatorname{Ker}\phi = \alpha B$ *is isomorphic to the projection B' of αB into C, and $C/B' \cong G/A$. Furthermore,*

(i) *A is isotype in G if and only if B' is isotype in C;*
(ii) *A is separative in G exactly if B' is separative in C;*
(iii) *G admits a $G(\aleph_0)$-family a separative subgroups over A (a separative chain from A to G) if and only if C has such a family over B'.*

Proof. The existence of a balanced-exact sequence with C and ϕ as stated is rather obvious; e.g. C in a balanced-projective resolution of G will do it (B is defined as the kernel of $A \oplus C \to G$). Form the commutative diagram

$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus C \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & G \arrow[r] & G/A \arrow[r] & 0,
\end{tikzcd}$$

with exact rows and middle column; the 3×3-lemma ensures that the third column is also exact. It is clear that $C/B' \cong G/A$ where $B' = \beta B$ (β is the projection map onto C).

As B (identified with αB) is balanced in $A \oplus C$, A is isotype (separative) in $G = (A \oplus C)/B$ if and only if $A \oplus B$ is isotype (separative) in $A \oplus C$. Because of $B \cap A = 0$, we have $B \cong \beta B = B' \leq C$. Hence $A \oplus B = A \oplus B'$. Thus $A \oplus B$ is isotype (separative) in $A \oplus C$ exactly if B' is isotype (separative) in C. (iii) follows in the same way. □

We can now deduce at once:

Theorem 7.14. *A p-group G admits a $G(\aleph_0)$-family a separative subgroups over its isotype subgroup A if and only if in a relative balanced-projective resolution* (11.11) *the kernel B is totally projective.* □

★ **Notes.** Theorem 7.2 can be complemented by adding that factor groups of totally projective groups modulo fully invariant subgroups are also totally projective. This too was proved independently by Fuchs–E. Walker (unpublished) (see [IAG]) and Linton [1].

Both the definition and most of the results on separativity are due to Hill [17] (he used different terminology: separability, which is used here in a different context). He also proved that the class of all balanced subgroups of reduced totally projective p-groups is closed under passing to direct sums, direct summands, and balanced subgroups. Moreover, for an ordinal σ, a group A belongs to this class exactly if both $p^\sigma A$ and $A/p^\sigma A$ do.

For uncountable cardinals κ, κ-separativity was studied by Fuchs–Hill [1] (in the definition of separativity replace 'countable' by κ) in order to determine the balanced-projective dimensions of p-groups. Several results in this section carry over almost *verbatim* to κ-separativity. If $0 \to B \to A \to C \to 0$ is a balanced-exact sequence with A a totally projective p-group, then B has an $H(\kappa)$-family of κ-separative subgroups if and only if C has an $H(\kappa^+)$-family of κ^+-separative subgroups. It was also proved that *the balanced-projective dimension of p-groups can be any integer* ≥ 0 *or* ∞; in particular, for each $n \geq 0$, any p^{ω_n}-high subgroup of a p-group A of cardinality $\aleph_n$ satisfying $p^{\omega_n}A \neq 0$ has balanced-projective dimension exactly n. (A corollary is that balanced subgroups in totally projective groups need not be totally projective.)

Hill [19] proved a strong isomorphism theorem on balanced subgroups of totally projective p-groups: If B, B' are balanced subgroups in the totally projective p-groups A, A' such that they have the same UK-invariants and satisfy $A/B \cong A'/B'$, then there is an isomorphism $A \to A'$ carrying B onto B'. Warfield [4] proved a criterion on the isomorphism of two isotype, λ-dense subgroups, C and C', of a totally projective p-group G of limit length λ. If $\dim G[p]/C[p] = \dim G[p]/C'[p]$, then there is an automorphism of G inducing an isomorphism $C \cong C'$.

Relative balanced-projective resolutions were introduced by Bican–Fuchs [Comm. Algebra **22**, 1031–1036 (1994)] for modules. They will play a more substantial role in Chapter 14.

Some classes of isotype subgroups of totally projective p-groups have well-developed theories. These include the classes of S-groups and A-groups. Let λ denote a limit ordinal not cofinal with ω. An isotype subgroup H of a totally projective p-group G is called a **λ-elementary S-group** if $G/H \cong \mathbb{Z}(p^\infty)$ and $G = H + p^\sigma G$ for all $\sigma < \lambda$. H is a **λ-elementary A-group** if G/H is (possibly non-reduced) totally projective and $p^\sigma(G/H) = (H + p^\sigma G)/H$ for all $\sigma < \lambda$. An S-group (A-group) is the direct sum of λ-elementary S-groups (A-groups) for various ordinals λ. S-groups can also be defined as torsion subgroups of Warfield groups. To classify S- and A-groups (within the mentioned class, of course), one needs one invariant in addition to the UK-invariants. The proofs are long and difficult. See Warfield [6], Hill [20], Hill–Megibben [6], as well as the literature quoted there.

Exercises

(1) Show that compatibility is reflexive, but not transitive.
(2) If $A \| B$ and $A \cap B = 0$, then $h(a+b) = \min\{h(a), h(b)\}$ for all $a \in A$ and $b \in B$.
(3) There exists a separative chain from A to G if there is a smooth chain $A = A_0 < \cdots < A_\alpha < \cdots < A_\tau = G$ of separative subgroups such that the cardinalities of the factor groups $A_{\alpha+1}/A_\alpha$ $(\alpha < \tau)$ are $\leq \aleph_1$.
(4) Let C be a subgroup of the p-group A such that $nA \leq C \leq A$ for some $n \in \mathbb{N}$. Then C is a direct sum of countable groups if and only if so is A.
(5) (Hill) Let A be a direct sum of countable groups, and suppose that the UK-invariants of A are $\leq \aleph_1$. An isotype subgroup of A is a direct sum of countable groups if and only if it is separative in A.
(6) If A is totally projective, then every endomorphism of $p^\sigma A$ extends to an endomorphism of A.

8 p^σ-Purity

There exist several versions of generalized purity, each of which is motivated by a particular feature of pure subgroups. In this section, we deal with one that depends on an arbitrarily chosen ordinal, and seems to be of special interest from the homological point of view.

Definition of p^σ-Purity Let σ denote an ordinal or ∞. The exact sequence $\mathfrak{e}$: $0 \to B \to A \to C \to 0$ is called p^σ**-pure-exact**, and the subgroup B of A p^σ**-pure** if

$$\mathfrak{e} \in p^\sigma \operatorname{Ext}(C, B).$$

Our study begins with listing a few easy consequences of the definition (Irwin–Walker–Walker [1], Nunke [3]).

(A) *For $n \in \mathbb{N}$, B is p^n-pure in A exactly if $p^k B = B \cap p^k A$ for each $k \leq n$.* This follows at once from Theorem 5.2(i). In particular, a p^1-pure subgroup is the same as a p-neat subgroup.

(B) *The p^ω-pure subgroups are exactly the p-pure subgroups in the old sense.*

(C) *If $\mathfrak{e}\colon 0 \to B \to A \to C \to 0$ is a p^σ-pure-exact sequence, then for every group G, the image of the connecting map* $\operatorname{Hom}(G, C) \to \operatorname{Ext}(G, B)$ *is contained in* $p^\sigma \operatorname{Ext}(G, B)$. Indeed, the connecting homomorphism carries $\xi \in \operatorname{Hom}(G, C)$ into $\mathfrak{e}\xi \in p^\sigma \operatorname{Ext}(G, B)$, as $\mathfrak{e}\xi$ is the image of $\mathfrak{e}$ also under the map $\operatorname{Ext}(C, B) \to \operatorname{Ext}(G, B)$ induced by ξ.

Similarly, for every group H, $\operatorname{Hom}(B, H)$ is mapped into $p^\sigma \operatorname{Ext}(C, H)$.

(D) *If $\mathfrak{e}$ is like in* (C), *and if $\beta : B \to B'$, $\gamma : C' \to C$ are arbitrary homomorphisms, then both sequences $\beta\mathfrak{e}$ and $\mathfrak{e}\gamma$ are p^σ-pure-exact.* This is routine.

Anticipating Theorem 8.2, we verify a generalization of Sect. 6(B) in Chapter 5:

(E) (Irwin–Walker–Walker [1]) *Let B be a p^σ-pure* (*and hence p^σ-balanced*) *subgroup in A such that A/B is divisible. Then $A/p^\rho A \cong B/p^\rho B$ for all $\rho < \sigma$.* Now B satisfies both $B + p^\rho A = \cap_{\nu<\rho}(B + p^\nu A)$ and $p^\rho B = B \cap p^\rho A$ for all $\rho < \sigma$. Hence from $B + pA = A$ by induction we obtain $B + p^\rho A = A$, thus $B/p^\rho B = B/(B \cap p^\rho A) \cong (B + p^\rho A)/p^\rho A = A/p^\rho A$ for all $\rho < \sigma$.

The main features of purity are retained by this generalization as is shown by (F)–(K) in addition to (C)–(E) [we skip the long proofs of (H) and (J)].

(F) *If B is p^σ-pure in A, and $H \leq B$, then B/H is p^σ-pure in A/H.* The canonical map $B \to B/H$ induces a homomorphism $\operatorname{Ext}(A/B, B) \to \operatorname{Ext}(A/B, B/H)$, which yields $p^\sigma \operatorname{Ext}(A/B, B) \to p^\sigma \operatorname{Ext}(A/B, B/H)$, proving the claim.

(G) *If B is p^σ-pure in A, then it is p^σ-pure in every subgroup H between B and A.* The injection $H/B \to A/B$ induces the homomorphism $p^\sigma \operatorname{Ext}(A/B, B) \to p^\sigma \operatorname{Ext}(H/B, B)$.

(H) *Assuming $H \leq B \leq A$, if H is p^σ-pure in B and B is p^σ-pure in A, then H is p^σ-pure in A.*

(J) *If H is p^σ-pure in A, and B/H in A/H, then H is p^σ-pure in A.*

(K) *The exact sequence $\mathfrak{e} : 0 \to B \to A \to C \to 0$ is p^∞-pure-exact if and only if it is torsion-splitting.* By definition, $\mathfrak{e}$ is p^∞-pure-exact if it belongs to $p^\infty \operatorname{Ext}(C, B)$. This Ext is a p-adic module, the last group is the divisible part of $\operatorname{Ext}(C, B)$. Hence the claim follows from Proposition 9.1 in Chapter 9.

p^σ-Balanced Subgroups A subgroup B of A is called p^σ**-isotype** if

$$p^\rho B = B \cap p^\rho A \quad \text{for all } \rho < \sigma,$$

and p^σ**-nice** if

$$p^\rho(A/B) = (p^\rho A + B)/B \quad \text{for all } \rho < \sigma.$$

A subgroup that is both p^σ-isotype and p^σ-nice will be called p^σ**-balanced**. It is straightforward to verify:

(a) *B is p^σ-isotype in A if and only if, for each $\rho < \sigma$, the map $B/p^\rho B \to A/p^\rho A$ induced by the inclusion is monic.*
(b) *B is p^σ-nice in A if and only if the map $A/p^\rho A \to C/p^\rho C$ induced by the surjection $A \to C = A/B$ is epic for all $\rho < \sigma$.*
(c) *B is p^σ-balanced in A if and only if the exact sequence $0 \to B \to A \to C \to 0$ implies the exactness of*

$$0 \to p^\rho B \to p^\rho A \to p^\rho C \to 0 \qquad \text{for all } \rho < \sigma.$$

A more substantial criterion for p^σ-balancedness is recorded in the next lemma.

Lemma 8.1. *A subgroup B of A is p^σ-balanced if and only if the exact sequence $0 \to B \to A \xrightarrow{\phi} C \to 0$ satisfies*

$$\phi(p^\rho A[p]) = p^\rho C[p] \qquad \text{for all } \rho < \sigma. \tag{11.12}$$

Proof. Suppose B is p^σ-balanced in A. For any $c \in p^\rho C[p]$ ($\rho < \sigma$), there is an $a \in p^\rho A$ satisfying $\phi a = c$. Now $pa \in p^{\rho+1}A \cap B = p^{\rho+1}B$, thus $pb = pa$ for a suitable $b \in p^\rho B$. Evidently, $a - b \in p^\rho A[p]$ is mapped by ϕ upon c, thus $p^\rho C[p] \leq \phi(p^\rho A[p])$. The inclusion in the opposite direction being trivial, (11.12) follows.

Conversely, assume (11.12) holds. The first stage is the verification of the inclusion $p^\rho C[p^k] \leq \phi(p^\rho A)$ for all $k \in \mathbb{N}$. Hypothesis (11.12) takes care of the case $k = 1$. Assume the inclusion holds for $k \geq 1$. If $c \in p^\rho C$ is of order p^{k+1}, then there is an $x \in A$ such that $\phi(x) = c$, and by induction hypothesis, some $y \in p^{\rho+1}A$ satisfies $\phi(y) = pc \in p^{\rho+1}C[p^k]$. Choose an $a_0 \in p^\rho A$ with $pa_0 = y$, and notice that $\phi(x - a_0) = c - \phi(a_0) \in p^\rho C[p]$. Consequently, $c - \phi(a_0) = \phi(a_1)$ for some $a_1 \in p^\rho A[p]$, and therefore, $c = \phi(a)$ with $a = a_0 + a_1 \in p^\rho A$. This shows that $\phi : p^\rho A \to p^\rho C$ is onto.

For the rest, we refer to the last part of the proof of Proposition 5.5. □

As every subgroup is p^n-nice ($n \in \mathbb{N}$), p^n-pure subgroups are also p^n-balanced. The same implication holds for all σ, but it is not as trivial.

Theorem 8.2.

(i) *Every p^σ-pure-exact sequence is p^σ-balanced-exact.*

(ii) *A p^σ-balanced-exact sequence $\mathfrak{e} : 0 \to B \to A \xrightarrow{\phi} C \to 0$ with C divisible is p^σ-pure-exact.*

Proof.

(i) Suppose $\mathfrak{e}$ is p^σ-pure-exact. Pick $c \in p^\rho C[p]$, and invoking Corollary 4.6 extend the isomorphism $p^\rho H_{\rho+1} = \langle g \rangle \to \langle c \rangle$ (where $H_{\rho+1}$ is a generalized Prüfer group and $g \mapsto c$) to a homomorphism $\gamma : H_{\rho+1} \to C$ (recall $p^\rho H_{\rho+1}$ is nice and $H_{\rho+1}/p^\rho H_{\rho+1}$ is simply presented). By the p^σ-pure-exactness of $\mathfrak{e}$ and the $p^{\rho+1}$-projectivity of $H_{\rho+1}$ [see Sect. 9(d)], there is $\psi : H_{\rho+1} \to A$ such that $\phi\psi = \gamma$. Thus $\psi(g) \in p^\rho A[p]$ maps upon c.

(ii) To prove the second claim, assume C is divisible, and $\mathfrak{e}$ is p^σ-balanced-exact. We induct on $\rho\,(\leq \sigma)$ to show that $\mathfrak{e} \in p^\rho \mathrm{Ext}(C, B)$. For $\rho = 1$, the claim is easily established. If ρ is a limit ordinal, then because of $p^\rho \operatorname{Ext}(C, B) = \bigcap_{\alpha<\rho} p^\alpha \operatorname{Ext}(C, B)$, there is nothing to prove. So let $\rho = \tau + 1$ and suppose $\phi(p^\alpha A[p]) = p^\alpha C[p] = C[p]$ for all $\alpha \leq \tau$. From the hypothesis it follows that there is a subgroup $H \leq p^\tau A[p]$ such that $A[p] = B[p] \oplus H$ and $\phi H = C[p]$. This leads to a commutative diagram

$$\begin{tikzcd}
\mathfrak{e}\colon\ 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & A \arrow[r, "\phi"] \arrow[d, "\delta"'] & C \arrow[r] \arrow[d, "\dot{p}"] & 0 \\
\mathfrak{e}'\colon\ 0 \arrow[r] & B \arrow[r, "\beta"] & A/H \arrow[r, "\gamma"] & C \arrow[r] & 0
\end{tikzcd}$$

where the middle (last) vertical map is the canonical one (multiplication by p), β is the obvious one, while $\gamma(a + H) = \phi(pa)$ for $a \in A$.

Since $\mathfrak{e} = p\mathfrak{e}'$, the proof will be complete if we can show that the bottom row satisfies condition (11.12), i.e. $\gamma(p^\alpha(A/H)[p]) = C[p]$ for all $\alpha \leq \tau$. Pick a $c \in C[p]$; by divisibility, there is a $d \in C$ such that $pd = c$. Now (11.12) implies that there exists an $x \in H \leq p^{\alpha+1}A[p]$ satisfying $\phi(x) = c$; let $a \in p^\alpha A$ be such that $pa = x$. Then $d - \phi(a) \in C[p]$, so $d - \phi(a) = \phi(y)$ for some $y \in H$. The element $a' = a + y$ satisfies $a' + H \in (p^\alpha A/H)[p]$ and $\gamma(a' + H) = p\phi(a) = pd = c$. □

We interrupt our discussion of p^σ-balancedness in order to verify an interesting corollary to Theorem 8.2 that improves on Lemma 5.1.

Theorem 8.3 (Nunke [4]). *p^σ-high subgroups of p-groups are $p^{\sigma+1}$-pure* (*and therefore isotype*).

Proof. As p^n-high subgroups ($n \in \mathbb{N}$) are summands, for the proof we may assume $\sigma \geq \omega$. Let H be p^σ-high in A. We claim $(A/H)[p] = (p^\rho A[p] + H)/H$ for each $\rho \leq \sigma$. To justify this, first note that the inclusion $\geq$ is obvious. So suppose $a \in A \setminus H$ satisfies $pa \in H$. By maximality, $\langle H, a \rangle \cap p^\sigma A[p] \neq 0$, whence $a \in p^\sigma A[p] + H \leq p^\rho A[p] + H$, in fact. Hence the exact sequence $0 \to H \to A \xrightarrow{\gamma} A/H \to 0$

satisfies (11.12) for $\sigma + 1$ (rather than for σ). Since $\sigma \geq \omega$, A/H is divisible, so the claim follows in view of Theorem 8.2. That H is then isotype is evident. □

The following example exhibits an interesting p^σ-pure subgroup.

Example 8.4. Consider an epimorphism $\xi : H_\sigma \to \mathbb{Z}(p^\infty)$ where H_σ is the generalized Prüfer group of length $\sigma \geq \omega$, and ξ maps a generator (in a simple presentation) of H_σ of order p^n upon the generator c_n of order p^n of $\mathbb{Z}(p^\infty)$. We claim that $M_\sigma = \operatorname{Ker}\xi$ is p^σ-pure in H_σ. To demonstrate that the exact sequence

$$0 \to M_\sigma \to H_\sigma \xrightarrow{\xi} \mathbb{Z}(p^\infty) \to 0$$

has property (11.12), note that since $\mathbb{Z}(p^\infty)$ is divisible, the right-hand side of (11.12) is just the socle of $\mathbb{Z}(p^\infty)$ for each $\rho < \sigma$. From the definitions of H_σ and the map ξ, it is clear that (11.12) is trivially satisfied whenever σ is a successor ordinal: the non-zero elements of height σ in H_σ are mapped upon non-zero elements of the socle of $\mathbb{Z}(p^\infty)$. If σ is a limit ordinal, then H_σ is the direct sum of groups H_ρ for $\rho < \sigma$, and the property is obvious. Hence the sequence is p^σ-balanced-, and so by Theorem 8.2(ii) also p^σ-pure-exact.

We continue with pointing out remarkable connections with Tor.

Lemma 8.5. *If the exact sequence* $0 \to B \to A \xrightarrow{\phi} C \to 0$ *is* p^σ*-balanced for some* $\sigma \geq \omega$*, then so is the induced exact sequence*

$$0 \to \operatorname{Tor}(X, B) \to \operatorname{Tor}(X, A) \xrightarrow{\phi_*} \operatorname{Tor}(X, C) \to 0$$

for every group X*.*

Proof. The exactness of this sequence is guaranteed by the p-purity of the first sequence (Theorem 3.1 in Chapter 8). Keeping Lemma 8.1 in mind, it suffices to show that this sequence satisfies condition (11.12). In view of Lemmas 4.2 and 4.3 in Chapter 8, $p^\sigma \operatorname{Tor}(X, A)[p] = \operatorname{Tor}(p^\sigma X[p], p^\sigma A[p])$; this group is isomorphic to the direct sum of κ copies of $p^\sigma A[p]$, where $\kappa = \dim(p^\sigma X[p])$. Similar assertion holds for $p^\sigma \operatorname{Tor}(X, C)[p]$, and by definition, the induced map carries copies of $p^\sigma A[p]$ upon corresponding copies of $p^\sigma C[p]$. Thus (11.12) for the given sequence implies that (11.12) holds for the sequence of the Tors. □

Proposition 8.6 (Keef [7]). *An exact sequence* $0 \to B \to A \to C \to 0$ *of* p*-groups is* p^σ*-pure for an infinite ordinal* σ *if and only if the induced sequence*

$$0 \to \operatorname{Tor}(H_\sigma, B) \to \operatorname{Tor}(H_\sigma, A) \to \operatorname{Tor}(H_\sigma, C) \to 0$$

is splitting (H_σ *is the generalized Prüfer group of length* σ)*.*

Proof. First suppose the given sequence is p^σ-pure. Then the sequence of the Tors is exact, since $\sigma \geq \omega$ (cf. Theorem 3.1 in Chapter 8). It is even p^σ-pure-exact, because by Theorem 3.8 in Chapter 9 $\operatorname{Ext}(\operatorname{Tor}(H_\sigma, C), \operatorname{Tor}(H_\sigma, B)) \cong \operatorname{Ext}(H_\sigma, \operatorname{Ext}(C, \operatorname{Tor}(H_\sigma, B))$, and H_σ is p^σ-projective (Sect. 9(d)). Taking into consideration that by Sect. 9(f) the Tors are p^σ-projective, the claim in one direction follows.

Conversely, assume that the sequence of Tors splits, so that there is a splitting map $\mathrm{Tor}(H_\sigma, C) \to \mathrm{Tor}(H_\sigma, A)$. The exact sequence in Example 8.4 yields a surjective map $\mathrm{Tor}(H_\sigma, A) \to \mathrm{Tor}(\mathbb{Z}(p^\infty), A) \cong A$. Then with the help of the composite of these two maps we form a commutative square on the right, and complete it to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}(M_\sigma, C) & \longrightarrow & \mathrm{Tor}(H_\sigma, C) & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the top row is by Example 8.4 and Lemma 8.5 p^σ-pure-exact (M_σ is from Example 8.4). Owing to (D), the same holds for the bottom row. ☐

***C_λ*-Groups** p^σ-purity plays an important role in the theory of C_λ-groups; these groups were introduced by Megibben [5]. Let λ denote a countable limit ordinal. A p-group A is called a **C_λ-group** if, for each ordinal $\sigma < \lambda$, $A/p^\sigma A$ is a direct sum of countable groups.

Example 8.7.

(i) Every p-group A is a C_ω-group, since $A/p^n A$ is Σ-cyclic for all $n < \omega$.
(ii) A direct sum of countable p-groups of lengths $< \lambda$ ($< \omega_1$) is a C_λ-group.

The following observations are obvious for a fixed λ.

(a) *Direct sums and summands of C_λ-groups are again C_λ-groups.*
(b) *Ulm factors of C_λ-groups are Σ-cyclic groups.*
(c) *If A is a C_λ-group, then $p^\sigma A$ is a $C_{\lambda'}$-group for some λ', and $A/p^\sigma A$ is a C_σ-group provided σ is a limit ordinal.*
(d) *p^λ-pure subgroups of a C_λ-group are likewise C_λ-groups.* If G is a p^λ-pure subgroup of a C_λ-group C, then it is p^λ-balanced [see Theorem 8.2(i)], so it satisfies $G \cap p^\sigma C = p^\sigma G$ for all $\sigma < \lambda$. Then $(G + p^\sigma C)/p^\sigma C$ is isotype in $C/p^\sigma C$, since

$$[(G + p^\sigma C) \cap p^\rho C]/p^\sigma C = [(G \cap p^\rho C) + p^\sigma C]/p^\sigma C = (p^\rho G + p^\sigma C)/p^\sigma C$$

for all $\rho < \sigma$. As $C/p^\sigma C$ is now a direct sum of countable groups, by Corollary 7.10, $G/p^\sigma G \cong (G + p^\sigma C)/p^\sigma C$ is also a direct sum of countable groups.

***λ*-Basic Subgroups** Let again λ be a countable limit ordinal. A subgroup B of a p-group A is said to be a **λ-basic subgroup** (Megibben [5]) if

(i) it is a direct sum of countable groups of lengths $< \lambda$,
(ii) it is p^λ-pure in A; and

(iii) A/B is a divisible group.

Since properties (ii) and (iii) imply that $A/p^\sigma A \cong B/p^\sigma B$, it is clear that the UK-invariants of B are determined by the UK-invariants of $A/p^\sigma A$. Hence we derive the uniqueness of λ-basic subgroups up to isomorphism, provided they exist.

Theorem 8.8 (Megibben [5]). *A p-group admits a λ-basic subgroup if and only if it is a C_λ-group.*

Proof. Let A be a p-group with λ-basic subgroup B. From the isomorphism $A/p^\sigma A \cong B/p^\sigma B$ the necessity is evident. Conversely, assume A is a C_λ-group. For each $\sigma < \lambda$, let K_σ denote a p^σ-high subgroup of A; there is no loss of generality in assuming that the K_σ form a chain. Our claim is that $B = \cup_{\sigma<\lambda} K_\sigma$ is a direct sum of countable groups. As each K_σ is isotype in A, the same holds for their union. Furthermore, K_σ is isomorphic to an isotype subgroup of a direct sum $A/p^\sigma A$ of countable groups, so Corollary 7.10 implies that K_σ is likewise a direct sum of countable groups. Thus we have a countable chain of isotype subgroups of B, each is a direct sum of countable groups of length $< \lambda$. It only remains to appeal to Corollary 7.11 to conclude that B satisfies condition (i).

The rest will follow from the obvious containment $A[p] \leq B[p] + p^\sigma A$ for all $\sigma < \lambda$. This, along with $B \cap pA = pB$, implies that A/B is divisible. It also implies that condition (11.12) holds for B for all $\sigma < \lambda$, so B is p^λ-balanced in A. Then (ii) follows from Theorem 8.2(ii). $\square$

★ **Notes.** p^σ-purity is a natural generalization of purity, and its importance lies also in its role which it had in leading to the concept of total projectivity. The properties we have not proved in details will not be needed in the rest of this volume.

Nunke [3] defines S-purity for a subfunctor S of the identity, and proves several theorems generalizing results on p^σ-purity. (In the definition, p^σ Ext is replaced by S-Ext.) For more relevant results, see Irwin–Walker–Walker [1] and Nunke [3].

Cutler [3] proves the existence of a p-group for which it is undecidable in ZFC whether or not all of its p^ω-high subgroups are isomorphic. Isomorphism of high subgroups is known, however, in special cases.

The theorems on C_λ-groups are straightforward generalizations of the classical theory. Megibben [5] unveiled several additional features of interest, like 'separability': every finite subset is contained in a countable direct summand, or that a p^λ-pure subgroup of length $< \lambda$ is a summand. Though the theory sheds more light on Kulikov's theorems on p-groups, so far it has not produced exciting results. The book Salce [S] contains a lot of information on C_λ-groups. Generalizations have been discussed by Crawley [4] and Albrecht [5] for uncountable limit λ. Linton [2] introduces *λ-large subgroups* as fully invariant subgroups L of A such that $L + B = A$ for all λ-basic B. If A is a C_λ-group, then A/L is simply presented.

A surprising result on C_{ω_1}-groups of length ω_1 is due to Keef [1]. He proved that the torsion product of two such groups is a direct sum of countable groups if and only if the so-called Kurepa hypothesis fails. (This principle, which holds in L, says that there is a tree of height ω_1 with countable levels and $\aleph_2$ branches. It is independent of ZFC+GCH.)

Exercises

(1) (Nunke) If $0 \to B \to A \to C \to 0$ is a p^σ-pure-exact sequence of p-groups, then $0 \to \mathrm{Tor}(B, X) \to \mathrm{Tor}(A, X) \to \mathrm{Tor}(C, X) \to 0$ is likewise p^σ-pure-exact.
(2) (Keef) An exact sequence $0 \to B \to A \to C \to 0$ is p^σ-pure-exact if and only if the natural map $\mathrm{Tor}(H_\sigma, C) \to C$ factors through $A \to C$.
(3) If C is isotype in A, then for every ordinal σ, $p^\sigma C$ is isotype in $p^\sigma A$.
(4) (Irwin–Walker–Walker) If B, C are subgroups in a p-group A such that $B \cap C$ is p^σ-pure in A, then C is p^σ-pure in $B + C$.
(5) An infinite subgroup of a C_λ-group is contained in a p^λ-balanced subgroup of the same cardinality.
(6) The UK-invariants $f_\rho(H)$ of a p^σ-high subgroup H of A are equal to the corresponding UK-invariants $f_\rho(A)$ of A provided $\rho < \sigma$.
(7) (Hill) The exact sequence $0 \to B \to A \to C \to 0$ is p^σ-pure-exact if $A[p] = B[p] + p^\rho A[p]$ holds for all $\rho < \sigma$. [Hint: if $\sigma \geq \omega$, C is divisible, apply Theorem 8.2(ii).]

9 The Functor p^σ

In this section we are interested in p-groups, and therefore we will restrict our discussion to p-groups, though sometimes we might get a more meaningful statement without this restriction.

We call a group A p^σ**-projective** (σ an ordinal or ∞) if

$$p^\sigma \,\mathrm{Ext}(A, G) = 0 \quad \text{for all } p\text{-groups } G.$$

In Sect. 6, we gave a slightly different definition by allowing G to be any group, but for p-groups A, this change does not make any difference. Indeed, this is obvious for $\sigma \in \mathbb{N}$, and for $\sigma \geq \omega$ it follows from the natural isomorphism $\mathrm{Pext}(A, G) \cong \mathrm{Pext}(A, tG)$ (Sect. 5 in Chapter 9, Exercise 7).

Example 9.1. Evidently, $A/p^\sigma A$ is p^σ-projective for every ordinal σ whenever A is a totally projective p-group.

p^σ-Projectivity We have the following easily verified properties.

(a) *A p^σ-projective group has the projective property with respect to p^σ-pure-exact sequences.* From the definition it is obvious that C is p^σ-projective if and only if all p^σ-pure-exact sequences of the form $\mathfrak{e}\colon 0 \to B \to A \to C \to 0$ are splitting. If C is arbitrary and $\phi\colon P \to C$ is any map from a p^σ-projective p-group P, then $\mathfrak{e}\phi$ is p^σ-pure-exact as well [see Sect. 8(D)], so splitting. Hence there exists a map $P \to A$ required to establish p^σ-projectivity.
(b) *Direct sums and summands of p^σ-projective groups are again p^σ-projective.*

(c) *A p-group is p^n-projective for an integer n if and only if it is p^n-bounded. It is p^ω-projective if and only if it is Σ-cyclic.* This follows at once from Theorem 4.3 in Chapter 5.

(d) *The generalized Prüfer group H_σ of length σ is p^σ-projective.* Once we know that H_σ is totally projective, this becomes obvious.

(e) *If in the exact sequence $0 \to G \to F \to A \to 0$, F is a Σ-cyclic p-group and G is p^n-bounded, then A is $p^{\omega+n}$-projective.* To prove this, observe that in the induced exact sequence $\mathrm{Hom}(G, \star) \to \mathrm{Ext}(A, \star) \to \mathrm{Ext}(F, \star) \to \mathrm{Ext}(G, \star)$ the groups at both ends are p^n-bounded and $p^\omega \mathrm{Ext}(F, \star) = 0$. Therefore $p^{\omega+n} \mathrm{Ext}(A, \star) = 0$ [cf. Sect. 1(A)–(B) in Chapter 10].

(f) *If A is p^σ-projective, then so is* $\mathrm{Tor}(A, C)$ *for every C.* In the natural isomorphism

$$\mathrm{Ext}(\mathrm{Tor}(A, C), B) \cong \mathrm{Ext}(A, \mathrm{Ext}(C, B))$$

(see Theorem 3.8 in Chapter 9), we have $p^\sigma \mathrm{Ext}(A, \mathrm{Ext}(C, B)) = 0$ for every B.

(g) *The length of a p^σ-projective p-group is at most σ.* (This has been proved in Sect. 6(B), here we give a different proof.) The claim is obvious whenever σ is an integer, so assume $\sigma \geq \omega$. From the p^σ-pure-exact sequence in Example 8.4 we derive the exact sequence $0 \to \mathrm{Tor}(M_\sigma, A) \to \mathrm{Tor}(H_\sigma, A) \to A \to 0$ which has to be p^σ-balanced as stated in Proposition 8.6. The length of $\mathrm{Tor}(H_\sigma, A)$ cannot exceed σ (see Lemma 4.2 in Chapter 8), hence the criterion (11.12) in Lemma 8.1 ensures that $p^\sigma A = 0$.

(h) *All torsion groups are p^∞-projective.* This follows from Theorem 6.5(ii) in Chapter 9.

Enough p^σ-Projectives By a **p^σ-projective resolution** of a p-group A is meant a p^σ-pure-exact sequence $0 \to G \to P \to A \to 0$ where P is a p^σ-projective p-group. (h) shows that p^∞-projective resolutions of p-groups are trivial, so we deal only with the case when σ is an ordinal.

Theorem 9.2 (Nunke [5]). *For each ordinal $\sigma \geq \omega$, every p-group A admits a p^σ-projective resolution.*

Proof. For $\sigma \geq \omega$, consider the exact sequence $0 \to \mathbb{Z} \to N_\sigma \to H_\sigma \to 0$ where N_σ is the Nunke group for the prime p (cf. Sect. 1 in Chapter 15). For any p-group A, there is an induced exact sequence

$$\mathfrak{e}\colon\ 0 \to \mathrm{Tor}(N_\sigma, A) \to \mathrm{Tor}(H_\sigma, A) \xrightarrow{\delta} A \to 0. \tag{11.13}$$

In fact, $\mathrm{Tor}(\mathbb{Z}, A) = 0$ holds in the induced long tensor-Tor exact sequence, and the surjective map $N_\sigma \otimes A \to H_\sigma \otimes A$ is actually an isomorphism, since the elements of the subgroup $\mathbb{Z}$ have infinite heights in N_σ, so in the tensor product they annihilate the p-group A.

We know from (f) that $\mathrm{Tor}(H_\sigma, A)$ is p^σ-projective, so in order to confirm that (11.13) is a p^σ-projective resolution of A, it only remains to verify that $\mathfrak{e} \in p^\sigma T$ where $T = \mathrm{Ext}(A, \mathrm{Tor}(N_\sigma, A))$. The map δ in (11.13) induces a homomorphism

$$\delta^* \colon T \to \operatorname{Ext}(\operatorname{Tor}(H_\sigma, A), \operatorname{Tor}(N_\sigma, A))$$
$$\cong \operatorname{Ext}(H_\sigma, \operatorname{Ext}(A, \operatorname{Tor}(N_\sigma, A))) = \operatorname{Ext}(H_\sigma, T).$$

δ was the natural map for A induced by the exact sequence $0 \to \mathbb{Z} \to N_\sigma \to H_\sigma \to 0$, so δ^* must also be the natural map for T. By Lemma 1.5 in Chapter 15, the kernel of δ^* is $p^\sigma T$. We now form the extension $\mathfrak{e}\delta$ *via* pull-back:

$$\begin{array}{rcccccccc}
\mathfrak{e}\delta\colon 0 & \longrightarrow & \operatorname{Tor}(N_\sigma, A) & \longrightarrow & G & \longrightarrow & \operatorname{Tor}(H_\sigma, A) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \delta} & & \\
\mathfrak{e}\colon 0 & \longrightarrow & \operatorname{Tor}(N_\sigma, A) & \longrightarrow & \operatorname{Tor}(H_\sigma, A) & \xrightarrow{\delta} & A & \longrightarrow & 0
\end{array}$$

Observe that the top row splits by Lemma 2.1 in Chapter 9 as the identity map of $\operatorname{Tor}(H_\sigma, A)$ furnishes a map between its two copies for the commutative triangle. This tells us that $\mathfrak{e} \in \operatorname{Ker}\delta^* = p^\sigma T$, as desired. $\square$

We are at once led to the following corollary.

Corollary 9.3 (Nunke [5]). *Assuming $\sigma \geq \omega$, a p-group A is p^σ-projective if and only if it is a summand of* $\operatorname{Tor}(H_\sigma, A)$. $\square$

Example 9.4. There exist p-groups that are not p^σ-projective for any σ. A good example is any reduced p-group A that is not starred, i.e. has larger cardinality than its basic subgroup. It is trivially true that such an A is not p^n-projective for any $n \leq \omega$. Let $\sigma \geq \omega$. In view of Lemma 4.8 in Chapter 8, all subgroups of $\operatorname{Tor}(H_\sigma, A)$ have the same cardinality as their basic subgroups. Therefore, A cannot be a summand of $\operatorname{Tor}(H_\sigma, A)$, though it were if it was p^σ-projective Corollary 9.3.

On p^σ-Projective p-Groups We record more useful results on p^σ-projectives.

Proposition 9.5 (Nunke [5]). *A $p^{\sigma+1}$-pure subgroup of a p^σ-projective p-group is a summand* (*and hence it is also p^σ-projective*).

Proof. Consider a $p^{\sigma+1}$-pure-exact sequence $0 \to C \to A \to A/C \to 0$ where A is p^σ-projective. In the induced exact sequence [cf. Sect. 8(C)]

$$\operatorname{Hom}(C, C) \xrightarrow{\alpha} p^\sigma \operatorname{Ext}(A/C, C) \to p^\sigma \operatorname{Ext}(A, C)$$

the last term is 0 in view of the hypothesis on A, thus α is an epimorphism. Since C is $p^{\sigma+1}$-pure, $\operatorname{Im}\alpha \leq p^{\sigma+1} \operatorname{Ext}(A/C, C)$ [see Sect. 8(C)], and therefore $p^\sigma \operatorname{Ext}(A/C, C) = p^{\sigma+1} \operatorname{Ext}(A/C, C)$ is a p-divisible group. But if A/C is a p-group, then Theorem 6.5 in Chapter 9 implies $\operatorname{Ext}(A/C, C)$ is reduced. Consequently, $p^\sigma \operatorname{Ext}(A/C, C) = 0$, and the given exact sequence splits. $\square$

Corollary 9.6 (Nunke [5]). *If the p-group A is not p^σ-projective, then* $\operatorname{Tor}(H_{\sigma+1}, A)$ *is $p^{\sigma+1}$-, but not p^σ-projective.*

Proof. If $\sigma \in \mathbb{N}$, then the claim is pretty clear. If $\sigma \geq \omega$, then consider the $p^{\sigma+1}$-pure-projective resolution $0 \to \mathrm{Tor}(N_{\sigma+1}, A) \to \mathrm{Tor}(H_{\sigma+1}, A) \to A \to 0$. If $\mathrm{Tor}(H_{\sigma+1}, A)$ were p^σ-projective, then the sequence would split (Proposition 9.5). Hence A would be p^σ-projective, contradicting the hypothesis. □

Example 9.7. There are separable p^σ-projective p-groups for every ordinal σ. In fact, if A is any separable p-group, then $\mathrm{Tor}(H_\sigma, A)$ is separable and p^σ-projective. Actually, if A is a group like the one in Example 9.4, then for distinct ordinals σ, $\mathrm{Tor}(H_\sigma, A)$ will supply groups with different levels of pure-projectivity, as is evident from Proposition 9.5.

Proposition 9.8 (Irwin–Walker–Walker [1]). *If σ is a limit ordinal, then a p^σ-projective p-group is a summand of a direct sum of p^ρ-projective groups for $\rho < \sigma$.*

Proof. Let P be p^σ-projective. For each $\rho < \sigma$, select a p^ρ-projective resolution $0 \to H_\rho \to P_\rho \xrightarrow{\phi_\rho} P \to 0$ with p^ρ-projective P_ρ. Form the exact sequence $0 \to K \to \oplus_{\rho<\sigma} P_\rho \xrightarrow{\phi} P \to 0$ with $\phi = \nabla(\oplus \phi_\rho)$, $K = \mathrm{Ker}\,\phi$. It is easily seen that this sequence is p^ρ-pure-exact for all $\rho < \sigma$, so it is also p^σ-pure-exact. Hence it splits, and the claim follows. □

***p^σ*-Injectivity** We now move on to consider the dual notion. A p-group A is called **p^σ-injective** (σ an ordinal or ∞) if

$$p^\sigma\,\mathrm{Ext}(G, A) = 0 \quad \text{for all } p\text{-groups } G. \tag{11.14}$$

It is **totally injective** if

$$p^\sigma\,\mathrm{Ext}(G, A/p^\sigma A) = 0 \quad \text{for all ordinals } \sigma \text{ and } p\text{-groups } G. \tag{11.15}$$

The following properties are easily verified from the definitions.

(A) *A p^σ-injective group has the injective property with respect to p^σ-pure-exact sequences.* The proof is dual to the proof in (a).
(B) *If A is a totally injective p-group, then so are all the factor groups $A/p^\sigma A$.*
(C) *Summands and torsion parts of products of p^σ-injective* (*totally injective*) *p-groups are again p^σ-injective* (*totally injective*).
(D) *A reduced p-group is p^n-injective if and only if it is p^n-bounded. The reduced p^ω-injective p-groups are precisely the torsion-complete groups.* This is a consequence of Theorem 3.6 in Chapter 10.
(E) *The length of a reduced p^σ-injective p-group cannot exceed σ.* If A has length $> \sigma$, then $A \cong t\,\mathrm{Ext}(\mathbb{Z}(p^\infty), A)$ shows that A cannot be p^σ-injective.
(F) *$t\,\mathrm{Ext}(C, A)$ is p^σ-injective for every C provided so is A.* This follows from the isomorphism $\mathrm{Ext}(B, \mathrm{Ext}(C, A)) \cong \mathrm{Ext}(\mathrm{Tor}(B, C), A)$, since the functor p^σ annihilates the right side, so the left side is 0 for every p-group B. (See Theorem 3.8 in Chapter 9.)
(G) *If C is a p^σ-projective p-group* (σ *an ordinal*), *then $t\,\mathrm{Ext}(C, A)$ is p^σ-injective for every group A.* This is clear in view of the isomorphism $\mathrm{Ext}(B, \mathrm{Ext}(C, A)) \cong \mathrm{Ext}(C, \mathrm{Ext}(B, A))$.

(H) *All p-groups are p^∞-injective.* This is obvious from Theorem 6.5(ii) in Chapter 9.

Example 9.9. A separable p-group is totally injective (p^σ-injective for an ordinal $\sigma \geq \omega$) if and only if it is torsion-complete. The total injectivity (the p^σ-injectivity) of a separable p-group A amounts to $p^\omega \operatorname{Ext}(X, A) = \operatorname{Pext}(X, A) = 0$ for all p-groups X. Because of Theorem 3.6 in Chapter 10, this holds for A exactly if A is torsion-complete.

One more result of interest on totally injective p-groups.

Proposition 9.10 (Fuchs–Salce [1]). *The Ulm factors of a reduced totally injective p-group are torsion-complete.*

Proof. For a totally injective p-group A, the exact sequence $0 \to A^\rho \to A \to A/A^\rho \to 0$ yields the exact sequence

$$0 = \operatorname{Hom}(\mathbb{Z}(p^\infty), A/A^\rho) \to \operatorname{Ext}(\mathbb{Z}(p^\infty), A^\rho) \to$$
$$\to \operatorname{Ext}(\mathbb{Z}(p^\infty), A)^\rho \to \operatorname{Ext}(\mathbb{Z}(p^\infty), A/A^\rho)^\rho = 0;$$

see Lemma 5.4 in Chapter 9. Consequently, $\operatorname{Ext}(\mathbb{Z}(p^\infty), A^\rho) \cong \operatorname{Ext}(\mathbb{Z}(p^\infty), A)^\rho$ holds for every ordinal ρ. Applying this isomorphism to the totally injective group $A/A^{\rho+1}$, and passing to the Ulm subgroup, we obtain the isomorphism $\operatorname{Ext}(\mathbb{Z}(p^\infty), A^\rho/A^{\rho+1})^1 \cong \operatorname{Ext}(\mathbb{Z}(p^\infty), A/A^{\rho+1})^{\rho+1} = 0$. Hence we conclude $\operatorname{Pext}(\mathbb{Z}(p^\infty), A^\rho/A^{\rho+1}) = 0$, so the ρth Ulm factor $A^\rho/A^{\rho+1}$ (for any ordinal ρ) is torsion-complete (Corollary 3.7 in Chapter 10). □

Finally, we mention without proof that *p^σ-injective resolutions exist for all $\sigma \geq \omega$.*

★ **Notes.** Since we have ignored torsion-free groups in the definition of p^σ-projectivity and -injectivity, some results sound different from those in the literature.

We have proved above that a p^σ-projective p-group behaves like a totally projective group in as much as for a limit ordinal σ, it is a summand of a direct sum of p^ρ-projectives for $\rho < \sigma$. Similar holds for p^σ-injectives, with direct sum replaced by torsion subgroup of the direct product.

p-groups A satisfying $p^\sigma \operatorname{Ext}(\mathbb{Z}(p^\infty), A/p^\sigma A) = 0$ for all σ were investigated by Salce [1]. His results are similar to the totally injective case.

Several authors discussed the p^λ-topology for limit ordinals λ, especially completions in this topology. See, e.g., Mines [1] and Dubois [1]. If $\operatorname{cf} \lambda > \omega$, the situation is different from the familiar metrizable topologies (e.g., Exercises 7 and 8).

Exercises

(1) (a) A p-group A is p^n-projective if and only if it is p^n-bounded.
(b) It is p^n-injective exactly if $A = B \oplus D$ where $p^n B = 0$ and D is divisible.

(2) (Irwin–Cellars–Snabb) A p^σ-projective p-group is embeddable in a totally projective p-group of length σ. [Hint: $A \to D$ implies $\operatorname{Tor}(A, H_\sigma) \to \oplus H_\sigma$ is monic, A a summand of Tor.]

(3) Let A be a p-group such that $A/p^\sigma A$ is p^σ-injective, and $p^\sigma A$ is p^n-bounded. Then A is $p^{\sigma+n}$-injective.
(4) A p-group A is p^σ-injective if and only if it is isomorphic to a summand of $t\,\mathrm{Ext}(H_\sigma, A)$.
(5) If σ is a limit ordinal, then a p^σ-injective p-group is a summand of the torsion subgroup of a direct product of p^ρ-injective groups for $\rho < \sigma$. [Hint: dualize the proof of Example 9.4.]
(6) (Fuchs–Salce) A p-group A satisfying $p^\sigma\,\mathrm{Ext}(\mathbb{Z}(p^\infty), A/p^\sigma A) = 0$ for all ordinals σ is called **almost totally injective**. Show that A is almost totally injective if and only if for an ordinal σ, both $p^\sigma A$ and $A/p^\sigma A$ are almost totally injective. [Hint: use the long exact sequence in Corollary 9.6.]
(7) (Waller) Let $A = \oplus_{i\in I} A_i$ be of length ω_1 where each A_i is a countable p-group. A is complete in the p^{ω_1}-topology.
(8) (Salce) There exists a p-group of length ω_1, whose completion in the p^{ω_1}-topology is not complete in its own p^{ω_1}-topology. [Hint: in a p^{ω_1}-balanced projective resolution of H_{ω_1+1}, take the completion of the kernel.]

10 $p^{\omega+n}$-Projective p-Groups

The initial interest in these groups was sparkled by Nunke's result that showed that they are composed of a Σ-cyclic and a bounded group. Moreover, the $p^{\omega+n}$-projective p-groups (for integers $n \geq 1$) share some properties which fail for p^σ-projectives with large σ.

Characterization We prove a main result of $p^{\omega+n}$-projective p-groups.

Theorem 10.1 (Nunke [5]). *For a p-group A, the following are equivalent:*

(i) *A is $p^{\omega+n}$-projective;*
(ii) *A contains a p^n-bounded subgroup P such that A/P is Σ-cyclic;*
(iii) *$A \cong F/G$ where F is a Σ-cyclic p-group and $p^n G = 0$.*

Proof.

(i) $\Rightarrow$ (ii) We start with the exact sequence

$$0 \to p^\omega H_{\omega+n} \to H_{\omega+n} \to H_\omega \to 0$$

which implies the exactness of

$$0 \to \mathrm{Tor}(p^\omega H_{\omega+n}, A) \to \mathrm{Tor}(H_{\omega+n}, A) \to \mathrm{Tor}(H_\omega, A).$$

In this sequence, the first Tor is p^n-bounded, and the third is Σ-cyclic. Hence $\mathrm{Tor}(H_{\omega+n}, A)$ is an extension of a p^n-bounded group P by a Σ-cyclic group. If A is $p^{\omega+n}$-projective, then it is a summand of this Tor (see Corollary 9.3), and as such it is likewise an extension of a p^n-bounded group by a Σ-cyclic group.

(ii) $\Rightarrow$ (iii) Let $\{a_i \mid i \in I\}$ be a set of representatives of generators of A mod P, and define $C = \oplus_{i\in I} \langle c_i \rangle$ as a Σ-cyclic group where we let c_i have the same order as a_i. Set $F = C \oplus P'$ with $P' \cong P$, so F is Σ-cyclic. Define an epimorphism $\phi : F \to A$ by sending c_i to a_i and mapping P' isomorphically upon P. It follows at once that $G = \operatorname{Ker} \phi$ is p^n-bounded.
(iii) $\Rightarrow$ (i) This implication was already proved in Sect. 9(e). □

Since property (ii) is inherited by subgroups, we can state:

Corollary 10.2. *Subgroups of $p^{\omega+n}$-projective p-groups are themselves $p^{\omega+n}$-projective.* □

It should be emphasized that the p^n-bounded subgroup P in condition (ii) is not uniquely determined by A, not even up to isomorphism. But we can claim that if P_1 and P_2 are p^n-bounded subgroups such that both A/P_1 and A/P_2 are Σ-cyclic, then $A/(P_1 \cap P_2)$ is again a Σ-cyclic group. In fact, it is a subgroup in the Σ-cyclic group $A/P_1 \oplus A/P_2$.

Example 10.3. For any subsocle S of a Σ-cyclic p-group A, the group A/S is $p^{\omega+1}$-projective. It can be separable or not, depending on the closure property of S.

The next result is a noteworthy generalization of Theorem 3.5 in Chapter 10.

Lemma 10.4 (Danchev [1]). *Let A be a separable p-group, and C a $p^{\omega+n}$-projective pure subgroup. If A/C is countable, then A too is $p^{\omega+n}$-projective.*

Proof. C contains a p^n-bounded subgroup P such that C/P is Σ-cyclic. The p-adic closure P^- of P in A is likewise p^n-bounded (Lemma 7.6 in Chapter 1), and since C/P is pure in A/P, and has no elements of infinite height, we have $C \cap P^- = P$. Again, C/P is pure in A/P, and it stays pure if we pass modulo $(A/P)^1 = P^-/P$. Thus $(C + P^-)/P^- \cong C/P$ is a pure Σ-cyclic subgroup of countable index in the separable p-group A/P^-, whence by Theorem 3.5 in Chapter 10 we conclude that A/P^- is Σ-cyclic, and thus A is $p^{\omega+n}$-projective. □

Isomorphism of $p^{\omega+n}$-Projective p-Groups Another noteworthy result on $p^{\omega+n}$-projectivity is an isomorphism theorem.

Theorem 10.5 (Fuchs [20]). *Two $p^{\omega+n}$-projective p-groups, A and C, are isomorphic if and only if there is a height-preserving isomorphism between $A[p^n]$ and $C[p^n]$.*

Proof. Only sufficiency requires verification. Let $\psi : A[p^n] \to C[p^n]$ denote a height-preserving isomorphism. In view of Theorem 10.1, there are subgroups $P \leq A[p^n]$ and $Q \leq C[p^n]$ such that A/P and C/Q are Σ-cyclic groups.

First, we show that P can be replaced by $P' = P \cap \psi^{-1}Q$ and Q by $Q' = \psi P \cap Q$, so that $\psi(P') = Q'$ can be assumed. We have to ascertain that A/P' and C/Q' are Σ-cyclic. Take into account that $(A/P')/(P/P') \cong A/P$ is Σ-cyclic, and so P/P' is nice in A/P', and the correspondence $x + P' \mapsto \psi x + Q$ ($x \in P$) is a monomorphism $\phi: P/P' \to C/Q$ not decreasing heights. Therefore, ϕ extends to a homomorphism $A/P' \to C/Q$. Hence there is a map $A/P' \to A/P \oplus C/Q$ which is easily seen to be monic. Thus A/P' is Σ-cyclic as a subgroup of a Σ-cyclic group.

We next show that the Hill invariants of A relative to P can be computed by using only elements of $A[p^n]$. For each ordinal σ, we have $(p^{\sigma+1}A + P) \cap p^{\sigma}A[p] = (p^{\sigma+1}A + P)[p] \cap p^{\sigma}A[p])$, therefore it is enough to prove that $(p^{\sigma+1}A + P)[p] \leq (p^{\sigma+1}A[p^n] + P)[p]$. But this is clear since if $p(a + x) = 0$ where $a \in p^{\sigma+1}A$ and $x \in P$, then $p^n a = -p^n x = 0$ implies $a \in p^{\sigma+1}A[p^n]$.

To finish the proof, we may argue under the assumption $\psi(P) = Q$. Manifestly, P is nice in A, and Q is nice in C. Since the Hill invariants can be computed in terms of the p^n-socles, those of P in A and of Q in C have to be equal. It only remains to appeal to Theorem 3.6 to conclude that ψ can be extended to an isomorphism $A \to C$. $\square$

It is interesting that whether or not a p-group A is $p^{\omega+n}$-projective is completely determined by $A[p^{n+1}]$ as a group valuated by heights in A (for valuated groups, cf. Sect. 3 in Chapter 15).

Proposition 10.6. *A $p^{\omega+n}$-projective p-group A can be reconstructed from the valuated group $A[p^{n+1}]$.*

Proof. Given A, consider $A[p^{n+1}]$ as valuated by the heights in A. For A to be $p^{\omega+n}$-projective, it is necessary that there exists a p^n-bounded subgroup P such that the socle of $A[p^{n+1}]/P$ is a free valuated vector space with integer values. To see that if such a P exists, then A is $p^{\omega+n}$-projective, observe that it is straightforward to construct an extension G of P by a Σ-cyclic group C such that the restriction to $C[p]$ is isometric to the socle of $A[p^{n+1}]/P$. This G is $p^{\omega+n}$-projective by construction. As P is nice in A, and the Hill invariants of P in A up to $n + 1$ are computable in $A[p^{n+1}]$, and are trivially 0 for $> n + 1$, from Theorem 3.6 we can conclude that $A \cong G$. $\square$

Ubiquity of $p^{\omega+1}$-Projective Subgroups The following result is a remarkable evidence of the ubiquity of proper $p^{\omega+1}$-projective p-groups. (By a **proper $p^{\omega+n}$-projective** group we mean a $p^{\omega+n}$-projective group that is not $p^{\omega+n-1}$-projective.)

Theorem 10.7 (Benabdallah–Irwin–Rafiq [1]). *Every separable p-group which is not Σ-cyclic contains a proper $p^{\omega+1}$-projective subgroup.*

Proof. Suppose A is a separable p-group that is not Σ-cyclic. Splitting off a bounded summand if necessary, we may assume that fin rk$(A) = |A|$. Let B be a lower basic subgroup of A, i.e. $|A/B| = |A|$. We can represent A/B as the countable union of divisible summands A_n/B where $A_0 = B$ and $|A/A_n| = |A|$ $(n < \omega)$. Then the A_n form a chain of pure subgroups with union A, so Theorem 5.5 in Chapter 3 guarantees the existence of an index $m \geq 1$ such that A_m is not Σ-cyclic.

Let $0 \to H \to F \to A_m \to 0$ be a pure-projective resolution of A_m where F is a Σ-cyclic p-group which is chosen such that fin rk$(F) = |A_m| \leq |A|$. Clearly, $F/(H[p])$ is a $p^{\omega+1}$-projective group which—because of $H[p] = H \cap F[p]$—embeds in the direct sum $F/H \oplus F/(F[p])$. The group $F/(H[p])$ is not Σ-cyclic, because otherwise the embedding $H[p] \to H$ viewed as a height-preserving map of a nice subgroup of F would extend to a homomorphism $\phi : F \to H$. It is readily checked that ϕ would then induce an automorphism of H, so that H were a summand of F—which is obviously not the case. Since A/A_m is divisible and has cardinality $|A|$, it contains a subgroup G/A_m isomorphic to the Σ-cyclic group $F/(F[p])$. As A_m is pure in G, we get $G = A_m \oplus C$ with $C \cong F/(F[p])$. Thus the proper $p^{\omega+1}$-projective group $F/(H[p])$ is isomorphic to a subgroup of G, and hence of A. $\square$

★ **Notes.** The 1962 New Mexico conference (the first US conference on abelian groups) recorded a substantial progress in the research of p^σ-purity and p^σ-projectivity, due to Irwin–Walker–Walker [1] and Nunke [3]. Some of their results have been discussed in this volume. That $p^{\omega+1}$-projective p-groups can be studied successfully *via* valuation was observed by Fuchs–Irwin [1]. It was shown that a separable proper $p^{\omega+1}$-projective p-group A admits a decomposition $A = C \oplus A'$ where C is Σ-cyclic of fin rk $C =$ fin rk A and $A' \cong A$. Cutler [2] proves that a $p^{\omega+n}$-projective p-group is determined in the class of all p-groups by its p^{n+1}-socle, but not by its p^n-socle. Cutler–Irwin–Snabb [1] show that if the reduced p-group A is not starred, then for every $n < \omega$, A contains a proper $p^{\omega+n}$-projective p-group. For more on $p^{\omega+n}$-projective p-groups, see Cutler–Missel [1]. $p^{\omega 2}$-projectives have been studied, but with not much success.

It is an old unsettled problem to describe extensions of Σ-cyclic by Σ-cyclic, and more generally, extensions of direct sums of countable groups by groups of the same kind. In the unbounded case such extensions need not share any projective property. Even if C is an essential subgroup of A, and if both C and A/C are Σ-cyclic p-groups, A need not be a direct sum of countable groups. In fact, let A be a separable $p^{\omega+n}$-projective p-group which is not Σ-cyclic, B a p^n-bounded subgroup with Σ-cyclic A/B. If the subgroup C of A is defined so as to have $C/B = (A/B)[p]$, then $p^{n+1}C = 0$. Hence C is Σ-cyclic, is essential in A, and $A/C \cong p(A/B)$ is Σ-cyclic. But A is not a direct sum of countable groups (and thus not totally projective), for otherwise—as a group with no elements of infinite height—it would be Σ-cyclic.

Fuchs–Salce [2] consider p-groups A, C of length $\rho + n$, where ρ is a limit ordinal and n is a positive integer. It is shown that $A \cong C$ provided there is an isomorphism $\phi : A/p^\rho A \to C/p^\rho C$ such that $\phi(A[p^n]/p^\rho A) = C[p^n]/p^\rho C$. Keef–Danchev [1] study the following generalization: call a p-group A *n-simply presented* if it contains a p^n-bounded subgroup P such that A/P is simply presented. They call an exact sequence $0 \to A \to B \to C \to 0$ *n-balanced* if it represents an element of $p^n \operatorname{Bext}(C, A)$, and C *n-balanced-projective* if $p^n \operatorname{Bext}(C, A) = 0$ for all A, and prove that the n-balanced-projective p-groups are precisely the summands of n-simply presented p-groups.

Salce [1] proves the dual of Theorem 10.1 for $p^{\omega+n}$-injectives, i.e. the equivalence of (i) *the p-group A is $p^{\omega+n}$-injective;* (ii) *A contains a p^n-bounded subgroup P such that A/P is torsion-complete;* (iii) $A \cong B/H$ *where B is torsion-complete and* $p^nH = 0$. He also verifies the isomorphy of $p^{\omega+n}$-injectives whose p^n-socles are isometric as valuated groups.

Exercises

(1) If a p-group A is $p^{\omega+n}$-projective, then so is A/A^1.
(2) (Benabdallah–Irwin–Lazaruk) If $0 \to H \to F \to A \to 0$ is a pure-exact sequence with F a Σ-cyclic p-group, then A is $p^{\omega+n}$-projective exactly if it is isomorphic to a summand of $F/H[p^n]$.
(3) (Benabdallah–Irwin–Lazaruk) A p-group A is $p^{\omega+n+1}$-projective exactly if it contains a subsocle S such that A/S is $p^{\omega+n}$-projective.
(4) (Benabdallah–Irwin–Lazaruk) If S is a subsocle of a p-group A that supports a pure subgroup of A, and if A/S is Σ-cyclic, then A is Σ-cyclic.
(5) (Keef) If A is a $p^{\omega+n}$-projective p-group, and ϕ is a height-non-decreasing homomorphism of $A[p^n]$ into a p-group C, then there is a homomorphism $A \to C$ which agrees with ϕ on the socle of A.

(6) (Danchev) A reduced p-group is $p^{\omega+n}$-projective if and only if it has a large subgroup that is $p^{\omega+n}$-projective.

11 Summable p-Groups

We continue the discussion of torsion groups with an interesting class which is somewhat more general than direct sums of countable p-groups. The study was initiated by Honda [3], and continued by Hill–Megibben [5] under the appropriate name 'summable groups.'

A p-group A is said to be **summable** if its socle $S = A[p]$ is a free valuated vector space. This means that $S = \oplus_\sigma S_\sigma$ where S_σ is a subgroup satisfying $p^\sigma A[p] = p^{\sigma+1}A[p] \oplus S_\sigma$. Hence every element $\neq 0$ of S_σ carries the same value σ.

Example 11.1. A Σ-cyclic p-group is summable, and so is the Prüfer group $H_{\omega+1}$ whose socle is isometric to $\oplus_{\sigma\leq\omega} S_\sigma$; here, $\dim S_\sigma = 1$ for each $\sigma \leq \omega$. Also, divisible p-groups are summable. On the other hand, no unbounded torsion-complete p-group is summable.

We mention the following properties of summability.

(A) *Countable p-groups are summable.* Their socles as countably valuated vector spaces are free Proposition 8.8 in Chapter 10.
(B) *Direct sums of summable p-groups are summable.* Hence *direct sums of countable p-groups are summable*.
(C) *Summands of summable p-groups are summable.* The socle of a summable p-group may be viewed as a direct sum of countable dimensional subspaces, so by Theorem 8.10 in Chapter 10, every summand has a socle of the same structure.
(D) *Countable isotype subgroups of summable p-groups are summable.*

The generalized Prüfer group $H_{\omega_1} = \oplus_{\sigma<\omega_1} H_\sigma$ is summable. However, no summable reduced p-group exists whose length exceeds ω_1.

Theorem 11.2 (Honda [3]). *A reduced summable p-group A satisfies $p^{\omega_1}A = 0$.*

Proof. Write $A[p] = S \oplus p^{\omega_1}A[p]$ where $S = \oplus_{\sigma<\omega_1} S_\sigma$, and all non-zero elements of S_σ have height σ. Working toward contradiction, assume there is an $a \in A$ of height ω_1. Choose a sequence $b_n \in A$ $(n < \omega)$ such that $pb_n = a$, and the heights $h(b_n) = \rho_n$ are increasing with n. Thus $b_{n+1} - b_n \in A[p]$ has non-zero coordinate in S_{ρ_n}, and we may even assume that it has 0 coordinate in every S_σ if $\sigma \geq \rho_{n+1}$. Since the ρ_n are countable ordinals, there is a $b \in A$ of countable height $\rho > \sup \rho_n$ satisfying $pb = a$. Then $b - b_0 = (b - b_n) + (b_n - b_{n-1}) + \cdots + (b_1 - b_0)$ shows that $b - b_0$ must have infinitely many non-zero coordinate in the decomposition of S, since $b_n - b_{n-1}$ (of height ρ_{n-1}) is the only term that has non-zero coordinate in $S_{\rho_{n-1}}$, except possibly for of $b - b_n$. However, this is impossible in a direct sum. $\square$

The preceding result is strengthened in (ii) of the following proposition.

Proposition 11.3.

(i) *An extension of a summable p-group by a countable p-group is again summable.*
(ii) (Hill–Megibben [5]) *The equation* $p^{\omega_1}\operatorname{Ext}(\mathbb{Z}(p^\infty), A) = 0$ *holds for every summable p-group A.*

Proof.

(i) Let G be an extension of the summable group A by the countable p-group X. There is a countable subgroup $C \leq A$ that contains all the relations of the extension; C may be assumed to be isotype. Evidently, G contains an extension H of C by X. Then $G[p]$ is obtained from $A[p]$ by replacing the free vector space $C[p]$ by the free $H[p]$ of countable dimension—thus freeness is preserved.
(ii) By way of contradiction, suppose $0 \to A \to G \xrightarrow{\gamma} \mathbb{Z}(p^\infty) \to 0$ is a non-splitting extension representing an element in $p^{\omega_1}\operatorname{Ext}(\mathbb{Z}(p^\infty), A)$. Thus this exact sequence is p^{ω_1}-pure, and hence p^{ω_1}-balanced by Theorem 8.2. This means that $\gamma(p^\sigma G[p]) = p^\sigma\mathbb{Z}(p^\infty)[p] \cong \mathbb{Z}(p)$ for all $\sigma < \omega_1$. This is impossible, because owing to (i), G is summable, and from the proof of (i) it is clear that $p^\sigma G[p] = p^\sigma A[p]$ holds for large enough σ. □

We should point out that the class of summable p-groups is strictly larger than the class of direct sums of countable p-groups. Indeed, for every ordinal σ with $\omega < \sigma \leq \omega_1$, there exist summable p-groups of length σ that are not direct sums of countable groups. Next we show that even ω_1-pure subgroups of summable groups need not be summable.

Example 11.4. Assume that $A = \oplus_\sigma H_{\sigma+1}$ where $H_{\sigma+1}$ runs over the generalized Prüfer groups of non-limit lengths $\sigma + 1 < \omega_1$; thus, A has length ω_1. For every σ, map $z_\sigma \in H_{\sigma+1}$ of height σ upon a generator of the socle of $\mathbb{Z}(p^\infty)$, and extend this map to obtain a homomorphism $\alpha : A \to \mathbb{Z}(p^\infty)$. As in Example 8.4, it follows that the arising exact sequence $0 \to M = \operatorname{Ker}\alpha \to A \to \mathbb{Z}(p^\infty) \to 0$ is ω_1-pure. Thus it represents a non-splitting extension in $p^{\omega_1}\operatorname{Ext}(\mathbb{Z}(p^\infty), M)$. In view of Proposition 11.3, M is not summable.

From this example we can derive a negative answer to a question which has stayed unanswered for several years: are subgroups of direct sums of countable p-groups again such direct sums? (See Example 7.1 where an explicit example was given.)

Lemma 11.5 (Nunke [5], Hill [5], Hill–Megibben [5]). *A subgroup of a direct sum of countable p-groups need not be a direct sum of countable groups.* □

★ **Notes.** Honda [3] started the study of summable p-groups under the name of *principal p-groups*. Summability plays an important role in generalizing the classical theory of p-groups to countable length; see Megibben [4]. Danchev [2] shows that if the p-group A has length $\leq \omega_1$, and $p^\sigma A$ and $A/p^\sigma A$ are both summable for some $\sigma < \omega_1$, then so is A.

Exercises

(1) A separable p-group is summable if and only if it is Σ-cyclic.

(2) (Hill–Megibben) Let A be a reduced p-group, and $A[p] = S \oplus p^{\omega_1}A[p]$. If the second summand is not 0, then S is not a summable socle.
(3) (Hill–Megibben) If A is a summable p-group, and H is a p^σ-high subgroup in A, then H is also summable.
(4) (Hill–Megibben) A summable p-group is a summand of every p-group in which it is p^{ω_1}-high. [Hint: Proposition 11.3(ii).]
(5) (Griffith) If G is a p^{ω_1}-high subgroup of the generalized Prüfer group H_{ω_1+1}, then G is isomorphic to an isotype subgroup of H_{ω_1+1}, but it is not summable. [Hint: Exercise 2.]
(6) (Salce) Let $\overline{B}$ be the torsion-complete p-group with the standard basic subgroup B. Then $G = \overline{B}/B[p]$ is a summable group of length $\omega + 1$ which fails to be totally projective.

12 Elongations of p-Groups

Let λ denote a limit ordinal, and E, G p-groups. If $G = p^\lambda A$ holds in the exact sequence $0 \to G \to A \to E \to 0$ (and so $E \cong A/p^\lambda A$), then we call A a **λ-elongation** of G by E (elongation has already been defined in Sect. 6). Needless to say, for given E, G, such an elongation exists only if the groups satisfy certain conditions.

Existence of Elongations We will state a necessary and sufficient condition for the existence of λ-elongations of p-groups. We require an ad hoc definition. If A is a p-group of length λ, then by its **transfinite final rank** we mean the cardinal tr-fin-rk$(A) = \min_{\sigma<\lambda} \operatorname{rk}(p^\sigma A[p])$. If $\lambda = \tau + 1$ is a successor ordinal, then tr-fin-rk$(A) = \operatorname{rk}(p^\tau A[p])$.

Lemma 12.1 (Nunke [7]). *Let E be a p-group of length λ, and G a reduced p-group with basic subgroup B. There exists a λ-elongation of G by E if and only if* $\operatorname{rk} B \leq$ tr-fin-rk(E).

Proof. In a λ-elongation A of $G \neq 0$ by E, for every basis element $b \in B$, and for every $\sigma < \lambda$, there is $x + G \in p^\sigma E[p]$ such that $px = b$. If $b' \in B$ is a different basis element, and $px' = b'$ for some $x' + G \in p^\sigma E[p]$, then $x - x' = g \in G$ is impossible. In fact, the equation $pg = b - b' \in B$ would contradict the purity of B in G. This shows that $p^\sigma E[p]$ must contain at least as many elements for every $\sigma < \lambda$ as $\operatorname{rk} B$.

Conversely, assume $\kappa = \operatorname{rk} B \leq$ tr-fin-rk(E) holds. Let H be the divisible hull of G, thus $D = H/G$ satisfies $D[p] \cong B/pB$, and therefore $\operatorname{rk} D[p] = \kappa$. As D is divisible, it is a straightforward arithmetical exercise to construct a map $\gamma : E \to D$ such that $\gamma(p^\sigma E[p]) = D[p]$ for all $\sigma < \lambda$. With such a γ, we form the pull-back diagram

$$\begin{array}{rccccccccc}
\mathfrak{e}\gamma\colon & 0 & \longrightarrow & G & \longrightarrow & A & \longrightarrow & E & \longrightarrow & 0 \\
& & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\
\mathfrak{e}\colon & 0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & D & \longrightarrow & 0,
\end{array}$$

and claim that A is a λ-elongation. This is an easy consequence of the fact that multiplication by p in H induces an isomorphism $D[p] \to B/pB$; indeed, the choice of γ guarantees that multiplication by p in A will furnish the basis elements of B with height λ in A. □

The second part of the preceding proof is essentially Kulikov's method of constructing elongations [3]. His choice for γ was a map whose kernel was isotype and dense in E.

Uniquely Elongating A p-group E has the **λ-Zippin property** if its length is λ, and if A, C are groups such that $A/p^\lambda A \cong E \cong C/p^\lambda C$, then every isomorphism that exists between $p^\lambda A$ and $p^\lambda C$ extends to a suitable isomorphism $A \cong C$. By Zippin's theorem 1.7, reduced countable p-groups E enjoy this property for countable λ.

We now focus our attention on ω-elongations; this is the most exciting case. Crawley [3] was the first to consider elongations, and raised the question as to which separable p-groups E have the property that all ω-elongations of $G \cong \mathbb{Z}(p)$ by E are isomorphic. This question turns out undecidable in ZFC. A weaker form asks for the isomorphism of all extensions of an arbitrary group G by E—this easier question is answered in the next theorem for separable groups.

Theorem 12.2 (Crawley [3], Hill–Megibben [4], Nunke [7], Warfield [5]). *A separable p-group E has the property that all of the ω-elongations of the same, arbitrarily selected p-group by E are isomorphic if and only if E is Σ-cyclic.*

Proof. Consider two ω-elongations, A_1 and A_2, of a group G by the Σ-cyclic p-group E. The identity map of G may be viewed as a height-preserving isomorphism between subgroups of the A_i. As E has no elements of infinite height, G is nice in A_i. Furthermore, a simple calculation shows that the nth Hill invariant of A_i relative to G is equal to the nth UK-invariant of E, so they are equal for A_1 and A_2. It remains to appeal to Theorem 3.6 to infer that the identity map of G extends to an isomorphism between the A_i.

For necessity, we prove that if E is a separable p-group, but not Σ-cyclic, then there is an elementary p-group G admitting non-isomorphic ω-elongations by E. There is no loss of generality in assuming that fin rk $E = |E| = \kappa$, where $\kappa \geq \aleph_1$. Let B be a lower basic subgroup of E, so that $|E/B| = \kappa$. Owing to the arguments in the proof of Theorem 10.7, E contains a pure subgroup C which is not Σ-cyclic such that $|E/C| = \kappa$. We now use the method described in Lemma 12.1 above to elongate an elementary p-group G of the size κ by E by using the socles of E/B and E/C, respectively. The resulting extensions are not isomorphic, since the socles

$B[p] \oplus G$ and $C[p] \oplus G$ are not isometric: the former is a free valuated vector space (G is free with value ω), but the latter is not. □

Crawley Groups A separable torsion group E is said to be a **Crawley group** if all ω-elongations of $G \cong \mathbb{Z}(p)$ by E are isomorphic. Megibben [7] proved that in ZFC it is undecidable whether or not Crawley groups of cardinality $\aleph_1$ are Σ-cyclic. We follow the Mekler–Shelah proof in solving the problem in the Constructible Universe without cardinality restriction.

In the following arguments, *H denotes an arbitrary, but fixed, p-group of cardinality $\leq \aleph_1$ such that $p^\omega H \cong \mathbb{Z}(p)$.* (For convenience, temporarily we may choose $H = H_{\omega+1}$, the Prüfer group of length $\omega + 1$, but later a different choice will be needed.)

Lemma 12.3 (Mekler–Shelah [1]). *Let E' be a separable p-group containing a subgroup E of countable index, and $0 \to G \to A \to E \to 0$ an ω-elongation with $G = \langle g \rangle$ of order p. If E'/E is not Σ-cyclic, then in the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & A & \longrightarrow & E & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \text{incl}} & & \\ 0 & \longrightarrow & G & \longrightarrow & A_i & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

groups A_i $(i = 1, 2)$ can be chosen such that any homomorphism $\phi : A \to H$ with $\phi(g) \neq 0$ extends to at most one of A_i.

Proof. The non-unique existence of the bottom exact sequence is a consequence of the fact that the induced map $\mathrm{Ext}(E', G) \to \mathrm{Ext}(E, G)$ is epic, but not necessarily monic. Suppose an elongation A_1 has been selected. As E'/E is countable, but not Σ-cyclic (so not separable), it has non-zero elements of infinite heights, i.e. there are elements $x_n \in E' \setminus E$ and $e_n \in E$ $(n < \omega)$ such that $px_0 = e_0, p^{n+1}x_{n+1} = x_0 + e_{n+1}$. To construct A_2, define an extension of G by $\langle E, x'_n\ (n < \omega)\rangle$ $(\leq E')$ subject to the relations $px'_0 = e_0 + g$, $p^{n+1}x'_{n+1} = x'_0 + e_{n+1}$, and then extend it to all of E' (this is possible as $\mathrm{Ext}(E', G) \to \mathrm{Ext}(E, G)$ is surjective).

Suppose that $\phi : A \to H$ extends to $\phi_i : A_i \to H$ for $i = 1, 2$. Then $p^{n+1}\phi_1(x_{n+1}) = \phi_1(x_0) + \phi(e_{n+1})$ and $p^{n+1}\phi_2(x'_{n+1}) = \phi_1(x'_0) + \phi(e_{n+1})$, as well as $p\phi_1(x_0) = \phi(e_0)$ and $p\phi_2(x'_0) = \phi(e_0) + \phi(g)$. Subtracting, we get

$$p^{n+1}(\phi_2(x'_{n+1}) - \phi_1(x_{n+1})) = \phi_2(x'_0) - \phi_1(x_0).$$

Hence $\phi_2(x'_0) - \phi_1(x_0) \in p^\omega H_{\omega+1}$. But $p(\phi_2(x'_0) - \phi_1(x_0)) = \phi(g) \neq 0$, a contradiction. □

Next, we extend the preceding lemma to the general case when the cardinality of E'/E is arbitrary. This process requires the assumption of the diamond principle.

Lemma 12.4 (Mekler–Shelah [1] (V = L)). *Let again E' be a separable p-group containing a subgroup E, and $0 \to G \to A \to E \to 0$ an ω-elongation with $G = \langle g \rangle$ of order p. Suppose E'/E is not Σ-cyclic, and $\phi\colon A \to H$ satisfies $\phi(g) \neq 0$.*

Then there is an elongation $0 \to G \to A' \to E' \to 0$, extending the given one, such that ϕ cannot be extended to A'.

Proof. Observe that the condition that E'/E is not Σ-cyclic is necessary for the existence of an elongation as required. For otherwise, every map $\phi\colon A \to H$ extends to A', as A is nice in A' and A'/A is Σ-cyclic.

We induct on the cardinality $\kappa = |E'/E|$. The limit case being clear, we may assume κ is regular, and also uncountable, and the claim holds for smaller cardinals Lemma 12.3. We will be done if we can find a subgroup C with $E < C < E'$ such that the corresponding elongation has the stated property. Therefore, we may assume for the rest of the proof that C/E is Σ-cyclic whenever $|C/E| < \kappa$, and hence C/E has a κ-filtration C_σ/E ($\sigma < \kappa$) with Σ-cyclic subgroups. As E'/E is not Σ-cyclic, the set

$$S = \{\sigma < \kappa \mid C_{\sigma+1}/C_\sigma \text{ is not } \Sigma\text{-cyclic}\}$$

must be stationary in κ. The diamond principle guarantees the existence of a set $\{\gamma_\sigma \mid \sigma \in S\}$ of functions $\gamma_\sigma : C_\sigma \to H$ such that, for every function $\gamma : E' \to H$, there is a $\sigma \in S$ satisfying $\gamma \restriction C_\sigma = \gamma_\sigma$.

We define a direct system of elongations $0 \to G \to A_\sigma \to C_\sigma \to 0$ for $\sigma < \kappa$ starting with the given one ($A_0 = A$). For the proof, an extension A_σ is viewed as being built on a subset on $G \times E'$. Only the step from σ to $\sigma + 1$ has to be specified. For each $\sigma < \kappa$, any extension of G by $C_{\sigma+1}$ that is a continuation of the extension by C_σ can be selected, except when $\sigma \in S$ is such that $\phi_\sigma : A_\sigma \to H$ is a homomorphism inducing γ_σ ($\sigma \in S$). In this exceptional case, we can apply the induction hypothesis (because $A_{\sigma+1}/A_\sigma$ is not Σ-cyclic) to get an extension $A_{\sigma+1}$ such that ϕ_σ does not extend to any map $A_{\sigma+1} \to H$.

We claim that the direct limit sequence $0 \to G \to A' \to E' \to 0$ is as required. Suppose, on the contrary, there is an extension $\phi' : A' \to H$ of ϕ. Evidently, ϕ' defines a function $\gamma\colon E' \to H$ whose restriction to some C_σ ($\sigma \in S$) equals γ_σ. But this is a contradiction to the choice of $A_{\sigma+1}$. □

Lemma 12.5 (Mekler–Shelah [1]). *Assuming $V = L$, a separable p-group E is Σ-cyclic exactly if every ω-elongation A of $\mathbb{Z}(p)$ by E admits a homomorphism $\phi\colon A \to H$ such that $\phi(p^\omega A) \neq 0$.*

Proof. If A is an ω-extension of $G \cong \mathbb{Z}(p)$ by a Σ-cyclic E, then a map $G \to p^\omega H$ extends to a homomorphism $A \to H$.

Conversely, suppose E has the stated property. Theorem 12.2 takes care of the case $|E| = \aleph_0$, so suppose that $|E| = \kappa > \aleph_0$, and that groups of smaller sizes with this property are Σ-cyclic. Hence all subgroups of E of cardinality $< \kappa$ are Σ-cyclic. If κ is a regular cardinal, then Lemma 12.4 implies that in a κ-filtration of E with Σ-cyclic subgroups the set S as defined in the preceding proof cannot be stationary. This means that E is Σ-cyclic. The rest follows from the singular compactness theorem. $\square$

After all this preparation we can now prove:

Theorem 12.6 (Mekler–Shelah [1]). *In the Constructible Universe, every Crawley group is Σ-cyclic.*

Proof. Let E be a separable p-group that is not Σ-cyclic. Owing to Lemma 12.5, we can choose an ω-elongation A of $\mathbb{Z}(p)$ by E, and a group G such that $p^\omega G \cong \mathbb{Z}(p)$ and $|G| \leq 2^{\aleph_0}(= \aleph_1)$ hold true, and there is a homomorphism $\phi : A \to G$ with $\phi(p^\omega A) \neq 0$. But by Lemma 12.4 there exists an ω-elongation A' of $\mathbb{Z}(p)$ by E such that no homomorphism $\psi : A' \to G$ exists with $\psi(p^\omega A') \neq 0$. Hence $A \cong A'$ is impossible, and consequently, E is not a Crawley group. $\square$

Megibben [7] proved: MA+ $\neg$ CH implies that all $\aleph_1$-separable p-groups of cardinality $\aleph_1$ are Crawley groups; by Theorem 7.1 in Chapter 10 such groups do exist in ZFC without being Σ-cyclic. Consequently, Crawley's problem is undecidable in ZFC. We forgo the details of the proof.

★ **Notes.** By Mekler–Shelah [1], it is consistent that there exists a Crawley group of cardinality $\aleph_1$ which is not $\aleph_1$-separable. Also, assuming V = L, there exist a separable, non-Σ-cyclic p-group E of cardinality $\aleph_1$ and elongations A_σ ($\sigma < 2^{\aleph_1}$) of $\mathbb{Z}(p)$ by E such that if $\phi : A_\rho \to A_\sigma$ ($\rho \neq \sigma$) is a homomorphism, then $\phi(p^\omega A_\rho) = 0$. Mekler [1] shows that it is consistent with GCH that there exist non-Σ-cyclic Crawley groups, so the Crawley problem is undecidable in ZFC+GCH.

Eklof–Huber–Mekler [1] define a p-group A **totally Crawley** if $A/p^\lambda A$ is Crawley for all limit ordinals λ. It is undecidable in ZFC that for countable lengths, every totally Crawley group is a direct sum of countable groups.

Warfield [5] points out that ω-elongations by non-Σ-cyclic groups may be abundant. For instance, there are $2^{2^{\aleph_0}}$ non-isomorphic ω-elongations by a separable p-group A of cardinality $2^{\aleph_0}$ with countable basic subgroup; the elongated group can be any reduced p-group that admits ω-elongations by A.

Exercises

(1) (Nunke) A totally projective p-group E of limit length λ is uniquely λ-elongating.
(2) Summands in a uniquely elongating group are likewise uniquely elongating provided they are of the same length.
(3) Let E be a p-group of length λ, a limit cardinal, and N a fully invariant subgroup of the same length. If E is uniquely λ-elongating, then so is N.
(4) (Richman) A separable p-group A is Crawley if and only if, for any two dense subgroups of codimension 1 in $A[p]$, one is carried into the other by an automorphism of A.
(5) (Nunke) Let $0 \to G \to A \to E \to 0$ be an exact sequence. The connecting map $\mathrm{Tor}(E, \mathbb{Z}(p)) \to G \otimes \mathbb{Z}(p)$ defines a homomorphism $\chi : E[p] \to G/pG$. If λ is

the length of E, then the given exact sequence is a λ-elongation if and only if $\chi(p^\sigma E[p]) = G/pG$ holds for all $\sigma < \lambda$.

(6) Suppose that A is a p-group such that its first Ulm subgroup A^1 is a direct sum of countable groups, and A/A^1 is Σ-cyclic. Then A is likewise a direct sum of countable groups.

Problems to Chapter 11

PROBLEM 11.1. Investigate subgroups G in totally projective p-groups (or more generally, in transitive groups) A such that each height-preserving endomorphism of G extends to an endomorphism of A.

PROBLEM 11.2. Are there test groups for p^σ-projectives ? and for p^σ-injectives?

PROBLEM 11.3. Which p-groups contain $p^{\omega+1}$-projective pure subgroups?

PROBLEM 11.4. Does there exist a $p^{\omega+1}$-projective p-group of cardinality κ that contains a copy of every $p^{\omega+1}$-projective p-group of cardinality $\leq \kappa$?

PROBLEM 11.5. (Cutler) Are the (separable) p-groups A, C isomorphic if, for every $n \in \mathbb{N}$, the subgroups $A[p^n]$ and $C[p^n]$ are isometric as valuated groups?

PROBLEM 11.6. Find a complete set of invariants for p-groups of finite Ulm length whose Ulm factors are torsion-complete.

Richman [3] considers the case when the first Ulm subgroup is an elementary group.

PROBLEM 11.7. Let

$$\underline{\mathrm{u}} = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$$

be a strictly increasing sequence of ordinals and symbols ∞. The p-group A is an $\underline{\mathrm{u}}$-*elongation* of G by E if $G = A(\underline{\mathrm{u}})$ and $A/G \cong E$.

(a) When do $\underline{\mathrm{u}}$-elongations exist?
(b) For which groups E are such elongations of a fixed G by E all isomorphic?

PROBLEM 11.8. Imitating the definition of pure-completeness, call a p-group A *isotype-complete* if every subsocle that is isometric to the subsocle of a (totally projective) p-group supports an isotype subgroup of A. Study isotype-complete p-groups.

PROBLEM 11.9. Suppose A is totally projective p-group contained in a p-group G. What can be said about the subgroups C of G if $C[p]$ and $A[p]$ are isometric as valuated vector spaces?

Chapter 12
Torsion-Free Groups

Abstract In this chapter we start the discussion of torsion-free groups. First, we deal with general properties along with the finite rank case, and delegate the in-depth theory of torsion-free groups of infinite rank to the next chapter.

After presenting the basic definitions and facts, we enter the study of balancedness, a stronger version of purity, which we have already met in the theory of torsion groups. Turning to the problem of direct decompositions, we start with the discussion of indecomposable groups; we do not restrict ourselves to the finite rank case as it seems more natural to deal with this important problem without rank restrictions. Concentrating on the finite rank case, the study of pathological direct decompositions is followed by positive results, the highlight being Lady's theorem about the finiteness of non-isomorphic direct decompositions.

Other aspects of direct decompositions are also discussed, including quasi- and near-homomorphisms. Finite rank dualities will also be dealt with.

1 Characteristic and Type: Finite Rank Groups

In this section, all groups are assumed to be torsion-free, unless stated otherwise. $p_1, p_2, \ldots, p_n, \ldots$ will denote the sequence of prime numbers.

Every torsion-free group A is a subgroup of a $\mathbb{Q}$-vector space V such that a maximal independent set in A is a basis of V. If $\{x_i \ (i \in I)\}$ is a maximal independent set in A, then every element of A depends on it, and therefore every element $a \in A$ can be written uniquely as

$$a = r_1x_1 + \cdots + r_kx_k \qquad (r_i \in \mathbb{Q}).$$

Note that only certain combinations of rational coefficients are allowed, depending on the group and the choice of the maximal independent set.

Characteristics The most typical features of elements in torsion-free groups are crystallized in the concept corresponding to height. Let A be a torsion-free group and $a \in A$. Recall that, for a prime p, the largest integer k with $p^k | a$, i.e., for which the equation $p^kx = a$ is solvable in A, is called the p-**height** $h_p(a)$ of a. If no such maximal integer k exists, then we set $h_p(a) = \infty$. The sequence of p-heights,

$$\chi(a) = (h_{p_1}(a), h_{p_2}(a), \ldots, h_{p_n}(a), \ldots),$$

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is called the **characteristic** or the **height-sequence** of a. We will write $\chi_A(a)$ if we wish to indicate the group A in which the characteristic was computed. (For torsion-free groups, transfinite heights are unnecessary: elements of height ω are in the p-divisible subgroup, so have height ∞.)

The following observations are immediate consequences.

(a) $\chi(-a) = \chi(a)$ for all $a \in A$.
(b) If $\chi(a) = (k_1, k_2, \dots, k_n \dots)$ (where $k_n \geq 0$ or $= \infty$), then $a \in A$ is divisible by the integer $m = p_1^{\ell_1} \cdots p_r^{\ell_r}$ if and only if $\ell_i \leq k_i$ for $i = 1, \dots, r$.
(c) If we agree that $\infty + 1 = \infty$, then

$$\chi(p_n a) = (h_{p_1}(a), h_{p_2}(a), \dots, h_{p_n}(a) + 1, \dots).$$

(d) Every sequence $(k_1, k_2, \dots, k_n \dots)$ of non-negative integers and symbols ∞ is a realizable characteristic. In the subgroup R of $\mathbb{Q}$ that is generated by all $p_n^{-\ell_n}$ with $\ell_n \leq k_n$, for all n, the element 1 will have this characteristic. With this R, we can write $Ra = \langle a \rangle_*$ if $\chi_A(a) = (k_1, k_2, \dots, k_n \dots)$ in a group A.

The set of all characteristics is partially ordered under the pointwise ordering; it becomes a complete, distributive lattice of the cardinality of the continuum $2^{\aleph_0}$. The lattice operation $\wedge$ is the point-wise minimum, and $\vee$ is the point-wise maximum. The minimum member of this lattice is the characteristic $(0, 0, \dots, 0, \dots)$ of $1 \in \mathbb{Z}$, while the maximum is the characteristic $(\infty, \infty, \dots, \infty, \dots)$ of every element in $\mathbb{Q}$.

(e) If $b, c \in A$, then $\chi(b + c) \geq \chi(b) \wedge \chi(c)$. Thus

$$A(\chi) = \{a \in A \mid \chi(a) \geq \chi\}$$

is for each characteristic χ a (fully invariant) subgroup of A. Furthermore, if $A = B \oplus C$ and $b \in B, c \in C$, then $\chi(b + c) = \chi(b) \wedge \chi(c)$.
(f) A subgroup B of a torsion-free group A is pure exactly if $\chi_B(b) = \chi_A(b)$ for all $b \in B$.
(g) If B is a pure subgroup of the torsion-free A, then the characteristic of a coset mod B is computed according to the rule

$$\chi_{A/B}(a + B) = \bigvee_{b \in B} \chi_A(a + b).$$

(h) If $\alpha : A \to C$ is a homomorphism between torsion-free groups, then $\chi_A(a) \leq \chi_C(\alpha a)$ for all $a \in A$.

For a torsion-free A, the inclusion map $\mathbb{Z} \to \mathbb{Q}$ induces an embedding $A \cong \mathbb{Z} \otimes A \to \mathbb{Q} \otimes A$. The latter group is torsion-free, moreover, a $\mathbb{Q}$-vector space; its dimension is equal to the **rank** $\mathrm{rk}(A)$ of A. If B is a pure subgroup of A, then A/B is torsion-free, its rank is called the **corank** of B in A.

Types Two characteristics, $(k_1, k_2, \ldots, k_n, \ldots)$ and $(\ell_1, \ell_2, \ldots, \ell_n, \ldots)$, are said to be **equivalent** if $k_n = \ell_n$ for almost all n such that both k_n and ℓ_n are finite whenever $k_n \neq \ell_n$. The equivalence classes of characteristics are called **types**. If $\chi(a)$ belongs to type $\mathbf{t}$, then we say a is **of type t**, and write $\mathbf{t}(a) = \mathbf{t}$, or more explicitly, $\mathbf{t}_A(a) = \mathbf{t}$ if we wish to emphasize that the type of a is computed in the group A. Keep in mind that $\mathbf{t}(a) = \mathbf{t}(ma)$ for all $m \in \mathbb{N}$.

A type can be represented by any member in its equivalence class; thus, we can write $\mathbf{t} = (k_1, k_2, \ldots, k_n, \ldots)$ if the right-hand side is a characteristic in the class of $\mathbf{t}$. The lattice order on the set of characteristics induces a lattice order on the set $\mathfrak{T}$ of types. Accordingly, $\mathbf{s} \leq \mathbf{t}$ for the types $\mathbf{s}, \mathbf{t}$ means that there are characteristics $(k_1, k_2, \ldots, k_n, \ldots)$ in $\mathbf{s}$ and $(\ell_1, \ell_2, \ldots, \ell_n, \ldots)$ in $\mathbf{t}$, such that $(k_1, k_2, \ldots, k_n, \ldots) \leq (\ell_1, \ell_2, \ldots, \ell_n, \ldots)$. $\mathfrak{T}$ is a (non-complete) distributive lattice.

(A) Linearly dependent elements a, b in a group A have the same type. In fact, then $ra = sb$ for some non-zero integers r, s, and thus $\mathbf{t}(a) = \mathbf{t}(ra) = \mathbf{t}(sb) = \mathbf{t}(b)$.
(B) If $b, c \in A$, then $\mathbf{t}(b + c) \geq \mathbf{t}(b) \wedge \mathbf{t}(c)$, and if $A = B \oplus C$ and $b \in B, c \in C$, then $\mathbf{t}(b + c) = \mathbf{t}(b) \wedge \mathbf{t}(c)$.
(C) For $\alpha : A \to B$ and $a \in A$, we always have $\mathbf{t}(a) \leq \mathbf{t}(\alpha a)$.
(D) A torsion-free group A is called **homogeneous** if all of its elements $\neq 0$ are of the same type $\mathbf{t}$. In this case we write $\mathbf{t}(A) = \mathbf{t}$, and call $\mathbf{t}$ the **type of** A. With a type $\mathbf{t}$ we associate four fully invariant subgroups of a torsion-free group A. The two major ones are:

$$A(\mathbf{t}) = \{a \in A \mid \mathbf{t}(a) \geq \mathbf{t}\} \quad \text{and} \quad A^\star(\mathbf{t}) = \langle a \in A \mid \mathbf{t}(a) > \mathbf{t}\rangle.$$

$A(\mathbf{t})$ is always a pure subgroup in A, but $A^\star(\mathbf{t})$ need not be pure in A. Its purification satisfies $A^\star(\mathbf{t})_* \leq A(\mathbf{t})$. The **Baer invariants** of a torsion-free group A are the torsion-free ranks of the factor groups $A(\mathbf{t})/A^\star(\mathbf{t})$. Thus the Baer invariants are given by a function from the set of types to the cardinal numbers. Less useful are the fully invariant subgroups that are defined in terms of a characteristic: $A(\chi) = \{a \in A \mid \chi(a) \geq \chi\}$ and $A^\star(\chi) = \langle a \in A \mid \chi(a) > \chi\rangle$.

The other two noteworthy fully invariant subgroups are:

$$A[\mathbf{t}] = \cap_{\mathbf{t}(A/G) \leq \mathbf{t}} G \quad \text{and} \quad A^\star[\mathbf{t}] = \cap_{\mathbf{t}(A/G) < \mathbf{t}} G$$

where $\operatorname{rk} A/G = 1$. The ranks of these fully invariant subgroups are the **Arnold–Vinsonhaler invariants** of A.

Rank One Groups The non-zero subgroups of $\mathbb{Q}$ (i.e., the **rational groups**) are the rank 1 groups. In a rank 1 group R, all non-zero elements have the same type, the **type** $\mathbf{t}(R)$ of R. Our next task is to prove the fundamental theorem on rank 1 torsion-free groups.

Theorem 1.1 (Baer [6]). *Two torsion-free groups of rank* 1 *are isomorphic exactly if they are of the same type. Every type is realized by a subgroup of* $\mathbb{Q}$.

Proof. Let A, B be rank one torsion-free groups of the same type. If $a \in A, b \in B$ are non-zero elements, then $\mathbf{t}(a) = \mathbf{t}(b)$ implies that there are non-zero integers m, n such that $\chi_A(ma) = \chi_B(nb)$. The correspondence $ma \mapsto nb$ extends to an isomorphism $A \to B$.

The second claim is the immediate consequence of (d) above. ☐

It is useful to know the cardinality of the lattice of types.

Proposition 1.2. *The cardinality of the set of non-isomorphic rank one torsion-free groups is the power of the continuum.*

Proof. The set of all characteristics is evidently of the power $2^{\aleph_0}$, so the set of types cannot have a larger cardinality. On the other hand, different characteristics consisting only of 0's and ∞'s represent different types, so the cardinality in question is at least $2^{\aleph_0}$. From Theorem 1.1 we know that every type is realized by a rational group. ☐

The **product** $\chi_1\chi_2$ of the characteristics $\chi_1 = (k_1, k_2, \dots, k_n, \dots)$ and $\chi_2 = (\ell_1, \ell_2, \dots, \ell_n, \dots)$ is defined as

$$\chi_1\chi_2 = (k_1 + \ell_1, k_2 + \ell_2, \dots, k_n + \ell_n, \dots),$$

where naturally the sum of ∞ and anything is ∞. A characteristic χ is **idempotent** (i.e., $\chi^2 = \chi$) exactly if, for every n, k_n is either 0 or ∞. If $\chi_1 \geq \chi_2$, then there is a unique largest characteristic χ' such that $\chi_2\chi' = \chi_1$; this will be denoted by $\chi' = \chi_1 : \chi_2$. The multiplication of characteristics is compatible with the equivalence relation introduced above, so we may speak of the product $\mathbf{t}_1\mathbf{t}_2$ and quotient $\mathbf{t}_1 : \mathbf{t}_2$ of the types $\mathbf{t}_1, \mathbf{t}_2$, and of an idempotent type $\mathbf{t}^2 = \mathbf{t}$. Type $\mathbf{t}_0 = \mathbf{t} : \mathbf{t}$ is the largest idempotent type $\leq \mathbf{t}$: to obtain $\mathbf{t}_0$ from $\mathbf{t} = (k_1, k_2, \dots, k_n, \dots)$, replace the finite k_n by 0.

Lemma 1.3. *If A and C are torsion-free groups of rank* 1, *then their tensor product $A \otimes C$ is again torsion-free of rank* 1 *and*

$$\mathbf{t}(A \otimes C) = \mathbf{t}(A) \cdot \mathbf{t}(C).$$

Proof. Theorem 3.5 in Chapter 8 implies that $A \otimes C$ is a subgroup of $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$, so it is of rank 1. Write $\chi(a_0) = (k_1, k_2, \dots, k_n, \dots)$ and $\chi(c_0) = (\ell_1, \ell_2, \dots, \ell_n, \dots)$ for $a_0 \in A, c_0 \in C$. Then $a_0 \otimes c_0 \in A \otimes C$ will certainly be divisible by p_n^m for all $m \leq k_n + \ell_n$, so $\chi(a_0 \otimes c_0) \geq \chi(a_0)\chi(c_0)$. The correspondence $(ra_0 \otimes sc_0) \mapsto rs$ (where $ra_0 \in A, sc_0 \in C$ and $r, s \in \mathbb{Q}$) is a bilinear map $A \times C \to R$, where R is the rational group with $\chi_R(1) = \chi_A(a_0)\chi_C(c_0)$. From the definition of the tensor product it is clear that $\chi(a_0 \otimes c_0) > \chi_R(1)$ is impossible. ☐

In the future, we shall need the following simple facts which are easy to verify directly.

Lemma 1.4. *Let R and S be rational groups of types* $\mathbf{t}$ *and* $\mathbf{s}$*, respectively. Then*

(i) $\operatorname{Hom}(R, S) = 0$ *unless* $\mathbf{t} \leq \mathbf{s}$*;*
(ii) *if* $\mathbf{t} \leq \mathbf{s}$*, then* $\operatorname{Hom}(R, S)$ *is a rational group of type* $\mathbf{s} : \mathbf{t}$*;*
(iii) $\operatorname{Hom}(R, R)$ *is a rational group of idempotent type* $\mathbf{t}_0 = \mathbf{t} : \mathbf{t}$*; and*
(iv) $\operatorname{Hom}(\operatorname{Hom}(R, R), R)$ *is a rational group of type* $\mathbf{t}$*.* □

Finite Rank Groups A torsion-free group A of finite rank n is a subgroup of an n-dimensional $\mathbb{Q}$-vector space V, so we can describe A by starting with an independent set $a_1, \ldots, a_n$, and then adjoining more elements of V as generators of A, responsible for divisibility properties.

Example 1.5. Let P be a set of primes, and p, q primes $\notin P$. Then

$$A = \langle p^{-2}a, q^{-\infty}b, P^{-1}(a + b)\rangle$$

is the group of rank 2 in which a, b are independent elements, and the additional generators are: $p^{-2}a, q^{-n}b$ for every $n \in \mathbb{N}$, and $r^{-1}(a + b)$ for every $r \in P$.

Type(A) will denote the **Typeset** of A, i.e. the set of types of the non-zero elements of A. Type(A) can be infinite even if rk(A) is finite. We also talk of Cotype(A) as the set of **cotypes**, i.e. types of rank one torsion-free factor groups of A; but this gives useful information only if A is of finite rank. IT(A) and OT(A) are the notations for the **inner** and **outer types** of A: IT(A) is defined as the intersection of all types $\mathbf{t}(a)$ for all $a \in A$, and OT(A) is the union of all types of rank 1 torsion-free factor groups of A. We have:

Lemma 1.6. *Let A be a torsion-free group of finite rank, $a_1, \ldots, a_n$ a maximal independent set in A, and A_i the pure subgroup of A generated by $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$. Then*

$$\mathrm{IT}(A) = \mathbf{t}(a_1) \wedge \ldots \wedge \mathbf{t}(a_n) \quad \text{and} \quad \mathrm{OT}(A) = \mathbf{t}(A/A_1) \vee \ldots \vee \mathbf{t}(A/A_n).$$

Proof. What we have to show for IT(A) is that, for every $a \in A$, the type $\mathbf{t}(a)$ is larger than or equal to the stated intersection. If we write $a = r_1a_1 + \cdots + r_na_n$ with $r_i \in \mathbb{Q}$, then it is clear that we must have $\mathbf{t}(a) \geq \mathbf{t}(r_1a_1) \wedge \ldots \wedge \mathbf{t}(r_na_n) = \mathbf{t}(a_1) \wedge \ldots \wedge \mathbf{t}(a_n)$.

For OT(A) we argue that, since $A_1 \cap \ldots \cap A_n = 0$, there is an embedding $A \to A/A_1 \oplus \cdots \oplus A/A_n$ in a direct sum of rank 1 groups, call it $\bar{A}$ for a moment. By definition, $\vee_i \mathbf{t}(A/A_i) \leq \mathrm{OT}(A)$. On the other hand, it is obvious that $\mathrm{OT}(A) \leq \mathrm{OT}(\bar{A})$. Since every homomorphism of a direct sum of rank 1 groups into $\mathbb{Q}$ carries each rank 1 summand either to 0 or isomorphically into $\mathbb{Q}$, it is evident that $\mathrm{OT}(\bar{A}) = \vee_i \mathbf{t}(A/A_i)$. □

Example 1.7. For a finite rank A, we have $\mathrm{IT}(A) = \mathrm{OT}(A)$ if and only if A is a direct sum of isomorphic rank 1 groups. The ‘if’ part is pretty obvious, while the converse is a consequence of Lemma 3.6.

If A is torsion-free of rank m, and if F is a free subgroup of rank m, then $T = A/F$ is a torsion group whose p-components T_p have ranks $\leq m$. Thus for each prime p, we can write:

$$T_p = \mathbb{Z}(p^{k_{p1}}) \oplus \cdots \oplus \mathbb{Z}(p^{k_{pm}}) \tag{12.1}$$

where the exponents k_{pi} are non-negative integers or symbols ∞ (note that there are exactly m summands, and $\mathbb{Z}(p^0) = 0$). Another choice of F can change only the finite exponents, and only in a finite number of p-components, into other finite exponents. The obvious equivalence class is called the **Richman type** of A; this is the same for quasi-isomorphic groups (see Sect. 9). It is easy to check that if we assume $k_{p_n1} \leq \cdots \leq k_{p_nm}$ ($\forall\, n \in \mathbb{N}$), then $\mathrm{IT}(A)$ is represented by the characteristic $(k_{p_11}, \ldots, k_{p_n1}, \ldots)$, and $\mathrm{OT}(A)$ by the characteristic $(k_{p_1m}, \ldots, k_{p_nm}, \ldots)$.

To close this topic, we show that the cardinality of the typeset is not bounded by the rank of the group. As a matter of fact, it can be infinite even for rank 2 groups (cf. Exercise 10).

Lemma 1.8 (Koehler [2], Mutzbauer [2]). *A finite set of types is the typeset of a torsion-free group if and only if it is closed under intersections.*

Proof. Let $T = \{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ be a set of types, closed under intersections. The direct sum of rational groups of these types has typeset T.

Conversely, suppose A has a finite typeset $T = \{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$. First we show that $\mathbf{t}_1 \wedge \ldots \wedge \mathbf{t}_k \in T$. Clearly, $A = \cup_i A(\mathbf{t}_i)$ (set-union) implies $\mathbb{Q}A = \cup_i \mathbb{Q}A(\mathbf{t}_i)$, thus the $\mathbb{Q}$-vector space $\mathbb{Q}A$ is the set-union of a finite number of subspaces. This can happen only if $\mathbb{Q}A = \mathbb{Q}A(\mathbf{t}_j)$ for some j, which means that this $\mathbf{t}_j$ is the intersection of all types in T. Next, let $\mathbf{t}$ be the intersection of some types in T. The typeset of the subgroup $A(\mathbf{t})$ consists of those $\mathbf{t}_i \in T$ for which $\mathbf{t} \leq \mathbf{t}_i$. By the preceding argument, $\mathbf{t}$ is a type of a subgroup in $A(\mathbf{t})$, so it belongs to T. $\square$

A type $\mathbf{t}$ is said to be an **extractable type** of the torsion-free group A if A has a rank 1 summand of type $\mathbf{t}$.

Homogeneous Groups Recall that 'homogeneous' means that every non-zero element is of the same type.

Proposition 1.9 (Nongxa [2]). *A homogeneous torsion-free group of type* $\mathbf{t}$ *is the tensor product of a rank one group of type* $\mathbf{t}$ *with a homogeneous torsion-free group of type* $(0, 0, \ldots, 0, \ldots)$.

Proof. First we deal with an idempotent type $\mathbf{t}$. Let R denote a rational group of type $\mathbf{t}$, and let Π_1 be the set of primes at which $\mathbf{t}$ is 0, and Π_2 the set of primes at which $\mathbf{t}$ is ∞. Choose an essential free subgroup F in the homogeneous torsion-free group A of type $\mathbf{t}$, and decompose the torsion group $A/F = A_1/F \oplus A_2/F$ where A_i/F is the Π_i-component of A/F. The exact sequence $0 \to A_1 \to A \to A/A_1 \cong A_2/F \to 0$ induces the exact sequence $0 \to R \otimes A_1 \to R \otimes A \cong A \to R \otimes A_2/F = 0$, whence $A \cong R \otimes A_1$ follows. The type of every element in A_1 must be $\leq \mathbf{t}$, and in view of the given decomposition of A/F, it cannot be larger than the type of $\mathbb{Z}$.

If $\mathbf{t}$ is an arbitrary type, and R is a rational group of type $\mathbf{t}$, then the group $A' = \operatorname{Hom}(R, A)$ is homogeneous of idempotent type $\mathbf{t}_0 = \mathbf{t} : \mathbf{t}$, and we have an isomorphism $R \otimes A' \cong A$. This, combined with the preceding paragraph, completes the proof. □

Subgroups Isomorphic to the Group It is evident that the subgroups nA $(n \in \mathbb{N})$ are isomorphic to A. Though several other subgroups may be isomorphic to A, in the finite rank case they are not far from being an integral multiple of the group. Indeed, we can claim:

Proposition 1.10 (Jónsson [2]). *If C is a subgroup of the finite rank torsion-free group A such that $C \cong A$, then $nA \leq C \leq A$ for some $n \in \mathbb{N}$.*

Proof. (E. Walker) Let $\eta : A \to C$ be an isomorphism, viewed as an element of $\operatorname{End} A$. It extends uniquely to the divisible hull D of A, so the extension (also denoted by η) is a linear transformation of the finite dimensional $\mathbb{Q}$-vector space D. By the Cayley–Hamilton theorem, it satisfies an equation of the form $n_k\eta^k + n_{k-1}\eta^{k-1} + \cdots + n_1\eta + n_0 = 0$ $(n_i \in \mathbb{Z})$. As η is monic, so left-cancellable in $\operatorname{End} A$, $n_0 \neq 0$ may be assumed. But then $-n_0A = \eta(n_k\eta^{k-1} + n_{k-1}\eta^{k-2} + \cdots + n_1)A \leq \eta A = C$, as claimed. □

★ **Notes.** It was Levi [1] who started the study of torsion-free groups; his work was practically ignored for a long time. He wrote a thesis to obtain *venia legendi*, i.e. the right to lecture at the university, but has never published his results in a journal. He was also the first to point out the connection of abelian groups with p-adic modules. Almost 20 years later, the duality theory of Pontryagin prompted a lot of interest in abelian groups, in particular in torsion-free groups, since some of their properties reflect faithfully on topological properties of compact abelian groups. Starting with the theory of p-adic modules discussed by Derry [1], Mal'cev [1] and Kurosh [2] developed a theory, based on the equivalence of matrices, in order to find structural results on torsion-free groups of finite rank. (As observed in [AG], the theory can easily be extended to the countable rank case.) The theory was, unfortunately, too difficult to apply in practice, as it translates a difficult problem to another difficult problem, without giving a better insight into the structure. (Many years later, Richman [5] followed a more transparent approach—as described above.) Campbell [1] tried to classify countable torsion-free groups, using matrices to describe divisibility; this does not seem a useful approach either.

The inner and outer types were introduced in Fuchs [14], and developed by Warfield [1]. Mutzbauer [2] shows that the sum of types in the chain $0 = A_0 < A_1 < \cdots < A_n = A$ of pure subgroups (the factor groups are of rank 1) of a finite rank torsion-free group A is independent of the chain.

Metelli [1] defines a torsion-free group A **t-bihomogeneous** if it satisfies both $A(\mathbf{t}) = A$ and $A[\mathbf{t}] = 0$. *Inter alia* she proves that countable bihomogeneous groups are completely decomposable an analogue of Theorem 4.3 in Chapter 13.

In [IAG] it was asked what can be said about finite rank torsion-free groups A, B if $\operatorname{Ext}(A, C) \cong \operatorname{Ext}(B, C)$ holds for all finite rank torsion-free groups C. Several papers dealt with the problem; we point out Krylov [4] where it was shown that then A and B have to be quasi-isomorphic provided they have no summand $\cong \mathbb{Z}$ or $\mathbb{Q}$. Goeters [1], Chekhlov–Krylov [1] investigate the dual problem: reduced finite rank torsion-free groups A, B satisfying $\operatorname{Ext}(C, A) \cong \operatorname{Ext}(C, B)$ for all reduced torsion-free C of finite rank. It is shown that then A and B have to be quasi-isomorphic.

The foundation of the modern theory of torsion-free groups is due to Baer; see his seminal paper [6]. In the present chapter, several fundamental results carry his name. We should also point out that Yakovlev [2] develops an innovative new method of dealing with finite rank torsion-free groups, based on the theory of integral representations. The study splits, for each prime p, into a Krull–Schmidt category.

Exercises

1. (a) Let A, B be rank 1 torsion-free groups. A is isomorphic to a subgroup of B if and only if $\mathbf{t}(A) \leq \mathbf{t}(B)$ if and only if $\operatorname{Hom}(A, B) \neq 0$.
   (b) Two rank 1 torsion-free groups, A and B, are isomorphic exactly if both $\operatorname{Hom}(A, B) \neq 0$ and $\operatorname{Hom}(B, A) \neq 0$.
2. (a) A group of rank 1 contains either only finitely many or continuously many pairwise non-isomorphic subgroups.
   (b) For every type $\mathbf{t} \neq (\infty, \dots, \infty, \dots)$, there exist countably many characteristics of the same type $\mathbf{t}$.
3. The completion $\tilde{R}$ of a rational group $R \neq \mathbb{Q}$ is isomorphic to a summand of $\tilde{\mathbb{Z}} \cong \prod_p J_p$.
4. Let G be a torsion-free group of rank 1. Find conditions on G to be written as $G \cong R \otimes \cdots \otimes R$, the nth 'tensor power' of a rank 1 torsion-free group R.
5. Give examples such that

   (a) the elements of type $>\mathbf{t}$ do not form (only generate) a subgroup of A;
   (b) $A^*(\mathbf{t})$ is not pure in A;
   (c) A contains elements of type $\mathbf{t}$ and $A^*(\mathbf{t}) = A(\mathbf{t})$.

6. (Baer) If A is of finite rank, then $\operatorname{Type}(A)$ satisfies both the maximum and the minimum conditions. [Hint: ranks of $A(\mathbf{t})$.]
7. (a) For a torsion-free group G and a type $\mathbf{t}$, $G(\mathbf{t})$ is the trace of a rank 1 torsion-free group of type $\mathbf{t}$ in G.
   (b) If C is pure in A, then $C(\mathbf{t})$ is pure in $A(\mathbf{t})$.
   (c) Let $0 \to A \to B \to C \to 0$ be a pure-exact sequence of finite rank groups. If $B = B(\mathbf{t})$ for some type $\mathbf{t}$, then also $A = A(\mathbf{t})$ and $C = C(\mathbf{t})$.
8. (Warfield) $\operatorname{rk} \operatorname{Hom}(A, C) = \operatorname{rk} A \cdot \operatorname{rk} C$ holds for finite rank torsion-free groups A, C if and only if $\operatorname{IT}(C) \geq \operatorname{OT}(A)$.
9. (Müller–Mutzbauer) The intersection of cotypes is in finite rank groups is equal to the inner type.
10. (Baer) There exist torsion-free groups of rank 2 whose typeset is infinite. [Hint: in $\mathbb{Q}a \oplus \mathbb{Q}b$ choose A such that $h_p(a + p_n b) = \infty$ or 0 according as $p = p_n$ or not.]
11. (a) Let B be a pure subgroup of rank 1 in the torsion-free group A of rank 2. Then $\mathbf{t}(B) \cdot \mathbf{t}(A/B)$ is an invariant of A: this product is independent of the choice of B in A.
    (b) Suppose that A is a homogeneous torsion-free group of rank 2. Show that all the cotypes are the same.
12. A *minimax group* is defined as an extension of a group with maximum condition on subgroups by a group with minimum condition on subgroups. Prove or disprove: a torsion-free group is minimax if and only if it is of finite rank.
13. (a) (Soifer [1]) A generating system S of A is **irreducible** if no proper subset of S generates A. Show that a rank 1 torsion-free group A has an irreducible

generating system if and only if it is cyclic or its type is not idempotent. [Hint: for sufficiency, find generators $p^{-k_p}q^{-n}$ such that in the type we have $k_p > 0$ an integer and $k_q = \infty$.]

(b) Use a similar trick to show that an uncountable group A admits an irreducible system of generators whenever $|A/pA| = |A|$ for some prime p.

14. (Höfling) (a) Let $\{\mathbf{t}_n\ (n < \omega)\}$ be a countable set of types none of which contains ∞. Then there is a type $\mathbf{t}$ such that $\mathbf{t} > \mathbf{t}_n$ for all n.

(b) There exists a chain containing $\aleph_1$ different types.

2 Balanced Subgroups

We now introduce a concept that plays a vital role in the theory of torsion-free groups.

Balancedness Let B be a pure subgroup of the torsion-free group A. An element $a \in A \setminus B$ is called **proper with respect to** B if

$$\chi(a + b) \leq \chi(a) \qquad \text{for all } b \in B.$$

Evidently, $a \in A$ is proper with respect to B if and only if $\chi_A(a) = \chi_{A/B}(a + B)$. It is a useful fact that then $\chi(a + b) = \chi(a) \wedge \chi(b)$ holds for all $b \in B$.

We say that B is **balanced** in A if every coset of B contains an element proper with respect to B. Summands are always balanced subgroups, but balancedness is a much stronger property than purity. If B is balanced in A, we call the exact sequence $0 \to B \to A \to A/B \to 0$ **balanced-exact**.

Example 2.1. (a) If B is a pure subgroup of A such that A/B is homogeneous of type $\mathbb{Z}$, then B is balanced in A.

(b) A simple non-trivial example for balancedness is given in a torsion-free group A whose elements have types $\geq \mathbf{t}$. If $A^\star(\mathbf{t})$ is pure in A, then it is a balanced subgroup.

(c) An indecomposable torsion-free group of rank 2 has no balanced subgroups except for the trivial ones.

In order to get better acquainted with balancedness, we show that it can be characterized in several ways.

Lemma 2.2. *A pure subgroup B of a torsion-free group A is balanced in A if and only if, for every $a \in A$ and for every countable subset $\{b_n\}_{n<\omega}$ of B there exists a $b \in B$ such that*

$$\chi(a + b_n) \leq \chi(b + b_n) \qquad \textit{for every } n < \omega. \tag{12.2}$$

Proof. Let B be balanced in A, and $a \in A$. For some $b \in B$, $a - b$ is proper with respect to B, i.e. $\chi(a - b) \geq \chi(a + x)$ for all $x \in B$. Then $\chi(b + x) \geq \min\{\chi(b - a), \chi(a + x)\} = \chi(a + x)$ for all $x \in B$.

Conversely, suppose that, for every $a \in A$ and for $\{b_n\}_{n<\omega} \subset B$, there is a $b \in B$ satisfying (12.2). Then $\chi(a-b) \geq \chi(a+b_n)$ for each b_n. If the set $\{b_n\}_{n<\omega}$ is chosen so as to satisfy $\chi_{A/B}(a+B) = \bigvee_{n<\omega} \chi(a+b_n)$, then $\chi_B(a-b) \geq \chi_{A/B}(a+B)$, i.e. $a-b$ is proper with respect to B. $\square$

The next lemma is probably the best tool for checking balancedness in most of the proofs.

Lemma 2.3. *An exact sequence of torsion-free groups is balanced-exact if and only if all rank one torsion-free groups have the projective property relative to it.*

Proof. Let $\mathfrak{e} : 0 \to B \to A \xrightarrow{\gamma} C \to 0$ be balanced-exact, and $\alpha : R \to C$ a homomorphism where R is a subgroup of $\mathbb{Q}$, say, with $1 \in R$. There is an $a \in A$ such that $\gamma a = \alpha 1$ and $\chi_C(\gamma a) = \chi_A(\alpha 1) \geq \chi_R(1)$. Hence the correspondence $1 \mapsto a$ extends to a homomorphism $\phi : R \to A$ such that $\gamma\phi = \alpha$.

Conversely, assume $\mathfrak{e}$ has the stated property, and let $x \in A$. Then the embedding $\alpha : R \to C$ (where $\gamma x = \alpha 1$) lifts to a homomorphism $\phi : R \to A$ such that $\gamma\phi = \alpha$. It is easy to check that $a = \phi 1 \in A$ is in the coset of $x + B$, and satisfies $\chi_A(a) = \chi_C(\alpha 1)$. $\square$

Some relevant properties of balancedness are reminiscent of those of purity.

Lemma 2.4 (Haimo [1]). *Let A and B be pure subgroups of the torsion-free group G such that $B \leq A$. Then we have:*

(a) *If B is balanced in G, then it is balanced in A as well.*
(b) *If B is balanced in A, and A is balanced in G, then B is balanced in G.*
(c) *If A is balanced in G, then A/B is balanced in G/B.*
(d) *If B is balanced in G, and A/B is balanced in G/B, then A is balanced in G.*

Proof. (a) is trivial. To prove (b), form the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & A/B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & G \arrow[r] \arrow[d] & G/B \arrow[r] \arrow[d] & 0 \\
 & & H \arrow[r, equal] \arrow[d] & H \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns, where the top row and the middle column are balanced-exact. Let R be a rational group containing 1 and $\alpha : R \to G/B$ a non-trivial map. Manifestly, α lifts to (A and hence to) G whenever $\operatorname{Im}\alpha$ is contained in A/B. If this is not the case, then α followed by $G/B \to H$ lifts to G in the middle column.

(c) By hypothesis, in the preceding diagram, the middle column is balanced-exact. Thus, any map $\alpha : R \to H$ lifts to G, and hence to G/B.

(d) Now in our diagram the middle row and the last column are assumed to be balanced-exact. Consequently, any map $\alpha : R \to H$ lifts to G/B, and hence to G. Thus the middle column is balanced-exact. □

Lemma 2.5. *For an exact sequence* $\mathfrak{e} : 0 \to B \to A \xrightarrow{\gamma} C \to 0$ *of torsion-free groups, the following are equivalent:*

(i) $\mathfrak{e}$ *is balanced-exact;*

(ii) *the induced sequence* $\mathfrak{e}(\chi) : 0 \to B(\chi) \to A(\chi) \xrightarrow{\gamma} C(\chi) \to 0$ *is exact for each characteristic* χ*;*

(iii) *the induced sequence* $\mathfrak{e}(\mathbf{t}) : 0 \to B(\mathbf{t}) \to A(\mathbf{t}) \xrightarrow{\gamma} C(\mathbf{t}) \to 0$ *is exact for every type* $\mathbf{t}$.

Proof. The exactness of the sequences (ii) and (iii) at the middle terms follows at once from the purity of B in A. Since (i) says that every $c \in C$ has a preimage $a \in A$ with $\chi_A(a) = \chi_C(c)$, it is clear that (i) is equivalent to (ii). That (ii) ⇒ (iii) is also evident.

To prove (iii) ⇒ (ii), let $\alpha : R \to C$ be a homomorphism, where R is a subgroup of $\mathbb{Q}$ of type $\mathbf{t}$. Then $\alpha R \leq C(\mathbf{t})$. We refer to Lemma 3.6 *infra*, to argue that $B(\mathbf{t})$ is a summand of $B(\mathbf{t}) + \alpha R$, so α lifts. □

Lemmas on Balanced Subgroups Balanced subgroups will play a most important role in the discussion of Butler groups in Chapter 14. Therefore, let us proceed by assembling here a few needed facts on balanced subgroups that will be needed in the sequel.

It is far from being a trivial task to prove the existence of balanced subgroups in large torsion-free groups.

Lemma 2.6 (Dugas–Hill–Rangaswamy [1]). *If C is any subgroup of the torsion-free A, then there exists a balanced subgroup B of A containing C such that* $|B| \leq |C|^{\aleph_0}$.

A torsion-free group that contains no balanced subgroups other than the trivial ones must have cardinality $\leq 2^{\aleph_0}$.

Proof. Ignoring a trivial case, assume $\kappa = |C|^{\aleph_0} \geq 2^{\aleph_0}$. Let X be a countable subset of C. For $a \in A$, let a_X denote the function from X into the lattice of all characteristics defined by setting

$$a_X(x) = \chi(a + x) \qquad (x \in X).$$

Consider the set S of all these functions (taken for all $a \in A$, and for all countable subsets $X \subseteq C$); since there are only $2^{\aleph_0}$ characteristics, S has cardinality at most $\kappa 2^{\aleph_0} = \kappa$. Thus there exists a subset W in A of cardinality $\leq \kappa$ such that $S =$

$\{a_X \mid a \in W,\ X$ a countable subset of $C\}$. Let C_1 be the pure subgroup generated by C and this set W. Repeat this process for C_1 in the place of C to obtain C_2, and continue transfinitely, taking unions at limit ordinals. In this way, we obtain a smooth chain of pure subgroups C_σ of A. After ω_1 steps, we get a pure subgroup $B = C_{\omega_1}$ of A. Since each C_σ $(\sigma < \omega_1)$ has cardinality $\leq \kappa$, also $|B| \leq \kappa$.

We claim that this B is balanced in A. Let $a + B$ $(a \in A)$ be a coset; its characteristic is $\bigvee_n \chi(a + b_n)$ for some $\{b_n\}_{n<\omega} \subseteq B$. There is a C_σ $(\sigma < \omega_1)$ such that all $b_n \in C_\sigma$. Construction guarantees the existence of $c \in C_{\alpha+1}$ satisfying $\chi(c + b_n) \geq \chi(a + b_n)$ for all $n < \omega$. Then $a - c \in a + B$ is proper with respect to B, so B is balanced in A.

The second statement is a simple corollary. □

It is easy to see that the union of a countable chain of balanced subgroups need not be balanced. However, for longer chains, we have:

Lemma 2.7. *Let* $0 = A_0 < A_1 < \cdots < A_\sigma < \cdots$ $(\sigma < \lambda)$ *be a* (*not necessarily smooth*) *well-ordered ascending chain of balanced subgroups of the torsion-free group* G. *If* $\operatorname{cf} \lambda \geq \omega_1$, *then the subgroup* $A = \bigcup_\sigma A_\sigma$ *is again balanced in* G.

Proof. Given $g \in G$, there is a countable subset $\{a_n\}_{n<\omega} \subseteq A$ such that $\chi_{G/A}(g + A) = \bigvee_{n<\omega} \chi(g + a_n)$. In view of the hypothesis $\operatorname{cf} \lambda \geq \omega_1$, there exists an index $\rho < \lambda$ with $\{a_n\}_{n<\omega} \subset A_\rho$. The balancedness of A_ρ implies that $\chi(g + b_\rho) = \chi_{G/A_\rho}(g + A_\rho)$ for some $b_\rho \in A_\rho$. Then

$$\chi(g + b_\rho) = \chi_{G/A_\rho}(g + A_\rho) \geq \bigvee_n \chi(g + a_n) = \chi_{G/A}(g + A),$$

establishing the claim. □

Lemma 2.8 (Bican–Fuchs [1]). *Suppose*

$$0 \to B_\sigma \to A_\sigma \xrightarrow{\gamma_\sigma} C_\sigma \to 0 \qquad (\sigma < \kappa)$$

is a well-ordered direct system of balanced-exact sequences where the C_σ *are reduced torsion-free groups. If the connecting maps* $\phi_{\rho\sigma} : C_\rho \to C_\sigma$ (*for all* $\rho < \sigma < \kappa$) *are monomorphisms with* $\operatorname{Im} \phi_{\rho\sigma}$ *pure in* C_σ, *then the direct limit* $0 \to B \to A \xrightarrow{\gamma} C \to 0$ *of the system is likewise a balanced-exact sequence.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & B_\sigma \arrow[r] \arrow[d] & A_\sigma \arrow[r, "\gamma_\sigma"] \arrow[d, "\alpha_\sigma"] & C_\sigma \arrow[r] \arrow[d, "\phi_\sigma"] & 0 \\
0 \arrow[r] & B \arrow[r] & A \arrow[r, "\gamma"] & C \arrow[r] & 0
\end{tikzcd}$$

Here α_σ and ϕ_σ stand for the canonical maps into the direct limits. To establish balancedness, let R be a rank 1 torsion-free group. Given η: $R \to C$ and $0 \neq x \in R$, there is a $\sigma < \kappa$ such that $\phi_\sigma x_\sigma = \eta x$ for some $x_\sigma \in C_\sigma$. Then the correspondence $x \mapsto x_\sigma$ extends to a homomorphism $\eta_\sigma \colon R \to C_\sigma$. Evidently, $\phi_\sigma \eta_\sigma = \eta$. By the balanced-exactness of the top row, there is a map $\psi_\sigma \colon R \to A_\sigma$ such that $\gamma_\sigma \psi_\sigma = \eta_\sigma$. Now $\alpha_\sigma \psi_\sigma \colon R \to A$ satisfies $\gamma(\alpha_\sigma \psi_\sigma) = (\gamma\alpha_\sigma)\psi_\sigma = (\phi_\sigma \gamma_\sigma)\psi_\sigma = \phi_\sigma(\gamma_\sigma \psi_\sigma) = \phi_\sigma \eta_\sigma = \eta$, as desired. $\square$

Groups of Balanced Extensions Let A, C be torsion-free groups. The equivalence classes of extensions of A by C in which A is balanced form a subgroup in $\mathrm{Ext}(C, A)$, called the **group of balanced extensions**, and denoted $\mathrm{Bext}^1(C, A)$. That this is in fact a subgroup of $\mathrm{Ext}(C, A)$ follows in the same way as for Ext, by making use of the next lemma. (We have the affix ^{1} to Bext, since there are higher Bexts, as we shall see later on.)

Lemma 2.9. *If $\mathfrak{e} : 0 \to B \xrightarrow{\alpha} A \xrightarrow{\gamma} C \to 0$ is a balanced-exact sequence, and $\phi : B \to B'$ and $\psi : C' \to C$ are homomorphisms, then the top and bottom rows in the commutative diagram*

$$\begin{array}{rccccccccc}
\mathfrak{e}\psi\colon & 0 & \longrightarrow & B & \longrightarrow & A'' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle\psi} & & \\
\mathfrak{e}\colon & 0 & \longrightarrow & B & \xrightarrow{\alpha} & A & \xrightarrow{\gamma} & C & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow & & \Big\| & & \\
\phi\mathfrak{e}\colon & 0 & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

are balanced-exact.

Proof. That rank 1 torsion-free groups have the projective property for the top row follows from the pull-back property, while the same for the bottom row is almost obvious. $\square$

We conclude that ϕ induces a map $\phi_* : \mathrm{Bext}^1(C, B) \to \mathrm{Bext}^1(C, B')$ and ψ induces $\psi^* : \mathrm{Bext}^1(C, B) \to \mathrm{Bext}^1(C', B)$. With the aid of these induced maps, we have:

Theorem 2.10. *Let $0 \to B \xrightarrow{\alpha} A \xrightarrow{\gamma} C \to 0$ be a balanced-exact sequence. Then for every torsion-free group G, the induced sequences*

$$0 \to \mathrm{Hom}(G, B) \xrightarrow{\alpha} \mathrm{Hom}(G, A) \xrightarrow{\gamma} \mathrm{Hom}(G, C) \to$$

$$\to \mathrm{Bext}^1(G, B) \xrightarrow{\alpha} \mathrm{Bext}^1(G, A) \xrightarrow{\gamma} \mathrm{Bext}^1(G, C) \to \dots$$

and

$$0 \to \mathrm{Hom}(C, G) \xrightarrow{\gamma} \mathrm{Hom}(A, G) \xrightarrow{\alpha} \mathrm{Hom}(B, G) \to$$

$$\to \mathrm{Bext}^1(C, G) \xrightarrow{\gamma} \mathrm{Bext}^1(A, G) \xrightarrow{\alpha} \mathrm{Bext}^1(B, G) \to \dots$$

are exact.

Proof. This follows from a general theorem on relative homological algebra; see, e.g., Mac Lane [M]. □

There are more functors Bext^i which continue the long exact sequences in the same fashion. As far as their definition is concerned, let us observe that if $0 \to B \to C \to G \to 0$ is a balanced-projective resolution of G (with completely decomposable C, see next section), then

$$\mathrm{Bext}^{i+1}(G, A) = \mathrm{Bext}^i(B, A)$$

for every torsion-free group A, and for every $i \geq 1$. This follows from the continuation of the exact sequences in the preceding theorem by making use of $\mathrm{Bext}^i(C, A) = 0$ for completely decomposable groups C. Thus with the aid of repeated balanced-projective resolutions the higher Bexts can be computed recursively.

★ **Notes.** The systematic study of balancedness started with Hunter [1], though several sporadic results on balancedness had been proved earlier under different or no names, see, e.g., Lyapin [1]. The first balanced-projective resolutions appear in Albrecht–Hill [1].

Specific subgroups of various kind were subjects of publications, but none of them turned out to be able to compete in importance with balancedness. For example, Bican [1] studied subgroups G of a torsion-free A, called **regular** to mean $\mathbf{t}_G(g) = \mathbf{t}_A(g)$ for all $g \in G$. Regular subgroups display a few interesting properties, and are useful in certain proofs. We also mention the dual concept: an exact sequence $0 \to B \to A \to C \to 0$ is said to be **cobalanced** if all rank 1 torsion-free groups R have the injective property relative to it, i.e. if the induced map $\mathrm{Hom}(A, R) \to \mathrm{Hom}(B, R)$ is always surjective. Cobalancedness has been investigated by several authors, see, e.g., Goeters [2]. So far, cobalancedness has not proved to be very useful.

Exercises

(1) Localization preserves balancedness.
(2) The group J_p of the p-adic integers (p any prime) is a torsion-free group of cardinality $2^{\aleph_0}$ that has only two balanced subgroups, *viz.* 0 and J_p.
(3) Let F be a free group, and G a pure subgroup such that F/G is homogeneous of type $(0, \dots, 0, \dots)$, but not free (e.g., an infinite direct product of copies of $\mathbb{Z}$). Show that G is balanced, but not a summand in F.
(4) (Kravchenko) If B is a balanced subgroup of A, then for all types $\mathbf{s}$ and $\mathbf{t}$, $(B + A(\mathbf{s})) \cap (B + A(\mathbf{t})) = B + A(\mathbf{s} \wedge \mathbf{t})$.
(5) Let $G = A + B$ be a torsion-free group, where A, B are pure subgroups of G. If $A \cap B$ is balanced in A, then B is balanced in G. [Hint: $G/B \cong A/(A \cap B)$

implies $\forall\, g \in G\, \exists\, a \in A$ with $g - a \in B$ and $\chi(a + (A \cap B)) = \chi(g + B)$; if $A \cap B$ balanced, then $\chi(a + (A \cap B)) = \chi(a + b)$ for a $b \in A \cap B$.]

(6) Let G_σ $(\sigma < \omega_1)$ be a smooth chain of pure subgroups of a torsion-free group G, and $G = \bigcup_{\sigma<\omega_1} G_\sigma$. If B is a subgroup of G such that $B \cap G_\sigma = B_\alpha$ is balanced in G_σ for each $\sigma < \omega_1$, then B is balanced in G.

(7) With the aid of the 3×3-lemma, prove that equivalent conditions for balancedness in Lemma 2.5 are that $0 \to B/B(\chi) \to G/G(\chi) \to C/C(\chi) \to 0$ is exact for each characteristic χ, and $0 \to B/B(\mathbf{t}) \to G/G(\mathbf{t}) \to C/C(\mathbf{t}) \to 0$ is exact for each type $\mathbf{t}$, respectively.

3 Completely Decomposable Groups

At this point it is appropriate to inquire about balanced-projectives and balanced-projective resolutions. A torsion-free group is **balanced-projective** if it has the projective property relative to all balanced-exact sequences $0 \to B \to A \to C \to 0$ of torsion-free groups.

A torsion-free group is called **completely decomposable** if it is a direct sum of rank 1 groups. Free groups and divisible torsion-free groups are the most common examples of completely decomposable groups. We intend to show that the completely decomposable groups have the projective property relative to balanced-exact sequences. Moreover, this projective property characterizes the completely decomposable groups.

Balanced-Projective Resolutions Our starting point is the following theorem; one part of its proof assumes subsequent results that do not depend on it.

Theorem 3.1 (C. Walker [3]). *The balanced-projective torsion-free groups are precisely the completely decomposable groups.*

Proof. In view of Lemma 2.3 it is obvious that rank one groups are balanced-projective. The projective property is inherited by direct sums, so the balanced-projectivity of completely decomposable groups is obvious. That only the completely decomposable groups are balanced-projective will follow by standard arguments, making use of Theorems 3.2 and 3.9 below. □

Balanced-projectivity is destined to play for torsion-free groups a similar role as for p-groups, as witnessed by the following analogue of Proposition 5.8 in Chapter 11.

Theorem 3.2. (i) *Every torsion-free group G has a balanced-projective resolution, i.e., it can be embedded in a balanced-exact sequence*

$$0 \to B \to C \to G \to 0, \tag{12.3}$$

where C is completely decomposable.

(ii) *If* $0 \to B' \to C' \to G \to 0$ *is another balanced-projective resolution of the same group, then*

$$B \oplus C' \cong B' \oplus C.$$

Proof. (i) Let R_α run over all rank one pure subgroups of G. For each α, pick a rank one group C_α isomorphic to R_α along with an isomorphism $\gamma_\alpha : C_\alpha \to R_\alpha$. Let C be the direct sum of all these C_α, one for every R_α, and define $\gamma : C \to G$ to act like γ_α on C_α. Then, evidently, the map γ is surjective. Its kernel must be balanced in C, because the criterion of Lemma 2.3 applies by construction.

(ii) (à la Schanuel's lemma) The pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "\theta"] \arrow[dl, dashed, "\phi"'] & 0 \\
0 \arrow[r] & B \arrow[r] & C \arrow[r, "\gamma"] & G \arrow[r] & 0
\end{tikzcd}$$

can be completed by a map $\phi : C' \to C$ such that $\theta = \gamma\phi$ by Theorem 3.1. This means that the top exact sequence is splitting, i.e. $H \cong B \oplus C'$. For reason of symmetry, $H \cong B' \oplus C$ as well. □

Imitating the definition of projective dimension of modules, we can introduce a dimension concept for balancedness. Accordingly, a group G is defined to have **balanced-projective dimension** n if n is the smallest integer for which

$$\mathrm{Bext}^{n+1}(G, A) = 0 \quad \text{for all groups } A.$$

We will indicate this by writing bpd $G = n$. A more explicit way to describe this dimension is to form a long balanced-exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to G \to 0$$

(with completely decomposable groups C_i) that is shortest of this kind. Thus bpd $G = 0$ if and only if G is completely decomposable. We will see later on in Lemma 1.9 in Chapter 14 that for every $n \geq 0$, there are finite rank torsion-free groups G with bpd $G = n$.

Relative Balanced-Projective Resolutions A useful device in the discussion of infinite rank Butler groups (Chapter 14) is a straightforward generalization of balanced-projective resolutions: the relative balanced-projective resolution introduced by Bican–Fuchs [2].

Let A be a pure subgroup of the torsion-free group G. We say that

$$0 \to K \to A \oplus C \xrightarrow{\phi} G \to 0 \tag{12.4}$$

is a **balanced-projective resolution of G relative to A** if this is a balanced-exact sequence with completely decomposable C such that $\phi \restriction A$ is the inclusion map $\alpha : A \to G$.

The existence of such a sequence is easily established: just take a balanced-projective resolution of G, say, $0 \to H \to C \xrightarrow{\psi} G \to 0$, and choose $\phi = \nabla(\psi \oplus \alpha) : C \oplus A \to G$ and $K = \operatorname{Ker} \phi$. The corresponding exact sequence must be balanced, because every rank 1 torsion-free group will have the projective property relative to it (Lemma 2.3). In order to get a better understanding of relative resolutions, let us complement this observation by adding

Lemma 3.3. *Using the same notations, let the map $\pi : K \to C$ be the injection $K \to A \oplus C$ followed by the projection into C. Then π is monic and $C/\pi K \cong G/A$.*

Proof. Since $\phi|A$ is monic, so is π. Noting that $A \oplus \pi K = A \oplus K$, we evidently have

$$C/\pi K \cong (A \oplus C)/(A \oplus \pi K) \cong ((A + C)/K)/((A + K)/K) \cong G/A,$$

establishing the claim. □

As far as the uniqueness of such relative resolutions is concerned, we can assert:

Proposition 3.4. *If* (12.4) *and* $0 \to K' \to A \oplus C' \xrightarrow{\psi} G \to 0$ *are two balanced-projective resolutions of G relative to the subgroup A (C' is completely decomposable), then $K \oplus C' \cong K' \oplus C$.*

Proof. Making use of the pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & A \oplus C \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & K' \arrow[r] & A \oplus C' \arrow[r, "\psi"] & G \arrow[r] & 0,
\end{tikzcd}$$

the proof is as for Theorem 3.2(ii). □

It is an immediate corollary that the group K is completely decomposable if and only if K' is completely decomposable.

Completely Decomposable Groups The following theorem provides a satisfactory structure theorem for completely decomposable groups.

Theorem 3.5 (Baer [6]). *Any two decompositions of a completely decomposable group into direct sums of rank one groups are isomorphic.*

Proof. Let $A = \oplus_{i \in I} A_i$ where the A_i are rational groups. It is readily seen that for any type $\mathbf{t}$, the subgroups $A(\mathbf{t})$ and $A^*(\mathbf{t})$ are the direct sums of those A_i that are of types $\geq \mathbf{t}$ and $> \mathbf{t}$, respectively. Thus $A_{\mathbf{t}} = A(\mathbf{t})/A^*(\mathbf{t})$ is isomorphic to the direct sum of those A_i which are exactly of type $\mathbf{t}$, and the rank of $A_{\mathbf{t}}$ is precisely the number of components A_i of type $\mathbf{t}$. Since $A_{\mathbf{t}}$ is defined independently of the given direct decomposition, the claim follows from the uniqueness of the rank. $\square$

As a consequence, for a completely decomposable group A, the ranks of the factor groups $A(\mathbf{t})/A^\star(\mathbf{t})$ for all types $\mathbf{t}$ (the Baer invariants of A) form a complete (and independent) set of invariants for A.

A most useful technical lemma (known as 'Baer's Lemma') is as follows.

Lemma 3.6 (Baer [6]). *Let B be a pure subgroup of the torsion-free group A such that every element of the coset $a + B$ $(a \in A \setminus B)$ has the same type in A as the coset has in A/B. Then B is a summand of $\langle B, a\rangle_*$.*

Proof. As $\chi_A(a) \leq \chi_{A/B}(a + B)$ and $\mathbf{t}(a) = \mathbf{t}(a + B)$, there exist a finite number of elements $a_1, \ldots, a_k \in a + B$ such that $\chi_A(a_1) \vee \ldots \vee \chi_A(a_k) = \chi_{A/B}(a + B)$. It is thus enough to show that if $a \equiv a'$ mod B and $\chi_A(a), \chi_A(a')$ are equivalent characteristics, then the coset $a + B$ contains an element g satisfying $\chi_A(g) \geq \chi_A(a) \vee \chi_A(a')$. Hypothesis implies the existence of relatively prime integers m, n such that $\chi_A(ma) = \chi_A(na')$. If u, v are integers such that $mu + nv = 1$, then $g = mua + nva'$ is $\equiv a$ mod B, and evidently satisfies $\chi_A(g) \geq \chi_A(ma) \geq \chi_A(a) \vee \chi_A(a')$. $\square$

We record two important consequences of Lemma 3.6.

Corollary 3.7. *In a finite rank homogeneous completely decomposable group, every pure subgroup is a summand.*

Proof. Let $A = \oplus_{i=1}^n A_i$ with rational groups A_i of the same type $\mathbf{t}$, and B a pure subgroup in A. The factor group A/B must be homogeneous of type $\mathbf{t}$, since every element of A/B is contained in the sum of the images of (finitely many) A_i. Owing to Lemma 3.6, for each $a \in A \setminus B$, B is a summand of $B_1 = \langle B, a\rangle_*$. Keep going, arguing the same way with B_1, until A is reached. $\square$

Proposition 3.8 (Baer [6]). *Let C be a pure subgroup of the torsion-free group A such that*

(i) *A/C is completely decomposable and homogeneous of type* $\mathbf{t}$;
(ii) *all the elements in $A \setminus C$ are of type* $\mathbf{t}$.

Then C is a summand of A.

Proof. Referring to Lemma 3.6, from (ii) we derive that C is a balanced subgroup of A. The rest follows from Lemma 2.3. $\square$

Our next result generalizes the classical theorem that subgroups of free groups are again free (Theorem 1.6 in Chapter 3). Its proof runs in much the same way.

Theorem 3.9 (Baer [6], Kolettis [2]). *In a homogeneous completely decomposable group, every subgroup that is homogeneous of the same type (in particular, every pure subgroup) is completely decomposable.*

Proof. Suppose $A = \oplus_\sigma A_\sigma$ with groups A_σ of rank 1 and of type $\mathbf{t}$, where the index σ ranges over the ordinals less than some ordinal τ. Define $G_\sigma = \oplus_{\rho<\sigma} A_\rho$, and $C_\sigma = C \cap G_\sigma$, where C is a $\mathbf{t}$-homogeneous subgroup. Clearly, $C_\sigma = C_{\sigma+1} \cap G_\sigma$, thus

$$C_{\sigma+1}/C_\sigma \cong (C_{\sigma+1} + G_\sigma)/G_\sigma \leq G_{\sigma+1}/G_\sigma \cong A_\sigma.$$

We claim that C_σ is a summand of $C_{\sigma+1}$. In fact, if $C_{\sigma+1} \neq C_\sigma$, then all the elements of $C_{\sigma+1}$ are of type $\mathbf{t}$, and the type of $C_{\sigma+1}/C_\sigma$ must be $\mathbf{t}$, since it is both an epic image of $C_{\sigma+1}$ and isomorphic to a subgroup of A_σ. Hence Lemma 3.6 implies that $C_{\sigma+1} = C_\sigma \oplus B_\sigma$ for some rank one B_σ. The B_σ's ($\sigma < \tau$) generate their direct sum, which must be all of C. □

Summands of Completely Decomposable Groups The next theorem is perhaps the most important result on complete decomposability.

Theorem 3.10 (Baer [6], Kulikov [4], Kaplansky [2]). *Direct summands of completely decomposable torsion-free groups are completely decomposable.*

Proof. We start with the case of finite typesets. Let $A = H_1 \oplus \cdots \oplus H_n = B \oplus C$, where each H_i is completely decomposable and homogeneous of type $\mathbf{t}_i$; assume that these types are different. We use induction on n. If $n = 1$, then A is homogeneous, and the complete decomposability of B and C follows from Theorem 3.9. Assume $n > 1$, and let $\mathbf{t}_n$ be a maximal type in the set $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$. Then H_n is fully invariant in A, so $H_n = A(\mathbf{t}_n) = B(\mathbf{t}_n) \oplus C(\mathbf{t}_n)$, whence $B = B(\mathbf{t}_n) \oplus B'$ and $C = C(\mathbf{t}_n) \oplus C'$ for some $B' \leq B$ and $C' \leq C$; here $B(\mathbf{t}_n)$ and $C(\mathbf{t}_n)$ are completely decomposable by case $n = 1$. We now have $A = H_n \oplus B' \oplus C'$, so $H_1 \oplus \cdots \oplus H_{n-1} \cong B' \oplus C'$. Induction hypothesis guarantees that B', C' are completely decomposable, and so are therefore B and C as well.

We interrupt our argument to verify the following lemma that contains a key technical element in our proof covering the infinite rank case.

Lemma 3.11. *Let $A = B \oplus C$ be a completely decomposable group. Then every finite set of elements of B is contained in a finite rank completely decomposable summand of B.*

Proof. First, suppose $A = B \oplus C = H \oplus G$ where $H = H_1 \oplus \cdots \oplus H_n$ and each H_i is homogeneous completely decomposable of different types $\mathbf{t}_i$. In addition, we assume that G has no summand of type $\mathbf{t}_i$. We want to show by induction on n that $H = (H \cap B) \oplus (H \cap C)$.

If $n = 1$ and $\mathbf{t}(H) = \mathbf{t}$, then omitting summands that are of types $> \mathbf{t}$, we obtain $\overline{A} = \overline{B} \oplus \overline{C} \cong H \oplus \overline{G}$, where bars indicate the remaining direct sum. Here H is a fully invariant subgroup, so $H = (H \cap \overline{B}) \oplus (H \cap \overline{C}) = (H \cap B) \oplus (H \cap C)$.

If $n > 1$, then arguing in the same way with the component H_n of a maximal type $\mathbf{t}_n$, we get a similar decomposition where $H_1 \oplus \cdots \oplus H_{n-1} \leq B' \oplus C'$. This permits us to apply the induction hypothesis to establish the claim. Once we have that H splits along B and C, we can conclude that $B = (H \cap B) \oplus B'$ and $C = (H \cap C) \oplus C'$ for some $B' \leq B, C' \leq C$, and $G \cong B' \oplus C'$.

A finite set of elements of B is contained in a summand of A of the form $H = H_1 \oplus \cdots \oplus H_n$ with H_i homogeneous completely decomposable. Write $A = H \oplus G$. The preceding argument shows that $H \cap B$ is a summand of B containing the given set. It is a summand of H, so it is completely decomposable by the proof above on the case of finite typesets. This completes the proof of the lemma. □

Resuming the proof of Theorem 3.10, our next step is to reduce the proof to the case of countable rank. This is done at once with an appeal to the rank version of Kaplansky's lemma 2.5 in Chapter 2. So we may assume that A is of countable rank. Let $b_1, \ldots, b_k, \ldots$ be a complete list of elements of B. In view of Lemma 3.11, B has a finite rank completely decomposable summand B_k containing $b_1, \ldots, b_k$ as well as a maximal independent set of B_{k-1}. Clearly, B is the union of the chain $B_1 \leq \cdots \leq B_k \leq \ldots$ where $B_{k+1} = B_k \oplus E_k$ for some summand E_k of B. It follows that $B = \oplus_{k<\omega} E_k$, where E_k (as a summand of B_{k+1}) is completely decomposable by the first part of the proof. Hence B is completely decomposable, indeed. □

Completely Decomposable Subgroups There is no reasonable criterion for a subgroup of a completely decomposable group to be again completely decomposable. But it seems interesting that when a pure subgroup is completely decomposable, then there is an entire chain of completely decomposable groups between the subgroup and the containing group.

Theorem 3.12 (Fuchs–Viljoen [2]). *Let A be a completely decomposable pure subgroup of the completely decomposable group C. Then there is a smooth chain*

$$A = B_0 < \cdots < B_\sigma < \cdots < B_\tau = C$$

of completely decomposable pure subgroups of C such that all the factor groups $B_{\sigma+1}/B_\sigma$ are countable.

Proof. Consider the pure-exact sequence $0 \to A \to C \xrightarrow{\phi} G \to 0$. A straightforward back-and-forth argument leads us to a smooth chain $0 = C_0 < \cdots < C_\sigma < \cdots < C_\tau = C$ such that, for each $\sigma < \tau$,

(i) C_σ is a summand of C;
(ii) $A_\sigma = A \cap C_\sigma$ is a summand of A;
(iii) ϕC_σ is pure in G; and
(iv) $C_{\sigma+1}/C_\sigma$ is countable.

As a consequence of (i) and (ii), $B_\sigma = A + C_\sigma$ is pure in C, and $B_{\sigma+1}/B_\sigma$ is countable. Thus B_σ is completely decomposable, because the summands in $B_\sigma = A + C_\sigma \cong A/A_\sigma \oplus C_\sigma$ are completely decomposable. □

It is obvious that any torsion-free group contains pure completely decomposable subgroups. The following result shows that such subgroups need not be too small.

Theorem 3.13 (Griffith [2]). *A reduced torsion-free group A contains a pure completely decomposable subgroup C such that $|A| \leq |C|^{\aleph_0}$.*

Proof. Partially order the set of all pure completely decomposable subgroups C_i of A, by declaring C_i less than C_j if C_i is a summand of C_j. Zorn's lemma guarantees that this set has maximal members under this partial order, and let C be one of them. Evidently, the closure C^- of C in the $\mathbb{Z}$-adic completion $\tilde{A}$ of A is pure in $\tilde{A}$ and is complete, and hence algebraically compact. Thus $\tilde{A} = C^- \oplus C'$ for some C'. Now A is pure in $\tilde{A}$, so $A \cap C^- \oplus A \cap C'$ is pure in A. If $A \cap C'$ were $\neq 0$, then it would contain a rank 1 pure subgroup X, and $C \oplus X$ would be a larger pure completely decomposable subgroup in A. Hence the projection of A in C^- is monic, and $|A| \leq |C^-| \leq |C|^{\aleph_0}$ follows (see (6.6) in Chapter 6). □

It should be emphasized that the groups C in the preceding lemma are not unique, not even up to isomorphism (Exercise 5). The only uniqueness we know of is that if they are of infinite rank, then their ranks are equal.

Chains of Completely Decomposable Groups Hill's theorem 7.7 in Chapter 3 on chains of free groups generalizes to homogenous completely decomposable groups.

Theorem 3.14 (Nongxa [1]). *The union A of a countable chain $0 = A_0 < \cdots < A_n < \cdots$ $(n < \omega)$ of pure, homogeneous completely decomposable subgroups of fixed type* **t** *is completely decomposable.*

Proof. The proof runs in the same line as the proof of the quoted theorem. The argument on projectivity is replaced by an appeal to Baer's lemma 3.6. □

Balanced-Injective Groups Once balanced-projectivity has been discussed, it is natural to have a look at the dual: the question of balanced-injectives. In the torsion case, balanced-injectivity led to an interesting new class of groups. Surprisingly, in the torsion-free case, we do not get anything new.

Theorem 3.15 (Hunter [1]). *In the category of torsion-free groups, the balanced-injectives are precisely the algebraically compact groups.*

Proof. Evidently, pure-injective groups are balanced-injective. For the converse, assume G is torsion-free and balanced-injective. Let $S = \mathbb{Z}^{(\kappa)}$ be a free group where the cardinal κ is chosen so as to satisfy $|G| \leq \kappa < \kappa^{\aleph_0} = 2^\kappa$; this can be done, see the note after Proposition 7.10 in Chapter 10. Then $|S| = \kappa$ and $|S^-| = \kappa^{\aleph_0}$, where S^- denotes the $\mathbb{Z}$-adic closure of S in $P = \mathbb{Z}^\kappa$. Every extension by S^- is a balanced-extension, therefore $\mathrm{Ext}(S^-, G) = \mathrm{Bext}(S^-, G)$ vanishes by the hypothesis on G. Furthermore, the exact sequence $0 \to S \to S^- \to \oplus_{\kappa^{\aleph_0}} \mathbb{Q} \to 0$ induces the exact sequence $\mathrm{Hom}(S, G) \to \mathrm{Ext}(\oplus_{\kappa^{\aleph_0}} \mathbb{Q}, G) \to \mathrm{Ext}(S^-, G) = 0$. Here $|\mathrm{Hom}(S, G)| = |G|^\kappa \leq \kappa^\kappa = 2^\kappa$, while

$$|\mathrm{Ext}(\oplus_{\kappa^{\aleph_0}} \mathbb{Q}, G)| \geq |\mathrm{Ext}(\mathbb{Q}, G)|^{\kappa^{\aleph_0}} \geq 2^{\kappa^{\aleph_0}} = 2^{2^\kappa} > 2^\kappa$$

whenever $\mathrm{Ext}(\mathbb{Q}, G) \neq 0$. This is a contradiction unless G is cotorsion. Thus G is cotorsion and torsion-free, so algebraically compact. $\square$

However, with the sole exception of algebraically compact groups, no torsion-free group A can be embedded in a balanced-injective group as a balanced subgroup. In fact, if A is balanced in $\tilde{A}$, then it must be a summand, since $\tilde{A}/A \cong \oplus \mathbb{Q}$.

★ **Notes.** The theory of completely decomposable groups was initiated by Baer [6]. He proved the main characterization Theorem 3.5. He also proved Theorem 3.10, but only for the case when the typeset satisfied the maximum condition. Kulikov [4] gave a long proof for the countable case. The final touch was given by Kaplansky who proved that summands of direct sums of countable groups are again of the same kind.

A homogeneous pure subgroup H of a completely decomposable group C need not be completely decomposable (though it is if C is homogeneous). However, H will be completely decomposable if it is *-pure in the sense that $(H^*(\mathbf{t}))_* = H \cap (C^*(\mathbf{t}))_*$ for all types $\mathbf{t}$; see Nongxa [2]. On the other hand, Kravchenko [1] found necessary and sufficient conditions on a completely decomposable group to have all pure subgroups completely decomposable. Hill–Megibben [7] pointed out that balanced subgroups of completely decomposable groups need not be completely decomposable again. As a matter of fact, the balanced-projective dimension of a torsion-free group can be any integer and infinity. See Nongxa–Rangaswamy–Vinsonhaler [1] for an exposition of balanced subgroups of completely decomposable groups.

Eklof [6] proves that a homogeneous torsion-free group of singular cardinality is completely decomposable, provided each of its subgroups of smaller cardinality is contained in a completely decomposable subgroup.

Baer's ubiquitous lemma 3.6 is a most useful device needed in numerous proofs. It is the source of several generalizations, discussed by Albrecht, Arnold, Goeters, and Lady. See, e.g., Proposition 12.4.

Exercises

(1) Any two direct decompositions of a completely decomposable group have isomorphic refinements.

(2) (a) The typeset of a completely decomposable group is a semilattice under the operation $\wedge$. It is a subsemilattice of the lattice of all types.

(b) Find the least upper bound for the cardinalities of typesets of completely decomposable groups of rank n.

(3) A countable homogeneous torsion-free group, in which all finite rank pure subgroups are completely decomposable, is completely decomposable.

(4) Let C be a pure subgroup of the torsion-free group A such that A/C is a homogeneous completely decomposable group of type $\mathbf{t}$. If $A = A(\mathbf{t}) + C$, then C is a summand of A. [Hint: Baer's lemma.]

(5) (Griffith) In the group $\tilde{\mathbb{Z}} \cong \prod_p J_p$, there are maximal pure completely decomposable subgroups of any finite rank ≥ 1 as well as of rank $\aleph_0$.

(6) (Procházka) A reduced torsion-free group A has the property that an isomorphism $A \cong C \oplus A/C$ holds for each pure subgroup C if and only if A is a completely decomposable group of finite rank such that the types of the rank 1 summands are totally ordered.

(7) (Procházka) Let C be a completely decomposable group such that the different types of its rank 1 summands are inversely well-ordered. Show that pure subgroups of C are completely decomposable.

(8) (Fuchs–Kertész–Szele) A torsion-free group has the property that all of its pure subgroups are summands if and only if its reduced part is completely decomposable homogeneous and of finite rank. [Hint: reduce to the finite rank case; use Corollary 3.7.]

(9) Assume $G = Q^{(p)} \oplus Q^{(q)}$ for different primes p, q. Show that each of the completely decomposable groups $A \cong G^{(\aleph_0)}$ and $C \cong \mathbb{Z} \oplus G^{(\aleph_0)}$ is isomorphic to a pure subgroup of the other, but A and C are not isomorphic

(10) (Rangaswamy) Let C be a completely decomposable subgroup of countable index in the torsion-free group A. There is a decomposition $A = C' \oplus B$ with $C' \leq C$ and $|B| = \aleph_0$.

(11) (Yakovlev) Let A be a finite rank completely decomposable group. If A has a homogeneous completely decomposable subgroup C and an endomorphism α such that $A = C + \alpha C + \cdots + \alpha^n C$ for some $n \in \mathbb{N}$, then A is also homogeneous. [Hint: $C^{n+1} \to A$ *via* $(c_0, \ldots, c_n) \mapsto c_0 + \alpha c_1 + \cdots + \alpha^n c_n$.]

(12) Prove a stronger version of Theorem 3.12 by using a $G(\aleph_0)$-family in place of a smooth chain.

4 Indecomposable Groups

One of the most important questions in abelian group theory is concerned with indecomposable groups. Recapitulating: a group is called **indecomposable** if it has no direct summands other than 0 and itself. Equivalently, it has no idempotent endomorphism different from 0 and 1. Among the torsion groups, only the cocyclic groups are indecomposable, while no proper mixed group is indecomposable (see Corollary 2.4 in Chapter 5). The situation is drastically different for torsion-free groups: *there exist arbitrarily large indecomposable torsion-free groups.* This is what we want to prove here.

Finite Rank Indecomposable Groups We start with the finite rank case. The simplest way of getting indecomposable groups even up to rank $2^{\aleph_0}$ is *via* the p-adic integers.

Theorem 4.1 (Baer [6]). *p-pure subgroups of the group J_p of the p-adic integers are indecomposable.*

Proof. Let A be a p-pure subgroup of J_p, and let $0 \neq \pi \in A$. If $\pi = s_k p^k + s_{k+1} p^{k+1} + \cdots$ $(s_k \neq 0)$ is the canonical form of π, then by p-purity, also $s_k + s_{k+1} p + \cdots \in A$. Thus A contains a p-adic unit, and therefore $A + pJ_p = J_p$. By purity, $pA = A \cap pJ_p$, whence

$$A/pA = A/(A \cap pJ_p) \cong (A + pJ_p)/pJ_p = J_p/pJ_p \cong \mathbb{Z}(p).$$

If we had $A = B \oplus C$ with $B, C \neq 0$, also p-pure in J_p, we would obtain the contradiction $A/pA \cong B/pB \oplus C/pC \cong \mathbb{Z}(p) \oplus \mathbb{Z}(p)$. □

Our next goal is to obtain more explicit examples for indecomposable groups. In the constructions, we will often use the practice—as a matter of convenience—of writing $p^{-\infty}a$ as an abbreviation for what would properly be written as an infinite set: $\{p^{-1}a, \dots, p^{-n}a, \dots\}$. Also, we will construct groups A by starting from a basis $\{a_1, \dots, a_n\}$ of a $\mathbb{Q}$-vector space V, and then specifying additional generators of A in V, or by starting from a direct sum, and then adjoining elements from its divisible hull, even without mentioning V. Unexplained letters, like a_n, b_n, c_n, will denote independent elements in some unspecified $\mathbb{Q}$-vector space; in any case, our notation should be self-explanatory.

Example 4.2. Every torsion-free group of rank 1 is indecomposable.

Example 4.3 (Bognár [1]). Consider a set $\{p_1 \dots, p_n\}$ of primes, and an integer m relatively prime to each p_i. For every i, let $E_i = \langle p_i^{-\infty} e_i \rangle$ and $G_i = \langle p_i^{-\infty} e_i, m^{-1} e_i \rangle$ be groups of rank 1; clearly, $E_i = mG_i$. Define A as a subgroup of $G_1 \oplus \dots \oplus G_n$ as

$$A = \langle E_1 \oplus \dots \oplus E_n, m^{-1}(e_1 + e_2), \dots, m^{-1}(e_1 + e_n) \rangle.$$

To show that A is indecomposable, suppose $A = B \oplus C$. Notice that $\mathrm{Hom}(E_i, G_j) = 0$ if $i \neq j$, since every element of E_i is divisible by every power of p_i, while G_j does not contain any such elements $\neq 0$. Therefore, the groups E_i are fully invariant in A, so we have $E_i = (B \cap E_i) \oplus (C \cap E_i)$. The group E_i (as a rank 1 group) is indecomposable, so either $E_i \leq B$ or $E_i \leq C$. Assume e.g. $E_1 \leq B$ and $E_i \leq C$ for some $i > 1$. We can write $m^{-1}(e_1 + e_i) = b + c$ with $b \in B, c \in C$. Then $e_1 = mb, e_i = mc$, which is impossible, since none of e_i is divisible in A by any prime divisor of m. This means that all of E_i are contained either in B or in C, and hence either $B = A$ or $C = A$.

Example 4.4. Modify the preceding example, choosing a prime q different from the p_i, and forming the group

$$G = \langle E_1 \oplus \dots \oplus E_n, q^{-\infty}(e_1 + e_2), \dots, q^{-\infty}(e_1 + e_n) \rangle.$$

The same proof applies to establish the indecomposability of this G.

Rigid Systems In the construction of indecomposable groups, the following concept is useful. A set $\{G_i\}_{i \in I}$ of torsion-free groups $\neq 0$ is said to be a **rigid system** if

$$\mathrm{Hom}(G_i, G_j) \cong \begin{cases} \text{a subgroup of } \mathbb{Q} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

That is, groups in a rigid system have no endomorphisms other than multiplications by rational numbers, and no non-trivial homomorphisms into each other. In particular, groups in a rigid system have no idempotent endomorphisms $\neq 0, 1$, thus they are necessarily indecomposable. A group G is called **rigid** if the singleton $\{G\}$ is a rigid system.

The simplest example for a rigid system is a set of rank 1 groups.

Proposition 4.5. *There exists a rigid system of torsion-free groups of rank* 1 *consisting of* $2^{\aleph_0}$ *groups.*

Proof. Suppose $p_1, q_1, \ldots, p_n, q_n, \ldots$ are different primes. We consider all sets $S_i = \{r_1, r_2, \ldots, r_n, \ldots\}$ where, for each n, either $r_n = p_n$ or $r_n = q_n$. Clearly, no S_i contains S_j if $i \neq j$; and i runs over an index set I of the power of the continuum. For each S_i, define A_i to be the rational group of type $\mathbf{t}_i = (k_1, \ldots, k_n, \ldots)$, where $k_n = \infty$ exactly if the nth prime belongs to S_i, and $k_n = 0$ otherwise. Then the $\mathbf{t}_i$ are pairwise incomparable, hence $\{A_i\}_{i \in I}$ is a rigid system. □

Evidently, Theorem 4.1 yields homogeneous indecomposable groups of any rank $\leq 2^{\aleph_0}$. More explicit examples are provided by the following theorem, which is a generalized version of the example by Pontryagin [1].

Lemma 4.6. *For every $r \geq 2$, there exists a rigid system of $2^{\aleph_0}$ torsion-free groups $\{A_i\}_{i \in I}$ of rank r such that* $\operatorname{End} A_i \cong \mathbb{Z}$ *for each $i \in I$. The groups A_i are homogeneous of type* $\mathbf{t}_0 = (0, \ldots, 0, \ldots)$.

Proof. For any prime p, a single group A of the desired kind will be defined as a subgroup of $G = \mathbb{Q}^{(p)} a_1 \oplus \cdots \oplus \mathbb{Q}^{(p)} a_r$, a completely decomposable rank r group. First we dispose of the case $r = 2$. Choose a p-adic unit $\pi = s_0 + s_1 p + \cdots + s_n p^n + \cdots$ $(0 \leq s_n < p)$ which is transcendental over $\mathbb{Z}$; such units exist in abundance, since the transcendence degree of J_p over the rationals is of the power of the continuum. Set

$$x_n = p^{-n}(a_1 + \pi_n a_2) \in G, \tag{12.5}$$

where $\pi_n = s_0 + s_1 p + \cdots + s_{n-1} p^{n-1}$ is the nth partial sum of the standard form of π. The subgroup $A \leq G$ is defined as

$$A = \langle a_1, a_2; x_1, x_2, \ldots, x_n, \ldots \rangle.$$

A is of rank 2, and we will now show that it is rigid with $\operatorname{End} A \cong \mathbb{Z}$, and homogeneous of the type $\mathbf{t}_0$ of $\mathbb{Z}$.

Since $p x_{n+1} = x_n + s_n a_2$ for every n, an element of A not in $B = \langle a_1, a_2 \rangle$ has the form $k x_n + k_2 a_2$ for some integers k, k_2 with $\gcd\{p, k\} = 1$, and for some $n \geq 1$. Furthermore, if $p^m(k x_n + k_2 a_2) = \ell a_2$ $(\ell \in \mathbb{Z})$, then substituting (12.5), the coefficient of a_1 must be 0. We conclude that $\langle a_2 \rangle$ is a (p-pure, and hence a) pure subgroup of A. To show that A is homogeneous of type $\mathbf{t}_0$, only divisibility by p^n needs to be checked. Assume $p^{-n}(k_1 a_1 + k_2 a_2) \in A$ for all $n > 0$; here $k_1 \neq 0$, since $p^n | a_2$ is impossible. Then also $p^{-n}(k_1 a_1 + k_2 a_2 - k_1 p^n x_n) \in A$, and substitution from (12.5) yields $p^n | k_1 - k_2 \pi_n$ for all n, owing to the purity of $\langle a_2 \rangle$ in A. Hence the sequence $\{k_2 - k_1 \pi_n\}_{n \leq \omega}$ is a 0-sequence, therefore, its limit $k_2 - k_1 \pi = 0$. But this means that π is a rational number, a contradiction. Consequently, the group A is homogeneous of type $\mathbf{t}_0$.

To show that every endomorphism $\eta \neq 0$ of A is a multiplication by a rational number, we do this (for convenience) for a multiple $m\eta$ of η (which we will denote also by η) to be able to use that $\eta B \leq B$. η is determined by the images

$$\eta(a_1) = t_{11} a_1 + t_{12} a_2, \quad \eta(a_2) = t_{21} a_1 + t_{22} a_2 \qquad (t_{ij} \in \mathbb{Z}).$$

Then we have

$$\eta(x_n) = p^{-n}[\eta(a_1) + \pi_n\eta(a_2)] = p^{-n}[t_{11}a_1 + t_{12}a_2 + \pi_n(t_{21}a_1 + t_{22}a_2)]$$
$$= k_nx_n + \ell_na_2 = p^{-n}k_n(a_1 + \pi_na_2) + \ell_na_2$$

for certain $k_n, \ell_n \in \mathbb{Z}$. Comparing coefficients on both sides, we obtain

$$t_{11} + \pi_nt_{21} = k_n \quad \text{and} \quad p^{-n}[t_{12} + \pi_nt_{22} - k_n\pi_n] = \ell_n.$$

It follows that the coefficient in the square brackets must be divisible by p^n. Letting $n \to \infty$, and setting $\xi = \lim k_n$, we are led to the equations

$$t_{11} + \pi t_{21} = \xi \quad \text{and} \quad t_{12} + \pi t_{22} = \xi\pi.$$

Thus $t_{12}+\pi t_{22} = \pi(t_{11}+\pi t_{21})$, or $t_{21}\pi^2+(t_{11}-t_{22})\pi-t_{12} = 0$. The last equation is of degree ≤ 2 in π, so by the transcendency of π we get $t_{11} = t_{22}$ and $t_{12} = t_{21} = 0$. This shows that η acts as a multiplication by the integer t_{11}. If we do not assume that $\eta B \leq B$, then we can only conclude that η is a multiplication by a rational number. But this must carry the pure subgroup $\langle a_2\rangle$ into itself, so it has to be an integer. It follows that $\operatorname{End} A \cong \mathbb{Z}$, as we wished to show.

If we want to construct a group of rank $r > 2$, then we select p-adic units $\pi_2, \dots, \pi_r$ which are algebraically independent over $\mathbb{Q}$. We define A as above as a subgroup in G by adjoining to the independent set $\{a_1, a_2, \dots, a_r\}$, for each $j = 2, \dots, r$, additional generators

$$x_{jn} = p^{-n}(a_1 + \pi_{jn}a_j) \in G \qquad (n < \omega,\ 2 \leq j \leq r),$$

where π_{jn} denotes the nth partial sum in the standard form of π_j. Then the subgroups $A_j = \langle a_1, a_j\rangle_*$ are fully invariant in A, any endomorphism of A acts as multiplication by an integer on each A_j, and since $\langle a_1\rangle$ is common to all of them, these integers must be the same for all the A_j.

Using more algebraically independent units, $\pi_2, \dots, \pi_r, \rho_2, \dots, \rho_r$, we can construct, besides A, also another group A' of rank r with the aid of $\rho_2, \dots, \rho_r$. A word-by-word repetition of the proof in the preceding paragraphs will convince us that the only $\phi : A \to A'$ must be the zero map. Since there are continuously many algebraically independent units in J_p, one can construct a rigid system of continuously many homogeneous groups of any finite rank $r \geq 2$ whose endomorphism rings are $\cong \mathbb{Z}$. □

Indecomposable Groups of Infinite Ranks The various methods used above open the door to a vast supply of large indecomposable groups. Our construction of such groups relies on the next result.

Lemma 4.7 (Fuchs [12]). *Let $\{E_0, E_i \ (i \in I)\}$ be a rigid system of torsion-free groups, and p_i $(i \in I)$ primes such that there are elements $u_i \in E_0$ and $v_i \in E_i$ not divisible by p_i. Then the group*

$$A = \langle E_0 \oplus (\oplus_{i \in I} E_i);\ p_i^{-1}(u_i + v_i)\ \forall\, i \in I \rangle$$

is indecomposable. The same holds if p_i^{-1} is replaced by $p_i^{-\infty}$ or by n_i^{-1} for integers n_i whose prime divisors divide neither u_i nor v_i.

Proof. The proof of indecomposability runs as before. Rigity ensures the full invariance of the subgroups E_0, E_i in A. Therefore, if $A = B \oplus C$, then $E_i = (B \cap E_i) \oplus (C \cap E_i)$ for $i \in I \cup \{0\}$. Hence the indecomposability of the E_i implies that each of E_i is contained entirely either in B or in C. Assume $E_0 \leq B$ and $E_j \leq C$ for some $j \in I$, and write $p_j^{-1}(u_j + v_j) = b + c$ $(b \in B, c \in C)$. Hence $u_j = p_j b$ and $v_j = p_j c$, which is impossible in E_j, and likewise in A. It follows that all of E_0, E_i belong to the same summand B or C, and A is indecomposable. □

The Blowing Up Lemma We keep in mind that our ultimate goal in this section is to establish the existence of arbitrarily large indecomposable groups. In order to get ready for the proof, we will need a powerful result on blowing up a small system of indecomposable groups to extremely large systems. First, a definition.

Let S denote a $\mathbb{Z}$-algebra (in our application, $\mathsf{S} = \mathbb{Z}$). A **fully rigid system** $\{G_X \mid X \subseteq I\}$ of groups for S is a set of groups G_X, indexed by the subsets X of a set I, such that

(i) $G_X \leq G_Y$ whenever $X \subseteq Y$, and
(ii) $\mathrm{Hom}(G_X, G_Y) \cong \mathsf{S}$ or 0 according as $X \subseteq Y$ or not.

(If S is a subring of $\mathbb{Q}$, then a fully rigid system for S is a rigid system in the sense above.) We quote the relevant result without proof.

Lemma 4.8 (Corner [8]). *Suppose there exists a fully rigid system $\{N_X \mid X \subseteq J\}$ of groups for the $\mathbb{Z}$-algebra S over a set J of at least six elements. If I is any infinite set of cardinality $\geq |N_J|$, then there exists a fully rigid system $\{M_X \mid X \subseteq I\}$ for S such that*

$$|M_X| = |I| \qquad \text{for all } X \subseteq I.$$

Evidently, all the groups in a fully rigid system are indecomposable if 0 and 1 are the only idempotents in S.

Arbitrarily Large Indecomposable Groups By virtue of the preceding lemma, in order to prove the existence of arbitrarily large indecomposable groups, it only remains to find a fully rigid system for $\mathbb{Z}$ over a set with at least six elements. This is what we are going to do in the next proof.

Theorem 4.9. (i) (Shelah [1]) *For any cardinal κ, there exist indecomposable torsion-free groups of cardinality κ.*

(ii) *For every infinite cardinal κ, there exist fully rigid systems of 2^κ indecomposable groups of cardinality κ with $\mathbb{Z}$ as endomorphism ring.*

Proof. If κ is a finite cardinal, then we refer to Theorem 4.1 or to Example 4.3. For infinite cardinals κ, we exhibit fully rigid systems for $\mathbb{Z}$ over finite sets (of any size) so that Lemma 4.8 can be applied for κ.

The group A of rank r constructed in Lemma 4.6 depends on the choice of the p-adic units, so let us indicate this by $A = A(\pi_2, \dots, \pi_r)$. Corner's proof shows that different choices of r algebraically independent p-adic units yield to a rigid set. Now, choose $I = \{\pi_2, \dots, \pi_r\}$ as our index set for $r \geq 7$, and for a subset $X \subseteq I$, the group $A(X)$ as our G_X to obtain a collection of groups. It should be clear from the proof above that this is a fully rigid system for $\mathbb{Z}$. $\square$

Since a set of cardinality κ has a set of 2^κ subsets none of which contains another, it is clear that, for every infinite cardinal κ, there is a rigid system consisting of 2^κ groups—as many groups as the cardinality of the set of all non-isomorphic groups of cardinality κ. We find the abundance of arbitrarily large indecomposable groups quite remarkable, especially because their sheer existence was in doubt for quite a while.

The following generalization of rigid systems turned out to be very useful. Charles [4] calls a set of groups A_i $(i \in I)$ a **semi-rigid system** if the index set I is partially ordered such that

$$\mathrm{Hom}(A_i, A_j) \neq 0 \qquad \text{if and only if } \; i \leq j \; \text{ in } I.$$

The typical example is the set of isomorphy classes of all rank 1 torsion-free groups. Several results proved for rigid systems can be extended to semi-rigid systems.

Purely Indecomposable Groups We now consider a property stronger than mere indecomposability. A group A is **purely indecomposable** if every pure subgroup of A is indecomposable.

Theorem 4.10 (Armstrong [1], Griffith [1]). *A reduced torsion-free purely indecomposable group A is isomorphic to a pure subgroup of $\tilde{\mathbb{Z}}$; thus $|A| \leq 2^{\aleph_0}$.*

Proof. If A is as stated, then its completion $\tilde{A}$ is a reduced algebraically compact group, so every $0 \neq a \in A$ embeds in a summand G of $\tilde{A}$ that is $\cong J_p$ for some p. Write $\tilde{A} = G \oplus H$, and note that both $G \cap A$ and $H \cap A$ are pure in A, and so is their direct sum. Hence necessarily $H \cap A = 0$. Thus $A \cong G \cap A$, a pure subgroup of J_p. $\square$

★ **Notes.** The first example of an indecomposable torsion-free group of rank ≥ 2 was constructed by Levi [1], but it seems that only a few people were aware of it (it was not published in any journal). Much later, Pontryagin [1] gave an example of rank 2, and subsequently, Kurosh [2] and Mal'cev [1] developed a theory of finite rank torsion-free groups based on Derry's paper [1] on p-adic modules, using equivalent matrices. They succeeded in proving the existence of indecomposable groups of any finite rank. Baer's elegant idea (see [6]) was a big step forward, it furnished us with a supply of indecomposable groups up to rank $2^{\aleph_0}$. In the early 1950s, Szele tried really hard to construct examples beyond the continuum without success, and he was willing to conjecture that the continuum was a barrier (which turned out to be the case indeed for purely indecomposable groups). In 1958/1959 (after Szele's death), Hulanicki [2], Sąsiada [1], and Fuchs [9] independently constructed indecomposable groups of cardinality $2^{2^{\aleph_0}}$. Corner [6] proved the existence up to the first inaccessible cardinal, correcting an unfortunate set-theoretical mistake in Fuchs [12]. Indecomposable groups even for some measurable (but not for all) cardinals were constructed in Fuchs [18]. The final step was furnished by Shelah [1] who got rid of the cardinality restrictions. His proof was an amazing technical tour de force. Corner's Blowing-up Lemma 4.8 relies on Shelah's ideas and technique. Of course, once the most powerful theorem on the existence of large groups with prescribed cotorsion-free endomorphism rings is established (Sect. 7 in Chapter 16), the existence of arbitrarily large indecomposable groups becomes a trivial corollary by choosing $\mathbb{Z}$ as endomorphism ring.

Assuming V = L, Eklof–Mekler [1] prove that for every regular, not weakly compact cardinal κ, there exist 2^κ non-isomorphic indecomposable strongly κ-free groups of cardinality κ. Dugas [1] shows that these exist with $\mathbb{Z}$ as endomorphism ring.

There are exciting results on **absolutely indecomposable** torsion-free groups. These are groups that are not only indecomposable, but they remain so under any generic extension of the universe, i.e. they do not decompose into a proper direct sum no matter what kind of forcing is applied. The existence of absolutely indecomposable groups has been established by Göbel–Shelah [5] up to the first Erdős cardinal (which seems to be the natural boundary for such groups).

The first rigid system of size $> 2^{\aleph_0}$ was constructed by de Groot [1]. The existence of arbitrarily large rigid sets of torsion-free groups raises the question whether or not there exists a rigid proper class. The existence of such a proper class is not provable in ZFC, see A. Kanamori–M. Magidor [Springer Lecture Notes **669**, 99–275 (1978)]. However, such a class can be constructed in L, as is shown by Dugas–Herden [1]. Göbel–Shelah [1] establish the existence of a semi-rigid proper class in ZFC.

Exercises

(1) (de Groot) Let $p, p_1, \dots, p_n$ denote distinct odd primes. Prove that the group $G = \langle p_1^{-\infty}a_1, \dots, p_n^{-\infty}a_n,\ p^{-1}(a_1 + \cdots + a_n)\rangle$ is indecomposable.
(2) The group $A = \langle B \oplus C,\ p^{-1}(b + c)\rangle$ (p any prime) is not indecomposable if B, C are rank 1 torsion-free groups with $\mathbf{t}(B) \leq \mathbf{t}(C)$.
(3) (a) The pure subgroups A, C of J_p are isomorphic if and only if there is a p-adic unit π such that multiplication by π yields an isomorphism $A \to C$. [Hint: J_p is algebraically compact, and $\operatorname{End} J_p \cong J_p$.]
    (b) Let $r \in \mathbb{N}$ or $r = \aleph_0$. J_p contains a rigid system (of size $2^{\aleph_0}$) of indecomposable pure subgroups of rank r.
(4) Examine the proof of Lemma 4.6, and show that it would be enough to assume that the p-adic integers π_j are quadratically independent.
(5) (Corner) For every integer $m \geq 2$, there is a torsion-free group A of rank m with the following properties:

    (i) all subgroups of rank $\leq m - 1$ are free;
    (ii) all torsion-free factor groups of rank 1 are divisible;
    (iii) $\operatorname{End} A \cong \mathbb{Z}$. [Hint: $x_n = p^{-n}(a_1 + \pi_{2n}a_2 + \cdots + \pi_{mn}a_m)$.]

(6) If a reduced group $A = B \oplus C$ contains a subgroup $G \cong J_p$, then either $G \leq B$ or $G \cap B = 0$. [Hint: $G \cap B$ is pure in G, so $G/(G \cap B)$ is divisible.]

(7) (de Groot) (a) A torsion-free group is purely indecomposable if any two independent elements have incomparable types.

(b) Let V be a $\mathbb{Q}$-vector space with basis $\{a_1, \dots, a_n, \dots\}$. Arrange the linear combinations $b = k_1 a_{n_1} + \dots + k_r a_{n_r}$ with $n_1 < \dots < n_r$ and integers $0 \neq k_i \in \mathbb{Z}, k_1 \in \mathbb{N}$, $\gcd\{k_1, \dots, k_r\} = 1$ in a sequence. To the jth b in the sequence assign the characteristic $(0, \dots, 0, \infty, 0, \dots)$ with ∞ only at the jth prime to obtain a subgroup A of V. Prove that A is purely indecomposable.

5 Pathological Direct Decompositions of Finite Rank Groups

From the point of view of direct decompositions, the existence of large indecomposable groups is a major difference between torsion and torsion-free groups. Another striking difference between them is the occurrence of various phenomena in the direct decompositions of torsion-free groups already in the finite rank case. The situation is more cumbersome than one might expect, and we shall spend some time to make it more transparent.

It is evident that a torsion-free group of finite rank decomposes into the direct sum of a finite number of indecomposable groups. The question we pose is this: is there any kind of uniqueness in these decompositions? The best way to describe the answer is to quote Kaplansky who claimed that *in this strange part of the subject, anything that can conceivably happen actually does happen*.

We will list several examples of pathological decompositions, involving lots of computations (in particular, divisibility considerations). Complicated technicalities are, however, unavoidable in view of the nature of the subject, and unfortunately, they only provide a very superficial insight into the structure of finite rank torsion-free groups.

Decompositions with the Same Number of Summands Our first theorem is a variant of Jónsson's original counterexample that set the stage for further strange examples. Recall: for a set P primes, we use $P^{-1}a$ as an abbreviation of the set $\{p^{-1}a \mid p \in P\}$.

Theorem 5.1. *There are rank* 3 *torsion-free groups that have many non-isomorphic direct decompositions into indecomposable summands.*

Proof. Let P, Q denote two infinite disjoint sets of primes, and $p > 3$ a prime $p \notin P \cup Q$. Define

$$A = \langle P^{-1}a \rangle \oplus \langle P^{-1}b,\, Q^{-1}c,\, p^{-1}(b + c) \rangle$$

with independent a, b, c. Pick $s \in \mathbb{Z}$ with $\gcd\{s, p\} = 1$, $s \not\equiv 0, \pm 1 \bmod p$, and find $r, t \in \mathbb{Z}$ satisfying $sr - pt = 1$. Then $a' = sa + tb, b' = pa + rb, c$ are independent, and we claim that

$$A' = \langle P^{-1}a' \rangle \oplus \langle P^{-1}b',\, Q^{-1}c,\, p^{-1}(sb'+c) \rangle$$

is equal to A. As $\langle P^{-1}a \rangle \oplus \langle P^{-1}b \rangle = \langle P^{-1}a' \rangle \oplus \langle P^{-1}b' \rangle$ is obvious, we prove $A' = A$ by showing that the given relations imply that $b + c$ is divisible by p if and only if $sb' + c$ is divisible by p. But this is rather obvious in view of $sb' + c = spa + srb + c = spa - ptb + (b + c)$. It is also clear that the rank 2 summands above are indecomposable. It remains to prove that these summands are not isomorphic. Any isomorphism ϕ between them must act as $b \mapsto \pm b'$ (i.e. $+$ or $-$) and $c \mapsto \pm c$, since ϕ must respect divisibility properties. But then $\phi(b + c) = \pm b' \pm c$, which is impossible, since these elements are not divisible by p in A. □

Even if we restrict ourselves to direct decompositions with a fixed number of indecomposable summands, the rank distribution of the summands can be arbitrary—this is shown by the following striking result.

Theorem 5.2 (Corner [1]). *Suppose n, k are the integers such that $1 \le k \le n$. There exists a torsion-free group A of rank n with the following property: to every partition $n = r_1 + \cdots + r_k$ of n into k integers $r_i \ge 1$, there is a decomposition $A = A_1 \oplus \cdots \oplus A_k$ where A_i is indecomposable of rank r_i.*

Proof. Let $p, p_1, \ldots, p_{n-k}, q_1, \ldots, q_{n-k}$ denote different primes. Select independent elements $a_1, \ldots, a_k, b_1, \ldots, b_{n-k}$, and define

$$A = \langle p^{-\infty}a_1, \ldots, p^{-\infty}a_k,\, p_1^{-\infty}b_1, \ldots, p_{n-k}^{-\infty}b_{n-k},$$
$$q_1^{-1}(a_1 + b_1), \ldots, q_{n-k}^{-1}(a_1 + b_{n-k}) \rangle.$$

To show that this group has the indicated property, let $n = r_1 + \cdots + r_k$ be a partition of n with $r_i \ge 1$. If $s_1, \ldots, s_k$ are integers such that $s_1 + \cdots + s_k = 1$, then the system

$$\begin{array}{lcr} s_1c_1 + \cdots & & +s_kc_k = a_1 \\ -c_1 + c_2 & & = a_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -c_1 & & +c_k = a_k \end{array}$$

with determinant 1 is solvable for $c_1, \ldots, c_k \in A$, and $\langle c_1, \ldots, c_k \rangle = \langle a_1, \ldots, a_k \rangle$ holds. Set $t_1 = 0$, and $t_i = (r_1 - 1) + \cdots + (r_{i-1} - 1)$ for $i = 2, \ldots, k$. Using these c_i, we define

$$A_i = \langle p^{-\infty}c_i,\, p_j^{-\infty}b_j,\, q_j^{-1}(c_i + b_j)\ (j = t_i + 1, \ldots, t_{i+1}) \rangle$$

for $i = 1, \ldots, k$. In view of

$$a_1 + b_j = (c_i + b_j) + s_1c_1 + \cdots + s_{i-1}c_{i-1} + (s_i - 1)c_i + \cdots + s_kc_k,$$

it is routine to check that if the numbers $s_1, \dots, s_k$ are chosen such that

$$s_i \equiv \begin{cases} 1 \mod q_j & \text{if } j = t_i + 1, \dots, t_{i+1}, \\ 0 \mod q_j & \text{otherwise,} \end{cases}$$

then the A_i will be subgroups of A, and every generator of A will be contained in $A_1 + \dots + A_k$. This yields at once $A = A_1 \oplus \dots \oplus A_k$ where by construction, $r(A_i) = r_i$, as desired. That the A_i are indecomposable follows as in Example 4.3. $\square$

For instance, the numbers s_i can be chosen as $s_i = m_{t_i+1}\hat{q}_{t_i+1} + \dots + m_{t_{i+1}}\hat{q}_{t_{i+1}}$, where $\hat{q}_j = q_1 \dots q_{j-1}q_{j+1} \dots q_{n-k}$ and $\sum_{j=1}^{k} m_j\hat{q}_j = 1$.

In our endeavor to find some sort of uniqueness under reasonable restrictions on the decompositions, we try comparing only decompositions with the same distribution of ranks. This is still not enough to ensure isomorphy.

Theorem 5.3 (Fuchs–Loonstra [1]). *Given any integer $m \geq 2$, there exist indecomposable groups, A and C, of rank 2 such that*

$$A \oplus \dots \oplus A \cong C \oplus \dots \oplus C \qquad (n \text{ summands})$$

if and only if $n \equiv 0 \mod m$.

Proof. We start with two disjoint, infinite sets P_1, P_2 of primes, and a prime p (to be suitably chosen later on) such that $p \notin P_1 \cup P_2$. We define the following rank 1 groups:

$$X_i = \langle P_1^{-1} x_i \rangle, \qquad Y_i = \langle P_2^{-1} y_i \rangle \quad (i = 1, \dots, n).$$

With independent x_i, y_i, we form the isomorphic indecomposable groups

$$A_i = \langle X_i \oplus Y_i,\, p^{-1}(x_i + y_i) \rangle \qquad (i = 1, \dots, n).$$

Next, we choose groups $U_i \cong X_i$ and $V_i \cong Y_i$ $(i = 1, \dots, n)$ and let $u_i \mapsto x_i$, $v_i \mapsto y_i$ $(u_i \in U_i, v_i \in V_i)$ under some fixed isomorphisms. For any choice of $k = 1, \dots, p-1$, we form the isomorphic groups

$$C_i = \langle U_i \oplus V_i,\, p^{-1}(u_i + kv_i) \rangle \qquad (i = 1, \dots, n)$$

which are also indecomposable.

Suppose $\phi : A = A_1 \oplus \dots \oplus A_m \to C = C_1 \oplus \dots \oplus C_m$ is an isomorphism. The choice of the groups X_i, Y_i assures that ϕ maps $X = X_1 \oplus \dots \oplus X_m$ and $Y = Y_1 \oplus \dots \oplus Y_m$ upon $U = U_1 \oplus \dots \oplus U_m$ and $V = V_1 \oplus \dots \oplus V_m$, respectively. In particular,

$$\phi(x_i) = \sum_{j=1}^{m} r_{ij} u_j \qquad \text{and} \qquad \phi(y_i) = \sum_{j=1}^{m} s_{ij} v_j, \tag{12.6}$$

where r_{ij}, s_{ij} are integers such that the matrices $\|r_{ij}\|$, $\|s_{ij}\|$ are invertible:

$$\det \|r_{ij}\| = \pm 1 \qquad \text{and} \qquad \det \|s_{ij}\| = \pm 1. \tag{12.7}$$

ϕ has to preserve divisibility by p, therefore

$$\phi(x_i + y_i) = \sum_{j=1}^{m} r_{ij}(u_j + kv_j) + \sum_{j=1}^{m} (s_{ij} - kr_{ij}) v_j$$

taken in combination with the independence of the v_j implies

$$s_{ij} \equiv kr_{ij} \bmod p \quad \text{for all } i, j. \tag{12.8}$$

On the other hand, (12.7) and (12.8) together imply that ϕ as given in (12.6) is in fact an isomorphism $A \to C$, since $p | x_i + y_i$ in A for all i implies $p | u_j + kv_j$ in C for all j, and *vice versa.*

It remains to choose the coefficients r_{ij}, s_{ij}, the prime p, and the integer k in a suitable way to avoid isomorphism for n summands if $m \nmid n$. From (12.7) and (12.8) we obtain $k^m \equiv \pm 1 \mod p$. On the other hand, in order to exclude isomorphy for $n < m$, k will be chosen so as to satisfy

$$k^m \equiv -1 \bmod p, \quad \text{but} \quad k^n \not\equiv \pm 1 \bmod p \ (n = 1, \ldots, m-1). \tag{12.9}$$

Observe that if k satisfies (12.9), then it satisfies $k^n \equiv \pm 1 \bmod p$ if and only if $n | m$. By Dirichlet's theorem on primes in arithmetic progressions, we can find a prime p such that $p \equiv 1 \bmod 2m$, and by a well-known theorem in number theory, there is a primitive root $t \bmod p$. If we set $k = t^{(p-1)/2m}$, then (12.9) is satisfied.

Let ℓ be an integer such that $k\ell \equiv 1 \bmod p$, and define two $m \times m$ matrices:

$$\begin{pmatrix} 1 & & & & & & & \\ \ell & 1 & & & & & & \\ 0 & \ell & 1 & & & & & \\ \ell & 1 & \ell & 1 & & & & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & \ell & 1 & \\ . & . & . & . & \ell & 1 & \ell & -1 \end{pmatrix}, \quad \begin{pmatrix} k & & & & & & & k' \\ 1 & k & & & & & & \\ 0 & 1 & k & & & & & \\ 1 & k & 1 & k & & & & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & k & 1 & k & \\ . & . & . & . & 1 & k & 1 & -k \end{pmatrix}.$$

In the first matrix, we have 1s in the main diagonal except for the last entry, below the diagonal we have alternately ℓ and 0 in the first column, ℓ and 1 in the other columns. The second matrix is obtained from the first by substituting k for 1 and

1 for ℓ throughout, and finally, putting $k' = (-1)^{m+1}(k^m + 1)$ in the upper right corner. A simple calculation is needed to check that if we choose the first matrix for $\|r_{ij}\|$, and the second matrix for $\|s_{ij}\|$, then all of (12.7), (12.8) and (12.9) will be satisfied. $\square$

Different Numbers of Summands The next result tells us that even the number of indecomposable summands can be different in the decompositions. (Here 'summand' will mean 'non-zero summand.')

Theorem 5.4 (Fuchs–Gräbe [1]). *For every integer $n \geq 1$, there exists a torsion-free group of rank $2n$ which has decompositions into the direct sum of m indecomposable summands for any m with $2 \leq m \leq n+1$.*

Proof. Let $p_1, \dots, p_n, q_2, \dots, q_n, r$ denote different primes. For $1 \leq k \leq n$, define the group

$$\begin{aligned} A_{nk} =& \langle p_1^{-\infty} a_1 \rangle \oplus \cdots \oplus \langle p_k^{-\infty} a_k \rangle \\ & \oplus \langle p_1^{-\infty} b_1, \dots, p_n^{-\infty} b_n,\ r^{-1} q_2^{-1}(b_1 - b_2), \dots, r^{-1} q_n^{-1}(b_1 - b_n) \rangle, \end{aligned}$$

where the summands are indecomposable. If s, t, u, v are integers such that $vs - ut = 1$, then set $c_i = sa_i + tb_i, d_i = ua_i + vb_i$ in order to get $\langle p_i^{-\infty} a_i \rangle \oplus \langle p_i^{-\infty} b_i \rangle = \langle p_i^{-\infty} c_i \rangle \oplus \langle p_i^{-\infty} d_i \rangle$ for $i = 1, \dots, k$. Define

$$\begin{aligned} C = & \langle p_1^{-\infty} c_1, \dots, p_k^{-\infty} c_k, p_{k+1}^{-\infty} b_{k+1}, \dots, p_n^{-\infty} b_n, r^{-1} q_i^{-1} u(c_1 - c_i), \\ & r^{-1} q_j^{-1}(uc_1 + b_j), r^{-1} q_j^{-1}(b_1 - b_j)\ (1 < i \leq k < j \leq n) \rangle, \\ D = & \langle p_1^{-\infty} d_1, \dots, p_k^{-\infty} d_k,\ r^{-1} q_i^{-1} s(d_1 - d_i)\ (1 < i \leq k) \rangle. \end{aligned}$$

These groups are indecomposable, and we want to choose s, t, u, v so as to have $A_{nk} = C \oplus D$. To ensure that $C \leq A_{nk}$, the divisibility relations

$$\begin{aligned} rq_i \mid u(c_1 - c_i) &= usa_1 - usa_i + ut(b_1 - b_i), \\ rq_j \mid uc_1 + b_j &= usa_1 + (ut+1)b_1 + (b_j - b_1) \end{aligned}$$

must hold for all indices $1 < i \leq k < j \leq n$. Thus $rq_i \mid us$ for $i = 2, \dots, n$, and $rq_j \mid (ut+1) = vs$ for $j = k+1, \dots, n$. Hence $\gcd\{u, v\} = 1$ implies $rq_j \mid s$. Thus, the conditions

$$rq_i \mid us \ \ (i = 2, \dots, k) \quad \text{and} \quad rq_j \mid s \ \ (j = k+1, \dots, n) \tag{12.10}$$

must be satisfied. For $D \leq A_{nk}$ to hold, we must have

$$rq_i \mid s(d_1 - d_i) = sua_1 - sua_i + sv(b_1 - b_i) \ \ (i = 2, \dots, k),$$

which are satisfied whenever (12.10) holds. Since $a_i = vc_i - td_i$ and $b_i = -uc_i + sd_i$ $(i = 1, \dots, k)$, we have

$$r^{-1}q_i^{-1}(b_1 - b_i) = -r^{-1}q_i^{-1}u(c_1 - c_i) + r^{-1}q_i^{-1}s(d_1 - d_i) \ \ (i = 2, \dots, k),$$
$$r^{-1}q_j^{-1}(b_1 - b_j) = -r^{-1}q_j^{-1}(uc_1 + b_j) + r^{-1}q_j^{-1}sd_1 \ \ (j = k + 1, \dots, n),$$

showing that $A_{nk} \leq C \oplus D$, i.e. $A_{nk} = C \oplus D$ provided (12.10) holds. It does if we set $u = q_2 \dots q_k$ and $s = r \cdot q_{k+1} \dots q_n$.

Now we choose $A = A_{nn}$ of rank $2n$, and observe that $A = A_{nk} \oplus \langle p_{k+1}^{-\infty} a_{k+1} \rangle \oplus \dots \oplus \langle p_n^{-\infty} a_n \rangle$ for each k, where by the above, the first component has a direct decomposition into two indecomposable summands. As $1 \leq k \leq n$, the number of indecomposable summands in direct decompositions of A could be any m provided $2 \leq m \leq n + 1$. □

No Cancellation As our final attempt in search of uniqueness we consider the cancellation property. We say that a finite rank torsion-free group A has the **cancellation property** if, for any finite rank torsion-free groups, G and H, $A \oplus G \cong A \oplus H$ implies $G \cong H$. An equivalent formulation is: $A \oplus G = A' \oplus H$ with $A \cong A'$ implies $G \cong H$. The next result shows that not all rank 1 groups share the cancellation property (though some do, see Sect. 2, Exercise 12 in Chapter 3).

Theorem 5.5 (Fuchs–Loonstra [1]). *For every integer $m \geq 1$, there exists a torsion-free group A of rank 3 which has m decompositions*

$$A = B_i \oplus C_i \qquad (i = 1, \dots, m)$$

such that $B_1 \cong \dots \cong B_m$ are of rank 1, while $C_1, \dots, C_m$ are pairwise non-isomorphic, indecomposable groups of rank 2.

Proof. We start off with two disjoint, infinite sets P_1, P_2 of primes, and a prime $p \notin P_1 \cup P_2$. We define the groups

$$B_1 = \langle P_1^{-1} b_1 \rangle, \quad X_1 = \langle P_1^{-1} x_1 \rangle, \quad Y = \langle P_2^{-1} y \rangle.$$

If b_1, x_1, y are independent, then with the indecomposable group $C_1 = \langle X_1 \oplus Y, p^{-1}(x_1 + y) \rangle$ we set

$$A = B_1 \oplus C_1.$$

Now choose integers q_i, r_i, s_i, t_i $(i = 2, \dots, m)$ such that $q_i t_i - r_i s_i = 1$, and let

$$b_i = q_i b_1 + s_i x_1, \quad x_i = r_i b_1 + t_i x_1 \quad (i = 2, \dots, m).$$

Then we have $B_i \oplus X_i = B_1 \oplus X_1$ for the pure subgroups $B_i = \langle b_i \rangle_*, X_i = \langle x_i \rangle_*$ isomorphic to B_1. We intend to choose B_i, X_i $(i \geq 2)$ such that, for some integers k_i we will have

$$A = B_i \oplus C_i \quad \text{with} \quad C_i = \langle X_i \oplus Y,\ p^{-1}(k_i x_i + y)\rangle \tag{12.11}$$

where the C_i's are non-isomorphic (and indecomposable). Then to ensure that $k_i x_i + y = k_i r_i b_1 + (k_i t_i - 1)x_1 + (x_1 + y)$ will be divisible by p, we must select k_i such that

$$k_i r_i \equiv 0 \quad \text{and} \quad k_i t_i \equiv 1 \mod p. \tag{12.12}$$

For any k_i with $1 < k_i < p$, we choose $r_i = -p, q_i = k_i$, and s_i, t_i so as to satisfy $q_i t_i - r_i s_i = 1$; then (12.12) holds, and hence so does (12.11).

We still have to ascertain that the C_i are pairwise non-isomorphic. Observe that the only automorphisms of the X_i and Y are multiplications by ± 1, thus any isomorphism $\phi : C_i \to C_j$ must map X_i upon X_j, and Y upon itself such that $\phi(x_i) = \pm x_j, \phi(y) = \pm y$. This means that we have $\phi(k_i x_i + y) = \pm(k_j x_j \pm y)$. The preservation of divisibility forces $k_j \equiv \pm k_i \mod p$. Consequently, if we choose $(k_1 = 1), k_2 = 2, \dots, k_m = m$ and a prime $p > 2m - 1$, then we guarantee that $k_j \not\equiv \pm k_i \mod p$ for $i \neq j$, and therefore, no two of the groups $C_1, \dots, C_m$ are isomorphic. $\square$

This theorem also shows that, for any rank $r \geq 3$, and for every integer m, there exist torsion-free groups of rank r which have at least m pairwise non-isomorphic decompositions into indecomposable summands. However, Lady's theorem 6.9 will show that no torsion-free group of finite rank admits infinitely many non-isomorphic decompositions.

Possible Rank Distributions The question of rank distributions in direct sums of finite rank torsion-free groups is too important to leave out completely. The problem was completely solved by Yakovlev and Blagoveshchenskaya in a satisfactory way. The proof is too long, it requires further structural information to be determined, so here we cannot do more than just stating the theorem without proof. As expected, the rank 1 summands play a special role.

Theorem 5.6 (Yakovlev [1], Blagoveshchenskaya–Yakovlev [1]). *Let*

$$n = r_1 + \cdots + r_k = s_1 + \cdots + s_\ell$$

be two partitions of the positive integer $n \geq 3$ ($r_i, s_j \in \mathbb{N}$). Let u and v denote the number of r_i and s_j, respectively, that are equal to 1.

A necessary and sufficient condition for the existence of a torsion-free group of rank n to admit a direct decomposition into indecomposable summands of ranks r_i, and also one with indecomposable summands of ranks s_j, is that

(i) *$r_i \leq n - v$ for all i, and $s_j \leq n - u$ for all j; and*
(ii) *if $r_i = n - v$ for some i, then exactly one $s_j \neq 1$ (and is equal to $n - v$), and if $s_j = n - u$ for some j, then exactly one $r_i \neq 1$ (and is equal to $n - u$).* $\square$

The proof of sufficiency is based on a clever construction of special indecomposable groups, called *flower groups*.

Another result deals with the precise spectrum of the numbers of indecomposable summands in possible direct decompositions. Permitted as well as forbidden numbers can be selected as desired. We quote it without its lengthy proof.

Theorem 5.7 (Fuchs–Gräbe [1]). *Let N be any finite collection of integers ≥ 2, and n the largest integer in N. There exists a torsion-free group of rank $\leq 2(n-1)$ which has a decomposition into the direct sum of k indecomposable summands $\neq 0$ if and only if $k \in N$.* □

★ **Notes.** After reading all the above strange decompositions that torsion-free groups of finite rank display, one might get the false impression that these are the norm rather than the exception. As a matter of fact, the first pathological example by Jónsson [1] was quite a surprise, and even some group theorists had doubts about the correctness of his numerical example. This was an apparent setback in the structure theory, though Jónsson himself tried to restore an order by introducing an equivalence relation weaker than isomorphism (see Theorem 9.9), shaking our belief in isomorphism as an exclusive principle of classification. Thanks to E. Walker [2], we know that the best way of viewing quasi-isomorphism is to interpret it as an isomorphism in an adequate quotient category. Quasi-isomorphism clarified the direct decompositions, but unfortunately it has not contributed too much to the structure problem: the results by Beaumont–Pierce [2], as well as numerous later theorems (e.g., Mutzbauer [1], Richman [2], etc.) on rank 2 groups are a convincing evidence that there is still a long way to go (if possible at all) to classify finite rank torsion-free groups even up to quasi-isomorphism. There is an extensive literature on pathological direct decompositions; one of the earliest is Jesmanowicz [1].

For more important material on direct decompositions of finite rank torsion-free groups, we refer to Faticoni [Fa].

Exercises

(1) A finite rank torsion-free group $A = B \oplus C$ may have a summand which is completely decomposable of rank $r > 0$, even if neither B nor C has rank one summands.

(2) There is no torsion-free group of rank 4, which can be written as a direct sum of two rank 1 groups, and an indecomposable rank 2 group, and also as a direct sum of a rank 1 and an indecomposable rank 3 group. [Hint: look at the types of the rank 1 summands.]

(3) Let $r_1, \dots, r_{k-1} \in \mathbb{N}$ and $n = r_1 + \cdots + r_{k-1}$. There exists a group A of rank $2n$ such that $A = A_1 \oplus \cdots \oplus A_{k-1} \oplus A_k = B \oplus C$, where A_i is indecomposable of rank r_i $(i = 1, \dots, k-1)$, while the groups A_k, B, C are indecomposable of rank n. [Hint: in the proof of Theorem 5.1, replace rank 1 groups by groups from a suitable rigid system.]

(4) (Fuchs–Loonstra) For every $m \geq 2$, there exist pairwise non-isomorphic indecomposable groups $B, C_1, \dots, C_m$ of rank 2 such that $B \oplus \cdots \oplus B \cong C_1 \oplus \cdots \oplus C_m$ (m summands B), but the direct sum of $n\,(< m)$ copies of B is not isomorphic to the direct sum of any n groups in the set $\{C_1, \dots, C_m\}$. [Hint: argue as in Theorem 5.3.]

(5) For every integer $n \geq 1$, there is an indecomposable group A of rank n such that $A \oplus B \cong A \oplus C$ for suitable non-isomorphic indecomposable groups B, C of finite rank. [Hint: Theorem 5.5.]
(6) (Sąsiada) If $X = \langle 5^{-\infty}x, 3^{-\infty}y, 2^{-1}(x+y)\rangle$, then every indecomposable summand of $X^{(n)}$ for any $n > 1$ is isomorphic to X. [Hint: any such summand must have a subgroup $\cong \langle 5^{-\infty}x, 3^{-\infty}y\rangle$ of index 2.]
(7) (a) (Corner) Let A be torsion-free of rank n, and B, C isomorphic pure subgroups of A of rank $n-1$. Then $A/B \cong A/C$. [Hint: reduce to $n = 2$, $B \neq C$, and calculate the torsion subgroup of $A/(B+C)$ in two ways.]
(b) Cancellation by a finite rank torsion-free group is permitted if the complements are of rank 1.

6 Direct Decompositions of Finite Rank Groups: Positive Results

We have seen that a torsion-free group of finite rank may have numerous non-isomorphic direct decompositions. Is there anything positive which can be stated about such decompositions? In particular, must such a group have but finitely many non-isomorphic such decompositions? Lady [3] has answered this question in the positive.

This is an important result, therefore we give a detailed proof, though from here on nearly everything is ring theory. The proof of the main Theorem 6.9 relies on several not so well-known facts from ring theory, in particular, on a deep result by Jordan–Zassenhaus that will be quoted without proof.

Lemmas on Rings and Modules In the next lemma, E stands for the endomorphism ring of A.

Lemma 6.1. *Let B and B' be direct summands of a group A, and let π, π' denote the projections onto them. Then $B \cong B'$ if and only if the right ideals $\pi\mathsf{E}$ and $\pi'\mathsf{E}$ are isomorphic as right E-modules.*

Proof. See the proof of the more complete Lemma 1.6 in Chapter 16. □

We break down the discussion of the required ring-theoretical background into a series of easy lemmas. The first lemma can be found in most text books on rings and modules. R will denote a ring with 1, and J its Jacobson radical.

Lemma 6.2 (Nakayama's Lemma). *If M is a finitely generated right R-module such that $M = M\mathsf{J}$, then $M = 0$.*

Proof. Let $\{x_1, \dots, x_k\}$ with $k \geq 1$ be a minimal generating system for M. By hypothesis, we have $x_1 = x_1r_1 + \cdots + x_kr_k$ with $r_i \in \mathsf{J}$. Since $1 - r_1$ is invertible in R, x_1 can be expressed as a linear combination of the other generators, contrary to the minimality of the generating system. Thus $k = 0$, and hence $M = 0$. □

A right ideal L of R is **nilpotent** if $\mathsf{L}^k = 0$ for some integer $k \geq 1$. The union of all nilpotent right ideals of R is a two-sided ideal N called the **nilpotent radical** of R. Always, $\mathsf{N} \leq \mathsf{J}$. We say that a ring R is torsion-free or of finite rank if its additive group has this property.

Lemma 6.3. *For a torsion-free ring* R *of finite rank, the nilpotent radical* N *is nilpotent and pure in* R.

Proof. It is immediate that the pure closure of a nilpotent ideal is likewise nilpotent; hence, N is pure in R. If L is a pure nilpotent right ideal of R of maximal rank, then for every nilpotent right ideal K of R, $\mathsf{L} + \mathsf{K}$ is nilpotent, and must be of the same rank as L. As L is pure, we obtain $\mathsf{L} + \mathsf{K} = \mathsf{L}$, i.e. $\mathsf{K} \leq \mathsf{L}$, and hence $\mathsf{N} \leq \mathsf{L}$. □

Lemma 6.4. *If* R *is a finite dimensional algebra over a field* F*, then* $\mathsf{J} = \mathsf{N}$*, and* R/J *is a semi-simple artinian ring.*

Proof. If n is such that J^n has (as an F-vector space) the smallest dimension among the powers of the Jacobson radical J, then $\mathsf{J}^{n+1} = \mathsf{J}^n$, and Nakayama's lemma implies $\mathsf{J}^n = 0$. Hence $\mathsf{J} = \mathsf{N}$, and the rest is classical. □

Lemma 6.5. *Let* N *be the nilpotent radical of the torsion-free ring* R *of finite rank, and* A, B *two finitely generated projective right* R*-modules. Then* $A \cong B$ *if and only if*

$$A/A\mathsf{N} \cong B/B\mathsf{N}.$$

Proof. An isomorphism $\phi : A \to B$ evidently induces an isomorphism $A/A\mathsf{N} \to B/B\mathsf{N}$. Conversely, assume $\psi : A/A\mathsf{N} \to B/B\mathsf{N}$ is an isomorphism. Consider the diagram

$$\begin{tikzcd} A \arrow[r, dashed, "\phi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A/A\mathsf{N} \arrow[r, "\psi"] & B/B\mathsf{N} \end{tikzcd}$$

where the vertical maps are the canonical ones; the existence of a map $\phi : A \to B$ making the square commute follows from the projectivity of A. As $\beta\phi(= \psi\alpha)$ is surjective, $\phi A + B\mathsf{N} = B$ follows. Applying Nakayama's lemma to $B/\phi A$ (with J = N), we obtain $\phi A = B$. Thus $A \cong B \oplus \operatorname{Ker}\phi$ by the projectivity of B. Hence $\operatorname{rk} A \geq \operatorname{rk} B$, and by symmetry, we obtain $\operatorname{rk} A = \operatorname{rk} B$. Therefore, $\operatorname{Ker}\phi = 0$, and ϕ is an isomorphism. □

Lemma 6.6. *A torsion-free ring* R *of finite rank* n *contains a subring* S *whose additive group is a free group of rank* n*. If* R *has an identity,* S *can be chosen to contain it.*

Proof. Let $\{x_1, \dots, x_n\}$ be a maximal independent set in the additive group of R. There are $t_{ijk} \in \mathbb{Q}$ such that $x_i x_j = t_{ij1}x_1 + \cdots + t_{ijn}x_n$. If $m \in \mathbb{Z}$ is a common denominator of the t_{ijk}, then the subgroup generated by the elements $y_i = mx_i$ is a subring S of R with free additive group. If R has identity, it can be adjoined to S. □

The Jordan–Zassenhaus Lemma So far we have managed quite well with the proofs, but now we have come to a deep result about the finiteness of certain right modules, without which it seems we cannot prove Lady's theorem. This result is fundamental in representation theory (we state it only for the ring $\mathbb{Z}$ rather than for Dedekind domains).

Theorem 6.7 (Jordan–Zassenhaus Lemma (for $\mathbb{Z}$)). *Let* R *be a finite-dimensional semisimple $\mathbb{Q}$-algebra, and* S *a subring of* R *such that the additive groups of* S *and* R *have the same rank. For any right* R*-module H, there are but finitely many non-isomorphic right* S*-submodules M of H such that MR $= H$.* □

This result fits perfectly to verify the following lemma.

Lemma 6.8. *A torsion-free ring* R *of finite rank has but a finite number of non-isomorphic left ideals that are summands of* R.

Proof. In view of Lemma 6.5, the proof can be reduced to the case when R has 0 radical. In this case $\mathbb{Q}$R is a semi-simple artinian $\mathbb{Q}$-algebra. Let S be a subring of $\mathbb{Q}$R as stated in Theorem 6.7. Evidently, S can be replaced by nS $\leq$ R with 1 adjoined, i.e. S $\leq$ R can be assumed. As a semi-simple artinian ring, $\mathbb{Q}$R has up to isomorphism but a finite number of right ideals. By Theorem 6.7, there exist only a finite number of non-isomorphic S-modules in $\mathbb{Q}$R, say, $M_1, \dots, M_t$.

If the right ideal L is a summand of R, then L $= \epsilon$R for an idempotent ϵ. For some integer $n > 0$, we have $n\epsilon \in$ S, and there is an S-isomorphism $n\epsilon$S $\to M_i$ for some i. This extends to an R-isomorphism $n\epsilon$R $\to M_i$R. As $n\epsilon$R $\cong \epsilon$R as right R-modules, the proof is complete. □

Lady's Theorem The lemma just proved, if combined with Lemma 6.1, leads us at once to the following major result.

Theorem 6.9 (Lady [3]). *A torsion-free group of finite rank can have but a finite number of non-isomorphic direct summands.* □

It seem sensible at this point to illustrate how the situation is drastically simplified if we move to J_p-modules. There exist only two non-isomorphic rank 1 torsion-free J_p-modules, viz. J_p and its field of quotients, $\mathbb{Q}_p^*$. Both are algebraically compact, and as a consequence, every finite rank torsion-free J_p-module is a direct sum of a finite number of copies of these two modules, so completely decomposable. (By the way, this also holds if the rank is countable.) Moreover, such direct decompositions are unique up to isomorphism—this is rather obvious in this situation, and also follows from the fact that these modules have the exchange property. We can claim that *the torsion-free J_p-modules of finite rank have the* **Krull–Schmidt property**.

★ **Notes.** The Lady's theorem is one of the most important results in the theory of torsion-free groups. A group-theoretical proof would be most welcome.

The Jordan–Zassenhaus lemma plays a prominent role in Representation Theory, see. e.g., C.W. Curtis–I. Reiner, *Representation Theory of Finite Groups and Associative Algebras,* Sect. 10. As a matter of fact, this lemma is intrinsically tied to the finiteness of non-isomorphic summands, see, e.g., Fuchs–Vámos [J. Algebra **230**, 730–748 (2000)].

Lots of interesting positive results on direct decompositions of finite rank torsion-free groups may be found in Faticoni [Fa], using categorical approach.

Exercises

(1) Let A be a torsion-free group of infinite rank whose endomorphism ring is of finite rank. Then

   (a) A is a direct sum of a finite number of indecomposable groups;
   (b) A has only a finite number of non-isomorphic direct decompositions.

(2) Let $A = B \oplus C$, where B, C are torsion-free of countable rank. A might have infinitely many non-isomorphic direct decompositions even if both B and C have but finitely many. [Hint: Theorem 1.1 of Chapter 13.]

7 Substitution Properties

The Substitution Property Another positive result for finite rank groups we wish to draw attention to is the substitution property. A group A has the **substitution property** if any pair of decompositions

$$G = A_1 \oplus B = A_2 \oplus C \qquad \text{with } A_1 \cong A \cong A_2$$

implies that there exists an $A' \leq G$ such that

$$G = A' \oplus B = A' \oplus C.$$

In other words, summands isomorphic to A can be replaced by the same subgroup, by the same copy of A, in both decompositions. (Notice the fine distinction between equality and isomorphism.)

We take note that in general, none of the exchange, the substitution, and the cancellation properties implies the other, with one exception, see (c). This is evidenced by the following counterexamples.

Example 7.1. The group $A = \oplus_{n<\omega} \mathbb{Z}(p^\infty)$ is injective, so it has the exchange property, but neither the substitution, nor the cancellation property. In fact, A cannot be cancelled in the direct sums $A \oplus \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty) \cong A \oplus \mathbb{Z}(p^\infty)$.

Example 7.2. The additive group of the ring R of rational numbers with denominators coprime to pq for primes $p \neq q$ does not have the exchange property, since its endomorphism ring R is not local (Theorem 6.2 in Chapter 6). But it does have the substitution property; this will follow from (d) below, since such an R has 1 in the stable range.

Example 7.3. The group $\mathbb{Z}$ has the cancellation, but neither the exchange nor the substitution property. No exchange property, since its endomorphism ring is not local (Theorem 6.2 in Chapter 6). For cancellation, see Sect. 2, Exercise 12 in Chapter 3. Let p be a prime, and $k \in \mathbb{N}$ such that $k \not\equiv -1, 0, 1 \bmod p$. Let $m, n \in \mathbb{N}$ be chosen such that $mk - np = 1$. Consider the free group $F = \langle x \rangle \oplus \langle y \rangle = \langle x' \rangle \oplus \langle y' \rangle$ where $x' = mx + ny, y' = px + ky$. If $z \in F$ satisfies $F = \langle z \rangle \oplus \langle y \rangle$, then it must be of the form $z = \pm x + \ell y$ for some $\ell \in \mathbb{N}$. But such a z can never satisfy $F = \langle z \rangle \oplus \langle y' \rangle$, because y is contained in this direct sum only if $k - \ell p = \pm 1$. Consequently, the substitution property fails for $\mathbb{Z}$.

(a) *If the groups A, B have the substitution property, then so does their direct sum $A \oplus B$.* This is straightforward.
(b) *Summands inherit the substitution property.* Let $A = B \oplus C$ have the substitution property, and let $G = B_1 \oplus H = B_2 \oplus K$ with $B_i \cong B$. Apply the hypothesis to $G \oplus C = (B_1 \oplus C) \oplus H = (B_2 \oplus C) \oplus K$ to find $A' \cong A$ such that $G \oplus C = A' \oplus H = A' \oplus K$. Then $B' = G \cap A'$ can replace both B_i in the direct sums.
(c) *For all groups, substitution implies cancellation.* What we have to show is that if $A' \oplus B = A' \oplus C$, then $B \cong C$—which is pretty evident.

Stable Range Let R be a ring with 1. A subset $\{a_1, \dots, a_n\}$ of R is called **right unimodular** if $\sum_{i=1}^{n} a_i\mathsf{R} = \mathsf{R}$. The integer $n > 0$ is **in the stable range** of R if, for every right unimodular subset $\{a_1, \dots, a_{n+1}\}$ of R, there exist $b_1, \dots, b_n \in \mathsf{R}$ such that the set

$$\{a_1 + a_{n+1}b_1, \dots, a_n + a_{n+1}b_n\}$$

is right unimodular. In particular, R **has 1 in the stable range** if $ax + by = 1$ $(a, b, x, y \in \mathsf{R})$ implies the existence of a $z \in \mathsf{R}$ such that $a + bz$ is a unit in R; equivalently, there is a $z' \in \mathsf{R}$ with $au + bz' = 1$ for a unit $u \in \mathsf{R}$. (It can be shown that this definition is left-right symmetric, but this fact is irrelevant for us.)

(d) It is a well-known theorem by H. Bass that 1 *is in the stable range of* R *whenever* R/J *is artinian*; here J denotes the Jacobson radical of R.

The principal result on the substitution property is as follows.

Theorem 7.4 (Warfield [9]). *A group has the substitution property if and only if its endomorphism ring has* 1 *in the stable range.*

Proof. This is a special case of the more general Theorem 7.8 that will be proved below. □

A noteworthy consequence of the preceding theorem is as follows.

Corollary 7.5. *Groups with semi-local endomorphism rings have the substitution property.*

Proof. Combine (d) with Theorem 7.4 to argue that semi-local rings have 1 in the stable range. □

Example 7.6. (a) A ring R of rational numbers which is p-divisible for almost all primes p is semi-local, so it has 1 in the stable range. A torsion-free group whose endomorphism ring is isomorphic to this R has the substitution property.

(b) 1 is not in the stable range of $\mathbb{Z}$: $2\mathbb{Z} + 5\mathbb{Z} = \mathbb{Z}$, but $2 + 5n = \pm 1$ for no $n \in \mathbb{Z}$.

In the following proof we use the fact that in the endomorphism ring E of a finite rank torsion-free group A, one-sided units are two-sided units: if $\gamma, \delta \in \mathsf{E}$ satisfy $\gamma\delta = \mathbf{1}_A$, then γ is monic (counting the ranks), so it can be cancelled on the left in $\gamma\delta\gamma = \gamma$, to get $\delta\gamma = \mathbf{1}_A$.

Proposition 7.7 (Arnold [A]). *Let A be a torsion-free group of finite rank.*

(i) *If 1 is in the stable range of $\mathsf{E} = \operatorname{End} A$, then for each $n \in \mathbb{N}$, every unit of $\mathsf{E}/n\mathsf{E}$ lifts to a unit of E.*

(ii) *If A is strongly indecomposable (see Sect. 9), and if every unit of $\mathsf{E}/n\mathsf{E}$ lifts to a unit of E for every $n \in \mathbb{N}$, then 1 is in the stable range of E.*

Proof. (i) If for $\alpha \in \mathsf{E}$, $\alpha + n\mathsf{E} \in \mathsf{E}/n\mathsf{E}$ is a unit, then $\alpha\beta + n\eta = 1$ for some $\beta, \eta \in \mathsf{E}$. By hypothesis, $\alpha + n\xi$ is a unit in E for some $\xi \in \mathsf{E}$.

(ii) We refer to Proposition 9.6 to claim that the quasi-endomorphism ring $\mathbb{Q}\mathsf{E}$ of a strongly indecomposable A is a local ring. Suppose $\alpha, \beta, \gamma, \delta \in \mathsf{E}$ are such that $\alpha\beta + \gamma\delta = 1$. If $\gamma\delta \in \mathsf{J}$ (Jacobson radical of $\mathbb{Q}\mathsf{E}$), then it is nilpotent (Lemma 6.4), and $\alpha\beta = 1 - \gamma\delta$ is a unit in E, so $\alpha = \alpha + \gamma \cdot 0$ is a unit in E. If $\gamma\delta \notin \mathsf{J}$, then γ is a unit of $\mathbb{Q}\mathsf{E}$, i.e. $\gamma\gamma' = n \neq 0$ for some $\gamma' \in \mathsf{E}, n \in \mathbb{Z}$. The artinian ring $\mathsf{E}/n\mathsf{E}$ has 1 in the stable range, so (d) implies that the same holds for E. $\square$

***n*-Substitution** A notable generalization of the substitution property is due to Warfield [9] who defined the ***n*-substitution property** ($n \in \mathbb{N}$) for a group A as follows: if

$$G = A_1 \oplus \cdots \oplus A_n \oplus B = A_0 \oplus C \qquad \text{with } A_i \cong A \ (i = 0, \ldots, n),$$

then there exists $A' \leq G$ such that

$$G = A' \oplus L \oplus B = A' \oplus C \qquad \text{with } A' \cong A,\ L \leq A_1 \oplus \cdots \oplus A_n. \tag{12.13}$$

Evidently, 1-substitution is identical with the substitution property discussed above.

In the next proof, the *ad hoc* notation $((a_1, \ldots, a_n))$ will be used for a column vector.

Theorem 7.8 (Warfield [9]). *A group enjoys the n-substitution property ($n \geq 1$) if and only if n is in the stable range of its endomorphism ring.*

Proof. Given A, suppose $\mathsf{E} = \operatorname{End} A$ has n in the stable range, and let $G = A_1 \oplus \cdots \oplus A_n \oplus B = A_0 \oplus C$ with $A_i \cong A$ for all i. We can write the projection $\pi : G \to A_0$ as $\pi = (\pi_1, \ldots, \pi_n, \beta)$ with $\pi_i : A_i \to A_0, \beta : B \to A_0$, and the injection $\rho : A_0 \to G$ as $\rho = ((\rho_1, \ldots, \rho_n, \gamma))$ with $\rho_i : A_0 \to A_i, \gamma : A_0 \to B$ (i.e. $\rho(a) = \rho_1(a) + \cdots + \rho_n(a) + \gamma(a)$ $(a \in A_0)$). Thus $1 = \pi\rho = \pi_1\rho_1 + \cdots + \pi_n\rho_n + \beta\gamma$. Applying the stable range condition, we get

$$(\pi_1 + \beta\gamma\delta_1)\xi_1 + \cdots + (\pi_n + \beta\gamma\delta_n)\xi_n = 1$$

for certain $\delta_i, \xi_i \in \mathsf{E}$. Define the maps

$$\phi = (\pi_1 + \beta\gamma\delta_1, \ldots, \pi_n + \beta\gamma\delta_n, 0)\colon\ G \to \operatorname{Im}\phi = A',$$
$$\psi = ((\xi_1, \ldots, \xi_n, \gamma(\delta_1\xi_1 + \cdots + \delta_n\xi_n)))\colon\ A' \to G,$$

so that $\phi\psi = 1$. Therefore, $G = A' \oplus \operatorname{Ker}\phi$. Since $B \le \operatorname{Ker}\phi$ manifestly, we can write $\operatorname{Ker}\phi = L \oplus B$ where $L = (A_1 \oplus \cdots \oplus A_n) \cap \operatorname{Ker}\phi$. Consequently, $G = A' \oplus L \oplus B$, and also $G = A' \oplus C$, because $\pi\psi = 1$ as well.

Conversely, assume A has the n-substitution property, and $\sum_{i=1}^{n+1} \pi_i\rho_i = 1 \in \mathsf{E}$ where $\pi_i, \rho_i \in \mathsf{E}$. Let $G = A_1 \oplus \cdots \oplus A_{n+1}$ with $A_i \cong A$, and consider the monomorphism $\rho = ((\rho_1, \ldots, \rho_{n+1})) : A \to G$. The map $\pi = (\pi_1, \ldots, \pi_{n+1}) : G \to A$ is a splitting map for ρ, since $\pi\rho = 1$; thus, $G = \rho A \oplus C$ with $C = \operatorname{Ker}\pi$. By hypothesis, there are $A' \cong A$ and $L < A_1 \oplus \cdots \oplus A_n$ such that (12.13) holds with $B = A_{n+1}$. The injection $\xi : A' \to G$ is of the form $\xi = ((\xi_1, \ldots, \xi_{n+1}))$ with $\xi_i \in \mathsf{E}$ satisfying $\sum_{i=1}^{n+1} \pi_i\xi_i = 1 \in \mathsf{E}$. Let $\tau : G \to A'$ with $\operatorname{Ker}\tau = L \oplus A_{n+1}$; then $\tau\xi = 1$, $\tau = (\tau_1, \ldots, \tau_n, 0)$ yields $\tau\xi = \tau_1\xi_1 + \cdots + \tau_n\xi_n + 0 = 1$. Hence

$$(\pi_1 + \pi_{n+1}\xi_{n+1}\tau_1)\xi_1 + \cdots + (\pi_n + \pi_{n+1}\xi_{n+1}\tau_n)\xi_n = 1,$$

establishing the claim. □

2 in Stable Range We end this section with a brief look at a remarkable feature of torsion-free groups of finite rank: a weaker version of the substitution property. The result relies on the following theorem that has its own independent interest.

Theorem 7.9 (Warfield [9]). *The endomorphism ring of any finite rank torsion-free group has* 2 *in the stable range.*

Proof. Let A be a torsion-free group of finite rank, and let $\pi_1\rho_1 + \pi_2\rho_2 + \pi_3\rho_3 = 1$ where $\pi_i, \rho_i \in \mathsf{E} = \operatorname{End} A$. First, assume that π_1 is a monic map, thus $\pi_1\mathsf{E}$ is of finite index in E, and so $\bar{\mathsf{E}} = \mathsf{E}/\pi_1\mathsf{E}$ is a finite ring (see Proposition 1.10). As such, it is artinian, so 1 is in the stable range, whence $\pi_2\rho_2 + (\pi_1\rho_1 + \pi_3\rho_3) \equiv 1 \mod \pi_1\mathsf{E}$ implies $\pi_2 + (\pi_1\rho_1 + \pi_3\rho_3)\gamma \equiv \epsilon \mod \pi_1\mathsf{E}$ for some $\gamma, \epsilon \in \mathsf{E}$ where the coset of ϵ is a unit in $\bar{\mathsf{E}}$. Hence

$$(\pi_2 + \pi_1\mathsf{E}) + (\pi_3\rho_3\gamma + \pi_1\mathsf{E}) = \epsilon + \pi_1\mathsf{E}.$$

If $\xi_2 \in \mathsf{E}$ satisfies $\epsilon\xi_2 - 1 \in \pi_1\mathsf{E}$, then $(\pi_2 + \pi_3\rho_3\gamma)\xi_2 + \pi_1\mathsf{E} = 1 + \pi_1\mathsf{E}$, which means that $1 = \pi_1\xi_1 + (\pi_2 + \pi_3\rho_3\gamma)\xi_2 = (\pi_1 + \pi_3 \cdot 0)\xi_1 + (\pi_2 + \pi_3\rho_3\gamma)\xi_2$ for some $\xi_1 \in \mathsf{E}$.

If π_1 is not monic, then we use the fact (that we will learn in Lemma 9.5) that $\mathbb{Q}A$ is an artinian $\mathbb{Q}$-algebra, and therefore 1 is in the stable range of $\mathbb{Q}\mathsf{E}$. Consequently,

$\pi_1' = \pi_1 + (\pi_2\rho_2 + \pi_3\rho_3)\beta$ is a unit in $\mathbb{Q}\mathsf{E}$ for some $\beta \in \mathsf{E}$, or, equivalently, π_1' is monic in E. Therefore, we can argue with π_1' in the same way as we did with π_1 before, and the rewritten equation $\pi_1'\rho_1 + \pi_2(\rho_2 - \rho_2\beta\rho_1) + \pi_3\rho_3(1-\beta\rho_1) = 1$ will lead us to

$$[\pi_1 + (\pi_2\rho_2 + \pi_3\rho_3)\beta]\,\xi_1 + [\pi_2 + \pi_3\rho_3(1-\beta\rho_1)\gamma]\,\xi_2 = 1$$

for some $\xi_1, \xi_2 \in \mathsf{E}$. This is the same as

$$[\pi_1 + \pi_3\rho_3(\beta - (1-\beta\rho_1)\gamma\rho_2)]\,\xi_1 + [\pi_2 + \pi_3\rho_3(1-\beta\rho_1)\gamma](\xi_2 + \rho_2\xi_1) = 1,$$

completing the proof. □

From Theorems 7.8 and 7.9 we can now conclude:

Corollary 7.10 (Warfield [9]). *Torsion-free groups of finite rank enjoy the 2-substitution property.* □

We will need the following simple observation.

Lemma 7.11 (Warfield [9]). *Suppose A is a torsion-free group of finite rank. If $A \oplus A \oplus X \cong A \oplus Y$ for certain groups X, Y, then there exists a subgroup L of $A \oplus A$ such that $A \oplus L \cong A \oplus A$ and $Y \cong L \oplus X$.*

Proof. Set $G = A \oplus A \oplus X = A' \oplus Y'$ with $A' \cong A, Y' \cong Y$. Applying the 2-substitution, we have A_0, L such that $G = A_0 \oplus L \oplus X = A_0 \oplus Y'$ with $L \leq A \oplus A$. Hence $Y \cong Y' \cong L \oplus X$ is clear. Also, $A \oplus A \cong A_0 \oplus L \cong A \oplus L$ follows as $A_0 \cong A' \cong A$. □

★ **Notes.** Prompted by a proof of Crawley [2], the substitution property was introduced in the author's paper [Monatshefte Math. **75**, 198–204 (1971)], where it was shown that a ring enjoys this property if and only if 1 is in the stable range (without using this terminology). Substitution was developed by Warfield [9], who defined also the n-substitution property, and showed that it is related to the stable range of the endomorphism ring.

Another noteworthy development is due to K.R. Goodearl [Pac. J. Math. **64**, 387–411 (1976)]. He says that A has *power-substitution* if $G = A_1 \oplus B = A_2 \oplus C$ with $A_i \cong A$ implies that $G^n = H \oplus B^n = H \oplus C^n$ for some $n \geq 1$, and some group H. He goes on to prove that if n is in the stable range of $\operatorname{End} A$, then A has power-substitution. (The converse fails as is shown by a counterexample $C(X)$ of continuous functions for a suitable compact Hausdorff space X.) Torsion-free groups of finite rank have the power-substitution property.

Exercises

(1) Give a detailed proof for the claim that $A = B \oplus C$ has the n-substitution property if and only if both B and C have it.

(2) (Warfield) Suppose A has the substitution property, and $G = A_1 \oplus B = A_2 \oplus C$ where $A_i \cong A$. Then there is a group $D \leq G$ such that $G = A_1 \oplus D = A_2 \oplus D$.

(3) (Warfield) Suppose A has the 2-substitution property. If $A \oplus B \cong A \oplus C$, and if B has a summand $\cong A$, then $B \cong C$.
(4) (Warfield) If A has the n-substitution property, then so does its endomorphism ring E as a left E-module.

8 Finite Rank *p*-Local Groups

By a ***p*-local** group is meant a group that is also a $\mathbb{Z}_{(p)}$-module; equivalently, a group that is uniquely q-divisible for every prime $q \neq p$. Thus there is only one prime to deal with in p-local groups. The restriction to p-local groups has quite a simplifying effect on the group structure; for instance, there are only two isomorphy classes of rank 1 torsion-free p-local groups: $\mathbb{Z}_{(p)}$ and $\mathbb{Q}$. Furthermore, like p-groups, p-local groups may contain non-zero q-basic subgroups only for $q = p$, so we may call it simply a basic subgroup.

Preliminaries on Local Groups We begin with important informations on p-local groups.

Lemma 8.1. *Let A be a p-local torsion-free group of finite rank. Then $p\,\mathrm{End}\,A$ is contained in the Jacobson radical J of the endomorphism ring* $\mathrm{End}\,A$*. Thus* $(\mathrm{End}\,A)/\mathsf{J}$ *is a finite semi-simple ring of characteristic p.*

Proof. We show that for every $\eta \in \mathrm{End}\,A$, $1 - p\eta$ is an automorphism of A. It is clear that if p does not divide $a \in A$, then it does not divide $(1 - p\eta)a$ either. Hence multiplication by $1 - p\eta$ preserves p-heights. Therefore, the pure subgroup $\mathrm{Ker}(1-p\eta)$ must be 0, thus $1-p\eta$ is monic, and $\mathrm{Im}(1-p\eta)$ is an essential subgroup of A. From the preservation of heights it follows that $\mathrm{Im}(1-p\eta)$ is pure in A, whence $\mathrm{Im}(1 - p\eta) = A$. Consequently, $1 - p\eta$ is an automorphism, indeed. □

Theorem 8.2. *A reduced finite rank torsion-free p-local group has semi-local endomorphism ring. Therefore, it enjoys both the substitution and the cancellation properties.*

Proof. By Lemma 8.1, E/J is finite, so E is semi-local (cf. also Theorem 5.11 in Chapter 16). Therefore, from Corollary 7.5 we deduce that A has the substitution, and hence also the cancellation property. □

Power Cancellation A group A is said to have **power cancellation** if an isomorphism $A^n \cong C^n$ for some $n \geq 2$ implies $A \cong C$.

In the next proof, we shall use notation and results from Sect. 12.

Proposition 8.3. *Torsion-free p-local groups of finite rank have the power cancellation property.*

Proof. Suppose that the finite rank torsion-free groups A, C are p-local, and satisfy $A^n = C^n$ for some $n \geq 2$. Evidently, $A, C \in \mathrm{add}(G)$. Denote by E the endomorphism ring of $G = A \oplus C$. In the Arnold–Lady category equivalence (Theorem 12.2),

$A \mapsto \mathrm{Hom}(G, A)$, where Hom is a projective right E-module. Thus $\mathrm{Hom}(G, A)^n = \mathrm{Hom}(G, C)^n$. Passing modulo the Jacobson radical,

$$[\mathrm{Hom}(G, A)/\mathsf{J}\,\mathrm{Hom}(G, A)]^n = [\mathrm{Hom}(G, C)/\mathsf{J}\,\mathrm{Hom}(G, C)]^n.$$

Since E/J is finite, so artinian, the category of E/J-modules enjoys the Krull-Schmidt property. Hence we conclude

$$\mathrm{Hom}(G, A)/\mathsf{J}\,\mathrm{Hom}(G, A) \cong \mathrm{Hom}(G, C)/\mathsf{J}\,\mathrm{Hom}(G, C).$$

We can argue as in Lemma 6.5 to prove the isomorphy of the projective right E-modules $\mathrm{Hom}(G, A)$ and $\mathrm{Hom}(G, C)$. Therefore, $\mathrm{Hom}(G, A) \otimes_{\mathsf{E}} G \cong A$ is isomorphic to $\mathrm{Hom}(G, C) \otimes_{\mathsf{E}} G \cong C$. □

Modules Over the *p*-Adic Integers The situation becomes extremely simple, even in the countable rank case, if we move to modules over the ring J_p of p-adic integers.

Theorem 8.4 (Derry [1], Kaplansky [2]). *Let M be a torsion-free J_p-module of countable rank. Then M is a direct sum of rank* 1 *modules, each isomorphic to J_p or to its quotient field $\mathbb{Q}_p^*$.*

Proof. Every rank 1 subgroup of M is contained in a pure rank 1 subgroup isomorphic to J_p or $\mathbb{Q}_p^*$. These are algebraically compact, so summands. The finite rank case is now obvious, while for countable rank the argument used in Pontryagin's theorem 7.1 in Chapter 3 establishes the claim. □

Let G be an arbitrary torsion-free group, and $G_{(p)} = \mathbb{Z}_{(p)} \otimes G$ its localization at the prime p. It is harmless to identify $G_{(p)}$ with the subgroup of the divisible hull D of G whose elements x satisfy $nx \in G$ for some $n \in \mathbb{N}$ coprime to p. We then have $G = \cap_p G_{(p)}$ where the intersection is taken in the injective hull of G (see Lemma 5.1 of Chapter 8).

If we embed the divisible hull D of G (which is a $\mathbb{Q}$-vector space) in the $\mathbb{Q}_p^*$-vector space $\mathbb{Q}_p^* \otimes D = \mathbb{Q}_p^* \otimes G$ *via* $d \mapsto 1 \otimes d$, then we get an embedding of G in $J_p \otimes G$ which we will denote by $G_{(p)}^*$.

Lemma 8.5. *For a torsion-free group G and a prime p,*

$$G_{(p)} = D \cap G_{(p)}^*,$$

where the intersection is computed in $\mathbb{Q}_p^* \otimes G$.

Proof. To prove the non-trivial part of the claim that $D \cap G_{(p)}^* \leq G_{(p)}$, let x belong to the intersection. If $S = \{g_i\}_{i \in I}$ is a maximal independent set in $G_{(p)}$, then we can write $x = \rho_1 g_1 + \cdots + \rho_m g_m$ ($\rho_i \in \mathbb{Q}_p^*, g_i \in S$). Since $x \in D$ and the coefficients of the g_i are uniqueley determined, we must have $\rho_i \in \mathbb{Q}$, whence $x \in G_{(p)}$ is immediate. □

Splitting Fields We are now restricting our considerations to p-local torsion-free groups A of *finite rank*. Besides $\operatorname{rk} A$, there are two significant numerical invariants for such an A: the p-corank $n = \operatorname{rk}^p A = \operatorname{rk}(A/pA) = \operatorname{rk} B$ and $m = \operatorname{rk} A/B$, where B is a basic subgroup of A. (In this section, the letters m, n will keep this meaning.) Tensoring the exact sequence $0 \to B \to A \to A/B \cong \mathbb{Q}^m \to 0$ with J_p, we obtain splitting:

$$A_{(p)}^* \cong (J_p \otimes B) \ \oplus \ (\mathbb{Q}_p^*)^m,$$

thus $A_{(p)}^*$ becomes completely decomposable over J_p. The Kurosh theory shows that there is no need to go as far as J_p (which has the transcendence degree of the continuum over $\mathbb{Z}_{(p)}$) in order to obtain such an attractive decomposition; indeed, we will see in a moment that a suitable finite extension will suffice.

We start with a particular maximal independent system in our group A. Extend a basis $\{b_1, \dots, b_n\}$ of B by elements $c_1, \dots, c_m \in A$ to have a maximal independent set in A (thus the cosets of $c_1, \dots, c_m$ form a basis of the $\mathbb{Q}$-vector space A/B). We choose representatives $x_{ij} \in A$ ($i = 1, \dots, m;\ j < \omega$) of the generators of A/B such that

$$x_{i0} = c_i,\ px_{i,j+1} \equiv x_{ij} \bmod B \qquad (i = 1, \dots, m;\ j < \omega).$$

Thus $px_{i,j+1} = x_{ij} + r_{i1j}b_1 + \cdots + r_{inj}b_n$ with uniquely determined integer coefficients r_{ikj}. If we express $p^{j+1}x_{i,j+1}$ in terms of x_{i0}, we then get

$$p^{j+1}x_{i,j+1} = \pi_{i1j}b_1 + \cdots + \pi_{inj}b_n + c_i \quad (j < \omega) \tag{12.14}$$

where π_{ikj} denotes the jth partial sum of a p-adic integer π_{ik} ($i = 1, \dots, m;\ k = 1, \dots, n$). Let $\mathbb{P} = \|\pi_{ik}\|$ be the $m \times n$ matrix with entries defined by Eq. (12.14), and $\mathsf{K}^\natural$ the subfield of $\mathbb{Q}_p^*$ obtained by adjoining all the π_{ik} to $\mathbb{Q}$ (*a priori* $\mathsf{K}^\natural$ also depends on the basis, but—as it will be clear from Theorem 8.6—it does not). Set $\mathsf{R}^\natural = J_p \cap \mathsf{K}^\natural$; this is a valuation domain. We now have

$$\mathsf{R}^\natural \otimes A = \mathsf{R}^\natural b_1 \oplus \cdots \oplus \mathsf{R}^\natural b_n \ \oplus \ \mathsf{K}^\natural d_1 \oplus \cdots \oplus \mathsf{K}^\natural d_m, \tag{12.15}$$

where $d_i = \pi_{i1}b_1 + \cdots + \pi_{in}b_n + c_i$ ($i = 1, \dots, m$).

We will say that a subfield K of $\mathbb{Q}_p^*$ is **a splitting field** for A if $\mathsf{R} \otimes A$ decomposes like (12.15), i.e. $\mathsf{R} \otimes A$ is a direct sum of cyclic and rank 1 divisible R-modules, where $\mathsf{R} = \mathsf{K} \cap J_p$.

We are looking for a minimal splitting field. We claim:

Theorem 8.6 (Szekeres [1]). *Given a p-local torsion-free group A of finite rank, there exists a uniquely determined subfield of the field of the p-adic numbers, which is the minimal splitting field for A, contained in any other splitting field for A.*

Proof. The argument above establishes the existence of splitting fields for A (which was not doubtful anyway). We now prove that the field $\mathsf{K}^\natural$ defined above is independent of the choice of the basis, and is in fact the minimal splitting field for A.

Let us choose another basic subgroup with a basis $b'_1, \dots, b'_n$, and extend it to a maximal independent set by adjoining elements $c'_1, \dots, c'_m$. The new set depends linearly over $\mathbb{Q}$ on the b_k, c_i, and therefore the coefficients of the b'_k in the expressions (12.13) for the c'_i, being uniquely determined, must belong to $\mathsf{K}^\natural$. This means that the splitting field K determined by the new choice is contained in $\mathsf{K}^\natural$. For reasons of symmetry, $\mathsf{K} = \mathsf{K}^\natural$ follows. Consequently, $\mathsf{K}^\natural$ is well defined by A, it must be its unique minimal splitting field contained in any splitting field of A. $\square$

It is now clear that *the* splitting field $\mathsf{K}^\natural$ is determined by the $m \times n$ matrix $\mathbb{P} = \|\pi_{ik}\|$, where the entries are from J_p. Actually, any choice of the matrix $\mathbb{P}$ defines a p-local torsion-free group of rank $n + m$ and p-corank n. To prove this, and also to show how to recapture A from the data provided by the π_{ik}, consider the following pull-back diagram

$$\begin{tikzcd}
 & & & \oplus_{k=1}^n \mathbb{Z}_{(p)} b_k \arrow[d] & \\
0 \arrow[r] & \oplus_{i=1}^m \mathbb{Z}_{(p)} c_i \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & \oplus_{k=1}^n \mathbb{Q} b_k \arrow[r] \arrow[d, "\mathsf{P}'\cdot\mathrm{can}"] & 0 \\
0 \arrow[r] & \oplus_{i=1}^m \mathbb{Z}_{(p)} c_i \arrow[r] & \oplus_{i=1}^m \mathbb{Q} c_i \arrow[r, "\mathrm{can}"] & \oplus_{i=1}^m \mathbb{Z}(p^\infty) \arrow[r] & 0
\end{tikzcd}$$

where we used the canonical map onto the factor group $\oplus_{k=1}^n \mathbb{Z}(p^\infty)$, followed by the map between the two divisible factor groups given by the transpose $\mathbb{P}'$ of $\mathbb{P}$, to define the vertical map on the right. Indeed, it is easily checked that the pull-back construction yields A as defined by the equations in (12.14).

Example 8.7. Consider the indecomposable group of rank r in Lemma 4.6. Its minimal splitting field is $\mathbb{Q}(\pi_2, \dots, \pi_r)$.

★ **Notes.** The category of p-local torsion-free groups of finite rank is not a Krull–Schmidt category. Arnold [6] points out that in order to have an example with two non-isomorphic direct decompositions, groups of ranks at least 10 are required.

Most of the theorems on p-local torsion-free groups carry over to torsion-free modules over discrete valuation domains, where the role of J_p is taken over by the completion of the domain, see Lady [6]. In several papers, Lady has developed a theory of splitting fields over Dedekind domains.

For those who want more results and more details, we refer to his papers [5]. Glaz–Vinsonhaler–Wickless [1] introduce splitting rings for p-local torsion-free groups, depending on the choice of a basic subgroup. For more on the local case, see Hill–Lane–Megibben [1].

It is of course a major problem to move from the local to the global case. The Derry–Kurosh–Malcev theory provides a reasonable method. For a discussion of the problem from the sheaf-theoretic point of view, see Rotman [4] and Turgi [1].

Exercises

(1) Show that quasi-isomorphic torsion-free groups of finite rank (next section) share splitting fields.
(2) Relate the splitting field of a direct sum to those of the summands.
(3) (Lady) If a field contains the splitting fields of A and C, then it also contains the splitting fields of both $A \otimes C$ and $\mathrm{Hom}(A, C)$.
(4) How is the p-adic completion $\tilde{A}$ related to $\mathsf{R}^\natural \otimes \mathsf{A}$?
(5) (Szekeres) Show that Theorem 8.6 also holds for groups of countable rank.

9 Quasi-Isomorphism

In view of the pathological examples of Sect. 5, one might abandon in despair the hope of finding any kind of uniqueness in the direct decompositions of finite rank torsion-free groups. Fortunately, there is good news. In order to remedy the situation, B. Jónsson came up with an ingenious idea: a weaker version of isomorphism that led to a kind of uniqueness of the direct decompositions. This is a drastic departure from the traditional approach, but the consequences are significant and far reaching, even though they follow rather quickly.

Quasi-Morphisms By a **quasi-homomorphism** $\phi : A \stackrel{\cdot}{\to} C$ between two finite rank torsion-free groups we mean a homomorphism of A into the divisible hull of C such that $n \operatorname{Im} \phi \leq C$ for some $n \in \mathbb{N}$. Thus the group of quasi-homomorphisms is

$$\mathbb{Q}\,\mathrm{Hom}(A, C) \cong \mathbb{Q} \otimes \mathrm{Hom}(A, C).$$

The groups A, C are **quasi-isomorphic**, written

$$A \sim C,$$

if there are quasi-homomorphisms $\phi : A \stackrel{\cdot}{\to} C$ and $\psi : C \stackrel{\cdot}{\to} A$ such that $\psi\phi = q\mathbf{1}_A$ and $\phi\psi = q'\mathbf{1}_C$ for some non-zero $q, q' \in \mathbb{Q}$. Then an integral multiple of ϕ is an isomorphism of A with a subgroup of C, and an integral multiple of ψ is an isomorphism of C with a subgroup of A. It is immediate that quasi-isomorphism is an equivalence relation in the class of finite rank torsion-free groups, weaker than isomorphism.

Actually, $\phi\psi = q\mathbf{1}_A$ implies $(\phi\psi)\phi = q\phi$ whence cancellation by the monic ϕ yields $\psi\phi = q\mathbf{1}_C$. Therefore, it suffices to assume only one half of (b) and (c) in

Lemma 9.1. *For the torsion-free groups A, C of finite rank the following are equivalent:*

(a) *A and C are quasi-isomorphic;*
(b) *there are integers $m, n > 0$ such that*

$$mA \leq C' \leq A \quad \textit{and} \quad nC \leq A' \leq C, \tag{12.16}$$

where $C' \cong C$ and $A' \cong A$.
(c) *A is isomorphic to a finite index subgroup of C, and C is isomorphic to a finite index subgroup of A.*

Proof. (a) $\Rightarrow$ (b) Assume $A \sim C$, i.e. there are maps ϕ, ψ as stated in the definition. Then there are $r, s \in \mathbb{N}$ such that $r\phi(A) \leq C$ and $s\psi(C) \leq A$. Thus $rsqA = rs\psi\phi(A) \leq s\psi(C) \leq A$ proves the first relation in (12.16). The second follows similarly.

(b) $\Rightarrow$ (c) This implication is trivial, because if mA is of finite index in A, and nC in C.

(c) $\Rightarrow$ (a) If (c) holds, then an isomorphism $\phi : A \to A' \leq C$ may be regarded as a quasi-homomorphism $A \dot{\to} C$, and an isomorphism $\psi : C \to C' \leq A$ as $C \dot{\to} A$. The composite map $\psi\phi : A \to A$ carries A isomorphically into itself, so by Proposition 1.10 it is multiplication by some $q \in \mathbb{N}$ times an automorphism α of A. Changing ϕ to $\alpha^{-1}\phi$, we get $\psi(\alpha^{-1}\phi) = q\mathbf{1}_A$, and hence also $(\alpha^{-1}\phi)\psi = q\mathbf{1}_C$. $\square$

Every subgroup of finite index in a rank 1 group is isomorphic to the group itself, therefore, *for torsion-free groups of rank* 1*, quasi-isomorphism implies isomorphism*. More generally, we have

Lemma 9.2 (Procházka [2]). *Assume A is a finite rank torsion-free group such that $|A/pA| \leq p$ for every prime p. If a torsion-free group C is quasi-isomorphic to A, then $A \cong C$.*

Proof. For the proof, there is no loss of generality in assuming that C is a subgroup of A of prime index p. Then $pA \leq C < A$ whence $|A/pA| \leq p$ implies that $C = pA \cong A$. $\square$

It is routine to check for all finite rank torsion-free groups A, B, C:

(A) If $A \sim B$, then Type(A)=Type(B).
(B) If D, D' are divisible groups, then $D \sim D'$ implies $D \cong D'$.
(C) If $B \sim C$, then $A \oplus B \sim A \oplus C$ for all A.
(D) If $A \sim B$, then for all C, we have both $\mathrm{Hom}(A, C) \sim \mathrm{Hom}(B, C)$ and $\mathrm{Hom}(C, A) \sim \mathrm{Hom}(C, B)$. Similarly for Ext and the tensor product.

Quasi-Direct Decompositions If A is quasi-isomorphic to the direct sum $C_1 \oplus \cdots \oplus C_n$, then we say that

$$A \sim C_1 \oplus \cdots \oplus C_n \tag{12.17}$$

is a **quasi-direct decomposition**, and the C_i are **quasi-summands** of A. A group that has only the trivial quasi-direct decomposition is called **strongly indecomposable**. Equation (12.17) and $A \sim C'_1 \oplus \cdots \oplus C'_m$ are viewed as quasi-isomorphic direct decompositions if $n = m$, and there is a bijection between the sets of their components that preserves quasi-isomorphism.

(E) Being a quasi-summand is a transitive property.
(F) If B is a quasi-summand of A, and X is quasi-isomorphic to a subgroup of A containing B, then X has a quasi-summand $\cong B$.

Example 9.3. The group A in Example 4.3 is quasi-equal to the direct sum $\oplus_n E_n$. Thus $A \sim \oplus_n E_n$ is a quasi-direct decomposition of A into strongly indecomposable summands, as stated in Theorem 9.9 below. On the other hand, the group in Example 4.4 is strongly indecomposable.

Example 9.4. Rigid groups are strongly indecomposable. For example, the groups in Lemma 4.6 are strongly indecomposable.

Let A be torsion-free group of finite rank, and D its divisible hull. A **quasi-endomorphism** of A is a homomorphism $\phi : A \to D$ such that $n\phi(A) \leq A$ for some integer $n > 0$. The quasi-endomorphisms of A form a subring $\mathbb{Q}\,\mathrm{End}\,A$ in the ring of all endomorphisms of D.

This definition immediately yields a number of useful results.

Lemma 9.5. *For a torsion-free group of finite rank A, the following hold:*

(i) $\mathbb{Q}\,\mathrm{End}\,A$ *is a left artinian* $\mathbb{Q}$*-algebra;*
(ii) *every idempotent* $\epsilon \in \mathbb{Q}\,\mathrm{End}\,A$ *yields a quasi-direct decomposition*

$$A \sim \epsilon A \oplus (1-\epsilon)A.$$

If $A \sim B \oplus C$*, then* $B \sim \epsilon A, C \sim (1-\epsilon)A$ *for an idempotent* $\epsilon \in \mathbb{Q}\,\mathrm{End}A$.

Proof. (i) $\mathbb{Q}\,\mathrm{End}\,A$ is clearly a $\mathbb{Q}$-vector space, since ϕ is a quasi-endomorphism exactly if $n\phi$ is an endomorphism for some $n > 0$. The rest follows from the finite dimensionality of $\mathbb{Q}\,\mathrm{End}\,A$.
(ii) The claim is an immediate consequence of the definitions. □

It is a good idea to keep in mind that A is strongly indecomposable if $nA \leq B \oplus C \leq A$ (for some integer $n > 0$, and subgroups B, C) implies that either $B = 0$ or else $C = 0$.

The finite rank torsion-free groups A share the property that their quasi-direct decompositions are in a bijective correspondence with the direct decompositions of their quasi-endomorphism rings $\mathbb{Q}\,\mathrm{End}\,A$ into left ideals (see Lemma 1.5). In our case, this correspondence is given by

$$A \sim \epsilon A \oplus (1-\epsilon)A \quad \leftrightarrow \quad \mathbb{Q}\,\mathrm{End}\,A = (\mathbb{Q}\,\mathrm{End}\,A)\epsilon \oplus (\mathbb{Q}\,\mathrm{End}\,A)(1-\epsilon),$$

for an idempotent $\epsilon \in \mathbb{Q}\,\mathrm{End}\,A$.

It is well known that an artinian ring with identity decomposes into a direct sum of a finite number of ideals $\neq 0$ which are local rings. Hence we obtain the following important information:

Proposition 9.6 (Reid [2]). *The quasi-endomorphism ring of a finite rank torsion-free group is local if and only if the group is strongly indecomposable.* $\square$

An analogue of the well-known Fitting's Lemma can be established in the finite rank situation.

Lemma 9.7. *Let η be an endomorphism of the finite rank torsion-free group A. Then there is an integer $n \geq 1$ and a quasi-direct decomposition*

$$A \sim \operatorname{Im} \eta^n \oplus \operatorname{Ker} \eta^n. \tag{12.18}$$

Proof. Choose $n \in \mathbb{N}$ such that $\operatorname{Im} \eta^n$ has the smallest rank. Then η^n is monic on $\operatorname{Im} \eta^n$, whence Proposition 1.10 implies that there exists $m \in \mathbb{N}$ such that $m \operatorname{Im} \eta^n \leq \operatorname{Im} \eta^{2n}$. Therefore, for any given $0 \neq a \in A$, there exists $b \in A$ such that $m\eta^n(a) = \eta^{2n}(b)$. Hence $ma = \eta^n(b) + (ma - \eta^n(b)) \in \operatorname{Im} \eta^n + \operatorname{Ker} \eta^n$. As η^n is monic on $\operatorname{Im} \eta^n$, we also have $\operatorname{Im} \eta^n \cap \operatorname{Ker} \eta^n = 0$, and the claim follows. $\square$

When Quasi-Isomorphism Implies Isomorphic Endomorphism Rings In general, quasi-isomorphic groups need not have isomorphic endomorphism rings, though they share simultaneously some relevant properties, like commutativity. An interesting special case when their endomorphism rings have to be isomorphic is recorded in the following result.

Lemma 9.8 (Arnold–Hunter–Richman [1]). *Assume that A, B are quasi-isomorphic finite rank torsion-free groups. If* $\operatorname{End} B$ *is a principal ideal domain, then the endomorphism rings* $\operatorname{End} A$ *and* $\operatorname{End} B$ *are isomorphic.*

Proof. It is clear that $\operatorname{End} A$ is an integral domain. Let $\alpha : A \dot{\to} B$ denote a quasi-isomorphism. The set $\mathsf{H}_\alpha = \alpha \operatorname{Hom}(B, A)$ is evidently a non-zero ideal in $\operatorname{End} B$. A non-zero $\eta \in \operatorname{End} A$ induces an $\operatorname{End} B$-endomorphism $\theta_\eta : \mathsf{H}_\alpha \to \mathsf{H}_\alpha$ such that $\alpha\beta \mapsto \alpha\eta\beta$ ($\beta \in \operatorname{Hom}(B, A)$). Indeed, if $\alpha\beta = 0$, then $(\beta\alpha)^2 = 0$, so also $\beta\alpha = 0$, thus $(\alpha\eta\beta)^2 = 0$ and $\alpha\eta\beta = 0$ as $\operatorname{End} B$ is a domain. In addition, there exists a u_η^α in the quotient field of $\operatorname{End} B$ such that $\theta_\eta(x) = u_\eta^\alpha x$ for all $x \in \mathsf{H}_\alpha$. Repeated application of θ_η yields $(u_\eta^\alpha)^k x \in \mathsf{H}_\alpha$ for all $k \in \mathbb{N}$, which can happen only if $u_\eta^\alpha \in \operatorname{End} B$ (a PID).

We claim that the correspondence $\eta \mapsto u_\eta^\alpha$ is a ring isomorphism $\phi_\alpha : \operatorname{End} A \to \operatorname{End} B$. It is evidently additive, and maps $\mathbf{1}_A$ upon $\mathbf{1}_B$. From

$$\phi_\alpha(\eta)\phi_\alpha(\xi)(\alpha\beta) = \phi_\alpha(\eta)(\alpha\xi\beta) = \alpha\eta\xi\beta = \phi_\alpha(\eta\xi)(\alpha\beta) \quad (\eta, \xi \in \operatorname{End} A)$$

we conclude that it preserves multiplication. Finally, if $\gamma : B \dot{\to} C$ is a quasi-isomorphism, then

$$\phi_{\gamma\alpha}(\eta)(\gamma\alpha\delta) = (\gamma\alpha)\eta\delta = \gamma\phi_\alpha(\eta)(\alpha\delta) = \phi_\gamma(\phi_\alpha(\eta))(\gamma\alpha\delta) \quad (\delta \in \mathrm{Hom}(C,A)),$$

thus $\phi_{\gamma\alpha} = \phi_\gamma\phi_\alpha$. In the special case $A = C$, θ_η is multiplication by η, so $u_\eta^{\gamma\alpha} = \eta$ and $\phi_{\gamma\alpha}$ is the identity. Consequently, $\phi_{\gamma\alpha}$ is the identity on $\mathrm{End}\, A$, and so ϕ_α has to be an isomorphism (interestingly, it is independent of the choice of α). □

Krull–Schmidt for Quasi-Direct Decompositions The main result on quasi-direct decompositions is our next theorem.

Theorem 9.9 (Jónsson [2]). *Let A be a torsion-free group of finite rank, and*

$$A \sim A_1 \oplus \cdots \oplus A_m, \qquad A \sim C_1 \oplus \cdots \oplus C_n$$

quasi-direct decompositions of A, where each of A_i and C_j is strongly indecomposable $\neq 0$. Then $m = n$, and after suitably rearranging the components, $A_i \sim C_i$ for $i = 1, \ldots, m$.

Proof. Groups with local endomorphism rings share the exchange property. It is not difficult to check that this also holds for local quasi-endomorphism rings in quasi-direct decompositions. Hence it is immediate that A_1 can replace one of the C_i, say, C_1, i.e. we have $A \sim A_1 \oplus C_2 \oplus \cdots \oplus C_n$. Passing mod A_1, we get $A_2 \oplus \cdots \oplus A_m \sim C_2 \oplus \cdots \oplus C_n$. Induction completes the proof. □

Once quasi-isomorphism proved to be a useful concept, the category of finite rank torsion-free groups under quasi-homomorphisms becomes an object of interest. The motivation, even the need for studying this category, was pointed out by E. Walker who interpreted the preceding theorem by considering it in this category. Accordingly, we now define the category $\mathcal{A}_0$ where the objects are the torsion-free groups of finite rank, and the morphisms are the quasi-homomorphisms, i.e.

$$\mathrm{Hom}_{\mathcal{A}_0}(A,B) = \mathbb{Q}\,\mathrm{Hom}(A,B)$$

for torsion-free groups A, B of finite rank. This category can also be viewed as the quotient category of the category of torsion-free groups of finite rank modulo the category of bounded groups. The theorems above tell us that $\mathcal{A}_0$ *is a Krull–Schmidt category where the indecomposable objects have local endomorphism rings.*

Procházka–Murley Groups We consider briefly a class of groups where quasi-isomorphism equals isomorphism. A torsion-free group M of finite rank is called a **Procházka–Murley group** if its p-corank is ≤ 1 for each prime p, i.e. $\mathrm{rk}(M/pM) \leq 1$ for every p.

Example 9.10. (a) Rank 1 torsion-free groups are Procházka–Murley groups.
(b) The same holds for finite rank pure subgroups of $\tilde{\mathbb{Z}} = \prod_p J_p$.

We can state immediately:

(α) *The class of Procházka–Murley groups is closed under taking pure subgroups, torsion-free homomorphic images, and tensor products.*
(β) *A torsion-free group of finite rank is a Procházka–Murley group exactly if all of its finite homomorphic images are cyclic.*

(γ) *If N is a pure subgroup of a Procházka–Murley group M, then for any prime p, either N or M/N is p-divisible.* Indeed, this follows from the exact sequence $0 \to N/pN \to M/pM \to (M/N)/p(M/N) \to 0$, where only the first two or the last two factor groups can be non-zero.

(δ) *The $\mathbb{Z}$-adic completion $\tilde{M}$ of a Procházka–Murley group M is isomorphic to a summand of $\tilde{\mathbb{Z}}$.* More precisely, isomorphic to the direct product of a copy of J_p for each prime p satisfying $pM < M$.

Lemma 9.11 (Murley [1]). *A torsion-free group M of finite rank is a Procházka–Murley group if and only if it shares the following properties:*

(i) *if N is quasi-isomorphic to M, then $N \cong M$; and*
(ii) *if N is a subgroup of M isomorphic to M, then $N = nM$ for some $n \in \mathbb{N}$.*

Proof. Suppose M is a Procházka–Murley group. If $N \sim M$, then there are an integer $k > 0$ and a subgroup $N' \cong N$ satisfying $kM \leq N' \leq M$. N'/kM is a subgroup of the cyclic group M/kM, so cyclic: $N'/kM = nM/kM$ for a divisor n of k. Then $N' = nM \cong M$, proving both (i) and (ii). Conversely, if M has properties (i) and (ii), then let $pM \leq N \leq M$ for a prime p. By (i), $N \cong M$, while (ii) implies that $N = nM \cong M$ for an $n \in \mathbb{N}$. Clearly, only $n = p$ is a possibility whenever $pM \neq M$. □

We can now derive the following theorem as a corollary to the preceding lemma.

Corollary 9.12 (Murley [1]). *Finite rank Procházka–Murley groups have the Krull-Schmidt property: if*

$$A = M_1 \oplus \cdots \oplus M_m = N_1 \oplus \cdots \oplus N_n$$

where the M_i and N_j are indecomposable Procházka–Murley groups, then $m = n$, and there is a rearrangement such that $M_i \cong N_i$ for $i = 1, \ldots, m$.

Proof. From Lemma 9.11(i) we infer that indecomposable Procházka–Murley groups are strongly indecomposable, thus the components in the given direct decompositions are strongly indecomposable. A simple appeal to Jónsson's theorem 9.9 completes the proof. □

We conclude with pointing out a remarkable feature of indecomposable Procházka–Murley groups.

Theorem 9.13 (Murley [1]). *Any endomorphism of an indecomposable Procházka–Murley group is an integer times an automorphism. The endomorphism ring is a principal ideal domain.*

Proof. Let M be an indecomposable Procházka–Murley group. Every endomorphism $\eta \neq 0$ extends uniquely to an endomorphism of its completion $\tilde{M}$ which is, by (δ), a product of p-adic integers J_p. By indecomposability, each prime occurs at most once, thus $\operatorname{End} \tilde{M}$ is isomorphic to the direct product of the rings of p-adic integers for the same primes. This is a commutative ring with no nilpotent elements, so the same holds for $\operatorname{End} M$. Lemmas 9.7 and 9.11(i) imply $M \cong \operatorname{Im}\eta^n \oplus \operatorname{Ker}\eta^n$ for some $n \in \mathbb{N}$. In view of indecomposability, one of the summands vanishes, so η is either monic, or nilpotent. But nilpotency is not an option, thus η is monic, and therefore, $\operatorname{Im}\eta = mM$ for some $m \in \mathbb{N}$, because of Lemma 9.11(ii). Hence $m^{-1}\eta$ is an automorphism of M. □

★ **Notes.** The concept of quasi-isomorphism is due to Jónsson [1]. It was Reid [3] who formulated the basic properties of quasi-direct decompositions for finite rank torsion-free groups. Quasi-isomorphism has been studied extensively, it is frequently the natural concept when isomorphism cannot be expected. Actually, the study of the p-corank ≤ 1 case was initiated by Procházka [2], a few years before Murley's paper [1] was published with more substantial results. We have chosen to break tradition in the terminology (Murley group) to give due credit to Procházka.

Faticoni [2] proves *inter alia* the following for a torsion-free group G that is quasi-isomorphic to the direct sum of a finite number of pairwise non-quasi-isomorphic, strongly indecomposable finite rank torsion-free groups: Every decomposition of G into the direct sum of indecomposable groups is unique up to isomorphism.

Kozhukhov [1] investigates finite rank torsion-free groups G which have no nilpotent endomorphisms $\neq 0$. Such a G has a unique finite index subgroup A with direct decomposition $A = A_1 \oplus \cdots \oplus A_n$, where the A_i are pure in G, are strongly indecomposable, and satisfy $\operatorname{Hom}(A_i, A_j) = 0$ $(i \neq j)$. Krylov [4] points out that if for the finite rank torsion-free groups A, B there is a finite set $\{C_1, \ldots, C_n\}$ of finite rank torsion-free groups such that $\operatorname{rk}^p \operatorname{Hom}(A, C_i) = \operatorname{rk}^p \operatorname{Hom}(B, C_i)$ for all primes p and for all i, then A, B are quasi-isomorphic.

There are various versions of notation for quasi-isomorphism. I propose to use $\sim$ for quasi-isomorphism and $\approx$ for near-isomorphism; these are close to the standard symbol $\cong$ for isomorphism, and it is easy to remember: $\cong \Rightarrow \approx \Rightarrow \sim$.

Exercises

(1) For every $n \in \mathbb{N}$, there exist strongly indecomposable torsion-free groups of rank n.

(2) Let A be a torsion-free group of finite rank. There exist at most countably many non-isomorphic groups that are quasi-isomorphic to A. [Hint: count the subgroups between nA and A.]

(3) If $A \sim C$, then $\operatorname{IT}(A) = \operatorname{IT}(C)$ and $\operatorname{OT}(A) = \operatorname{OT}(C)$.

(4) (Procházka) Suppose $A \sim B$ are finite rank torsion-free groups. Then $\operatorname{Ext}(A, C) \cong \operatorname{Ext}(B, C)$ for every group C.

(5) If B, C are pure subgroups in a finite rank torsion-free group A such that $B \cap C = 0$ and $A/B \sim C$, then $A \sim B \oplus C$.

(6) A finite rank torsion-free group A is Procházka–Murley if and only if every subgroup of finite index is isomorphic to A.

(7) (Procházka) Every rank 2 quotient-divisible group (see Sect. 11) is a Procházka–Murley group.

(8) (Murley) The Kaplansky test problems for Procházka–Murley groups yield positive results.

(9) (Murley) (a) Let M, N be reduced Procházka–Murley groups such that, for every prime p, only one of $pM < M$ and $pN < N$ holds. Then $\mathrm{Hom}(M, N) = 0 = \mathrm{Hom}(N, M)$.

(b) In any direct decomposition $M = M_1 \oplus \cdots \oplus M_k$ of a reduced Procházka–Murley group M, the summands are fully invariant.

(10) Let A be as in Corollary 9.12. If a pure subgroup of A is a finite direct sum of groups quasi-isomorphic to the A_i, then it is a summand of A. [Hint: induction.]

10 Near-Isomorphism

From Jónsson's theorem 9.9 we can derive that if $A \oplus B \cong A \oplus C$ holds for finite rank torsion-free groups A, B, C, then B and C are quasi-isomorphic. In fact, passing to quasi-isomorphism and decomposing into direct sums of strongly indecomposable groups, Theorem 9.9 applies. It turns out that a stronger statement can be made: B and C are near-isomorphic in the sense defined below.

The Category $\mathcal{A}_p$ Following Arnold [A], we introduce the category $\mathcal{A}_p$ for each prime p. The objects in $\mathcal{A}_p$ are the finite rank torsion-free groups, while

$$\mathrm{Hom}_{\mathcal{A}_p}(A, C) = \mathbb{Z}_{(p)} \otimes \mathrm{Hom}(A, C) = \mathrm{Hom}(A, C)_{(p)},$$

i.e. the localization of $\mathrm{Hom}(A, C)$ at p.

Theorem 10.1 (Lady [4], Arnold [A]). *The following conditions are equivalent for the reduced torsion-free groups A and C of finite rank:*

(i) *for every prime p, C contains a subgroup isomorphic to A, whose index is coprime to p, and vice versa;*

(ii) *for every prime p, there exist an integer k coprime to p as well as maps $\alpha_p \colon A \to C$ and $\gamma_p \colon C \to A$ such that*

$$\gamma_p \alpha_p = k\mathbf{1}_A \quad \textit{and} \quad \alpha_p \gamma_p = k\mathbf{1}_C;$$

(iii) *A and C are quasi-isomorphic as well as $\mathcal{A}_p$-isomorphic for each prime p;*

(iv) *for every square-free integer m, there are maps $\rho_m : A \to C$ and $\sigma_m : C \to A$ such that*

$$\sigma_m \rho_m \equiv \mathbf{1}_A \bmod m\,\mathrm{End}\,A \quad \text{and} \quad \rho_m \sigma_m \equiv \mathbf{1}_C \bmod m\,\mathrm{End}\,C. \tag{12.19}$$

Proof. (i) $\Rightarrow$ (ii) Condition (i) means that for each p there is a map α_p carrying A isomorphically onto a subgroup of C, say, of index k, coprime to p. Thus $kC \leq \alpha_p A \leq C$. If we denote by $\gamma_p : C \to A$ the composite map of multiplication by k

followed by α_p^{-1}, then we have $\alpha_p\gamma_p = k\mathbf{1}_C$. Hence $\alpha_p\gamma_p\alpha_p = k\alpha_p$, and since α_p is necessarily monic, we can cancel it on the left to obtain $\gamma_p\alpha_p = k\mathbf{1}_A$.

(ii) $\Rightarrow$ (iii) The maps α_p, γ_p in (ii) are evidently monic, so $A \sim C$. They define uniquely $\mathcal{A}_p$-maps between A and C, which satisfy the same equations. As k is a unit mod p, (ii) implies that A, C are $\mathcal{A}_p$-isomorphic.

(iii) $\Rightarrow$ (iv) Condition (iii) implies that there exist $\rho_p \in \mathrm{Hom}_{\mathcal{A}_p}(A, C)$ and $\sigma_p \in \mathrm{Hom}_{\mathcal{A}_p}(C, A)$ that are inverse to each other mod p. Hence (iv) is obvious if m is a prime. To complete the proof, we show that if there are maps ρ_m, σ_m and ρ_n, σ_n satisfying (12.19) with $\gcd\{m, n\} = 1$, then there exist also ρ_{mn}, σ_{mn} as required.

Let $mx + ny = 1$ for $x, y \in \mathbb{Z}$, and set $\rho = mx\rho_n + ny\rho_m, \sigma = mx\sigma_n + ny\sigma_m$. Then using $mx = m^2x^2 + mnxy$ and $ny = mnxy + n^2y^2$, as well as $\rho_m\sigma_m \equiv \mathbf{1}_C \bmod m \,\mathrm{End}\, C$ and $\rho_n\sigma_n \equiv \mathbf{1}_C \bmod n \,\mathrm{End}\, C$, we obtain

$$\begin{aligned}\rho\sigma &\equiv m^2x^2\rho_n\sigma_n + n^2y^2\rho_m\sigma_m \equiv mx\rho_n\sigma_n + ny\rho_m\sigma_m \\ &\equiv mx\mathbf{1}_C + ny\mathbf{1}_C \equiv \mathbf{1}_C \bmod mn \,\mathrm{End}\, C.\end{aligned}$$

Similarly, $\sigma\rho \equiv \mathbf{1}_A \bmod mn \,\mathrm{End} A$, thus $\rho_{mn} = \rho, \sigma_{mn} = \sigma$ are as desired.

(iv) $\Rightarrow$ (i) First we show that there is a square-free $m \in \mathbb{N}$ such that $mX \neq X$ for every non-zero subgroup X of $B = A \oplus C$. If there is a prime p such that B contains no p-divisible subgroup, then $m = p$ is a good choice. Otherwise, pick a prime p for which B is not p-divisible. The p-divisible subgroup D of B has a smaller rank than B, so by induction we can assume the existence of an $n \in \mathbb{N}$ such that $nX \neq X$ for all subgroups $X \leq D$. Then $m = pn$ is as desired.

Using the integer m selected in the preceding paragraph, we next show that for the corresponding maps ρ_m, σ_m, the composite $\sigma_m\rho_m$ is monic. (iv) claims that $\sigma_m\rho_m = \mathbf{1}_A + m\eta$ for some $\eta \in \mathrm{End} A$. If $a \in \mathrm{Ker}\, \sigma_m\rho_m$, then $a = -m\eta(a)$, thus $\mathrm{Ker}\, \sigma_m\rho_m$ is contained in the m-divisible subgroup of A, so it is 0. Hence $\mathrm{Im}\, \sigma_m\rho_m \cong A$, so it is of finite index in A (Proposition 1.10). Furthermore, $\sigma_m\rho_m = \mathbf{1}_A + m\eta$ implies that $\mathrm{Im}\, \sigma_m\rho_m$ is m-pure in A, thus there is a $k \in \mathbb{N}$, coprime to m, such that $kA \leq \sigma_m\rho_m A \leq \sigma_m C \leq A$. Analogously, $\rho_m\sigma_m$ is monic and $k'C \leq \rho_m A < C$ for some k' coprime to m. Thus A, C satisfy (i). $\square$

Making use of the notation $\mathcal{A}_0$ introduced in the preceding section, (iii) can be rephrased by saying that A and C are both $\mathcal{A}_0$- and $\mathcal{A}_p$-isomorphic for all p.

From the proof it is clear that in (i), one-sided condition would suffice. It also follows:

Lemma 10.2. *The preceding theorem holds even if in conditions* (i), (ii)*, the prime p is replaced by an arbitrary square-free integer m.* $\square$

Near-Isomorphism Call the finite rank torsion-free groups A and C **near-isomorphic** or **of the same genus**, in notation:

$$A \approx C,$$

if they satisfy the equivalent conditions of Theorem 10.1. Thus

$$\text{isomorphism} \Rightarrow \text{near-isomorphism} \Rightarrow \text{quasi-isomorphism}.$$

We will see that none of the arrows is reversible.

(A) *Near-isomorphism is an equivalence relation.* Symmetry is clear from Theorem 10.1, while transitivity is obvious.
(B) *$A \approx B$ if and only if their divisible parts are isomorphic and their reduced parts are near-isomorphic.*
(C) *$A \approx B$ implies that $A_{(p)} \cong C_{(p)}$ for each prime p.*
(D) *For torsion-free groups of finite rank, we have $A_1 \oplus \cdots \oplus A_n \approx B_1 \oplus \cdots \oplus B_n$ whenever $A_i \approx B_i$ for $i = 1, \dots, n$.*
(E) *If a finite rank group A is p-divisible for almost all primes p, then $A \approx C$ implies $A \cong C$.* Let n denote the product of primes for which A is not divisible. In view of Lemma 10.2, there are $k \in \mathbb{N}$ coprime to n and an injection $\phi: C \to A$ such that $kA \leq \phi C \leq A$. As $kA = A$, ϕ is an isomorphism.
(F) *The objects in the category $\mathcal{A}_p$ have the cancellation property.* This is a consequence of Theorem 8.2.

Example 10.3. (a) The rank 2 summands A_i, C_i in the proof of Theorem 5.3 are near-isomorphic. If the coefficient k is chosen there to be coprime to p, then the arising groups are non-isomorphic. (The reader might enjoy in examining all the summands in Sect. 5 for near-isomorphism.)

(b) Let p_1, p_2, q be different primes. With independent elements e_1, e_2, consider the groups

$$A_1 = \langle p_1^{-\infty} e_1, p_2^{-\infty} e_2, q^{-1}(e_1 + e_2)\rangle,$$

$$A_2 = \langle p_1^{-\infty} e_1, p_2^{-\infty} e_2, q^{-1}(e_1 + k e_2)\rangle,$$

$$B = \langle p_1^{-\infty} e_1, p_2^{-\infty} e_2, q^{-2}(e_1 + e_2)\rangle,$$

where $k \not\equiv 1 \bmod q$ and $\gcd\{k, q\} = 1$. All three groups are quasi-isomorphic, but pairwise non-isomorphic. In addition, $A_1 \approx A_2$ (easy to check). However, A_1 is not near-isomorphic to B, as the index of any embedding of A_1 in B must be divisible by the prime q.

Near-Isomorphism and Cancellation The following theorem is an important result on near-isomorphism, and a good reason for its study.

Theorem 10.4 (Lady [4]). *Let $G = A \oplus B = A' \oplus C$ be two direct decompositions of a torsion-free group of finite rank. If $A \approx A'$, then $B \approx C$, i.e. also B and C are near-isomorphic.*

Proof. Consider the given decompositions in the categories $\mathcal{A}_0$ and $\mathcal{A}_p$. Cancellation holds in all these categories where A and A' are isomorphic (Lemma 9.1). Hence B and C are also isomorphic in these categories, i.e. they are near-isomorphic. $\square$

Our next aim is to relate near-isomorphism to power-cancellation. We require a lemma that points out another relevant property of near-isomorphism.

Lemma 10.5 (Lady [4]). *Suppose that A_i, B_i, C $(i = 1, \dots, k)$ are near-isomorphic finite rank torsion-free groups. Then*

(i) *there is a group D near-isomorphic to C such that*

$$G = A_1 \oplus \cdots \oplus A_k \oplus C \cong B_1 \oplus \cdots \oplus B_k \oplus D;$$

(ii) *there are but a finite number of non-isomorphic groups near-isomorphic to A_i.*

Proof. (i) To settle the $k = 1$ case, we prove that, for near-isomorphic A, B, C, the group $G = A \oplus C$ has a summand $\cong B$. We can find subgroups A', C' of B such that $A' \cong A$, $C' \cong C$, and the indices $[B : A'] = m$ and $[B : C'] = n$ are coprime. There are integers r, s satisfying $rm + sn = 1$. Then the monic map $b \mapsto (rmb, snb) \in A' \oplus C' \cong G$ followed by $\psi : A' \oplus C' \to A' + C' = B$ is the identity on B. Hence $G \cong B \oplus D$ for some D. From Theorem 10.4 it follows that D is near-isomorphic to C.

Suppose $A_1 \oplus \cdots \oplus A_k \oplus C \cong B_1 \oplus \cdots \oplus B_k \oplus D_k$ for some $D_k \approx C$. By the case $k = 1$, there is a $D_{k+1} \approx C$ such that $A_{k+1} \oplus D_k \cong B_{k+1} \oplus D_{k+1}$. Then $A_1 \oplus \cdots \oplus A_{k+1} \oplus C \cong B_1 \oplus \cdots \oplus B_{k+1} \oplus D_{k+1}$, and (i) follows.

(ii) An obvious consequence of (i) is that every group B near-isomorphic to A is isomorphic to a summand of $A \oplus A$. This direct sum is a finite rank group, so Lady's theorem 6.9 applies: it has but finitely many non-isomorpic summands. □

It is worthwhile observing that the number of non-isomorphic, near-isomorphic finite rank torsion-free groups can be an arbitrarily large integer. This is evident, e.g., from Theorem 5.5.

Theorem 10.6 (Arnold [5]). *Finite rank torsion-free groups A, C are near-isomorphic if and only if $A^k \cong C^k$ holds for some integer $k \geq 1$.*

Proof. First, suppose that $A^k \cong C^k$ for some $k \in \mathbb{N}$. We argue as in Theorem 10.4 to conclude that $A \approx C$.

Conversely, assume $A \approx C$. By Lemma 10.5(i), for every $n > 1$, there is $B_n \approx C$ such that $A^n \cong C^{n-1} \oplus B_n$. From Lemma 10.5(ii) we infer that there exist integers $n < m$ with $B_n \cong B_m$. Hence

$$A^m \cong C^{m-1} \oplus B_m \cong C^{m-n} \oplus C^{n-1} \oplus B_n \cong C^{m-n} \oplus A^n.$$

We appeal to the 2-substitution property, in particular, to Lemma 7.11 to find a group $L\,(\leq A \oplus A)$ satisfying $L \oplus A \cong A \oplus A$ such that

$$L \oplus A^{m-2} \cong C^{m-n} \oplus A^{n-1}.$$

We can rewrite the last isomorphism as $A^{m-1} \cong C^{m-n} \oplus A^{n-1}$. If we repeat this argument $n-1$ more times, then we arrive at the desired conclusion: $A^{m-n} \cong C^{m-n}$. □

The preceding theorem, taken in combination with Theorem 10.4, gives the following interesting result on the cancellations of finite rank torsion-free groups: $A \oplus B = A' \oplus C$ *with* $A \cong A'$ *implies that* $B^k \cong C^k$ *holds for some* $k \in \mathbb{N}$.

★ **Notes.** The idea of near-isomorphism originates in Representation Theory. In the theory of lattices over orders over Dedekind domains, it is equivalent to being of the same genus. It was Lady who introduced this notion into the theory of finite rank torsion-free groups.

In contrast to quasi-isomorphism, indecomposability is preserved under near-isomorphism; see Arnold [3]. For near-isomorphism of Butler-groups, see Arnold–Dugas [2]. O'Meara–Vinsonhaler [1] prove a very surprising result: there exist non-isomorphic torsion-free groups A, C of finite rank such that $A^n \cong C^n$ for all $n \geq 2$.

The cancellation property was introduced by Jónsson–Tarski [Notre Dame Math. Lectures **5** (1947)]. It became an ubiquitous topic in the theory of modules. For abelian groups, it originated from a question by Kaplansky, answered by E. Walker [1], Honda [2], and Cohn [1], that $\mathbb{Z}$ was cancellative (see Sect. 2, Exercise 11 in Chapter 3), and continued with Fuchs–Loonstra [1] where the rational groups with the cancellation property were characterized. See Stelzer [1] for a systematic treatment in the finite rank case. Blazhenov [1] succeeded in getting a complete solution: *a finite rank torsion-free group A has the cancellation property if and only if $A = F \oplus B$, where F is free and B satisfies the following conditions:* (i) *for each $n \in \mathbb{N}$, units of* $\operatorname{End} B/n \operatorname{End} B$ *lift to units of* $\operatorname{End} B$*; and* (ii) *the quasi-summands of B satisfy the so-called weak Eichler condition.* Arnold [A] has definite results concerning *self-cancellation*: $A \oplus A \cong A \oplus B$ implies $A \cong B$, while O'Meara–Vinsonhaler [1] discuss *separative cancellation*: $A \oplus A \cong A \oplus B \cong B \oplus B$ implies $A \cong B$.

Exercises

(1) Let B, C be subgroups of the finite rank torsion-free group A such that $B \cap C = 0$. If B is pure in A and $A/B \approx C$, then $A \approx B \oplus C$.
(2) Decide whether or not $B \approx C$ implies $\operatorname{Hom}(A, B) \approx \operatorname{Hom}(A, C)$.
(3) If $A \approx C$ for finite rank torsion-free groups, then the trace of A in C is all of C.
(4) Let A, C be near-isomorphic finite rank torsion-free groups, and A_1, A_2 subgroups of A isomorphic to C with coprime indices. Prove that $A_1 \cap A_2 \approx C$.
(5) Let A, B, C be finite rank torsion-free groups such that $pA = A$ for almost all primes. Then $A \oplus B \cong A \oplus C$ implies $B \cong C$.
(6) (Fuchs–Loonstra) A rational group R has the cancellation property if and only if, for every $n \in \mathbb{N}$, each automorphism of R/nR lifts to an automorphism of R.
(7) Assume B, C are subgroups of A such that there are $\beta, \gamma \in \operatorname{End} A$ for which $\beta \restriction B$ and $\gamma \restriction C$ are inverse isomorphisms between B and C. If A has the cancellation property, then $A/B \cong A/C$. [Hint: consider the push-out of $\beta : B \to A$ and the injection $B \to A$.]
(8) A group with the exchange property has the cancellation property if and only if isomorphic summands have isomorphic complements.
(9) (Arnold) A group A is said to have **self-cancellation** if $A \oplus A' \cong A \oplus A$ implies $A' \cong A$.

 (a) Summands inherit self-cancellation.
 (b) Let A be torsion-free and of finite rank. If $A \cong C \oplus C$, then A has self-cancellation. [Hint: from $C^2 \oplus A' \cong C^4$ argue as in Theorem 10.4.]

11 Dualities for Finite Rank Groups

The main objective of this section is to discuss dualities for finite rank torsion-free groups. There exist several duality theories which are of independent interest and deserve particular attention, though they cannot compete in applicability or in depth with dualities implemented by maps into injective objects. Our exposition is centered around the Warfield and the Arnold dualities. Both are limited in applications, but have a number of interesting features that are worthwhile exploring.

Maps from and Into Rank 1 Groups First, we collect information about homomorphisms from and into groups of rank 1. In this section, our standing hypothesis is that all groups are torsion-free and of finite rank (unless stated otherwise), and A is of rank 1 with $\operatorname{End} A = \mathsf{R} < \mathbb{Q}$. We observe that $\operatorname{Hom}(B, C)$, $B \otimes C$ are (both left and right) R-modules if either B or C is an R-module. Also, it is useful to keep in mind that, if C is a torsion-free R-module, and if B is a pure subgroup of C, then both B and C/B are R-modules. Also, if both X and Y are R-modules, then $\operatorname{Hom}(X, Y) = \operatorname{Hom}_{\mathsf{R}}(X, Y)$, i.e. all group homomorphisms between R-modules are automatically R-homomorphisms.

The following two lemmas provide relevant information needed in the dualities.

Lemma 11.1 (Warfield [1]). *Let A be as stated above. For a torsion-free group G of finite rank, the homomorphism*

$$\phi\colon\ G \to \operatorname{Hom}(A, A \otimes G)$$

given by $\phi(g)(a) = a \otimes g$ $(a \in A, g \in G)$ is injective. It is an isomorphism exactly if G is an R*-module.*

Proof. Since $\operatorname{rk} A = 1$, $\operatorname{rk}(A \otimes G) = \operatorname{rk} G$, and manifestly $\operatorname{rk} \operatorname{Hom}(A, A \otimes G) = \operatorname{rk}(A \otimes G)$, so ϕ is injective. If ϕ is an isomorphism, then G ought to be an R-module, as the Hom is such a module.

Conversely, assume G is an R-module. As $\phi(G)$ is essential in Hom (by counting the ranks), for an $\eta : A \to A \otimes G$ we must have an integer $m \neq 0$ such that $m\eta(a) = a \otimes g$ for some $g \in G$ and for all $a \in A$. We can get rid of a prime power divisor p^k of m for which $p\mathsf{R} = \mathsf{R}$, by replacing g by $g/(p^k) \in G$. If $p^k \mid m$ and $p\mathsf{R} \neq \mathsf{R}$, then choosing $a \in A \setminus pA$, $p^k \mid a \otimes g$ implies $p^k \mid g$. Thus g can be replaced by some $g' \in G$ such that m is not divisible by p^k. In this way, at the end of this process, we get $\eta(a) = a \otimes g'$, establishing the surjectivity of ϕ. $\square$

Lemma 11.2 (Warfield [1]). *With the notation of* Lemma 11.1*, the homomorphism*

$$\psi\colon\ \operatorname{Hom}(A, G) \otimes A \to G$$

given by $\psi(\eta \otimes a) = \eta(a)$ $(\eta \in \operatorname{Hom}(A, G), a \in A)$ is monic. Furthermore, $\operatorname{Im} \psi = \operatorname{Tr}_A(G)$*, the trace of A in G.*

Proof. Suppose $\psi(\eta \otimes a) = 0$, i.e. $\eta(a) = 0$ for some $\eta \in \mathrm{Hom}(A, G)$ and $0 \neq a \in A$. Then by torsion-freeness, $\eta(A) = 0$, and $\eta = 0$. The claim about the trace is straightforward. □

Torsionless and Reflexive Groups Interesting questions arise if we repeat homomorphisms into the same group A. (Recall A has rank 1, but $\ncong \mathbb{Q}$.) Once the group A has been fixed, we may use unambiguously the notation $G^* = \mathrm{Hom}(G, A)$. Observe:

(a) $A^* \cong \mathsf{R}^+$ and $(\mathsf{R}^+)^* \cong A$.
(b) If G is of finite rank, then always $\mathrm{rk}\, G^* \leq \mathrm{rk}\, G$.
(c) $(G \oplus H)^* = G^* \oplus H^*$.
(d) If $G \to H$ is an epimorphism, then the induced map $H^* \to G^*$ is monic.

Consider the canonical homomorphism

$$\theta_G\colon\ G \to G^{**} = \mathrm{Hom}(\mathrm{Hom}(G, A), A) = \mathrm{Hom}(G^*, A)$$

defined as $\theta_G(g)(\eta) = \eta(g)$ $(g \in G, \eta \in G^*)$ for G of arbitrary finite rank. Adopting the customary terminology, G will be called A**-reflexive** if θ_G is an isomorphism, and A**-torsionless** in case θ_G is a monomorphism. Note that a homomorphism $\xi : G \to H$ induces a map $\xi^* : H^* \to G^*$, and hence a map $\xi^{**} : G^{**} \to H^{**}$.

The following lemma is a simple, but fundamental prelude to the theorem below.

Lemma 11.3. *The following conditions are equivalent for a torsion-free group G of finite rank n:*

(i) *G is A-torsionless;*
(ii) $G^* = \mathrm{Hom}(G, A)$ *has rank n;*
(iii) $\mathrm{OT}(G) \leq \mathbf{t}(A)$*;*
(iv) *G is isomorphic to a subgroup of $A^n = A \oplus \cdots \oplus A$.*

Proof. (i) ⇒ (ii) That $\mathrm{rk}\, G^* \leq \mathrm{rk}\, G$ is obvious. On the other hand, $\mathrm{rk}\, G^{**}$ can equal $\mathrm{rk}\, G$ only if $\mathrm{rk}\, G^* = \mathrm{rk}\, G$.

(ii) ⇒ (iii) $\mathrm{OT}(G) > \mathbf{t}(A)$ would mean that there exists a corank 1 pure subgroup B of G such that $\mathrm{Hom}(G/B, A) = 0$. Then in the exact sequence $0 = \mathrm{Hom}(G/B, A) \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(B, A)$ the last Hom has rank $\leq n - 1$, and hence so does $\mathrm{Hom}(G, A)$, in contradiction to (ii).

(iii) ⇒ (iv) Select pure subgroups $B_1, \ldots, B_n$ in G, each of rank $n - 1$, such that $\bigcap_i B_i = 0$. Then G embeds in $\oplus_i G/B_i$, which in turn embeds in A^n.

(iv) ⇒ (i) Given a monomorphism $\eta\colon G \to A^n$, for each $0 \neq g \in G$, there exists a canonical projection $\pi_i : A^n \to A$ such that $\pi_i(\eta(g)) \neq 0$. This shows that the canonical homomorphism θ_G is injective. □

The embedding (iv) provides a framework for an analysis of A-torsionless groups, besides it allows us to confine the study to R-submodules of A^n where $\mathsf{R} = \mathrm{End}\, A$.

The next proposition is concerned with the closure properties of the class of A-torsionless R-modules.

Lemma 11.4. *The class of A-torsionless* R*-modules of finite rank is closed under finite direct sums, pure subgroups and torsion-free quotients.*

Proof. Let B be a pure subgroup of an A-torsionless group G. A moment's reflection on the ranks of the groups in the exact sequence $0 \to \mathrm{Hom}(G/B, A) \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(B, A)$ convinces us that, in view of Lemma 11.3(ii), both B and G/B are A-torsionless. $\square$

Example 11.5. A completely decomposable group $G = G_1 \oplus \cdots \oplus G_n$ of finite rank is A-torsionless if and only if all the rank 1 summands have types $\leq \mathbf{t}(A)$. G is A-reflexive if and only if $\mathbf{t}(G_i) = \mathbf{t}(A)$ for all i.

We also have the pleasant fact about dual groups: G^* is always A-reflexive.

Lemma 11.6. *For a torsion-free group G of finite rank, the map $\theta_{G^*} : G^* \to G^{***}$ has an inverse:*

$$\mu \colon G^{***} \to G^* \qquad \text{where } \mu(\xi)(g) = \xi(\theta_G(g)) \ \ (g \in G,\ \xi \in G^{***}).$$

Proof. Only a simple calculation is needed right from the definitions:

$$\mu(\theta_{G^*}(\eta))(g) = \theta_{G^*}(\eta)(\theta_G(g)) = \theta_G(\eta) = \eta(g) \qquad (\eta \in G^*, g \in G)$$

whence $\mu\theta_{G^*} = \mathbf{1}_{G^*}$ is immediate. Since G^{***} cannot have a larger rank than G^*, also $\theta_{G^*}\mu = \mathbf{1}_{G^{***}}$, thus the maps θ_{G^*} and μ are inverse isomorphisms. $\square$

Dualities Let $\mathcal{C}$ and $\mathcal{D}$ denote two categories. The contravariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ define a **duality** between $\mathcal{C}$ and $\mathcal{D}$ if

$$GF \cong \mathbf{1}_{\mathcal{C}} \quad \text{and} \quad FG \cong \mathbf{1}_{\mathcal{D}}.$$

Here we mean isomorphism in the sense that $GF(C) \cong C$ naturally for all $C \in \mathcal{C}$. Note that the functors F, G are exact. If $\mathcal{C} = \mathcal{D}$ and $F = G$, then we say F is a **duality functor** on $\mathcal{C}$.

Motivated by the well-known duality in Linear Algebra for finite dimensional vector spaces over a field, we consider dualities for full subcategories of finite rank torsion-free groups defined in terms of a fixed rank 1 torsion-free group A. Always $\mathcal{C} = \mathcal{D}$ is assumed, and the functors are $F = G = \mathrm{Hom}(*, A)$. The group $C^* = \mathrm{Hom}(C, A)$ will be called the A**-dual** of C. The question we are interested in is concerned with subcategories $\mathcal{C}$, for which this is a genuine duality, i.e., when the map θ_C of C into its A**-bidual** C^{**} is an isomorphism for all $C \in \mathcal{C}$.

Warfield Duality A torsion-free R-module C is said to be **locally free (over** R) if the localization $C_{(p)} = C \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module for all primes p with $p\mathsf{R} \neq \mathsf{R}$. Thus the localization of C at a prime p is either a divisible group or a free $\mathbb{Z}_{(p)}$-module according as $p\mathsf{R} = \mathsf{R}$ or not.

Lemma 11.7. *Let A and B be torsion-free groups, A of rank one, and B of finite rank. Then $C = \mathrm{Hom}(B, A)$ is locally free over $\mathsf{R} = \mathrm{End}\, A$.*

Proof. Clearly, $pC = C \neq 0$ if and only if $pA = A$. If $pA \neq A$, then there are natural injections $C_{(p)} \to \mathrm{Hom}(B,A)_{(p)} \to \mathrm{Hom}(B_{(p)}, A_{(p)})$ where $A_{(p)} \cong \mathbb{Z}_{(p)}$. Let X denote a basic submodule of $B_{(p)}$. As $B_{(p)}/X$ is divisible, but $A_{(p)}$ is not, the restriction map $\mathrm{Hom}(B_{(p)}, A_{(p)}) \to \mathrm{Hom}(X, A_{(p)})$ is injective. Since X and $A_{(p)}$ are free over $\mathbb{Z}_{(p)}$, so is $\mathrm{Hom}(X, A_{(p)})$. Therefore, $C_{(p)}$ is $\mathbb{Z}_{(p)}$-free as a submodule of a free $\mathbb{Z}_{(p)}$-module. □

This brings us to Warfield duality:

Theorem 11.8 (Warfield [1]). *For A and R as above, define $\mathcal{C}_A$ as the full subcategory of the category of torsion-free groups of finite rank whose objects are the locally free R-modules C satisfying* $\mathrm{OT}(C) \leq \mathbf{t}(A)$. *The category $\mathcal{C}_A$ admits a duality defined by the functor* $\mathrm{Hom}(*, A)$.

Proof. We show that a torsion-free R-module C of finite rank is A-reflexive if and only if (i) C is locally free over R, and (ii) $\mathrm{OT}(C) \leq \mathbf{t}(A)$.

If C is A-reflexive, then it is a locally free R-module by Lemma 11.7, and satisfies (ii) because of Lemma 11.3.

Conversely, assume C has properties (i) and (ii). Then Lemma 11.3 guarantees that C is A-torsionless. It also follows that $\mathrm{Im}\,\theta_C$ is an essential subgroup in C^{**}, so it only remains to verify its purity in C^{**}. Suppose the map $\xi : \mathrm{Hom}(C,A) \to A$ is divisible by a prime p in C^{**}. Then $\mathrm{Im}\,\xi \leq pA$ holds. In the non-trivial case when $pA \neq A$, we localize at p to get $\xi_p : \mathrm{Hom}(C_{(p)}, A_{(p)}) \to pA_{(p)}$. In view of (i), $C_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module, and therefore the same holds for $\mathrm{Hom}(C_{(p)}, A_{(p)})$. Consequently ξ_p must be divisible by p. Then ξ is divisible by p in $\mathrm{Im}\,\theta_C$. □

Arnold Duality A more sophisticated duality for a different subclass of finite rank torsion-free groups was established by Arnold [1]. This duality involves the quotient-divisible groups, introduced by Beaumont–Pierce [2]. The Arnold duality works only up to quasi-isomorphism.

A torsion-free group A of finite rank is said to be **quotient-divisible** if it contains a free essential subgroup F such that A/F is a divisible (torsion) group. We observe right away:

(A) *If A is quotient-divisible, and F' is any free subgroup of the same rank as A, then A/F' is a direct sum of a bounded and a divisible torsion group.*
(B) *Direct sums and pure subgroups of quotient-divisible groups are again quotient-divisible.*
(C) *A group that is quasi-isomorphic to a quotient-divisible group is likewise quotient-divisible.*

Example 11.9. (a) Pure subgroups of J_p as well as finite rank local torsion-free groups are quotient-divisible.
(b) The groups constructed in the proof of Lemma 4.6 are quotient-divisible.

Let $\mathcal{Q}$ denote the category of finite rank quotient-divisible torsion-free groups with quasi-homomorphisms as morphisms. Then we can state:

Theorem 11.10 (Arnold [1]). *There is a duality functor $D\colon \mathcal{Q} \to \mathcal{Q}$ that preserves ranks, and for every prime p satisfies:*

$$\mathrm{rk}^p D(A) = \mathrm{rk} A - \mathrm{rk}^p A.$$

Proof. First we deal with the p-local case, and describe how the duality works there. Any p-local group A is necessarily quotient-divisible, so we can use the representation of A as a pull-back, given after Theorem 8.6. The dual $D(A)$ is obtained in a similar fashion, switching the roles of the bottom and the right-hand sequences, and using the matrix $\mathbb{P}$ (Sect. 8):

$$\begin{array}{ccccccccc}
 & & & & \oplus_{k=1}^n \mathbb{Z}_{(p)}\, b_k & = & \oplus_{k=1}^n \mathbb{Z}_{(p)}\, b_k & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^m \mathbb{Z}_{(p)} c_i & \longrightarrow & D(A) & \longrightarrow & \oplus_{k=1}^n \mathbb{Q} b_k & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \mathrm{can}} & & \\
0 & \longrightarrow & \oplus_{i=1}^m \mathbb{Z}_{(p)} c_i & \longrightarrow & \oplus_{i=1}^m \mathbb{Q} c_i & \xrightarrow{\mathsf{P}\cdot\mathrm{can}} & \oplus_{k=1}^n \mathbb{Z}(p^\infty) & \longrightarrow & 0
\end{array}$$

In order to verify that this correspondence preserves quasi-isomorphisms if different elements b_k, c_i are selected, it suffices to refer to our claim in Sect. 8 that such a change keeps the quasi-isomorphy classes fixed.

In the global case, the duality is defined as the intersection of the locally defined duals, so it is defined in terms of the matrices $\mathbb{P}_p$, one for each prime p. A switch to a different basis can change the isomorphy class for a finite number of primes only, so the intersection of the modified local duals will yield a quasi-isomorphic dual. □

Fomin Duality More general dualities are also available in the literature, above all the Fomin and the Vinsonhaler–Wickless dualities. It is evident that if larger classes of groups are included in a duality, then some pleasant features are lost in the generalization, and the duality becomes less usable, though it can be instructive by revealing unnoticed features.

Fomin [2] defines a category in terms of the Richman type of the group ((12.19) of Sect. 1). Suppose $\mathbf{r} = (\dots, k_p, \dots)$ and $\mathbf{s} = (\dots, \ell_p, \dots)$ are fixed types (represented by characteristics) such that $\mathbf{r} \leq \mathbf{s}$. The objects of the category $\mathcal{D}_{\mathbf{r}}^{\mathbf{s}}$ are the torsion-free groups A of finite rank such that in their Richman types, for each prime p, all the exponents are equal either to k_p or to ℓ_p. Thus a p-component looks like

$$T_p \cong \mathbb{Z}(p^{k_p})^{r_p} \oplus \mathbb{Z}(p^{\ell_p})^{s_p} \qquad (0 \leq k_p \leq \ell_p)$$

where r_p, s_p are non-negative integers with sum $\mathrm{rk} A$. It is understood that the morphisms in this category $\mathcal{D}_{\mathbf{r}}^{\mathbf{s}}$ are the quasi-homomorphisms.

The definition of the dual group is similar to the one employed for the Arnold duality, e.g. for the dual, the ranks r_p and s_p switch roles. The precise definition is not simple, and since we are not going to use this duality, we do not elaborate, and refer to Fomin [2] for details.

Example 11.11. (a) If $\mathbf{r} = (0, \ldots, 0, \ldots)$ (the type of $\mathbb{Z}$) and $\mathbf{s} = (\infty, \ldots, \infty, \ldots)$ (the type of $\mathbb{Q}$), then the objects of the category $\mathcal{D}_{\mathbf{r}}^{\mathbf{s}}$ are the quotient-divisible groups.
(b) If $\mathbf{s}$ is the type of $\mathbb{Z}_{(p)}$, and $\mathbf{r}$ is as in (a), then the p-local groups are in $\mathcal{D}_{\mathbf{r}}^{\mathbf{s}}$.
(c) For any type $\mathbf{t}$, the objects in $\mathcal{D}_{\mathbf{t}}^{\mathbf{t}}$ are homogeneous of type $\mathbf{t}$.

Vinsonhaler–Wickless [4] formulate a more general duality that includes the dualities mentioned above as special cases. They write (not uniquely) a torsion-free group A of finite rank as $A = B + C$, where B is locally free, and C is quotient-divisible. Now the duality assigns $A^* = A(B) + W(C)$ to A, where $A(B)$ is the Arnold dual of B, and $W(C)$ denotes the Warfield dual of C. It turns out that A^* is unique up to quasi-isomorphism.

★ **Notes.** The Warfield duality has wider appeal in module theory, see Bazzoni–Salce [J. Algebra **185**, 836–868 (1996)] for a far reaching generalization. Goeters [3] extends the Warfield duality by allowing A to be chosen from a larger class of finite rank groups. Mader–Mutzbauer–Vinsonhaler [1] establish a duality for the category of almost completely decomposable finite rank groups, where the categorical isomorphism is near-isomorphism.

Faticoni–Goeters–Vinsonhaler–Wickless [1] prove the following interesting result: let $\mathcal{C}, \mathcal{D}$ be full subcategories of torsion-free groups of finite rank, such that $\mathcal{C}$ is closed under isomorphisms, and taking rank 1 pure subgroups. If there exist functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ implementing a duality, then both F and G ought to be naturally equivalent to the functor $\operatorname{Hom}(*, A)$ for some rank 1 torsion-free group A.

Fomin–Wickless [1] extend the notion of quotient-divisibility to mixed groups, and show that the category of all quotient-divisible groups is dual to the category of torsion-free groups.

Krylov [3] studies homomorphism groups into rank 1 groups. For infinite rank groups, an extensive study is presented, including duality as well as $\mathbb{Z}$-reflexivity. For groups of infinite ranks, duality has been the subject of a large number of publications. The theory requires advanced set theory. There are fascinating examples, e.g. $\aleph_1$-free groups with trivial duals (Pabst [1]). The reader is referred to Mekler–Schlitt [1]. See Eklof–Mekler [EM] for an excellent presentation.

Exercises

(1) (Warfield) Let A be a rank 1 group with endomorphism ring R. A finite rank torsion-free group H is of the form $H \cong \operatorname{Hom}(G, A)$ for some G exactly if H is locally free over R.
(2) (a) If $G = H \oplus K$, then $G^* = H^* \oplus K^*$.
(b) Direct sums and summands of A-reflexive groups are A-reflexive.
(3) If θ_G is a surjective map, then G^* is reflexive.
(4) Let G be an A-reflexive torsion-free group of finite rank. The endomorphism rings of G and G^* are anti-isomorphic.
(5) Find the Arnold dual of a rank 2 pure subgroup of J_p.

12 More on Finite Rank Groups

The Arnold–Lady Category Equivalence Let A be a torsion-free group of finite rank. We define two categories. One is $\operatorname{add}(A)$: the full subcategory of $\mathcal{A}b$ whose objects are the finite direct sums of copies of A, and their summands. The other is proj(E) where $\mathsf{E} = \operatorname{End} A$; this is the full subcategory of the category of right E-modules whose objects are the finitely generated projective modules. The morphisms are $\mathbb{Z}$- and E-isomorphisms, respectively.

Example 12.1. Let A be a strongly indecomposable torsion-free group of finite rank. Then the objects in add(A) are the finite direct sums of groups near-isomorphic to A.

Theorem 12.2 (Arnold–Lady [1]). *For a torsion-free group A of finite rank, the categories* add(A) *and* proj(E) *are equivalent, as witnessed by the inverse functors*

$$\operatorname{Hom}(A, \star)\colon \operatorname{add}(A) \to \operatorname{proj}(\mathsf{E}) \quad \textit{and} \quad \star \otimes_{\mathsf{E}} A\colon \operatorname{proj}(\mathsf{E}) \to \operatorname{add}(A).$$

Proof. It is clear that $F = \operatorname{Hom}(A, \star)$ and $G = \star \otimes_{\mathsf{E}} A$ are functors between the indicated categories. We show that there exists a natural transformation ϕ from GF to the identity functor of add(A) given by

$$\phi_C\colon \operatorname{Hom}(A, C) \otimes_{\mathsf{E}} A \to C \qquad (\phi_C(\eta \otimes a) = \eta(a))$$

where $C \in \operatorname{add}(A))$, $a \in A$, $\eta \in \operatorname{Hom}(A, C)$. Also, a natural transformation ψ from the identity functor of proj(E) to FG such that

$$\psi_P\colon P \to \operatorname{Hom}(A, P \otimes_{\mathsf{E}} A) \qquad (\psi_P(\xi)(a) = \xi \otimes a)$$

where $P \in \operatorname{proj}(\mathsf{E})$, $\xi \in P$, $a \in A$. It is evident that if C is a finite direct sum of copies of A, then ϕ_C is an isomorphism, and if P is a finitely generated free E-module, then ψ_P is an isomorphism. The same holds for summands, and since the isomorphisms are natural, the proof is complete. □

In a special case we do not have to restrict ourselves in add(A) to finite direct sums of copies of A and in proj(E) to finitely generated projective E-modules.

Corollary 12.3 (Arnold–Murley [1]). *If A is self-small, then the category of* add(A) *is equivalent to the category* proj(E) *of projective right E-modules, even if the extended categories are considered.*

Proof. This is rather obvious, since if A is self-small, then $\operatorname{Hom}(A, *)$ commutes with infinite direct sums as well. □

This theory provides a fresh point of view of known facts on rank 1 groups, besides extending several useful results from rank 1 to the finite rank case. As a sample, we prove one such result.

The idea behind condition (i) in Proposition 12.4 was inspired by Baer's lemma: in case $A = C$ is of rank 1 and of type $\mathbf{t}$, Lemma 3.6 claims that the exact sequence in (i) is splitting. (The symbol $S_A(G)$ in the next result stands for the trace $\operatorname{Tr}_A(G) = \langle \zeta(A) \mid \zeta : A \to G \rangle$ of A in G.)

Proposition 12.4 (Arnold–Lady [1]). *The following are equivalent for a torsion-free group A of finite rank:*

(i) *an exact sequence* $0 \to B \to G \xrightarrow{\gamma} C \to 0$ *of finite rank torsion-free groups is splitting if* $G = B + S_A(G)$ *and* $C \in \operatorname{add}(A)$*;*
(ii) $\mathsf{I}A \neq A$ *for all proper right ideals* I *of* $\mathsf{E} = \operatorname{End} A$.

Proof. (i) $\Rightarrow$ (ii) Assume the right ideal I of E satisfies $\mathsf{I}A = A$, so the map $\mu : \mathsf{I} \otimes_{\mathsf{E}} A \to \mathsf{I}\,A = A$ is epic. Choose a free right E-module P with an epimorphism $\pi : P \to \mathsf{I}$. Then $\nu = \mu(\pi \otimes \mathbf{1}_A) : P \otimes A \to A$ is likewise epic. Since $S_A(P \otimes A) = P \otimes A$, (i) implies that ν splits. Consequently, the induced map $P \cong \operatorname{Hom}(A, P \otimes A) \to \operatorname{Hom}(A, A) \cong \mathsf{E}$ is also splitting, thus epic. We now calculate: if $\xi \in P$, $a \in A$, then $\theta_P(\xi)(a) = \xi \otimes a \mapsto \pi(\xi) \otimes a \mapsto \pi(\xi)(a)$. Hence $\{\pi(\xi) \mid \xi \in P\} = \mathsf{E}$ which along with $\pi(\xi) \in \mathsf{I}$ implies $\mathsf{I} = \mathsf{E}$. Thus $\mathsf{I}A = A$ only if $\mathsf{I} = \mathsf{E}$.

(ii) $\Rightarrow$ (i) It suffices to prove (i) for $C = A$, because then a splitting map for a direct sum of copies of A can be obtained individually for the summands. Define the right ideal $\mathsf{I} = \{\gamma\delta \mid \delta : A \to G\}$; by the hypothesis on G, it satisfies $\mathsf{I}A = A$. Hence (ii) implies $\mathsf{I} = \mathsf{E}$. This means there is $\delta : A \to G$ such that $\gamma\delta = \mathbf{1}_A$, i.e. the exact sequence in (i) splits. □

Irreducible Torsion-Free Groups The concept of irreducibility was introduced by Reid [3]. He called a torsion-free group G **irreducible** if every non-zero fully invariant pure subgroup of G is equal to G. Equivalently, if every non-trivial $\operatorname{End}(G)$-submodule M of G is an essential subgroup of G (i.e. G/M is torsion).

(A) *Irreducibility is invariant under quasi-isomorphism.*
(B) *Pure subgroups of irreducible groups are irreducible.*
(C) *An irreducible torsion-free group G is homogeneous.* In fact, if $\mathbf{t}$ is a type of an element in G, then $G(\mathbf{t}) = G$, and this is true for every type in G.

Example 12.5. A strongly indecomposable torsion-free group G of finite rank is irreducible: its quasi-endomorphism ring $\mathbb{Q}\operatorname{End} G$ is a division ring, and therefore all non-zero fully invariant subgroups are essential in G.

It is not difficult to characterize irreducible torsion-free groups of finite rank up to quasi-isomorphism.

Theorem 12.6 (Reid [3]). *For a torsion-free group G of finite rank the following are equivalent.*

(i) *G is irreducible;*
(ii) *G is quasi-isomorphic to a direct sum of a finite number of copies of a strongly indecomposable, irreducible group H of finite rank;*
(iii) $\mathbb{Q}\operatorname{End} G$ *is a full matrix ring over a division ring* D.

Proof. (i) $\Leftrightarrow$ (ii) Let G be irreducible. We have $G \sim \oplus_{i=1}^n G_i$ with strongly indecomposable groups G_i (Theorem 9.9). The fully invariant subgroup generated by any of the G_i must include an essential subgroup of each of the other summands. This being mutual, all the G_i must be quasi-isomorphic. Conversely, (ii) implies that a G_i contains the $\operatorname{End} H$-submodule generated by any of its non-zero submodules as an essential submodule. Furthermore, there are monic maps $G_i \to G_j$ for all i, j.

(ii) $\Leftrightarrow$ (iii) If $G \sim \oplus_{i=1}^n G_i$ with quasi-isomorphic strongly indecomposable summands, then the quasi-endomorphism rings of the components must be isomorphic division rings. Hence (iii) follows. Conversely, (iii) implies that there are primitive idempotents $\epsilon_i \in \mathbb{Q}\operatorname{End} G$ $(i = 1, \ldots, n)$ such that $G \sim \oplus_{i=1}^n \epsilon_i G$ with strongly indecomposable summands $G_i = \epsilon_i G$. That $\mathbb{Q}\operatorname{End} G$ is a full matrix ring over a division ring D shows that $\mathbb{Q}\operatorname{Hom}(G_i, G_j) \cong \mathsf{D}$ for all $i \neq j$, so the G_i's are quasi-isomorphic. □

★ **Notes.** Arnold [2] considers strongly homogeneous finite rank groups (any pure rank 1 subgroup can be carried by a suitable automorphism onto any other such subgroup), and proves that they are exactly the groups of the form $\mathsf{R} \otimes H$, where H is homogeneous completely decomposable, and R is a ring in an algebraic numbers field such that every element of R is an integral multiple of a unit. For a finite rank torsion-free group G, Reid [3] defines the **pseudo-socle** S as the pure subgroup generated by all non-zero minimal pure fully invariant subgroups of G. *Inter alia*, he shows that $G = S$ if and only if $\mathbb{Q}\operatorname{End} G$ is a semi-simple artinian ring.

Albrecht [3] extends several results to groups of infinite rank, but he imposes extra conditions on their endomorphism rings.

In these final notes to Chapter 12, let us make additional comments.

There are numerous publications on the theory of torsion-free groups of finite rank, dealing with various questions concerning a smaller or a larger subclass. Needless to say, we have made no attempt to be exhaustive in our presentation. We had to draw a line somewhere, and tried to be selective so that the ideas that we believe to be central to the theory would stand out.

At this time, the ultimate goal of finding a classification of finite rank torsion-free groups in terms of reasonable invariants is still beyond our reach, and will very likely remain so for quite a while. Eklof points out that in L all countable groups are "classifiable," since a definable well-ordering of the universe can be used, and a canonical member of each isomorphy class can be chosen. Of course, this is not a kind of classification that one has in mind. There has been considerable effort to find reasonable invariants, without much success. The amount of open questions is staggering. New evidence has begun to emerge that questions the feasibility of a classification theory. The aim of these remarkable investigations is to verify the conjecture that no reasonable classification is possible for this class of groups. The closest one could get so far is the introduction of levels of difficulty in the classification problem, and the characterization of the set-theoretical complexity of any candidate for complete invariants, providing strong evidence against classification. (Besides finite rank torsion-free groups, also separable p-groups of cardinality $\aleph_1$ as well as $\aleph_1$-separable torsion-free groups of size $\aleph_1$ have been investigated.) The approach is by forming a standard Borel space, and studying a Borel equivalence relation on the space—in this way, the complexity of the classification problem can be measured. It has been proved (that we believe to know anyway from experience) that no classification is possible in the same fashion as the rank 1 groups were classified *via* their types. More precisely, it is not provable in ZFC that the class of torsion-free groups of rank 2 is classifiable in terms of recursive functions (Melles). For more details, see papers especially by G. Hjorth, A.S. Kechris, G. Melles, and S. Thomas. The interesting survey by Thomas [1] is highly recommended.

Exercises

(1) (Arnold–Lady) Every $C \in \operatorname{add}(A)$ is isomorphic to a direct sum of copies of the finite rank torsion-free group A if and only if each finitely generated projective right E-module is free.

(2) (Reid) An irreducible torsion-free group of finite rank is strongly indecomposable if and only if its quasi-endomorphism ring is a division ring.

(3) Define the *level* of a pure fully invariant subgroup of a finite rank torsion-free group A as follows: $\{0\}$ is of level 0, and a pure fully invariant subgroup of A is of level n if it contains properly a level $n-1$ fully invariant pure subgroup of A, but none of higher level. Show that for every $n \in \mathbb{N}$, there is a group A_n that has a level n pure fully invariant subgroup.

Problems to Chapter 12

PROBLEM 12.1. Let G, H be torsion-free groups of finite rank. Call H an *immediate extension* of G if $G \leq H$, and $G(\mathbf{t})/G^*(\mathbf{t}) \cong H(\mathbf{t})/H^*(\mathbf{t})$ for all types $\mathbf{t}$, where the isomorphism is induced by the injection map. Study immediate extensions.

PROBLEM 12.2. When are two torsion-free groups of rank $n \geq 3$ quasi-isomorphic if they have the same collection of rank $n-1$ pure subgroups?

PROBLEM 12.3. (Krylov–Tuganbaev). Characterize torsion-free groups in which every balanced subgroup is a summand.

PROBLEM 12.4. Characterize finite rank torsion-free groups for which quasi-isomorphism and near-isomorphism are equivalent.

PROBLEM 12.5. Investigate the category of torsion-free groups modulo finite rank groups. Hom is taken modulo homomorphisms with finite rank images.

PROBLEM 12.6. (Warfield). Find cancellation rings, i.e. rings R such that every group A with $\operatorname{End} A \cong \mathsf{R}$ has the cancellation property.

PROBLEM 12.7. When does $A \oplus B \cong A \oplus C$ imply $\operatorname{End} B \cong \operatorname{End} C$? ($A, B, C$ are torsion-free of finite rank.)

Chapter 13
Torsion-Free Groups of Infinite Rank

Abstract This chapter continues the theme of torsion-free groups, this time for the infinite rank case. There is no shortage of relevant results.

After a short discussion of direct decompositions of countable torsion-free groups, we enter the study of slender groups which display remarkable phenomena. We provide the main results on this class of groups. Much can be said about separable and vector groups. These seem theoretically close to completely decomposable groups, but are less tractable, and so more challenging. The measurable case is quite interesting.

The theory of torsion-free groups would not be satisfactorily dealt with without the discussion of the Whitehead problem. For a quarter of century this was the main open problem in abelian groups. We will give a detailed proof of its undecidability, mimicking Shelah's epoch-making solution. We show that the answers are different in the constructible universe, and in a model of set theory with Martin's Axiom and the denial of CH.

1 Direct Decompositions of Infinite Rank Groups

In our discussions of torsion-free groups of infinite rank, the first program is to study their direct decompositions. One can say quite a bit about their strange decompositions already in the countable rank case, and we will concentrate on this cardinality.

In contrast to the finite rank case, countable rank groups need not be direct sums of indecomposable groups. Some rather paradoxical behavior will be shown in the theorems that follow. In view of our experience with finite rank groups, it should not be a surprise that nearly everything conceivable can occur in the countable case as well.

Decompositions into Indecomposable Summands First we consider the case when the groups are better behaved, and admit decompositions into indecomposable summands. The first question we take up is concerned with the number of summands.

Theorem 1.1 (Corner [1]). *There exists a torsion-free group A that has two decompositions*

$$A = B \oplus C = \bigoplus_{n \in \mathbb{Z}} E_n,$$

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-319-19422-6_13

where B and C are indecomposable of rank $\aleph_0$, while the groups E_n are indecomposable of rank 2*.*

Proof. Let $\{p_n\}_n, \{q_n\}_n, \{r_n\}_n$ be three, pairwise disjoint, infinite sets of primes, indexed by $n \in \mathbb{Z}$. With independent elements b_n, c_n define

$$B = \langle p_n^{-\infty} b_n,\ q_n^{-1}(b_n + b_{n+1})\ \forall\ n \in \mathbb{Z}\rangle,$$
$$C = \langle p_n^{-\infty} c_n,\ r_n^{-1}(c_n + c_{n+1})\ \forall\ n \in \mathbb{Z}\rangle.$$

Both B and C are of rank $\aleph_0$ and indecomposable. Choose integers k_n (to be specified later), and with $u_n = (1 + k_n)b_n - k_n c_n$, $v_n = k_n b_n + (1 - k_n)c_n$ define

$$E_n = \langle p_n^{-\infty} u_n,\ p_{n+1}^{-\infty} v_{n+1},\ q_n^{-1} r_n^{-1}(u_n + v_{n+1})\rangle \quad (\forall\ n \in \mathbb{Z}).$$

The groups E_n are indecomposable of rank 2. It is straightforward to check that $\langle b_n\rangle_* \oplus \langle c_n\rangle_* = \langle u_n\rangle_* \oplus \langle v_n\rangle_*$. From

$$u_n + v_{n+1} = (b_n + b_{n+1}) + k_n(b_n - c_n) + (k_{n+1} - 1)(b_{n+1} - c_{n+1})$$

it is apparent that in order to have $q_n \mid u_n + v_{n+1}$ it is necessary to choose the k_n such that $k_n \equiv 0$ and $k_{n+1} \equiv 1 \mod q_n$. Similarly, from

$$u_n + v_{n+1} = (c_n + c_{n+1}) + (1 + k_n)(b_n - c_n) + k_{n+1}(b_{n+1} - c_{n+1})$$

it follows that for $r_n \mid u_n + v_{n+1}$ the k_n must satisfy $k_{n+1} \equiv 0$ and $k_n \equiv -1 \mod r_n$. Thus if we choose the numbers k_n such that

$$k_n \equiv \begin{cases} 0 & \mod q_n r_{n-1}, \\ 1 & \mod q_{n-1}, \\ -1 & \mod r_n, \end{cases}$$

then the E_n will be subgroups of $A = B \oplus C$. Moreover, if the k_n are (and elementary number theory tells us that they can be) chosen in this way, then $q_n \mid u_n + v_{n+1}$ will imply $q_n \mid b_n + b_{n+1}$, and $r_n \mid u_n + v_{n+1}$ will imply $r_n \mid c_n + c_{n+1}$ in the direct sum $\oplus_{n\in\mathbb{Z}} E_n$, proving that $A = \oplus_{n\in\mathbb{Z}} E_n$. □

We observe the interesting fact that indecomposable summands of infinite direct sums of finite rank groups need not be of finite rank.

The method of Theorem 5.1 in Chapter 12 can be extended, with minor modifications, to yield an analogous conclusion in the countable rank case.

Theorem 1.2. *There exist torsion-free groups G that have decompositions $G = A \oplus B = C \oplus D$, where B, C, D are indecomposable of rank $\aleph_0$, and A is completely decomposable of rank $\aleph_0$.*

Proof. We start with selecting different primes p, q, and p_n ($n \in \mathbb{N}$). For independent a_n, b_n define

$$A = \oplus_{n=1}^{\infty} \langle p_n^{-\infty} a_n \rangle \quad \text{and} \quad B = \langle p_n^{-\infty} b_n,\, p^{-1}q^{-1}(b_n - b_{n+1}) \;\forall\, n \in \mathbb{N} \rangle.$$

Then A is completely decomposable, and reasoning as usual we infer that B is indecomposable. We set $G = A \oplus B$. Next we choose integers s, t such that $ps - qt = 1$. With $c_n = pa_n + tb_n, d_n = qa_n + sb_n$, we set

$$C = \langle p_n^{-\infty} c_n,\, p^{-1}(c_n - c_{n+1}) \;\forall\, n \in \mathbb{N} \rangle,$$
$$D = \langle p_n^{-\infty} d_n,\, q^{-1}(d_n - d_{n+1}) \;\forall\, n \in \mathbb{N} \rangle.$$

These groups are indecomposable. The equality $A \oplus B = C \oplus D$ follows as in the proof of Theorem 5.1 in Chapter 12. □

Also, Theorem 5.2 in Chapter 12 has an analogue in the countable case.

Theorem 1.3 (Corner [1]). *There exists a torsion-free group A of countable rank such that, for every choice of a sequence $r_1, \dots, r_n, \dots$ of positive integers with infinitely many $r_n > 1$, there exist indecomposable groups A_n of rank r_n such that $A = \oplus_{n \in \mathbb{N}} A_n$.*

Proof. Choose different primes p, p_n, q_n ($n \in \mathbb{N}$), and independent elements u_n, x_n ($n \in \mathbb{N}$), and define

$$A = \oplus_{n \in \mathbb{N}} B_n \quad \text{where} \quad B_n = \langle p^{-\infty} u_n,\, p_n^{-\infty} x_n,\, q_n^{-1}(u_n + x_n) \rangle.$$

Given a sequence $\{r_n\}_n$, let m denote the number of 1's in this sequence ($0 \le m \le \aleph_0$), and let $s_1, \dots, s_n, \dots$ be the list of those r_n that are > 1. Finally, set $t_0 = 0, t_n = s_1 + \dots + s_n - n$. Clearly,

$$A = \oplus_{n \in \mathbb{N}} C_n \quad \text{where} \quad C_n = B_{t_{2n-2}+1} \oplus \dots \oplus B_{t_{2n}};$$

C_n is of rank $2s_{2n-1} + 2s_{2n} - 2$. By the proof of Theorem 5.2 in Chapter 12, C_n decomposes into the direct sum of an indecomposable group X_n of rank $s_{2n-1} + s_{2n} - 1$ (of the same form as A_i in the proof of the quoted theorem) and $s_{2n-1} + s_{2n} - 1$ groups of rank 1 (all of the form $\langle p^{-\infty} v_n \rangle$). We regroup the summands: we let Y_n be the direct sum of X_n and a rank 1 group, and keep m rank 1 groups separately from the X_n. Applying the quoted theorem once again, we rewrite Y_n as a direct sum of two indecomposable groups of ranks s_{2n-1} and s_{2n}. In this way, we get a decomposition of A into indecomposable summands exactly m of which are of rank 1, while the rest are of ranks $s_1, \dots, s_n, \dots$. □

This remarkable group admits continuously many non-isomorphic direct decompositions into indecomposable summands; in fact, there exist continuously many distinct infinite subsets of the positive integers ≥ 2, and owing to this theorem each subset gives rise to a direct decomposition into indecomposable summands whose ranks are exactly the elements in the selected subset. (We could have admitted repeated use of the same rank, but there was no need for it.)

The next theorem shows that a countable group may have continuously many non-isomorphic indecomposable summands even if it is a direct sum of only two indecomposable groups.

Theorem 1.4 (Fuchs [17]). *There exists a torsion-free group A of rank $\aleph_0$ such that*

$$A = B_j \oplus C_j \quad \textit{with} \quad B_j \cong C_j$$

holds for continuously many, pairwise non-isomorphic indecomposable groups B_j of countable rank.

Proof. Let P_i, Q_i ($i \in \mathbb{N}$) be pairwise disjoint, infinite sets of primes and p, q, r distinct odd primes not in their union. With independent elements a_i, b_i, c_i, d_i ($i \in \mathbb{N}$) we define

$$\begin{aligned} B =& \langle P_i^{-1}a_i, Q_i^{-1}b_i, p^{-1}(a_i + a_{i+1}), q^{-1}(b_i + b_{i+1}), r^{-1}(a_i + b_i) \ \forall\ i \in \mathbb{N}\rangle, \\ C =& \langle P_i^{-1}c_i, Q_i^{-1}d_i, p^{-1}(c_i + c_{i+1}), q^{-1}(d_i + d_{i+1}), r^{-1}(c_i + d_i) \ \forall\ i \in \mathbb{N}\rangle. \end{aligned}$$

Then B and C are isomorphic indecomposable groups, and set $A = B \oplus C$, For each i choose an integer k_i (to be specified later), and let

$$u_i = a_i, \ \ v_i = k_ib_i + (k_i^2 - 1)d_i, \ \ x_i = k_ia_i + c_i, \ \ y_i = b_i + k_id_i$$

for all $i \in \mathbb{N}$. The aim is to choose the integers k_i such that $A = U \oplus X$ where

$$\begin{aligned} U =& \langle P_i^{-1}u_i, Q_i^{-1}v_i, p^{-1}(u_i + u_{i+1}), q^{-1}(v_i + v_{i+1}), r^{-1}(u_i + k_iv_i) \ \forall\ i \in \mathbb{N}\rangle, \\ X =& \langle P_i^{-1}x_i, Q_i^{-1}y_i, p^{-1}(x_i + x_{i+1}), q^{-1}(y_i + y_{i+1}), r^{-1}(x_i + k_iy_i) \ \forall\ i \in \mathbb{N}\rangle. \end{aligned}$$

Investigating divisibility by p, q, and r, our usual technique shows that for $A = U \oplus X$ it is necessary and sufficient that the k_i be subject to the conditions:

$$k_i \equiv k_{i+1} \bmod pq \quad \text{and} \quad k_i^2 \equiv 1 \bmod r \quad \text{for all } i \in \mathbb{N}. \tag{13.1}$$

Pick an integer ℓ such that $\ell \equiv 1 \bmod pq$ and $\ell \equiv -1 \bmod r$. If, for each i, we choose $k_i = 1$ or $k_i = \ell$, then the sequence of such k_i will satisfy (13.1), so we get a decomposition $A = U \oplus X$ with indecomposable $U \cong X$.

We fix $k_1 = 1$, and show that if the sequence $k_2, \dots, k_i, \dots$ differs from the sequence $k'_2, \dots, k'_i, \dots$ (i.e., for at least one i, $k_i \neq k'_i$), then the corresponding groups U and U' are not isomorphic. Manifestly, divisibility consideration shows that an isomorphism $\phi : U \to U'$ must act on the generators as $u_i \mapsto \pm u'_i$ and $v_i \mapsto \pm v'_i$. From

$$p \mid u_i + u_{i+1} \mapsto \pm(u'_i \pm u'_{i+1}), \quad q \mid v_i + v_{i+1} \mapsto \pm(v'_i \pm v'_{i+1}),$$
$$r \mid u_1 + v_1 \mapsto \pm(u'_1 \pm v'_1)$$

we infer that the signs of u'_i and v'_i should be throughout the same, say $+$. Therefore, $u_i \mapsto u'_i, v_i \mapsto v'_i$ implies $r \mid u_i + k_i v_i \mapsto u'_i + k_i v'_i$ for every i. This is impossible if one of k_i, k'_i is 1 and the other is ℓ, because $\ell \not\equiv 1 \mod r$. To complete the proof, it only remains to observe that there are continuously many ways of choosing the sequence $k_2, \ldots, k_n, \ldots$. □

Superdecomposability So far we have dealt with groups which had direct decompositions with indecomposable summands. The next result establishes the existence of groups for any infinite cardinal without indecomposable summands $\neq 0$.

A group that has no indecomposable summands other than 0 will be called **superdecomposable**.

Theorem 1.5 (Corner [1]). *There exist superdecomposable torsion-free groups of any infinite rank.*

Proof. The proof relies on the existence theorem of groups with prescribed endomorphism rings; see Sect. 7 in Chapter 16. We therefore begin the proof with introducing our ring R.

Let κ denote an infinite cardinal, which will be viewed as the set of ordinals $< \kappa$, so it makes sense to form subsets. We define a monoid T to consist of the finite subsets of κ with multiplication defined as

$$\sigma \cdot \tau = \sigma \cup \tau$$

for all $\sigma, \tau \in T$. Evidently, the empty set is the identity of T.

Define $\mathsf{A} = \mathbb{Z}[T]$ as the semigroup algebra of T over $\mathbb{Z}$; thus, $\mathsf{A} = \bigoplus_{\sigma \in T} \mathbb{Z}\sigma$ with free additive group. The completion $\tilde{\mathsf{A}}$ of A in the $\mathbb{Z}$-adic topology is a $\mathbb{Z}$-algebra whose elements $a \neq 0$ may be written as countable sums $a = \sum_{i<\omega} r_i \sigma_i$ with $r_i \in \tilde{\mathbb{Z}}, \sigma_i \in T$, where for every $n \in \mathbb{N}$ almost all coefficients r_i are divisible by $n!$. By the *support* $[a]$ of a we mean the set $\{\sigma_i \mid r_i \neq 0\} \subseteq T$; this is always a countable subset of T.

Our claim is that every $\mathbb{Z}$-algebra R between the algebras A and $\tilde{\mathsf{A}}$ for any infinite cardinal κ is superdecomposable as an algebra. To prove this, consider an algebra summand $\mathsf{S} \neq 0$ of R with projection $e : \mathsf{R} \to \mathsf{S}$. It is enough to show that the idempotent e is not primitive. We distinguish two cases.

Case 1. If there is an ordinal $\alpha \in \kappa$ not contained in any $\sigma \in [e]$, then $\{\alpha\} \in \mathsf{R}$ is an idempotent satisfying $e\{\alpha\} \neq 0$. It also satisfies $e\{\alpha\} \neq e$, since $\tau \cup \alpha \notin [e]$ for any $\tau \in [e]$. The elements $e\{\alpha\}$ and $e - e\{\alpha\}$ are non-zero orthogonal idempotents in R with sum e, so e cannot be primitive.

Case 2. If no such ordinal α exists, then we must have $\kappa = \aleph_0$ and $\kappa = \bigcup[e]$. Write $e = \sum_{\sigma\in[e]} r_\sigma \sigma = \sum_{\sigma\in T} r_\sigma \sigma \in \mathsf{R}$ ($r_\sigma \in \tilde{\mathbb{Z}}$) with $r_\sigma = 0$ for all $\sigma \in T \setminus [e]$. If $e\{\alpha\} = e$, then $\sum_{\sigma\in T} r_\sigma(\sigma \cup \{\alpha\}) = \sum_{\sigma\in T} r_\sigma \sigma$. Pick $\tau \in [e]$ with $r_\tau \neq 0$. If $\alpha \notin \tau$, then the comparison of the coefficients of $\tau \cup \{\alpha\} \in T$ on both sides yields $r_\tau + r_{\tau\cup\{\alpha\}} = r_{\tau\cup\{\alpha\}}$. Hence $r_\tau = 0$, contradicting the choice of τ. Consequently, $e\{\alpha\} \neq e$ for all $\alpha \in \kappa$.

By way of contradiction, suppose that $e\{\alpha\} = 0$ for all $\alpha \in \kappa \setminus [\tau]$. Then $\sum_{\sigma\in T} r_\sigma(\sigma \cup \{\alpha\}) = 0$, where the coefficient of $\tau \cup \{\alpha\}$ is $r_\tau + r_{\tau\cup\{\alpha\}} = 0$. Thus $r_{\tau\cup\{\alpha\}} = -r_\tau$ for all $\alpha \in \kappa \setminus [\tau]$, which is impossible. It follows that there is always an $\alpha \in \kappa$ satisfying $e\{\alpha\} \neq 0$ (in addition to $e\{\alpha\} \neq e$). We have completed the proof that R has no primitive idempotent.

Since the additive group of R is a free group, by Theorem 7.1 in Chapter 16 there exist groups A of cardinalities $\geq \kappa$ with $\operatorname{End} A \cong \mathsf{R}$. Such an A is superdecomposable, since its endomorphism ring contains no primitive idempotents. $\square$

Groups Isomorphic to Certain Powers of Their Own It is easy to find torsion-free groups A that are isomorphic to $A \oplus A$, and hence to the direct sum of any finite or countably infinite copies of itself; e.g. infinite rank free or divisible groups. But it is a not an easy task to construct an A that is isomorphic to $A \oplus A \oplus A$, but not to $A \oplus A$.

Theorem 1.6 (Corner [3]). *For every positive integer m, there exists a countable torsion-free group A such that $A^{n_1} \cong A^{n_2}$ ($n_i \in \mathbb{N}$) if and only if $n_1 \equiv n_2 \mod m$.*

Proof. We remark at the outset that A has the stated property if and only if $A \cong A \oplus \cdots \oplus A$ with $m + 1$, but not for a fewer number (≥ 2) of summands. Indeed, the non-trivial "if" part follows by observing that one has merely to add a certain number of copies of A to both sides of an isomorphism, and cancel m summands A whenever there are more than m summands.

The construction of the group A is based on Theorem 3.3 in Chapter 16. Accordingly, first we define a ring R (on a countable free group) that will be the endomorphism ring of our group.

Let T be a monoid with 0, generated by symbols ρ_i, σ_i ($i = 0, 1, \ldots, m$) subject to the relations

$$\rho_j \sigma_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let S denote the semigroup ring of T over $\mathbb{Z}$ such that $0 \in T$ is identified with $0 \in \mathsf{S}$. The additive group of S is free where the different non-zero products of the ρ_j and σ_i form a basis. Every such product is of the form

$$\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}\rho_{j_\ell}\dots\rho_{j_2}\rho_{j_1}, \tag{13.2}$$

where $k, \ell \geq 0$, and the indices i, j are taken from the set $\{0, 1, \dots, m\}$. The principal ideal L generated by $\tau = 1 - \sigma_0\rho_0 - \dots - \sigma_m\rho_m$ in S is additively generated by the elements of the form

$$\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}\tau\rho_{j_\ell}\dots\rho_{j_2}\rho_{j_1} \tag{13.3}$$

since $\rho_i\tau = 0 = \tau\sigma_i$ for all i. It is a trivial observation that if the basis elements (13.2) with $k, \ell \geq 1$ and $i_k = j_\ell = 0$ are replaced by the elements of the form (13.3), then the new system remains an additive basis for S. Consequently, the additive group of the ring $\mathsf{R} = \mathsf{S}/\mathsf{L}$ is free (and countable).

Our next step is to define a group homomorphism $\psi : \mathsf{S} \to \mathbb{Z}(m)$ which vanishes on L and satisfies

$$\psi(1) = \mathbf{1}_m \quad \text{and} \quad \psi(\xi\eta) = \psi(\eta\xi) \quad \text{for all } \xi, \eta \in \mathsf{S}.$$

Let ψ map all the generators (13.2) to 0 except

$$\psi(\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}\rho_{i_k}\dots\rho_{i_2}\rho_{i_1}) = \mathbf{1}_m \quad \text{for all } k \in \mathbb{N}.$$

Thus ψ trivially vanishes on the elements (13.3) except when $k = \ell$ and $i_t = j_t$ for $t = 0, 1, \dots, k$ in which case its value is $1_m - (m+1)1_m$, which is likewise 0 in $\mathbb{Z}(m)$. It remains to verify $\psi(\xi\eta) = \psi(\eta\xi)$ when ξ is of the form (13.2), and η is ρ_i or σ_i; this is left to the reader as an exercise.

In view of Theorem 3.3 in Chapter 16, there is a countable torsion-free group A whose endomorphism ring is isomorphic to the ring R; for convenience, we assume $\operatorname{End} A = \mathsf{R}$. Then $\sigma_i\rho_i = \epsilon_i$ $(i = 0, 1, \dots, m)$ are pairwise orthogonal idempotents with sum 1, thus $A = \epsilon_0 A \oplus \dots \oplus \epsilon_m A$. But $\sigma_i\rho_i = \epsilon_i$ and $\rho_i\sigma_i = 1$ imply $\epsilon_i A \cong A$; hence A is isomorphic to the direct sum of $m+1$ copies of itself.

Now suppose A is isomorphic to the direct sum of $n+1$ copies of itself $(0 < n < m)$. In view of Lemma 1.6 in Chapter 16, this means the existence of $\alpha, \beta \in \operatorname{End} A$ such that $\alpha\beta = \mathbf{1}$ and $\beta\alpha = \epsilon_0 + \dots + \epsilon_n$. But $\mathbf{1}_m = \psi(\alpha\beta) = \psi(\beta\alpha) = \psi(\epsilon_0 + \dots + \epsilon_n) = (n+1)\,\mathbf{1}_m$ holds for no n if $0 < n < m$. □

★ **Notes.** A different kind of pathological decomposition was pointed out by Fuchs–Gräbe [1]. Given any (finite or infinite) set N of integers > 1, there is a countable torsion-free group A_N that has a decomposition into the direct sum of n non-zero indecomposable summands if and only if $n \in N$.

Our Theorem 1.4 can be extended to groups of higher cardinalities κ: there is a torsion-free group A of cardinality κ admitting 2^κ non-isomorphic direct decompositions $A = B_i \oplus C_i$ with indecomposable $B_i \cong C_i$. Also, there are groups A with the same kind of decompositions where all the B_i are isomorphic, and the C_i are non-isomorphic (Fuchs [17]). The proofs rely on the existence of rigid systems of cardinality κ. A consequence is that there exist connected compact abelian groups of cardinality 2^κ admitting 2^κ non-isomorphic closed summands which are algebraically all isomorphic.

Honestly, for uncountable torsion-free groups, anomaly in direct decompositions is less exciting, since their size makes it easier to use fancy constructions. However, there are certain limitations. For instance, Kaplansky's theorem that summands of direct sums of countable groups are again direct sums of countable groups makes it impossible to have $A = B \oplus C = \oplus_i A_i$ with indecomposable summands, where $|A| = \aleph_1$ and the A_i are of finite or countable ranks. A surprising example is a torsion-free group which is both the countable direct sum and the countable direct product of copies of the very same group; see Eda's Proposition 4.9.

Göbel–Ziegler [1] prove that, for every infinite cardinal λ, there exist groups of cardinality λ that decompose into the direct sum of κ non-zero summands for each $\kappa < \lambda$, but not to λ non-zero summands. For each infinite cardinal κ, Dugas–Göbel [5] establish the existence of torsion-free groups such that every non-zero summand decomposes into the direct sum of κ non-zero subgroups. Answering a question by Sabbagh [1] and Eklof–Shelah [1] proved the existence, for every $m \in \mathbb{N}$, of a torsion-free group A of cardinality $2^{\aleph_0}$ such that $A \oplus \mathbb{Z}^n \cong A$ holds if and only if m divides n. Countable superdecomposable groups were constructed by Krylov [2] and Benabdallah–Birtz [1], using different methods. Corner [8] proves the existence of $A = B \oplus C = D \oplus E$ of any size $\kappa \geq \aleph_0$, where A, B are indecomposable, and each summand of E decomposes into κ non-trivial summands.

Blass [1] investigates when a group A has the property that, for every subgroup B, the size of the divisible subgroup of A/B does not exceed $|B|$. He proves that it suffices to check this for countable B; then it holds for all $|B| < \aleph_\omega$. Also, it is consistent with ZFC that it fails for $\aleph_\omega$, but holds for many larger sizes.

Exercises

(1) Every summand of the group A in the proof of Theorem 1.5 is fully invariant.

(2) (Corner)

  (a) Let A be a direct sum of infinitely many indecomposable groups of finite rank. If almost all ranks are 1, then in every decomposition of A into indecomposable groups, almost all components must be of rank 1. [Hint: A is completely decomposable modulo a subgroup of finite rank.]

  (b) Conclude that in Theorem 1.3 the condition that infinitely many r_i be > 1 is relevant.

  (c) Part (a) fails to hold if the components are not all of finite rank.

(3) (Corner) There are countable torsion-free groups A, B, C such that $A \cong B \oplus C$, $B \cong A \oplus C$, but A and B are not isomorphic. [Hint: choose $m = 2$ in Theorem 1.6.]

(4) (Corner) There exists a countable torsion-free group A such that A is isomorphic to the direct sum of any finite number of, but not to the direct sum of infinitely many copies of itself.

(5) If A is superdecomposable, then so are the groups $A^{(\kappa)}$ for every cardinal κ. [Hint: direct sums of groups in Theorem 1.5; prove this for a finite sum, a countable sum, and apply Theorem 2.5 in Chapter 2.]

(6) A torsion-free group of infinite rank is a subdirect product of copies of $\mathbb{Q}$.
(7) (Reid) Every torsion-free group of infinite rank is a sum of two free groups.

2 Slender Groups

A remarkable class of torsion-free groups was discovered by J. Łoś in the 1950s: the class of slender groups. Without further ado, we embark on their very attractive theory.

Let P denote the direct product of a countable set of infinite cyclic groups, and S its subgroup that is the direct sum of the same groups:

$$P = \prod_{n=1}^{\infty} \langle e_n \rangle, \quad S = \bigoplus_{n=1}^{\infty} \langle e_n \rangle, \quad \text{where } \langle e_n \rangle \cong \mathbb{Z}.$$

The elements of P can be written as formal infinite sums: $x = \sum_{n=1}^{\infty} k_n e_n$, or as infinite vectors, like $x = (k_1 e_1, \ldots, k_n e_n, \ldots)$, or else simply as $x = (k_1, \ldots, k_n, \ldots)$, where $k_n \in \mathbb{Z}$.

Slenderness A group G is called **slender** if, for every homomorphism $\eta : P \to G$, we have $\eta e_n = 0$ for almost all indices n. From this definition it is evident that the following hold true:

(a) *Subgroups of slender groups are slender.*
(b) *The group P is not slender.*
(c) *No injective group $\neq 0$ is slender, so slender groups are reduced.*
(d) *The group J_p of p-adic integers, and more generally, an algebraically compact group $\neq 0$, is not slender.* In fact, the free group S can be mapped into J_p by a homomorphism ϕ such that $\phi e_n \neq 0$ for all n. Since S is pure in P, and J_p is algebraically compact, ϕ extends to a map $\eta \colon P \to J_p$ with $\eta e_n \neq 0$ for all n.
(e) *Slender groups are torsion-free.* The argument in (d) works for $\mathbb{Z}(p)$ (for any prime p), since it is algebraically compact. A slender group cannot contain a non-slender $\mathbb{Z}(p)$, so it must be torsion-free.

The following lemma is a consequence of our remarks.

Lemma 2.1. *If $\eta : P \to G$ with G slender, then $\eta S = 0$ implies $\eta = 0$. Homomorphic image of P in a slender group is a finitely generated free group.*

Proof. If η is as stated, then $\operatorname{Im} \eta$ is an epic image of the algebraically compact group P/S, so it is a cotorsion group. A torsion-free cotorsion group is algebraically compact. No such group can be a subgroup of G, hence $\operatorname{Im} \eta = 0$. The second claim is a simple corollary to the first, using the definition of slenderness. $\square$

A brief analysis of the preceding lemma makes it apparent that every homomorphism η of P into a slender group G is completely determined by its restriction $\eta \restriction S$. As a consequence, we can write

$$\eta\left(\sum_{n=1}^{\infty} k_n e_n\right) = \sum_{n=1}^{\infty} k_n(\eta e_n) \in G,$$

where in the last sum almost all terms are 0.

An important observation is contained in the following simple lemma.

Lemma 2.2. *The class of slender groups is closed under taking extensions.*

Proof. Let G be an extension of the slender group A by the slender group B, and let $\eta : P \to G$. Only finitely many ηe_n are contained in A, and the same holds for $G/A \cong B$ and the images of η followed by the canonical map. Therefore, almost all $\eta e_n = 0$, and G is slender as well. □

One can exhibit a large number of examples of slender groups by making use of the following two lemmas.

Lemma 2.3 (Sąsiada [2]). *A torsion-free group of cardinality less than the continuum is slender if and only if it is reduced.*

Proof. Because of (c), it suffices to verify the "if" part of the statement. Let G be reduced torsion-free of cardinality $< 2^{\aleph_0}$, and suppose $\eta : P \to G$ is a homomorphism such that $\eta e_n \neq 0$ for infinitely many indices n. Omitting the groups $\langle e_n \rangle$ with $\eta e_n = 0$, we may assume $\eta e_n \neq 0$ for all n. Being reduced means $\cap_{k \in \mathbb{N}} kG = 0$, thus we can find an increasing sequence $1 = k_1 < \cdots < k_n < \cdots$ of integers such that $\eta(k_n! e_n) \notin k_{n+1}G$ for each n. Consider the set of all $(r_1, \dots, r_n, \dots) \in P$ where $r_n = 0$ or $r_n = k_n!$; this set is of the power of the continuum, therefore there exist at least two elements in this set which have under η the same image in G. Their difference is of the form $a = (s_1, \dots, s_n, \dots)$ where $s_n = 0$ or $\pm k_n!$, and clearly $\eta a = 0$. We claim that this is impossible. Indeed, if j is the first index with $s_j \neq 0$, then $\eta(s_j e_j) \notin k_{j+1}G$, but

$$\eta(s_j e_j) = -\eta(0, \dots, 0, s_{j+1}, \dots) \in k_{j+1}G,$$

since on the right side every coordinate is divisible by k_{j+1}. □

Lemma 2.4 (Fuchs [AG]). *Direct sums of slender groups are slender.*

Proof. The claim is obvious for finite direct sums. Let G_i $(i \in I)$ be slender groups, and $G = \oplus_{i \in I} G_i$ with coordinate projections $\pi_i : G \to G_i$. Given $\phi : P \to G$, Lemma 2.1 implies that each $\pi_i \eta P$ is a finitely generated free subgroup of G_i, and $\operatorname{Im} \eta \leq \oplus_{i \in I} \pi_i \eta P$. Here but a finite number of $\pi_i \eta P$ can be different from 0, since otherwise P had a forbidden homomorphism into a countable free group, contradicting Lemma 2.3. Therefore, almost all $\pi_i \eta = 0$, and η may be viewed as a map from P into the direct sum of finitely many G_i, say, into the slender group $G_1 \oplus \cdots \oplus G_m$. Hence almost all of $\eta e_n (n \in \mathbb{N})$ vanish, which means that the direct sum G is slender. □

Example 2.5. The following groups are slender: $\mathbb{Z}$, and more generally, every proper subgroup of $\mathbb{Q}$, and direct sums of these groups: the reduced completely decomposable groups. Also, direct sums of any number of reduced countable torsion-free groups A_i. On the other hand, $\mathbb{Q}, P$ are not slender.

Monotone Subgroups Though the group P is not slender, it contains a lot of uncountable slender subgroups, as is shown by the next theorem.

A subgroup M of $P = \prod_{n\in\mathbb{N}}\langle e_n\rangle$ is called a **monotone** subgroup (Specker [1]) if it satisfies:

(i) $\sum_{n\in\mathbb{N}} k_n e_n \in M$ implies that also $\sum_{n\in\mathbb{N}} m_n e_n \in M$ where $m_n = \max(1, |k_1|, \dots, |k_n|)$;
(ii) $\sum_{n\in\mathbb{N}} k_n e_n \in M$ and $|\ell_n| \le |k_n|$ $(n \in \mathbb{N})$ imply $\sum_{n\in\mathbb{N}} \ell_n e_n \in M$.

Example 2.6.

(a) Every monotone subgroup contains the smallest monotone subgroup B consisting of all $\sum_{n\in\mathbb{N}} k_n e_n$ with $|k_n|$ bounded.
(b) The infinite sums $\sum_{n\in\mathbb{N}} k_n e_n$, for which there is a $c \in \mathbb{N}$ such that the coefficients satisfy $|k_n| \le cn!$, form a monotone subgroup.

Interestingly, monotone subgroups are slender with a single exception: the obvious one.

Theorem 2.7 (G. Reid [1]). *All monotone subgroups of the product $P = \prod_{n\in\mathbb{N}}\langle e_n\rangle$ are slender, with the exception of P itself.*

Proof. Let M be a monotone subgroup of P, properly contained in P, and assume there is a homomorphism $\psi : P \to M$ such that $\psi e_n \ne 0$ for infinitely many indices n. We may assume that $\psi e_n \ne 0$ for all n. Infinite cyclic groups are slender, so for each k, there are but finitely many indices n for which the coordinate of ψe_n in $\langle e_k\rangle$ is non-zero. In view of this, we may even drop more summands from P to have $\psi e_n = (0, \dots, 0, t_{n,i_n}, t_{n,i_n+1}, \dots)$ with $t_{n,i_n} \ne 0$ for a strictly increasing sequence $i_1 < \cdots < i_n < \cdots$ of indices. Pick an $x = (\ell_1, \dots, \ell_n, \dots) \in P \setminus M$ with $1 < \ell_1 \le \cdots \le \ell_n \le \cdots$, and set $y = \sum_{n\in\mathbb{N}} s_n e_n \in P$ by letting $s_n = \pm\ell_{i_n}$ where the sign is to be chosen so as *not* to have $s_1 t_{1,i_n} + \cdots + s_{n-1} t_{n-1,i_n}$ and $s_n t_{n,i_n}$ opposite signs. As ψ is determined by the ψe_n, we conclude that $\psi y = (r_1, r_2, \dots, r_n, \dots)$ where $r_n = s_1 t_{1n} + \cdots + s_{n-1} t_{n-1,n} + s_n t_{n,n}$. But $|r_{i_n}| \ge |s_n t_{n,i_n}| \ge \ell_{i_n}$ and $x \notin M$ imply $\psi y \notin M$, a contradiction. Hence M is slender, indeed. $\square$

Main Theorem by Łoś The results so far are still incomplete inasmuch as we lack information on homomorphisms of uncountable products into slender groups. Strangely enough, measurable cardinals enter into the picture.

A main result on slender groups is the Łoś–Eda theorem. First we prove Łoś' theorem for non-measurable cardinals, and then we move to the general case.

For an index set I of cardinality κ, we will use the *ad hoc* notations: $P_\kappa = \prod_{i\in I} A_i$ and $S_\kappa = \oplus_{i\in I} A_i$ for the direct product and the direct sum, respectively, of arbitrary non-zero torsion-free groups A_i $(i \in I)$.

Theorem 2.8 (Łoś). *Let G be a slender group, and $\eta\colon P_\kappa \to G$. Then*

(i) $\eta A_i = 0$ *for almost all i,*
(ii) *if κ is not measurable and $\eta S_\kappa = 0$, then $\eta = 0$.*

Proof.

(i) Suppose, by way of contradiction, that $\eta A_i \neq 0$ holds for infinitely many indices, say, for $i = i_1, \ldots, i_n, \ldots$. Then choosing $e_n \in A_{i_n}$ with $\eta e_n \neq 0$, the restriction of η to $P' = \prod_{n=1}^{\infty} \langle e_n \rangle (\leq P_\kappa)$ would be a forbidden homomorphism into a slender group.
(ii) This is also an indirect proof, but more sophisticated. Suppose that $\eta S_\kappa = 0$, but there exists an $a \in P_\kappa$ with $\eta a \neq 0$. We introduce a G-valued "measure" ν on the subsets of I, by using the selected element a. For a subset J of I, we set $\nu(J) = \eta a_J$, where a_J has the same coordinates as a for indices $j \in J$, and 0 coordinates elsewhere. It is clear that for pairwise disjoint subsets J, K of I, we have $\nu(J \cup K) = \eta(a_{J\cup K}) = \eta(a_J + a_K) = \nu(J) + \nu(K)$, so that ν is additive. In order to convince ourselves that it is moreover countably additive, let $J_1, \ldots, J_n, \ldots$ be a countable set of pairwise disjoint subsets of I, and J_0 the complement of their union J in I. Then $P^* = \prod_{k=0}^{\infty} \langle a_{J_k} \rangle$ is a subgroup of P_κ. Applying the definition of slenderness to $\eta \restriction P^* : P^* \to G$, we conclude that almost all of $\eta a_{J_k} = \nu(J_k) = 0$. This shows that ν is countably additive. An appeal to Lemma 6.4 in Chapter 2 shows that I is measurable, completing the proof. $\square$

If P_κ is equipped with the product topology (keeping the components discrete), then condition (i) in Theorem 2.8 amounts to that every homomorphism of P_κ into a (discrete) slender group is continuous. We should point out that (ii) is the best possible result in the sense that for measurable cardinals κ, $\eta S_\kappa = 0$ no longer implies $\eta = 0$. This is illustrated by the following example, generalizing Łoś' original example.

Example 2.9. For a measurable index set I, there are non-trivial homomorphisms η of $P^* = \mathbb{Z}^I$ into any torsion-free group $G \neq 0$ such that $\eta S^* = 0$ where $S^* = \mathbb{Z}^{(I)}$. To prove this, we choose any $0 \neq g \in G$. Let μ be a $\{0,1\}$-valued measure on I, and for $a = (\ldots, k_i, \ldots) \in P^*$ define pairwise disjoint subsets $X_n(a) \subseteq I$ $(n \in \mathbb{Z})$ via $X_n(a) = \{i \in I \mid k_i = n\}$. Then the union of these subsets is I, hence exactly one of them, say, $X_m(a)$ has measure 1, and the rest have measure 0. We then set $\eta a = mg$, thus $\eta P^* \neq 0$. From the properties of μ it is readily checked that η preserves addition, and $\eta S^* = 0$.

At this point let us pause and list a couple of corollaries to the last theorem.

Corollary 2.10. *Let G be a slender group, and A_i $(i \in I)$ torsion-free groups. If the index set I is non-measurable, then there is a natural isomorphism*

$$\mathrm{Hom}\Big(\prod\nolimits_{i\in I} A_i, G\Big) \cong \oplus_{i\in I} \mathrm{Hom}(A_i, G).$$

Proof. From Theorem 2.8 we know that every element η of the left-hand side Hom is actually a map from a finite sum $A_{i_1} \oplus \cdots \oplus A_{i_k}$ into G (where the index set depends on η). Hence the claim is evident. $\square$

The following corollary exhibits an interesting duality between direct sums and products of infinite cyclic groups. It shows that direct sums and products of $\mathbb{Z}$ over a non-measurable index set are $\mathbb{Z}$-reflexive.

Corollary 2.11 (Zeeman [1]). *For a non-measurable set I,*

$$\operatorname{Hom}(\oplus_{i\in I}\mathbb{Z},\mathbb{Z}) \cong \prod_{i\in I}\mathbb{Z} \quad \textit{and} \quad \operatorname{Hom}\left(\prod_{i\in I}\mathbb{Z},\mathbb{Z}\right) \cong \oplus_{i\in I}\mathbb{Z}$$

are natural isomorphisms.

Proof. These isomorphisms are special cases of Theorem 1.7 in Chapter 7 and Corollary 2.10, respectively. $\square$

Corollary 2.12. *If I is a non-measurable set, then every summand of $\mathbb{Z}^I$ is likewise a product.*

Proof. If $\mathbb{Z}^I = A \oplus B$, then $\mathbb{Z}^{(I)} = \operatorname{Hom}(A,\mathbb{Z}) \oplus \operatorname{Hom}(B,\mathbb{Z})$, where the summands are free groups (Corollary 2.11). Hence $A \cong \operatorname{Hom}(\operatorname{Hom}(A,\mathbb{Z}),\mathbb{Z})$ is also a product. $\square$

Main Theorem by Eda Next we turn our attention to the generalization of Theorem 2.8 to arbitrary cardinalities. We will not give a detailed proof for the main result (Theorem 2.14), but prove a crucial lemma which will help to appreciate the originality and the cleverness of the idea involved.

Lemma 2.13 (Eda [1,2]). *Let G be a slender group, and A_i $(i \in I)$ a family of torsion-free groups, $|I| = \kappa$. For every homomorphism*

$$\eta\colon\ P_\kappa = \prod\nolimits_{i\in I} A_i \ \to G$$

there exists a finite number of $\aleph_1$-complete ultrafilters $\mathcal{U}_1,\dots,\mathcal{U}_k$ on κ such that for each $a \in P_\kappa$, $\operatorname{supp} a \notin \mathcal{U}_j$ for $j = 1,\dots,k$ implies $\eta(a) = 0$.

Proof. For $a \in P_\kappa$, and for a subset J of I define $a_J \in P_\kappa$ to have its ith coordinate equal to the ith coordinate a_i of a or to 0 according as $i \in J$ or not. Let P_J denote the set of all $a \in P_\kappa$ with $\operatorname{supp} a \subset J$.

Consider the set $S = \{J \subset I \mid \eta(P_J) \neq 0\}$. We claim that there are no infinitely many pairwise disjoint sets in S. In fact, if the sets $J_1,\dots,J_n,\dots$ were pairwise disjoint members in S and $\eta(a_{J_n}) \neq 0$ for each n, then η would induce a forbidden homomorphism $\prod_{n<\omega}\langle a_{J_n}\rangle \to G$.

We now restrict the set S to

$$S^* = \{J \subset I \mid \forall\ K \subset J \text{ exactly one of } \eta(P_K) \neq 0,\ \eta(P_{J\setminus K}) \neq 0 \text{ holds}\}.$$

For a fixed $J \in S^*$, define

$$\mathcal{U} = \{X \subset I \mid \eta(P_{X \cap J}) \neq 0\}.$$

Claim I. $\mathcal{U}$ is an ω_1-complete ultrafilter.

(a) The empty set does not belong to $\mathcal{U}$. Subsets of I containing a member of $\mathcal{U}$ also belong to $\mathcal{U}$.
(b) *$\mathcal{U}$ is a filter.* Suppose $X, Y \in \mathcal{U}$, but $X \cap Y \notin \mathcal{U}$; without loss of generality, we may assume that X, Y are subsets of J. Then for some $a \in P_\kappa$, $0 \neq \eta(a_X) = \eta(a_{X\cap Y}) + \eta(a_{X\setminus Y})$, where the first term vanishes by hypothesis. Hence $\eta(P_{X\setminus Y}) \neq 0$, and analogously, $\eta(P_{Y\setminus X}) \neq 0$. But this contradicts $J \in S^*$, because for the choice of $K = Y \setminus X$ both $\eta(P_K) \neq 0$ and $\eta(P_{J\setminus K}) \neq 0$ would hold.
(c) *$\mathcal{U}$ is an ultrafilter.* For, if $X \notin \mathcal{U}$ and $a \in P_J$ is such that $\eta(a) \neq 0$, then $0 \neq \eta(a) = \eta(a_{X\cap J}) + \eta(a_{J\setminus X}) = \eta(a_{J\setminus X})$, so $I \setminus X$ belongs to $\mathcal{U}$.
(d) *$\mathcal{U}$ is $\aleph_1$-complete.* Let the subsets $X_1, \dots, X_n, \dots$ of I belong to $\mathcal{U}$ such that their intersection is empty; here again, we may assume that they are all subsets of J, and moreover, $X_1 \supset \cdots \supset X_n \supset \cdots$. For each n, pick $a_n \in P_{X_n}$ such that $\eta(a_n) \neq 0$, and define a map $\phi : P \to G$ via $\phi(\dots, k_n, \dots) = \eta(\sum_{n<\omega} k_n a_n)$. The existence of such a ϕ contradicts the slenderness of G.

Next choose $\{J_1, \dots, J_k\}$, a maximal set of pairwise disjoint sets in S^*, and denote by $\mathcal{U}_j$ the ultrafilter defined by J_j for $j = 1, \dots, k$, as described above.

Claim II. If K denotes the complement of the set union $J_1 \cup \cdots \cup J_k$ in I, then $\eta(P_K) = 0$.

Assume the contrary, i.e. $K \in S$. By the maximal choice of the $J_j \in S^*$, K cannot belong to S^*, so there is a subset X_1 of K such that both $\eta(P_{X_1}) \neq 0$ and $\eta(P_{K\setminus X_1}) \neq 0$. Suppose that we have found pairwise disjoint subsets $X_1, \dots, X_n$ of K such that for each $i = 1, \dots, n$, both $\eta(P_{X_i}) \neq 0$ and $\eta(P_{K\setminus(X_1\cup\cdots\cup X_i)}) \neq 0$. To find an X_{n+1} with similar features disjoint from all of $X_1, \dots, X_n$, just iterate the same argument with K replaced by $K \setminus (X_1 \cup \cdots \cup X_n)$. If we choose for each n an a_{X_n} such that $\eta(a_{X_n}) \neq 0$, then η induces a forbidden homomorphism $\prod_{n<\omega} \langle a_{X_n} \rangle \to G$.

Hence it is clear that the $\mathcal{U}_j$ are as claimed. □

Now we state the theorem, and refer to the proof to Eklof–Mekler [ME], Section III.3.

Theorem 2.14 (Eda [1]). *Suppose that $\{A_i \mid i \in I\}$ is a family of torsion-free groups, and G is a slender group. There is a natural isomorphism*

$$\operatorname{Hom}\left(\prod A_i, G\right) \cong \oplus_{\mathcal{U}} \operatorname{Hom}\left(\left(\prod A_i\right)/\mathcal{U}, G\right),$$

where $\mathcal{U}$ runs over all $\aleph_1$-complete ultrafilters on the index set I. □

We refer to the combined Theorems 2.8 and 2.14 as the *Łoś-Eda theorem.*

★ **Notes.** The roots of the theory of slender groups are in Specker [1] where it is proved under CH that $\mathrm{Hom}(\mathbb{Z}^{\aleph_0}, \mathbb{Z}) \cong \mathbb{Z}^{(\aleph_0)}$, establishing the slenderness of $\mathbb{Z}$. (He was motivated by the search for the structure of the first cohomology group of an infinite complex.) Specker's result was extended by Ehrenfeucht–Łoś [1] and Zeeman [1]. The theory of slender groups is the creation of J. Łoś, his theory was published in Fuchs [AG]. Two main theorems, Theorems 2.8 and 3.5, are due to Łoś and Nunke [2], respectively.

Eda [1] proved that $\mathrm{Hom}(\mathbb{Z}^{\kappa}, \mathbb{Z})$ is a free group for all cardinals κ. It is an undecidable problem whether or not the group P could be replaced by one of its subgroups of smaller cardinality to use as a test group for slenderness. He also discussed extension of Theorem 2.8 to the Boolean powers of $\mathbb{Z}$, and to sheaves.

The theory of monotone subgroups was initiated by Specker [1]. Irwin–Snabb [1] discussed monotone subgroups defined by analytic functions with integral coefficients. Göbel–Wald [1] introduced M-slenderness (replacing P in the definition by a monotone subgroup M), and proved that a consequence of Martin's Axiom is that there are as many as $2^{2^{\aleph_0}}$ inequivalent notions of slenderness. They also proved that M/S is algebraically compact for every monotone M.

For $\mathbb{Z}$-reflexive groups, see Huber [2], and above all Eklof–Mekler [EM]. Göbel–Shelah [4] use Martin's Axiom to construct reflexive subgroups of the Baer–Specker group.

Exercises

(1) A torsion-free group G is slender if and only if, for all $\eta : P \to G$, $\mathrm{Im}\,\eta$ is finitely generated. [Hint: finitely generated groups are slender.]

(2) Show that if x_n $(n = 1, 2, \dots)$ are elements of P, then an infinite sum $x = \sum_{n=1}^{\infty} s_n x_n$ with $s_n \in \mathbb{Z}$ makes sense in P if and only if, for each positive integer i, almost all of $s_n x_n$ have vanishing ith coordinates.

(3) (Nunke) Give a different proof that J_p is not slender by showing that $\sum_{n=1}^{\infty} k_n e_n \mapsto \sum_{n=1}^{\infty} p^{n-1} k_n$ defines a non-zero homomorphism $P \to J_p$.

(4) (a) Prove Lemma 2.1 by selecting any $x_1 = (k_1, \dots, k_n, \dots) \in P \setminus S$ with $\eta x_1 \neq 0$, and showing that η induces a forbidden homomorphism of $\prod \langle x_n \rangle \to G$, where $x_n = x_1 - k_1 e_1 - \dots - k_{n-1} e_{n-1}$.
(b) P can be represented in many ways as a direct product of cyclic groups.

(5) (T. Yen) Let P_0 denote the subgroup of P for which P_0/S is the maximal divisible subgroup of P/S. Slenderness can be tested by using P_0 in place of P. [Hint: $(a_1, \dots, a_n, \dots) \mapsto (a_1, \dots, n! a_n, \dots)$ is a monic map $P \to P_0$.]

(6) (a) Let $P' = \mathbb{Z}^{\kappa}$ with a non-measurable cardinal κ. Show that every summand of P' is again a product of infinite cyclic groups.
(b) In any direct decomposition of P', there are but a finite number of non-zero summands. If $\kappa \geq \aleph_0$, then one of the summands is $\cong P'$.

(7) (Łoś) Let G_i $(i \in I)$ be slender groups, where I is a non-measurable infinite index set. Then $\prod_{i \in I} G_i$ has no proper summand containing $\oplus_{i \in I} G_i$.

(8) An infinite product of reduced countable torsion-free groups can never be a direct sum of countable groups.

(9) (Fuchs) Let G be a countable slender group, and M a monotone subgroup of P, containing some unbounded vectors. Each homomorphism $\eta : M \to G$ satisfies $\eta e_n = 0$ for almost all n, and η is determined by $\eta \restriction S$.

(10) (Specker) Let M_1 and M_2 be monotone subgroups of P, and $\psi : M_1 \to M_2$ a homomorphism such that $S \leq \operatorname{Im}\psi$. Then $M_1 \leq M_2$.
(11) (Specker) The set of non-isomorphic monotone subgroups of P has cardinality $2^{2^{\aleph_0}}$. [Hint: find continuously many elements in P such that none of them is contained in the monotone subgroup generated by the rest.]
(12) (Eklof–Mekler) Let G denote the subgroup of $\mathbb{Z}^\kappa$ consisting of elements with countable support. Show that $\operatorname{Hom}(G, \mathbb{Z}) \cong \mathbb{Z}^{(\kappa)}$. [Hint: every countable subset of G is contained in a subgroup $\cong \mathbb{Z}^{\aleph_0}$.]

3 Characterizations of Slender Groups

This section is devoted to the discussion of the characterizations of slenderness, both in topological terms and by the absence of certain subgroups. The center stage is now taken by topology as the proofs draw heavily on (rather elementary) topological arguments.

We always think of $P = \prod_{n=1}^{\infty} \langle e_n \rangle$ as being equipped with the product topology where each $\langle e_n \rangle$ carries the discrete topology. This is a metrizable linear topology, in which P is complete, and $S = \oplus_{n=1}^{\infty} \langle e_n \rangle$ is a dense subgroup of P. We observe right away that if C is any subgroup of P, then its completion $\breve{C}$ in the induced topology is the same as its closure in P. Now $\breve{C}$ consists of all $x \in P$ with the following property: for every $n \in \mathbb{N}$, there is $c_n \in C$ such that the first n coordinates of $x - c_n$ are 0.

Topological Results No further preparation is needed for the following topological characterization of slenderness.

Theorem 3.1 (Heinlein [1], De Marco–Orsatti [1]). *A reduced torsion-free group is slender if and only if it contains no subgroup that admits a complete non-discrete metrizable linear topology.*

Proof. To prove necessity, let G be a subgroup of the torsion-free group A, and U_n $(n < \omega)$ a strictly descending chain of subgroups that forms a base of neighborhoods of 0 in a complete non-discrete metrizable linear topology of G. Define a map $\eta : S \to G$ by setting $\eta(e_n) = g_n$, where g_n is any element of $U_n \setminus U_{n+1}$. The universal property of completions guarantees that η extends uniquely to a continuous homomorphism $\psi : P \to G$. Such a map contradicts slenderness, showing that G, and hence A, cannot be slender.

Conversely, suppose A is reduced, but not slender. Then there is a map $\eta \colon P \to A$ such that $\eta e_n \neq 0$ for all n. Thus A contains the subgroup $G = \operatorname{Im}\eta \cong P/K$ where $K = \operatorname{Ker}\eta$. If $K = 0$, then $G \cong P$, so G admits a stated kind of topology. Now, if $K \neq 0$, then its closure in P is its completion $\breve{K}$ in the topology inherited from P. If $\breve{K} = K$, then P/K is the epic image of a complete metrizable group, and as such it is complete in the induced topology. If $\breve{K} > K$, then by Proposition 6.8 in Chapter 9 $\breve{K}/K$ is cotorsion; it is moreover algebraically compact, since it is torsion-free.

Evidently, it is reduced; therefore, by Corollary 3.6 in Chapter 6 it must contain a copy of a J_p, which is obviously compact in its (metrizable) p-adic topology. $\square$

It is remarkable that slender groups can be characterized in an unusual way as groups not containing certain specific types of subgroups. This is Nunke's theorem that we are going to prove next. First, we examine more closely what the completions of $\mathbb{Z}$ are like in linear topologies.

Lemma 3.2. *The completion $\breve{\mathbb{Z}}$ of $\mathbb{Z}$ in any non-discrete, linear topology is a compact group. If $\breve{\mathbb{Z}}$ is torsion-free, then it contains a copy of J_p for some prime p.*

Proof. The open sets about 0 in any non-discrete linear topology of $\mathbb{Z}$ are necessarily finite index subgroups. Thus the completion $\breve{\mathbb{Z}}$ (as a closed subgroup in the direct product of finite groups) is necessarily compact. It is obviously reduced, and again we invoke (Corollary 3.6 in Chapter 6) to argue that it must contain a copy of some J_p whenever it is torsion-free. $\square$

Example 3.3.

(a) If we choose the subgroups $p^n\mathbb{Z}$ $(n < \omega)$ as a base of neighborhoods about 0, then $\tilde{\mathbb{Z}} \cong J_p$.
(b) If the subgroups $n\mathbb{Z}$ with square-free numbers n are chosen as a base of neighborhoods, then $\breve{\mathbb{Z}} \cong \prod_p \mathbb{Z}(p)$.

Lemma 3.4. *Let A be a reduced torsion-free group that is complete in some non-discrete metrizable linear topology. Then it contains a copy of P or a copy of J_p for some prime p.*

Proof. Suppose that A contains a cyclic group $C \cong \mathbb{Z}$ that is not discrete in the induced topology. Then it also contains its completion $\breve{C}$, and Lemma 3.2 implies that it must contain a copy of some J_p.

In the remaining case, all cyclic subgroups of A are discrete in the induced topology. Let $V_1 > \cdots > V_n > \cdots$ be a base of neighborhoods at 0. For any $0 \neq c_1 \in A$, $\langle c_1\rangle$ carries the discrete topology, hence there exists $n_1 \in \mathbb{N}$ such that $\langle c_1\rangle \cap V_{n_1} = 0$. Pick any $0 \neq c_2 \in V_{n_1}$, and then an $n_2 > n_1$ such that $\langle c_2\rangle \cap V_{n_2} = 0$. Continuing in the same fashion, we construct a sequence c_i $(i \in \mathbb{N})$ of elements in A, and a strictly descending chain V_{n_i} $(i \in \mathbb{N})$ of open subgroups. The c_i generate the direct sum $C = \oplus_{i\in\mathbb{N}} \langle c_i\rangle \cong S$, and a quick comparison of the topologies shows that this is even a topological isomorphism. It follows that the closure of C is its completion, which is evidently $\cong P$. $\square$

Nunke's Theorem on Slenderness The interest in slender groups was strongly influenced by the light shed by the following theorem which gives a most powerful characterization of slender groups.

Theorem 3.5 (Nunke [2]). *A group is slender exactly if it does not contain any subgroup isomorphic to $\mathbb{Q}, P, J_p$, or $\mathbb{Z}(p)$ for any prime p.*

Proof. The necessity being obvious (the indicated groups are not slender), assume that no subgroup of the torsion-free group G is isomorphic to $\mathbb{Q}, P$ or J_p. Thus G is reduced, so by Lemma 3.4 G does not contain any subgroup that is complete in any metrizable linear topology. An appeal to Theorem 3.1 completes the proof. $\square$

The argument on the preceding theorem rested on the homomorphic images of P in torsion-free groups. Let us find out in general what epic images of P might have. (Actually, the insight obtained from the next proposition was the key that eventually led to Theorem 3.5.)

Proposition 3.6 (Nunke [2]). *Every epic image of P is the direct sum of a cotorsion group, and a direct product of at most countably many infinite cyclic groups.*

Proof. For a group A, we use the notation $A^* = \mathrm{Hom}(A, \mathbb{Z})$. This Hom is always a subgroup of a product of infinite cyclic groups. In particular, for a free group S of countable rank, we have $S^* \cong P$ and $P^* \cong S$.

If A is a subgroup of P, then the exact sequence $0 \to A \to P \to P/A \to 0$ induces the exact sequence $0 \to (P/A)^* \to P^* \to A^*$ where P^* is a countable free group. Let B denote the image of P^* in A^*. As A^* (and hence B) is contained in a product of infinite cyclic groups, and B is countable, by Theorem 8.2 in Chapter 3, B is a free group, consequently, $P^* \cong B \oplus (P/A)^*$. Hence $P \cong P^{**} \cong B^* \oplus (P/A)^{**}$, where both summands are products of cyclic groups, and $A \leq A^{**} \leq B^*$. Thus $P/A \cong B^*/A \oplus (P/A)^{**}$, and we want to show that B^*/A is cotorsion.

First, $(B^*/A)^* = 0$, because the canonical map $(P/A)^* \to (P/A)^{***}$ is an isomorphism (Corollary 2.11), and the projection onto the second summand in $(P/A)^* \cong (B^*/A)^* \oplus (P/A)^{***}$ is the natural map.

As $B^* \cong P$, in order to complete the proof, it suffices to show that P/A is cotorsion whenever $(P/A)^* = 0$. Let $\breve{A}$ denote the closure of A in P, where P is equipped with the product topology. Then $\breve{A}$ is complete in the induced topology, so Proposition 6.8 in Chapter 9 implies that $\breve{A}/A$ is cotorsion. $(P/\breve{A})^* = 0$ ensures that, for every neighborhood U of 0 in P, $U \cap \breve{A}$ is essential in U. Thus the groups $(U + \breve{A})/\breve{A}$ form a base of neighborhoods of 0 in $P/\breve{A}$. The factor group $P/\breve{A}$ is also complete in the induced topology, and it follows that $P/\breve{A}$ is the inverse limit of the finite groups $P/(U + \breve{A})$, and as such it is a compact group. To conclude, it remains to observe that an extension P/A of a cotorsion group $\breve{A}/A$ by a cotorsion group $P/\breve{A}$ is itself cotorsion. □

By making use of this theorem, we can retrieve our earlier result (Theorem 3.5) as an immediate corollary. It is also easy to derive Lemmas 2.3 and 2.4. As another application of Nunke's theorem we prove the theorem below.

Chains with Slender Factors By Lemma 2.2, the class of slender groups is closed under finite chains, i.e. if all factors A_{i+1}/A_i of a chain $0 = A_0 < A_1 < \cdots < A_k$ ($k \in \mathbb{N}$) are slender, then so is A_k. We now want to replace finite chains by smooth chains with no restriction on length.

Theorem 3.7 (Fuchs–Göbel [1]). *Let*

$$0 = A_0 < A_1 < \cdots < A_\sigma < \cdots \quad (\sigma < \tau) \tag{13.4}$$

be a smooth chain of groups where all the factor groups $A_{\sigma+1}/A_\sigma$ $(\sigma < \tau)$ are slender. Then the union $A = \cup_{\sigma<\tau} A_\sigma$ of the chain is likewise slender.

Proof. We first observe that all the groups A_σ are pure in A because of the torsion-freeness of the factor groups. The following proof is by induction on the length τ of the chain.

Case 1: $\tau = n$ is an integer. We refer to the observation just before the theorem.

Case 2: $\tau = \omega$, i.e. we have a chain $0 = A_0 < A_1 < \cdots < A_k < \cdots$ $(k < \omega)$, and $A = \cup_{k<\omega} A_k$. Case 1 implies that all the A_k are slender groups. It suffices to show that A cannot contain any copy of the groups $\mathbb{Q}, J_p$, or P.

(a) Since none of the subgroups A_k contains a copy of $\mathbb{Q}$, neither can A. By the same token, A/A_k contains no copy of $\mathbb{Q}$.

(b) Suppose, by way of contradiction, that A contains a copy of J_p for some prime p. J_p is the completion in the p-adic topology of the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$. Let $a \in A$ denote the element corresponding to $1 \in \mathbb{Z}_{(p)}$. There is an index k such that $a \in A_{k+1} \setminus A_k$, thus $\mathbb{Z}_{(p)}$ embeds in A_{k+1}/A_k as a subgroup. As $J_p/\mathbb{Z}_{(p)}$ is divisible, and by (a) A/A_{k+1} fails to contain any copy of $\mathbb{Q}$, it follows that J_p is contained already in A_{k+1}/A_k. But this is impossible, since by hypothesis this factor group is slender. Similarly, A/A_k contains no copy of J_p for any prime p.

Combining parts (a) and (b), we conclude that no factor A/A_k can contain any non-zero cotorsion group.

(c) Again, arguing by contradiction, assume that there is a monomorphism $\eta : P = \prod_{n\in\mathbb{N}} \langle e_n \rangle \to A$. It suffices to consider the case where no A_k contains infinitely many of the ηe_n. Now η followed by the canonical maps $A \to A/A_k$ yields a homomorphism

$$\bar{\eta} : P \to \prod_{k<\omega} A/A_k.$$

For each $x \in P$, there exists an index $k < \omega$ such that $\eta x \in A_k$, which means that necessarily $\operatorname{Im} \bar{\eta} \leq \oplus_{k<\omega} A/A_k$. In view of Theorem 6.5 in Chapter 2 there must exist an index $m < \omega$ such that $\operatorname{Im} \bar{\eta} \leq A/A_1 \oplus \cdots \oplus A/A_m$. Consequently, $\operatorname{Im} \eta \leq A_{m+1}$, in contradiction to the slenderness of A_{m+1}.

The proofs in (a)–(c) establish the claim for chains of length ω.

Case 3: $\tau > \omega$. The arguments above in (a) and (b) apply to arbitrary τ, thus it remains to settle the case about P. So assume that for chains of smaller lengths the claim holds, and by way of contradiction, suppose that A contains a copy of P.

If the length τ is a successor ordinal, $\tau = \sigma + 1$, then we invoke Lemma 2.2. If τ is a limit ordinal, and if all the generators $e_n \in P$ are contained in A_σ for some ordinal $\sigma < \tau$, then the proof in Case 2 should convince us that then also $P \leq A_\sigma$. If τ is a limit ordinal and cf $\tau \neq \omega$, then such a σ (a successor ordinal or a limit ordinal of cofinality ω) must exist. In this case the induction hypothesis implies that it is impossible that a copy of P is contained in A.

Finally, if τ is cofinal with ω, and if there does not exist any index $\sigma < \tau$ with $e_n \in A_\sigma$ for all n, then we select a countable subchain $0 = A_0 \leq A_{\alpha_1} \leq \cdots \leq A_{\alpha_k} \leq \cdots$ with union A such that we have $e_n \in \bigcup_{k<\omega} A_{\alpha_k}$ for all $n < \omega$. The induction hypothesis guarantees that all the factor groups $A_{\alpha_k}/A_{\alpha_{k-1}}$ are slender, so an appeal to Case 2 leads us to the desired conclusion: the union A must be slender. □

Cotorsion-Free Groups We close this section with drawing attention to another class of groups which is more general than slender groups, and which earned importance in the construction of groups with prescribed endomorphism rings.

A group A is said to be **cotorsion-free** if it does not contain any cotorsion subgroup $\neq 0$. These groups can be characterized as follows.

Theorem 3.8. *The following are equivalent for a group A:*

(i) *A is cotorsion-free;*
(ii) *A contains no copy of $\mathbb{Q}, J_p$ or $\mathbb{Z}(p)$ for any prime p;*
(iii) $\mathrm{Hom}(\tilde{\mathbb{Z}}, A) = 0$.

Proof. A cotorsion-free group contains none of the listed groups as a subgroup, hence (i) and (ii) are equivalent by Theorem 3.5. The completion $\tilde{\mathbb{Z}}$ is a compact group, all of its homomorphic images are cotorsion, and each of the groups in (ii) is an epic image. Thus (iii) is equivalent to (ii). □

★ **Notes.** Actually, topology is more profoundly tied to slenderness than it appears from the results above. Namely, if we consider, for an arbitrary index set I, the group $\mathbb{Z}^I$ equipped with the product topology, then we can ask when every homomorphism $\mathbb{Z}^I \to \mathbb{Z}$ is continuous. The answer is not difficult: this is the case exactly if I is non-measurable. (For measurable I, a non-continuous map is exhibited in Example 2.9).

This idea is used by Kruchkov [1] who selects a topological group T, and considers the class of groups G with the property that every group homomorphism $\phi\colon T \to G$ is continuous. The choice $T = \mathbb{Z}^{\aleph_0}$ (the Baer-Specker group) with the product topology on discrete components yields the class of slender groups. The thin p-groups are obtained if T is chosen as the torsion subgroup of $\prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$ furnished with the product topology. The groups G are characterized for special choices of T.

Eda has several interesting papers generalizing almost all aspects of slender groups. For instance, he proved the following generalization of Nunke's theorem: for an arbitrary cardinal κ, every closed subgroup of $\prod_\kappa^{\aleph_0} \mathbb{Z}$ (consisting of vectors with countable support in the direct product) is isomorphic to $\prod_\mu^{\aleph_0} \mathbb{Z}$ for some cardinal μ.

Fuchs–Göbel [1] also show that the *class* of slender groups cannot be obtained from any *set* of slender groups by taking repeatedly unions of chains of the mentioned kind (including direct sums) and passing to subgroups.

As already pointed out above, rings with cotorsion-free additive groups are important in constructing groups with prescribed endomorphism rings. See Sect. 7 in Chapter 16.

Exercises

(1) An $\aleph_1$-free group is slender if and only if it has no subgroup $\cong P$.
(2) (Nunke) Equip P with the product topology where the components $\langle e_n \rangle$ carry the discrete topology. Then: (a) all the endomorphisms of P are continuous; (b) the topology of P is independent of the way P is represented as a direct product of infinite cyclic groups.
(3) If G is a dense subgroup of P, then P/G is cotorsion.
(4) A subgroup of P that is closed in the product topology of P is a product of infinite cyclic groups. [Hint: Proof of Proposition 3.6.]
(5) Suppose X is a product in P. Then P/X is a product of finite and infinite cyclic groups.
(6) Show that J_p does not contain any subgroup $\cong P$.
(7) Let A be a torsion-free group that contains a slender subgroup G such that A/G is a reduced torsion group. Show that A is slender. [Hint: $\operatorname{Im}\phi\eta$ is reduced torsion cotorsion (Proposition 3.6), so bounded; $\eta : P \to A$, $\phi : A \to A/G$.]
(8) A countable group is cotorsion-free if and only if it is slender.

4 Separable Groups

Our next topic is a class of groups that is more general than completely decomposable groups, but behave "locally" as if they were completely decomposable.

A torsion-free group A is called **separable** (Baer [6]) if every finite subset of elements of A is contained in a completely decomposable summand of A. Clearly, this summand may then be assumed to be of finite rank. Recall: if A has a rank 1 summand of type $\mathbf{t}$, then we say $\mathbf{t}$ is an extractable type in A.

Example 4.1. The proof of Theorem 8.2 in Chapter 3 shows that the Baer-Specker group $\mathbb{Z}^{\aleph_0}$ is separable, but not completely decomposable. The same holds for the generalized Baer-Specker group $\mathbb{Z}^{\kappa}$ for any infinite cardinal κ.

Properties of Separability Some key properties of separability are listed below.

(A) *A group is separable if and only if its reduced part is separable.*
(B) *Direct sums of separable groups are separable* (but their direct products are not necessarily).
(C) *If A is separable, then for every type $\mathbf{t}$, $A(\mathbf{t})$ and $A^*(\mathbf{t})$ are pure, separable subgroups of A.*
(D) *If A is separable, and $\mathbf{t}$ is any type, then $A(\mathbf{t})/A^*(\mathbf{t})$ is separable.* This will be a consequence of the following lemma.

Lemma 4.2.

(i) *Fully invariant subgroups of a separable group are separable.*
(ii) *Factor groups of a separable group modulo fully invariant pure subgroups are separable.*

Proof.

(i) Let C be a fully invariant subgroup in the separable group A, and let $c_1, \dots, c_k \in C$. By definition, there is a decomposition $A = A_1 \oplus \cdots \oplus A_n \oplus A'$ where A_i are of rank 1 and $c_1, \dots, c_k \in A_1 \oplus \cdots \oplus A_n$. By full invariance, $C = (C \cap A_1) \oplus \cdots \oplus (C \cap A_n) \oplus (C \cap A')$ where each $C \cap A_i$ is 0 or of rank 1, and the c_j are contained in their direct sum.

(ii) Now C is pure, so A/C is torsion-free. Given $a_1 + C, \dots, a_k + C$, there is a direct decomposition $A = A_1 \oplus \cdots \oplus A_n \oplus A'$ where A_i are of rank 1 and $a_1, \dots, a_k \in A_1 \oplus \cdots \oplus A_n$. Then $A/C = (A_1 + C)/C \oplus \cdots \oplus (A_n + C)/C \oplus (A' + C)/C$ where the first n summands are either 0 or of rank 1, and their direct sum contains the given cosets mod C. □

Main Results on Separable Groups The next result tells us that only the uncountable separable groups yield something new.

Theorem 4.3 (Baer [6]). *A countable separable torsion-free group is completely decomposable.*

Proof. Let A be a countable separable group, and $\{a_1, \dots, a_n, \dots\}$ a generating system of A. There is a finite rank completely decomposable summand G_1 of A that contains a_1. Assume that we have an ascending chain $G_1 \leq \cdots \leq G_n$ of finite rank completely decomposable summands of A such that $a_1, \dots, a_i \in G_i$ for $i = 1, \dots, n$. There is then a completely decomposable summand G_{n+1} of A which contains both (a maximal independent set of) G_n and a_{n+1}. The union of the chain of the G_n ($n \in \mathbb{N}$) is evidently A. Setting $B_1 = G_1$ and $G_{n+1} = G_n \oplus B_{n+1}$ for $n \geq 1$, we obtain $A = \oplus_{n=1}^{\infty} B_n$. Here B_n is completely decomposable as a summand of the completely decomposable group G_{n+1}, see Theorem 3.10 in Chapter 12. □

Once we know that summands of completely decomposable torsion-free groups are completely decomposable (Theorem 3.10 in Chapter 12), it is not difficult to prove the analogous result for separable groups.

Theorem 4.4 (Fuchs [16]). *Summands of separable torsion-free groups are again separable.*

Proof. Let G be a separable torsion-free group, and $G = A \oplus B$ a direct decomposition. We have to show that every finite subset S of A is contained in a finite rank completely decomposable summand H of A.

Let G_0 be a finite rank completely decomposable summand of G containing S; there are finite rank pure subgroups A_0, B_0 of A and B, respectively, such that $G_0 \leq A_0 \oplus B_0$. There is a finite rank completely decomposable summand G_1 of G that contains a maximal independent set in $A_0 \oplus B_0$, and hence it contains both A_0 and B_0. Furthermore, there are finite rank pure subgroups A_1, B_1 of A and B, respectively, satisfying $G_1 \leq A_1 \oplus B_1$. Continuing this way, we obtain an ascending chain

$$G_0 \leq A_0 \oplus B_0 \leq G_1 \leq A_1 \oplus B_1 \leq \cdots \leq G_n \leq A_n \oplus B_n \leq \cdots \quad (n < \omega),$$

where the G_n are finite rank completely decomposable summands of G, and A_n, B_n are finite rank pure subgroups of A, B. The union G' of this chain is a countable rank subgroup of G, which is completely decomposable as the union of the chain of completely decomposable subgroups G_n where every group in the chain is a summand in each of the following ones with completely decomposable complements. Moreover, by construction,

$$G' = A' \oplus B' \quad \text{where } A' = \bigcup_n A_n, \ B' = \bigcup_n B_n.$$

A', B' are completely decomposable as summands of the completely decomposable group G'. Therefore, S is contained in a finite rank completely decomposable summand H of A'. Then H is a summand of G', and since $H \leq G_k < G'$ for some $k < \omega$, H is a summand of G_k, so also of G, and hence of A. As a summand of G_k, H is completely decomposable. □

The case when the rank 1 summands are of the same type deserves special attention.

Lemma 4.5 (Baer [6]).

(i) *A homogeneous torsion-free group is separable if and only if every finite rank pure subgroup is a summand.*
(ii) *Pure subgroups of homogeneous separable groups are separable.*

Proof.

(i) Let C be a finite rank pure subgroup in the homogeneous separable group A. It can be embedded in a finite rank completely decomposable summand B of A. By homogeneity, Corollary 3.7 in Chapter 12 implies C is a summand of B, and hence of A. Conversely, if pure subgroups of finite rank are summands of A, then every finite rank pure subgroup of C is a summand of C, and it follows at once that C is completely decomposable.
(ii) Let again C be a finite rank pure subgroup in the homogeneous separable group A. If $c_1, \dots, c_n \in C$, then $\langle c_1, \dots, c_n \rangle_*$ is by (i) a summand of A, and hence of C. (i) also implies C separable. □

Example 4.6 (Nunke). Subgroups of homogeneous separable groups that are not pure need not be separable. As usual, set $P = \prod_{n \in \mathbb{N}} \langle e_n \rangle$ and $S = \oplus_{n \in \mathbb{N}} \langle e_n \rangle$. To see that the group $A = pP + S$ for any prime p is not separable, we show that the pure subgroup $\langle x \rangle$ where $x = (pe_1, \dots, pe_n, \dots)$ cannot be a summand. If $\pi : A \to \langle x \rangle$ were the projection, then $\pi e_i = 0$ for all $i > n$ for some $n \in \mathbb{N}$. Then $x = \pi x = \pi(pe_1 + \dots + pe_n) = p(\pi e_1 + \dots + \pi e_n)$ would be divisible by p in A, a contradiction.

Separability of Direct Products The following two results are concerned with the separability of vector groups (see next section).

Lemma 4.7. *For an infinite index set I, let $A = \prod_{i \in I} A_i$ where the A_i are of rank* 1 *and of fixed type* $\mathbf{t}$ ($\neq \mathbf{t}(\mathbb{Q})$). *$A$ is separable if and only if* $\mathbf{t}$ *is idempotent, in which case A is homogeneous of type* $\mathbf{t}$.

Proof. Let R be the subring of $\mathbb{Q}$ whose additive group is of idempotent type $\mathbf{t}$. Then A of type $\mathbf{t}$ may be viewed as a module over the principal ideal domain R. The argument in Lemma 4.5 carries over *verbatim* to provide a proof of the "if" part.

To verify the converse, let $\mathbf{t}$ be a non-idempotent type represented by $R < \mathbb{Q}$, and p_j ($j < \omega$) a list of primes with $p_jR \neq R$. For each j select an $i_j \in I$ and an $a_j \in A_{i_j}$ such that a_j is not divisible by p_j. The indices i_j may be chosen different. Then consider the infinite vector $x = (0, \dots, a_j, \dots, 0, \dots) \in A$, where the i_jth coordinate is a_j for every $j < \omega$, while all other coordinates are 0. Such an x has type $< \mathbf{t}$. $\square$

The foregoing lemma provides a clue how to construct homogeneous separable groups of idempotent types $\mathbf{t}$: take any pure subgroup of a direct product of rank 1 groups of type $\mathbf{t}$. If the type is not idempotent, then a somewhat more difficult procedure is needed, but it yields all homogeneous separable groups.

Proposition 4.8. *A homogeneous torsion-free group of type* $\mathbf{t}(\neq \mathbf{t}(\mathbb{Q}))$ *is separable if and only if it is isomorphic to a pure subgroup of some*

$$G = \left(\prod\nolimits_{i \in I} A_i\right)(\mathbf{t}),$$

where all groups A_i *are rational groups of type* $\mathbf{t}$.

Proof. Let A be separable and homogeneous of type $\mathbf{t}$. For every $a \in A$, the pure subgroup $\langle a \rangle_*$ is a summand of A as is shown in Lemma 4.5, so the projections $\pi_a : A \to \langle a \rangle_*$ give rise to an embedding $\phi : A \to \prod_{0 \neq a \in A} \langle a \rangle_*$ of A in the direct product of rank one groups of type $\mathbf{t}$. By construction, it must be a pure embedding with image in the subgroup of elements of type $\geq \mathbf{t}$.

In order to show that the group G in the theorem is separable, represent $\mathbf{t}$ as a subgroup R of $\mathbb{Q}$, and pick elements $a_i \in A_i$ corresponding to $1 \in R$ under arbitrary isomorphisms $A_i \cong R$. All $0 \neq g \in G$ have type $\mathbf{t}$, since $> \mathbf{t}$ is impossible. We write $g = (\dots, r_ia_i, \dots)$ with $r_i \in R$. Set $r_i = s_in_i$ ($s_i, n_i \in \mathbb{Q}$) where the numerator and denominator of s_i contain only primes p with $pR = R$, while n_i has no such primes. If g corresponds to 1 in an isomorphism $\langle g \rangle_* \cong R$, then the n_i have to be integers. If there is an index $j \in I$ with $n_j = \pm 1$, then clearly $G = \langle g \rangle_* \oplus H_j$ where H_j is G intersected with the product of the A_i with $i \neq j$. If there is no such j, then we proceed as in the proof of Theorem 8.2 in Chapter 3, and argue that there are elements $x_1, \dots, x_k \in G$ such that $\prod_{i \in I} A_i = \langle x_1 \rangle_* \oplus \dots \oplus \langle x_k \rangle_* \oplus H'$, where $g \in \langle x_1 \rangle_* \oplus \dots \oplus \langle x_k \rangle_*$ and H' is the product of almost all of the A_i. We continue with an induction on n, to find a finite rank completely decomposable summand X of $\prod_{i \in I} A_i$ containing preassigned elements $g_1, \dots, g_n \in G$. By Lemma 4.5, $X \cap G$ is a completely decomposable summand of G, proving that G is separable. $\square$

Separable torsion-free groups are abundant. A rich source of interesting examples is the theory of groups of continuous functions of a topological space into the discrete group $\mathbb{Z}$. To make good this claim, we will prove Proposition 4.9 below.

In the following result, $C(T,\mathbb{Z})$ denotes the group of continuous functions from the topological space T into the discrete group $\mathbb{Z}$. We focus on the case when T is the set of rational numbers in an open interval of the real line (in the interval topology). Clearly, for every open interval (a,b) with irrational endpoints $a < b$, there is an isomorphism $C((a,b)\cap\mathbb{Q},\mathbb{Z}) \cong C(\mathbb{Q},\mathbb{Z})$. The same holds if $a = -\infty$ or $b = \infty$.

Proposition 4.9 (Eda [2]). *The group $C(\mathbb{Q},\mathbb{Z})$ is separable, and is isomorphic both to $C(\mathbb{Q},\mathbb{Z})^{\aleph_0}$ and to $C(\mathbb{Q},\mathbb{Z})^{(\aleph_0)}$.*

Proof. Let δ denote an irrational number, and let $I_n = (n+\delta, n+1+\delta)\cap\mathbb{Q}$, i.e. the set of rational numbers in the given interval. Then $\mathbb{Q} = \cup_{n\in\mathbb{Z}} I_n$ is a disjoint union, and therefore

$$C(\mathbb{Q},\mathbb{Z}) = \prod_{n\in\mathbb{Z}} C(I_n,\mathbb{Z}) \cong C(\mathbb{Q},\mathbb{Z})^{\mathbb{Z}}.$$

Let $\delta_0 > \cdots > \delta_i > \cdots$ be a decreasing sequence of positive irrational numbers converging to 0. Define

$$\begin{aligned} J_0 &= ((-\infty,-\delta_0)\cup(\delta_0,\infty))\cap\mathbb{Q},\\ J_i &= ((-\delta_i,-\delta_{i+1})\cup(\delta_{i+1},\delta_i))\cap\mathbb{Q} \quad \forall i\in\mathbb{N}. \end{aligned}$$

We claim that the subgroup $C_0 = \{f \in C(\mathbb{Q},\mathbb{Z}) \mid f(0) = 0\}$ is equal to $\oplus_{i<\omega} C(J_i,\mathbb{Z})$. Indeed, the inclusion $\geq$ is obvious. For the converse, note that $C_0 \leq \prod_{i<\omega} C(J_i,\mathbb{Z})$, but by continuity, for an $f \in C_0$ there is a $k \in \mathbb{N}$ such that $f \restriction J_i = 0$ for all $i > k$. Hence $C_0 = \oplus_{i<\omega} C(J_i,\mathbb{Z}) \cong C(\mathbb{Q},\mathbb{Z})^{(\omega)}$, and $C(\mathbb{Q},\mathbb{Z}) = \mathbb{Z}\oplus C_0 \cong \mathbb{Z}\oplus C(\mathbb{Q},\mathbb{Z})^{(\omega)}$. We obtain

$$C(\mathbb{Q},\mathbb{Z}) \cong \mathbb{Z}\oplus(\mathbb{Z}\oplus C(\mathbb{Q},\mathbb{Z}))^{(\omega)} \cong (\mathbb{Z}\oplus C(\mathbb{Q},\mathbb{Z}))^{(\omega)} \cong C(\mathbb{Q},\mathbb{Z})^{(\omega)}.$$

The separability of $C(\mathbb{Q},\mathbb{Z})$ follows immediately from the observation that it is a pure subgroup of the homogeneous separable group $\mathbb{Z}^{\mathbb{Q}}$. ☐

Dual Groups We intend to draw attention to a large variety of separable torsion-free groups arising from the so-called dual groups. A^* will denote the group $\mathrm{Hom}(A,\mathbb{Z})$ which will be called the **dual group of** A. If we say that G is a **dual group**, then we mean that $G \cong \mathrm{Hom}(A,\mathbb{Z})$ for some group A.

Lemma 4.10. *A dual group is torsion-free, homogeneous, and separable. Its type is* $\mathbf{t}(\mathbb{Z})$.

Proof. It is pretty obvious that every dual group is a subgroup in a direct product of infinite cyclic groups. ☐

Example 4.11. For non-measurable cardinality, free groups are dual groups, and so are the direct products of infinite cyclic groups.

Example 4.12. There are in general many groups with the same dual. For instance, $\mathbb{Z}^{(\aleph_0)}$ is the dual of $\mathbb{Z}^{\aleph_0}$, and also the dual of the $\mathbb{Z}$-adic closure of $\mathbb{Z}^{(\aleph_0)}$ in $\mathbb{Z}^{\aleph_0}$.

If G is a dual group, then so are G^κ and $G^{(\kappa)}$ for any non-measurable cardinal κ. The **Reid class** of dual groups is generated by $\mathbb{Z}$, using alternately direct sum and non-measurable direct product constructions: $\oplus\,\mathbb{Z}, \prod\mathbb{Z}, \prod(\oplus\,\mathbb{Z}), \oplus(\prod\mathbb{Z})$, $\oplus(\prod(\oplus\mathbb{Z}))$, etc. It was an interesting challenging question raised by G. Reid whether the groups in this class are all different. Now it seems clear that they are indeed different. Dual groups that do not belong to this class have been subjects of extensive research; see Eklof–Mekler [EM].

$\aleph_1$-Separability There is an immediate generalization of separability to higher cardinals κ. A group A is called **κ-separable** if every subset of A whose cardinality is $< \kappa$ is contained in a completely decomposable summand of A. Thus $\aleph_0$-separability is just ordinary separability. Our focus is on $\aleph_1$-separability.

Example 4.13. An $\aleph_1$-separable torsion-free group A of cardinality $\aleph_1$ has a filtration $\{A_\sigma \mid \sigma < \omega_1\}$ such that $A_{\sigma+1}$ is a countable completely decomposable summand of A for each $\sigma < \omega_1$. Using this description, one can construct $\aleph_1$-separable groups that are not completely decomposable.

For a pure subgroup X of a $\mathbf{t}$-homogeneous torsion-free group A we use the *ad hoc* notation $X^\dagger = \cap_{\phi:A\to R,\,\phi X=0} \operatorname{Ker}\phi$ where R is a rational group of type $\mathbf{t}$. Observe that $X \leq X^\dagger$, and $X^\dagger$ is pure in A.

Lemma 4.14. *A homogeneous torsion-free group A is $\aleph_1$-separable if and only if for every countable subgroup X, $X^\dagger$ is a countable completely decomposable summand of A.*

Proof. Suppose A is $\mathbf{t}$-homogeneous, $\aleph_1$-separable, and X is a countable subgroup. By definition, A has a countable completely decomposable summand E such that $X \leq E$. It is obvious that $X \leq X^\dagger \leq E$. Evidently, $E/X^\dagger$ is a countable pure subgroup of the $\mathbf{t}$-homogeneous separable group $A/X^\dagger$, thus it is completely decomposable of type $\mathbf{t}$. By Baer's lemma, $X^\dagger$ is a summand of E, and therefore $X^\dagger$ is a completely decomposable summand of A. The converse implication is trivial. □

Proposition 4.15. *Summands of homogeneous $\aleph_1$-separable torsion-free groups are again $\aleph_1$-separable.*

Proof. If $A = B \oplus C$ is such a group, and if X is a countable subgroup of B, then obviously $X^\dagger \leq B$. Hence $X^\dagger$ is a summand of B. □

Coseparable Groups A group A will be called **κ-coseparable** if every pure subgroup of corank $< \kappa$ contains a summand of A whose complement is completely decomposable of rank $< \kappa$. Our main interest lies in the $\kappa = \aleph_1$ case.

Lemma 4.16. *A torsion-free group all of whose pure subgroups are $\aleph_1$-separable is $\aleph_1$-coseparable.*

Proof. Assume that the pure subgroups of A are $\aleph_1$-separable. Let B be pure in A such that A/B is of countable rank, so $A = B + C$ for a countable C. B is $\aleph_1$-separable, thus $B = G \oplus H$ where $B \cap C \leq G$, and G is completely decomposable of countable rank. The subgroup $C + G$ is countable, and clearly $A = (C + G) + H$. If

$c+g = h \in H$ with $c \in C, g \in G$, then $c = h-g \in C \cap B \leq G$, so $h \in G \cap H = 0$. This proves $A = (C + G) \oplus H$. Here $H \leq B$, and the countable summand $C + G$ must be completely decomposable, since A is $\aleph_1$-separable. □

Theorem 4.17 (Griffith [3]). *A group A satisfies* $\mathrm{Ext}(A, S) = 0$ *for a countable free group S if and only if it is $\aleph_1$-free and $\aleph_1$-coseparable.*

Proof. Suppose A satisfies $\mathrm{Ext}(A, S) = 0$. Every countable subgroup X of A also satisfies $\mathrm{Ext}(X, S) = 0$, and therefore the free presentation $0 \to S \to F \to X \to 0$ of X with $|F| = \aleph_0$ splits. This means X is free, and the $\aleph_1$-freeness of A follows. To verify $\aleph_1$-coseparability, let A be uncountable, and B a pure subgroup of countable index in A. Then there is a countable (and thus free) subgroup $H \leq A$ such that $A = B + H$. Form the exact sequence $0 \to C \to B \oplus H \to A \to 0$, where $B \oplus H$ denotes the outer direct sum and $C = \{(c, -c) \mid c \in B \cap H\}$. C is countable and free, so $C \cong S$, and hypothesis implies splitting: $B \oplus H = A \oplus C'$ with $C' \cong C$. Let $\phi : A \to H$ be the obvious projection; then, $\mathrm{Im}\,\phi$ is free, thus $A = \mathrm{Ker}\,\phi \oplus A'$ with a countable free A'. Since obviously $\mathrm{Ker}\,\phi \leq B$, we can conclude that A is $\aleph_1$-coseparable.

Conversely, let A be $\aleph_1$-free and $\aleph_1$-coseparable. Let $\mathfrak{e}: 0 \to S \to H \xrightarrow{\alpha} A \to 0$ represent an element of $\mathrm{Ext}(A, S)$; we may view $S \leq H$. Let B be a subgroup of H maximal with respect to $B \cap S = 0$. Then B is of countable index in H, so $|A/\alpha B| \leq \aleph_0$, and hence there is a summand $G \leq \alpha B$ such that $A = G \oplus F$ with a countable (free) F. Let $C = \alpha^{-1}G \cap B, D = \alpha^{-1}F$. The exact sequence $0 \to S \to D \xrightarrow{\alpha} F \to 0$ splits, so $D = S \oplus F'$ where $F' \cong F$. We have $(C + F') \cap S = 0$, since $c+x = s$ $(c \in C, x \in F', s \in S)$ implies $c = s-x \in C \cap D = C \cap (C \cap D) = C \cap S \leq B \cap S = 0$; then also $s = x = 0$. From $\alpha(C + F') = (G \cap \alpha B) + F = G + F = A$ we obtain $H = (C + F') + S$. Therefore, S is a summand of H, i.e. the sequence $\mathfrak{e}$ splits, and $\mathrm{Ext}(A, S) = 0$. □

★ **Notes.** A word of caution: in some publications (especially where only separable groups of the type of $\mathbb{Z}$ are discussed) "separability" means that every finite set of elements is embeddable in a *free* summand of finite rank.

Cornelius [1] has an interesting observation: for the separability of A, it is enough to know that every single element of A is contained in a finite rank completely decomposable summand of A. This is in fact an unexpected result.

That summands of separable groups are again separable has several proofs, and a number of generalizations. We mention here that Rangaswamy [4], Hill–Megibben [7] introduce far-reaching generalizations of separability, proving various theorems similar to those discussed above. Their results show how more complex group structures can be approached by using the right concepts. A number of remarkable theorems on $\aleph_1$-separable groups ($\aleph_1$-free case) can be found in Eklof–Mekler [EM]; they also discuss filtration equivalence which seems to be quite a significant concept. Dugas–Irwin [1] investigate $\aleph_1$-separable subgroups in the Baer–Specker group $\mathbb{Z}^{\aleph_0}$, and show *inter alia* that it is undecidable in ZFC if every $\aleph_1$-separable $\aleph_1$-free group of cardinality $\aleph_1$ is embeddable in $\mathbb{Z}^{\aleph_0}$. In the proof, the same set-theoretical models are used as we will in Sects. 6 and 7. Blass–Irwin [1] have interesting results on subgroups of the Baer-Specker group $\mathbb{Z}^{\aleph_0}$.

It was conjectured that every reflexive group is of the form $\mathrm{Hom}(C, \mathbb{Z})$ for some C, but was disproved by Eda–Ohta [1]. Albrecht [4] discusses A-reflexivity when A is a non-measurable slender group.

Rychkov [4] considers families $\{A_i \mid i < 2^\kappa\}$ of κ-separable torsion-free groups for an uncountable regular cardinal κ that is not weakly compact. Assuming V = L, he establishes the existence of such families satisfying $|\operatorname{Im}\phi| < \kappa$ for each homomorphism $\phi : A_i \to A_j$ with $i \neq j$ in I. Grinshpon–Krylov [1] study the lattice of fully invariant subgroups of separable torsion-free groups. This lattice—as expected—is determined by the extractable types of the group. Grinshpon–Krylov [1] describe the fully transitive separable groups. For the transitivity of completely decomposable and separable groups, see also Metelli [2].

Hill–Megibben [7] generalize the notion of separability by defining K-groups. The class of these groups is closed under direct sums and direct summands, they display several useful properties similar to separable groups. By making use of this concept, knice subgroups are introduced, and torsion-free groups are studied that admit $H(\aleph_0)$-families of knice subgroups.

There is a considerable body of literature on dual groups. Their in-depth discussion requires sophisticated set-theoretical machinery; we refer to [EM] for many interesting results. See also Mekler–Schlitt [1] where dual groups are discussed from the logical point of view. Göbel–Pokutta [1] construct dual groups in $\mathbb{Z}^{\aleph_0}$ using MA+ $\neg$CH. It was for a while an open problem if the groups in the Reid classes were all different. The problem was solved by Zimmermann-Huisgen [1], Ivanov [3], and Eda [3] in the affirmative.

Nunke–Rotman [1] show that every integral singular cohomology group is of the form $\operatorname{Ext}(G,\mathbb{Z}) \oplus \operatorname{Hom}(H,\mathbb{Z})$, where G, H can be arbitrary groups. Thus they are direct sums of a cotorsion group and a dual group.

Coseparability is more difficult to deal with, and the reader might wonder why we have not given any explicit examples for non-free coseparable groups. The reason is simple: it is undecidable in ZFC whether or not all coseparable groups are free. We do not wish to enter into the discussion of these groups, just mention that Chase [Pac. J. Math. **12**, 847–854 (1962)] proved that CH implies that there exist non-free coseparable groups of cardinality $\aleph_1$, while Mekler–Shelah [3] showed that coseparable groups of cardinality $\aleph_1$ are free in the model of ZFC with $\aleph_2$ Cohen reals added.

It is an interesting fact that all $\aleph_1$-coseparable groups are reflexive (Huber [2]).

Coseparability can also be defined to depend on a cardinality. Usually the definition is restricted to κ-free groups. Accordingly, A is κ-**coseparable** if it is $\kappa + \aleph_1$-free and every pure subgroup of corank $< \kappa$ contains a summand of A of corank $< \kappa$.

Exercises

(1) (Baer) A torsion-free group A is separable if and only if

 (i) every finite subset of A belongs to a direct summand that is the direct sum of homogeneous groups; and

 (ii) $A(\mathbf{t})/A^*(\mathbf{t})$ is separable (in the usual sense) for every type $\mathbf{t}$. [Hint: summands in (i) are separable.]

(2) For every group A, the group $\operatorname{Hom}(A,\mathbb{Z})$ is separable.

(3) The tensor product of two separable groups is separable.

(4) A homogeneous separable torsion-free group A may have fully invariant subgroups other than nA $(n = 0, 1, 2, \ldots)$.

(5) Fully invariant subgroups of $\aleph_1$-separable torsion-free groups are likewise $\aleph_1$-separable.

(6) (Sąsiada) There are non-isomorphic dual groups that are isomorphic to pure subgroups of each other. [Hint: $\mathbb{Z}^{\aleph_0}$ and $\mathbb{Z}^{(\aleph_0)} \oplus \mathbb{Z}^{\aleph_0}$.]

(7) Show that there are dual groups of large non-measurable cardinalities that are isomorphic to their own duals. [Hint: $A \oplus A^*$.]

(8) (Metelli)

(a) Summands as well as direct sums of $\aleph_0$-coseparable groups are again $\aleph_0$-coseparable.

(b) If A is $\aleph_0$-coseparable, and C is a pure fully invariant subgroup of A, then A/C is likewise $\aleph_0$-coseparable.

(9) (Metelli) Call a torsion-free group A **t-bihomogeneous** if $A(\mathbf{t}) = A$ and $A[\mathbf{t}] = 0$. Show that

(a) A is $\mathbf{t}$-bihomogeneous if and only if it is a $\mathbf{t}$-homogeneous subgroup of a $\mathbf{t}$-homogeneous separable group;

(b) countable bihomogeneous groups are completely decomposable.

(10) The group $\mathrm{Hom}(G, \mathbb{Z})$ need not be coseparable even if G is coseparable.

(11) (Griffith) Generalize Theorem 4.17 as follows. Let F be a free group of infinite rank κ. A group G satisfies $\mathrm{Ext}(G, F) = 0$ if and only if (i) it is κ^+-free; and (ii) every subgroup of G of index $\leq \kappa$ contains a summand of G of index $\leq \kappa$. [Hint: same proof.]

5 Vector Groups

An important class of torsion-free groups is the class of vector groups. So far insufficient attention has been paid to it, though there are a number of noteworthy features which are worthwhile exploring. As a note of caution, we should point out at the outset that one has to be very careful with the statements, since some results depend on non-measurability, while some others hold without any cardinal restrictions.

By a **vector group** we mean a direct product of torsion-free groups of rank 1, i.e. a group $V = \prod_{i \in I} R_i$ where the R_i are rational groups and I is an arbitrary index set. (Usually, we assume $R_i \ncong \mathbb{Q}$.)

Basic Results on Vector Groups First, we will consider vector groups $V = \prod_{i \in I} R_i$ such that all the components R_i have the same type $\mathbf{t}$; that is, $V \cong R^I$ where R is a rational group of type $\mathbf{t}$. Such a group V will be called an **elementary vector group of type t**.

Elementary vector groups need not be homogeneous; in fact,

Lemma 5.1. *An elementary vector group $V = \prod_{i \in I} R_i$ of type* $\mathbf{t}$ *is homogeneous if and only if* $\mathbf{t}$ *is an idempotent type.*

Proof. This follows from Lemma 4.7. □

Example 5.2. The generalized Baer-Specker group $\mathbb{Z}^\kappa$ for any infinite cardinal κ is a homogeneous vector group. Its type is $\mathbf{t}(\mathbb{Z})$.

Example 5.3. Let R denote the group of rational numbers with square-free denominators. The group $R^{\aleph_0}$ is an elementary vector group, but it is not homogeneous: the vector $(2^{-1}, \dots, p^{-1} \dots) \in R^{\aleph_0}$ has characteristic $(0, \dots, 0, \dots)$.

Let $V = \prod_{i \in I} R_i$ be an arbitrary vector group where the groups R_i are torsion-free of rank 1. For a fixed type $\mathbf{t}$, let $V_{\mathbf{t}}$ denote the direct product of those R_i which are exactly of type $\mathbf{t}$. Thus

$$V = \prod_{\mathbf{t}} V_{\mathbf{t}}$$

with elementary vector group components, $\mathbf{t}$ running over all types.

In the study of vector groups where the rank 1 components are of distinct types, it is essential to get information about their rank 1 epic images, in particular, about the extractable types. The next lemma (valid for all cardinals) gives valuable information about this situation.

Lemma 5.4.

(a) *If the vector group $V = \prod_{i \in I} R_i$ has a non-trivial homomorphism into a rational group $R \neq 0$, then*

$$\mathbf{t}(R_i) \leq \mathbf{t}(R) \quad \textit{for some index } i.$$

(b) (Mishina [1], Eda [1]) *A rank one summand of the vector group $V = \prod_{i \in I} R_i$ is isomorphic to some R_i.*

Proof.

(a) If $R = \mathbb{Q}$, there is nothing to prove. So assume R is slender, and let $\eta \colon V \to R$ be a non-zero homomorphism. Let $v \in V$ be such that $\eta v \neq 0$, and write $v = (\dots, v_i, \dots)$ with $v_i \in R_i$. Using this v, we collect all the components R_i for which the characteristic $\chi(v_i) = \chi$ is fixed: $V_\chi = \prod_{\chi(v_i) = \chi} R_i$. In this way, we obtain a decomposition $V = \prod_\chi V_\chi$. The last product has at most continuously many non-zero components, so V is a product over a non-measurable index set. By the slenderness of R, there are but finitely many V_χ, say of indices $\chi_1, \dots, \chi_n$, whose images under η are non-zero, while the product of the rest of the V_χ is mapped by η to 0. One of the coordinates $w_1, \dots, w_n$ of v in the last decomposition of V has a non-zero image under η, since evidently $\eta v = \eta w_1 + \dots + \eta w_n$. From $\chi(\eta w_i) \geq \chi(w_i)$ we conclude that the type of R is $\geq$ the type of one of $w_1, \dots, w_n$ which is the type of some R_i.

(b) Without loss of generality, we may assume that V is reduced. From the arguments in (a) it is clear that a rank 1 summand R of V must be a summand of some $V_{\chi_1} \oplus \dots \oplus V_{\chi_n}$. Assume n is chosen minimal, and let $\mathbf{t}_1, \dots, \mathbf{t}_n$ denote the corresponding (not necessarily different) types. Then by (a) we have $\mathbf{t}(R) \geq \mathbf{t}_j$ for every $j \in \{1, \dots, k\}$. On the other hand, R projects non-trivially into a component in V_{χ_j}, so $\mathbf{t}(R) \leq \mathbf{t}_j$ for all j. Hence $\mathbf{t}(R) = \mathbf{t}_j$ for some j. □

Assertion (b) can be rephrased by saying that *the only extractable types of a vector group $V = \prod_{i \in I} R_i$ are the types $\mathbf{t}(R_i)$ $(i \in I)$.*

Isomorphism of Vector Groups At this point the natural question is: when are two vector groups isomorphic? To answer this question, we prove the following theorem. We denote by $\mathbf{t}_0$ the largest idempotent type with $\mathbf{t}_0 \leq \mathbf{t}$, i.e. $\mathbf{t}_0 = \mathbf{t} : \mathbf{t}$.

Theorem 5.5. *Suppose*

$$V = \prod_{\mathbf{t}} V_{\mathbf{t}} \quad \textit{and} \quad W = \prod_{\mathbf{t}} W_{\mathbf{t}}$$

are vector groups where $V_{\mathbf{t}}$ and $W_{\mathbf{t}}$ are elementary vector groups of type $\mathbf{t}$, and $\mathbf{t}$ runs over the different types.

(i) (Sąsiada [3]) *$V \cong W$ if and only if $V_{\mathbf{t}} \cong W_{\mathbf{t}}$ for every type $\mathbf{t}$.*
(ii) *If $V_{\mathbf{t}} = \prod_{i\in I} R_i \cong W_{\mathbf{t}} = \prod_{j\in J} R_j$ with a type $\mathbf{t} \not\cong \mathbf{t}(\mathbb{Q})$, then $|I| = |J|$ whenever I or J is non-measurable.*

Proof.

(i) The "if" part being obvious, assume there is an isomorphism $\phi : V \to W$. Let $\pi_{\mathbf{t}} : V \to V_{\mathbf{t}}$ and $\rho_{\mathbf{t}} : W \to W_{\mathbf{t}}$ denote the coordinate projections in the two direct products. By Lemma 5.4, every homomorphism of $V_{\mathbf{t}}$ into $W_{\mathbf{s}}$ is trivial unless the types satisfy $\mathbf{s} \geq \mathbf{t}$. Therefore, $\phi V_{\mathbf{t}} \leq \prod_{\mathbf{s}\geq\mathbf{t}} W_{\mathbf{s}}$. If $v \in V_{\mathbf{t}}$, then $\phi v = w + w'$ with $w \in W_{\mathbf{t}}$ and $w' \in \prod_{\mathbf{s}>\mathbf{t}} W_{\mathbf{s}}$. Again by Lemma 5.4, the last group has only the trivial homomorphisms into $V_{\mathbf{t}}$ and into $W_{\mathbf{t}}$, whence $\pi_{\mathbf{t}}\phi^{-1}w' = 0 = \rho_{\mathbf{t}}w'$ follows. This shows that $v = \pi_{\mathbf{t}}v = \pi_{\mathbf{t}}\phi^{-1}w$ and $w = \rho_{\mathbf{t}}w = \rho_{\mathbf{t}}\phi v$; consequently, the map $\pi_{\mathbf{t}}\phi^{-1}\rho_{\mathbf{t}}\phi$ is the identity on $V_{\mathbf{t}}$. Changing the roles of $V_{\mathbf{t}}$ and $W_{\mathbf{t}}$, the stated isomorphism is immediate.
(ii) Let R and R_0 be rational groups of types $\mathbf{t}$ and $\mathbf{t}_0$, respectively. In view of Corollary 2.10, we have $\mathrm{Hom}(V_{\mathbf{t}}, R) \cong \oplus_{i\in I}\mathrm{Hom}(R_i, R) = \oplus_{i\in I}R_0$ and $\mathrm{Hom}(W_{\mathbf{t}}, R) \cong \oplus_{j\in J}R_0$. Both are completely decomposable groups whence $V_{\mathbf{t}} \cong W_{\mathbf{t}}$ implies $|I| = |J|$. ((ii) holds also for measurable index sets, see Theorem 6.1 below.) □

Summands of Non-measurable Vector Groups One of the most interesting facts on vector groups is concerned with direct summands in the non-measurable case.

Theorem 5.6 (O'Neill [2]).

(i) *Summands of reduced vector groups are direct products of summands of elementary vector groups of different types.*
(ii) (Balcerzyk–Bialynicki-Birula–Łoś [1]) *Summands of reduced non-measurable vector groups are again vector groups.*

Proof.

(i) Let $V = \prod_{\mathbf{t}} V_{\mathbf{t}}$ with elementary vector groups $V_{\mathbf{t}}$ of type $\mathbf{t}$. The projections in this decomposition will be denoted by $\pi_{\mathbf{t}}$. For each type $\mathbf{t}$, the subgroups

$$V^{\mathbf{t}} = \prod_{\mathbf{s}\geq\mathbf{t}} V_{\mathbf{s}} \quad \text{and} \quad V^{\mathbf{t}^*} = \prod_{\mathbf{s}>\mathbf{t}} V_{\mathbf{s}}$$

are fully invariant, and satisfy $V^{\mathbf{t}} = V_{\mathbf{t}} \oplus V^{\mathbf{t}^*}$.

Suppose $V = A \oplus B$ with projections α, β. In view of full invariance, for each type $\mathbf{t}$, we have $V^{\mathbf{t}} = A^{\mathbf{t}} \oplus B^{\mathbf{t}}$ and $V^{\mathbf{t}^*} = A^{\mathbf{t}^*} \oplus B^{\mathbf{t}^*}$ where $A^{\mathbf{t}} = A \cap V^{\mathbf{t}}$, $A^{\mathbf{t}^*} = A \cap V^{\mathbf{t}^*}$ (and the same for B). Hence $A^{\mathbf{t}} = A_{\mathbf{t}} \oplus A^{\mathbf{t}^*}$ and $B^{\mathbf{t}} = B_{\mathbf{t}} \oplus B^{\mathbf{t}^*}$, where $A_{\mathbf{t}} = A^{\mathbf{t}} \cap (V_{\mathbf{t}} \oplus B^{\mathbf{t}^*})$ and $B_{\mathbf{t}} = B^{\mathbf{t}} \cap (V_{\mathbf{t}} \oplus A^{\mathbf{t}^*})$. Manifestly, we have $V_{\mathbf{t}} = \pi_{\mathbf{t}} A_{\mathbf{t}} \oplus \pi_{\mathbf{t}} B_{\mathbf{t}}$, so we can write $V = \prod_{\mathbf{t}} V_{\mathbf{t}} = \prod_{\mathbf{t}} \pi_{\mathbf{t}} A_{\mathbf{t}} \oplus \prod_{\mathbf{t}} \pi_{\mathbf{t}} B_{\mathbf{t}}$. If $a = v + b \in A_{\mathbf{t}}$ with $v \in V_{\mathbf{t}}$ and $b \in B^{\mathbf{t}^*}$, then $\pi_{\mathbf{t}} a = v$ and $a = \alpha v$ imply that $\alpha \pi_{\mathbf{t}} a = a$, i.e. $\alpha \pi_{\mathbf{t}}$ is the identity on $A_{\mathbf{t}}$.

Let $\gamma \in \operatorname{End} V$ denote the projection of V onto $A' = \prod_{\mathbf{t}} \pi_{\mathbf{t}} A_{\mathbf{t}}$. Consider the map $\gamma - \alpha\gamma : V \to V$. From what has been proved it follows that this map vanishes on each $\pi_{\mathbf{t}} A_{\mathbf{t}}$, thus also on their direct sum. The direct product of the $\pi_{\mathbf{t}} A_{\mathbf{t}}$ is over at most $2^{\aleph_0}$ summands, thus it is over a non-measurable index set. Since the R_i are slender, the projection of the direct product A' into each R_i is also 0, i.e. $(\gamma - \alpha\gamma)A' = 0$. Therefore, $A' = \gamma A' = \alpha\gamma A' \le A$. Similarly, $B' = \prod_{\mathbf{t}} \pi_{\mathbf{t}} B_{\mathbf{t}} \le B$. Evidently, A', B' cannot be proper subgroups in A, B, since $A \oplus B = V = A' \oplus B'$. Hence we derive the isomorphism $A \cong \prod_{\mathbf{t}} \pi_{\mathbf{t}} A_{\mathbf{t}}$.

(ii) For the second claim, we first show that every summand of an elementary vector group $V = \prod_{i \in I} R_i$ of type $\mathbf{t}$ is again a vector group, provided the index set I is non-measurable, and $\mathbf{t}$ is different from the type of $\mathbb{Q}$. Let $V = A \oplus B$, and R, R_0 rational groups of types $\mathbf{t}, \mathbf{t}_0$, respectively. By the slenderness of R, we have $\operatorname{Hom}(V, R) \cong \oplus_{i \in I} \operatorname{Hom}(R_i, R) \cong \oplus_{i \in I} R_0$. Hence $V \cong \operatorname{Hom}(\operatorname{Hom}(V, R), R)$. Observe that $\operatorname{Hom}(A, R)$ and $\operatorname{Hom}(B, R)$—as summands of a completely decomposable group—are likewise completely decomposable. They are complementary summands in $\operatorname{Hom}(V, R)$, so $\operatorname{Hom}(\operatorname{Hom}(A, R), R)$, and $\operatorname{Hom}(\operatorname{Hom}(B, R), R)$ are products of copies of R and complementary summands in $\operatorname{Hom}(\operatorname{Hom}(V, R), R)$. By the naturality of the isomorphisms, these groups are isomorphic to A, B and V, respectively. Thus for non-measurable vector groups, in the decomposition used in the proof of (i), each $\pi_{\mathbf{t}} A_{\mathbf{t}}$ is a vector group. Hence $A \cong \prod_{\mathbf{t}} \pi_{\mathbf{t}} A_{\mathbf{t}}$ is likewise a vector group. $\square$

Separability of Vector Groups We do not intend to pursue the exploration of vector groups much further, but we feel we should mention the main result on their separability. Separability of vector groups in general has been successfully investigated by Mishina [2] and Król [1]. According to their result, a vector group $V = \prod_{i \in I} R_i$ is separable if and only if the types $\mathbf{t}(R_i) = \mathbf{t}_i$ satisfy the following criteria:

(i) the descending chain condition holds for the types $\mathbf{t}_i$;
(ii) every set of incomparable types $\mathbf{t}_i$ is finite; and
(iii) if in some type $\mathbf{t}_j$, non-zero finite heights occur at an infinite set N of primes, then only finitely many types $\mathbf{t}_i$ exists which are $\ge \mathbf{t}_j$ and contain infinitely many non-zero finite heights at infinitely many primes in N.

★ **Notes.** Though in this section we have concentrated on the non-measurable case, we tried to prove the results, if possible, to include measurable vector groups as well.

Mishina [1] proves that every slender summand of a vector group is a finite direct sum of its summands. As pointed out above, separable vector groups have been characterized by Mishina [2] and Król [1] independently. (I wish to apologize for quoting their result incorrectly in [IAG].) See also Król–Sąsiada [1]. Albrecht–Hill [2] provide a better proof.

Huber [1] investigates homomorphisms $\phi: R^I \to R^J$ for rational groups R and non-measurable index sets I, J. The kernel and cokernels are characterized: they are of the form $\operatorname{Ker}\phi = \operatorname{Hom}(A, R)$ and $\operatorname{Coker}\phi = R^K \oplus \operatorname{Ext}(A, R)$ for some group A and index set K. Ivanov [7] shows that Kaplansky's test problems have positive solutions for non-measurable vector groups.

Exercises

(1) For non-measurable I, derive Lemma 5.4(b) directly from Corollary 2.10, without referring to (a).
(2) (Mishina, Łoś) A vector group over an infinite index set is completely decomposable if and only if almost all components are $\cong \mathbb{Q}$.
(3) A reduced vector group contains no cotorsion subgroups $\neq 0$.
(4) Let R be a rational group of idempotent type, and κ a non-measurable cardinal. There exists a group A of cardinality $\geq \kappa$ such that $\operatorname{Hom}(A, R) \cong A$.
(5) (Beaumont) Let $V = \prod_{i \in I} R_i$ be an elementary vector group of type $\mathbf{t}$ with infinite I. Show that V contains elements of any type $\mathbf{s}$ satisfying $\mathbf{t}_0 \leq \mathbf{s} \leq \mathbf{t}$. (Here, as above, $\mathbf{t}_0 = \mathbf{t} : \mathbf{t}$.)
(6) Prove that $(\mathbb{Z}^{\aleph_0})^n \cong \mathbb{Z}^{\aleph_0}$ for each $n \in \mathbb{N}$, but $(\mathbb{Z}^{\aleph_0})^{(\aleph_0)} \not\cong \mathbb{Z}^{\aleph_0}$.
(7) (Ivanov) Two non-measurable vector groups are isomorphic if each is isomorphic to a summand of the other.

6 Powers of $\mathbb{Z}$ of Measurable Cardinalities

The discussion of vector groups above suggests that there is a fundamental difference between the behavior of vector groups of measurable and non-measurable cardinalities. Our study would be incomplete without mentioning some results that are critical in understanding why non-measurability was so relevant an assumption in the above theorems. Already the measurable powers $\mathbb{Z}^I$ display phenomena whose understanding requires heavy set-theoretical machinery. To study $\mathbb{Z}^I$ fully in detail would take us too far afield, so we restrict ourselves to pointing out some typical results.

In this section, we are working in a model of set theory in which measurable cardinals exist. We will use the symbol μ for the smallest measurable cardinal, and the notation P_κ for the κth power $\mathbb{Z}^\kappa$ of $\mathbb{Z}$ which will also be written more explicitly as

$$P_\kappa = \prod_{i<\kappa} \langle e_i \rangle \quad \text{where} \quad \langle e_i \rangle \cong \mathbb{Z}$$

with projection maps $\pi_i : P_\kappa \to \langle e_i \rangle$. As usual, we set $S_\kappa = \oplus_{i<\kappa}\langle e_i\rangle$. An element of P_κ will be written in the form $x = \sum_{i<\kappa} n_i e_i$ with $n_i \in \mathbb{Z}$.

Let $\mathcal{U}$ be a (non-principal) $\aleph_1$-complete ultrafilter on the power set $\mathcal{P}(I)$ of an index set I of cardinality κ. The homomorphism $\eta_{\mathcal{U}} : P_\kappa \to \mathbb{Z}$ is defined as follows. For $x \in P_\kappa$, let $I_n = \{i \in I \mid \pi_i(x) = ne_i\}$ for $n \in \mathbb{Z}$, so that $\{I_n \mid n \in \mathbb{Z}\}$ is a partition of I. If m is the unique integer such that $I_m \in \mathcal{U}$, then we set $\eta_{\mathcal{U}}(x) = m$. It is straightforward to check that $\eta_{\mathcal{U}}$ is a genuine homomorphism.

It is known that if the cardinal κ is measurable, then there are at least 2^κ κ-complete ultrafilters on κ.

Uniqueness of κ in P_κ If we consider powers $\mathbb{Z}^\kappa$, then probably one of the first questions that comes to mind is to what extent κ is determined. We verify the uniqueness of this cardinality in a more general setting.

Theorem 6.1 (Eda [1]). *Let $G \neq 0$ denote a slender group. $G^I \cong G^J$ holds for infinite sets I, J if and only if $|I| = |J|$.*

Proof. To verify the "only if" part, let $\phi : G^I \to G^J$ be an isomorphism, and $\pi_j : G^J \to G$ the jth projection map. Pick a $0 \neq g \in G$, and for each $i \in I$, let $g_i \in G^I$ be the vector with 0 coordinates everywhere except with g as its ith coordinate. As G is slender, for any fixed $j \in J$, the set of $i \in I$ satisfying $\pi_j(\phi(g_i)) \neq 0$ must be finite. Consequently, $|I| \leq |J|$, and by symmetry, we obtain the desired equality. $\square$

Odd Decompositions of $\mathbb{Z}^\kappa$ In order to illustrate how drastically the measurable case differs from the non-measurable situation, we exhibit examples showing that for a measurable cardinal κ, P_κ admits non-trivial direct decompositions where S_κ is contained in one of the summands.

Example 6.2 (O'Neill [3]).

(a) Let κ be a measurable cardinal, and let $\mathcal{U}$ denote a non-principal $\aleph_1$-complete ultrafilter on the power set $\mathcal{P}(\kappa)$. Then $P_\kappa = A \oplus \langle e \rangle$, where $A = \{x \in P_\kappa \mid \operatorname{supp} x \notin \mathcal{U}\}$, and $e \in P_\kappa$ is the vector whose ith coordinate is e_i for each $i < \kappa$. All the basis vectors e_i belong to A, so the projection π of P_κ onto $\langle e \rangle$ satisfies $\pi S_\kappa = 0$, though $\pi P_\kappa \neq 0$.

(b) More generally, let $S = \{\mathcal{U}_i \mid i < \kappa\}$ be a set of non-principal $\aleph_1$-complete ultrafilters on $\mathcal{P}(\kappa)$ such that, for each $i < \kappa$, there is an $X_i \subseteq \kappa$ satisfying: $X_i \in \mathcal{U}_j$ exactly if $i = j$. (The existence of such a set S is established in set theory.) Setting $x_j = \sum_{i \in X_j} e_i$, we have a decomposition

$$P_\kappa = B \oplus \prod_{j<\kappa} \langle x_j \rangle \quad \text{where } B = \{x \in P_\kappa \mid \operatorname{supp} x \notin \cup_{i<\kappa}\mathcal{U}_i\}.$$

Summable Subsets A subset X of P_κ is said to be **summable** (relative to the given decomposition of P_κ) if, for each $i < \kappa$, there are but a finite number of elements $x \in X$ such that $\pi_i(x) \neq 0$. The subset X will be called **completely independent** if $\sum_{x \in X} n_x x = 0$ ($n_x \in \mathbb{Z}$) implies $n_x = 0$ for all $x \in X$ (O'Neill).

Example 6.3.

(a) For each $i < \kappa$, let $x_i = e_i + \cdots + e_{i+n} \in P_\kappa$ with a fixed integer n. Then $\{x_i \mid i < \kappa\}$ is a completely independent summable subset in P_κ.

(b) The set $\{e_i\ (i < \kappa);\ e = \sum_{i<\kappa} e_i\}$ is summable, independent, but not completely independent.

A few properties of summable sets are recorded in (A)–(E).

(A) *Subsets of summable sets are summable. An infinite subset of P_κ is summable if and only if its countable subsets are summable.*

(B) *If X is a summable subset in P_κ, then any infinite linear combination $\sum_{x\in X} n_x x$ with $n_x \in \mathbb{Z}$ represents an element of P_κ.*

(C) *If X is a completely independent summable set, then $\prod_{x\in X} \langle x\rangle$ is a subgroup of P_κ.*

(D) *Let $X = \{x_j \mid j \in J\}$ be a completely independent summable subset in $P_\kappa = A \oplus B$. The non-zero projections of elements of X in A form a summable subset.* Indeed, if $\alpha : P_\kappa \to A$ is the projection map, then by (C), for each $i \in I$, only finitely many indices $j \in J$ satisfy $\pi_i\alpha(x_j) \neq 0$.

(E) *If $P_\kappa = A \oplus B$, and $X = \{x_j \mid j \in J\}$ is an infinite summable subset in A, then there is a completely independent summable subset $Y \subseteq A$ such that $|Y| = |X|$ and $\prod_{y\in Y} \langle y\rangle$ is a summand of P_κ.* For the proof we may assume that each $\langle x_j\rangle$ is pure (and hence a summand) in A. For every $x \in X$, there is a finite subset $S_x \subset \kappa$ such that $x = u_x + v_x$ where $\langle u_x\rangle$ is a summand of $\oplus_{i\in S_x}\langle e_i\rangle$ and $\pi_i(v_x) = 0$ for all $i \in S_x$. Thus

$$P_\kappa = \langle u_x\rangle \oplus C_x \oplus \prod\nolimits_{i\in\kappa\setminus S_x} \langle e_i\rangle = \langle x\rangle \oplus C_x \oplus \prod\nolimits_{i\in\kappa\setminus S_x} \langle e_i\rangle$$

for some finitely generated $C_x \leq \oplus_{i\in S_x}\langle e_i\rangle$. Consider a maximal set M of pairwise disjoint finite sets S_x with $x \in X$. Since X was infinite, we have $|M| = |X|$. It is clear that the set Y of indices x for all $S_x \in M$ will be a completely independent summable subset in A. This yields a decomposition

$$P_\kappa = \prod\nolimits_{y\in Y} \langle y\rangle \oplus \prod\nolimits_{y\in Y} C_y \oplus \prod\nolimits_{i\in\kappa\setminus S} \langle e_i\rangle,$$

where $S = \cup_{y\in Y} S_y$. (Note that it is *not* claimed that $\prod_{y\in Y} \langle y\rangle \leq A$.)

Measurable Vector Groups So far, it has not been proved that summands of measurable vector groups are again of the same sort. The difficulty lies in extending Theorem 5.6 to elementary vector groups of size $> \mu$. However, we will show that one of the summands is a vector group.

We require some preliminary lemmas. Let η be an endomorphism of P_κ such that

$$\eta(e_i) = e_i \quad \text{for each } i < \kappa. \tag{13.5}$$

By the Łoś-Eda Theorem 2.14, for each $j < \kappa$, there exists a finite set F_j of $\aleph_1$-complete ultrafilters on κ such that if $S \subseteq \kappa$ and $S \notin \mathcal{U}$ for any $\mathcal{U} \in F_j$, then $\pi_j\eta(\prod_{i\in S} \langle e_i\rangle) = 0$.

In the following arguments we have to deal separately with the ultrafilters that contain subsets of smaller cardinalities. For each $j < \kappa$, let N_j denote the set of those $\mathcal{U} \in F_j$ that contain also subsets of cardinality $< \kappa$, and let $M_j = F_j \setminus N_j$. In each $\mathcal{U} \in N_j$, we choose an element (i.e., a subset of κ) of minimal cardinality, and denote by X_j the union of these minimal subsets. Visibly, $j \in X_j$, $|X_j| < \kappa$, and if $S \subset \kappa$ is disjoint from X_j, then $S \notin \mathcal{U}$ for each $\mathcal{U} \in N_j$. Finally, we let $M = \cup_{j \in \kappa} M_j$. As every M_j is finite, $|M| \leq \kappa$ is manifest.

We claim that there exists a $T \subset \kappa$ such that $|T| = \kappa$, and $T \notin \mathcal{U}$ for any $\mathcal{U} \in M$. By Lemma 5.7 in Chapter 1, there exists a set $\mathcal{C}$ of subsets of κ such that $|\mathcal{C}| > \kappa$, $|X| = \kappa$ for each $X \in \mathcal{C}$, and $|X \cap Y| < \kappa$ whenever $X \neq Y$ in $\mathcal{C}$. If $X \neq Y$ are in $\mathcal{C}$, then X and Y cannot both be in the same $\mathcal{U} \in M$, since then $X \cap Y \in \mathcal{U}$, in contradiction to the definition of M. From $|M| < |\mathcal{C}|$ it is clear that there is a $T \in \mathcal{C}$ not contained in any member of M.

The following lemma holds for all infinite cardinals, but we prove it only for regular cardinals.

Lemma 6.4 (O'Neill [4]). *Let κ be a regular cardinal. If $\eta \in \operatorname{End} P_\kappa$ satisfies (13.5), then there is a subset K of κ of cardinality κ such that*

$$\pi_k \eta(\prod_{k<i \in K} \langle e_i \rangle) = 0 \quad \textit{for each } k \in K.$$

Proof. We keep the notation used above. Suppose $j < \kappa$, and for each ordinal $i < j$ we have already chosen an element $\phi(i) \in T \setminus \cup_{\ell < i} X_\ell$. Since $|X_\ell| < \kappa$ and κ is regular, we have $|T \setminus \cup_{\ell < j} X_\ell| = \kappa$. Pick any element $\phi(j)$ in this set, and define $K = \{\phi(i) \in T \mid i < \kappa\}$; thus, $|K| = \kappa$. For simplicity, rename $\phi(i) < \kappa$ as $i < \kappa$. Suppose now that $j \in K$ and $S_j = \{i \in K \mid i > j\}$. Evidently, $S_j \subseteq T$, so S_j is not contained in any $\mathcal{U} \in M_j$. In view of the choice of the elements $\phi(i)$, S_j is not in any $\mathcal{U} \in F_j = M_j \cup N_j$ either. Hence $\pi_k \eta(\prod_{k<i \in K} \langle e_i \rangle) = 0$ for $k \in K$. $\square$

Lemma 6.5 (O'Neill [4]). *Let η be an endomorphism of P_κ such that (13.5) holds. If*

$$\pi_j \eta(\prod_{i>j} \langle e_i \rangle) = 0 \quad \textit{for every } j < \kappa,$$

then η is an isomorphism.

Proof. First, η is injective. For, if $\eta(x) = 0$, but $x = \sum_{i<\kappa} n_i e_i \neq 0$, then let j be the first index with $n_j \neq 0$. Now $\pi_j \eta(x) = \pi_j \eta(n_j e_j) + \pi_j \eta(\prod_{i>j} \langle e_i \rangle) = n_j e_j \neq 0$, a contradiction.

To see that η is surjective, pick $x = \sum_{i<\kappa} n_i e_i \in P_\kappa$. Let $j < \kappa$, and assume that for all $k < j$ we have found $m_k \in \mathbb{Z}$ such that for all $i < j$, $\pi_i \eta(\prod_{k \leq i} m_k e_k) = n_i e_i$. Now $\pi_j \eta(\prod_{k<j} m_k e_k) = t_j e_j$ for some $t_j \in \mathbb{Z}$. Setting $m_j = n_j - t_j$, we have $\pi_j \eta(\prod_{k \leq j} m_k e_k) = n_j e_j$. In this way, we can find integers m_i for all $i < \kappa$. Now $y = \sum_{i<\kappa} m_i e_i \in P_\kappa$ satisfies $\eta(y) = x$, since $\pi_j \eta(y) = \pi_j \eta(\sum_{k \leq j} m_k e_k) + \pi_j \eta(\sum_{i>j} m_i e_i) = n_j e_j$. $\square$

Lemma 6.6 (O'Neill [4]). *Let $P_\kappa = A \oplus B$ for an infinite cardinal κ. Then P_κ has a decomposition*

$$P_\kappa = \prod_{j \in J} \langle c_j \rangle \oplus E,$$

where $c_j \neq 0$, $|J| = \kappa$, and either A or B contains all the c_j.

Proof. Let $\alpha : P_\kappa \to A$ be the natural projection, and $\alpha_i = \pi_i \alpha$. Let J denote either the set $\{i < \kappa \mid \alpha(e_i) \neq 0\}$ or the set $\{i < \kappa \mid (1 - \alpha)(e_i) \neq 0\}$ whichever has cardinality κ; say, J is the first set. As $\langle e_i \rangle$ is a slender group, the subset $\{\alpha(e_i) \mid i \in J\}$ is summable (see (D)). An appeal to (E) completes the proof. $\square$

We are now prepared to prove the two main results of this section.

Theorem 6.7 (O'Neill [3]). *Suppose κ is an infinite cardinal, and $\mathbb{Z}^\kappa = A \oplus B$. Then either*

$$A \cong \mathbb{Z}^\kappa \quad \text{or} \quad B \cong \mathbb{Z}^\kappa.$$

Proof. In view of Lemma 6.6, we may assume $P_\kappa = \prod_{j \in J} \langle a_j \rangle \oplus E$ where $|J| = \kappa$ and $a_j \in A$ for all $j \in J$. Let $\rho_j : P_\kappa \to \langle a_j \rangle$, $\alpha : P_\kappa \to A$ and $\gamma : P_\kappa \to \prod_{j \in J} \langle a_j \rangle$ denote the natural projections. Consider the composite map $\eta = \gamma\alpha : \prod_{j \in J} \langle a_j \rangle \to \prod_{j \in J} \langle a_j \rangle$.

First, let κ be a regular cardinal. In view of Lemma 6.4, there is a subset $K \subseteq J$ of cardinality κ such that $\rho_j \eta(\prod_{j < k \in K} \langle a_k \rangle) = 0$ for all $j \in K$. Next write $\prod_{j \in J} \langle a_j \rangle = \prod_{k \in K} \langle a_k \rangle \oplus \prod_{j \in J \setminus K} \langle a_j \rangle$, and denote by ξ the projection to the first summand. Lemma 6.5 (applied to $\prod_{k \in K} \langle a_k \rangle \cong P_\kappa$) tells us that the composite map $\xi\eta : \prod_{k \in K} \langle a_k \rangle \to \prod_{j \in J} \langle a_j \rangle \to \prod_{k \in K} \langle a_k \rangle$ is an isomorphism. In view of this isomorphism, we can conclude that $C = \prod_{k \in K} \langle a_k \rangle$ is a summand: $A = C \oplus C'$. Thus A has a summand $C \cong P_\kappa$. The rest is easy: $A \cong P_\kappa \oplus C' \cong (C \oplus C' \oplus B)^{\aleph_0} \oplus C' \cong (C \oplus C' \oplus B)^{\aleph_0} \cong (A \oplus B)^{\aleph_0} \cong P_\kappa$.

If κ is singular, then the arguments in the preceding paragraph apply to every regular cardinal $\lambda < \kappa$. Thus, for every such λ, selecting a subset of κ, A contains a summand $C_\lambda \cong \mathbb{Z}^\lambda$. If we carefully check the maps involved, we find that they are independent of the choice of the subsets, so they can be combined to obtain a summand $C \cong \mathbb{Z}^\kappa$, as before. The conclusion is the same: $A \cong \mathbb{Z}^\kappa$. $\square$

Theorem 6.8 (O'Neill [4]). *For the first measurable cardinal μ, every summand of $\mathbb{Z}^\mu$ is again a power of $\mathbb{Z}$.*

Proof. From Theorem 6.7 we know that if $P_\mu = A \oplus B$ (with projections α, β), then one of the components, say A, is isomorphic to P_μ. First, suppose $|B| < \mu$. Define an equivalence relation on the index set by declaring i and j equivalent if and only if $\pi_i(b) = \pi_j(b)$ for each $b \in B$. This gives a partition of the index set into $< \mu$ equivalence classes X_j $(j \in J)$, and B is evidently a summand of $\prod_{j \in J} \langle \sum_{i \in X_j} e_i \rangle$ which is a non-measurable vector group. Therefore, by Theorem 5.6, B is a vector group.

If $|B| = \mu$, then $\{\beta e_i \mid i < \kappa\} \subset B$ is a summable set of cardinality κ. Application of (E) yields a completely independent summable set of the same cardinality in B, and the arguments in the proof of Theorem 6.7 for B (rather than for A) complete the proof. □

Subgroup in Large Product The final topic in this section is concerned with relatively small subgroups in $\mathbb{Z}^I$ for arbitrary index set I.

Proposition 6.9 (O'Neill [5]). *Let A denote a subgroup of cardinality κ in the product $\mathbb{Z}^I$. Then A is contained in a subgroup C of $\mathbb{Z}^I$ such that $|C| \leq 2^\kappa$.*

Proof. Assume $\kappa < |I|$, and let $a = (\dots, a_i, \dots) \in A$. Define $i, j \in I$ equivalent if $a_i = a_j$ for all $a \in A$. This gives rise to a partition Π of I where i, j belong to the same equivalence class $u \in \Pi$ if they are equivalent. Each u defines a homomorphism $\pi_u : A \to \mathbb{Z}$ by $\pi_u(a) = a_i$ if $i \in u$. Clearly, $|\Pi| \leq |\mathbb{Z}^\kappa| = 2^\kappa$. Let $e_u \in \mathbb{Z}^I$ denote the vector whose ith coordinate is 1 or 0 according as $i \in u$ or not. Let J be a subset of I that contains one representative from each u. Then $\mathbb{Z}^I = C \oplus \mathbb{Z}^{I \setminus J}$ where the subgroup $C = \prod_{u \in \Pi} \langle e_u \rangle$ contains A. □

★ **Notes.** Eda [1] extends Lemma 5.4 by showing that every slender summand of a product $\prod_{i \in I} A_i$ of non-measurable torsion-free groups A_i is isomorphic to a summand of a direct sum of finitely many A_i.

Fink [1] calls a set $F = \{\mathcal{U}_i \mid i \in I\}$ of $\aleph_1$-complete ultrafilters on the power set $P(\kappa)$ *completely independent* if there is a partition $\kappa = \cup_{i \in I} X_i$ such that $X_i \in \mathcal{U}_i \setminus \cup_{j \neq i} \mathcal{U}_j$. He shows that if $\mathbb{Z}^\kappa = A \oplus B$, and if the set F of ultrafilters associated with the natural projection $\alpha : \mathbb{Z}^\kappa \to A$ is completely independent, then $A \cong \mathbb{Z}^{|F|}$. The results on $\mathbb{Z}^\kappa$ can be modified and extended to powers of any rational group $< \mathbb{Q}$. Another result of interest is due to Huber [1]: assuming V = L, for a rational group $R < \mathbb{Q}$, a subgroup A of a power R^I is a summand if and only if R^I/A is likewise a power of R.

Exercises

(1) A summable subset generates a free subgroup.
(2) A subset of P_κ that is summable relative to a decomposition of P_κ is also summable relative to any other decomposition.
(3) (a) Every element $x \in P_\kappa$ can be written in the form $x = \sum_{n \in \mathbb{Z}} n e_{X_n}$, where the X_n ($n \in \mathbb{Z}$) are disjoint subsets of I, and $e_{X_n} = \sum_{i \in X_n} e_i$;
    (b) for an $\aleph_1$-complete ultrafilter $\mathcal{U}$ on the power set $\mathcal{P}(\kappa)$, there is exactly one index $m \in \mathbb{Z}$ such that $X_m \in \mathcal{U}$.
(4) Let A be a summand of P_κ, κ any infinite cardinal. Then $A \oplus P_\kappa \cong P_\kappa$.
(5) If $G \neq 0$ is a slender group, then G^I is isomorphic to a subgroup of G^J if and only if $|I| \leq |J|$.
(6) (O'Neill) Let $X = \{x_j \mid j \in J\} \subset P_\kappa$ be a summable subset. Write $x_j = \sum_{n \in \mathbb{Z}} n e_{Y_{jn}}$ where $\{Y_{jn} \mid n \in \mathbb{Z}\}$ is a partition of I for each $j \in J$, and $e_K = \sum_{k \in K} e_k$ for $K \subseteq I$. Then $\{e_{Y_{jn}} \mid j \in J, n \in \mathbb{Z}\}$ is a summable subset.

(7) (a) (O'Neill) A subgroup of P_κ of cardinality λ is contained in a summand of P_κ that is $\cong P_{\kappa'}$ with $\kappa' \leq 2^\lambda$.
(b) Derive from (a) that a non-measurable summand of any P_κ is a product.

7 Whitehead Groups If V = L

By a **Whitehead group** or, in short, by a **W-group** (Rotman) is meant a group A satisfying

$$\mathrm{Ext}(A, \mathbb{Z}) = 0,$$

i.e. every extension of $\mathbb{Z}$ by A is splitting. Evidently, free groups are W-groups, and Whitehead's question was whether or not the converse is true, i.e. *are all W-groups free?* After several unsuccessful attempts by various algebraists, it was S. Shelah who took the bold step of approaching the problem in a revolutionary way, and found a flabbergasting answer to this question: *it is undecidable in ZFC,* i.e. it can be neither proved nor refuted within the axiom system ZFC. We direct our efforts towards a proof of this result, without exploring consequences or related problems.

But how can one prove that a problem is undecidable? An "easy" way would be to show that it is equivalent to a known undecidable problem—this approach is hard to come by in a case when no such problem exists that is related. It remains to follow the standard road to proving undecidability: to create two models of set theory, one in which the answer is positive, and another one which leads to a negative answer. Such models are constructed by adjoining appropriate new axioms to the existing axioms of ZFC, new axioms that are consistent with the old axioms. To carry out this task for the Whitehead problem, we follow Shelah [1], and first assume Jensen's Diamond Principle (see Sect. 4 in Chapter 1) to obtain an affirmative answer; then we move to a model of ZFC with Martin's Axiom and $\neg$CH adjoined, in which the answer will be in the negative.

Elementary Results on W-Groups The proofs are not easy and involve lots of technicalities. One has to understand well what kind of influence the added axiom has on the problem. We devote this section and the next one to the presentation of a complete proof of undecidability.

In order to assemble basic facts, we list the following properties which can easily be derived staying in the realm of ZFC.

(a) *Subgroups of W-groups are again W-groups.* If B is a subgroup of a W-group A, then the inclusion map $B \to A$ induces an epimorphism $0 = \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(B, \mathbb{Z})$.
(b) *Direct sums of W-groups are again W-groups.* This follows at once from the isomorphism $\mathrm{Ext}(\oplus_{i \in I} A_i, \mathbb{Z}) \cong \prod_{i \in I} \mathrm{Ext}(A_i, \mathbb{Z})$.

(c) *The class of W-groups is closed under extension: if A, C are W-groups in the exact sequence* $0 \to A \to B \to C \to 0$, *then so is* B. This is immediate consequence of the exactness of the induced sequence $0 = \mathrm{Ext}(C, \mathbb{Z}) \to \mathrm{Ext}(B, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) = 0$.
(d) *W-groups are torsion-free.* In view of (a), it suffices to show that a cyclic group $\mathbb{Z}(p)$ is not a W-group. But this follows at once from the observation that $\mathbb{Z}$ is a non-splitting extension of $p\mathbb{Z} \cong \mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}(p)$.
(e) *A W-group of finite rank is free.* Let A be a torsion-free group of finite rank n, and F a free subgroup of rank n; thus, A/F is a torsion group. If A is not finitely generated, then A/F is infinite, and in this case $\mathrm{Ext}(A/F, \mathbb{Z})$ has to be (at least) of the power of the continuum. This follows at once from Corollaries 3.6 in Chapter 9 and 1.8 in Chapter 7, but it is easy to prove it directly. (In fact, if A/F contains a copy of $\mathbb{Z}(p^\infty)$ for some prime p, then $\mathrm{Ext}(A/F, \mathbb{Z})$ contains $\mathrm{Ext}(\mathbb{Z}(p^\infty), \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong J_p$. If A/F contains no copy of $\mathbb{Z}(p^\infty)$ for any prime p, then A/F has an infinite socle, so $\mathrm{Ext}(A/F, \mathbb{Z})$ maps upon an infinite product $\prod_p \mathrm{Ext}(\mathbb{Z}(p), \mathbb{Z}) \cong \prod_p \mathbb{Z}(p)$, and therefore its cardinality is $\geq 2^{\aleph_0}$.) Consider the exact sequence $0 \to F \to A \to A/F \to 0$; it induces the exact sequence $\mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Ext}(A/F, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z})$. Here $\mathrm{Hom}(F, \mathbb{Z}) = \oplus \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \oplus \mathbb{Z}$ is finitely generated free, so countable. Hence if A is a W-group, i.e. if $\mathrm{Ext}(A, \mathbb{Z}) = 0$, then $\mathrm{Ext}(A/F, \mathbb{Z})$ cannot be uncountable. Thus then A/F must be finite, so A is finitely generated free.
(f) (Rotman [2]) *W-groups are separable.* Let F be a finite rank pure subgroup of a W-group A. The exact sequence in (e) implies that there is an epimorphism $\mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Ext}(A/F, \mathbb{Z})$. In our case, Hom is finitely generated, while the Ext is divisible (see Sect. 3(F) in Chapter 9), so it must be 0. Hence also $\mathrm{Ext}(A/F, F) = 0$, i.e. F is a summand of A.

We have come to the first major result of this section.

Theorem 7.1 (Ehrenfeucht [1]). *Countable Whitehead groups are free. Hence all Whitehead groups are $\aleph_1$-free.*

Proof. In view of (f), countable W-groups are completely decomposable (cf. Theorem 4.3). By (e), their finite rank subgroups are free, so they are free as well. □

Uncountable W-Groups We now move to the case of W-groups of uncountable cardinalities, at this point, still no additional axiom is required. Obstacles to extend Theorem 7.1 quickly present themselves already at cardinality $\aleph_1$. Since it suffices to prove undecidability at $\aleph_1$, we will focus our attention on W-groups of this cardinality, but will prove results on all uncountable regular cardinals whenever no extra effort is required.

We shall need the following lemmas.

Lemma 7.2. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of torsion-free groups, where B is a* W*-group, but C is not. There is a homomorphism* $\eta : A \to \mathbb{Z}$ *that fails to extend to a homomorphism* $B \to \mathbb{Z}$.

Proof. Form the exact sequence $\mathrm{Hom}(B, \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Ext}(C, \mathbb{Z}) \to \mathrm{Ext}(B, \mathbb{Z}) = 0$. By hypothesis, the first Ext does not vanish, so the map between the two Homs is not epic. Hence the claim is obvious. □

Lemma 7.3 (Shelah [1]). *As before, let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be an exact sequence where* B *is a* W*-group, but* C *is not. Consider the splitting exact sequences in the rows of the following diagram where* π, π' *denote the projections onto the second summands. Let* σ *be the obvious injection map* $A \to \mathbb{Z} \oplus A$*. Then there is a homomorphism* $\chi : \mathbb{Z} \oplus A \to \mathbb{Z} \oplus B$ *making the diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z} \oplus A \arrow[r, "\pi"] \arrow[d, "\chi"'] & A \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus B \arrow[r, "\pi'"] & B \arrow[r] & 0
\end{tikzcd}$$

commute such that there is no splitting map $\tau : B \to \mathbb{Z} \oplus B$ *for* π' *with* $\chi\sigma = \tau\alpha$.

Proof. Let η be the map as stated in the preceding lemma. Define χ to map $(n, a) \in \mathbb{Z} \oplus A$ onto $(n + \eta a, \alpha a) \in \mathbb{Z} \oplus B$. It is readily checked that—due to the choice of η—there is no splitting map $\tau : B \to \mathbb{Z} \oplus B$ with the required property. □

Consequences of V = L The stage is now set to move to a model of set theory in which the Axiom of Constructibility holds. Our first order of business is to discuss its immediate consequences concerning W-groups. The reader is reminded that the Diamond Principle $\diamondsuit$ is a consequence of V = L.

In the next proof, it will be convenient to consider an extension G of $\mathbb{Z}$ by A to be a group on the set $\mathbb{Z} \times A$. Accordingly, if $\{A_\alpha\}_{\alpha<\kappa}$ is a filtration of A, then $\{\mathbb{Z} \times A_\alpha\}_{\alpha<\kappa}$ is a filtration of G.

Lemma 7.4 (Shelah [1]). *Assuming* $\diamondsuit$*, let* κ *be an uncountable regular cardinal, and* A *a torsion-free group of cardinality* κ*. Suppose that* A *has a filtration* $\{A_\alpha\}_{\alpha<\kappa}$ *where*

(i) *each* A_α *has cardinality* $< \kappa$;
(ii) *each* A_α *is a pure* W*-subgroup of* A;
(iii) $E = \{\alpha < \kappa \mid A_{\alpha+1}/A_\alpha$ *is not a* W*-group*$\}$ *is stationary in* κ.

Then A *is not a Whitehead group.*

Proof. Hypothesis $\diamondsuit$ implies that there is a family $\{g_\alpha\}_{\alpha\in E}$ of functions $g_\alpha : A_\alpha \to \mathbb{Z} \times A_\alpha$ $(\alpha \in E)$ such that, for every function $g : A \to \mathbb{Z} \times A$, the set $E' = \{\alpha \in E \mid g \restriction A_\alpha = g_\alpha\}$ is stationary in κ.

We construct a non-split exact sequence $\mathfrak{e} : 0 \to \mathbb{Z} \to G \xrightarrow{\pi} A \to 0$ as a direct limit of splitting exact sequences $\mathfrak{e}_\alpha : 0 \to \mathbb{Z} \to G_\alpha \xrightarrow{\pi_\alpha} A_\alpha \to 0$ for $\alpha < \kappa$ such that the underlying set for G_α is $\mathbb{Z} \times A_\alpha$, and whenever $\delta < \alpha$, there is a commutative diagram

$$\begin{array}{cccccccccc} \mathfrak{e}_\delta\colon & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & G_\delta & \xrightarrow{\pi_\delta} & A_\delta & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow & & \\ \mathfrak{e}_\alpha\colon & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & G_\alpha & \xrightarrow{\pi_\alpha} & A_\alpha & \longrightarrow & 0 \end{array} \tag{13.6}$$

where the right vertical map is the inclusion map. Let $\alpha < \kappa$, and assume that the exact sequences $\mathfrak{e}_\beta$ have been defined for all $\beta < \alpha$ such that all the required diagrams commute.

Case 1. If α is a limit ordinal, then $\mathfrak{e}_\alpha$ is defined as the direct limit of the exact sequences $\mathfrak{e}_\beta$ with $\beta < \alpha$. This is a splitting sequence, since A_α is a W-group.

Case 2. Let $\alpha = \delta + 1$. If $\delta \notin E$ or if g_δ is not a splitting map for π_δ, then let $\mathfrak{e}_\alpha : 0 \to \mathbb{Z} \to G_\alpha \xrightarrow{\pi_\alpha} A_\alpha \to 0$ be an extension of $\mathfrak{e}_\delta$ with any choice of $G_\delta \to G_\alpha$ such that (13.6) commutes, and the underlying set for G_α is $\mathbb{Z} \times A_\alpha$. $\mathfrak{e}_\alpha$ is a splitting exact sequence because of condition (ii).

Case 3. Let $\alpha = \delta + 1, \delta \in E$, and assume that $g_\delta : A_\delta \to \mathbb{Z} \times A_\delta$ (= the underlying set for G_δ) is a splitting map for π_δ. From Lemma 7.3 we conclude that there is an extension $\mathfrak{e}_\alpha : 0 \to \mathbb{Z} \to G_\alpha \xrightarrow{\pi_\alpha} A_\alpha \to 0$ with a commutative diagram (13.6) such that there is no splitting map $\tau : A_\alpha \to G_\alpha$ for π_α matching π_δ. Again, we view G_α to be a group on the set $\mathbb{Z} \times A_\alpha$.

We now define $\mathfrak{e} : 0 \to \mathbb{Z} \to G \xrightarrow{\pi} A \to 0$ as the direct limit of the splitting exact sequences $\mathfrak{e}_\alpha : 0 \to \mathbb{Z} \to G_\alpha \xrightarrow{\pi_\alpha} A_\alpha \to 0$ for $\alpha < \kappa$ where $G = \mathbb{Z} \times A$ as sets. By way of contradiction, assume there is a splitting homomorphism $g : A \to G$ for π. Note that we must have $g(A_\alpha) \leq G_\alpha$ for every $\alpha < \kappa$. By the choice of the g_α, there is a $\delta \in E$ (actually, stationarily many of them) such that $g \restriction A_\delta = g_\delta$. This means that $\mathfrak{e}_{\delta+1}$ has been constructed according to Case 3 above. Since $g \restriction A_{\delta+1}$ is both a splitting map for $\mathfrak{e}_{\delta+1}$ and an extension of $g \restriction A_\delta$ (which is impossible by construction), we reached a contradiction to the existence of g. Thus A cannot be a W-group. $\square$

We shall now state an auxiliary lemma that contains an important step in the proof of Theorem 7.6.

Lemma 7.5. *Assume $\diamondsuit$. A* W*-group A of uncountable regular cardinality κ satisfies:*

($\star$) *Every subgroup B of cardinality $< \kappa$ is contained in a pure subgroup C of cardinality $< \kappa$ such that D/C is a* W*-group for every subgroup D of cardinality $< \kappa$ satisfying $C < D < A$.*

Proof. Assume toward a contradiction that ($\star$) fails for A, i.e. there is a (pure) subgroup B of cardinality $< \kappa$ such that, for every pure subgroup $C > B$ of cardinality $< \kappa$, we can find a subgroup $D > C$ of cardinality $< \kappa$ such that D/C is not a W-group. Define a smooth chain $\{B_\alpha\}_{\alpha<\kappa}$ of pure subgroups in A as follows. Start with $B_0 = B$. If $\alpha < \kappa$ and B_β has been defined for all $\beta < \alpha$, then

1. if α is a limit ordinal, then set $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$;
2. if $\alpha = \delta + 1$, then using $(^\star)$, take for B_α a pure subgroup of A of cardinality $< \kappa$ that contains B_δ such that B_α/B_δ is not a W-group.

As a subgroup of A, $B' = \bigcup_{\alpha<\kappa} B_\alpha$ is a W-group. On the other hand, the preceding lemma with $E = \kappa$ applies, showing that B' cannot be a W-group. We have reached a contradiction, so $(^\star)$ holds. □

We are now able to prove:

Theorem 7.6 (Shelah [1]). *Assume again $\diamond$. Let κ be an uncountable regular cardinal, and A a Whitehead group of cardinality κ. If all Whitehead groups of cardinalities $< \kappa$ are free, then A is free.*

Proof. We use $(^\star)$ in Lemma 7.5 as a guidance to create a κ-filtration $\{A_\alpha\}_{\alpha<\kappa}$ of A (now D/C is a W-group of cardinality $< \kappa$, thus free). To start with, choose a maximal independent set $\{a_\alpha\}_{\alpha<\kappa}$ in A. Set $A_0 = 0$, and suppose that $\alpha < \kappa$ and the pure groups A_β have been defined for all $\beta < \alpha$ such that

(i) they are of cardinalities $< \kappa$,
(ii) they form a smooth chain, and
(iii) A_β contains a_γ for all $\gamma < \beta$.

Then for a limit ordinal α, set $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, as required. In case α is a successor ordinal, say, $\alpha = \delta + 1$, then let B be the pure subgroup in A that is generated by A_δ and a_δ, and choose for A_α the subgroup C obtained by applying $(^\star)$ to this B.

Since A is a W-group, the set $E = \{\alpha < \kappa \mid A_{\alpha+1}/A_\alpha \text{ is not free}\}$ cannot be stationary in κ, as is obvious in view of Lemma 7.4. Thus we can be sure that there is a cub F in κ which does not intersect E; write $F = \{f(\alpha)\}_{\alpha<\kappa}$. To finish the proof it suffices to show that $\{A_{f(\alpha)}\}_{\alpha<\kappa}$ is a filtration of A with free factor groups $A_{f(\alpha+1)}/A_{f(\alpha)}$. But this follows at once from that $A_{f(\alpha)+1}/A_{f(\alpha)}$ is free by construction, and that $A_{f(\alpha+1)}/A_{f(\alpha)+1}$ is free as $A_{f(\alpha)+1}$ is a C in the application of $(^\star)$. □

We have obtained enough information on our way toward the proof of the main theorem of this section.

Theorem 7.7 (Shelah [1]). *In the Constructible Universe, Whitehead groups of any cardinality are free.*

Proof. We induct on the cardinality κ of the W-group A. Theorem 7.1 takes care of the countable case, so assume that A is a W-group of uncountable cardinality κ, and all W-groups of cardinalities $< \kappa$ are free. If κ is a regular cardinal, then Theorem 7.6 guarantees that A is free. If κ is a singular cardinal, then the freeness of A is a consequence of the singular compactness theorem (Theorem 9.2 in Chapter 3). □

★ **Notes.** According to A. Ehrenfeucht, J.H.C. Whitehead raised the problem in 1952 during his visit to Warsaw. Unaware of the solution by K. Stein [Math. Annalen **123**, 201–222 (1951)] of an equivalent problem in the countable case (applied to the second Cousin problem), Ehrenfeucht [1] published his solution for countable groups in the equivalent form: if H is a subgroup of a countable free group F such that every $\phi\colon H \to \mathbb{Z}$ extends to a $\psi\colon F \to \mathbb{Z}$, then H is a summand of F.

It should also be mentioned that J. Dixmier [Bull. Sci. Math. France **81**, 3–48 (1957)] unveils an intimate relation between the Pontryagin duals of Whitehead groups, and the arcwise (locally arcwise) connected locally compact abelian groups.

Exercises

(1) Let A be torsion-free of finite rank. If A is not free, then $|\operatorname{Ext}(A,\mathbb{Z})| = 2^{\aleph_0}$.
(2) A group A is free if it satisfies $\operatorname{Ext}(A,F) = 0$ for all free groups F.
(3) For a reduced group A, the following are equivalent:
   (a) $\operatorname{Ext}(A,\mathbb{Z})$ is torsion-free;
   (b) A has the projective property relative to the exact sequences $0 \to n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ for all integers n. [Hint: apply the functor $\operatorname{Hom}(A,*)$ to this exact sequence.]
(4) In every model of ZFC, the class of W-groups is closed under isomorphisms, direct sums, summands, and extensions.

8 Whitehead Groups Under Martin's Axiom

Having proved that W-groups are free in some model of ZFC, in order to verify the claimed undecidability of the Whitehead problem, the issue is to find a suitable hypothesis which, if added to the axioms of ZFC, will provide us examples of non-free W-groups. As we have already mentioned above, such a hypothesis is Martin's Axiom (MA), which we have formulated in Sect. 4 of Chapter 1. (Application of MA makes sense only for uncountable cardinals $< 2^{\aleph_0}$, so we will have to deny CH.)

Consequences of Martin's Axiom Consequently, we place ourselves in the following setting: ZFC + MA + ¬CH, and we are in search for an example of a non-free W-group in this model of set theory. Of course, we are free to use the results which we have proved in the preceding section for every model of ZFC.

We turn to the proof of the crucial lemma which already requires the full strength of Martin's Axiom. The hard work goes into showing that the selected poset has the requisite properties.

Lemma 8.1. *Assume* ZFC+MA+¬CH. *Any separable $\aleph_1$-free group of cardinality $\aleph_1$ that satisfies condition $(^\star)$ in Lemma 7.5 for $\kappa = \aleph_1$ is a* W-*group.*

Proof. Suppose A is a separable $\aleph_1$-free group of cardinality $\aleph_1$ (this cardinality is now less than the continuum) satisfying $(^\star)$, and let

$$\mathfrak{e}\colon 0 \to \mathbb{Z} \to G \xrightarrow{\pi} A \to 0$$

be an exact sequence. We want to show that $\mathfrak{e}$ is splitting by establishing the existence of a map $\delta\colon A \to G$ satisfying $\pi\delta = \mathbf{1}_A$.

To get prepared for the application of MA, we have to select a suitable partially ordered set. This will be the set P of all pairs (S, δ_S) where S is a finitely generated pure subgroup (thus a summand) of A, and $\delta_S\colon S \to G$ satisfies $\pi\delta_S = \mathbf{1}_S$. Define a partial order on P in the obvious way:

$$(S, \delta_S) \le (S', \delta_{S'})$$

should mean that $S \le S'$ and $\delta_S = \delta_{S'} \restriction S$.

Let us point out right away that every finitely generated pure subgroup S of A occurs in a pair $(S, \delta_S) \in P$, since every S is a free summand of A. However, there are but countably many pairs $(S, \delta_S) \in P$ with the same S; in fact, the action of δ_S is completely determined on the generators of S, and the image of an element of S under δ_S always belongs to the same coset mod $\mathbb{Z}$. Hence it should be clear that, for a fixed S, δ_S can act in at most countably many different ways.

In order to be able to apply MA, we have to study the upper bound situation in P. If both S and S' are pure subgroups of finite rank in A, then S is a summand of the purification $S^\star$ of $S + S'$ which is a summand of A. Now if (S, δ_S), (S', δ'_S) are in P, then $\delta_S\colon S \to G$ and $\delta'_S\colon S' \to G$ have a common extension δ' to $S + S'$ exactly if they coincide on $S \cap S'$. As $\pi\delta' = \mathbf{1}_{S+S'}$, it is easy to see that δ' extends uniquely to a map $\delta_{S^\star}\colon S^\star \to G$ such that $\pi\delta_{S^\star} = \mathbf{1}_{S^\star}$. Hence we conclude that for every choice of $a \in A$, the set

$$C_a = \{(S, \delta_S) \in P \mid a \in S\}$$

is cofinal in P. If we can show that P satisfies the countable antichain condition, then we can appeal to MA to obtain a directed subset D of P that intersects every C_a. This will define a map $\delta : A \to G$ satisfying $\pi\delta = \mathbf{1}_A$ if we set $\delta a = \delta_S a$ for $a \in A$ whenever $(S, \delta_S) \in D \cap C_a$. From the fact that D is directed it follows that δ is in fact a homomorphism, and thus it is a splitting map—and the proof will then be finished.

Therefore, it remains to verify the countable antichain condition on P. Suppose $Q = \{(S_\nu, \delta_\nu) \mid \nu < \omega_1\}$ is an uncountable subset of P. We want to show that Q (or a subset of Q) contains two elements which have a common upper bound in P. This will be accomplished in three steps.

Step 1. There is a pure free subgroup A' of A that contains S_ν for uncountably many $(S_\nu, \delta_\nu) \in Q$. To verify this claim, note that each S_ν has finite rank, so there is an integer m such that uncountably many S_ν have rank m; moreover, by dropping to a subset of Q, we may assume that all of S_ν have the same rank m. Pick a pure subgroup T of A of possibly highest rank that is contained in uncountably many S_ν (maybe $T = 0$). By a remark above, $\delta_\nu \restriction T$ must be one of at most countably many maps, so we may as well suppose that the $\delta_\nu \restriction T$ are equal for all $\nu < \omega_1$. T is a summand of A, so $S_\nu = T \oplus U_\nu$ for a subgroup U_ν of A. Set $A_0 = T$, and suppose that, for some $\nu < \omega_1$, we have defined a smooth chain A_μ of countable pure free subgroups of A, along with ordinals $\sigma_{\mu+1}$, such that $U_{\sigma_{\mu+1}} \leq A_{\mu+1}$ for all $\mu < \nu$. If ν is a limit ordinal, we let A_ν be the union of all A_μ with $\mu < \nu$. If $\nu = \gamma + 1$, then appeal to (*), and pick a countable $\aleph_1$-pure subgroup B_γ of A that contains A_γ. There is an index σ_ν such that $\sigma_\nu > \sigma_{\mu+1}$ for all $\mu < \nu$ and $U_{\sigma_\nu} \cap B_\gamma = 0$; otherwise, we get a contradiction to the maximal choice of T. Define A_ν as the purification of $A_\gamma + U_{\sigma_\nu}$ in A. Now $U_{\sigma_\nu} \cap B_\gamma = 0$ implies $A_\nu \cap B_\gamma = A_\gamma$, so A_ν / A_γ is free as it is isomorphic to a countable subgroup of A/B_γ. Completing this induction for all $\nu < \omega_1$, we arrive at $A' = \bigcup_{\nu<\omega_1} A_\nu$ and an uncountable subset $Q' = \{(S_{\sigma_{\nu+1}}, \delta_{\sigma_{\nu+1}}) \mid \nu < \omega_1\}$ of Q. Evidently, A' contains all $S_{\sigma_{\nu+1}}$ with $\nu < \omega_1$, and is free, since the factors in the chain are free.

Step 2. Let $Q = \{(S_\nu, \delta_\nu) \mid \nu < \omega_1\}$ be an uncountable subset of P such that every S_ν ($\nu < \omega_1$) is contained in a pure free subgroup A' of A. If X is a basis of A', then there is a subset $Q' = \{(S'_\nu, \delta'_\nu) \mid \nu < \omega_1\}$ of P such that $(S_\nu, \delta_\nu) \leq (S'_\nu, \delta'_\nu)$ for each $\nu < \omega_1$, and each S'_ν is generated by a subset of X. This follows from the observation that every (S, δ) can be extended to an (S'_ν, δ'_ν) such that S' is generated by basis elements in X.

Step 3. If every S_ν ($\nu < \omega_1$) in Q (as in Step 2) is generated by a subset of a basis X of a pure free subgroup A' of A, then Q contains two pairs with a common upper bound in P.

As in Step 1, we may assume that all S_ν are of the same rank, and all contain a summand T such that all the $\delta_\nu \restriction T$ are equal for $\nu < \omega_1$. Pick any $(S_0, \delta_0) \in Q$; evidently, $T \neq S_0$. By the choice of T, for every pure subgroup T' generated by a subset of X with $T < T' \leq S_0$, there are at most countably many $\nu < \omega_1$ with $T' \leq S_\nu$. Also, there are but countably many such T', so after omitting all these (S_ν, δ_ν) from Q, there are still uncountably many pairs left. If $(S_\mu, \delta_\mu) \in Q$ is one of these, then necessarily $S_0 \cap S_\mu = T$. This guarantees that δ_0 and δ_μ have a common extension to $S_0 + S_\mu$. This group is pure in A', and hence in A, as it is generated by a subset of X.

This completes the proofs of the countable antichain condition and the lemma. □

The Basic Example For the proof of the main theorem we have to ascertain the existence of a non-free example for Lemma 8.1; this is furnished by the following lemma (valid in ZFC).

Lemma 8.2. *There exists a separable torsion-free group of cardinality $\aleph_1$ which satisfies (*), but is not free.*

Proof. Follow the proof of Theorem 8.9 in Chapter 3 in constructing a group F of cardinality $\aleph_1$ that is the union of a smooth chain F_σ ($\sigma < \omega_1$) groups satisfying

(i) F_σ is a countable free group for all $\sigma < \omega_1$;
(ii) $F_{\sigma+1}/F_\sigma \cong \mathbb{Q}$ if $\sigma < \omega_1$ is a limit ordinal;
(iii) F_ρ/F_σ is free for all $\sigma < \rho < \omega_1$ if σ is a successor ordinal.

Such an F is $\aleph_1$-free, but not free, and in addition it satisfies (*). □

We now formulate our findings in

Theorem 8.3 (Shelah [1] (ZFC+MA+ ¬CH)). *There exist Whitehead groups of cardinality $\aleph_1$ which are not free.* □

Undecidability of the Whitehead Problem From Theorems 7.7 and 8.3 we now conclude:

Corollary 8.4 (Shelah [1]). *The Whitehead problem is undecidable in* ZFC. □

★ **Notes.** The history of W-groups is replete with unsuccessful attempts, furnishing meager information about the groups. The solution of the Whitehead problem in 1973 by Shelah was the first undecidable result in abelian group theory, and became the starting point of a new era in the theory, dominated by set-theoretical methods. The discovery of several unsolvable problems followed suit. The methods were extended to module theory, and today there is a vast amount of literature dealing with problems that can be traced back to the Whitehead problem, as witnessed by the excellent monograph by Eklof–Mekler [EM].

For a relatively easy proof for the undecidability, Eklof [3] is recommended.

Numerous publications deal with variants of the Whitehead problem, imposing various conditions on the model of set theory. For example Shelah [3], where CH is assumed.

Rychkov [2] considers the problem of vanishing Pext that is reminiscent to the Whitehead problem. Assume A, C are groups satisfying $|A| < |C| = \kappa$. If κ is a regular cardinal, and C_ν $(\nu < \kappa)$ is a filtration of C with pure subgroups, then $\mathrm{Pext}(C, A) = 0$ holds exactly if $\mathrm{Pext}(C_\nu, A) = 0$ and $\mathrm{Pext}(C_{\nu+1}/C_\nu, A) = 0$ for all $\nu < \kappa$. He also establishes a kind of singular compactness result by proving that if κ is singular, then $\mathrm{Pext}(C, A) = 0$ if and only if $\mathrm{Pext}(X, A) = 0$ for all pure subgroups X of C of cardinality $< \kappa$. On the other hand, assuming ZFC+MA+ ¬CH, if $|A| = \aleph_0$, and if C is of cardinality $\aleph_1$ such that every countable subgroup is contained in a countable Σ-cyclic direct summand, then $\mathrm{Pext}(C, A) = 0$.

Exercises

(1) (Chase) If $|\operatorname{Ext}(G, \mathbb{Z})| < 2^{\aleph_0}$, then G is a direct sum of a finite group and an $\aleph_1$-free group. [Hint: argue as in Sect. 7(e), $t(G/F)$ must be finite.]
(2) (Rotman, Nunke) W-groups are slender. [Hint: $\operatorname{Ext}(\mathbb{Z}^{\aleph_0}, \mathbb{Z}) \neq 0$; refer to algebraic compactness of P/S.]
(3) (Griffith) An $\aleph_1$-free $\aleph_1$-coseparable torsion-free group is a W-group. [Hint: Theorem 4.17.]
(4) Prove that MA+¬CH implies that for every uncountable cardinal κ there exist non-free Whitehead groups.
(5) What can be said about a group A which has the property that every homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ lifts to a map $A \to \mathbb{Q}$?

Problems to Chapter 13

PROBLEM 13.1. Are summands of $\aleph_n$-free $\aleph_n$-separable torsion-free groups again $\aleph_n$-separable?

In the case $n = 1$, the answer is given in a few special cases in Mekler [2].

PROBLEM 13.2. Let $\kappa < \lambda$ denote infinite cardinals, and A the subgroup in $\mathbb{Z}^\lambda$ that consists of vectors whose supports are of cardinality $< \kappa$. Is every summand of A of the same form? In particular, if κ is measurable, and λ is not.

PROBLEM 13.3. When is the tensor product of two reflexive groups again reflexive?

PROBLEM 13.4. (Eklof–Mekler) Does there exist a reflexive group of measurable cardinality?

PROBLEM 13.5. Characterize vector groups A such that $A = B \oplus C$ implies that either $B \cong A$ or $C \cong A$.

PROBLEM 13.6. (J. Martinez) Which torsion-free groups admit archimedean lattice-order?

Chapter 14
Butler Groups

Abstract The theory of Butler groups is one of the most elaborate branches of abelian groups. We devote a whole chapter to its study. This may seem excessive, since Butler groups are very special, but the fact that the results and the methods provide more than a superficial glimpse into a fascinating theory (the most extensive one today on torsion-free groups of arbitrary rank) is a compelling reason for including a broader discussion.

In the finite rank case, Butler groups are torsion-free generated by a finite number of rank 1 groups. It is very tempting to think that a kind of finite generation makes them susceptible to a satisfactory classification; however, so far no comprehensive theory has emerged, though many encouraging results are available. We prove the basic results on them, but we are unable to dig deeply into the theory without getting involved in overcomplicated details. An important part of the theory aims at finding more tractable classes of finite rank Butler groups. The fast developing, very successful theory of almost completely decomposable groups is well documented in Mader[Ma].

We investigate more thoroughly Butler groups of large cardinalities; their theory has gained considerable popularity in the last quarter of the twentieth century, and even today they remain under intense scrutiny. It is an illuminating experience to see the effect of set theory on their algebraic structure.

1 Finite Rank Butler Groups

The only class of finite rank torsion-free groups that is larger than the class of completely decomposable groups and has a well developed theory is the class of Butler groups. This class is more manageable than finite rank torsion-free groups in general, but still general enough to generate a variety of challenging problems. This is still a fertile research area. The theory originates from Butler [1] and Bican [3], using different approaches. The fundamental facts on this class of groups are due to them.

Basic Properties of Butler Groups After these prefatory remarks, we now state the precise definition. A torsion-free group of finite rank is called a **Butler group** if it is a pure subgroup of a completely decomposable group (of finite rank).

Evidently, all completely decomposable groups of finite rank, in particular, all rank 1 groups, are Butler groups. The main interest lies, of course, in Butler groups that fail to be completely decomposable.

Example 1.1. Let A be the direct sum: $A = A_1 \oplus A_2 \oplus A_3$ where each A_i is torsion-free of rank 1 such that they contain elements $a_i \in A_i$ of characteristics $(\infty, \infty, 0, 0, \dots)$, $(\infty, 0, \infty, 0, 0, \dots)$,

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and $(0, \infty, \infty, 0, 0, \dots)$, respectively. The pure subgroup generated by the elements $a_1 - a_2$, $a_2 - a_3, a_3 - a_1$ is a rank two indecomposable Butler group.

Example 1.2. Let $\{q, p_1, \dots, p_n\}$ be a set of primes, and V a $\mathbb{Q}$-vector space with basis $\{e_1, \dots, e_n\}$. Then the group

$$G = \langle p_1^{-\infty} e_1, \dots, p_n^{-\infty} e_n, q^{-1}(e_1 + e_2), \dots, q^{-1}(e_1 + e_n) \rangle \leq V$$

is an indecomposable Butler group of rank n. (If we replace q^{-1} by $q^{-\infty}$, the arising group is still Butler.)

Example 1.3. No pure subgroup of rank $n \geq 2$ in the group J_p of the p-adic integers is a Butler group.

In a completely decomposable group $A = A_1 \oplus \cdots \oplus A_n$, where each A_i is of rank 1, the types of elements are evidently finite intersections of types of the A_i, so Type(A) is finite. Since the typeset of a pure subgroup is always a subset of the typeset of the group, it follows at once that *finite rank Butler groups have finite typesets.*

Butler's Theorem The next theorem will show that there is an equivalent way of defining a Butler group. The two definitions are used interchangeably.

Let $A = B_1 + \cdots + B_m$ be a torsion-free group where the B_j's are subgroups of rank one. This is equivalent to saying that A is an epimorphic image of the outer direct sum $A' = B_1 \oplus \cdots \oplus B_m$, which is a completely decomposable torsion-free group of finite rank. Here we may assume without loss of generality that the subgroups B_j are pure in A (they can always be replaced by their purifications), thus A is generated by a finite set of rank 1 pure subgroups.

The classical result on Butler groups reads as follows.

Theorem 1.4 (Butler [1]). *For a torsion-free group A the following are equivalent:*

(i) *A is a pure subgroup of a completely decomposable group of finite rank;*
(ii) *A is an epimorphic image of a completely decomposable group of finite rank.*

Proof. (i) $\Rightarrow$ (ii) Let A be a pure subgroup of a completely decomposable group $C = C_1 \oplus \cdots \oplus C_n$, where each C_i is of rank 1. For an $a \in A$ write $a = c_1 + \cdots + c_n$ with $c_i \in C_i$, and define the support of $a \in A$ as usual by $\operatorname{supp} a = \{i \mid c_i \neq 0\}$. Note that if $S = \operatorname{supp} a = \operatorname{supp} b$ (with $a, b \neq 0$ in A), then there are $s, t \in \mathbb{N}$ such that $\operatorname{supp}(sa - tb)$ is properly contained in S. Hence $sa = tb$ if S is minimal. This shows that, for every minimal support S, $\{a \in A \mid \operatorname{supp} a = S\} \cup 0$ is a rank 1 pure subgroup B_S of A.

Let $\{B_j \mid j \in J\}$ be the set of all rank one pure subgroups of A whose elements have minimal supports. Evidently, this is a finite set, say, $J = \{1, \dots, m\}$. In order to verify (ii) for A, it suffices to show that every $x \in A$ belongs to $B = B_1 + \cdots + B_m$. Clearly, for each $x \in A$ there is a $y \in A$ such that $\operatorname{supp} y$ is minimal and is a subset of $\operatorname{supp} x$. In view of the purity of A in C, for any given prime p, we can find an $i \in \operatorname{supp} y$ such that the ith coordinates x_i and y_i of x and y have the same heights at p. Then there are integers s, t with $(p, s) = 1$ such that $\operatorname{supp}(sx - ty)$ is a proper subset of $\operatorname{supp} x$. Using induction, we may assume $sx - ty \in B$. Thus $y \in B$ implies $sx \in B$, and so A/B is a torsion group. If we had an $x \in A \setminus B$ satisfying $px \in B$ for some prime p, then the existence of an integer s (prime to p) with $sx \in B$ would imply $x \in B$, a contradiction. Hence $A = B$, and (ii) follows.

(ii) $\Rightarrow$ (i) Suppose $A = B_1 + \cdots + B_m$ where the B_j are different pure subgroups of rank 1. If $m = 1$, there is nothing to prove. So assume $m > 1$, and the claim holds for groups with smaller m. Form the factor groups $X_j = A/B_j$, and consider the map

$$\phi : A \to X = X_1 \oplus \cdots \oplus X_m$$

defined by $\phi(a) = (a + B_1, \ldots, a + B_m)$. Since $\operatorname{rk} A > 1$, the intersection of two different B_j is 0, so ϕ is monic. Assume, by way of contradiction, that $\phi(A)$ is not pure in X, i.e., there are a prime p and an element $a \in A \setminus pA$ with $\phi(a) = p(x_1 + B_1, \ldots, x_m + B_m)$ for some $x_j \in A$. Then $b_j = a - px_j \in B_j \setminus pB_j$ for each j. Since the B_j are of rank 1, pB_j and b_j generate B_j, and so pA and a (or any of the b_j) generate A. Hence A/pA is cyclic of order p. This implies that none of the B_j is p-divisible, but every $X_j = A/B_j$ is p-divisible. This is impossible if the rank of A is > 1, since X_i is generated by the images of the B_j with $j \neq i$, and since a finite number of not p-divisible rank one subgroups cannot generate a p-divisible subgroup. Hence we conclude that ϕ embeds A as a pure subgroup in X. Again, X_i being generated by the images of the B_j with $j \neq i$, the induction hypothesis applies to the X_i: they are pure subgroups in completely decomposable groups of finite rank. This property carries over to their direct sum X, and hence it follows that A is likewise pure in a completely decomposable group. □

An easy consequence worthwhile recording is as follows.

Corollary 1.5. *A homogeneous finite rank Butler group is completely decomposable.*

Proof. A homogeneous finite rank Butler group B is an epic image of a finite rank completely decomposable group C that may be chosen to be homogeneous of the same type. The kernel of such an epimorphism $C \to B$ is a pure subgroup, so by Corollary 3.7 in Chapter 12 it is a summand of C. Therefore, Theorem 3.10 in Chapter 12 implies that B is likewise completely decomposable. □

The following claim is immediate from the definition and Theorem 1.4.

Lemma 1.6. *The class of finite rank Butler groups is closed under the formation of finite direct sums, pure subgroups and torsion-free epimorphic images.* □

In a Butler group B, of importance are the critical types. The factor group $B(\mathbf{t})/B^\star(\mathbf{t})_*$ is 0 or homogeneous of type $\mathbf{t}$ (we will see that it is completely decomposable). $\mathbf{t}$ is a **critical type** of B if this factor group is $\neq 0$. Evidently, the critical types of a completely decomposable group are the types of its rank 1 summands. The critical types play an important role in the following characterization of Butler groups.

Theorem 1.7 (Butler [1]). *A finite rank torsion-free group A is a Butler group if and only if it satisfies the following conditions:*

(a) Type(A) *is finite;*
(b) *for each critical type* $\mathbf{t}$, $A^\star(\mathbf{t})$ *has finite index in its purification* $A^\star(\mathbf{t})_*$;
(c) *for each critical type* $\mathbf{t}$, *there is a direct decomposition*

$$A(\mathbf{t}) = A_\mathbf{t} \oplus A^\star(\mathbf{t})_* \tag{14.1}$$

where $A_\mathbf{t}$ *is a homogeneous completely decomposable group of type* $\mathbf{t}$.

Proof. First suppose A is a Butler group. (a) has been observed above. As a pure subgroup of A, $A(\mathbf{t})$ is a Butler group, and hence $A_\mathbf{t}$ is likewise Butler. Since a homogeneous Butler group must be completely decomposable (see Corollary 1.5)(c) follows in view of Lemma 3.6 in Chapter 12. Considering that $A^\star(\mathbf{t})_*$ is a Butler group, we have $A^\star(\mathbf{t})_* = A_1 + \cdots + A_k + A^\star(\mathbf{t})$ for certain rank one pure subgroups A_i of A. If none of A_i can be dropped, then the A_i are all of type $\mathbf{t}$, and their intersections with $A^\star(\mathbf{t})$ are likewise of type $\mathbf{t}$. Hence each $A_i \cap A^\star(\mathbf{t})_*$ is of finite index in A_i, and therefore $A^\star(\mathbf{t})$ is of finite index in $A^\star(\mathbf{t})_*$.

Conversely, let A satisfy (a)–(c). (b) implies that $A^\star(\mathbf{t})_* = F_\mathbf{t} + A^\star(\mathbf{t})$ for a finitely generated free subgroup $F_\mathbf{t}$. Hence the subgroup $A' = \sum_\mathbf{t} A_\mathbf{t} + \sum_\mathbf{t} F_\mathbf{t}$ is by (a) a finite sum of rank one groups, so it is a Butler group. If $A \setminus A'$ is not empty, then choose an element $a \in A \setminus A'$ whose type $\mathbf{s}$ is maximal among the types of elements in $A \setminus A'$. Then

$$a \in A(\mathbf{s}) = A_\mathbf{s} \oplus A^\star(\mathbf{s})_* = A_\mathbf{s} \oplus (F_\mathbf{s} + A^\star(\mathbf{s})).$$

By the maximal choice of $\mathbf{s}$, we have $A^\star(\mathbf{s}) \le A'$, so also $A(\mathbf{s}) \le A'$, in contradiction to the choice of a. $\square$

Bican's Theorem The following remarkable characterization of Butler groups relies on the partition of primes. If Δ is a set of primes, then the **localization** $\mathbb{Z}_\Delta$ of $\mathbb{Z}$ at Δ is the group of all rational numbers whose denominators are relatively prime to every prime in Δ. It should be observed that an immediate consequence of the definition is that localizations of Butler groups are likewise Butler groups.

Theorem 1.8 (Bican [3]). *A finite rank torsion-free group B is a Butler group if and only if there is a partition*

$$\Pi = \Pi_1 \cup \cdots \cup \Pi_k$$

of the set Π of prime numbers such that for each ℓ ($\ell = 1, \ldots, k$), the tensor product $B \otimes \mathbb{Z}_{\Pi_\ell}$ (localization of B at Π_ℓ) is a completely decomposable group with totally ordered typeset.

Proof. Let B be a Butler group, viewed as a pure subgroup of the completely decomposable group $C = C_1 \oplus \cdots \oplus C_n$ where the C_i are regarded as subgroups of $\mathbb{Q}$ containing 1. Let Π_{ij} $(1 \leq i < j \leq n)$ denote the set of primes p for which $h_p(1)$ is not larger in C_i than in C_j, and define $\{\Pi_1, \dots, \Pi_k\}$ as the set of all non-empty intersections $\bigcap_{i<j} P_{ij}$ where, for each pair $i < j$, P_{ij} is equal either to Π_{ij} or to the complement of Π_{ij} in Π. For each ℓ, the subgroups $C_{i\ell} = C_i \otimes \mathbb{Z}_{\Pi_\ell}$ $(i = 1, \dots, n)$ of $\mathbb{Q}$ are totally ordered by inclusion, whence it follows that for each ℓ, $B \otimes \mathbb{Z}_{\Pi_\ell}$ (as a pure subgroup of $C \otimes \mathbb{Z}_{\Pi_\ell}$) has a totally ordered typeset. Such a Butler group has to be completely decomposable, since in its decomposition (14.1) with its minimal type $\mathbf{t}$, the first summand is completely decomposable, and so is the second by an induction hypothesis.

Conversely, it suffices to show that $B = \bigcap_{\ell=1}^{k}(B \otimes \mathbb{Z}_{\Pi_\ell})$ is isomorphic to a pure subgroup of the completely decomposable group $\oplus_{\ell=1}^{k}(B \otimes \mathbb{Z}_{\Pi_\ell})$, or more generally, $X_1 \cap \cdots \cap X_k$ is isomorphic to a pure subgroup of $X_1 \oplus \cdots \oplus X_k$ whenever $X_1, \dots, X_k$ are subgroups of a given torsion-free group. For $k = 2$ this is obvious from the exactness of the sequence $0 \to X_1 \cap X_2 \to X_1 \oplus X_2 \to X_1 + X_2 \to 0$, and the general case follows by a straightforward induction on k. □

Balanced-Projective Dimensions In the next theorem, which provides an upper bound for the balanced-projective dimension (bpd) of a Butler group (for the definition of bpd see Sect. 3 in Chapter 12), we consider strictly increasing sequences $\mathbf{t}_1 < \cdots < \mathbf{t}_n$ of types.

Lemma 1.9 (Arnold–Vinsonhaler [1]). *Let B be a finite rank Butler group, and n the maximal length of strictly increasing sequences of types in B. Then* bpd $B \leq n - 1$.

Proof. We argue by induction on n. $n = 1$ means that B is homogeneous, and therefore completely decomposable by Corollary 1.5; then bpd $B = 0$. Suppose $n > 1$, and let $0 \to A \to C \to B \to 0$ be a balanced-projective resolution of B, where C is completely decomposable. Hence we have $\mathrm{Type}(A) \leq \mathrm{Type}(C) \leq \mathrm{Type}(B)$; the second inequality holds if C is chosen with minimal typeset. Setting $A = C' \oplus A'$ where C' is completely decomposable, and A' has no rank 1 summand, we claim that $\mathrm{Type}(A')$ cannot contain any $\mathbf{t}_0$ that is maximal in $\mathrm{Type}(B)$. For if X is a pure rank 1 subgroup of A' of maximal type $\mathbf{t}_0$, then X is pure in C, so it must be a summand of $C(\mathbf{t}_0)$. It is then a summand of C, and hence of A', contradicting the choice of A'. Consequently, the maximal length of increasing sequences of types in A' is strictly less than $n-1$, so bpd $A' \leq n-2$ by induction hypothesis. As bpd $B =$ bpd $A + 1 =$ bpd $A' + 1$ (by standard dimension arguments), the assertion follows. □

Arnold–Vinsonhaler [1] exhibit an example for a Butler group B with type chains of maximal length n and bpd $B = n - 1$.

Example 1.10. Let $p_1 < p_2 < p_3$ be primes, and $\mathbb{Z}_{(p_i,\dots)} < \mathbb{Q}$ the localization of $\mathbb{Z}$ at the primes listed in the subscript. Define B as the kernel of the map $\beta : \mathbb{Z}_{(p_1)} \oplus \mathbb{Z}_{(p_2)} \oplus \mathbb{Z}_{(p_3)} \to \mathbb{Q}$, acting as an embedding on each summand. Clearly, $\operatorname{rk} B = 2$, and B is not completely decomposable. One can check that the sequence

$$0 \to \mathbb{Z}_{(p_1,p_2,p_3)} \xrightarrow{\alpha} \mathbb{Z}_{(p_2,p_3)} \oplus \mathbb{Z}_{(p_1,p_3)} \oplus \mathbb{Z}_{(p_1,p_2)} \longrightarrow B \to 0$$

is balanced-exact where $\alpha(x) = (-x, x, -x)$. It follows that $\operatorname{bpd} B = 1$.

Almost Completely Decomposable Groups Lady [1] defines an **almost completely decomposable group** as a finite extension of a finite rank completely decomposable group. These groups form a class of well treatable Butler groups. They have a fast growing, well-developed theory; for details, we refer to the very informative treatise Mader [Ma]. Here we only prove a typical result.

Let A be a finite rank Butler group. For each critical type $\mathbf{t}$, choose a completely decomposable subgroup $R_{\mathbf{t}} (= A_{\mathbf{t}})$ in the decomposition (1), and set $R = \oplus R_{\mathbf{t}}$, summation over the critical types. Such an R is called a **regulating subgroup**.

Theorem 1.11 (Lady [1]). *In an almost completely decomposable group, all regulating subgroups are isomorphic completely decomposable groups, and have the same finite index.*

Proof. Suppose R and S are regulating subgroups in the almost completely decomposable group A. By definition, they are completely decomposable, and since they have the same rank as A, they are isomorphic. The equality of the indices $|A : R|$ and $|A : S|$ is proved by induction on the number of critical types. Let $\mathbf{s}$ be a minimal type, and let $R' = \oplus_{\mathbf{t} \neq \mathbf{s}} R_{\mathbf{t}}$, so $R = R_{\mathbf{s}} \oplus R'$, and similarly, $S = S_{\mathbf{s}} \oplus S'$, and set $A' = A^{\star}(\mathbf{s})_*$. In view of the equality $|A : R| = |A : (R_{\mathbf{s}} \oplus A')| \cdot |(R_{\mathbf{s}} \oplus A') : R| = |A : (A_{\mathbf{s}} + A')| \cdot |A' : R'|$ and the same for S, the claim follows, as R', S' are regulating subgroups in A', so $|A' : R'| = |A' : S'|$ by induction hypothesis. ☐

Representation by Posets It would be a mistake to leave the theory of finite rank Butler groups without at least mentioning its close connection with the representation of partially ordered sets. This is an important branch in the theory of finite rank Butler groups.

Let S be a poset (partially ordered set), and K a field. By a **K-representation** of S we mean a K-vector space V along with subspaces V_i $(i \in S)$ such that $i \leq j$ in S implies $V_i \leq V_j$. The K-representations $\mathbf{V} = [V; V_i\ (i \in S)]$ of S form an (additive) category $\operatorname{Rep}(\mathsf{K}, S)$ if we define a morphism

$$\mathbf{V} = [V; V_i\ (i \in S)] \to \mathbf{W} = [W; W_i\ (i \in S)]$$

as a K-linear map $\phi : V \to W$ satisfying $\phi(V_i) \leq W_i$ for all $i \in S$.

Let B be a finite rank Butler group, and $T = \text{Type}(B)$. Consider the set T_0 of join-irreducible types $\mathbf{t} \in T$ under the reverse(!) ordering. Recall that $\mathbf{t} \in T$ is join-irreducible if $\mathbf{t} = \mathbf{t}_1 \vee \mathbf{t}_2$ $(\mathbf{t}_i \in T)$ implies $\mathbf{t} = \mathbf{t}_1$ or $\mathbf{t} = \mathbf{t}_2$. In the present case, $\mathsf{K} = \mathbb{Q}$, and the canonical $\mathbb{Q}$-representation of B is given by the correspondence

$$B \mapsto [\mathbb{Q} \otimes B;\ \mathbb{Q} \otimes B(\mathbf{t}) \mid \mathbf{t} \in T_0].$$

This defines a functor F_{T_0} from the quasi-isomorphism category of Butler groups with typeset in T to the category $\text{Rep}(\mathbb{Q}, T_0)$. It was already shown by Butler [1] that F_{T_0} is an isomorphism. We observe that $\text{Rep}(\mathbb{Q}, S)$ is a pre-abelian category with finite direct sums.

A plethora of interesting results are available for these representations; we refer to Arnold [A1], Arnold–Dugas [3] for details.

Special Butler Groups It is not our task to develop a full-fledged theory of finite rank Butler groups. Our program is less ambitious, but the theory is so extensive and fascinating that a couple of additional remarks has to be made. First, there are several treatable subclasses that deserve special attention. Needless to say, none of these subclasses offers a theory as impressive or as complete as that of completely decomposable groups. One of these classes consists of the **almost completely decomposable** groups mentioned above.

When it comes to investigating the relationship between being an image or a pure subgroup of a completely decomposable group, then the classes of $\mathcal{B}^{(m)}$-groups are in the center of attention. In the **Metelli classification** of finite rank Butler groups, a group B is said to be a $\mathcal{B}^{(m)}$**-group** if it is the quotient of a finite rank completely decomposable group by a pure subgroup of rank $m \geq 0$. In particular, $\mathcal{B}^{(0)}$ is the class of completely decomposable groups of finite rank, and obviously, we have a strictly increasing chain $\mathcal{B}^{(0)} \subset \mathcal{B}^{(1)} \subset \cdots \subset \mathcal{B}^{(m)} \subset \ldots$ of classes. So far only the $\mathcal{B}^{(1)}$-groups have been studied extensively. A more correct picture of finite rank Butler groups is likely to emerge when we will be able to find the right tools to deal with $\mathcal{B}^{(m)}$-groups for $m > 1$. The dual classification puts a Butler group B in class $\mathcal{B}_{(m)}$ if it is the kernel of a homomorphism of a finite rank completely decomposable group into the direct sum of m copies of $\mathbb{Q}$.

★ **Notes.** The groups discussed in this section were named by L. Lady *Butler groups* to credit Butler who initiated their study in 1965, thereby sawing the seeds of a rich and fascinating theory (Butler called them *diagrammatic groups*). These groups were rediscovered and discussed from the point of view of finite rank 1 generation by Bican later in 1980. Cf. also Koehler [1] in the rank 2 case.

Regulating subgroups play an important role in the theory of almost completely decomposable groups. The intersection of all regulating subgroups is the **regulator**, a fully invariant completely decomposable subgroup of finite index (Burkhardt [1]). For almost completely decomposable groups, Mader [Ma] is highly recommended.

Richman [6] defines a duality for Butler groups: the category of finite rank Butler groups whose typeset is contained in a fixed finite distributive lattice T is dual to the category where the typeset T' is anti-isomorphic to T. The duality is implemented by valuated homomorphisms into vector spaces over $\mathbb{Q}$. Another kind of duality was discussed by Arnold–Vinsonhaler [4], using groups A_i

that are rational groups of types in T. The kernels of maps $A_1 \oplus \cdots \oplus A_n \to \mathbb{Q}$ are the duals to the cokernels of diagonal embeddings $\cap_{i=1}^n A_i \to A_1 \oplus \cdots \oplus A_n$, called **bracket groups** $\mathcal{G}[A_1, \ldots, A_n]$.

The quest for complete sets of invariants has not been successful for Butler groups in general, but more or less satisfactory results have been obtained in very special cases. An in-depth study of $\mathcal{B}_{(1)}$-groups started with Richman [5] in a special case, and continued in the papers Arnold–Vinsonhaler [5], Goeters–Ullery–Vinsonhaler [1], and Hill–Megibben [10]. The study of $\mathcal{B}^{(1)}$-groups was initiated in Arnold–Vinsonhaler [4], continued by W.Y. Lee [1], Fuchs–Metelli [1], Metelli [3, 4], and developed in several papers by De Vivo–Metelli [1, 2], where the remarkable connection to matrices with $0, 1$ entries is explored in depth. Cf. also Yom [1] for a discussion of both cases. Arnold–Vinsonhaler [5] succeed in establishing complete sets of invariants for strongly indecomposable $\mathcal{B}^{(1)}$- and $\mathcal{B}_{(1)}$-groups. Vinsonhaler–Wallutis–Wickless [1] have encouraging results on a special subclass of $\mathcal{B}^{(2)}$-groups.

In a different vein, classes $K(n)$ ($n < \omega$) of finite rank Butler groups were defined by Kravchenko [2]: $K(0)$ is the class of all Butler groups. If $0 \to A \to C \to B \to 0$ is a balanced-exact sequence with completely decomposable C, then let $A \in K(n+1)$ provided $B \in K(n)$. For more on this classification, see Nongxa–Vinsonhaler [1]. Certainly, the study of finite rank Butler groups has a great deal of potential.

Arnold–Vinsonhaler [6] is an excellent survey. Butler groups have been studied from the constructive point of view by Richman [7] and Mines–Vinsonhaler [1]. For the classification problem, see Thomas [2].

Exercises

(1) Fully invariant subgroups of Butler groups are Butler groups.
(2) (Arnold) Balanced extensions of Butler groups by Butler groups are again Butler groups.
(3) A torsion-free group of finite rank with finite typeset need not be a Butler group.
(4) If A, B are Butler subgroups of a torsion-free group G, then both $A + B$ and $A \cap B$ are Butler groups. [Hint: $0 \to A \cap B \to A \oplus B \to A + B \to 0$.]
(5) Let B be a finite rank Butler group, and $\mathbf{t}$ a type. Then $B(\mathbf{t}) = B(\mathbf{t'})$ for some type $\mathbf{t'} \in \mathrm{Type}(B)$.
(6) Let A be a corank 1 pure subgroup of the finite rank Butler group B. Then $\mathbf{t}(B/A) = \sup \mathbf{t}(b)$ for $b \in B \setminus A$.
(7) Give a counterexample to the claim that a finite rank torsion-free group B is Butler if $B \otimes \mathbb{Z}_{(p)}$ is completely decomposable for every prime p.
(8) A Butler group with linearly ordered typeset is completely decomposable.
(9) The Metelli classes $\mathcal{B}^{(m)}$ are strictly increasing with increasing m.
(10) The tensor product of two finite rank Butler groups is again a Butler group.
(11) Homomorphism groups between finite rank Butler groups are Butler groups. In particular, the additive group of the endomorphism ring of a finite rank Butler group is again a Butler group.
(12) (Giovannitti) Let p, q denote different primes. Show that $\mathrm{Bext}^1(\mathbb{Q}, \mathbb{Z}_{(p,q)}) = 0$, and there is an exact sequence $0 \to \mathbb{Z}_{(p,q)} \to \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(q)} \to \mathbb{Q} \to 0$ (not balanced).

2 Prebalanced and Decent Subgroups

The further discussion of Butler groups is greatly enhanced by the introduction of new concepts. We examine briefly three salient properties which prove relevant to Butler groups both in the finite and in the infinite rank cases: prebalanced and decent subgroups, as well as subgroups with the torsion extension property (TEP).

Balanced subgroups play a decisive role in the theory of torsion-free groups, but the homological machinery they provide is unsatisfactory for the theory of Butler groups. The reason is that finite rank Butler groups need not contain any non-obvious balanced subgroup. However, there is a natural generalization of balancedness (due to Richman [6]) that can furnish us with a sought-after homological machinery. We also define the companion notion of decent subgroup (Albrecht–Hill [1]).

Prebalanced Subgroups Let A be a subgroup of a (not necessarily torsion-free) group G with torsion-free quotient G/A. A is said to be **prebalanced** in G if the following condition is satisfied: for each rank 1 pure subgroup H/A of G/A, there is a finite rank Butler subgroup B of G such that $H = A + B$.

Furthermore, A is called **decent** in G if the same conclusion holds whenever H/A is any finite rank pure subgroup of G/A. Evidently, decent subgroups are prebalanced, but balanced subgroups need not be decent as is shown by (c) in the following example.

Example 2.1.

(a) Every pure subgroup in a finite rank Butler group B is decent, and hence prebalanced: if $C < C'$ are pure subgroups of B with finite rk C'/C, then $C' = C + C'$ with C' a finite rank Butler group.

(b) The indecomposable group A in Example 4.3 in Chapter 12 has no proper balanced subgroups $\neq 0$, but all of its proper pure subgroups are prebalanced and decent.

(c) Let C be any finite rank torsion-free group that is not Butler; e.g. of Pontryagin type (Lemma 4.6 in Chapter 12). If $0 \to A \to G \to C \to 0$ is a balanced-projective resolution of C, then A is (pre)balanced, but not decent in G.

It might be helpful if we insert the following remarks before proceeding with the discussion.

(A) *A pure-exact sequence* $0 \to A \to G \to J \to 0$ *of torsion-free groups with* $\mathrm{rk}\, J = 1$ *is prebalanced exactly if there are a finite rank completely decomposable group* $X = X_1 \oplus \cdots \oplus X_n$ $(\mathrm{rk}\, X_i = 1)$ *and an epimorphism* $\xi : X \to J$ *that lifts to a homomorphism* $\eta : X \to G$. This follows at once from the definition.

(B) *A pure subgroup A of the torsion-free group G is prebalanced if and only if, for each* $g \in G$ *there is a finite subset* $\{a_1, \ldots, a_n\} \subseteq A$ *such that*

$$\langle A, g\rangle_* = A + \langle g + a_1\rangle_* + \cdots + \langle g + a_n\rangle_*.$$

That is, $\chi_{G/A}(g + A)$ is of the same type as $\chi_G(g + a_1) \vee \cdots \vee \chi_G(g + a_n)$. As $\chi_{G/A}(g + A)$ is the supremum of all $\chi_G(g + a)$ with $a \in A$, we can increase the subset of the a_i by adjoining finitely many elements $a_{n+1}, \ldots, a_{n+k} \in A$ to conclude that

$$\chi_{G/A}(g + A) = \chi_G(g + a_1) \vee \cdots \vee \chi_G(g + a_{n+k}).$$

Thus if A is prebalanced in G, then $\chi_{G/A}(g + A)$ is the join of a finite number of characteristics $\chi_{G/A}(g + a_i)$ $(a_i \in A)$.

(C) *A subgroup A of corank* 1 *in a torsion-free group G is prebalanced if and only if, in the relative balanced-projective resolution* (Sect. 3 in Chapter 12)

$$0 \to K \to A \oplus C \to G \to 0$$

the completely decomposable group C can be chosen to be of finite rank. For a proof, it suffices to take into consideration that G is generated by A and the image of C; the latter is a Butler group provided $\operatorname{rk} C < \infty$.

We leave the proof of Lemma 2.2 as an Exercise 2.

Lemma 2.2. *Let $B \leq A$ be pure subgroups of the torsion-free group G. Then the following holds:*

(i) *If B is prebalanced in G, then it is prebalanced in A.*
(ii) *If B is prebalanced in A, and A is prebalanced in G, then B is a prebalanced subgroup in G.*
(iii) *If A is prebalanced in G, then so is A/B in G/B.*
(iv) *If B is prebalanced in G, and A/B is prebalanced in G/B, then A is prebalanced in G.* □

Prebalanced-Exact Sequences An immediate consequence of this lemma is that prebalanced-exact sequences form a 'proper class': in the group $\operatorname{Ext}(C, A)$ of extensions of A by a torsion-free group C, the extensions represented by prebalanced-exact sequences form a subgroup; it will be denoted by $\operatorname{PBext}^1(C, A)$. Evidently, it contains $\operatorname{Bext}^1(C, A)$, the group of balanced extensions, discussed in Sect. 2 in Chapter 12.

This brings us to the exact sequence of central importance. The maps between the PBexts are precisely the restrictions of induced maps between the Exts. Like for Bext, here we have higher functors PBext^n to restore exactness to the right, but we will ignore them for $n > 1$.

Theorem 2.3. *Let $0 \to A \to B \to C \to 0$ be a prebalanced-exact sequence of torsion-free groups. For a torsion-free G and arbitrary H, we have the following exact sequences:*

$$0 \to \operatorname{Hom}(G, A) \to \operatorname{Hom}(G, B) \to \operatorname{Hom}(G, C) \to$$
$$\to \operatorname{PBext}^1(G, A) \to \operatorname{PBext}^1(G, B) \to \operatorname{PBext}^1(G, C) \to \ldots$$

and

$$0 \to \mathrm{Hom}(C,H) \to \mathrm{Hom}(B,H) \to \mathrm{Hom}(A,H) \to$$
$$\to \mathrm{PBext}^1(C,H) \to \mathrm{PBext}^1(B,H) \to \mathrm{PBext}^1(A,H) \to \dots .$$

Proof. We refer to Mac Lane[M] for relative homological algebra. □

Unfortunately, there are not enough prebalanced-projectives (only the free groups are such), and therefore we are unable to form prebalanced-projective resolutions. This deficiency, however, does not influence much the usefulness of this notion in the theory of Butler groups: we can always rely on balanced-projective resolutions.

More information about PBext is furnished by the following lemma.

Lemma 2.4 (Giovannitti [1]). *Let G be a torsion-free group. Then* $\mathrm{PBext}^1(G,B)$ *is a divisible subgroup of* $\mathrm{Ext}(G,B)$ *that contains the torsion part of* $\mathrm{Ext}(G,B)$.

Proof. The group property of $\mathrm{PBext}^1(G,B)$ has been established above. By Sect. 3(F) in Chapter 9, $\mathrm{Ext}(G,B)$ is divisible. To complete the proof, it suffices to show that if $m\mathfrak{e} \in \mathrm{PBext}^1(G,B)$ for some $\mathfrak{e} \in \mathrm{Ext}(G,B)$ and integer $m > 0$, then $\mathfrak{e} \in \mathrm{PBext}^1(G,B)$ likewise (this will include the $m\mathfrak{e} = 0$ case). Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
 & & & X \arrow[r, "\xi"] \arrow[d, "\eta"'] & J \arrow[d, "\mathrm{inj}"] & \\
m\mathfrak{e}\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & A' \arrow[r, "\alpha'"] \arrow[d, "\phi"'] & G \arrow[r] \arrow[d, "\dot{m}"] & 0 \\
\mathfrak{e}\colon & 0 \arrow[r] & B \arrow[r] & A \arrow[r, "\alpha"] & G \arrow[r] & 0,
\end{tikzcd}$$

and suppose that the upper row is prebalanced-exact. Let J be a rank 1 pure subgroup of G. By (A), we find a finite rank completely decomposable group X, and an epimorphism $\xi : X \to J$ that factors through A', i.e. $\alpha'\eta = \xi$ for some $\eta : X \to A'$. Now $\alpha\phi\eta$ maps X onto mJ which is of finite index in J. Thus, using a finitely generated free group F, we can extend $\alpha\phi\eta : X \to mJ$ to a surjective map $X \oplus F \to J$. This map clearly lifts to $X \oplus F \to A$, proving the prebalanced-exactness of $\mathfrak{e}$. □

Though the group $\mathrm{PBext}^1(C,A)$ is in general much larger than its subgroup $\mathrm{Bext}^1(C,A)$, surprisingly, for torsion groups A these two groups coincide. We give here a trivial proof for this frequently needed fact.

Lemma 2.5 (Fuchs–Metelli [3]). *A prebalanced-exact sequence*

$$0 \to T \to G \to C \to 0$$

with C torsion-free is necessarily balanced-exact whenever T is torsion.

Proof. We apply the definition of prebalancedness to $A = T$ and to a pure subgroup H of G such that $T < H$ and $\operatorname{rk} H/T = 1$. Then the sum $H = T + B$ (for a Butler group B) must be direct, since $T \cap B = 0$. Here B has to be of rank 1, so T is balanced in H (and hence in G). □

This lemma is applied to record a remarkable property of Butler groups. The proof for countable rank is as simple as for the finite rank case.

Proposition 2.6. *Every pure subgroup B of a countable completely decomposable group satisfies* $\operatorname{Bext}^1(B, T) = 0$ *for all torsion groups T.*

Proof. Let B be as stated, and $0 = B_0 < B_1 < \cdots < B_n < \ldots$ a chain of pure subgroups with union B where $\operatorname{rk} B_{n+1}/B_n = 1$ $(n < \omega)$. Then the B_n are Butler groups, and $0 \to B_n \to B_{n+1} \to B_{n+1}/B_n \to 0$ is a prebalanced-exact sequence, so Theorem 2.3 yields the exactness of

$$\operatorname{Bext}^1(B_{n+1}/B_n, T) \to \operatorname{Bext}^1(B_{n+1}, T) \to \operatorname{Bext}^1(B_n, T)$$

for every torsion group T, where we have applied Lemma 2.5 by switching PBext to Bext. The first Bext vanishes since B_{n+1}/B_n is of rank 1. A simple induction on n leads to $\operatorname{Bext}^1(B_n, T) = 0$ for $n < \omega$. Like in the proof of Lemma 4.1 in Chapter 9, hence we can conclude that $\operatorname{Bext}^1(B, T) = 0$ for all T. □

Decent Subgroups On several occasions, the stronger version of prebalancedness will be required: decent subgroups. Again, we leave the proof to the reader that *decent subgroups enjoy the same properties* (Lemma 2.2) *as balancedness and prebalancedness* (Exercise 3).

We have already observed that decent subgroups are prebalanced. However, for the converse an additional condition is needed, as is clear from the following result, where by a **finitely Butler** group we mean a group all of whose finite rank pure subgroups are Butler groups.

Lemma 2.7 (Fuchs–Viljoen [1]). *A pure subgroup A of a torsion-free group G is decent if and only if*

(i) *A is prebalanced in G, and*
(ii) *G/A is finitely Butler.*

Proof. Suppose A is decent in G. Then (i) is obvious. To check (ii), let C/A be a finite rank pure subgroup of G/A, and H a finite rank Butler group such that $C = A + H$. Then $C/A \cong H/(A \cap H)$ is Butler, as guaranteed by Theorem 1.4.

Conversely, assume (i) and (ii) are satisfied, and let C/A be a finite rank pure subgroup of G/A. By (ii), it is a Butler group, so $C/A = C_1/A + \cdots + C_k/A$ with rank 1 subgroups C_i/A. By (i), for each i, $C_i = A + B_i$ with a finite rank Butler group B_i. But then $C = A + B$ where $B = B_1 + \cdots + B_n$ is a finite rank Butler group. □

★ **Notes.** Prebalancedness was introduced by Richman [6] under the name 'semi-balancedness,' and rediscovered by Fuchs–Viljoen [1]. Its homological properties were explored by Bican–Fuchs [2]. Giovannitti [2] investigates, for Butler groups B, C, certain subgroups of $\mathrm{PBext}(C, B)$ that are defined in terms of lattices of types.

Easy examples show that the union of an ascending chain of prebalanced subgroups need not be prebalanced. All these counterexamples are essentially countable unions, because the following holds. Suppose that $0 = B_0 < B_1 < \cdots < B_\sigma < \ldots$ $(\sigma < \tau)$ is a smooth properly ascending chain of prebalanced subgroups of the group G. The union $B = \bigcup_{\sigma<\tau} B_\sigma$ is prebalanced in G whenever $\mathrm{cf}\,\tau \geq \omega_1$ (cp. Lemma 2.7 in Chapter 12).

The notion of **regular subgroup** is due to Bican [2]: regularity means that the elements have the same types in the subgroup as in the entire group.

Exercises

(1) Let $0 \to B_\sigma \to G_\sigma \xrightarrow{\alpha_\sigma} C_\sigma \to 0$ $(\sigma < \tau)$ be a direct system of prebalanced-exact sequences such that the connecting maps $\phi_{\sigma\rho}\colon C_\sigma \to C_\rho$ $(\sigma < \rho < \tau)$ are all monic with $\mathrm{Im}\ \phi_{\sigma\rho}$ pure in C_ρ. Then the direct limit of the system is likewise prebalanced-exact.

(2) Prove Lemma 2.2.

(3) Let $B < A$ be pure subgroups in the torsion-free group G.

  (a) If B is decent in G, then it is decent in A.
  (b) If B is decent in A, and A is decent in G, then B is decent in G.
  (c) If A is decent in G, then A/B is decent in G/B.
  (d) If B is decent in G, and A/B is decent in G/B, then A is decent in G.

(4) Prove the analogue of (C) for decent subgroups.

(5) Give an explicit example of

  (a) a pure subgroup that is not prebalanced;
  (b) a prebalanced subgroup that is not decent; and
  (c) a prebalanced subgroup that is not balanced.

(6) Extend (Lemma 2.5) to the case when T is a decent subgroup.

(7) (Kravchenko) Assume B is a finite rank Butler group. An exact sequence $0 \to A \to B \to C \to 0$ is balanced-exact if and only if, for all types $\mathbf{t}$, $A + B(\mathbf{t})$ is pure in B.

3 The Torsion Extension Property

We turn our attention to a third vital ingredient in our study, the torsion extension property. Unlike prebalancedness, it is not a generalization of balancedness.

TEP-Subgroups A subgroup A of a torsion-free group G is said to be a **TEP-subgroup**, or **to have the TEP in** G, if every homomorphism $A \to T$ (T any torsion group) extends to a homomorphism $G \to T$; that is, if the induced map

$\mathrm{Hom}(G, T) \to \mathrm{Hom}(A, T)$ is surjective for all torsion T. This is easily seen to be equivalent to the following: for every subgroup C of A for which A/C is torsion, A/C is a summand of G/C.

Observe that only a pure subgroup can be a TEP-subgroup. In fact, if A is not pure in G, then there is a prime p such that $pA < A \cap pG$. It is then easy to find a homomorphism of A into the cyclic group $\mathbb{Z}(p)$ of order p which cannot be extended to G.

Needless to say, direct summands are TEP-subgroups. A few elementary remarks on the new concept are in order.

Lemma 3.1. *Let $B < A$ be pure subgroups of the torsion-free group G. Then the following holds:*

(i) *If B is* TEP *in G, then it is* TEP *in A.*
(ii) *If B is* TEP *in A, and A is* TEP *in G, then B is a* TEP-*subgroup in G.*
(iii) *If A is* TEP *in G, then so is A/B in G/B.*
(iv) *If B is* TEP *in G, and A/B is* TEP *in G/B, then A is* TEP *in G.*

Proof. All claims are straightforward with the exception of (iv). So let T be a torsion group, and $\alpha : A \to T$ a homomorphism. By hypothesis, the restriction $\alpha \restriction B : B \to T$ extends to a homomorphism $\gamma : G \to T$. Now $\gamma - \alpha$ induces a map $A/B \to T$, which extends to a homomorphism $\delta : G/B \to T$. If $\phi : G \to G/B$ denotes the natural map, then $\gamma - \delta\phi : G \to T$ satisfies $(\gamma - \delta\phi)a = \gamma a - (\gamma - \alpha)a = \alpha a$ for every $a \in A$. ◻

Example 3.2.

(a) Let G be a finite rank torsion-free group that is not Butler, and $0 \to A \to C \to G \to 0$ a balanced-projective resolution of G, with completely decomposable C. The subgroup A is balanced in C, but fails to have the TEP, as it will be evident from Lemma 3.3 below.
(b) Pure subgroups of finite rank Butler groups are TEP-subgroups, since all mixed factor groups are splitting. This is made precise in Corollary 3.6.

The Basic Lemma Most significant is the property of finite rank Butler groups which is the content of the next theorem. Parts of its proof are separated as preliminary lemmas; they cover countable groups as well; at one point in the proof of Theorem 3.5 we will need this more general result.

Lemma 3.3 (Dugas–Rangaswamy [1]). *Let $0 \to A \to G \to B \to 0$ be a prebalanced-exact sequence, where G satisfies* $\mathrm{Bext}^1(G, T) = 0$ *for all torsion groups T. Then* $\mathrm{Bext}^1(B, T) = 0$ *holds for all torsion T if and only if A has* TEP *in G.*

Proof. The given sequence induces the sequence $\mathrm{Hom}(G, T) \to \mathrm{Hom}(A, T) \to \mathrm{Bext}^1(B, T) \to \mathrm{Bext}^1(G, T) = 0$ which is exact for any torsion T (Theorem 2.3); we have already replaced PBext by Bext as permitted by Lemma 2.5. The map between the two Bexts is monic for each torsion T exactly if A is TEP in G. ◻

The key technical ingredient in the proof of Theorem 3.5 is the following lemma.

Lemma 3.4 (Dugas–Rangaswamy [2]). *Suppose G is a countable torsion-free group such that* $\mathrm{Bext}^1(G,T) = 0$ *for all torsion T. Every* TEP-*subgroup in G that is of finite corank is decent in G.*

Proof. Assume A is a pure subgroup of finite corank in G. Let $\mathcal{G} = \{J_i \mid i < \omega\}$ be the set of all rank 1 pure subgroups in G, and E a finitely generated free subgroup of G such that G/K is a torsion group, where $K = A + E$. Let $\beta: G \to G/K$ denote the canonical projection. Given a family $\{n_i \mid i < \omega\}$ of positive integers, for each i define $C_i = J_i/n_i(J_i \cap K)$, and γ_i as the composite map

$$\gamma_i\colon\ C_i = J_i/n_i(J_i \cap K) \to J_i/(J_i \cap K) \to (K + J_i)/K \to G/K$$

(all maps are canonical). Finally, set

$$C = \oplus_{i<\omega} C_i, \quad \gamma = \oplus_{i<\omega} \gamma_i : C \to G/K, \quad \text{and} \quad T = \operatorname{Ker}\gamma.$$

Let H be the pull-back of γ and β in the diagram

$$\begin{array}{ccccccccccc}
\mathfrak{e}\beta\colon & 0 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\
\mathfrak{e}\colon & 0 & \longrightarrow & T & \longrightarrow & C & \xrightarrow{\gamma} & G/K & \longrightarrow & 0.
\end{array}$$

To see that $\mathfrak{e}\beta$ is balanced-exact, take any map $\rho : J \to G$ where J is torsion-free of rank 1. Clearly, $\beta\rho$ maps J into some $\gamma_i C_i \leq G/K$, and hence ρ lifts to a homomorphism $\xi : J \to C$ such that $\gamma\xi = \beta\rho$. Then by the pull-back property, there exists a map $J \to H$ establishing the balanced-exactness of $\mathfrak{e}\beta$. In view of the hypothesis on G, the top row splits, giving rise to a morphism $\alpha : G \to C$ such that $\gamma\alpha = \beta$.

Next suppose A is TEP-subgroup in G. Since $\alpha A \leq T$, the map $\alpha \restriction A$ extends to a homomorphism $\alpha' : G \to T$. Clearly, $\alpha \restriction K$ need not coincide with $\alpha' \restriction K$. But if $\zeta : T \to T/F$ is the canonical projection with kernel $F = (\alpha - \alpha')E$, then $\zeta(\alpha \restriction K) = (\zeta\alpha') \restriction K$. We conclude that in the commutative diagram

$$\begin{array}{ccccccccccc}
 & 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & G/K & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle\alpha\restriction K} & & \downarrow{\scriptstyle\alpha} & & \| & & \\
\mathfrak{e}\colon & 0 & \longrightarrow & T & \longrightarrow & C & \xrightarrow{\gamma} & G/K & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle\zeta} & & \downarrow & & \| & & \\
\zeta\mathfrak{e}\colon & 0 & \longrightarrow & T/F & \longrightarrow & C/F & \xrightarrow{\delta} & G/K & \longrightarrow & 0
\end{array}$$

with exact rows and columns the bottom sequence splits, thus there is a map $\eta : G/K \to C/F$ with $\delta\eta = \mathbf{1}_{G/K}$.

If $G/K = (K + J_1 + \cdots + J_j)/K$ holds for some $j < \omega$, then $G = A + B$ with a Butler group $B = E + J_1 + \cdots + J_j$, and we are done. Otherwise, for each $i < \omega$ we can choose a $g_i \in G$ with $g_i \notin H_i = (K + J_1 + \cdots + J_i)/K$, say, of order q_i. We show that this leads to a contradiction.

We regress to the construction of C, and choose the integers n_i in a particular way. With the aid the q_i just defined, we set $n_i = q_1 \cdots q_i$ for all $i < \omega$. As F is a finite group, we have $F \le C_1 \oplus \cdots \oplus C_j$ for some $j < \omega$. For such a j, let $\eta(g_j) = c_{j+1} + \cdots + c_k + F$ ($c_i \in C_i$). From $q_j g_j = 0$ we obtain $q_j\eta(g_j) = q_j(c_{j+1} + \cdots + c_k) \in F$, whence $q_j(c_{j+1} + \cdots + c_k) = 0$ follows. By construction, $q_j | n_i$ for each $i \ge j$, whence $q_j c_i = 0$ implies

$$c_i \in C_i[n_i] = (J_i \cap K)/n_i(J_i \cap K) = \operatorname{Ker}\gamma_i \le \operatorname{Ker}\gamma.$$

Then $g_j = \delta(c_{j+1}+\cdots+c_k+F) = \gamma(c_{j+1}+\cdots+c_k) \in H_j$, an obvious contradiction. $\square$

Butler Groups and Bext After all this preparation, it is easy to verify the following theorem whose extension to the infinite case reshaped the theory of Butler groups.

Theorem 3.5 (Bican [2], Bican–Salce [1]). *A finite rank torsion-free group B is a Butler group if and only if it satisfies*

$$\operatorname{Bext}^1(B, T) = 0 \qquad \textit{for all torsion groups } T.$$

Proof. The necessity is a special case of Proposition 2.6. Conversely, assume that B is of finite rank and satisfies $\operatorname{Bext}^1(B, T) = 0$ for all torsion T. Consider a balanced-projective resolution $0 \to A \to C \to B \to 0$ of B. Here C is a countable completely decomposable group, and by Lemma 3.3 B satisfies $\operatorname{Bext}^1(B, T) = 0$ for all torsion T only if A is a TEP-subgroup of C. Lemma 3.4 applies to conclude that A is then decent in C. An appeal to Lemma 2.7 leads us to the conclusion that $B \cong C/A$ is a (finitely) Butler group, completing the proof. $\square$

We give an alternative proof for the necessity part of the preceding theorem. Let B be a finite rank Butler group, and $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ a partition of the set of primes such that the groups $B \otimes \mathbb{Z}_{\Pi_\ell}$ are completely decomposable groups (cf. Theorem 1.8). If $0 \to T \to A \to B \to 0$ is a balanced-exact sequence with T torsion, then, for every $\ell = 1, \ldots, k$, the sequence

$$0 \to T \otimes \mathbb{Z}_{\Pi_\ell} \to A \otimes \mathbb{Z}_{\Pi_\ell} \to B \otimes \mathbb{Z}_{\Pi_\ell} \to 0$$

is also balanced-exact, so it splits. If $\phi_\ell : A \otimes \mathbb{Z}_{\Pi_\ell} \to T \otimes \mathbb{Z}_{\Pi_\ell}$ is a splitting map for the inclusion $T \otimes \mathbb{Z}_{\Pi_\ell} \to A \otimes \mathbb{Z}_{\Pi_\ell}$, then the composition

$$A \to \oplus_{\ell=1}^{k} A \otimes \mathbb{Z}_{\Pi_\ell} \xrightarrow{\phi} \oplus_{\ell=1}^{k} T \otimes \mathbb{Z}_{\Pi_\ell} = T$$

is a splitting map for the inclusion $T \to A$, where $\phi = \phi_1 \oplus \cdots \oplus \phi_k$.

Finally, we prove a corollary. If B is a finite rank Butler group, then for each pure subgroup C, the factor group B/C is a Butler group, so $\operatorname{Bext}^1(B/C, T) = 0$. From Lemma 3.3 we conclude:

Corollary 3.6 (Bican [3]). *In finite rank Butler groups, all pure subgroups are* TEP-*subgroups.* □

★ **Notes.** That finite rank Butler groups have trivial balanced extensions was observed by Bican [2], using complicated computations, and the idea of homological characterization in terms of Bext is due to Bican–Salce [1]. This started an extensive study of infinite rank Butler groups that was in the center of investigations for more than two decades in the twentieth century. See the Notes to Sect. 1.

The Torsion Extension Property was observed by Procházka [1], and used by Bican [3] in an equivalent form. It was more fully developed by Dugas–Rangaswamy [1] and Dugas–Hill–Rangaswamy [1].

Exercises

(1) Supply the details for the proofs of (i)–(iii) in Lemma 3.1.
(2) The condition "all pure subgroups in A are TEP-subgroups" does not imply that a finite rank torsion-free group A has to be a Butler group. [Hint: Pontryagin's example for a rank two indecomposable group.]
(3) (a) A completely decomposable group of countable rank contains pure subgroups that do not have the TEP.
    (b) Every torsion-free group of infinite rank contains a pure subgroup that is not a TEP-subgroup.
(4) For torsion-free groups A, C, the extensions of A by C in which A is a TEP-subgroup form a subgroup of $\operatorname{Ext}(C, A)$.
(5) (a) In every extension of $\mathbb{Z}$ by a torsion-free group, $\mathbb{Z}$ is a TEP-subgroup.
    (b) Let R denote the group of rationals with square-free denominators. Except for the splitting extension, none of the extensions of R by $\mathbb{Q}$ contains R as a TEP-subgroup.

4 Countable Butler Groups

Bican–Salce [1] introduced two generalizations of finite rank Butler groups to groups of arbitrary ranks. There is no general agreement which of these should be called Butler groups, so we follow the convention, and call them B_1- and B_2-groups, respectively.

***B_1*- and *B_2*-Groups** We will say that a torsion-free group B (of any rank) is a B_1**-group** if

$$\mathrm{Bext}^1(B,T) = 0 \qquad \text{for all torsion groups } T; \tag{14.2}$$

and a B_2**-group** if there is a smooth chain of pure subgroups,

$$0 = B_0 < B_1 < \cdots < B_\sigma < \cdots < B_\tau = \cup_{\sigma<\tau} B_\sigma, \tag{14.3}$$

with finite rank factors such that, for each $\sigma < \tau$, B_σ is decent in $B_{\sigma+1}$, i.e. $B_{\sigma+1} = B_\sigma + G_\sigma$ for some finite rank (pure) Butler subgroup G_σ. Note that the factor groups $B_{\sigma+1}/B_\sigma$ in (14.3) are finite rank Butler groups.

There is a third concept which we have already introduced before and is relevant to the theory: a torsion-free group all of whose finite rank pure subgroups are Butler groups is called **finitely Butler**.

An important remark is called for regarding terminology. My opinion is that it is not justified to call infinite rank groups satisfying condition (14.2) or (14.3) Butler groups: Butler's contribution is immense, but limited to the finite rank case. It is tempting to rename them *Bican–Salce groups*, since Bican and Salce brought the idea of balancedness into the picture. However, I hesitated to change the adopted terminology, because this would cause difficulty in relating our presentation to the respective literature. Notwithstanding, I feel strongly that the terminology in the infinite rank case is incorrect. (May I quote Ovid: *Video meliora proboque, deteriora sequor.*)

Example 4.1. The union B of an ascending chain $0 = B_0 < B_1 < \cdots < B_n < \ldots$ $(n < \omega)$ of finite rank Butler groups is a B_2-group provided each B_n is pure in B_{n+1}. B is evidently finitely Butler, and from Theorem 4.2 below it will follow that it is also a B_1-group.

Countable Butler Groups Focusing now our attention on the countable case, we prove the following main result which summarizes the precise relation between the three concepts under consideration.

Theorem 4.2 (Bican–Salce [1]). *For a countable rank torsion-free group B the following are equivalent:*

(i) *B is finitely Butler;*
(ii) *B is a B_2-group;*
(iii) *B is a B_1-group.*

Proof. (i) ⇒ (ii) B is the union of a countable chain $0 = B_0 < B_1 < \cdots < B_n < \ldots$ of pure subgroups where B_n is of rank n. By hypothesis, they are Butler groups. As B_n is decent in B_{n+1}, (ii) holds for B.

(ii) $\Rightarrow$ (iii) B is the union of a countable chain (as in (i)), where the factor groups are finite rank Butler groups, so by Theorem 3.5 they satisfy (iii). We argue just as in the proof of Lemma 4.1 in Chapter 9, using the fact that T stays balanced in its extension by B_{n+1}, if B_n factored out. Thus B satisfies (iii).

(iii) $\Rightarrow$ (i) Let $0 \to A \to C \to B \to 0$ be a balanced-projective resolution of the B_1-group B, where C is a countable completely decomposable group. Pick a finite rank pure subgroup G of B, and form the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where the right-hand square is a pull-back. H is finitely Butler as a pure subgroup of the completely decomposable group C, hence it is a B_1-group by the first two steps of this proof. Owing to Lemma 3.3, A has TEP in C, so it is a TEP-subgroup of finite corank in H. Lemma 3.4 implies that A is decent in H, and hence $G \cong H/A$ is a Butler group. □

This theorem gives a clear picture of the groups under consideration in the countable case. As a consequence, there is no ambiguity in calling a *countable* group a **Butler group** to mean either a B_1- or a B_2-group, or else a finitely Butler group.

Since the property of being finitely Butler is evidently inherited by pure subgroups, we obtain at once:

Corollary 4.3 (Bican–Salce [1]). *Pure subgroups of countable Butler groups are themselves Butler groups.* □

In Corollary 3.6 we have seen that pure subgroups of finite rank Butler groups are TEP-subgroups. Now we raise the question as to when a pure subgroup of a countable Butler group is a TEP-subgroup. Posing this question gives us the opportunity of gaining a better insight into the interplay of three important concepts.

Corollary 4.4 (Dugas–Rangaswamy [1]). *Let B be a countable Butler group, and A a pure subgroup of B. The following are equivalent:*

(a) *A is a* TEP-*subgroup of B;*
(b) *A is decent in B;*
(c) *A is prebalanced in B and B/A is a Butler group.*

Proof. Assuming (a), let B' be pure in B such that $A \leq B'$ and B'/A is of finite rank. Then B' is Butler by Corollary 4.3, and (b) follows from Lemma 3.4. Because of Theorem 4.2, the equivalence of (b) and (c) is a consequence of Lemma 2.7. If A satisfies (c), then Lemma 3.3 implies that also (a) holds. □

★ **Notes.** Countable Butler groups are important transitions from finite rank Butler groups to the general case: they share some of the relevant properties of the finite rank groups, most of which get lost for large cardinalities as we shall see in the following sections. We would like to make it clear: countable Butler groups are not necessarily pure subgroups in completely decomposable groups. Indeed, Arnold–Rangaswamy [1] give an example of a countable rank Butler group that fails to be a pure subgroup in any completely decomposable group.

Exercises

(1) The union of a chain of pure finitely Butler subgroups is also finitely Butler.
(2) A TEP-subgroup in a B_2-group is a B_2-group.
(3) (Bican–El Bashir) Let A be a torsion-free group, and $A = B + C$ where B, C are B_1-groups. If $B \cap C$ is TEP in B, then A is also a B_1-group.
(4) Let $0 = G_0 < G_1 < \cdots < G_n < \ldots$ be a countable chain of countable Butler groups such that each link is pure in the next one. Then the union $G = \bigcup_{n<\omega} G_n$ is likewise a Butler group. [Hint: Theorem 4.2.]
(5) (Dugas–Rangaswamy) A pure subgroup of a Butler group of countable rank is a decent subgroup if and only if it is a TEP-subgroup.
(6) The tensor product of two finitely Butler groups is again finitely Butler.

5 B_1- and B_2-Groups

In the preceding section, we have defined B_1- and B_2-groups, and now we wish to learn more about them without cardinality restriction. So forget about countability, and refer to a transfinite chain of pure subgroups in the B_2-group B,

$$0 = B_0 < B_1 < \cdots < B_\sigma < \cdots < B_\tau = \cup_{\sigma<\tau} B_\sigma = B \tag{14.4}$$

where the factors are of finite rank, and B_σ is decent in $B_{\sigma+1}$ $(\sigma < \tau)$.

The following claims are evident from the definitions.

(A) *Completely decomposable groups are both B_1-, B_2-groups, and also finitely Butler.*
(B) *Direct sums of B_1- (B_2)-groups (finitely Butler groups) are again of the same kind.*
(C) *Summands of B_1-groups (finitely Butler groups) are again B_1-groups (finitely Butler).* (For the same for B_2-groups, see Corollary 5.5.)
(D) *For each $\sigma < \tau$, both the subgroups B_σ and the factor groups B/B_σ in (14.4) are B_2-groups.* For factor groups, this is a consequence of the preservation of decency when passing to the stated factor group.
(E) *A torsion-free group B is a B_1-group exactly if it has the projective property relative to all balanced-exact sequences $0 \to T \to G \to C \to 0$ with T torsion and C torsion-free.* This is simply a reformulation of the definition of B_1-groups.

(F) *Homogeneous B_2-groups are completely decomposable.* Indeed, if the finite rank Butler pure subgroups G_σ in the definition of B_2-groups (recall: $B_{\sigma+1} = B_\sigma + G_\sigma$ in (14.4)) are homogeneous of the same type $\mathbf{t}$, then by Corollary 1.5 they are completely decomposable. Furthermore, then the intersection $B_\sigma \cap G_\sigma$ is a completely decomposable summand in G_σ, $G_\sigma = (B_\sigma \cap G_\sigma) \oplus H_\sigma$ for some H_σ which is likewise $\mathbf{t}$-homogeneous completely decomposable. Therefore, $B_{\sigma+1} = B_\sigma \oplus H_\sigma$, and B is the direct sum of the H_σ $(\sigma < \tau)$.

(G) *Prebalanced extensions of B_2-groups by B_2-groups are B_2-groups.* By Lemma 2.7, a prebalanced A is decent if G/A is B_2, so a chain of decent subgroups of A can be continued by such a chain of G/A to reach G.

(H) *Let $0 \to A \to B \to C \to 0$ be a prebalanced-exact sequence, and B a B_1-group. C is a B_1-group exactly if A is* TEP *in B.* Cp. Lemma 3.3.

If we wish, we may admit in (14.4) factors of countable rank:

Lemma 5.1 (Dugas–Rangaswamy [1]). *A torsion-free group B of cardinality κ is a B_2-group if and only if it admits a smooth chain* (14.4) *such that*

(i) *$B_{\sigma+1}/B_\sigma$ is countable for every $\sigma < \kappa$;*
(ii) *B_σ is decent in $B_{\sigma+1}$ (or in B) for every $\sigma < \kappa$.*

Proof. Note that by transfinite induction it follows that all B_σ in (14.4) are decent in the B_2-group B.

Necessity being obvious, in order to prove sufficiency suppose that (14.4) is a smooth chain satisfying (i)–(ii). There is a finite or countably infinite chain of pure subgroups between B_σ and $B_{\sigma+1}$, say, $B_\sigma = C_0 < C_1 < \cdots < C_n < \cdots < C_\omega = B_{\sigma+1}$ with rank 1 factors C_{n+1}/C_n. If X_n is a finite rank Butler group with $C_{n+1} = B_\sigma + X_n$, then also $C_{n+1} = C_n + X_n$, thus C_n is decent in C_{n+1}. Hence any chain (14.4) with (i)–(ii) can be refined to a chain as required in the definition of B_2-groups. □

Prebalancedness is particularly suited to deal with B_1-groups as well.

Lemma 5.2. *Prebalanced extensions of B_1-groups by B_1-groups are B_1-groups. Moreover, if* (14.4) *is a chain in B such that*

(i) *each subgroup B_σ is prebalanced in its immediate successor, and*
(ii) *all the factors $B_{\sigma+1}/B_\sigma$ are B_1-groups,*

then B is likewise a B_1-group.

Proof. If $0 \to A \to B \to C \to 0$ is a prebalanced-exact sequence of torsion-free groups, then for a torsion group T, the induced sequence $\mathrm{Bext}^1(C,T) \to \mathrm{Bext}^1(B,T) \to \mathrm{Bext}^1(A,T)$ is exact by Theorem 2.3; in view of Lemma 2.5, we could replace PBext by Bext. If A and C are B_1-groups, then the two extremal Bexts vanish, so $\mathrm{Bext}^1(B,T) = 0$ for all torsion T.

The second claim follows from the first if B is the union of a finite chain. For infinite chains, we argue as in Eklof's Lemma 4.1 in Chapter 9. □

The next result deals with the relation between B_1- and B_2-groups in general. One direction the implication is rather easy.

Theorem 5.3. *B_2-groups of any rank are B_1-groups.*

Proof. For a countable chain, the claim follows from Proposition 2.6. For transfinite chains, we again refer to the proof of Lemma 4.1 in Chapter 9. □

(J) *In the chain* (14.4)*, the decent subgroups B_σ are* TEP*-subgroups of the B_2-group B.* This is an immediate consequence of Lemma 3.3 in view of the preceding theorem.

Characterizations of B_2-Groups The next result establishes a larger supply of decent subgroups in B_2-groups.

Theorem 5.4 (Albrecht–Hill [1]). *A torsion-free group G is a B_2-group if and only if it admits an $H(\aleph_0)$-family $\mathcal{B}$ of decent subgroups. Equivalently, it has an $H(\aleph_0)$-family of decent* TEP*-subgroups.*

Furthermore, if $A, B \in \mathcal{B}$ and $B < A$, then A/B is a B_2-group.

Proof. One way the claim is obvious, since from an $H(\aleph_0)$-family of decent subgroups we can extract a smooth chain with countable factors which are B_2-groups. Such a chain can be refined to a desired chain with finite rank factors.

For the converse, assume that G is a B_2-group. We appeal to Theorem 5.5 in Chapter 1 to obtain an $H(\aleph_0)$-family $\mathcal{B}$, and to its proof to argue that, for blocked subsets S, the subgroups $G(S)$ are decent and TEP-subgroups in G. So let $C/G(S)$ be a finite rank pure subgroup of $G/G(S)$. There is a finite blocked subset S' that contains representatives of a maximal independent set in $C/G(S)$. Then $C = G(S) + (C \cap G(S'))$, where the intersection is a pure subgroup of $G(S')$, so a finite rank Butler group. Thus $G(S)$ is decent in G. It is also a TEP-subgroup, because by creating an increasing sequence of blocked subgroups, we can find a chain like (14.4) passing through $G(S)$ and climbing up to G; finally, we refer to (D).

The last argument implies that all the factors in $\mathcal{B}$ are B_2-groups. □

Equipped with this theorem, we can derive a not so obvious corollary (that we could not claim in (C)).

Corollary 5.5. *Summands of B_2-groups are B_2-groups.*

Proof. Let $A = G \oplus H$ be a direct decomposition of a B_2-group, and $\mathcal{B}$ an $H(\aleph_0)$-family of decent subgroups of A. Define the subfamily

$$\mathcal{B}^\star = \{B \in \mathcal{B} \mid B = B \cap G \oplus B \cap H\}.$$

In view of Lemma 5.4 in Chapter 1, this is an $H(\aleph_0)$-family of decent subgroups of A, and the set $\{B \cap G \mid B \in \mathcal{B}^\star\}$ is an $H(\aleph_0)$-family of decent subgroups of G. □

It might help the reader to navigate a passage through the theory of general Butler groups, through this rich and fascinating subject, if we explain briefly our approach and our limited goal, before getting involved more sophisticated arguments. A principal aim of the theory is to understand the fine distinction between B_1- and B_2-groups of arbitrary cardinalities. It turns out that the question whether or not the classes of B_1- and B_2-groups are identical has no definite answer: it is undecidable in ZFC. We have shown above that B_2-groups are always B_1-groups, and will provide a necessary and sufficient condition under which B_1-groups are B_2-groups (see Theorem 8.2 below). Though it is of no practical use, still it will be discussed, because it is instrumental in understanding why CH is a sufficient condition (Theorem 9.9). However, the proof that *it is undecidable in ZFC whether or not all B_1-groups are B_2-groups* would require a significant amount of preparations, which is beyond the scope of this book.

The equivalence problem of B_1- and B_2-groups is intrinsically tied to another, almost equally important question as to when $\mathrm{Bext}^2(G, T)$ vanishes for all torsion-free G and all torsion T. To answer this question, we will give a couple of equivalent conditions in Proposition 8.8. A third relevant question that is concerned with the subgroup problem will be addressed briefly: for which subgroups is the B_2-property hereditary?

★ **Notes.** It is still not known if $\mathrm{Bext}^1(B, T) = 0$ for all direct sums T of finite cyclic groups implies that B is a B_1-group (problem raised by Arnold). To answer the question as to whether or not $\mathrm{Bext}^1(B, T) = 0$ for all *countable* T entails that B is a B_1-group, we point out that this is undecidable in ZFC (Dugas–Rangaswamy [1]). Indeed, choose B homogeneous of type $\mathbb{Z}$, then $\mathrm{Bext}^1(B, T) = \mathrm{Ext}^1(B, T)$, so B is a Baer group, and for such groups the problem is undecidable in ZFC, see Sect. 2 in Chapter 15.

Exercises

(1) In a B_2-group, every countable subgroup is contained in a countable TEP-subgroup.

(2) Let $0 \to A \to B \to C \to 0$ be a prebalanced-exact sequence of torsion-free groups. A is a B_1-group if both B and C are B_1-groups. [Hint: (H).]

(3) (a) Prebalanced extension of a finitely Butler group by another such group is finitely Butler.
   (b) Let $0 = C_0 < C_1 < \cdots < C_n < \dots$ $(n < \omega)$ be a sequence of finitely Butler groups. If C_n is prebalanced in C_{n+1} for each $n < \omega$, then the union $C = \cup C_n$ is also finitely Butler.

(4) Corank 1 prebalanced subgroups are decent.

(5) (Dugas–Hill–Rangaswamy) A torsion-free group G of cardinality $\aleph_1$ is a B_2-group if it admits a smooth chain of TEP-subgroups with countable quotients.

(6) Let $0 \to A \to C \to B \to 0$ be a balanced-projective resolution of a B_1-group B. If A is a B_2-group, then B is finitely Butler. [Hint: if G/A is finite rank pure in $C/A \cong B$, then A has TEP in G, apply Lemma 3.4, A is decent in G; thus G/A is Butler.]

6 Solid Subgroups

We now step into the more complicated realm of uncountable Butler groups. We need additional tools to understand what happens for larger cardinalities. As it turns out, the crucial device in approaching the problem is a kind of chain that is more general than the one in the definition of B_2-groups. Our choice is based on a concept that is an extension of prebalancedness to arbitrary cardinals. We shall need only the countable version, so our focus will be on such extensions.

We start with a simple observation. Let G be a torsion-free group, and A a pure subgroup of corank 1 in G. Consider the types $\mathbf{t}(J)$ of those rank 1 pure subgroups J of G which are not contained in A (thus disjoint from A). In the lattice $\mathfrak{T}$ of all types, we form the lattice ideal $\mathcal{I}_{G|A}$ generated by all these types $\mathbf{t}(J)$. Thus $\mathcal{I}_{G|A}$ consists of all types $\mathbf{t}$ satisfying $\mathbf{t} \leq \mathbf{t}(J_1) \vee \cdots \vee \mathbf{t}(J_m)$ for some finite set $\{J_1, \ldots, J_m\}$ of rank 1 pure subgroups disjoint from A. As the cardinality of $\mathfrak{T}$ is the power of the continuum, every ideal of $\mathfrak{T}$ can be generated by at most $2^{\aleph_0}$ types.

Example 6.1.

(a) If A is prebalanced in G, then $\mathcal{I}_{G|A}$ is finitely generated.
(b) For each κ ($\aleph_0 \leq \kappa \leq 2^{\aleph_0}$), $\mathfrak{T}$ contains an ideal $\mathcal{I}$ that can only be generated by exactly κ types. Indeed, consider a set Σ of almost disjoint countable subsets $\{S_\sigma \mid \sigma < \kappa\}$ of the set Π of primes. (The existence of such a Σ was proved in Lemma 5.6 in Chapter 1.) For each $S_\sigma \in \Sigma$, define the type $\mathbf{t}_\sigma = (k_2, \ldots, k_p, \ldots)$ by setting $k_p = \infty$ or 0 according as $p \in S_\sigma$ or $p \notin S_\sigma$. It is easily seen that

$$\mathbf{t}_\sigma \leq \mathbf{t}_{\sigma_1} \vee \ldots \vee \mathbf{t}_{\sigma_k} \quad \text{if and only if} \quad \sigma \in \{\sigma_1, \ldots, \sigma_k\}.$$

Consequently, in the lattice $\mathfrak{T}$ of all types, the ideal $\mathcal{I}$ generated by all $\mathbf{t}_\sigma$ with $\sigma < \kappa$ is as desired.

Solid Subgroups Let A be a corank 1 pure subgroup in the torsion-free group G. A is said to be **solid** in G, if the lattice ideal $\mathcal{I}_{G|A}$ can be countably generated. More explicitly, A is solid in G means that there is a countable set $\mathcal{S} = \{J_n \mid n < \omega\}$ of rank 1 pure subgroups in G, but not in A, such that for every rank 1 pure subgroup J of G, not in A, we can find a finite subset $\{J_{n_1}, \ldots, J_{n_k}\} \subset \mathcal{S}$ such that $\mathbf{t}(J) \leq \mathbf{t}(J_{n_1}) \vee \cdots \vee \mathbf{t}(J_{n_k})$.

Example 6.2.

(i) Decent subgroups are solid.
(ii) If the typeset of a torsion-free group G is countable, then every pure subgroup of G is solid.

Let us point out right away that, if G, A are as before, then we have always $G = A + \sum_{n<\omega} J_n$ where the groups J_n are rank 1 pure subgroups of G not in A. Indeed, the characteristic of a coset $g + A \in G/A$ is the supremum of the characteristics of suitably chosen countably many elements in the coset (for each prime p, the correct p-height need to be obtained). But this does not mean that the ideal $\mathcal{I}_{G|A}$ is at most countably generated. A counterexample is given to substantiate this claim.

Example 6.3. Let A denote a free abelian group of rank $\aleph_1$, say, with basis $\{a_\sigma \mid \sigma < \omega_1\}$, and $\mathcal{I}$ an ideal of $\mathfrak{T}$ that is $\aleph_1$-generated. For each $\mathbf{t}_\sigma \in \mathcal{I}$ select a characteristic $\chi_\sigma \in \mathbf{t}_\sigma$. If $B = \mathbb{Z}b$ is an infinite cyclic group, define G as the subgroup of the divisible hull of $A \oplus B$ generated by A, B and by elements making $a_\sigma + b$ of characteristic χ_σ. Then $\mathcal{I}_{G|A} = \mathcal{I}$ is not countably generated.

In some proofs it is useful to have an element-wise criterion available for solid subgroups. Clearly, A is solid in G if for each $g \in G \setminus A$ there exists a subset

$\{a_n \mid n < \omega\}$ of A (to which we shall refer as a **solid set for** g **over** A) along with non-zero integers r_n such that for each $a \in A$

$$\mathbf{t}(g + a) \leq \mathbf{t}(r_1 g + a_{n_1}) \vee \cdots \vee \mathbf{t}(r_k g + a_{n_k})$$

for some $k < \omega$.

We should note that the use of the r_n is unavoidable (Rangaswamy). Indeed, let $G = \mathbb{Q}^{(p)}x \oplus \mathbb{Q}^{(q)}y$ with primes $p \neq q$, and $A = \mathbb{Z}(x + y)$ a pure subgroup. If $g = p^{-1}x$, then $h_q(g + A) = \infty$, but for no $a \in A$ is $g + a$ divisible by all powers of q.

The following characterization of solid subgroups will be required.

Lemma 6.4. *A pure subgroup A of corank one in the torsion-free group G is solid in G if and only if $G = A + H$ where H is a countable pure subgroup of G such that every $g \in G \setminus A$ satisfies*

$$\mathbf{t}(g) \leq \mathbf{t}(x_1) \vee \cdots \vee \mathbf{t}(x_n)$$

for some finite set $\{x_1, \ldots, x_n\} \subset H \setminus A$.

Proof. Let $\{g_n\}_{n<\omega}$ be a complete set of representatives of G mod A. If A is solid in G, then choose H as the purification of the subgroup generated by all g_n and all solid sets for the g_n. Conversely, if H is as stated, then the set $\{a \in A \mid \exists\, r \in \mathbb{N}$ with $rg + a \in H\}$ is a solid set for $g \in G \setminus A$. □

If G/A is torsion-free of arbitrary rank, then we define A to be **solid** in G whenever A is solid in every pure subgroup C of G that contains A as a corank 1 subgroup. Solidity will be most relevant in our future discussions, we list some of the main properties in the following two lemmas. For proofs we refer to Exercise 3.

Lemma 6.5 (Bican–Fuchs [2]). *Let $B < A$ be pure subgroups of the torsion-free group G. Then the following holds:*

(a) *If B is solid in G, then it is solid in A.*
(b) *If B is solid in A, and A is solid in G, then B is solid in G.*
(c) *Let B be prebalanced in G. A is solid in G if and only if A/B is solid in G/B.* □

Other properties of importance are as follows.

Lemma 6.6 (Bican–Fuchs [2]).

(i) *Assume $B < A$ are pure subgroups of the torsion-free group G with A/B countable. If B is solid in G, then so is A as well.*
(ii) *The union of a countable ascending chain of solid subgroups is likewise solid.*

Proof.

(i) Let $\{a_i\}_{i<\omega}$ be a complete set of representatives of A mod B. Given $g \in G \setminus A$, for each of the countably many $g + a_i$, select a solid set $\{b_{i1}, \ldots, b_{ik}, \ldots\} \subset B$. A simple calculation convinces us that $\{a_i + b_{i1}, \ldots, a_i + b_{ik}, \ldots\ (i < \omega)\} \subset A$ is a solid set for g over A.

(ii) Let $A_1 < \cdots < A_n < \ldots$ be a countable ascending chain of solid subgroups of G, and let A denote their union. If $g \in G \setminus A$, then a moment's reflection shows that the union of the solid sets for g over A_n (for each n) is a solid set for g over A. □

The following generalization of Lemma 3.4 will be needed.

Lemma 6.7. *Assume G is finitely Butler, and A is a solid* TEP-*subgroup in G. Then A is decent in G.*

Proof. The proof is identical with Lemma 3.4, only a single adjustment is needed: if $\mathrm{rk}(G/A) = 1$, then choose for $\mathcal{G}$ the set of rank one pure subgroups in G generating the ideal $\mathcal{I}_{G|A}$. □

Solid Subgroups in Completely Decomposable Group The following result reveals a close connection between solidity and the property of being a B_2-subgroup.

Theorem 6.8 (Fuchs–Metelli–Rangaswamy[1]). *Suppose G is a corank one pure subgroup of the completely decomposable group $C = \oplus_{\sigma<\tau} C_\sigma$ (each C_σ is of rank 1).*

G is a B_2-group if and only if the lattice ideal $\mathcal{I}$ generated by the types $\mathbf{t}(C_\sigma)$ with $C_\sigma \not\leq G$ can be countably generated, i.e. G is solid in C.

Proof. Let G be as stated, and $\phi : C \to C/G = R$ the canonical map where R is a rational group containing 1. For the proof we may assume that G does not contain any C_σ. For $\nu < \tau$, we set

$$C(\nu) = \oplus_{\sigma<\nu} C_\sigma \quad \text{and} \quad G(\nu) = G \cap C(\nu).$$

For each $\sigma < \tau$, pick $c_\sigma \in C_\sigma$ such that $\phi(c_\sigma) = 1 \in R$, and let $C_{\sigma\rho} = \langle c_\sigma - c_\rho \rangle_*$ for $\sigma < \rho < \tau$. Evidently, $C_{\sigma\rho}$ is of type $\mathbf{t}(C_\sigma) \wedge \mathbf{t}(C_\rho)$, and G is generated by all the $C_{\sigma\rho}$. Next we change the well-ordering of the C_σ by first listing those countably many C_σ that are needed to generate C along with G, i.e. $G + C(\omega) = C$.

For the proof of sufficiency, suppose the ideal $\mathcal{I}$ is countably generated. We may in addition assume that the well-ordering is chosen such that the types $\mathbf{t}(C_\sigma)$ with $\sigma < \omega$ generate $\mathcal{I}$. Then from ω on, every $G(\nu)$ is prebalanced in $G(\nu + 1)$. As $G(\omega)$ is a countable B_2-group, the chain $G(\omega) < \cdots < G(\sigma) < \ldots$ $(\sigma < \tau)$ ensures that G is likewise a B_2-group.

To prove necessity, let $\tau > \omega$ and G a corank 1 pure B_2-subgroup of C with $G \cap C_\sigma = 0$ for all σ. By Theorem 5.4, G has an $H(\aleph_0)$-family $\mathcal{F}$ of decent subgroups. Let F_1 be a countable member of $\mathcal{F}$ that contains $G(\omega)$. There is a countable index set I_1 such that $F_1 < \oplus_{\sigma\in I_1} C_\sigma$. We can find a countable $F_2 \in \mathcal{F}$ containing $G \cap \oplus_{\sigma\in I_1} C_\alpha$, and then a countable index set I_2 such that $I_1 \subset I_2$ and $F_2 < \oplus_{\sigma\in I_2} C_\sigma$. Continuing this back-and-forth argument, after ω steps we take unions to obtain an $F \in \mathcal{F}$ which is clearly equal to $G \cap \oplus_{\sigma\in I} C_\sigma$ for a countable index set I.

Again changing the well-ordering if necessary, we may in addition assume that there is a countable ordinal δ for which

$$F = G \cap \oplus_{\sigma \in I} C_\sigma = G(\delta).$$

This subgroup being prebalanced in G, we infer that for each $\nu \in C$ with $\nu \geq \delta$ there are a finite number of indices $\mu_1, \dots, \mu_k < \nu$ such that $G(\nu + 1) = G(\nu) + C_{\nu\mu_1} + \cdots + C_{\nu\mu_k}$. Then

$$\mathbf{t}(C_\nu) = \mathbf{t}(G(\nu+1)/G(\nu)) = (\mathbf{t}(C_\nu) \wedge \mathbf{t}(C_{\mu_1})) \vee \cdots \vee (\mathbf{t}(C_\nu) \wedge \mathbf{t}(C_{\mu_k}))$$

hence $\mathbf{t}(C_\nu) \leq \mathbf{t}(C_{\mu_1}) \vee \cdots \vee \mathbf{t}(C_{\mu_k})$ follows. Since the generation of $G(\nu)$ involves C_σ only for $\sigma < \nu$, this inequality remains true no matter how C_ν is selected from the set of the remaining C_σ (i.e., $\sigma \geq \nu$). Hence the ideal $\mathcal{I}$ is in fact generated by the types $\mathbf{t}(C_\sigma)$ with $\sigma < \delta$. $\square$

Example 6.9. If in the preceding theorem, the subgroup G is solid in C, then it is an example of a solid subgroup that is not prebalanced.

We will use now relative balanced-projective resolutions (Sect. 3 in Chapter 12).

Proposition 6.10. *For a corank* 1 *pure subgroup A of the torsion-free group G, the following are equivalent:*

(i) *A is solid in G;*
(ii) *there is a relative balanced-projective resolution*

$$0 \to K \to A \oplus C \xrightarrow{\phi} G \to 0 \tag{14.5}$$

with a countable completely decomposable group C;
(iii) *the kernel K is a B_2-group in a prebalanced-exact sequence of the form* (14.5) *where C is completely decomposable, and $\phi \restriction A = \mathbf{1}_A$.*

Proof. (i) $\Leftrightarrow$ (ii) If H is as stated in Lemma 6.4, then let $0 \to B \to C \xrightarrow{\phi} H \to 0$ be a balanced-projective resolution of H with countable C. In this case, $\nabla(\mathbf{1}_A \oplus \phi)$: $A \oplus C \to G$ yields a sought-after resolution. Conversely, if C is as in (ii), then the purification H of its image in G satisfies Lemma 6.4.

(iii) $\Rightarrow$ (i) If (14.4) is as stated in (iii), then starting from the middle row, we form the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & A & = & A & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & A \oplus C & \xrightarrow{\phi} & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & G/A & \longrightarrow & 0,
\end{array}$$

where the bottom row is obtained by factoring out A from $A \oplus C$ and from G; this is allowed because ϕ is the identity on A. (We have also used the fact that $A \cap K = 0$.) As $\operatorname{rk} G/A = 1$, K is a corank 1 pure subgroup in the completely decomposable group C. By Theorem 6.8, K is a B_2-group if and only if the $\mathfrak{T}$-ideal $\mathcal{I}$ generated by the rank 1 summands of C can be countably generated. As the summands of C are coming from those rank 1 subgroups of G that are disjoint from A, this ideal $\mathcal{I}$ is countably generated if and only if A is solid in G.

(i) $\Rightarrow$ (iii) If A is solid in G, then choose (14.4) to be a relative balanced-projective resolution of G over A. The final argument in the preceding paragraph convinces us that K must be a B_2-group. □

Example 6.11 (Bican [1]). The following group G is pure in a completely decomposable group, but is not B_2. Consider the set of types $\mathbf{t}_\sigma$ ($\sigma < 2^{\aleph_0}$) defined in Example 6.1 with the aid of an almost disjoint set of primes. Select a rank 1 group C_σ of type $\mathbf{t}_\sigma$, and let $C = \oplus_{\sigma<\omega_1} C_\sigma$. If G is a pure subgroup in C of corank 1 such that it does not contain any C_σ, then it is homogeneous of type $(0, \ldots, 0, \ldots)$, but is not a B_2-group (Theorem 6.8). Evidently, G is not solid in C.

★ **Notes.** Solid subgroups were introduced under the name of $\aleph_0$-prebalanced subgroups by Bican–Fuchs [2], where κ-balancedness was also discussed for cardinals κ with $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. Here we have adopted a more colloquial terminology for the only case $\kappa = \aleph_0$ treated here. Earlier literature on infinite Butler groups used subgroups called 'separable in the sense of Hill' or 'separative' which probably are a bit easier to define than solid subgroups: a pure subgroup A is **separative** in G if, for every $g \in G$, there is a countable subset $X \subset A$ (depending on g) such that for any $a \in A$ there is an $x \in X$ satisfying $\chi(g + a) \leq \chi(g + x)$ (cf. Sect. 7 in Chapter 11). Thus separative subgroups are solid. We do not spend time here to sort out the precise relation between separative and solid subgroups, we just wish to point out that assuming ¬CH, there exist torsion-free groups of cardinality $\aleph_2$ with solid, but no separative chains; see Dugas–Thomé [1].

Exercises

(1) Suppose that χ is the supremum of the characteristics $\{\chi_i \mid i \in I\}$. Then there is a countable subset J in I such that $\chi = \sup\{\chi_i \mid i \in J\}$.
(2) Let $C = \oplus_p \mathbb{Q}^{(p)}$ for all primes p, and $\phi : C \to \mathbb{Q}$ such that $\phi \restriction \mathbb{Q}^{(p)}$ is the embedding. Use the idea of Theorem 6.8 to show that $G = \operatorname{Ker} \phi$ is prebalanced, but not decent in C.
(3) Give detailed proof for Lemma 6.5.

(4) A corank 1 solid subgroup of a B_1-group is a B_1-group.
(5) Assuming CH, a corank 1 pure subgroup of a B_2-group is a B_2-group if and only if it is a solid subgroup.

7 Solid Chains

The solid exact sequences fail to form a proper class, consequently, they do not support a relative homological machinery. However, solid chains will provide us with powerful tools in the study of Butler groups of large cardinalities.

Solid Chains Let A denote a pure subgroup of the torsion-free group G. By a **solid chain from** A **to** G we understand a smooth chain

$$A = G_0 < G_1 < \ldots < G_\sigma < \ldots < G_\tau = G \tag{14.6}$$

of solid subgroups of G, where all the factors $G_{\sigma+1}/G_\sigma$ are torsion-free of rank one. We will say that G **admits a solid chain** if there is a solid chain from 0 to G.

(a) *All B_2-groups admit solid chains.* This is obvious, as decent subgroups are solid.
(b) *If in the chain* (14.6)*, the subgroups G_σ of G are solid, and the factors $G_{\sigma+1}/G_\sigma$ have cardinalities $\leq \aleph_1$, then the chain can be refined to a solid chain.* In fact, countable extensions of solid subgroups are again solid (see Lemma 6.6(i)), so we can refine the factor groups to factors of rank 1.
(c) *Torsion-free groups of cardinalities $\leq \aleph_1$ admit solid chains.*
(d) *If* Type(G) *is countable, then G has a solid chain from every pure subgroup up to G.*

Lemma 7.1. *Let A be a pure subgroup of a completely decomposable group C. If the cardinality of* Type(A) *is $\leq \aleph_1$, then A admits a solid chain.*

Proof. Without loss of generality, we may assume that C has typeset of cardinality $\leq \aleph_1$. Such a C is the union of a smooth chain of completely decomposable summands with countable typesets. The intersections of these summands with A provide a solid chain for A. □

If we deny CH, then we can give an example of a torsion-free group of cardinality $\aleph_2$ that does not admit a solid chain; see Lemma 8.6 below.

Existence of Solid Chains The following result is crucial for the rest of our discussion of Butler groups.

Theorem 7.2 (Fuchs [21]). *Let A be a pure subgroup of the torsion-free group G. The following conditions are equivalent:*

(i) *there exists a solid chain from* A *to* G;
(ii) *there is a balanced-exact sequence*

$$0 \to B \to A \oplus C \xrightarrow{\phi} G \to 0 \tag{14.7}$$

where C *is completely decomposable,* $\phi \restriction A$ *is the inclusion, and* B *is a* B_2*-group;*
(iii) *the same as* (ii) *with prebalanced-exact* (14.7).

Proof. (i) ⇒ (ii) We place ourselves in the following setting: we have a solid chain $A = G_0 < \cdots < G_\sigma < G_{\sigma+1} < \cdots < G$ with factors of rank 1. We build a relative balanced-projective resolution of G with the aid of this chain as follows: from a relative balanced-projective resolution $0 \to B_\sigma \to A \oplus C_\sigma \to G_\sigma \to 0$ of G_σ we form a relative balanced-projective resolution $0 \to B_{\sigma+1} \to A \oplus C_{\sigma+1} \to G_{\sigma+1} \to 0$ of $G_{\sigma+1}$ by choosing $C_{\sigma+1} = C_\sigma \oplus C^*$ for a suitable completely decomposable group C^* and mapping C^* into $G_{\sigma+1}$ such that in an arbitrarily chosen direct decomposition of C^*, no rank 1 summand maps into G_σ.

In this way, we obtain commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_\sigma & \longrightarrow & A \oplus C_\sigma & \longrightarrow & G_\sigma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{\sigma+1} & \longrightarrow & A \oplus C_\sigma \oplus C^* & \longrightarrow & G_{\sigma+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{\sigma+1}/B_\sigma & \longrightarrow & C^* & \longrightarrow & G_{\sigma+1}/G_\sigma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Evidently, B_σ is balanced in $B_{\sigma+1}$, since it is balanced in $A \oplus C_\sigma$ which is a summand of $A \oplus C_\sigma \oplus C^*$. It is routine to check that at limit ordinals the direct limits provide relative balanced-projective resolutions.

Now, if G_σ is solid in $G_{\sigma+1}$, then the ideal $\mathcal{I}$ generated by the types of summands in C^* has a countable set of generators, thus $|C^*| = \aleph_0$ may be assumed. In the diagram, $B_{\sigma+1}/B_\sigma$ is a corank one pure subgroup of the countable completely decomposable group C^*, so Theorem 6.8 guarantees that $B_{\sigma+1}/B_\sigma$ is a B_2-group. Consequently, B is the union of a smooth chain $0 = B_0 < \cdots < B_\sigma < B_{\sigma+1} < \dots$ of subgroups such that each is balanced in its successor, and each of the factors

$B_{\sigma+1}/B_\sigma$ admits a chain of decent subgroups. These chains lift to B creating a desired chain for B, so (ii) holds even with balanced (14.7).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Suppose (14.7) is as stated in (iii). B admits an $H(\aleph_0)$-family $\mathcal{B}$ of decent TEP-subgroups (Theorem 5.4). C is a completely decomposable group, so it has an $H(\aleph_0)$-family $\mathcal{C}$ of summands; furthermore, by (5.2), the torsion-free group G/A has an $H(\aleph_0)$-family $\mathcal{G}$ of pure subgroups. Dropping to suitable subfamilies (by the usual straightforward, but tedious back-and-forth argument), filtrations of B, C can be found, and one for G from A, such that the sequences $0 \to B_\sigma \to A \oplus C_\sigma \to G_\sigma \to 0$ (where $B_\sigma \in \mathcal{B}$, $C_\sigma \in \mathcal{C}$ and $G_\sigma \in \mathcal{G}$) are all prebalanced-exact, and the factor groups $C_{\sigma+1}/C_\sigma$ are all countable. Setting $C_{\sigma+1} = C_\sigma \oplus C^*$ (with countable completely decomposable C^*), consider the commutative diagram with prebalanced-exact rows

$$\begin{tikzcd}
& B_\sigma \arrow[r, equal] \arrow[d] & B_\sigma \arrow[d] & & \\
0 \arrow[r] & B_{\sigma+1} \arrow[r] \arrow[d] & A \oplus C_\sigma \oplus C^* \arrow[r] \arrow[d] & G_{\sigma+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & G_\sigma \oplus C^* \arrow[r] & G_{\sigma+1} \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are monic in the first set, and epic in the second set; the first column is exact by the 3×3-lemma. Since $B_\sigma, B_{\sigma+1} \in \mathcal{B}$, it follows from Theorem 5.4 that $K \cong B_{\sigma+1}/B_\sigma$ is a B_2-group. The bottom row is like the one in Proposition 6.10, so we can conclude that G_σ is solid in $G_{\sigma+1}$, and the proof is complete. $\square$

The special case $A = 0$ in the preceding theorem leads at once to a criterion we have been looking for:

Theorem 7.3. *A torsion-free group G admits a solid chain if and only if there is a prebalanced-exact sequence $0 \to B \to C \to G \to 0$ (where C is a B_2-group) in which B is a B_2-group.*

Proof. If G admits a solid chain, then by Theorem 7.2 there is a balanced-exact sequence with torsion-free C and B_2-group B. Conversely, let B, C be B_2-groups in the bottom prebalanced-exact sequence. Using a surjective map $\phi : C' \to C$ from a completely decomposable C' with balanced kernel, we form the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\phi} & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & G & \longrightarrow & 0.
\end{array}$$

Ker ϕ is balanced in C', and hence also in B'. By Theorem 7.2 it is a B_2-group, and thus B' is B_2 as a balanced extension of the B_2-group Ker ϕ by B'. By Lemma 2.2(iv), the top row is prebalanced-exact. Hence Theorem 7.2 implies that G admits a solid chain. □

We have a noteworthy corollary to this theorem:

Corollary 7.4. *If a torsion-free group admits a solid chain, then its summands also admit solid chains.*

Proof. Suppose $G = G_1 \oplus G_2$ has a solid chain. Choose balanced-projective resolutions $0 \to A_i \to C_i \to G_i \to 0$ $(i = 1, 2)$, where the C_i are completely decomposable. Then $0 \to A_1 \oplus A_2 \to C_1 \oplus C_2 \to G \to 0$ is a balanced-projective resolution of G. By Theorem 7.3, $A_1 \oplus A_2$ is a B_2-group, so by Corollary 5.5 the groups A_i are also B_2-groups. Apply Theorem 7.3 again to conclude that the groups G_i admit solid chains. □

Next we verify an analogue of Lemma 6.5.

Lemma 7.5. *Suppose $B < A$ are pure subgroups of the torsion-free G. Then the following holds:*

(a) *If there are solid chains from B to A and from A to G, then there is also one from B to G.*
(b) *Assume B is balanced in G. There is a solid chain from A to G if and only if there is one from A/B to G/B.*

Proof. The proofs are similar for both claims, making use of Theorem 7.2. We give details for (b) only.

(b) Starting from the relative balanced-exact resolution in the top row, we factor out B to obtain the bottom row in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & A \oplus C & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & A/B \oplus C & \longrightarrow & G/B & \longrightarrow & 0.
\end{array}$$

As B is balanced in G, the bottom row is also balanced-exact, so it is a relative balanced-exact resolution of A/B in G/B. Both solid chains in question exist exactly if K is a B_2-group (Theorem 7.2). □

The following lemma is a crucial ingredient in the proof of Corollary 7.7.

Lemma 7.6 (Bican–Fuchs [2]). *Suppose* $0 \to T \to G \to A \to 0$ *is a balanced-exact sequence where* T *is a torsion group. If* A *is a subgroup in a torsion-free group* B *such that there is a solid chain leading from* A *up to* B*, then there exists a commutative diagram with balanced-exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Proof. The claim is equivalent to the statement that the map $\mathrm{Bext}^1(B, T) \to \mathrm{Bext}^1(A, T)$ induced by the inclusion $A \to B$ is surjective. Consider a relative balanced-projective resolution $0 \to K \to A \oplus C \to B \to 0$, where C is completely decomposable. The existence of a solid chain from A to B implies that K is a B_2-group. In the induced exact sequence

$$\mathrm{Bext}^1(B, T) \to \mathrm{Bext}^1(A \oplus C, T) = \mathrm{Bext}^1(A, T) \to \mathrm{Bext}^1(K, T)$$

(for any torsion group T) the last term vanishes in view of Theorem 5.3. The balancedness of the bottom sequence follows straightforwardly (it is easier if we add $\oplus C$ to both G and A). Hence the claim follows. □

A sufficient criterion for a pure subgroup of a B_1-group to be again such a group can be given immediately.

Corollary 7.7. *Let* B *be a* B_1*-group, and* A *a pure subgroup of* B*. If there is a solid chain from* A *to* B*, then* A *is likewise a* B_1*-group.*

Proof. Suppose there is a solid chain from A to B. Lemma 7.6 implies that any balanced-exact sequence $0 \to T \to G \to A \to 0$ can be embedded in a commutative diagram like the one in Lemma 7.6 with balanced-exact bottom row. Now, if B is a B_1-group, then the bottom row splits, and hence so does the top row. Consequently, A is a B_1-group. □

Absolutely Solid Groups As a brief excursion, we consider the following concept. Call a torsion-free group **absolutely solid** if it is solid in every torsion-free group in which it is contained as a pure subgroup. The proofs of a few basic properties of this notion are delegated to the exercises. Clearly, all countable torsion-free groups are absolutely solid. We are interested in more exciting examples: in B_2-groups.

Theorem 7.8 (Rangaswamy [7], Bican–Fuchs [2]). B_2*-groups are absolutely solid.*

Proof. Let B be a B_2-group, and G a torsion-free group containing B as a pure subgroup. For the proof that B is solid in G, we may assume without loss of generality that G/B is of rank 1, so there is a countable pure subgroup S of G such that $G = B + S$. By enlarging S if necessary, we may assume that $S \cap B$ is decent in B; see Theorem 5.4. The isomorphism $G/S \cong B/(S \cap B)$ convinces us that G/S is a B_2-group, and as such it admits a solid chain. This chain lifts to a solid chain from S to G, as S is countable, so solid in G. Thus, G admits a solid chain. By Theorem 7.3, the kernel H in a balanced-projective resolution $0 \to H \to A \to B \to 0$ of B (with completely decomposable A) is a B_2-group; owing to Sect. 5(H), it is also a TEP-subgroup in A. After choosing a relative balanced-projective resolution $0 \to K \to B \oplus C \to G \to 0$ of B in G, we form the commutative diagram

$$\begin{tikzcd}
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & A \oplus C \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & B \oplus C \arrow[r] & G \arrow[r] & 0,
\end{tikzcd}$$

where the vertical arrows are monic in the first set, and epic in the second set. Since the bottom row and the middle column are balanced-exact, the middle row is also balanced-exact, and hence it is a balanced-projective resolution of G. As G admits solid chains, L must be a B_2-group. Therefore, K is a B_1-group as the quotient of a B_2-group by a balanced TEP-subgroup (cp. Sect. 5(H)). Moreover, it is a B_2-group, as H is a B_2-group (see Exercise 6). It remains to refer to Theorem 7.2 to conclude that B is solid in G. □

★ **Notes.** Theorem 7.8 was proved by Bican–Fuchs [2] for B_1-groups in L, and the general version by Rangaswamy [7].

Fuchs–Rangaswamy [3] show that the union of a countable chain of pure B_2-subgroups is again a B_2-group. The same holds for smooth chains of length ω_1 provided the subgroups in the chain are decent and of cardinalities $\leq \aleph_1$. Bican–Rangaswamy [1] extend the results to longer chains under appropriate conditions on the factor groups. See also Bican–Rangaswamy–Vinsonhaler [1].

Exercises

(1) A torsion-free group of cardinality $\leq \aleph_1$ admits a solid chain.
(2) The direct sum of groups with solid chains also has a solid chain.
(3) Furnish the details of proof in Lemma 7.5.
(4) If a torsion-free group has a solid chain, then it also has an $H(\aleph_0)$-family of solid subgroups.
(5) If the torsion-free group G contains a pure B_2-subgroup B of infinite index κ, then it contains a pure subgroup S of cardinality κ such that G/S is a B_2-group. [Hint: Theorem 7.2.]
(6) Show that direct sums and summands of absolutely solid groups are likewise absolutely solid.

(7) A solid subgroup A of an absolutely solid group G is likewise absolutely solid. [Hint: push-out for $A \to G$ and $0 \to A \to H \to H/A \to 0$.]
(8) Any extension of an absolutely solid group by a countable group is again absolutely solid.

8 Butler Groups of Uncountable Ranks

We are at last on the final route toward a necessary and sufficient condition that makes a B_1-group into a B_2-group. Armed with the arsenal of solid subgroups, the only missing ingredient is the following key lemma which is a recast of Lemma 4.4 in Chapter 9.

Lemma 8.1 (Dugas–Hill–Rangaswamy [1]). *Let κ be an uncountable regular cardinal, and $0 = C_0 < C_1 < \cdots < C_\sigma < \ldots$ $(\sigma < \kappa)$ a smooth chain of pure subgroups of the torsion-free group C such that*

(a) $\bigcup_{\sigma<\kappa} C_\sigma = C$;
(b) $|C_\sigma| < \kappa$ *for all* $\sigma < \kappa$;
(c) *for each* $\sigma < \kappa$, C_σ *is a* B_1*-group.*

If C is a B_1-group, then the set

$$E = \{\sigma < \kappa \mid \exists\, \rho > \sigma \textit{ such that } C_\sigma \textit{ is not a } \mathrm{TEP}\textit{–subgroup in } C_\rho\}$$

is not stationary in κ.

Proof. Apply Lemma 4.4 in Chapter 9 (see the notation there) when the B_σ are torsion groups, noting that the condition on Ext in the definition of the set S amounts to that the map $\mathrm{Hom}(C_{\sigma+1}, B_\sigma) \to \mathrm{Hom}(C_\sigma, B_\sigma)$ is not surjective; see the proof there. Surjectivity fails in the present case for suitable B_σ whenever C_σ is not TEP in $C_{\sigma+1}$. Consequently, Lemma 4.4 in Chapter 9 implies what we wish to prove. □

When B_1 Implies B_2 We are going to use this lemma to prove the following theorem that is one of the cornerstones in our investigation of the relation between B_1- and B_2-groups.

Theorem 8.2 (Fuchs [21]). *A B_1-group is a B_2-group if and only if it admits a solid chain.*

Proof. Necessity is evident, but the verification of sufficiency requires a delicate argument. Let B be a B_1-group of cardinality κ, and $0 = B_0 < \cdots < B_\sigma < B_{\sigma+1} < \ldots$ $(\sigma < \kappa)$ a solid chain leading up to B. It is clear that this chain contains a κ-filtration of the group B; we drop to such a filtration, but keep the notation. Observe that all the subgroups B_σ in the chain are B_1-groups—as it is manifest in view of Corollary 7.7.

The claim is verified by transfinite induction on κ. If $\kappa = \aleph_0$: we know that countable B_1-groups are B_2-groups (Theorem 4.2). So let $\kappa > \aleph_0$.

First, dispose of the case of regular cardinals κ. The preceding lemma assures the existence of a cub S in κ such that B_σ is a TEP-subgroup for all $\sigma \in S$. Omitting all the B_σ with $\sigma \notin S$, we relabel the indices of the remaining B_σ by ordinals $< \kappa$. The new smooth chain contains only B_1-subgroups which are TEP in B, and so by Sect. 5(H) even all the factors are B_1-groups. We appeal to Lemma 6.7 to argue that B_σ is decent (and hence prebalanced) in $B_{\sigma+1}$ for every $\sigma < \kappa$, thus segments of the original solid chain provides solid chains for the factors [Lemma 6.5(c)]. Therefore, the factors (being of cardinalities $< \kappa$) are B_2-groups by induction hypothesis. Decency being transitive, we can insert a chain of decent subgroups (available in the factor B_2-groups) between the links. This yields a smooth chain of decent subgroups from 0 all the way up to B, and we can conclude that B itself is a B_2-group.

We turn to the consideration of the singular cardinal case. Let B be a B_1-group of singular cardinality κ with a solid chain, say, with rank 1 factors. By Corollary 7.7, all the subgroups in the chain are B_1-groups, and hence B_2-groups by induction hypothesis. We refer to Theorem 5.5 in Chapter 1 to conclude that B has an $H(\aleph_0)$-family $\mathcal{F}$ of subgroups each of which has a smooth chain of decent subgroups. At this point, we have an ample supply of B_2-subgroups in B of cardinalities $< \kappa$ which makes the application of Theorem 9.3 in Chapter 3 possible. Using the notation introduced there, choose $\mu = \aleph_0$. Now $\mathcal{F}$ consists of B_2-groups, and a 'basis' of a $G \in \mathcal{F}$ is a smooth chain of decent subgroups. A subgroup $H < G$ is a 'free factor' of G if both H and G/H belong to $\mathcal{F}$. It is straightforward, but a bit lengthy, to check that our situation is covered by Theorem 9.3 in Chapter 3; we leave the details to interested readers. □

Unfortunately, the preceding result fails to give any clue on how to build a solid chain in a B_1-group. More can be done if CH is assumed, but then the problem would be oversimplified, and would become less exciting. We have no explanation for the reason why the B_1-property forces certain solid chains to become decent chains.

A quick consequence of the preceding theorem is recorded for homogeneous groups.

Corollary 8.3. *Homogeneous B_1-groups are completely decomposable.*

Proof. Homogeneous groups obviously have solid chains, so by Theorem 8.2 they are B_2-groups provided they are B_1-groups. The rest follows from Sect. 5(F). □

A more relevant result is obtained if we combine Theorem 8.2 with Sect. 5(D):

Corollary 8.4. *A B_1-group whose typeset has cardinality $\leq \aleph_0$ is a B_2-group.* □

Another useful corollary to Theorem 8.2 is as follows.

Corollary 8.5. *Let $0 \to A \to B \to C \to 0$ be a prebalanced-exact sequence where B is a B_2-, and A, C are B_1-groups. A is B_2 if and only if C is.*

Proof. From Theorem 7.3 it follows that A is a B_2-group if and only if C has a solid chain. As C is B_1, this is the case exactly if C is a B_2-group. □

The Vanishing of Bext^2 We turn our attention to the other principal problem. The following two results are preludes to our main result (Theorem 8.8) on Bext^2; they are concerned with the case when CH fails. Under the hypothesis ¬CH, they provide an explicit mechanism for constructing groups without solid chains.

The next lemma is a natural continuation of Theorem 6.8: it has the same setup, only the cardinality is assumed to be $> \aleph_1$, but $\leq 2^{\aleph_0}$. For its proof, we will need the existence of two sets, Σ and Σ', of the cardinality of the continuum, whose members are almost disjoint subsets of a countable set S with properties listed in Lemma 5.8 in Chapter 1. We may assume that the elements of S are prime numbers. A set X in Σ or in Σ' defines a rational group of type $\mathbf{t} = (k_2, k_3, \ldots, k_p, \ldots)$ where $k_p = 1$ or 0 according as $p \in X$ or not.

Lemma 8.6 (Fuchs [21]). *Suppose that* $2^{\aleph_0} > \aleph_1$. *Let*

$$C = \bigoplus_{\sigma < \omega_2} C_\sigma$$

be a completely decomposable group where the set of types of the rank one summands C_σ *is the union of the sets of types defined by the almost disjoint sets* Σ *and* Σ' *of cardinality* $\aleph_2$. *Then no corank* 1 *subgroup of* C *that contains none of the* C_σ *admits a solid chain.*

Proof. The notations $C(\nu), G(\nu), C_{\sigma\rho}$ defined in the proof of Theorem 6.8 will have the same meaning here, the only difference is that the range of indices is now ω_2. We may suppose that the well-ordering of the C_σ is done alternately from the two systems Σ and Σ'.

Consider the chain $\{G(\nu)\}$ for $\nu < \omega_2$. If $\nu > \omega_1$, then $G(\nu)$ is not solid in G; as a matter of fact, not even in $G(\nu + 1)$. In fact, the lattice ideal generated by the types $\mathbf{t}(C_\sigma) \wedge \mathbf{t}(C_\nu)$ $(\sigma < \nu)$($\aleph_1$ of them) is $\aleph_1$-generated. For, if the type $\mathbf{t}_\sigma$ does not belong to the same system (Σ or Σ') as $\mathbf{t}_\nu$, then $\mathbf{t}_\nu \wedge \mathbf{t}_\sigma \leq (\mathbf{t}_\nu \wedge \mathbf{t}_{\sigma_1}) \vee \cdots \vee (\mathbf{t}_\nu \wedge \mathbf{t}_{\sigma_k})$ can hold only if $\sigma \in \{\sigma_1, \ldots, \sigma_k\}$, while adjoining intersections of $\mathbf{t}_\nu$ with types from the system containing $\mathbf{t}_\nu$ does not change the type of the union at all.

Assume the claim is false: G has a solid chain. The cardinal $\aleph_2$ being regular, there would then exist a cub E in ω_2 such that $G(\nu)$ is solid in G for all $\nu \in E$. But this is absurd, so G cannot have a solid chain. □

As an immediate consequence of the preceding results, we verify:

Proposition 8.7 (Dugas–Thomé [1]). *If* $2^{\aleph_0} > \aleph_1$, *then there exist a torsion-free group* H *and a torsion group* T *such that*

$$\mathrm{Bext}^2(H, T) \neq 0.$$

In other words, if in a model V *of* ZFC, *we have* $\mathrm{Bext}^2(G, T) = 0$ *for all torsion-free groups* G *and torsion groups* T, *then* CH *must hold in* V.

Proof. Hypothesis ¬CH implies that some corank 1 subgroup H (of cardinality $\aleph_2$) of C in Lemma 8.6 fails to admit a solid chain. Such a group H is generated by the rank 1 subgroups $C_{\sigma\rho}$, so choosing groups $X_{\sigma\rho} \cong C_{\sigma\rho}$ (where $\mathbf{t}(X_{\sigma\rho}) =$

$\mathbf{t}_\sigma \wedge \mathbf{t}_\rho$) with fixed isomorphisms $\phi_{\sigma\rho} : X_{\sigma\rho} \to C_{\sigma\rho}$, we set $X = \oplus X_{\sigma\rho}$. There is a balanced-projective resolution $0 \to K \to X \xrightarrow{\phi} H \to 0$ with $\phi = \oplus \phi_{\sigma\rho}$; hence $\mathrm{Bext}^2(H, T) \cong \mathrm{Bext}^1(K, T)$. By virtue of Theorem 7.2, the kernel K is not a B_2-group. Taking into account that by construction, any intersection of different types $\mathbf{t}(X_{\sigma\rho})$ is equal to the type of $\mathbb{Z}$, it follows that K is homogeneous of type of $\mathbb{Z}$. A homogeneous K is a B_1-group if and only if it is B_2. Consequently, $\mathrm{Bext}^1(K, T) \neq 0$ must hold for a suitable torsion group T. □

At this point we pause to look at the picture that emerges from the analysis. The following theorem will shed more light on the connection between the vanishing of Bext^2 and the existence of solid chains.

Theorem 8.8. *In any model of* ZFC, *the following are equivalent:*

(i) $\mathrm{Bext}^2(G, T) = 0$ *for all torsion-free G and torsion T.*
(ii) *Every torsion-free group admits a solid chain.*
(iii) *Balanced subgroups of completely decomposable groups are B_2-groups.*

Either of (i)–(iii) *implies the Continuum Hypothesis.*

Proof. Before entering into the proof, we point out that (i) is equivalent to saying that in a balanced-projective resolution $0 \to B \to C \to G \to 0$ (with completely decomposable C) of any torsion-free group G, the kernel B is a B_1-group. This is an obvious consequence of the exact sequence $0 = \mathrm{Bext}^1(C, T) \to \mathrm{Bext}^1(B, T) \to \mathrm{Bext}^2(G, T) \to \mathrm{Bext}^2(C, T) = 0$.

(i) ⇒ (ii) Assuming (i), from Lemma 8.6 we deduce that CH holds, and therefore by Lemma 7.1 the kernels of balanced-projective resolutions admit solid chains. An appeal to Theorem 7.2 implies (ii).

(ii) ⇔ (iii) is an immediate consequence of Theorem 7.3.

(iii) ⇒ (i) Evidently, (iii) implies that in balanced-projective resolutions the kernels B are B_2-groups. Then $\mathrm{Bext}^1(B, T) = 0$ always for kernels B, and thus $\mathrm{Bext}^2(G, T) = 0$ for all G and for all torsion T.

As far as CH is concerned, we have already observed above that it follows from (i). □

To answer the speculation about whether the stated conditions (i)–(iii) follow from CH, we point out that not even GCH implies the vanishing of Bext^2. In fact, Magidor–Shelah [2] succeeded in establishing the existence of a model of ZFC with GCH in which (i) fails. However, the hypothesis V = L turns out to suffice to have $\mathrm{Bext}^2(G, T) = 0$ for all torsion-free G and torsion T (see Fuchs–Magidor [1]). Consequently, in the constructible universe L, the properties B_1 and B_2 are equivalent.

Rangaswamy's Criterion for B_2-Groups Nevertheless, CH has a remarkable consequence.

Corollary 8.9 (Rangaswamy [6]). *Assuming* CH*, a torsion-free group B is a B_2-group if and only if*

$$\mathrm{Bext}^1(B,T)=0 \quad \textit{and} \quad \mathrm{Bext}^2(B,T)=0$$

for all torsion groups T.

Proof. If B is a B_2-group, then it admits a solid chain, so $\mathrm{Bext}^2(B,T) = 0$ by Theorem 8.8, and the stated condition is necessary. Conversely, the vanishing of the two Bexts implies that B is a B_1-group with a solid chain, thus—in view of Theorem 8.2—it is a B_2-group. □

It is now time to think of the higher Bexts as well. The only relevant information we have is that if B is a B_2-group, then $\mathrm{Bext}^i(B,T) = 0$ for all $i \geq 1$, and for all torsion groups T. This is a simple consequence of the long Hom-Bext exact sequence, using induction on i.

Subgroups of B_2-Groups As far as the subgroup problem is concerned, it boils down to that of the existence of solid chains, this time from the subgroup to the group. The following theorem improves on Corollary 7.7.

Theorem 8.10 (Fuchs [21]). *For a pure subgroup A of a B_2-group G, these are equivalent:*

(i) *A is a B_2-group;*
(ii) *there is a solid chain from A to G.*

Proof. (i) ⇒ (ii) Let $0 \to B \to A \oplus C \to G \to 0$ be a balanced-projective resolution of G relative to A, where C is completely decomposable. This gives rise to the exact sequence

$$0=\mathrm{Bext}^1(G,T) \to \mathrm{Bext}^1(A,T) \to \mathrm{Bext}^1(B,T) \to \mathrm{Bext}^2(G,T)=0$$

for each torsion T; the terms involving G vanish, G being a B_2-group. Hence A is a B_1-group exactly if so is B. Now, if A is a B_2-group, then so is $A \oplus C$, and Corollary 8.5 implies that B is likewise a B_2-group. Hence (ii) follows from Theorem 7.3.

(ii) ⇒ (i) Hypothesis implies that B is a B_2-group, and so the same holds for $A \oplus C$ as a balanced extension of B_2-groups. Summands preserve B_2-property, so (i) follows. □

★ **Notes.** The theory of Butler groups culminates in the undecidability statement that in ZFC alone it is impossible to prove or disprove that B_1-groups are B_2-groups. The very sophisticated arguments by Shelah–Strüngmann [1] exceed the scope of this book; however, a remark is called for regarding the method of proof. They start with a model of ZFC in which GCH holds, and use forcing with Cohen reals for $< \kappa$ (where $\kappa \geq \aleph_4$) in order to extend the model. A group is constructed, and it is shown that in the extended model, the group is a B_1-, but not a B_2-group.

Most important developments in the problem of B_1-B_2 relations are recent papers by Mader–Strüngmann [1] and Strüngmann [3]. In the joint paper, a new class of groups, called **Hawaiian groups**, are introduced to provide examples for B_1- and B_2-groups of arbitrary cardinalities. Let $\kappa \leq 2^{\aleph_0}$ be a cardinal. A κ-Hawaiian group is built on a completely decomposable subgroup with additional data in the form of κ new generators satisfying appropriate conditions. Strüngmann

[3] shows that for certain data the Hawaiian groups are B_1-, but not B_2-groups, when adding κ Cohen reals to the universe. In some models of ZFC with $2^{\aleph_0} = \aleph_4$, he obtains indecomposable B_1-groups that are pure subgroups in completely decomposable groups, and have endomorphism rings $\cong \mathbb{Z}$, without having the B_2-property.

Let us indicate briefly the milestones in the history of Butler groups of arbitrary cardinalities. As it has already been mentioned, it started with Bican–Salce [1] who defined these groups, and proved that B_2-groups are B_1-groups without cardinality restrictions. They succeeded in showing that the converse holds in the countable case. The converse was established for groups of cardinality $\aleph_1$ by Albrecht–Hill [1]. For higher cardinalities, the proof of the coincidence of the two classes requires additional set-theoretical hypothesis, and it was proved up to $\aleph_\omega$ assuming CH by Dugas–Hill–Rangaswamy [1]. Rangaswamy [6] obtained the homological characterization (Theorem 8.9). Restricting the size of the groups, rather than expanding the ZFC models, Fuchs [21] showed that a torsion-free B of cardinality $\aleph_n$ ($n \geq 1$) is a B_2-group if and only if it satisfies $\mathrm{Bext}^i(B, T) = 0$ for all $i \leq n + 1$. Dugas–Thomé [1] proved that $\mathrm{Bext}^2(G, T)$ need not vanish for all torsion-free G if CH is denied. Magidor–Shelah [2] have the stronger claim that $\mathrm{Bext}^2(G, T) \neq 0$ may hold even if GCH is assumed. A result by Fuchs–Magidor [1] states: in the constructible universe, the class of B_1-groups coincides with the class of B_2-groups, and $\mathrm{Bext}^2(G, T) = 0$ holds for all torsion-free groups G and torsion groups T. The proof makes use of the so-called Box Principle, a set-theoretical device, known to be a consequence of the hypothesis V = L. As mentioned above, the heralded independence result in ZFC is due to Shelah–Strüngmann [1].

Exercises

(1) There are finitely Butler groups of cardinality $\aleph_1$ which fail to be B_1-groups. [Hint: union of free groups which is not free.]

(2) Suppose A is a TEP-subgroup of the finitely Butler group G. If A is a B_2-group, then it is decent in G.

(3) Assume

$$\begin{tikzcd} 0 \arrow[r] & B' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d, "\gamma"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & G \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram with exact rows, where G is a B_1-group and γ is epic with $\mathrm{Ker}\,\gamma$ prebalanced in G'. Then the top row has TEP exactly if the bottom row has it.

(4) (a) Let B be a B_1-group of cardinality $\leq \aleph_1$. Show that it is a B_2-group by proving that it has a solid chain.
(b) A B_1-group that has a smooth chain of prebalanced subgroups with factors of cardinalities $\leq \aleph_1$ is a B_2-group.

(5) (Fuchs–Rangaswamy) A pure subgroup of a completely decomposable group is a B_2-group provided it is a B_1-group.

(6) (Fuchs–Rangaswamy) Derive Griffith's Theorem 2.1 in Chapter 15 from Theorem 8.10 by writing a Baer group B as $B = F/H$ with free groups F, H. Then argue that the Hom-Ext sequence implies that H is TEP in F, so B is a B_2-group. Finite rank subgroups of B are free, so B is homogeneous, and therefore free.

9 More on Infinite Butler Groups

A few additional results of interest on Butler groups are worthwhile recording.

Rigid Systems of B_2-Groups One is inclined to think that B_2-groups —being so special—possess more transparent structures than torsion-free groups in general. This is probably true, but in spite of this, they can display several unusual features known for general torsion-free groups. For instance, on one hand, there exist indecomposable B_2-groups of arbitrarily large cardinalities, and on the other hand, there exist countable superdecomposable B_2-groups. As one of our final projects on Butler groups, we discuss next the question of large indecomposable B_2-groups. We will not only establish the existence of such groups, but will moreover show that there is an ample supply of them.

To begin with, we construct a fully rigid system of finite rank Butler groups for $\mathbb{Z}$ on the index set $I = \{1, \dots, n\}$ for any integer $n \geq 2$.

Let $\{A_{-1}, A_0, \dots, A_n\}$ be a set of rank one torsion-free groups; the type of A_i will be denoted by $\mathbf{t}_i$. Assume that the types in the set

$$\{\mathbf{t}_{ij} = \mathbf{t}_i \wedge \mathbf{t}_j \mid i \neq j;\ i, j = -1, 0, \dots, n\}$$

are pairwise incomparable. Fix any map $\phi\colon\ A_{-1} \oplus A_0 \oplus \cdots \oplus A_n \to \mathbb{Q}$ such that $\phi(A_i) \neq 0$ for all i. We claim: $H = \operatorname{Ker}\phi$ is a strongly indecomposable Butler group of rank $n+1$ whose endomorphism ring is isomorphic to a subring of $\mathbb{Q}$. The rank 1 subgroups $H(\mathbf{t}_{ij})$ are fully invariant in H, so any $\eta \in \operatorname{End} H$ acts as a multiplication by a rational number r_{ij} on $H(\mathbf{t}_{ij})$. For distinct indices i, j, k, $H(\mathbf{t}_{ik})$ intersects $H(\mathbf{t}_{ij}) \oplus H(\,\mathbf{t}_{jk})$ non-trivially, thus we must have $r_{ij} = r_{jk} = r_{ik}$, proving the claim.

$\operatorname{End} H$ will be $\cong \mathbb{Z}$, if e.g. the types are chosen such that $\mathbb{Z}$ is the endomorphism ring of a rank 1 group of type $\mathbf{t}_{-1} \wedge \mathbf{t}_0$.

It is easy to find types of the desired kind. For instance, choose

$$\mathbf{t}_{-1} = (\infty, 0, 1, 1, 1, \dots) \quad \text{and} \quad \mathbf{t}_0 = (0, \infty, 1, 1, 1, \dots),$$

while for $i = 1, 2, \dots, n$, let $\mathbf{t}_i$ be the type which starts with ∞, ∞ and continues with a periodically repeating string of the form $(1, \dots, 1, 0, 1, \dots, 1)$ of length n, with a single 0 in the ith place.

For a subset $X \subseteq I = \{1, \dots, n\}$, define the group H_X as the kernel of the restriction of ϕ to the direct sum $\oplus A_i$ with $i \in \{-1, 0\} \cup X$. Then $H_I = H$, and the subgroups H_X are fully invariant in H. It is readily seen that the set $\{H_X \mid X \subseteq I\}$ is in fact a fully rigid system for $\mathbb{Z}$. We have all the ingredients for the application of Corner's Lemma 4.8 in Chapter 12 to derive (Exercise 3):

Theorem 9.1 (Fuchs–Metelli [2]). *For every infinite cardinal κ, there exists a fully rigid system consisting of 2^κ (pairwise non-quasi-isomorphic) B_2-groups of cardinality κ each of which has endomorphism ring isomorphic to $\mathbb{Z}$.* □

Superdecomposable Butler Groups We now turn our attention to the dual problem: the existence of superdecomposable Butler groups. Recall that a superdecomposable group was defined to be a group which had no indecomposable summands except for 0. (The superdecomposable groups constructed in Theorem 1.5 in Chapter 13 are not Butler.)

The method we adopt here is based on the construction of a special tree. Define a tree T of length ω whose nth level T_n contains 2^n vertices:

$$T_n = \{(n, 0), (n, 1), \dots, (n, 2^n - 1)\}.$$

The edges are directed, connecting vertex (n, m) to vertices $(n + 1, 2m)$ and $(n + 1, 2m + 1)$ for all $0 \leq m < 2^n$. To each vertex (n, m) of $T = \cup_{n<\omega} T_n$ assign a torsion-free group A_{nm}, and to each directed edge a monomorphism

$$\phi^0_{nm}\colon\ A_{nm} \to A_{n+1,2m}, \qquad \phi^1_{nm}\colon\ A_{nm} \to A_{n+1,2m+1}.$$

Next, for each $n < \omega$ define the group $G_n = \bigoplus_{(n,m)\in T_n} A_{nm}$, and the map $\phi_n\colon G_n \to G_{n+1}$ such that its restriction to A_{nm} is the map

$$\phi^0_{nm} \oplus \phi^1_{nm}\colon\ A_{nm} \to A_{n+1,2m} \oplus A_{n+1,2m+1}.$$

Let G denote the limit of the arising direct system $\{G_n \mid \phi_n\ (n < \omega)\}$. All the canonical maps $\psi_n\colon G_n \to G$ are monic, so the A_{nm} and the G_n may be identified with their images in the direct limit G.

Lemma 9.2 (Benabdallah–Birtz [1], Meinel [1]). *The direct limit G is a superdecomposable group if $\{A_{nm} \mid (n, m) \in T_n\}$ is a family of non-trivial torsion-free groups satisfying the following two conditions:*

(i) $\mathrm{Hom}(A_{nm}, A_{ij}) = 0$ *if there is no directed path from vertex (n, m) to vertex (i, j);*
(ii) *for every homomorphism $\eta\colon A_{nm} \to G$ there is an integer k such that η factors through a map $\eta_k\colon A_{nm} \to G_k$, i.e., $\eta = \psi_k\eta_k$.*

Proof. Suppose we have a collection of torsion-free groups A_{nm} satisfying (i)–(ii). Define B_{nm} ($n < \omega$; $m = 0, \dots, 2^n - 1$) as the union of the A_{ij} for all vertices (i, j) above (n, m) (including (n, m) itself), which are connected to (n, m) by a directed path. Just as the A_{ij}, these B_{nm} may be viewed as subgroups of G. It is readily checked:

(a) $B_{00} = G$. Furthermore, $B_{nm} \cap G_n = A_{nm}$.
(b) $B_{nm} = B_{n+1,2m} \oplus B_{n+1,2m+1}$ for all $(n, m) \in T$, as is clear from the way A_{nm} was embedded in the direct sum $A_{n+1,2m} \oplus A_{n+1,2m+1}$.
(c) Each B_{nm} is fully invariant in G. Indeed, this is an immediate consequence of conditions (i) and (ii).
(d) The intersection of the B_{nm} for vertices (n, m) in a branch $Y = \{(n, i_n) \mid n < \omega\}$ of T is 0 (where $i_{n+1} = 2i_n$ or $2i_n + 1$). In fact, let $y \in G$ belong to each

$B_{n,i_n} \in Y$. There is an $n < \omega$ with $y \in G_n$, thus $y \in B_{n,i_n} \cap G_n = A_{n,i_n}$. Also, $y \in B_{n+1,i_{n+1}} \cap G_{n+1} = A_{n+1,i_{n+1}}$, whence $y \in A_{n,i_n} \cap A_{n+1,i_{n+1}} = 0$.

To verify the lemma, suppose C is an indecomposable summand of G, say, $G = C \oplus D$ for some $D \leq G$. From (a)–(c) we derive $G = B_{00} = B_{10} \oplus B_{11}$, whence $B_{1i} = (B_{1i} \cap C) \oplus (B_{1i} \cap D)$ for $i = 1, 2$. Thus $C = (B_{10} \cap C) \oplus (B_{11} \cap C)$. By indecomposability, either $C \leq B_{10}$ or $C \leq B_{11}$, and so C is a summand of one of them. In the first alternative, $C = (B_{20} \cap C) \oplus (B_{21} \cap C)$, so again, C is a summand of either B_{20} or B_{21}. Continuing in the obvious fashion, we see that C is contained in each B_{n,i_n} for vertices (n, i_n) in a branch of T. Hence (d) implies $C = 0$. ☐

We are now prepared to prove:

Theorem 9.3 (Dugas–Thomé [2], Fuchs–Metelli [2]). *There exist superdecomposable Butler groups of countable rank.*

Proof. We want to find finite rank Butler groups A_{nm} satisfying conditions (i) and (ii). We do not have to pay attention to (ii) as it is automatically satisfied by groups of finite rank.

Let $\{\mathbf{t}_{-1}, \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_n, \ldots\}$ be a countable set of types subject to the conditions that the types $\mathbf{t}_i \wedge \mathbf{t}_j$ $(i \neq j)$ are pairwise incomparable and $\mathbb{Z}$ is the endomorphism ring of a rank 1 group of type $\mathbf{t}_{-1} \wedge \mathbf{t}_0$. These conditions are fulfilled, e.g., if we define $\mathbf{t}_{-1}$ and $\mathbf{t}_0$ to have 1's at even numbered places, and alternately ∞ and 0 at odd numbered places:

$$\mathbf{t}_{-1} = (\infty, 1, 0, 1, \infty, 1, 0, 1, \ldots) \quad \text{and} \quad \mathbf{t}_0 = (0, 1, \infty, 1, 0, 1, \infty, 1, 0, \ldots),$$

while $\mathbf{t}_n$ $(n \geq 1)$ has 0 at even numbered, and ∞ at odd numbered places, except for the $2n-1$st and $2n$th odd numbered places, where 0's are chosen:

$$\mathbf{t}_1 = (0, 0, 0, 0, \infty, 0, \infty, 0, \infty, \ldots), \mathbf{t}_2 = (\infty, 0, \infty, 0, 0, 0, 0, 0, \infty, 0, \ldots), \ldots$$

We proceed to select torsion-free groups A_n $(n = -1, 0, 1, 2, \ldots)$ of rank one such that $\mathbf{t}(A_n) = \mathbf{t}_n$, and monomorphisms $\phi_n : A_n \to \mathbb{Q}$. Define a bijection $f : \omega \to T$ with $f(0) = (0, 0)$. For $(n, m) \in T$, denote by X_{nm} the set of vertices of T which lie on the path from $(0, 0)$ to (n, m). Using the maps ϕ_n, define A_{nm} as the group H_I constructed above with the aid of the A_i with $i \in I = \{-1\} \cup \{f^{-1}(X_{nm})\}$.

If there is no directed path connecting (n, m) and (i, j), then A_{nm} and A_{ij} are incomparable members of the fully rigid system consisting of the subgroups of H_J on the index set $J = \{-1\} \cup \{f^{-1}(X_{nm} \cup X_{ij})\}$, hence condition (i) will be satisfied. For ϕ^0_{nm}, ϕ^1_{nm} choose embeddings of A_{nm} in $A_{n+1,2m}, A_{n+1,2m+1}$ as pure subgroups. Then the groups G_n will be finite rank Butler groups, and the map $\phi_n : G_n \to G_{n+1}$ yields a pure embedding of G_n in G_{n+1}. Since the union of a countable chain of pure finite rank Butler subgroups is a Butler group, the proof is completed. ☐

★ **Notes.** The construction for Theorem 9.1 shows that the groups can be chosen so as to have their typesets contained in a lattice generated by eight elements. Arnold–Dugas [1] succeeded in sharpening this result to five generators; they use free groups with distinguished subgroups to derive their result. The same paper contains several results on Butler groups with finite typesets.

Blagoveshchenskaya–Göbel–Strüngmann [1] investigate a new class of groups: epic images of local almost completely decomposable groups. A generalization of near-isomorphism to infinite ranks is used as the basis of classification.

Exercises

(1) Any direct decomposition of the group G in the proof of Theorem 9.3 has only a finite number of summands $\neq 0$.
(2) Every endomorphism of G in Theorem 9.3 is determined by its action on A_{00}.
(3) Prove that the groups H_X referred to in Theorem 9.1 are B_2-groups. [Hint: they are finite sums of B_2-groups tensored by free groups.]

Problems to Chapter 14

PROBLEM 14.1 (Metelli). Find invariants for the quasi-isomorphy classes of $\mathcal{B}^{(n)}$-groups, especially in the $n = 2$ case.

PROBLEM 14.2. Determine the projective dimensions of finite rank Butler groups over their endomorphism rings.

PROBLEM 14.3. When is the tensor product of two B_1-groups again a B_1-group?

PROBLEM 14.4. When is a finitely Butler group a B_2-group? Is it if it has a solid chain?

PROBLEM 14.5. Characterize inverse limits of (finite rank) Butler groups.

Chapter 15
Mixed Groups

Abstract Mixed groups form the most general class of abelian groups, and traditionally, the main question is to find out how the torsion and torsion-free parts are put together to create a mixed group.

For a long time the theory of mixed groups was a neglected area; papers dealing with mixed groups were embarrassingly sparse. The reason was perhaps that no powerful methods were available to deal with more intricate situations. The enormous complexity of their structure made an in-depth study of mixed groups virtually impossible. As a result, the main concern was to find out when a group was not an honest mixed group, but a splitting group: a direct sum of a torsion and a torsion-free group. The characterizations of those torsion and the torsion-free groups that force splitting were the main results on mixed groups up to the 1970s.

The theory was suddenly revitalized: the impetus was the revision of the traditional view as an extension of a torsion group by a torsion-free group. Mixed groups could also be regarded in the diagonally opposite way as an extension of a torsion-free group by a torsion group. This point of view, inspired by J. Rotman, turned out a more profound approach to mixed groups, though there was no canonical way of treating them from this angle. New life was brought to the study of mixed groups after it was observed, in addition, that newly developed methods in the theory of torsion groups may be used, *mutatis mutandis*, to deal with some special interesting mixed groups. The theory of valuated groups—as the new vehicle for research—was developed under the leadership of the New Mexico group theorists.

The quintessence of the new approach, first systematically applied by Warfield, is the idea of ignoring temporarily the torsion subgroup, but at the same time enriching a free subgroup of maximal rank with the height functions (as valuations) of its elements. In this way, relevant information about the absent torsion subgroup is incorporated into the free subgroup considered, which sufficed in interesting special cases.

In the course of developing a theory of mixed groups, we start with the conventional splitting problem. We then introduce the machinery needed for the discussion of tractable classes of mixed groups. Some tools developed for torsion and torsion-free groups need to be readjusted, and new tools have to be acquired to deal with the new entities. Existence and extension of homomorphisms between mixed groups are in the center of discussion. The highlights are the structure theorems on simply presented and Warfield groups.

1 Splitting Mixed Groups

It is easy to find examples for mixed groups: just take the direct sum of a torsion and a torsion-free group. But there exist non-splitting mixed groups, i.e. mixed groups that do not split into such a direct sum, and needless to say, we are mainly concerned with this kind of mixed group.

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Non-Splitting Mixed Groups We begin with examples of non-splitting mixed groups. Our first example provides a stunningly simple method of manufacturing non-splitting mixed groups, while the other two are key examples exhibiting non-splitting mixed groups of torsion-free rank 1.

Example 1.1. Let T be an unbounded reduced torsion group. Then $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) = T^\bullet$ is a non-splitting reduced mixed group (the cotorsion hull of T) with T as torsion subgroup.

Example 1.2. Let $p_1, \dots, p_n, \dots$ denote different primes, and define

$$T_1 = \oplus_{n=1}^{\infty} \langle a_n \rangle \qquad \text{with } o(a_n) = p_n.$$

Thus T_1 is the torsion subgroup of the direct product $B = \prod_{n=1}^{\infty} \langle a_n \rangle$, a mixed group. Let $b_0 = (a_1, \dots, a_n, \dots) \in B$. For $i \neq n$, the equation $p_n x = a_i$ is uniquely solvable in $\langle a_i \rangle$, thus B contains a unique element b_n such that $p_n b_n = (a_1, \dots, a_{n-1}, 0, a_{n+1}, \dots) = b_0 - a_n$. T_1 is the torsion part of the group $A_1 = \langle T_1, b_1, \dots, b_n, \dots \rangle$ such that A_1/T_1 is torsion-free of rank 1. We claim A_1 is not splitting. Otherwise, we had $A_1 = T_1 \oplus G$ for some $G < A_1$, so $b_n = t_n + g_n$ with $t_n \in T_1, g_n \in G$ for each $n < \omega$. Thus $p_n t_n + p_n g_n = p_n b_n = b_0 - a_n = (t_0 - a_n) + g_0$. Equating the T_1-coordinates we get $p_n t_n = t_0 - a_n$ for $n = 1, 2, \dots$. There is a prime p_j for which the equation $p_j x = t_0$ is solvable in T_1, and if $x = t \in T_1$ is a solution, then we get $p_j(t - t_j) = a_j$, an obvious contradiction.

Example 1.3. Let p be any prime, and set

$$T_2 = \oplus_{n=1}^{\infty} \langle a_n \rangle \qquad \text{with } o(a_n) = p^{2n}.$$

Now we define $b_n = (0, \dots, 0, a_n, pa_{n+1}, p^2 a_{n+2}, \dots) \in \prod_{n=1}^{\infty} \langle a_n \rangle$. These b_n are of infinite order, and satisfy $pb_{n+1} = b_n - a_n$ $(n = 1, 2, \dots)$. It is readily checked that T_2 is the torsion part of the p-local group $A_2 = \langle T_2, b_1, \dots, b_n, \dots \rangle$ such that A_2/T_2 is p-divisible torsion-free of rank 1. If $A_2 = T_2 \oplus G$ held for some $G < A_2$, then G would be a p-divisible subgroup of A_2, contrary to the fact that $\prod_{n=1}^{\infty} \langle a_n \rangle$ has no such subgroup $\neq 0$. Consequently, A_2 is not splitting.

One nice aspect of Example 1.1 is that the method employed there can be used to show that every non-splitting mixed group of torsion-free rank 1 can be found in the cotorsion hull of its torsion subgroup.

Proposition 1.4. *Let A be a reduced, non-splitting mixed group of torsion-free rank 1, and $T = t(A)$. The canonical map $T \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ extends to a monomorphism $A \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$.*

Proof. As A/T is torsion-free and $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ is cotorsion, Theorem 9.2 in Chapter 9 allows us to extend the canonical map $T \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ to a homomorphism $\phi : A \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$. Evidently, $\mathrm{Ker}\,\phi$ is torsion-free and reduced. Hence, if $\mathrm{Ker}\,\phi \neq 0$, then $\mathrm{Im}\,\phi$ must have a torsion subgroup $> T$. But this is impossible. □

We stop for a moment to interpret this result as saying in effect that the cotorsion hull of a reduced torsion group T is a largest mixed group with torsion part T that has no non-trivial direct decomposition where T is contained in one of the summands.

Nunke Groups By making use of the generalized Prüfer p-groups H_σ (which will now be denoted as $H_\sigma(p)$ to indicate dependence on p), we introduce non-splitting mixed groups of torsion-free rank 1, following Nunke [3]. The **Nunke group $N_\sigma(p)$ of length σ** is obtained as follows.

For every prime p and every ordinal σ, a mixed group $N_\sigma(p)$ is constructed such that the sequence

$$0 \to \mathbb{Z}_\sigma \to N_\sigma(p) \to H_\sigma(p) \to 0 \tag{15.1}$$

is exact, where $\mathbb{Z}_\sigma$ is an infinite cyclic group, and the groups $N_\sigma(p)$ satisfy the following conditions:

(i) $p^\sigma N_\sigma(p) = \mathbb{Z}_\sigma$,
(ii) $N_{\sigma+1}(p) = pN_\sigma(p)$ and $\mathbb{Z}_{\sigma+1} = p\mathbb{Z}_\sigma$, and
(iii) for a limit ordinal σ, $N_\sigma(p)/\mathbb{Z}_\sigma \cong \oplus_{\rho<\sigma} N_\rho(p)/\mathbb{Z}_\rho$.

The construction is an imitation of the definition of the H_σ in Sect. 1 in Chapter 10, the only difference being that the starting point is not a cyclic group of order p, but an infinite cyclic group. The details are left to the reader.

We will refer several times to the following important observation.

Lemma 1.5 (Nunke [5]). *For every group A, the exact sequence* (15.1) *induces the long exact sequence*

$$\begin{aligned} 0 \ \to\ & \mathrm{Hom}(H_\sigma(p), A) \to \mathrm{Hom}(N_\sigma(p), A) \to \mathrm{Hom}(\mathbb{Z}_\sigma, A) \cong A \xrightarrow{\delta} \\ \xrightarrow{\delta}\ & \mathrm{Ext}(H_\sigma(p), A) \to \mathrm{Ext}(N_\sigma(p), A) \to 0 \end{aligned}$$

where $\mathrm{Ker}\,\delta = p^\sigma A$.

Proof. We get the sequence from Theorem 2.3 in Chapter 9. The image of the map between the last two Homs comes from the restrictions of maps $N_\sigma(p) \to A$ to $\mathbb{Z}_\sigma = p^\sigma N_\sigma(p)$, so it is contained in $p^\sigma A$. On the other hand, as $H_\sigma(p)$ is totally projective, every $\mathbb{Z}_\sigma \to p^\sigma A$ extends to a map $N_\sigma(p) \to A$. □

Splitting Criteria We continue with the discussion of conditions under which a mixed group A **splits**, i.e. $A = T \oplus G$, where $T = t(A)$ is unique, while the torsion-free $G \cong A/T$ is unique up to isomorphism.

First, we ask whether a mixed group ought to split if all of its localizations at primes are splitting; the answer is "no:" Example 1.2 is a counterexample (see also Examples 2.2 and 2.3). However, a fixed homomorphism all of whose localizations are splitting maps is itself a splitting map.

The next theorem is a frequently quoted characterization of those torsion groups that force the splitting of mixed groups whenever they appear as torsion parts.

Theorem 1.6 (Baer [4], S. Fomin [1]). *A torsion group T satisfies* $\mathrm{Ext}(G, T) = 0$ *for all torsion-free groups G if and only if $T = B \oplus D$ for a bounded group B and a divisible group D.*

Proof. Considering that divisible subgroups are always summands, it suffices to deal with reduced groups T. Our Example 1.1 above shows that T ought to be bounded to have the stated property. On the other hand, if T is bounded, then because of its purity it is a summand in every mixed group in which it is the torsion subgroup. □

The dual problem of characterizing the torsion-free groups G such that extensions of torsion groups by G are always splitting is more difficult; it will be discussed in the next section.

There are several necessary and sufficient criteria for splitting, but none is totally satisfactory. One of the best is given in the next theorem.

Theorem 1.7 (May [2]). *A mixed group A is splitting if and only if it is the union of a countable chain*

$$0 = A_0 \leq A_1 \leq \cdots \leq A_n \leq \dots$$

of subgroups such that, for all $n < \omega$, the torsion subgroup tA_n is bounded and $t(A/A_n) = (tA + A_n)/A_n$.

Proof. If A is splitting, $A = tA \oplus G$, then the subgroups $A_n = A[n!] \oplus G$ satisfy the stated conditions. Conversely, suppose A has subgroups A_n as stated. Then $A_n = tA_n \oplus G_n$ with torsion-free G_n $(n < \omega)$. We are done if we can choose the G_n to form a chain, because then we can put $G = \cup_{n<\omega} G_n$. Given a partial chain $0 = G_0 \leq \cdots \leq G_n$, we argue

$$\begin{aligned} A_{n+1}/(tA_{n+1} \oplus G_n) &= A_{n+1}/[(tA \oplus G_n) \cap A_{n+1}] \cong (tA + A_{n+1})/(tA + A_n) \cong \\ &\cong [(tA + A_{n+1})/A_n]/[(tA + A_n)/A_n] \cong [(tA + A_{n+1})/A_n]/t(A/A_n). \end{aligned}$$

The last factor group is torsion-free, thus $tA_{n+1} \oplus G_n$ is pure in A_{n+1}. It follows that $(tA_{n+1} \oplus G_n)/G_n$ is a bounded pure subgroup in A_{n+1}/G_n, thus

$$A_{n+1}/G_n = (tA_{n+1} \oplus G_n)/G_n \oplus G_{n+1}/G_n$$

for some $G_{n+1} \geq G_n$. Therefore, $A_{n+1} = tA_{n+1} \oplus G_{n+1}$. □

Quasi-Splitting We turn our attention to mixed groups that are close to being splitting in the following sense. A group A is **quasi-splitting** if it has a splitting subgroup B such that $nA \leq B \leq A$ for some integer $n > 0$. Note that if $B = S \oplus G$, where S is torsion and G is torsion-free, then $B + T = T \oplus G$ also splits for the torsion part T of A. Consequently, in the definition of quasi-splitting we may assume that B has the same torsion subgroup as A. The next example exhibits a quasi-splitting mixed group that fails to split.

Example 1.8. Let $T = \oplus_{n=1}^{\infty} \mathbb{Z}(p^n)$ and $G = \tilde{F}$, the p-adic completion of a free group F of countable rank. Then G is algebraically compact such that $G/pG \cong T/pT$, so there is an epimorphism $\eta\colon G \to T/pT$ with kernel pG. Let A be the pull-back in the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & pT & \longrightarrow & A & \longrightarrow & G & \longrightarrow & 0 \\ & & \Vert & & \downarrow & & \downarrow{\scriptstyle\eta} & & \\ 0 & \longrightarrow & pT & \longrightarrow & T & \xrightarrow{\text{can}} & T/pT & \longrightarrow & 0, \end{array}$$

so A may be thought of as the set of all $(t, g) \in T \oplus G$ with $\eta g = t + pT$. Then $pA \leq pT \oplus pG \leq A$, so A is quasi-splitting. If A were splitting, then there would exist a map $\xi : G \to T$ such that $\eta g = \xi g + pT$ for all $g \in G$. But ξG is cotorsion and reduced torsion, so it must be of bounded order. Therefore, the image of ξG in T/pT would be finite, while ηG is infinite. Thus A cannot split.

Next we prove a lemma in a more general setting than needed for the proof of Theorem 1.10. Let C be a group and $n > 0$ an integer. Multiplication by n in C factors as $C \xrightarrow{\mu} nC \xrightarrow{\nu} C$ where $\mu : c \mapsto nc$ and ν is the inclusion map. Given the bottom exact sequence, we build a commutative diagram with exact rows:

$$\begin{array}{lcccccccc}
\mathfrak{e}\nu\mu\colon 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \mu} & & \\
\mathfrak{e}\nu\colon 0 & \longrightarrow & A & \xrightarrow{\alpha} & nB + A & \xrightarrow{\beta} & nC & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \nu} & & \\
\mathfrak{e}\colon 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0.
\end{array}$$

Lemma 1.9 (C. Walker [1]). *If $\mathfrak{e}\nu$ splits, then $\mathfrak{e}$ represents an element of* $\mathrm{Ext}(C, A)[n]$. *If $C[n] = 0$, then the converse is also true.*

Proof. Observe that $\mathfrak{e}\nu\mu = n\mathfrak{e}$. Now, if $\mathfrak{e}\nu$ splits, then so does $n\mathfrak{e}$, and therefore $\mathfrak{e}$ belongs to $\mathrm{Ext}(C, A)[n]$. Conversely, $\mathfrak{e} \in \mathrm{Ext}(C, A)[n]$ means that $\mathfrak{e}$ is contained in the kernel of the endomorphism of $\mathrm{Ext}(C, A)$ induced by multiplication by n in C. Therefore, if $C[n] = 0$, then in view of Lemma 5.1(iii) in Chapter 9 this is equivalent to the splitting of $\mathfrak{e}\nu$. $\square$

The following theorem identifies the torsion subgroup of $\mathrm{Ext}(G, T)$ as the set of quasi-splitting extensions.

Theorem 1.10 (C. Walker [1]). *A mixed group A is quasi-splitting if and only if the exact sequence*

$$0 \to T \to A \to G \to 0 \tag{15.2}$$

(*where $T = t(A)$ and G is torsion-free*) *represents an element of finite order in* $\mathrm{Ext}(G, T)$.

Proof. If this exact sequence represents an element of finite order in $\mathrm{Ext}(G, T)$, then the second part of the preceding lemma implies that the exact sequence

$$0 \to T \to nA + T \to (nA + T)/T \to 0 \tag{15.3}$$

is splitting. Since $nA \leq nA + T \leq A$, this means that A is quasi-splitting.

Conversely, assume A is quasi-splitting, say, $n > 0$ is an integer such that some subgroup $B \leq A$ with $nA + T \leq B$ is splitting. Then T is a summand of $nA + T$, and (15.3) is a splitting sequence. From Lemma 1.9 we conclude that then (15.2) represents an element of $\mathrm{Ext}(G, T)[n]$. □

Non-Splitting Groups Close to Splitting From a different point of view, we might regard a non-splitting mixed group to be close to being splitting if all of its subgroups of smaller cardinalities are splitting. Concerning such groups we have the following result.

Proposition 1.11. *If, for the cardinal κ, there is a non-free κ-free group of cardinality κ, then there exists a non-splitting mixed group of cardinality κ all of whose subgroups of smaller cardinalities are splitting.*

Proof. Let A be κ-free, non-free group of cardinality κ. Theorem 2.1 in the next section implies that A is not a Baer group, thus there exists a torsion group T with $\mathrm{Ext}(A, T) \neq 0$. Lemma 2.4 shows that $|T| \leq \kappa$ may be assumed. Choose a mixed group M which is a representative of a non-zero element in this Ext. Then every subgroup N of M of cardinality $< \kappa$ must be splitting, since $N/(N \cap T) \cong (N+T)/T$ is a free group. □

In particular, in view of Corollary 8.10 in Chapter 3, we can conclude that for every integer $n > 1$ there exist mixed groups of cardinality $\aleph_n$ which do not split, but all of their subgroups of cardinalities $\leq \aleph_{n-1}$ are splitting.

★ **Notes.** It was Levi [1] who constructed the first non-splitting mixed group. It took nearly two decades until S. Fomin [1] and Baer [4] found a sufficient and a precise condition, respectively, for a torsion group to force splitting. The theory of quasi-splitting is due primarily to C. Walker [1].

Several authors, including A. Fomin [3], investigate quotient-divisible mixed groups. For more on these groups, see Albrecht–Breaz–Vinsonhaler–Wickless [1].

The obvious question as to when non-equivalent extensions are isomorphic has not been thoroughly investigated. Exception is Mader [3] who studied isomorphic extensions of a torsion group by a torsion-free group.

Exercises

(1) Summands of splitting groups are splitting.
(2) (Procházka) If A is a mixed group such that, for some $n \in \mathbb{N}$, the subgroup nA splits, then A also splits.
(3) (a) (S. Fomin) $A = \langle a_1, \dots, a_n, \dots \mid pa_1 = \cdots = p^n a_n = \dots \rangle$ is a non-splitting mixed group.
 (b) (Kulikov) The mixed group $A = \langle a_1, \dots, a_n, \dots \mid p^2(a_1 - pa_2) = \cdots = p^{2n}(a_n - pa_{n+1}) = \cdots = 0 \rangle$ does not split.

(4) (Oppelt) Let $T = T_1 \oplus \cdots \oplus T_n$ be a torsion group with p_i-components T_i, and let A be a mixed group with torsion part T. A splits if and only if, for every i, the group $G/(T_1 \oplus \cdots \oplus T_{i-1} \oplus T_{i+1} \oplus \cdots \oplus T_n)$ splits.

(5) (Corner) For a torsion group T, these are equivalent:

   (a) T is an endomorphic image of every mixed group with torsion part T;
   (b) T has a basic subgroup that is an endomorphic image of every mixed group in which T is the torsion part;
   (c) $T = B \oplus D$ with bounded B and divisible D.

(6) There exist epimorphisms $A_1 \to T_1$ and $A_2 \to T_2$ (see Examples 1.2 and 1.3). [Hint: $a_i \mapsto 0, b_0 \mapsto 0, b_n \mapsto a_n$, and $a_{4n} \mapsto a_n, a_{4n+i} \mapsto 0$ for $i = 1, 2, 3$.]

(7) (Mishina) Let A be a mixed group with no elements of order 2. A is splitting if it has an automorphism which acts as multiplication by -1 on its torsion part T, and induces the identity map on A/T.

(8) (Stratton) Let A be a mixed group, and F a pure free subgroup of finite rank. A is splitting exactly if A/F is splitting. [Hint: the canonical $A \to A/F$ yields an isomorphism between torsion groups.]

(9) Give an example showing that the torsion part need not be an endomorphic image of a mixed group. [Hint: Example 1.1, or B in Example 1.2.]

(10) (Strüngmann) Let T be a torsion group with only a finite number of non-trivial bounded p-components (no restriction on the unbounded ones). There is a countable completely decomposable group C such that every countable torsion-free group G with $\mathrm{Ext}(G, T) = 0$ embeds in C. [Hint: try $C = R^{(\aleph_0)}$ with $R < \mathbb{Q}$, $h_p(1) = \infty$ for $\{p \mid T_p$ not unbounded$\}$.]

(11) (Joubert–Ohlhoff–Schoeman) A mixed group A with torsion subgroup T is quasi-splitting if and only if there is $n \in \mathbb{N}$ such that the map $\dot{n} : T \to T$ extends to $A \to T$.

2 Baer Groups are Free

We now attack the problem dual to the one settled in Theorem 1.6 above: *which are the torsion-free groups G such that every mixed group A whose torsion subgroup T satisfies $A/T \cong G$ is splitting?* This problem was posed by Baer [4], and stayed open for 30 years, until it was solved by Griffith [4].

Baer Groups Actually, we will deal with this problem in a slightly more general form by not assuming in advance that G is torsion-free. Accordingly, we say that B is a **Baer group** if

$$\mathrm{Ext}(B, T) = 0 \qquad \text{for all torsion groups } T.$$

Evidently, free groups are Baer groups, and our goal is to show that there are no other Baer groups.

We start with a list of a few elementary properties.

(a) *Subgroups of Baer groups are Baer groups.* This follows from the fact that if $C \to B$ is monic, then the induced map $\mathrm{Ext}(B, T) \to \mathrm{Ext}(C, T)$ is epic.
(b) *A Baer group B must be torsion-free.* In fact, if B contains a cyclic subgroup C of order p for some prime p, then from the exactness of the sequence $\mathrm{Ext}(B, C) \to \mathrm{Ext}(C, C) \to 0$ and from $\mathrm{Ext}(C, C) \neq 0$ we conclude $\mathrm{Ext}(B, C) \neq 0$, so such a B cannot be Baer.
(c) *A finite rank Baer group is free.* Let T_0 denote the direct sum of countably many copies of the group $\oplus_p \oplus_{k<\omega} \mathbb{Z}(p^k)$ (summation over all primes). Suppose B is torsion-free of finite rank n, and F is a free subgroup of rank n in B. If B is not finitely generated, then B/F is a countably infinite torsion group, in which case $\mathrm{Ext}(B/F, T_0)$ has to be of the power of the continuum. In fact, if B/F contains a copy of $\mathbb{Z}(p^\infty)$ for some prime p, then $\mathrm{Ext}(B/F, T_0)$ contains a copy of the p-adic integers. If B/F is infinite, but contains no copy of $\mathbb{Z}(p^\infty)$, then B/F has an infinite socle, so $\mathrm{Ext}(B/F, T_0)$ maps upon an infinite product of cyclic groups of (not necessarily different) prime orders.

    The exact sequence $0 \to F \to B \to B/F \to 0$ induces the exact sequence $\mathrm{Hom}(F, T_0) \to \mathrm{Ext}(B/F, T_0) \to \mathrm{Ext}(B, T_0)$. Here $\mathrm{Hom}(F, T_0) = \oplus_n \mathrm{Hom}(\mathbb{Z}, T_0) = \oplus_n T_0$ is a countable group. Hence if $\mathrm{Ext}(B, T_0) = 0$, then $\mathrm{Ext}(B/F, T_0)$ cannot be uncountable. This implies that B/F must be finite, thus B is finitely generated, and hence free.
(d) *Countable Baer groups are free.* By Pontryagin's criterion, it suffices to show that their finite rank subgroups are free. These subgroups are Baer by (a), so the claim follows from (c).

Only Free Groups are Baer We can now verify the main result on Baer groups.

Theorem 2.1 (Griffith [4]). *A group is a Baer group if and only if it is free.*

Proof. Only the necessity requires a proof. So assume B is a Baer group, say, of cardinality κ. To start a transfinite induction on the cardinality, refer to (d) for the case $\kappa = \aleph_0$. Assume κ is an uncountable cardinal, and Baer groups of cardinalities $< \kappa$ are free. We distinguish two cases according as κ is a regular or a singular cardinal.

Case I: *κ is a regular cardinal.* Pick a filtration $\{B_\sigma\}_{\sigma<\kappa}$ of B with pure subgroups of cardinalities $< \kappa$. By (a), the B_σ are Baer groups. We now apply Lemma 4.4 in Chapter 9 with large Σ-cyclic groups A_ν to conclude that there is a cub C in κ such that for $\sigma \in C$, B_ρ/B_σ is a Baer group of cardinality $< \kappa$ for all $\rho > \sigma$ (cf. Lemma 2.4), and hence free by the induction hypothesis. Keeping only the B_σ with $\sigma \in C$, we obtain a subfiltration $\{B_\sigma\}_{\sigma\in C}$ where all the factor groups are free: this cannot be anything else than a filtration of a free group. Thus B is free.

Case II: *κ is a singular cardinal.* In view of (a) and the induction hypothesis, B is a κ-free group. The singular compactness theorem 9.2 in Chapter 3 guarantees that B is free. $\square$

We note in passing that a group G, satisfying $\mathrm{Ext}(G,T) = 0$ for all p-groups T and all primes p, need not be a Baer group. This is demonstrated by the following examples.

Example 2.2. A prototype example is the group R of rational numbers with square-free denominators. The exact sequence $0 \to \langle p^{-1}\mathbb{Z}\rangle \to R \to \oplus_{q\neq p}\mathbb{Z}(q) \to 0$ (summation over all primes $q \neq p$) induces the exact sequence $0 = \prod_q \mathrm{Ext}(\mathbb{Z}(q),T) \to \mathrm{Ext}(R,T) \to \mathrm{Ext}(\langle p^{-1}\mathbb{Z}\rangle,T) = 0$ for any p-group T, showing that $\mathrm{Ext}(R,T) = 0$ for every p-group T and for every prime p. However, R is not a Baer group.

Example 2.3. There is a non-free $\aleph_1$-free group A, for which $\mathrm{Ext}(A,T) = 0$ holds for all p-groups T, for every prime p. We use the construction described in Theorem 8.9 in Chapter 3 to get a group A of cardinality $\aleph_1$ as the union of a smooth chain $\{F_\sigma\ (\sigma < \omega_1)\}$ of countable free groups. If σ is a successor ordinal, then we put $F_{\sigma+1} = F_\sigma \oplus \mathbb{Z}$, and if σ is a limit ordinal, then we let $F_\sigma = \cup_{\rho<\sigma}F_\rho$ (which will be free by Theorem 7.5 in Chapter 3). Finally, if $\sigma = \rho + 1$ for a limit ordinal ρ, then define F_σ to be countable free and to fit into the exact sequence $0 \to F_\rho \to F_{\rho+1} \to R \to 0$ (free resolution of R) where R is the group of rational numbers with square-free denominators. Then $A = \cup_{\sigma<\omega_1}F_\sigma$ is $\aleph_1$-free, and Theorem 7.5 in Chapter 3 proves that it is not free. Using the fact that $\mathrm{Ext}(R,T) = 0$ for p-groups T (any p), Lemma 4.1 in Chapter 9 shows $\mathrm{Ext}(A,T) = 0$ holds for all p-groups T, for any prime p.

Test Groups for Baer Groups We now ask the question: *is there a 'test group' for Baer groups?* In other words, does there exist a single torsion group $\dot{T}$ such that $\mathrm{Ext}(B,\dot{T}) = 0$ implies that B is a Baer group? If we limit the size of B, then a direct sum of cyclic groups of adequate size is such a test group—as is shown by the next lemma, but there is no such group that works in ZFC for all groups B. However, the situation changes drastically in the constructible universe: in L, there exist test groups even of countable cardinality; see Theorem 2.5.

Lemma 2.4. *A group B of cardinality $\leq \kappa$ is a Baer group provided that* $\mathrm{Ext}(B,T_\kappa) = 0$ *holds for the torsion group*

$$T_\kappa = \oplus_\kappa[\oplus_p \oplus_{k<\omega} \mathbb{Z}(p^k)].$$

Proof. Assume B has cardinality $\leq \kappa$ and satisfies $\mathrm{Ext}(B,T_\kappa) = 0$. To prove B is a Baer group, we first show that $\mathrm{Ext}(B,S) = 0$ for all Σ-cyclic groups S. The factor set defining an extension of S by B has at most κ different elements, so it is contained in a summand of S of cardinality $\leq \kappa$. This summand is isomorphic to a summand of T_κ, hence hypothesis implies that such a factor set ought to be a transformation set.

Once we know that $\mathrm{Ext}(B,S) = 0$ for all Σ-cyclic torsion groups S, then we consider a pure-projective resolution $0 \to U \to S \to T \to 0$ of an arbitrary torsion T; here S is torsion and Σ-cyclic. The exactness of $\mathrm{Ext}(B,S) \to \mathrm{Ext}(B,T) \to 0$ implies that B is a Baer group. $\square$

Next we intend to show that in the constructible universe, there do exist test groups for the Baer property.

Theorem 2.5 ((V = L) (Eklof [4])). *Let $\dot{T}$ denote the direct sum of countably many copies of the group $\oplus_p \oplus_{k<\omega} \mathbb{Z}(p^k)$. A group B is a Baer group if and only if* $\mathrm{Ext}(B,\dot{T}) = 0$.

Proof. Necessity being obvious, assume that B satisfies $\mathrm{Ext}(B, \dot{T}) = 0$. We induct on the cardinality κ of the group B.

Case I: *κ is a regular cardinal.* If $\kappa = \aleph_0$, the claim follows from Lemma 2.4. So suppose κ is uncountable, and B satisfies $\mathrm{Ext}(B, \dot{T}) = 0$. Then $\mathrm{Ext}(B', \dot{T}) = 0$ for all subgroups $B' \leq B$, so by induction, all subgroups B' of cardinalities $< \kappa$ are Baer, thus there is a smooth chain $0 = B_0 < B_1 < \cdots < B_\sigma < \ldots$ of Baer groups with union B such that $|B_\sigma| < \kappa$ for all $\sigma < \kappa$. Then $\mathrm{Ext}(B_\sigma, \dot{T}) = 0$ for all σ, so—in view of the hypothesis V = L—we can appeal to Lemma 4.3 in Chapter 9 to argue that there is a cub $C \subset \kappa$ such that $\mathrm{Ext}(B_\sigma/B_\rho, \dot{T}) = 0$ for all $\rho < \sigma$ in C. By induction hypothesis, then $\mathrm{Ext}(B_\sigma/B_\rho, T) = 0$ holds for all torsion T. It remains to refer to Lemma 4.3 in Chapter 9 to derive that also $\mathrm{Ext}(B, T) = 0$. Thus B is a Baer group.

Case II: *κ is a singular cardinal.* We argue as in the proof of Theorem 2.1. □

Eklof [4] has shown that Theorem 2.5 is false if the hypothesis V = L is dropped: it is undecidable in ZFC that a group B satisfying $\mathrm{Ext}(B, \dot{T}) = 0$ is a Baer group.

★ **Notes.** As mentioned above, the problem on Baer groups was open for 30 years; during this time, several partial results were published. Finally, Griffith [4] solved the problem using an excellent idea. First he constructed, for every infinite cardinal κ, a mixed group G_κ such that all torsion-free subgroups of G_κ were free, but G_κ modulo its torsion part T_κ (which was like the one in Lemma 2.4 above) was divisible of cardinality κ. Then he considered the exact sequence $0 \to T_\kappa \to G_\kappa \to G_\kappa/T_\kappa \to 0$ which induced the exact sequence

$$\mathrm{Hom}(B, G_\kappa) \to \mathrm{Hom}(B, G_\kappa/T_\kappa) \to \mathrm{Ext}(B, T_\kappa) = 0$$

for a Baer group B. He argued that since the map between the Homs is epic, if B embeds in G_κ/T_κ, then it also embeds in G_κ. Consequently, it must be free.

A more difficult problem is to describe the pairs (F, T) of torsion-free F and torsion T that satisfy $\mathrm{Ext}(F, T) = 0$. So far the best approach is *via* **Baer cotorsion-pairs** $(\mathcal{F}, \mathcal{T})$. They are defined to consist of a class $\mathcal{F}$ of torsion-free and a class $\mathcal{T}$ of torsion groups such that $\mathrm{Ext}(F, T) = 0$ for all $F \in \mathcal{F}, T \in \mathcal{T}$, and these classes are saturated. Assuming V = L, Strüngmann [2] classified these pairs in terms of sets P of primes with $\oplus_{p \in P} \mathbb{Z}(p) \in \mathcal{T}$ along with sets Q of primes p with $\oplus_{n<\omega} \mathbb{Z}(p^n) \in \mathcal{T}$. In another paper [1], he considered the class $\mathcal{C}_G$ of all torsion groups T which satisfy $\mathrm{Ext}(G, T) = 0$ for a torsion-free group G. He showed that if G is countable, then $\mathcal{C}_G = \mathcal{C}_C$ for a suitable countable completely decomposable group C. Again assuming V = L, he also proved the existence of λ-**universal groups** G for every torsion group T that has but finitely many bounded p-components $\neq 0$. Here λ-universal for a T means that $|G| = \lambda$, $\mathrm{Ext}(G, T) = 0$, and every group H of cardinality $\leq \lambda$ that satisfies $\mathrm{Ext}(H, T) = 0$ embeds in G.

Under the hypothesis V = L, Eklof [4] shows that, for a torsion group T and for a torsion-free group G of singular cardinality λ, $\mathrm{Ext}(G, T) = 0$ holds if and only if $\mathrm{Ext}(C, T) = 0$ for all subgroups $C < G$ of cardinality $< \lambda$.

Kaplansky raised the question of Baer modules B over integral domains R: *which R-modules B satisfy* $\mathrm{Ext}^1_R(B, T) = 0$ *for all torsion modules T*? It took much more time than for abelian groups to get the complete solution. After sporadic publications by various authors dealing with special cases, Eklof–Fuchs–Shelah [Trans. Amer. Math. Soc. **322**, 547–560 (1990)] reduced the problem to countably generated modules B, and finally, Angeleri-Hügel–Bazzoni–Herbera [ibid. **360**, 2409–2421 (2008)] succeeded in proving that countable Baer modules are projective. Thus all Baer modules over any integral domain are projective.

One does not need a sharp eye to notice the difference when we ask for groups that force Ext to vanish for a class of groups (like in the Baer problem) or just for a single group (Whitehead problem). The Baer problem is an excellent example to demonstrate the difference when we admit all torsion groups or just one of fixed (countable) cardinality. The common thread in both cases is that advanced set-theoretical methods are required, but the latter case is more prone to undecidability.

Exercises

(1) (Sąsiada) Let B be a basic subgroup of a torsion group T, and G a torsion-free group.

   (a) Show that $\mathrm{Ext}(G,T) = 0$ if and only if $\mathrm{Ext}(G,B) = 0$.
   (b) Conclude that in studying pairs T, G for which $\mathrm{Ext}(G,T) = 0$, it suffices to consider for T only Σ-cyclic torsion groups.

(2) Let A be a mixed group with torsion part T such that A/T is a separable torsion-free group, not completely decomposable. A need not split even if all of its subgroups split that contain T and have finite torsion-free rank.
(3) (Griffith) If G satisfies $\mathrm{Ext}(G,S) = 0$ for all p-groups S, then $\mathrm{Ext}(G,T)$ is torsion-free for all torsion groups T. [Hint: if $T_p = 0$, then multiplication by p is an automorphism on T, and hence on Ext.]
(4) (Griffith) Let G be a torsion-free group such that every extension of every torsion group by G is quasi-splitting. Then G is free.
(5) (Strüngmann) Let G be torsion-free of finite rank. There is a rational group R such that, for all torsion groups T, $\mathrm{Ext}(G,T) = 0$ is equivalent to $\mathrm{Ext}(R,T) = 0$. [Hint: outer type.]
(6) (Baer) Let T and G be torsion and torsion-free groups, respectively, such that $\mathrm{Ext}(G,T) = 0$. Then:

   (a) Suppose $p_1, \dots, p_n, \dots$ is an infinite sequence of different primes satisfying $p_nT < T$ for all n. G cannot contain any pure finite rank subgroup S such that G/S has elements $\neq 0$ divisible by all the p_n.
   (b) If for some prime p, the p-basic subgroup of T is unbounded, then G contains no pure subgroup S of finite rank such that G/S has elements $\neq 0$ of infinite p-height.

(7) (Baer) If G in the preceding exercise is countable, then conditions (a) and (b) are necessary and sufficient for $\mathrm{Ext}(G,T) = 0$ for any torsion T. [Hint: write G as the union of a chain of finite rank pure subgroups, and use induction.]
(8) (Strüngmann) Let G be countable and torsion-free. $\mathrm{Ext}(G,T) = 0$ for some torsion T if and only if $\mathrm{Ext}(H,T) = 0$ for all finite rank subgroups H of G. [Hint: Lemma 4.1 in Chapter 9.]
(9) (Pext version of Baer's problem) A group G satisfying $\mathrm{Pext}(G,T) = 0$ for all torsion groups T is a Σ-cyclic group. [Hint: pure-projective resolution.]

(10) (Megibben) Call a mixed group **separable** if every finite set of elements is contained in a direct sum of a group with minimum condition and a completely decomposable torsion-free group.

  (a) A is separable if and only if so are tA, A/tA, and tA is balanced in A.
  (b) A separable mixed group need not be splitting. [Hint: extend a torsion by an $\aleph_1$-free.]

(11) (Megibben) A torsion-free group A satisfies $\mathrm{Bext}(A, T) = \mathrm{Ext}(A, T)$ for all torsion groups T if and only if it is homogeneous of type $(0, \dots, 0, \dots)$.

3 Valuated Groups. Height-Matrices

The fundamental properties of heights are formalized in the concept which we are going to study here: valuation. Valuated groups, as we shall see soon, are indispensable tools in the theory of mixed groups. Their theory was developed in the fourth quarter of the last century.

Ordinal Valuation Consider the class Γ of ordinals with the symbol ∞ adjoined as the maximum element, and let p be an arbitrary, but fixed prime. Let v_p be a function on the group A to Γ subject to the following conditions (we agree that $\sigma < \infty$ for all $\sigma \in \Gamma$):

(i) $v_p(a) \in \Gamma$ for all $a \in A$; $v_p(0) = \infty$;
(ii) $v_p(a + b) \geq \min\{v_p(a), v_p(b)\}$ for all $a, b \in A$;
(iii) $v_p(pa) > v_p(a)$ for all $a \in A \setminus \{0\}$;
(iv) $v_p(na) = v_p(a)$ for all $a \in A$ and all integers n prime to p.

In this case, v_p will be called a ***p*-valuation** of A. A ***p*-valuated group** is a group together with a p-valuation v_p.

A subgroup C of a p-valuated group A carries the inherited valuation. The factor group A/C is equipped with the supremum valuation, i.e. to the coset $a+C$ we assign the value $\sup_{c \in C} v_p(a + c)$. In this way, the canonical homomorphism $\phi\colon A \to A/C$ becomes a valuated morphism in the sense explained below. (Actually, the assigned values in A/C are the smallest for which ϕ is a genuine morphism.)

Example 3.1. An immediate example for a p-valuation is the p-height function $h_p(a)$ on a group A. From the definition of $h_p(a)$ it is immediate that $v_p(a) \geq h_p(a)$ holds for any p-valuation v_p of $a \in A$.

Example 3.2. A similar example is obtained if A is embedded as a subgroup in a larger group G, and for $a \in A$, $v_p(a)$ is defined as the p-height $h_p(a)$, this time computed in G. (We shall see below in Theorem 3.6 that actually every p-valuation arises in this way by a suitable embedding.)

Example 3.3. Let $\sigma_0 < \sigma_1 < \dots < \sigma_n < \dots$ be any strictly increasing sequence of ordinals. We define a p-valuation on an infinite cyclic group $\langle g \rangle$ by setting $v_p(mp^n g) = \sigma_n$ for each $n \in \mathbb{N}$ and for every integer m prime to p. $\langle g \rangle$ is freely p-valuated (see below) if σ_0 is arbitrarily chosen and $\sigma_n = \sigma_0 + n$ for each $n < \omega$ (thus the values are totally determined once $v_p(g)$ is fixed for the generator g).

Example 3.4. If A_i $(i \in I)$ are p-valuated groups, then their direct sum (i.e., coproduct) $\bigoplus A_i$ and their direct product $\prod A_i$ become p-valuated groups if we set $v_p(a) = \inf\{v_p(a_i) \mid i \in \text{ supp } a\}$ where $a = (\ldots, a_i, \ldots)$ $(a_i \in A_i)$. (There is no danger of confusion by denoting the p-valuation in all occurring groups by the same symbol v_p.)

Category $\mathcal{V}_p$ For p-valuated groups A and B, a **morphism** $\phi\colon A \to B$ is a group homomorphism satisfying

$$v_p(\phi(a)) \geq v_p(a) \qquad \text{for all } a \in A.$$

An **isometry** $\psi : A \to B$ between valuated groups is a group isomorphism that preserves p-values (i.e. $v_p(\psi(a)) = v_p(a)$ holds for all $a \in A$), and an **embedding** is an injective group homomorphism that preserves p-values.

For the elements a of a p-valuated group whose orders are finite and prime to p, we have necessarily $v_p(a) = \infty$ (this follows from comparing conditions (iii) and (iv) in the definition). Consequently, no p-valuation can give any useful information about such elements. Therefore, it is reasonable to disregard them, and focus our attention on p-local groups, i.e., on $\mathbb{Z}_{(p)}$-modules. Thus we will be dealing with the **category** $\mathcal{V}_p$ of p-local p-valuated groups with the indicated morphisms.

By a **free p-valuated group** F is meant a free group $F = \oplus\langle a_i\rangle$ where each infinite cyclic group $\langle a_i\rangle$ carries a free p-valuation; this means—we repeat—that $v_p(a_i) = \sigma_i$ can be arbitrary for each $i \in I$, and then $v_p(p^n a_i) = \sigma_i + n$ for all $n < \omega$. The valuation on F is defined by taking infima of the values of the coordinates (like in Example 3.4 above). In other words, F is the direct sum of freely p-valuated infinite cyclic groups.

Lemma 3.5. *For every p-valuated p-local group A there is a free p-valuated group F such that A is isometric to the p-valuated quotient F/H for some subgroup H of F.*

Proof. For every $0 \neq a \in A$, choose a freely p-valuated infinite cyclic group $\langle x_a\rangle$ such that $v_p(x_a) = v_p(a)$, and let F be the p-valuated direct sum of all the $\langle x_a\rangle$. Then the surjective mapping $\phi\colon F \to A$ (induced by $x_a \mapsto a$) is easily seen not to increase values. Therefore, the induced map $F/\operatorname{Ker}\phi \to A$ must be an isometry. ☐

Note that $\operatorname{Ker}\phi$ in the last proof is a p-nice subgroup of F in the sense to be defined in the next section; i.e. each coset of F mod $\operatorname{Ker}\phi$ contains an element that has the same value as the coset.

Theorem 3.6 (Richman–Walker [5]). *Let $\mathcal{V}_p$ be the category defined above. There exist a functor T: $\mathcal{V}_p \to \mathcal{V}_p$ and a natural transformation $\mathbf{1} \rightsquigarrow T$ such that, for each group $A \in \mathcal{V}_p$ with p-valuation v_p,*

(a) $T(A)$ *has the p-height valuation h_p;*
(b) *A is a p-nice subgroup of $T(A)$, and $h_p(a) = v_p(a)$ for $a \in A$;*
(c) $T(A)/A$ *is a totally projective p-group.*

Proof. For every $0 \neq a \in A$, consider the Nunke group $N^a_\sigma(p)$ where $\sigma = v_p(a)$, and let $x_a \in N^a_\sigma(p)$ denote its generator of infinite order (of height σ). As $\langle x_a\rangle$ is p-nice in $N^a_\sigma(p)$, $F = \oplus_{0\neq a\in A}\langle x_a\rangle$ will be a p-nice subgroup of $G = \oplus_{0\neq a\in A} N^a_\sigma(p)$.

We denote by $\psi : F \to A$ the map induced by $x_a \mapsto a$, and set $H = \operatorname{Ker}\psi$. Define $T(A) = G/H$, so that $A = F/H$ is a subgroup in $T(A)$. If $T(A)$ is furnished with the height valuation, then—as is straightforward to check—the restriction to A will be identical with v_p. The p-niceness of A in $T(A)$ follows from that of F in G. The functorial behavior of the process is evident from the definition. (c) is an obvious consequence of the definition of Nunke groups. □

Global Valuation Turning to the global case, we introduce a valuation where the values are no longer ordinals and symbols ∞, but countable sequences $\chi = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots)$ of ordinals and ∞ (like characteristics for torsion-free groups). Suppose A has p-valuation v_p for each prime p. For $a \in A$, we set

$$\chi(a) = (v_2(a), v_3(a), \dots, v_{p_n}(a), \dots)$$

where p_n stands for the nth prime. The point-wise ordering makes the collection of these sequences a lattice-ordered class with $(0, 0, \dots, 0, \dots)$ as minimum and with $(\infty, \infty, \dots, \infty, \dots)$ as maximum element. This 'global valuation' χ (which we might call 'characteristic') satisfies conditions (i)–(iv) listed above for p-valuations.

The characteristics χ (as above) and $\chi' = (\sigma'_1, \sigma'_2, \dots, \sigma'_n, \dots)$ will be regarded **equivalent**, in notation: $\chi \sim \chi'$, if $\sigma_n = \sigma'_n$ for almost all n. For instance, $\chi(nx) \sim \chi(x)$ holds for all $n \in \mathbb{N}$. We will write $[\chi]$ for the equivalence class of the characteristic χ. The collection of these $[\chi]$ carries a lattice-order.

Let $C = \langle x\rangle$ be an infinite cyclic group that is furnished with a free global valuation, given by an arbitrarily chosen characteristic $\chi(x) = (\sigma_2, \sigma_3, \dots, \sigma_p, \dots)$. The cyclic groups $C = \langle x\rangle$ and $C' = \langle x'\rangle$ are of the **same type** if the characteristics $\chi(x)$ and $\chi(x')$ are equivalent.

Height-Matrices In mixed groups A, we have to consider the overall divisibility; this can be accomplished by listing the indicators of an element for all primes. Let $p_1, \dots, p_n, \dots$ denote the sequence of the prime numbers in order of magnitude. With an element $a \in A$ we associate the $\omega \times \omega$ **height-matrix**

$$\mathbb{H}(a) = \begin{pmatrix} h_{p_1}(a) & h_{p_1}(p_1 a) & \dots & h_{p_1}(p_1^k a) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ h_{p_n}(a) & h_{p_n}(p_n a) & \dots & h_{p_n}(p_n^k a) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \underline{\mathrm{u}}_{p_1}(a) \\ \dots \\ \underline{\mathrm{u}}_{p_n}(a) \\ \dots \end{pmatrix} = \|\sigma_{nk}\|;$$

(see Rotman [1], Megibben [3], Myshkin [1]) where the first column is $\chi(a)$, and the nth row represents the p_n-indicator of a. Keep in mind that the entry $\sigma_{nk} \in \mathbb{H}(a)$ in the (n, k)-position records the p_n-height of $p_n^k a$, for all $n \in \mathbb{N}$ and $k < \omega$.

The class of height-matrices is lattice-ordered in the obvious way:

$$\mathbb{H} = \|\sigma_{nk}\| \leq \|\sigma'_{nk}\| = \mathbb{H}'$$

means that $\sigma_{nk} \leq \sigma'_{nk}$ for all $n, k < \omega$. Thus the lattice operations $\wedge, \vee$ between height-matrices are the point-wise minima and maxima.

The following observations are evident from the definition.

(a) $\mathbb{H}(-a) = \mathbb{H}(a)$ for all $a \in A$.
(b) $\mathbb{H}(p_n a)$ is obtained from $\mathbb{H}(a)$ by deleting the first entry in the nth row, and shifting the rest of the row to the left.
(c) If a is of finite order, then almost all entries in $\mathbb{H}(a)$ are ∞.
(d) Every entry in $\mathbb{H}(a)$ is ∞ if and only if a is contained in the divisible subgroup of A.
(e) $p_1^{k_1} \cdots p_n^{k_n} a$ is divisible by $m = p_1^{\ell_1} \cdots p_n^{\ell_n}$ exactly if $\ell_i \leq \sigma_{ik_i}$ for $i = 1, \ldots, n$.
(f) If $A = B \oplus C$ and $a = b + c$ ($b \in B, c \in C$), then $\mathbb{H}(a) = \mathbb{H}(b) \wedge \mathbb{H}(c)$.

Example 3.7. Let $B = \oplus_{n=1}^{\infty} \langle b_n \rangle$ with $o(b_n) = p^{2n-1}$, and $A = \tilde{B}$, the p-adic completion of B. Let $j_0 < \cdots < j_k < \ldots$ denote a strictly increasing sequence of non-negative integers. Our claim is that A contains an element a with $h_p(p^k a) = j_k$ for every $k < \omega$ if and only if a gap $j_k + 1 < j_{k+1}$ implies that j_k is even. To verify, we argue as in Lemma 1.1 in Chapter 10.

Equivalence of Height-Matrices If the elements a, b of infinite order in the mixed group A satisfy $ra = sb$ for some integers $r, s \neq 0$, then from (b) it is easily seen that the nth rows of $\mathbb{H}(a) = \|\sigma_{nk}\|$ and $\mathbb{H}(b) = \|\rho_{nk}\|$ can differ only if $p_n | rs$. If this is the case for a prime p_n, then there must exist integers $i, j \geq 0$ such that

$$\sigma_{n,i+k} = \rho_{n,j+k} \qquad \text{for all } k < \omega. \tag{15.4}$$

In view of this, we define two $\omega \times \omega$ matrices $\mathbb{M} = \|\sigma_{nk}\|$ and $\mathbb{N} = \|\rho_{nk}\|$ **equivalent** (in notation: $\mathbb{M} \sim \mathbb{N}$) if, for almost all n, the nth rows are identical, and for each distinct row there exist integers $i, j \geq 0$ (depending on the row) such that (15.4) holds. Equivalently, if $p\mathbb{H}$ for a prime p and a height-matrix $\mathbb{H}$ is defined as a matrix obtained from $\mathbb{H}$ by shifting the p-row one position to the left, and $n\mathbb{H}$ for $n \in \mathbb{N}$ by induction, then $\mathbb{M} \sim \mathbb{N}$ means that there are $m, n \in \mathbb{N}$ satisfying $m\mathbb{M} = n\mathbb{N}$.

We note that the height-matrices of elements in the same coset modulo the torsion subgroup are always equivalent.

Characterization of Height-Matrices One of the fundamental questions relating height-matrices is concerned with conditions under which a matrix of ordinals and symbols ∞ can be realized as a height-matrix in some mixed group. The answer is not difficult, but it is not easy either. (Recall that we have agreed that $\infty < \infty$.)

Theorem 3.8 (Megibben [3], Rotman [1]). *An $\omega \times \omega$ matrix $\mathbb{M} = \|\sigma_{nk}\|$ consisting of ordinals and symbols ∞ is the height-matrix of an element in some mixed group A if and only if, for every n, it satisfies*

$$\sigma_{n0} < \sigma_{n1} < \cdots < \sigma_{nk} < \ldots .$$

Proof. Necessity being evident, we turn to the proof of sufficiency. Given the matrix $\mathbb{M} = \|\sigma_{nk}\|$ satisfying the stated condition, the strategy is to build a mixed group of torsion-free rank 1 by equipping the elements of an infinite cyclic group with the correct p-heights, for every prime p. This is first done separately for each prime.

Thus for each prime p_n and integer $k \geq 0$, we select an infinite cyclic group, and call a_{nk} its generator. We build a group A_{nk} around $\langle a_{nk} \rangle$. If $\sigma_{nk} = \infty$, then let A_{nk} be the rational group with $A_{nk}/\langle a_{nk} \rangle \cong \mathbb{Z}(p_n^\infty)$. If σ_{nk} is an ordinal, then let $A_{nk} \cong N_{\sigma_{nk}}(p_n)$ be the Nunke group (Sect. 1), with a_{nk} as the generator of height σ_{nk} in its infinite cyclic subgroup. Choose an extra infinite cyclic group $\langle a \rangle$, and define A by the push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{n,k}\langle a_{n,k}\rangle \arrow[r] \arrow[d, "\delta"'] & \oplus_{n,k} A_{n,k} \arrow[r] \arrow[d] & \oplus_{n,k} A_{n,k}/\langle a_{n,k}\rangle \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \langle a \rangle \arrow[r, "\alpha"] & A \arrow[r] & \oplus_{n,k} A_{n,k}/\langle a_{n,k}\rangle \arrow[r] & 0,
\end{tikzcd}$$

where δ is defined *via* $\delta : a_{n,k} \mapsto p_n^k a$ for all $n \in \mathbb{N}, k < \omega$. It is rather obvious that the height-matrix of $\alpha(a) \in A$ is $\geq \mathbb{M}$, and it is easy to see that equality must hold. For instance, if we factor out $\langle p_n \alpha(a) \rangle$ from A, then we get a huge direct sum of generalized Prüfer groups $H_\rho(p_n)$'s or $\mathbb{Z}(p_n^\infty)$, and $\alpha(a)$ lies in a single summand, not effected by the other summands, and therefore it will carry the correct p_n-height. □

With every (mixed) group A and every height-matrix $\mathbb{H}$ we associate two fully invariant subgroups; they are defined as

$$A(\mathbb{H}) = \{a \in A \mid \mathbb{H}(a) \geq \mathbb{H}\} \quad \text{and} \quad A^*(\mathbb{H}) = \langle a \in A \mid \mathbb{H}(a) > \mathbb{H} \rangle.$$

Obviously, $A^*(\mathbb{H}) \leq A(\mathbb{H})$, and $A(\mathbb{H}) \leq A(\mathbb{H}')$ whenever $\mathbb{H} \geq \mathbb{H}'$. In general, there exist also other fully invariant subgroups in A.

Example 3.9. Let A be a non-splitting mixed group such that all of the p-components of tA are unbounded. Then the minimal height-matrix $\mathbb{H}$ for which $tA \leq A(\mathbb{H})$ holds is $\mathbb{H} = \|\sigma_{nk}\|$ where $\sigma_{nk} = k$ for all n. For this height-matrix, we have $A(\mathbb{H}) = A$.

★ **Notes.** By now it should be clear that besides the order, the height is the most valuable information about a group element, and valuation is just a natural abstract version of height. The theory of valuation of groups developed slowly. Rotman [2] observed that a finite rank torsion-free group is determined by an essential free subgroup furnished with the height-valuation. Richman–Walker [5] is the pioneering work on valuations. Undoubtedly, much of the credit for a systematic theory of valuated groups goes to the New Mexico group, above all to Hunter, Richman, and E. Walker.

A few more words about $\mathcal{V}_p$. It is a pre-abelian category, i.e. it is additive, and morphisms have kernels and cokernels. An exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ makes perfectly good sense if it means that α is an embedding, and β induces an isometry $B/A \cong C$ (thus A is like a balanced subgroup). There are enough projectives (the free valuated groups) and injectives (algebraically compact groups with height-valuation). Interested reader should consult Richman–Walker [5] for further information.

Theorem 3.6 generalizes the 'crude existence theorem' by Rotman–Yen [1] in which A was an infinite cyclic group.

Valuated groups might be the source of easily tractable invariants that would help in the future to obtain reasonable classification of groups in some classes. We are spoiled by the tremendous success of numerical and cardinal invariants in the traditional structure theorems, and perhaps do not try hard enough to obtain other kinds of more complicated, but still simply recognizable invariants. We have to learn more about valuation, and look for invariants in the category $\mathcal{V}_p$ of p-valuated p-local groups.

It might, perhaps, be appropriate to mention another kind of valuation used in the literature: the 'coset valuation.' Let C be a subgroup of A. The p-height of the coset $a + C$ in A/C is calculated ordinarily as $\sup_{c \in C} h_p(a + c)$ (or ∞). In some situations, it is desirable to distinguish according as this supremum is equal to one of $h_p(a + c)$ (i.e., the coset contains an element proper with respect to C) or not. This can be done, e.g. by defining the coset valuation as follows: $k_p(a + C) = \sup_{c \in C} (h_p(a + c) + 1)$.

Grinshpon–Krylov [1] published an extensive study of transitivity and full transitivity in mixed groups. A is **fully transitive** if for all $x, y \in A$ with $\mathbb{H}(x) \leq \mathbb{H}(y)$, there is an $\eta \in \operatorname{End} A$ such that $\eta(x) = y$. *Inter alia*, the full transitivity of direct sums is investigated. See also Misyakov [1] and Grinshpon–Misyakov [1].

Let us mention here some literature on universal embeddings. A group U is (*purely*) *universal for a set S* of groups, if $U \in S$ and every group in S embeds as a (pure) subgroup in U. Kojman–Shelah [1] consider various classes of interest; their main set-theoretic tools are club guessing sequences.

Exercises

(1) Let A be p-local, and v_p a p-valuation in A. For $\sigma \in \Gamma$, define $A(\sigma) = \{a \in A \mid v_p(a) \geq \sigma\}$. Show that the collection of the $A(\sigma)$ determine v_p.
(2) Let A be a p-local mixed group. The indicators of elements in A satisfy the gap condition (see Sect. 1 in Chapter 10).
(3) Prove the inequality for height-matrices: $\mathbb{H}(a + b) \geq \mathbb{H}(a) \wedge \mathbb{H}(b)$ for all $a, b \in A$.
(4) Define $A = \langle a_0, a_1, \dots, a_n, \dots \rangle$ subject to the relations $p^n a_0 = p^{k_n} a_n$ for $n \geq 1$, where $0 < 1 < k_1 < \cdots < k_n < \dots$ are integers. Describe the height-matrices $\mathbb{H}(a_0)$ and $\mathbb{H}(a_n)$.
(5) $a \in A$ belongs to the σth Ulm subgroup A^σ of A if and only if $\mathbb{H}(a) = \|\sigma_{nk}\|$ satisfies $\sigma_{n0} \geq \omega\sigma$ for all n.
(6) Examine what can happen to a height-matrix when passing (a) to a subgroup; (b) to a factor group.
(7) (Megibben) Let A be a countable mixed group of torsion-free rank 1, $a \in A$ of order ∞. The following conditions on $\mathbb{H}(a)$ are necessary and sufficient for A to split:
    (a) almost every row is free of gaps;
    (b) no row has infinitely many gaps;
    (c) if a row contains entries $\geq \omega$, then it also contains ∞.
(8) Find a necessary and sufficient condition on the height-matrix to belong to a quasi-splitting mixed group of torsion-free rank 1.

4 Nice, Isotype, and Balanced Subgroups

Before embarking on the discussion of a variety of mixed groups, now away from splitting questions, we should agree to keep without modification those concepts and notations used for torsion and torsion-free groups that make sense for the mixed case. Accordingly, e.g. Ulm subgroups, UK-invariants (separately for each prime) as well as Hill invariants will be used without explanation.

However, we have to adapt to mixed groups certain concepts developed for torsion groups. We take a similar approach for the extended concepts, and since some proofs are almost, if not totally, identical to the torsion case, we will just wave hands, and refer to the appropriate places for verification.

Nice Subgroups As before, for a prime p, by a **p-nice subgroup** of A is meant a subgroup N such that

$$(p^\sigma A + N)/N = p^\sigma(A/N) \qquad \text{for all ordinals } \sigma. \tag{15.5}$$

The p-niceness of N is equivalent to the representability of every coset of N in A by an element of the p-height of the coset. Such a representative a is called **p-proper with respect to** N. It satisfies

$$h_p(a) \geq h_p(a + x) \qquad \text{for all } x \in N.$$

A subgroup N of A will be called **nice in** A if, for every prime p, the localization $N_{(p)} = N \otimes \mathbb{Z}_{(p)}$ is p-nice in $A_{(p)} = A \otimes \mathbb{Z}_{(p)}$.

Next we list some relevant properties of niceness (p-niceness).

(a) *Finite subgroups are always nice.*
(b) *Let N_i be a subgroup of the group A_i, for $i \in I$. The subgroup $\oplus_{i \in I} N_i$ is nice (p-nice) in $\oplus_{i \in I} A_i$ if and only if N_i is nice (p-nice) in A_i for all $i \in I$.*
(c) *If N is contained in a subgroup B of A, then N need not be nice in B even if N is nice in A.*
(d) *Niceness is not transitive in general.* See Sect. 2(E) in Chapter 11.

Lemma 4.1. *Let $N \leq M \leq A$ where N is nice in A. Then M is nice in A if and only if M/N is nice in A/N.*

Proof. We refer to the proof of Lemma 2.4 in Chapter 11. □

The next lemma was known for a while for J_p-modules, valid for finitely generated G. For groups, only a weaker version can be established.

Lemma 4.2 (Rotman [1], Megibben [3], Wallace [1]). *Let A be a mixed group of torsion-free rank 1, and $G = \langle g \rangle$ an infinite cyclic subgroup. If A/G is a p-group, then G is p-nice in A.*

Proof. We prove that cosets contain elements proper with respect to G. Let $a \in A \setminus G$ satisfy $p^t a = p^s rg$ for some $t \geq 1, s \geq 0$, and $\gcd\{r, p\} = 1$. Suppose there is a strictly increasing sequence $h_p(a + n_i g)$ $(i < \omega)$ of ordinals, where $n_i \in \mathbb{Z}$ and $n_0 = 0$. By the triangle inequality, $h_p(n_i g) = h_p(a)$, so there is $k \in \mathbb{N}$ such that $n_i = p^k m_i$ with $\gcd(m_i, p) = 1$ for all i. If $s \neq k + t$, then

$$h_p(a + n_i g) < h_p(p^t a + p^{k+t} m_i g) = h_p(p^s rg + p^{k+t} m_i g) = h_p(p^j g)$$

where $j = \min\{s, k + t\}$. Since $h_p((n_{i+1} - n_i)g) = h_p(a + n_i g)$ is increasing with i, it is clear that the coefficients $n_{i+1} - n_i$ are divisible by increasing powers of p, so it is impossible to have $h_p(a + n_i g) < h_p(p^j g)$ for all i. Consequently, there is a maximal height in the set $\{h_p(a + n_i g) \mid i < \omega\}$, and the coset $a + \langle g \rangle$ contains a proper element. If $s = k + t$, then we replace a by $a - p^k rg$ in our argument, so $p^t a = 0$ may be assumed. In this case, $h_p(a + n_i g) < h_p(p^{k+t} m_i g) = h_p(p^{k+t} g)$ leads to a similar contradiction. $\square$

Isotype Subgroups A straightforward generalization of isotypeness in p-groups is the following concept for mixed groups. A subgroup G of the (mixed) group A is called p**-isotype** if

$$p^\sigma G = G \cap p^\sigma A \quad \text{for all ordinals } \sigma. \tag{15.6}$$

G is **isotype** if it is p-isotype for each prime p.

Isotype subgroups have properties (a)–(d) listed in Sect. 5 in Chapter 11.

Example 4.3 (Megibben [3]). A subgroup G of A with torsion-free factor group A/G is isotype in A. In view of the definition of $p^\sigma A$ for limit ordinals σ, for a transfinite induction it will suffice to prove that if (15.6) holds for an ordinal σ, then it also holds for $\sigma + 1$. So pick $g \in G \cap p^{\sigma+1} A$, and let $a \in p^\sigma A$ satisfy $pa = g$. By the torsion-freeness of A/G, we have $a \in G$, thus $g \in p^{\sigma+1} G$.

The following lemma is readily verified from the definitions.

Lemma 4.4. *If C is p-isotype in A, then*

$$C(\underline{u}) = C \cap A(\underline{u}) \qquad \text{for all } p\text{-indicators } \underline{u},$$

and if C is isotype in A, then

$$C(\mathbb{H}) = C \cap A(\mathbb{H}) \qquad \text{for all height-matrices } \mathbb{H}.$$

$\square$

Balanced Subgroups A short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is called **balanced-exact** if the induced sequence

$$0 \to A(\chi) \xrightarrow{\alpha} B(\chi) \xrightarrow{\beta} C(\chi) \to 0 \tag{15.7}$$

is exact for every characteristic χ; we recall: $A(\chi) = \{a \in A \mid \chi(a) \geq \chi\}$. (15.7) is called $\mathbb{H}$**-exact** if, for every height-matrix $\mathbb{H}$, the induced sequence

$$0 \to A(\mathbb{H}) \to B(\mathbb{H}) \xrightarrow{\beta} C(\mathbb{H}) \to 0 \tag{15.8}$$

is exact. $\mathbb{H}$-exact sequences are evidently balanced-exact. However, the converse is false (see *infra*), though they are equivalent for torsion and torsion-free groups. In fact, for p-groups this follows from Proposition 5.5 in Chapter 11, while for torsion-free groups, we observe that height-matrices are determined by their first columns, i.e. by the characteristics.

A useful criterion for balanced-exactness is given in the next lemma.

Lemma 4.5. *The sequence* $0 \to A \to B \xrightarrow{\beta} C \to 0$ *is balanced-exact if and only if*

(i) $\beta(B(\chi)) = C(\chi)$ *for every characteristic* χ*, and*
(ii) $\beta(p^\sigma B[p]) = p^\sigma C[p]$ *for every prime* p *and ordinal* σ*.*

Proof. Condition (i) asserts that (15.7) is exact at $C(\chi)$. The rest of the proof is the same as in Proposition 5.5 in Chapter 11. □

An entirely analogous proof applies to verify:

Lemma 4.6. *The sequence* $0 \to A \to B \xrightarrow{\beta} C \to 0$ *is* $\mathbb{H}$*-exact if and only if*

(i) $\beta(B(\mathbb{H})) = C(\mathbb{H})$ *for every height-matrix* $\mathbb{H}$*, and*
(ii) $\beta(p^\sigma B[p]) = p^\sigma C[p]$ *for every prime* p *and every ordinal* σ*.* □

★ **Notes.** This section contains three fundamental concepts needed in the study of mixed groups. Our discussion above shows the overwhelming influence of p-groups on these concepts.

A brief comment on the definition of niceness. Naturally, one is tempted to define global niceness as p-niceness for every prime p. The bad news is that under this definition, e.g. $p\mathbb{Z}$ would *not* be nice in $\mathbb{Z}$ (the elements of $\mathbb{Z}/p\mathbb{Z}$ have infinite q-heights for every prime $q \neq p$). The good news is that this can easily be corrected: localizations provide a useful concept (so that $p\mathbb{Z}$ is nice in $\mathbb{Z}$).

For mixed groups over a complete discrete valuation domain (like J_p) the situation is more favorable: they are more tractable as we work with one prime only, and the important Lemma 4.2 holds more generally for finitely generated submodules G (Rotman [1], Rotman–Yen [1], and C.M. Bang [Proc. Amer. Math. Soc. **28**, 381–388 (1971)]).

Exercises

(1) $p^\sigma A$ is p-nice in a mixed group A for every ordinal σ.
(2) Prove that isotypeness is transitive also for mixed groups.
(3) If C is an isotype subgroup of the mixed group A, then the height-matrix of any $c \in C$ is the same whether computed in C or in A.
(4) If $0 \to A \to B \to C \to 0$ is a balanced-exact sequence, then the induced sequence $0 \to D_A \to D_B \to D_C \to 0$ of the divisible subgroups is splitting exact. [Hint: choose χ in (15.7) properly.]
(5) (Irwin–Walker–Walker) An exact sequence $0 \to A \to B \to C \to 0$ represents an element of $p^\infty \operatorname{Ext}(C, A)$ if and only if the induced sequence $0 \to A_p \to B_p \to C_p \to 0$ of p-components is splitting exact.

(6) For every increasing sequence $\sigma_0 < \sigma_1 < \cdots < \sigma_n < \ldots$ of ordinals, there is a $\mathbb{Z}_{(p)}$-module A which contains an element x of infinite order such that

(a) $h_p(p^n x) = \sigma_n$ for each $n < \omega$;
(b) $X = \mathbb{Z}_{(p)}x$ is a nice submodule of A;
(c) A/X is a totally projective p-group of length $\sup_n \sigma_n$. [Hint: Theorem 3.8.]

5 Mixed Groups of Torsion-Free Rank One

At this point we reduce generality, and restrict our considerations to mixed groups of torsion-free rank 1. The reason for our focusing on this special case lies in our wanting to learn the basics of how an infinite cyclic group and a torsion group blend to form a tractable mixed group. With the help of height-matrices and UK-invariants, a satisfactory structure theorem will be obtained for a class of such groups without much effort (see Theorem 5.2). Already this result will give the flavor of both the beauty and the difficulties in dealing with mixed groups.

The first thing we observe is that in a mixed group of torsion-free rank 1, any two elements of infinite order are dependent, so their height-matrices are equivalent. Consequently, there is a well-defined equivalence class of matrices associated with such a group A; we shall denote it by $\mathbb{H}(A)$, and call it **the height-matrix of** A.

In what follows, we will often deal with height-preserving isomorphisms $\phi : G \to H$ between subgroups G of A and H of C, i.e. $h_p(\phi g) = h_p(g)$ for all $g \in G$ and for all primes p, where the heights are computed in the containing groups C and A, respectively. This situation will be crystalized in introducing a category in Sect. 8.

The following lemma provides an important step in the proof of Theorem 5.2 inasmuch as it tells how to compute the Hill invariants relative to an infinite cyclic subgroup from the UK-invariants of the group. Recall that $f_\sigma(A)$ and $f_\sigma(A, G)$ denote the σth UK-invariant of A for a given prime, and its σth Hill invariant with respect to the subgroup G, respectively.

Lemma 5.1. *Let A be a group of torsion-free rank 1, and G an infinite cyclic subgroup such that A/G is a p-group. Then*

$$f_\sigma(A) = \begin{cases} f_\sigma(A, G) + 1 & \text{if the } p\text{–row of } \mathbb{H}(A) \text{ has a gap at } \sigma, \\ f_\sigma(A, G) & \text{if there is no such gap.} \end{cases}$$

Proof. Recall that we have $f_\sigma(A, G) = p^\sigma A[p]/G(\sigma)$, and suppose that $G(\sigma) = (p^{\sigma+1}A + G) \cap p^\sigma A[p] \neq 0$, i.e. there is $g \in G$ such that $h_p(g) = \sigma$ and $a + g \in p^\sigma A[p]$ for some $a \in p^{\sigma+1}A$. Then $pg = -pa \in p^{\sigma+2}A$, thus g represents a gap in the p-row at σ. Evidently, dim $G(\sigma) \leq 1$, showing that $f_\sigma(A) = f_\sigma(A, G) + 1$. Conversely, if the p-row of $\mathbb{H}(g)$ has a gap at σ, then the reverse argument yields $G(\sigma) \neq 0$. Hence the claim follows. $\square$

We have come to the classification theorem on mixed groups of torsion-free rank 1. It was proved by Rotman [2], Megibben [3], Myshkin [1] for countable groups, and generalized by Warfield [7].

Theorem 5.2 (Warfield [7]). *Let A and C be mixed groups of torsion-free rank* 1*, G in A and H in C infinite cyclic subgroups such that A/G and C/H are totally projective torsion groups. $A \cong C$ if and only if*

(i) *the UK-invariants of A and C are equal for all primes p; and*
(ii) *the height-matrices $\mathbb{H}(A)$ and $\mathbb{H}(C)$ are equivalent.*

Proof. It is obvious that the stated conditions are necessary. To verify sufficiency, assume (i)–(ii). By (ii), there are $a \in G$ and $c \in H$, both of infinite order with the same height-matrix, thus $a \mapsto c$ induces a height-preserving isomorphism $\phi' : \langle a \rangle \to \langle c \rangle$.

Let $A(p)$ denote the subgroup of A such that $A(p)/\langle a \rangle = (A/\langle a \rangle)_p$; thus $A(p)/\langle a \rangle$ is a totally projective p-group (observe that A/G and A/pG are simultaneously totally projective). Let $C(p)$ have similar meaning for C. In view of Lemma 4.2, $\langle a \rangle$ is p-nice in $A(p)$, and $\langle c \rangle$ is p-nice in $C(p)$. Furthermore, because of the equality of height-matrices, Lemma 5.1 guarantees that the relative UK-invariants $f_\sigma(A(p), \langle a \rangle)$ and $f_\sigma(C(p), \langle c \rangle)$ are the same for all p and σ. Thus we have all the ingredients to apply Lemma 4.5 in Chapter 11, and to conclude that there is an isomorphism $\phi_p : A(p) \to C(p)$ extending ϕ'. The difficult work having been done, it only remains to paste together the isomorphisms ϕ_p for all primes. This can be accomplished by the commutative diagram

$$\begin{tikzcd} \oplus_p A(p) \arrow[r, "\alpha"] \arrow[d, "\oplus\phi_p"'] & A \arrow[d, dashed, "\phi"] \\ \oplus_p C(p) \arrow[r, "\gamma"] & C \end{tikzcd}$$

where the maps α, γ are induced by the component-wise embeddings, and ϕ is uniquely determined by the commutativity of the diagram. It is straightforward to verify that ϕ is an isomorphism. □

It is, perhaps, worthwhile pointing out that the last theorem may fail whenever the factor groups A/G and C/H are not totally projective; this is demonstrated by the following example.

Example 5.3 (Megibben [3]). Let B be an unbounded countable direct sum of cyclic p-groups, and $\overline{B}$ its torsion-completion. Let T be a pure subgroup of $\overline{B}$ containing B such that $\overline{B}/T \cong \mathbb{Z}(p^\infty)$. The exact sequence $0 \to \mathrm{Hom}(\mathbb{Z}, T) \cong T \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \to \mathrm{Ext}(\mathbb{Q}, T) \cong \oplus\mathbb{Q} \to 0$ and the reducedness of the first Ext imply that $\mathrm{Ext}(\mathbb{Z}(p^\infty), T)$ contains a mixed group G such that $T < G$, $G/T \cong \mathbb{Q}$ and $G^1 \neq 0$. In a similar fashion, we obtain a subgroup H of $\mathrm{Ext}(\mathbb{Z}(p^\infty), B)$ such that $B < H$, $H/B \cong \mathbb{Q}$ and $H^1 \neq 0$. There are elements $g \in G, h \in H$ of infinite order such that $h_p(p^k g) = \omega + k = h_p(p^k h)$ for $k < \omega$, so that $A = G \oplus B$ and $C = H \oplus T$ have equivalent height-matrices $\mathbb{H}(A) \sim \mathbb{H}(C)$, besides being of torsion-free rank 1, and having the same UK-invariants. In order to justify the claim that $A \ncong C$, we prove:

$$\text{(a) } G/G^1 \cong \overline{B}, \quad \text{(b) } H/H^1 \cong B, \quad \text{(c) } B \oplus T \ncong B \oplus \overline{B}.$$

(a) follows from the fact that if T is pure in a separable p-group E such that $E/T \cong \mathbb{Z}(p^\infty)$, then $E \cong \overline{B}$ (Sect. 3 in Chapter 10, Exercise 15). Since H/H^1 is a countable separable p-group, and B is isomorphic to a pure subgroup with $\mathbb{Z}(p^\infty)$ as quotient, (b) follows from Theorem 3.5 in Chapter 10. Finally, the proof of Theorem 9.5 in Chapter 10 shows that $B \oplus T \cong B \oplus \overline{B}$ could hold only if $B \oplus T$ had a summand isomorphic to $\overline{B}$, which is evidently not the case.

We observe that in Theorem 5.2 the assumption that G and H were infinite cyclic groups was used in the proof only in the form that they were nice and isomorphic. Consequently, the proof works also for the following more general result, it needs no adaptation.

Theorem 5.4. *Let A and C be mixed groups, $G < A$ and $H < C$ nice subgroups such that*

(i) *there is a height-preserving isomorphism $\phi: G \to H$; and*
(ii) *A/G and C/H are isomorphic totally projective torsion groups.*

Then there is an isomorphism $\phi^*: A \to C$ such that $\phi^* \restriction G = \phi$. □

★ **Notes.** The first relevant result on the structure of non-splitting mixed groups is due to Kaplansky–Mackey [1] who proved that two countably generated modules of torsion-free rank 1 over a complete discrete valuation domain are isomorphic if and only if they have equivalent characteristics and the same UK-invariants. The completeness hypothesis was later removed by Megibben [3], making the Kaplansky–Mackey result an honest abelian group theorem. Rotman–Yen [1] proved the isomorphy of countably generated modules of finite torsion-free ranks over a complete discrete valuation domain, under the hypothesis that they have the same height-matrices and the same UK-invariants. Stratton [1] showed that the Rotman–Yen result fails if the completeness condition is abandoned. Wallace [1] proves that two p-local groups of torsion-free rank 1 with totally projective torsion subgroups are isomorphic exactly if they have equivalent height-matrices and the same UK-invariants.

For countable mixed groups of torsion-free rank 1, the isomorphism theorem was proved independently by Rotman [2], Megibben [3], Myshkin [1].

Files [2] considers transitivity and full transitivity for mixed groups of torsion-free rank 1. Such a group enjoys both properties whenever its torsion subgroup is separable. Transitive properties more generally were dealt with by Grinshpon–Krylov [1]; in this systematic study, a wealth of interesting results on mixed groups was proved. Self-small mixed groups of finite torsion-free ranks were discussed by Albrecht [6] (in particular, the A-projective groups), also by Albrecht–Breaz–Wickless [1].

Exercises

(1) (Pierce) Let A be a p-group of length τ. There exists a mixed group G such that $p^\tau G \cong \mathbb{Z}$ and $G/p^\tau G \cong A$.

(2) (Megibben) Prove Theorem 5.2 by replacing (i) by the condition that $t(A) \cong t(C)$ are torsion-complete p-groups.
(3) Two countable mixed groups of torsion-free rank 1 are isomorphic if each is isomorphic to an isotype subgroup of the other.
(4) (Rotman) Assume A, C are countable mixed groups of torsion-free rank 1, and T is a torsion group with finite UK-invariants. Then $A \oplus T \cong C \oplus T$ implies $A \cong C$.
(5) (Rotman) Again, let A and C be countable mixed groups of torsion-free rank 1. If $A \oplus A \cong C \oplus C$, then $A \cong C$.
(6) Find relations between the height-matrix and the UK-invariants of a countable mixed group of torsion-free rank 1.
(7) (Warfield) Verify the one-sided version of Theorem 5.4 for a homomorphism $\phi^* : A \to C$. Now $\phi : G \to H$ is a homomorphism not decreasing heights, and only the existence of a homomorphism $A/G \to C/H$ is assumed.

6 Simply Presented Mixed Groups

Having gotten acquainted with mixed groups of torsion-free rank 1, we move to groups of higher torsion-free ranks. Our next concern is the theory of simple presented mixed groups which combines two prominent theories at an elevated level: those on simply presented p-groups, and on completely decomposable torsion-free groups. It requires new machinery to deal with some unexpected, if not recondite, aspects.

A group A was called **simply presented** if it could be defined in terms of generators and relations in such a way that each relation involves at most two generators. Now the relations are of the form $nx = 0$ or $mx = ny$ with generators x, y and $m, n \in \mathbb{Z}$. Just as in the case of p-groups, no generality is lost by assuming that all the relations are of the form $px = 0$ or $px = y$ where x, y are generators, and p is a prime (not fixed). Henceforth it will be assumed that *the simple presentations are faithful* in the sense defined in Sect. 3 in Chapter 11, and 'generator' means one in a faithful simple presentation.

Example 6.1. (a) The Nunke groups N_σ are simply presented mixed groups of torsion-free rank 1.
(b) A faithfully simply presented group is torsion if, for every generator x, there are a sequence $x = x_0, \dots, x_k$ of generators, and a sequence of (not necessarily distinct) primes $p_0, \dots, p_k$ such that $p_{i-1}x_{i-1} = x_i$ for $i = 1, \dots, k$ and $p_k x_k = 0$.
(c) We repeat: a torsion-free group is simply presented if and only if it is completely decomposable.

Our discussion starts with a most informative lemma.

Lemma 6.2 (Warfield [4]). *A simply presented group is the direct sum of simply presented groups whose torsion-free rank is at most* 1.

Proof. Let X be the set of generators in a faithful simple presentation of A, and $0 \to H \to F \xrightarrow{\phi} A \to 0$ an exact sequence, where F is the free group on X, and ϕ acts as the identity map of X. An equivalence relation is defined for the elements of X such that $x \sim y$ means that there are integers $n, m \in \mathbb{Z}$ for which $m\phi(x) = n\phi(y) \neq 0$ holds in A. If V_i $(i \in I)$ denote the different equivalence classes, then every $\langle V_i \rangle$ is a summand of F, and evidently $F = \oplus_{i \in I} \langle V_i \rangle$. The subgroup H is generated by the elements of the form $mx - ny \in F$, each of which is contained in some $\langle V_i \rangle$. Therefore, $A \cong F/H \cong \oplus_{i \in I} \langle V_i \rangle / (\langle V_i \rangle \cap H)$, where the summands are either torsion or of torsion-free rank 1. □

A few useful facts are collected in the following statements. (The claims bear more than a passing resemblance to the torsion case.)

(a) *If A is simply presented on the set X, then for a subset $Y \subset X$, both $\langle Y \rangle$ and $\langle X \setminus Y \rangle$ are simply presented.*
(b) *If A is simply presented on the set X, then $\langle Y \rangle$ is a nice subgroup for every $Y \subset X$.* Because of Lemma 6.2, it suffices to prove this for $\mathrm{rk}_0 A \leq 1$. If $o(y) = \infty$ $(y \in Y)$, then $A/\langle y \rangle$ is simply presented torsion, so $\langle Y \rangle / \langle y \rangle$ is nice in it, as it is p-nice for each prime (see Lemma 3.4 in Chapter 11). Since by Lemma 4.2 $\langle y \rangle$ is nice in A, from Lemma 4.1 we conclude that $\langle Y \rangle$ is nice in A.
(c) *For every height-matrix $\mathbb{H}$, there exists a simply presented group whose height-matrix is equivalent to $\mathbb{H}$.* A close examination of the proof of Theorem 3.8 shows that the mixed group constructed there is simply presented.
(d) *The torsion subgroup of a simply presented group need not be simply presented.* This is illustrated by the following example.

Example 6.3 (Warfield). Consider the Nunke group $G = N_{\omega_1}(p)$; thus $p^{\omega_1}G \cong \mathbb{Z}$ and $G/p^{\omega_1}G$ is a simply presented p-group. Since $G/tG \cong Q^{(p)}$, tG embeds in the totally projective p-group $G' = G/p^{\omega_1}G$ as a dense isotype subgroup H with cokernel $\cong \mathbb{Z}(p^\infty)$. G' is of length ω_1, so $G' = \oplus_{i \in I} G_i$ holds with countable summands G_i. To see that H cannot be simply presented, observe that $H[p] < G'[p]$, and by density $G'[p] = H[p] + p^\sigma G'[p]$ for all $\sigma < \omega_1$. If H were simply presented, then $H = \oplus_{j \in J} H_j$ would hold with countable H_j. Then $G' = H + (\oplus_{i \in I'} G_i)$ for a countable index set I'. If $\sigma < \omega_1$ is the length of $\oplus_{i \in I'} G_i$, then $p^\sigma G'[p] = p^\sigma H[p]$ whence $G'[p] = H[p]$ would follow, a contradiction.

Theorem 6.4 (Warfield [4]).

(i) *A torsion summand of a simply presented mixed group is simply presented.*
(ii) *A mixed group A of torsion-free rank 1 is a summand of a simply presented group if and only if there is an element $a \in A$ of infinite order such that $A/\langle a \rangle$ is a simply presented torsion group.*

Proof. (i) Let $G = T \oplus K$ be a simply presented mixed group where T is a torsion group. From Lemma 6.2 we know that $G = \oplus_{i \in I} G_i$ where $\mathrm{rk}_0(G_i) = 1$ for every $i \in I$. Let $g_i \in G_i$ be a generator of infinite order; it can be replaced by an integral multiple, so we can choose $g_i \in K$. Then $G/F \cong T \oplus (K/F)$ where F denotes the (free) subgroup of G generated by the g_i $(i \in I)$. Hence T is a summand of the simply presented torsion group G/F, so a reference to Lemma 4.2 in Chapter 11 confirms that T is simply presented.

(ii) For the proof of sufficiency, assume $a \in A$ is as stated. In view of Theorem 3.8, there exists a simply presented group G with $A(\mathbb{H}) = G(\mathbb{H})$. Choose a simply presented torsion group T with large UK-invariants so that the UK-invariants of $A' = A \oplus T$ and $G' = G \oplus T$ are equal to those of T. Since the height-matrices have not changed by adding a torsion summand, we can appeal to Theorem 5.2 to conclude $A' \cong G'$. It follows that A is isomorphic to a summand of the simply presented G'.

For necessity, assume $G = A \oplus C$ is a simply presented group where $\mathrm{rk}_0(A) = 1$. Let G_i, g_i and F have the same meaning as in the proof of (i). Pick $0 \neq a \in A \cap F$, and select a finite subset $J \subseteq I$ such that $a \in \oplus_{i \in J} \langle g_i \rangle = F'$. Putting $G' = \oplus_{i \in J} G_i$, obviously, G'/F' is simply presented and torsion. As $F'' = \langle a \rangle \oplus (C \cap F')$ has finite index in F', G'/F'' is also simply presented. It follows that $A/\langle a \rangle$ is isomorphic to a summand of the simply presented group $G'/F'' \oplus \oplus_{i \in I \setminus J} G_i$. An appeal to (i) completes the proof. □

It should be pointed out that Theorem 6.4(ii) becomes false if 'summands of' is deleted. In fact, not all summands of simply presented mixed groups are simply presented; see Example 7.1.

One might expect simply presented mixed groups to have the same kind of classification as their torsion and torsion-free counterparts. However, as our next example demonstrates, no such result can exist.

Example 6.5 (Warfield [4]). There exist simply presented groups A, B, C, D of torsion-free rank 1 with inequivalent height-matrices such that

$$A \oplus B \cong C \oplus D.$$

In this example, the groups are countable and have no elements $\neq 0$ of infinite height.

To define A, B, start with $a \in A, b \in B$ of infinite orders. Adjoin generators in order to have: $h_2(2^n a) = h_3(3^n a) = 2n$ or $2n - 1$ according as n is even or odd; $h_2(2^n b) = h_3(3^n b) = 2n$ for all n; and $h_p(p^n a) = h_p(p^n b) = n$ for all primes $p \geq 5$. By Theorem 3.8, such A, B of torsion-free rank 1 do exist. In the group $G = A \oplus B$, choose $c = 9a + b, d = 8a + b$. We then have $h_2(2^n c) = h_2(2^n a)$, $h_2(2^n d) = h_2(2^n b)$ and $h_3(3^n c) = h_3(3^n b)$, $h_3(3^n d) = h_3(3^n a)$, while $h_p(p^n c) = h_p(p^n d) = n$ for $p \geq 5$.

The height-matrices $\mathbb{H}(a), \mathbb{H}(b), \mathbb{H}(c), \mathbb{H}(d)$ are inequivalent. It is a long, but rather straightforward calculation to verify that there exist subgroups C, D with $c \in C, d \in D$ and $G = C \oplus D$. (Actually, it would suffice to prove that $\{c, d\}$ is a decomposition basis for G (see below), because then it is easy to show that G has the stated decomposition.)

★ **Notes.** Following the triumph with the classification of simply presented torsion groups, the focus shifted to torsion-free and mixed groups. It was clear that one should attempt to adjust the powerful and most successful methods acquired in the torsion case to mixed groups. Mixed groups whose torsion subgroups are classifiable were the obvious targets, though this seemed a remote possibility. The change in viewing a mixed group as an extension of a torsion-free group by a torsion one, albeit not canonically, opened the door to a successful application of the arsenal. It is fortunate that Warfield took the lead in laying down the foundations of a theory for tractable mixed groups and in developing it into a remarkable theory. I think it is amazing that the UK-invariants of the ignored torsion part can be coded as a valuation in a torsion-free group.

Exercises

(1) Every simply presented mixed group admits an $H(\aleph_0)$-family of nice subgroups.
(2) Determine the height-matrix of the global Nunke group N_σ.
(3) (a) Find the height-matrices of the summands in Example 6.5.
    (b) Check that $\mathbb{H}(a) + \mathbb{H}(b) = \mathbb{H}(c) + \mathbb{H}(d)$.

7 Warfield Groups

The study of the so-called Warfield groups is one of the most challenging tasks in the theory of mixed groups. They are slightly more general than the simply presented groups, but their theory is less transparent.

Let A be a mixed group. The natural homomorphism $\alpha_p : A \to A_{(p)}$ maps $a \in A$ to $a \otimes 1 \in A_{(p)} = A \otimes \mathbb{Z}_{(p)}$. The maps α_p combined for all primes p define an embedding $\alpha : A \to \prod_p A_{(p)}$. If F is a free subgroup of A such that A/F is torsion, then $\alpha(A)/\alpha(F)$ is the torsion subgroup of $\prod_p A_{(p)}/\alpha(F)$. Hence we deduce that the cokernel of α is torsion-free and divisible.

(a) *If the torsion-free rank of A is* 1*, and $a \in A$ is of infinite order such that $A/\langle a\rangle$ is a totally projective p-group, then for every $b \in A$ of infinite order, $A/\langle b\rangle$ is also totally projective.* This follows from the linear dependence of a and b, and from the total projectivity of an extension of a finite group $\langle a\rangle/\langle ma\rangle$ by a totally projective p-group $A/\langle a\rangle$ (Lemma 7.12 in Chapter 11).
(b) *If A is of torsion-free rank* 1*, and $a \in A$ is of infinite order, then $A/\langle a\rangle$ is totally projective whenever the torsion subgroup tA is a totally projective p-group.* In fact, tA is isomorphic to a subgroup of countable index in $A/\langle a\rangle$, and a p-group is itself totally projective if it has a totally projective subgroup of countable index (see Lemma 7.12 in Chapter 11).

Warfield Groups By a **Warfield group** is meant a summand of a simply presented group. Thus simply presented torsion groups as well as completely decomposable torsion-free groups are Warfield (but we usually think of mixed groups when Warfield groups are mentioned). That not all mixed Warfield groups are simply presented was first observed by Rotman–Yen [1] (in a different context); we furnish the following counterexample due to Warfield [7].

Example 7.1. Consider the $\mathbb{Z}_{(p)}$-module A generated by the symbols a_n $(n < \omega)$ subject to the defining relations $p^{2n}a_n = p^n a_0$ $(n < \omega)$. Its torsion subgroup $tA = \oplus_{n<\omega} \mathbb{Z}_{(p)}(a_n - pa_{n+1})$ satisfies $A/tA \cong \mathbb{Q}$. The p-indicator of a_0 is $\underline{u} = (0, 2, 4, \dots)$, while for other primes the indicators are constant ∞. A is simply presented as a group.

Suppose $A = B \oplus C$ where $\mathrm{rk}_0\, B = 1$ and C is a p-group. The indicator of any $b \in B$ of infinite order is equivalent to $\underline{u}$ above, whence the gap condition implies that B ought to contain, for almost all n, elements of order p and of height $2n$. As $tA = (tA \cap B) \oplus C$, it is manifest that C must be finite.

Define G to be generated by the symbols g, a_n $(n < \omega)$ with the defining relations

$$pg = 0,\ \ p^{2n+1}(a_n - pa_{n+1}) = g\ \ (n < \omega)$$

to satisfy

$$p^{\omega}G = \langle g\rangle \cong \mathbb{Z}(p) \quad \text{and} \quad G/p^{\omega}G \cong A.$$

Evidently G is Warfield (since $G/\langle a_0\rangle$ is a countable p-group, so simply presented), and the claim is that G is not simply presented. For, suppose by way of contradiction that it is. Then $G = H \oplus K$, where H is simply presented of torsion-free rank 1 and K is torsion. From the properties of A, it follows that $p^{\omega}G \leq K$ would imply that $K/p^{\omega}G$ is finite, which is impossible because $p^{\omega}K = p^{\omega}G$ in this case. Therefore, by full invariance, $p^{\omega}G \leq H$. If x, y are generators of H, and $o(x) = \infty$, then the p-height of $y + \langle x\rangle$ in $H/\langle x\rangle$ is equal to $h_p(y)$. It follows that $H/\langle x\rangle$ admits a simple presentation such that all the generators are of finite heights; thus, $p^{\omega}(H/\langle x\rangle) = 0$, i.e. $p^{\omega}H \leq \langle x\rangle$. But this is impossible, as x is of infinite order, while $p^{\omega}G$ has order p. Consequently, H cannot be simply presented.

It is fairly easy to describe Warfield groups of torsion-free rank 1 in various ways.

Theorem 7.2 (Warfield [8]). *For a group A of torsion-free rank* 1 *the following conditions are equivalent:*

(i) *A is a Warfield group;*
(ii) *A is an extension of $\mathbb{Z}$ by a simply presented torsion group;*
(iii) *A has an $H(\aleph_0)$-family of nice subgroups.*

Proof. (i) $\Leftrightarrow$ (ii) This is proved in Theorem 6.4.

(ii) $\Rightarrow$ (iii) Let A be an extension of an infinite cyclic group $\langle a\rangle$ by a simply presented torsion group B. From Lemma 4.2 we derive that $\langle a\rangle$ is nice in A. We also know from Theorem 5.9 in Chapter 11 that B admits an $H(\aleph_0)$-family $\mathcal{N}$ of nice subgroups. By Lemma 4.2, $\langle a\rangle \otimes \mathbb{Z}_{(p)}$ is p-nice for each p, hence $\langle a\rangle$ is nice in A. It remains to refer to Lemma 4.1 to argue that the extensions of $\langle a\rangle$ by the members of $\mathcal{N}$ will be an $H(\aleph_0)$-family of nice subgroups in A.

(iii) $\Rightarrow$ (i) Suppose A has an $H(\aleph_0)$-family $\mathcal{N}$ of nice subgroups. Factoring out a (nice) infinite cyclic subgroup $\langle a\rangle$ from the members of $\mathcal{N}$ that contain $\langle a\rangle$, we get an $H(\aleph_0)$-family of nice subgroups in $A/\langle a\rangle$, so this factor group is totally projective. We can find a simply presented group C of torsion-free rank 1 with $\mathbb{H}(C) = \mathbb{H}(A)$. Furthermore, there exists a torsion group T with large UK-invariants such that the groups T, $A \oplus T$, $C \oplus T$ all have the same UK-invariants. Applying Theorem 5.2 to $A \oplus T$ and $C \oplus T$, the theorem is proved. $\square$

Decomposition Subgroups For Warfield groups of higher torsion-free ranks, a more sophisticated approach is required, as the situation is by far more complicated. The role of a single infinite cyclic group is replaced by a rank-wise maximal free subgroup. These free subgroups are not arbitrary, they are very special subgroups, absolutely fundamental in the study of Warfield groups.

The definition will be stated for arbitrary mixed groups A. A free subgroup F of A is said to be a **decomposition subgroup** and a basis $X = \{x_i\}_{i\in I}$ of F a **decomposition basis** if

(i) A/F is torsion; i.e. X is a maximal independent set of elements of infinite order;
(ii) for any prime p and for any finite subset J of I, we have

$$h_p(\sum_{i \in J} n_i x_i) = \min_{i \in J} h_p(n_i x_i) \qquad (n_i \in \mathbb{Z}).$$

If, in addition, F is a nice subgroup, it will be called a **nice decomposition subgroup.** We observe:

(A) *All groups of torsion-free rank* 1 *have decomposition subgroups, and so do their direct sums.*
(B) *If* $X = \{x_i\}_{i \in I}$ *is a decomposition basis for* A*, and if* $m_i \neq 0$ *are integers, then* $Y = \{m_i x_i\}_{i \in I}$ *is likewise a decomposition basis,* called a **subordinate** to X.
(C) *A subordinate to a nice decomposition basis is likewise nice.* The conclusion follows from the fact that if Y is a subordinate to a decomposition basis X, then $\langle Y \rangle$ is nice in $\langle X \rangle$.

Example 7.3. A simply presented group admits a nice decomposition subgroup: decompose the group as in Lemma 6.2, and form the direct sum of infinite cyclic groups, one from each non-torsion summand.

Two decomposition bases are considered **equivalent** if there is a bijection between the sets such that corresponding elements have equivalent height-matrices. That in general decomposition bases are inequivalent is shown by Example 6.5. Though decomposition bases are always nice if they are finite (Exercise 3), they need not be nice in general.

The next result has a long sophisticated proof, we record it without giving details.

Lemma 7.4 (Hunter–Richman [1]). *Every decomposition basis of a Warfield group* A *has a nice subordinate* Y *such that* $A/\langle Y \rangle$ *is totally projective torsion.* □

The next result is also fundamental, it is proved by using a general decomposition theorem on certain direct sums in additive categories. We quote it without proof.

Theorem 7.5 (Arnold–Hunter–Richman [1]). *Summands of a group with decomposition basis have decomposition bases.* □

We do not want to go through the detailed proofs of Lemma 7.4 and Theorem 7.5, we would rather concentrate on the principles that make the theory work.

Next, we characterize Warfield groups in terms of decomposition bases.

Theorem 7.6. *A group* A *is Warfield if and only if it contains a nice decomposition basis* X *such that* $A/\langle X \rangle$ *is totally projective torsion.*

Proof. First suppose A has a nice decomposition basis X as stated. There exists a simply presented group C whose decomposition basis Y is equivalent to X. By making use of a totally projective torsion group T with large UK-invariants, we argue as in the last part of the proof of Theorem 7.2 to conclude that A is a summand of the simply presented $C \oplus T$.

Conversely, let $A \oplus B = C$ where $C = \oplus_{i \in I} C_i$ is simply presented with $\mathrm{rk}_0\, C_i = 1$, and assume $Z = \{c_i \mid c_i \in C_i\}$ is a nice decomposition basis for C. By Theorem 7.5, both A and B have decomposition bases, say, X and Y, respectively. Then $X \cup Y$ is a decomposition basis for C, and Lemma 7.4 implies that it has a subordinate $X' \cup Y'$ ($X' \subseteq X, Y' \subseteq Y$) that generates a nice subgroup of C with totally projective $C/\langle X' \cup Y' \rangle$. The last group is clearly isomorphic to $A/\langle X' \rangle \oplus B/\langle Y' \rangle$, so X' is a nice decomposition basis in A with totally projective $A/\langle X' \rangle$. □

Warfield Invariants In order to classify local Warfield groups of arbitrary cardinalities, an additional cardinal invariant is required. We follow Stanton [1] in introducing the Warfield invariants in general. The advantage of this method is that the definition is not restricted to Warfield groups.

Let A be p-local, for a fixed p, and let $\underline{u} = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ denote a strictly increasing sequence of ordinals (now no ∞ is permitted). As before, we set $A(\underline{u}) = \{a \in A \mid h_p(p^n a) \geq \sigma_n\}$, and define

$$A^*(\underline{u}) = \langle a \in A(\underline{u}) \mid h_p(p^n a) > \sigma_n \text{ for infinitely many } n \rangle.$$

The elements in the definition of $A(\underline{u})$ form a subgroup, but this is not the case for $A^*(\underline{u})$, in general. Both are fully invariant subgroups of A, and obviously, $tA \leq A^*(\underline{u})$ for all $\underline{u}$. It is equally clear that $pA(\underline{u}) \leq A^*(\underline{u})$, whence we see that $A(\underline{u})/A^*(\underline{u})$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space.

For $n < \omega$, the map

$$\phi_n : A(\underline{u})/A^*(\underline{u}) \to A(p^n\underline{u})/A^*(p^n\underline{u})$$

acting as $a + A^*(\underline{u}) \mapsto p^n a + A^*(p^n\underline{u})$ ($a \in A(\underline{u})$) is a well-defined linear map between vector spaces. We claim ϕ_n is monic for each n; for the proof, the case $n = 1$ will suffice. Let $a + A^*(\underline{u}) \in \operatorname{Ker} \phi_1$ where $a \in A(\underline{u})$. To verify that then $a \in A^*(\underline{u})$, consider $pa = \sum_{j=1}^m r_j x_j$ ($r_j \in \mathbb{Z}$) where x_j are generators of $A^*(p\underline{u})$, thus $h_p(p^k x_j) \geq \sigma_{k+1}$ for all $k < \omega$ with strict inequalities for infinitely many k's. We can find $y_j \in A$ of height $\geq \sigma_0$ such that $py_j = x_j$. Now $p\underline{u}(y_j) = \underline{u}(x_j)$ implies $y_j \in A^*(\underline{u})$. Taking the containment $a - \sum_{j=1}^m r_j y_j \in tA \leq A^*(\underline{u})$ into account, $a = (a - \sum_{j=1}^m r_j y_j) + \sum_{j=1}^m r_j y_j \in A^*(\underline{u})$ follows.

Recall that $\underline{u} \sim \underline{v}$ means $p^n\underline{u} = p^m\underline{v}$ for some $n, m < \omega$. Hence we conclude that the direct limit

$$\varinjlim_{n<\omega} A(p^n\underline{u})/A^*(p^n\underline{u}) = w_A(\underline{u})$$

(with the ϕ_n as connecting maps) depends only on the equivalence class $[\underline{u}]$ of indicators, and not on the individual indicators. This vector space (or its dimension) is called the $\underline{u}$th **Warfield invariant** of A. For the indicator $\underline{w} = (\infty, \dots, \infty, \dots)$, $A(\underline{w})$ is the divisible part D of A, and $A^*(\underline{w}) = 0$, so that the $\underline{w}$th Warfield invariant is the (dimension of) D.

Lemma 7.7 (Stanton [1]). *Let A be a reduced p-local group.*

(i) *If the torsion-free rank of A is* 1*, then* $w_A(\underline{u}) = \mathrm{rk}_0\, A(\underline{u})$*, i.e. it is* 0 *or* 1.
(ii) *If* $A = \oplus_{i \in I} A_i$*, then* $w_A(\underline{u}) = \oplus_{i \in I} w_{A_i}(\underline{u})$.

Proof. (i) Suppose A has torsion-free rank 1, and $\underline{u}(a) = (\sigma_0, \dots, \sigma_n, \dots)$ for an $a \in A$ of infinite order. If $b \in A$ but $\notin A^*(\underline{u})$, then because of $\underline{u}(a) \sim \underline{u}(b)$ there is $m \in \mathbb{N}$ such that $p^m a = kp^m b$ for some integer k prime to p. Now $a - kb \in tA \leq A^*(\underline{u})$ implies $a + A^*(\underline{u}) = kb + A^*(\underline{u})$. This proves that $A(\underline{u})/A^*(\underline{u}) \cong \mathbb{Z}(p)$. For $A(p^n\underline{u})/A^*(p^n\underline{u})$ we have the same proof and the same conclusion, so $w_A(\underline{u}) \leq 1$. On the other hand, if the elements of infinite order in A have characteristics $\underline{v} \not\sim \underline{u}$, then $A(\underline{u}) = 0$ and $w_A(\underline{u}) = 0$.
(ii) This is a simple consequence of the definition and (i). □

Thus for Warfield groups, we have two sets of invariants: the UK- and the Warfield invariants, both are cardinal numbers.

Isomorphism of Local Warfield Groups All the groundwork has been laid for the classification of Warfield groups in the local case.

Theorem 7.8. *The p-local Warfield groups A, C are isomorphic if and only if*

(i) *they have the same UK-invariants;*
(ii) *their Warfield invariants are equal.*

Proof. Only sufficiency requires a proof. So assume A, C satisfy (i)–(ii). Owing to Theorem 7.6, decomposition bases X in A and Y in C can be selected such that $\langle X \rangle$ is nice in A, $\langle Y \rangle$ is nice in C, and $A/\langle X \rangle, C/\langle Y \rangle$ are totally projective. From Lemma 7.7 it is clear that condition (ii) implies the equality of the cardinalities of the sets of elements in X and Y whose indicators belong to a given equivalence class $[\underline{u}]$. Hence we can replace X, Y by subordinates, if necessary, (we keep the notations X, Y for the subordinates) to obtain a bijection $f : X \to Y$ such that $\underline{u}(x) = \underline{u}(f(x))$ for all $x \in X$. By (C), these subordinates generate nice subgroups, and as an extension of a nice totally projective subgroup by a totally projective group is again totally projective, it is evident that the total projectivity of the quotients $A/\langle X \rangle, C/\langle Y \rangle$ is preserved. Refer to an obvious extension of Lemma 5.1 to higher ranks, and argue that the equality of the relative invariants $f_\sigma(A, \langle X \rangle)$ and $f_\sigma(C, \langle Y \rangle)$ is a consequence of (i). Finally, we invoke Theorem 5.4 to complete the proof. □

The situation is far less transparent in the global case. The classification of global Warfield groups is not possible in terms of equivalence of matrices only, as is evident from Example 6.5. Stanton [2] introduced a coarser equivalence relation for height-matrices which he called 'compatibility,' but I prefer to use 'quasi-equivalence' instead to stress its strong parallelism to quasi-isomorphism of torsion-free groups. Accordingly, we will call the height-matrices $\mathbb{M}$ and $\mathbb{N}$ **quasi-equivalent** if there exist positive integers m, n such that

$$m\mathbb{M} \geq \mathbb{N} \quad \text{and} \quad n\mathbb{N} \geq \mathbb{M}.$$

By making use of the equivalence classes under this relation (in addition to the equivalence classes of matrices), Stanton introduces new invariants as a vehicle for a classification of global Warfield groups.

Example 7.9. Consider the groups A, B, C, D defined in Example 6.5. All their height-matrices are quasi-equivalent. For instance, $6\,\mathbb{H}(C) \geq \mathbb{H}(A)$ and $6\,\mathbb{H}(A) \geq \mathbb{H}(C)$.

In this section, the reader has been burdened with unproved propositions that have been utilized in subsequent proofs. This is not a usual practice in this volume, and I apologize for having done so. Mixed groups have enormously rich structure, some theorems require not only delicate reasoning, but also lots of technical calculations, consuming lots of pages. In spite of the missing details (for which we refer to the literature), I hope that the material presented above lays down the groundwork and conveys properly a feeling for the theory of mixed groups. A streamlined version of the existing results would be most welcome, it certainly calls for further investigations in order to simplify and to extend the results.

★ **Notes.** Each structure theorem should be accompanied with an existence theorem. Such a theorem on Warfield groups was provided by Hunter–Richman–Walker [1]; its proof requires delicate arguments on cardinals. Conditions on the UK- and Warfield invariants are stated to ensure the existence of a Warfield group for a prescribed nice decomposition basis. It is also shown that a direct sum of infinitely many copies of any local Warfield group is simply presented, and every local Warfield group is the direct sum of a simply presented group and a Warfield group of countable torsion-free rank.

Arnold–Hunter–Richman [1] proved that a finite torsion-free rank summand of a group with decomposition basis also has such a basis. For arbitrary summands, the proof is due to Stanton [3] and Arnold–Hunter–Richman [1]. Hunter–Richman [1] construct a valuated free group G with decomposition bases X, Y such that no subgroup of G is generated both by a subordinate to X and by a subordinate to Y. An important characterization of Warfield groups is due to Hill–Megibben [9]. If the torsion-free rank is > 1, then the existence of an $H(\aleph_0)$-family of nice subgroups is not enough to get a Warfield group, unless the existence of a decomposition basis is taken for granted. Alternately, the notion of niceness can be strengthened to so-called *knice* subgroups; then Warfield groups are characterized as groups admitting $H(\aleph_0)$-families of knice subgroups. In terms of a nice composition series, Moore [1] and Loth [2] establish different kinds of characterization of Warfield groups. In the paper Hill–Megibben [8], there are several interesting results on the existence of automorphisms in Warfield groups, carrying elements to elements with the same height-matrix. In his dissertation, Stanton proves that if A is a Warfield group, then for some torsion T, $A \oplus T$ is a direct sum of groups of torsion-free rank 1.

In a different vein, Warfield [8] considered cancellation for countable mixed groups of finite torsion-free ranks. Results analogous to those in Sect. 10 in Chapter 12 are established. Hill–Ullery [1] gave necessary and sufficient conditions on an isotype subgroup of a local Warfield group to be likewise Warfield; the results were generalized by Megibben–Ullery [1] to the global case. Loth [1] characterized the Pontryagin duals of Warfield groups.

A final comment on Warfield groups is in order. The best approach to their theory is undoubtedly *via* valuated groups as demonstrated by Richman–Walker [5]: a free subgroup with valuation is standing for the entire group. As a consequence, the study boils down to the investigation of valuated free groups. This approach is easy to understand, but not so easy to work with, because there is no canonical way to it, and besides, a less tractable non-abelian category is involved. We did not discuss Warfield groups from this point of view to avoid entering into an unexplored area.

The interested reader's attention is called to a series of papers by R. Göbel, C. Jacoby, K. Leistner, P. Loth, and L. Strüngmann, discussing model theoretic versions of some topics discussed here. E.g. in paper [1] by the last four authors, a classification theorem is proved using cardinal invariants that were deduced from the UK- and Warfield-invariants.

Exercises

(1) Every countably generated group of torsion-free rank 1 is a summand of a simply presented group.
(2) (Hunter–Richman–Walker) Let C be a subgroup of A such that A/C is torsion. Then A and C have the same Warfield invariants.
(3) (Hunter–Richman) Every finite subset Y in a decomposition basis generates a nice subgroup. [Hint: localize, induct on the size of Y using Lemma 4.2 and 4.1.]
(4) Given $\underline{u}$, construct a Warfield group with $w(\underline{u}) = 1$.
(5) (Jarisch–Mutzbauer–Toubassi) Let A be a p-local group such that tA is Σ-cyclic. Then A is Warfield if and only if $A/p^\omega A$ is simply presented and $p^\omega A$ is completely decomposable.

8 The Categories WALK and WARF

We proceed to consider two categories that can help a better understanding of the behavior of mixed groups. Their definition is based on the excellent idea that we have already seen to work effectively: ignoring torsion subgroups, but retaining the information on divisibility in the surviving free subgroup.

Category WALK The category WALK was introduced by E. Walker. Its objects are groups, its morphisms are ordinary group homomorphisms, with the proviso that two maps: $\alpha, \beta : A \to C$ are viewed equal whenever $\mathrm{Im}(\alpha - \beta)$ is a torsion group. In other words, a homomorphism $\alpha : A \to C$ with $\mathrm{Im}\,\alpha \leq tC$ is 0 in this category:

$$\mathrm{Hom}_{WALK}(A, C) = \mathrm{Hom}(A, C)/\,\mathrm{Hom}(A, tC).$$

Apply the Hom functor to the exact sequence $0 \to tC \to C \to C/tC \to 0$ to get $0 \to \mathrm{Hom}(A, tC) \to \mathrm{Hom}(A, C) \to \mathrm{Hom}(A, C/tC)$. This shows that $\mathrm{Hom}_{WALK}(A, C)$ is isomorphic to a subgroup of $\mathrm{Hom}(A, C/tC) = \mathrm{Hom}(A/tA, C/tC)$, i.e. it is a torsion-free group.

It is easily checked that WALK is an additive category with kernels and infinite direct sums. WALK-isomorphism can be described in traditional terms.

Lemma 8.1. *The groups A, C are isomorphic in* WALK, $A \cong_{WALK} C$, *if and only if there exist torsion groups X, Y such that*

$$A \oplus X \cong C \oplus Y.$$

Proof. Observe that in WALK, the injection $A \to A \oplus X$ and the projection $A \oplus X \to A$ are inverse morphisms provided that X is a torsion group. Hence the stated group isomorphism implies that we have $A \cong_{WALK} A \oplus X \cong C \oplus Y \cong_{WALK} C$.

Conversely, assume $A \cong_{WALK} C$. This means that there are homomorphisms: $\alpha : A \to C$ and $\gamma : C \to A$ such that $\alpha\gamma = \mathbf{1}_C + \eta$ and $\gamma\alpha = \mathbf{1}_A + \xi$ where $\eta \in \mathrm{Hom}(C, tC)$, $\xi \in \mathrm{Hom}(A, tA)$. Define G by the pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & tC \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{can}"] \arrow[dl, dashed, "\alpha"'] & 0 \\
0 \arrow[r] & tC \arrow[r] & C \arrow[r, "\mathrm{can}"] & C/tC \cong A/tA \arrow[r] & 0
\end{tikzcd}$$

The map α renders the lower triangle commutative, so the top row splits, thus $G \cong A \oplus tC$. By symmetry, $G \cong C \oplus tA$, completing the proof. $\square$

Actually, there is no loss of generality in assuming $X = Y$ in Lemma 8.1: if they are not isomorphic, then both can be replaced by $(X \oplus Y)^{(\aleph_0)}$.

Category WARF In the category WARF introduced by Warfield, the objects are groups, and a morphism $\alpha : A \to C$ between two objects is an honest homomorphism from a subgroup B of A into C such that

(i) A/B is a torsion group, and
(ii) $\mathbb{H}_A(b) \leq \mathbb{H}_C(\alpha b)$ for all $b \in B$ of infinite order.

The morphisms $\alpha, \beta : A \to C$ are considered equal if they have the same domain, and $\mathrm{Im}(\alpha - \beta)$ is a torsion group.

To be more specific, we give a precise description of Hom_{WARF}. For a subgroup B of A, let $\mathrm{Hom}(A|B, C)$ denote the subgroup of $\mathrm{Hom}(B, C)$ consisting of all homomorphisms $\alpha : B \to C$ (in $\mathcal{A}b$) that do not increase heights, computed in A and C, respectively. It is clear that an injective map $\beta : B' \to B$ induces a group homomorphism $\beta' : \mathrm{Hom}(A|B, C) \to \mathrm{Hom}(A|B', C)$ *via* $\beta'(\alpha) = \alpha\beta$. It follows that

$$\mathrm{Hom}_{WARF}(A, C) \cong \varinjlim \mathrm{Hom}(A|B, C)$$

where the direct limit is taken over all $B \leq A$ for which A/B is torsion.

It is easy to tell what the WARF-isomorphy of two arbitrary mixed groups A and C means: this is just another way of saying that there exist subgroups $B \leq A$ and $D \leq C$ such that $A/B, C/D$ are torsion and there is a height-preserving group isomorphism (i.e., in $\mathcal{A}b$) between B and D.

Note that WARF is an additive category with kernels and infinite sums. It fails to admit cokernels, so it is not an abelian category.

The definition of this category is motivated by the following lemmas.

Lemma 8.2. *Two mixed groups, A and C, of torsion-free rank* 1 *are isomorphic in* WARF *if and only if* $\mathbb{H}(A) \sim \mathbb{H}(C)$.

Proof. For sufficiency, note that the equivalence of height-matrices means that there are $a \in A$ and $c \in C$ such that $\mathbb{H}(a) = \mathbb{H}(c)$. The homomorphisms $\alpha\colon \langle a\rangle \to C$ and $\gamma\colon \langle c\rangle \to A$ (where $\alpha(a) = c,\ \gamma(c) = a$) are clearly inverse morphisms in WARF. Conversely, if there exist morphisms $\alpha\colon A \to C,\ \gamma\colon C \to A$ that are inverse to each other in WARF, then there exist $a \in A, c \in C$ with $\mathbb{H}(a) \le \mathbb{H}(c)$ and $a' \in A$ with $\mathbb{H}(a') \ge \mathbb{H}(c)$. Hence $\mathbb{H}(a) \sim \mathbb{H}(a')$, and the claim follows. □

Lemma 8.3. *A group has a decomposition basis if and only if it is* WARF-*isomorphic to a simply presented group.*

Proof. Suppose $X = \{x_i\}_{i\in I}$ is a decomposition basis of A. There exist simply presented groups C_i of torsion-free rank 1 such that some $c_i \in C_i$ satisfy $\mathbb{H}(c_i) = \mathbb{H}(x_i)$. The group $C = \oplus_{i\in I} C_i$ is then WARF-isomorphic to A *via* the correspondence $c_i \mapsto x_i$.

Conversely, let A be WARF-isomorphic to $C = \oplus_{i\in I} C_i$ where each C_i is simply presented of torsion-free rank 1. Let $Y = \{c_i\}_{i\in I}$ with $c_i \in C_i$ be a decomposition basis of C. By definition, there are subgroups $B \le A$, $D \le C$, and a height-preserving isomorphism $\phi : B \to D$, where A/B, C/D are torsion. Replace Y by a subordinate to get $Y \subset D$. Then $x_i = \phi^{-1}(c_i)$ is a decomposition basis of A. □

Some direct decompositions are unique in WARF.

Theorem 8.4. *Let A be a p-local simply presented group, and*

$$A = \oplus_{i\in I} A_i = \oplus_{i\in I} C_i$$

direct decompositions where A_i, C_i are of torsion-free rank 1. *There is a permutation π of the index set I such that*

$$A_i \cong_{WARF} C_{\pi(i)} \quad \textit{for all } i \in I.$$

Proof. In the local case, the WARF-isomorphism of groups of torsion-free rank 1 means that the associated indicators are equivalent; this follows from Lemma 8.2. For the equivalence class of $\underline{u}$, the number of components A_i (also the number of C_i) with indicators $\sim \underline{u}$ is given by the Warfield-invariant $w_A(\underline{u})$ as stated in Lemma 7.7. (The maximum indicator $\underline{u} = (\infty, \infty, \dots)$ requires a different, but trivial argument.) □

★ **Notes.** The idea of regarding groups equivalent by making them isomorphic *via* adding suitable summands goes back to an old paper by R. Baer. For torsion summands, this was first applied by Rotman–Yen [1] before Walker formalized the idea by defining an honest category.

The WALK- and WARF-isomorphisms are in general different; however, for countably generated J_p-modules of finite torsion-free ranks, they coincide. A category of mixed groups similar to WALK (defined in an earlier paper) was investigated by Yakovlev [3] with the purpose of transferring results on decompositions of torsion-free groups to mixed groups. Franzen–Goldsmith

[1] construct a full embedding of the category of reduced torsion-free groups in WALK to derive new results on mixed groups.

Evidently, when dealing with Warfield groups, there is an apparent advantage in interpreting the results in the category WARF. This simplifies and clarifies the situation, which is good enough motivation for introducing this category. It is safe to say that an additional benefit is that it facilitates to unveil further features of the groups. However, one should not forget that it cannot tell the whole story about the groups themselves. That the category WARF is so useful in the local case is due to the fact that in this category p-local groups of torsion-free rank 1 have local endomorphism rings. In contrast, in the global situation, the endomorphism rings are principal ideal domains, that makes the objects less tractable.

Exercises

(1) Give a detailed proof that WALK is an additive category with infinite direct sums, and that morphisms have kernels, but not necessarily cokernels.
(2) Prove that WARF is an additive category with infinite direct sums.
(3) (Warfield) Show that, in the category WARF, the small objects are the groups of finite torsion-free rank.
(4) (Warfield) Two countable groups of finite torsion-free rank are isomorphic if and only if they are WALK-isomorphic and have the same UK-invariants.

9 Projective Properties of Warfield Groups

We now embark on a brief study of the projective properties of Warfield groups: they can be described as projectives relative to a class of short exact sequences. This is entirely analogous to the balanced-projectivity of totally projective p-groups, discussed in Sect. 5 in Chapter 11.

$\mathbb{H}$-Projectivity An exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ was called $\mathbb{H}$-exact if the induced sequence

$$0 \to A(\mathbb{H}) \xrightarrow{\alpha} B(\mathbb{H}) \xrightarrow{\beta} C(\mathbb{H}) \to 0$$

was exact for all height-matrices $\mathbb{H}$. Accordingly, a group G will be called **$\mathbb{H}$-projective** if it has the projective property with respect to all $\mathbb{H}$-exact sequences; i.e. the map $\mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C)$ induced by β is an epimorphism.

(a) We observe right away that *for a p-group G, $\mathbb{H}$-projectivity is identical with balanced-projectivity.* Indeed, all the maps from G land in the p-components, the sequence of the p-components is balanced-exact, and the rest follows from Theorem 5.9 in Chapter 11.

(b) It is obvious that $\mathbb{H}$-*projective groups share the general properties typical for any class of projective objects, like preservation under arbitrary direct sums and summands.*

A result that is very special for this class of groups is the following.

Theorem 9.1 (Warfield [7]).

(i) *All simply presented groups are* $\mathbb{H}$-*projective.*

(ii) *An exact sequence relative to which every simply presented group* (*of torsion-free rank* 1) *has the projective property is* $\mathbb{H}$-*exact.*

Proof. (i) Owing to Lemma 6.2 it suffices to show that a simply presented group G of torsion-free rank ≤ 1 is $\mathbb{H}$-projective. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an $\mathbb{H}$-exact sequence, and $\phi : G \to C$. The torsion case has been settled by (a) above, so assume $g \in G$ is a generator of infinite order, $\langle g \rangle$ is nice in G, and $G/\langle g \rangle$ is simply presented torsion. By $\mathbb{H}$-exactness, there is a $b \in B$ such that $\beta b = \phi g$ and $\mathbb{H}(b) = \mathbb{H}(\phi g)$. The correspondence $g \mapsto b$ extends to a homomorphism $\gamma : G \to B$. Now $\phi - \beta\gamma$ induces a homomorphism $\phi' \colon G/\langle g \rangle \to C$, so by the torsion case there is some $\delta' : G/\langle g \rangle \to B$ such that $\beta\delta' = \phi'$. Hence if $\delta \colon G \to B$ denotes the canonical map followed by δ', then $\delta + \gamma \colon G \to B$ satisfies $\beta(\gamma + \delta) = \phi$, as desired.

(ii) Suppose that all simply presented groups of torsion-free rank 1 are projective relative to the exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. If $c \in C$ is of infinite order, then let G be simply presented of torsion-free rank 1 whose generator g of infinite order satisfies $\mathbb{H}(g) = \mathbb{H}(c)$; such a G exists by virtue of Sect. 6(c). The correspondence $g \mapsto c$ extends to a homomorphism $\gamma : G \to C$, and hypothesis implies the existence of a map $\phi : G \to B$ satisfying $\beta\phi = \gamma$. It is clear that $\mathbb{H}(\phi g) = \mathbb{H}(c)$, which shows that β maps $B(\mathbb{H})$ onto $C(\mathbb{H})$. If $c \in C[p]$, then choose a simply presented G of torsion-free rank 1 whose generator g satisfies $h_p(g) = h_p(c)$. The same argument leads to the conclusion that $\beta(p^\sigma B[p]) = p^\sigma C[p]$. An appeal to Lemma 4.6 completes the proof. □

$\mathbb{H}$-Projectivity and Warfield Groups To show that there are enough $\mathbb{H}$-projective groups, we prove:

Theorem 9.2 (Warfield [7]). *Every group A can be embedded in an* $\mathbb{H}$-*exact sequence* $0 \to K \to G \xrightarrow{\alpha} A \to 0$ *where G is* $\mathbb{H}$-*projective* (*simply presented*).

Proof. Let $f : A \to X$ be a bijection between A and a set X. Define the group G to be generated by X, subject exactly to those relations that hold between the corresponding elements in A and involve at most two elements. Then G is simply presented, and f^{-1} induces an epimorphism $\alpha : G \to A$. We let $K = \operatorname{Ker}\alpha$. To show that the sequence at hand is $\mathbb{H}$-exact, observe that $h_p(f(a)) = h_p(a)$ for all $a \in A$ and for all p, as it follows by a straightforward induction using the fact that a relation $pb = a$ between $a, b \in A$ implies $pf(b) = f(a)$ in G. Hence we conclude that $\mathbb{H}(f(a)) = \mathbb{H}(a)$ for all $a \in A$, and thus $\alpha(G(\mathbb{H})) = A(\mathbb{H})$. Furthermore, the generator $f(a)$ in G has the same height as a in A, and if $o(a) = p$, then the order of

$f(a)$ in G is also p. Consequently, $\alpha(p^\sigma G[p]) = p^\sigma A[p]$ holds for all p and σ, so the sequence is $\mathbb{H}$-exact (Lemma 4.6). □

We are now prepared for an important characterization of $\mathbb{H}$-projective groups.

Theorem 9.3 (Warfield [7]). *A group is $\mathbb{H}$-projective if and only if it is a Warfield group.*

Proof. In view of Theorem 9.1 and the definition of Warfield groups, the 'if' part is evident. It remains to show that an $\mathbb{H}$-projective group G is Warfield. The proof is standard. By Theorem 9.2, G embeds in an $\mathbb{H}$-exact sequence $0 \to K \to B \to G \to 0$ where B is simply presented. $\mathbb{H}$-projectivity forces splitting, thus G is isomorphic to a summand of B, i.e. G is Warfield. □

Balanced-Projectivity Example 6.5 makes it evident that no good classification is expected for global Warfield groups. However, there is a more tractable subclass that is more conveniently defined *via* its projective property, and that includes both the torsion and the torsion-free simply presented groups. And more importantly, luckily, this subclass does admit a satisfactory classification.

The exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ was defined to be balanced-exact if the induced sequence

$$0 \to A(\chi) \xrightarrow{\alpha} B(\chi) \xrightarrow{\beta} C(\chi) \to 0$$

was exact for all characteristics χ. A group G that has the projective property for all balanced-exact sequences will be called **balanced-projective**.

Lemma 9.4. *Balanced-projective groups are $\mathbb{H}$-projective.*

Proof. This follows from the fact that $\mathbb{H}$-exact sequences are balanced-exact. (Theorem 9.10 will show that the reverse implication fails.) □

Example 9.5. For torsion as well as for torsion-free groups, balanced-projectivity as defined above coincides with balanced-projectivity as defined in previous chapters. This is evident for torsion-free groups, and follows from Theorem 5.9 in Chapter 11 for torsion groups.

Example 9.6. Divisible groups are balanced-projective.

Given a characteristic $\chi = (\sigma_2, \sigma_3, \dots, \sigma_p, \dots)$ (where for each prime p the entry σ_p is an ordinal or the symbol ∞), let $\mathbb{Z}_\pi$ denote the localization of $\mathbb{Z}$ at the set π of primes p for which $\sigma_p \neq \infty$ (this notation for π will be used in the balance of this section). We define a group M_χ as a group of torsion-free rank 1 with an element x such that $M_\chi(\chi) = \mathbb{Z}_\pi x$, $M_\chi/\mathbb{Z}_\pi x$ is a simply presented reduced torsion group, and $\mathbb{H}(x)$ is the "gapless" height-matrix

$$\mathbb{H}_\chi = \begin{pmatrix} \sigma_2 & \sigma_2+1 & \dots & \sigma_2+k & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_p & \sigma_p+1 & \dots & \sigma_p+k & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \|\sigma_p + k\|.$$

Lemma 9.7. *The totally projective torsion groups as well as the groups M_χ are balanced-projective.*

Proof. It suffices to verify the claim for M_χ. The proof of Theorem 9.1 applies almost *verbatim*. □

The groups M_χ are called **χ-elementary balanced-projective**. Thus a Warfield group of torsion-free rank 1 is elementary balanced-projective if and only if its height-matrix is equivalent to a gapless matrix.

Example 9.8. (a) If $\chi = (\sigma_2, \sigma_3, \dots, \sigma_p, \dots)$ where each σ_p is either a finite ordinal or the symbol ∞, then M_χ is a rational group.

(b) If all $\sigma_p = 0$ with the exception of a single σ_q (which is an arbitrary ordinal), then we get the q-local Nunke group of length σ_q.

Characterization of Balanced-Projective Groups The existence of balanced-projective resolutions follows easily by standard arguments.

Lemma 9.9 (Warfield [7]). *For every group A, there is a balanced-exact sequence $0 \to K \to G \xrightarrow{\gamma} A \to 0$ where G is a direct sum of a totally projective torsion group and elementary balanced-projective mixed groups.*

Proof. By Proposition 5.8 in Chapter 11, the torsion subgroup tA admits a balanced-projective resolution $0 \to K' \to G' \xrightarrow{\gamma'} tA \to 0$ where G' is totally projective torsion. For each $a \in A$ of infinite order, select an elementary balanced projective group $G_a = M_{\chi(a)}$ along with a homomorphism $\gamma_a : G_a \to A$, mapping a generator of the infinite cyclic $G_a(\chi(a))$ upon a. We continue by setting $G = G' \oplus \oplus_{a \in A \setminus tA} G_a$ and $\gamma = \gamma' \oplus \oplus_a \gamma_a$, to obtain an exact sequence as stated above, where γ is surjective and does not increase heights.

It remains to verify balancedness. We argue that Lemma 4.5(i) is fairly obvious from the construction, while in Lemma 4.5(ii) only the torsion part is involved for which G' was selected so as to satisfy balancedness. □

Hence we can derive at once:

Theorem 9.10. *A (mixed) group is balanced-projective exactly if it is a summand of a direct sum of a totally projective torsion group and χ-elementary balanced-projective groups for various characteristics χ.* □

★ **Notes.** $\mathbb{H}$-projectivity was studied by Warfield [4] under the name *sequentially-pure-projectivity*. For any mixed group, $\mathbb{H}$-projective dimension makes sense. They were studied by Megibben–Ullery [2] who gave upper bounds for the $\mathbb{H}$-projective dimensions of certain groups.

The theory of balanced-projective groups was developed by Warfield [7]; the global case was investigated by Wick [2]. Warfield introduced the global heights $h = \prod_p p^{\sigma_p}$, and used the functorial notation hA for our $A(\chi)$. Lane [1] characterized p-local balanced-projective groups in terms of $H(\aleph_0)$-families of so-called K-nice subgroups, and in his paper [2], he obtained definitive results on the balanced-projective dimension of p-local groups. Lane–Megibben [1] extended these results to the global case.

Exercises

(1) (Warfield) A group A is WARF-isomorphic to an elementary balanced-projective group exactly if it is of torsion-free rank 1, and contains an element a such that $\chi(na) = n\chi(a)$ for all $n \in \mathbb{N}$.
(2) (Warfield) A is WARF-isomorphic to a direct sum of an elementary balanced-projective groups if and only if it contains a free subgroup $F = \oplus\langle x_i \rangle$ such that A/F is torsion, and for all finite subsets $\{x_1, \ldots, x_k\}$ and for all p

$$h_p(n_1x_1 + \cdots + n_kx_k) = \min\{h_p(n_1x_1), \ldots, h_p(n_kx_k)\}.$$

(3) For every characteristic χ there exists a χ-elementary balanced-projective group. [Hint: start with $\mathbb{Z}_\pi x$ and furnish x with the correct heights.]
(4) If A is a χ-elementary balanced-projective group, then $\mathrm{End}_{WARF} A \cong \mathbb{Z}_\pi$.

Problems to Chapter 15

PROBLEM 15.1. Using the theory of totally projective p-groups and height-matrices, develop a theory of mixed groups A such that $t(A)$ is totally projective, and $A/t(A)$ is completely decomposable (or divisible).

PROBLEM 15.2. Are two mixed groups of torsion-free rank 1 necessarily isomorphic if their endomorphism rings (automorphism groups) are isomorphic and the torsion-free parts are $\cong \mathbb{Q}$?

PROBLEM 15.3. Are there interesting groups that can be presented such that every generator occurs in at most n relations for a fixed $n \in \mathbb{N}$?

Examples for $n = 2$ include divisible groups and completely decomposable torsion-free groups.

Chapter 16
Endomorphism Rings

Abstract With an abelian group A one associates the ring End A of its endomorphisms. This is an associative ring with 1 which frequently reflects several relevant features of the group. Information about direct decompositions is certainly stored in this ring. It is quite challenging to unveil hidden relations between a group and its endomorphism ring.

Quite a lot of information is available for the endomorphism rings of p-groups. A celebrated theorem by Baer and Kaplansky shows that torsion groups with isomorphic endomorphism rings ought to be isomorphic. Moreover, the endomorphism rings of separable p-groups can be characterized ring-theoretically. However, in the torsion-free case, we can offer nothing anywhere near as informative or complete as for torsion groups. As a matter of fact, on one hand, there exists a large variety of non-isomorphic torsion-free groups (even of finite rank) with isomorphic endomorphism rings, and on the other hand, endomorphism rings of torsion-free groups seem to be quite general: every countable rank reduced torsion-free ring with identity appears as an endomorphism ring of a torsion-free group. The situation is not much better even if we involve the finite topology. The mixed case is of course more difficult to survey.

Though the problem concerning the relations between group and ring properties has attracted much attention, our current knowledge is still far from being satisfactory. The main obstacle to developing a feasible in-depth theory is probably the lack of correspondence between relevant group and relevant ring properties. However, there is a great variety of examples of groups with interesting endomorphism rings, and we will list a few which we think are more interesting. In some cases we have to be satisfied with just stating the results in order to avoid tiresome ring-theoretical arguments. In some proofs, however, we had no choice but to refer to results on rings which can be found in most textbooks on graduate algebra. There are very good surveys on endomorphism rings by Russian algebraists, e.g. Krylov–Tuganbaev [1], and especially, the book by Krylov–Mikhalev–Tuganbaev [KMT].

In this chapter, we require some, but no more than a reasonable acquaintance with standard facts on associative rings.

1 Endomorphism Rings

Rings of Endomorphisms It is a familiar fact that the endomorphisms $\alpha, \beta, \dots$ of an abelian group A form a ring under the addition and multiplication of homomorphisms:

$$(\alpha + \beta)a = \alpha a + \beta a \quad \text{and} \quad (\alpha\beta)a = \alpha(\beta a) \quad (\forall a \in A).$$

L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-319-19422-6_16

The **endomorphism ring** $\operatorname{End} A$ of A is an associative ring with identity. By abuse of notation, we will use the same symbol for the endomorphism ring and for its additive group: the **endomorphism group**.

Example 1.1.

(a) If $A = \langle e \rangle \cong \mathbb{Z}$, then every $\alpha \in \operatorname{End} A$ is completely determined by αe. Since e is a free generator, every correspondence $e \mapsto ne$ for any $n \in \mathbb{Z}$ extends to an endomorphism. The operations with endomorphisms are like with integers, so $\operatorname{End} A \cong \mathbb{Z}$ (ring isomorphism).
(b) A similar argument shows that if A is a cyclic group of order m, then we have $\operatorname{End} A \cong \mathbb{Z}/m\mathbb{Z}$ (as rings).

Example 1.2.

(a) From Example 1.4 in Chapter 7 we obtain the isomorphism $\operatorname{End}(\mathbb{Z}(p^\infty)) \cong J_p$.
(b) $\operatorname{End}(\mathbb{Q}/\mathbb{Z}) \cong \prod_p J_p = \tilde{\mathbb{Z}}$. This follows from (a).

Example 1.3. Example 1.6 in Chapter 7 shows that $\operatorname{End} J_p \cong J_p$ for every prime p.

Example 1.4. Let R denote a rational group, $1 \in R$. Here again, an endomorphism α is fully determined by $\alpha 1 = r \in R$, so α is simply a multiplication by the rational number $r \in R$. As endomorphisms respect divisibility, $\dot{r}$ can be an endomorphism of R only if every prime factor p of the denominator of r (in its reduced form) satisfies $pR = R$. That the converse is also true is seen immediately. Thus $\operatorname{End} R$ is a subring of $\mathbb{Q}$ whose type is the largest idempotent type $\leq \mathbf{t}(R)$, i.e. $\mathbf{t}(R) : \mathbf{t}(R)$.

We continue with a few elementary observations.

(A) *A group isomorphism $\phi : A \to C$ induces a ring isomorphism $\phi^* : \operatorname{End} A \to \operatorname{End} C$ via $\phi^* : \alpha \mapsto \phi\alpha\phi^{-1}$.*
(B) *Suppose $A = B \oplus C$ with $\epsilon : A \to B$ the projection map. Then the identification $\operatorname{End} B = \epsilon(\operatorname{End} A)\epsilon$ can be made.* Indeed, for $\alpha \in \operatorname{End} A$, $\epsilon\alpha\epsilon \in \operatorname{End} B$, while if $\beta \in \operatorname{End} B$, then $\beta = \epsilon\beta\epsilon$ may be regarded as an element in $\operatorname{End} A$.
(C) *If ϵ is a central idempotent in $\operatorname{End} A$, then ϵA is a fully invariant summand of A.*
(D) *If $A = A_1 \oplus \cdots \oplus A_n$ is a direct decomposition with fully invariant summands, then*

$$\operatorname{End} A \cong \operatorname{End} A_1 \oplus \cdots \oplus \operatorname{End} A_n.$$

If the summands are not fully invariant, then we get only a matrix representation, see Proposition 1.14.

An idempotent $\epsilon \neq 0$ is said to be **primitive** if it cannot be written as a sum of two non-zero orthogonal idempotents. The following claim is rather obvious in view of (D).

(E) *For an idempotent $\epsilon \neq 0$ of $\operatorname{End} A$, the summand ϵA is indecomposable if and only if ϵ is a primitive idempotent.*

We have already made frequent use of the fact that direct decompositions correspond to idempotent endomorphisms. This interplay between direct decompositions and endomorphisms is constantly used. At this point, we insert the following lemma that will be indispensable throughout.

Lemma 1.5. *There is a bijection between the finite direct decompositions $A = A_1 \oplus \cdots \oplus A_n$ of a group A, and decompositions of* $\operatorname{End} A$ *into finite direct sums of left ideals,*

$$\operatorname{End} A = \mathsf{L}_1 \oplus \cdots \oplus \mathsf{L}_n. \tag{16.1}$$

If $A_i = \epsilon_i A$ with pairwise orthogonal idempotents ϵ_i, then $\mathsf{L}_i = (\operatorname{End} A)\epsilon_i$ for $i = 1, \ldots, n$.

Proof. Suppose $A = \epsilon_1 A \oplus \cdots \oplus \epsilon_n A$ with mutually orthogonal idempotents ϵ_i. The well-known Peirce decomposition of $\operatorname{End} A$ yields $\operatorname{End} A = (\operatorname{End} A)\epsilon_1 \oplus \cdots \oplus (\operatorname{End} A)\epsilon_n$. Conversely, if (16.1) holds with left ideals L_i of $\operatorname{End} A$, then—as is readily checked—we have $\mathsf{L}_i = (\operatorname{End} A)\epsilon_i$ where ϵ_i is the ith coordinate of the identity of $\operatorname{End} A$. These ϵ_i are orthogonal idempotents with sum 1, hence $A = \epsilon_1 A \oplus \cdots \oplus \epsilon_n A$ follows at once. It is pretty clear that the indicated correspondence is a bijection. □

As far as isomorphic summands are concerned, the basic information is recorded in the next lemma.

Lemma 1.6. *Suppose that B_1, B_2 are summands of A, corresponding to idempotents $\epsilon_1, \epsilon_2 \in \mathsf{E} = \operatorname{End} A$. The following are equivalent:*

(i) $B_1 \cong B_2$;
(ii) *there exist $\alpha, \beta \in \mathsf{E}$ such that $\alpha = \epsilon_1\alpha\epsilon_2, \beta = \epsilon_2\beta\epsilon_1$, and $\alpha\beta = \epsilon_1$, $\beta\alpha = \epsilon_2$;*
(iii) $\mathsf{E}\epsilon_1 \cong \mathsf{E}\epsilon_2$ *as left* E*-modules;*
(iv) $\epsilon_1\mathsf{E} \cong \epsilon_2\mathsf{E}$ *as right* E*-modules.*

Proof.

(i) ⇒ (iii) Let $\gamma : B_1 \to B_2$ and $\delta : B_2 \to B_1$ be inverse isomorphisms. Thus $\gamma\delta = \epsilon_2$ and $\delta\epsilon_2 = \delta$ imply that $\mathsf{E}\delta = \mathsf{E}\epsilon_2$, because each of δ and ϵ_2 is contained in the left E-ideal generated by the other. Multiplication by δ and γ yield $\mathsf{E}\epsilon_1 \to \mathsf{E}\epsilon_1\delta \to \mathsf{E}\delta = \mathsf{E}\epsilon_2 \to \mathsf{E}\delta\gamma = \mathsf{E}\epsilon_1$. The composite is multiplication by ϵ_1 which is the identity on $\mathsf{E}\epsilon_1$, so multiplication by γ is an isomorphism.

(iii) ⇔ (ii) Choose an isomorphism $\phi : \mathsf{E}\epsilon_1 \to \mathsf{E}\epsilon_2$, and let $\phi(\epsilon_1) = \alpha$, $\phi^{-1}(\epsilon_2) = \beta$ with $\alpha, \beta \in \mathsf{E}$. Then $\alpha = \epsilon_1\alpha\epsilon_2, \beta = \epsilon_2\beta\epsilon_1$, and also $\phi^{-1}\phi : \epsilon_1 \mapsto \alpha = \alpha\epsilon_2 \mapsto \alpha\beta$, thus $\alpha\beta = \epsilon_1$; similarly, $\beta\alpha = \epsilon_2$. Conversely, if $\alpha, \beta \in \mathsf{E}$ are as stated, then $\mathsf{E}\alpha = \mathsf{E}\epsilon_2$. Thus $\epsilon_1 \mapsto \alpha$ induces an epimorphism $\phi : \mathsf{E}\epsilon_1 \to \mathsf{E}\epsilon_2$. ϕ is an isomorphism, because if $\phi(\rho\epsilon_1) = \rho\alpha = 0$ for some $\rho \in \mathsf{E}$, then also $\rho\epsilon_1 = \rho\alpha\beta = 0$.

(ii) ⇔ (iv) Condition (ii) is left-right symmetric, so by the last proof (ii) is also equivalent to $\epsilon_1\mathsf{E} \cong \epsilon_2\mathsf{E}$.

(iv) ⇒ (i) Let $\phi : \epsilon_1\mathsf{E} \to \epsilon_2\mathsf{E}$ be an isomorphism. Define a map $B_1 \to B_2$ as $\epsilon_1(a) \to \phi(\epsilon_1)(a)$ $(a \in A)$. It has the obvious inverse, so it has to be an isomorphism. □

In the next result we are referring to the completion $\tilde{A}$ and the cotorsion hull $A^\bullet$ of a group A.

Proposition 1.7.

(i) *Let A be a group with $A^1 = 0$ (i.e., Hausdorff in the $\mathbb{Z}$-adic topology). For every $\eta \in \operatorname{End} A$, there is a unique $\tilde{\eta} \in \operatorname{End} \tilde{A}$ such that $\tilde{\eta} \restriction A = \eta$.*
(ii) *For a reduced torsion group T, there is a natural isomorphism $\operatorname{End} T \cong \operatorname{End} T^\bullet$ between the endomorphism rings.*

Proof.

(i) The hypothesis $A^1 = 0$ allows us to regard A as a pure subgroup of $\tilde{A}$. By the pure-injectivity of $\tilde{A}$, η can be extended to an $\tilde{\eta} : \tilde{A} \to \tilde{A}$ which must be unique in view of the density of A in $\tilde{A}$.
(ii) The exact sequence $0 \to T \to T^\bullet \to D \to 0$ with torsion-free divisible D induces the exact sequence $0 = \operatorname{Hom}(D, T^\bullet) \to \operatorname{Hom}(T^\bullet, T^\bullet) \to \operatorname{Hom}(T, T^\bullet) \to \operatorname{Ext}(D, T^\bullet) = 0$. Hence and from the obvious $\operatorname{Hom}(T, T^\bullet) \cong \operatorname{Hom}(T, T)$ the claim is evident. □

Inessential Endomorphisms When studying the endomorphisms of reduced p-groups, we always have to deal with endomorphisms to and from cyclic summands. These as well as the small endomorphisms are 'inessential' endomorphisms: they do not reveal much about the group structure. The relevant information about the group encoded in the endomorphism ring is actually in the other endomorphisms. Also, certain torsion-free groups (like separable groups) admit endomorphisms that provide hardly any information about group. The idea of formalizing this phenomenon is due to Corner–Göbel [1]. We define this concept in the local case.

Let A be a reduced p-local group with $A^1 = 0$, and B a basic subgroup of A. Evidently, $B \le A \le \tilde{B}$, and every $\eta \in \operatorname{End} A$ extends uniquely to an $\tilde{\eta} \in \operatorname{End} \tilde{B}$. Now $\eta \in \operatorname{End} A$ is called **inessential** if $\tilde{\eta}(\tilde{B}) \le A$. The inessential endomorphisms form an ideal of $\operatorname{End} A$, denoted $\operatorname{Ines} A$.

Example 1.8. Let $A \cong (J_p)^{(\mathbb{N})}$. Thus $B \cong (\mathbb{Z}_{(p)})^{(\mathbb{N})}$ and $\tilde{B} < (J_p)^{\mathbb{N}}$. In this case, $\operatorname{Ines} A$ consists of those $\eta \in \operatorname{End} A$ for which $\operatorname{Im} \eta$ is contained in a finite direct sum of the J_p.

Example 1.9. Let A be a separable p-group, and B its basic subgroup. Now the ideal $\operatorname{Ines} A$ coincides with set of the small endomorphisms (Sect. 3 in Chapter 7).

If $\eta \in \operatorname{End}_s A$, then for every $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $\eta(p^n A[p^k]) = 0$. Then also $\tilde{\eta}(p^n \tilde{B}[p^k]) = 0$, thus $\tilde{\eta}$ is also small. By Sect. 3(E) in Chapter 7, $B + \operatorname{Ker} \tilde{\eta} = \tilde{B}$, so $\tilde{\eta}(\tilde{B}) = \tilde{\eta}(B) = \eta(B) \le A$. Thus $\operatorname{End}_s A \le \operatorname{Ines} A$.

Suppose η is not small, i.e. for some $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$, $\eta(p^n A[p^k]) \neq 0$. We can select independent elements $a_i \in A$ of orders $\le p^k$ and of increasing heights n_i such that $\eta a_i \neq 0$. We may, in addition, assume that the elements $\eta(a_i)$ are also independent. Then the sum $c = \sum_{i<\omega} a_i$ converges in $\tilde{B}$, but $\tilde{\eta}(c) \neq 0$. Consequently, $\operatorname{Ines} A \le \operatorname{End}_s A$, and so $\operatorname{Ines} A = \operatorname{End}_s A$.

Annihilator Ideals in End A There is a fundamental correspondence between certain subgroups of a group, and certain left ideals of its endomorphism ring.

For a subgroup G of the group A, we set

$$G^{\perp} = \{\eta \in \operatorname{End} A \mid \eta(g) = 0 \ \forall g \in G\},$$

and for a left ideal L of $\operatorname{End} A$, we define

$$\mathsf{L}^{\perp} = \{a \in A \mid \eta(a) = 0 \ \forall \eta \in \mathsf{L}\}.$$

Evidently, $G^{\perp}$ is a left ideal in $\operatorname{End} A$, called **annihilator left ideal**, and $\mathsf{L}^{\perp}$ is a subgroup of A, called **kernel subgroup**.

(a) For every subgroup $G \leq A$, we have $G^{\perp\perp\perp} = G^{\perp}$, and $\mathsf{L}^{\perp\perp\perp} = \mathsf{L}^{\perp}$ for every left ideal L in $\operatorname{End} A$.
(b) A subgroup $G \leq A$ is a kernel subgroup exactly if $G^{\perp\perp} = G$, and a left ideal L is an annihilator left ideal if and only if $\mathsf{L}^{\perp\perp} = \mathsf{L}$.
(c) The correspondences $G \mapsto G^{\perp}$ and $\mathsf{L} \mapsto \mathsf{L}^{\perp}$ are inclusion reversing inverse maps between the set of kernel subgroups of A and the set of annihilator left ideals of $\operatorname{End} A$.

Example 1.10.

(a) Consider the group $\mathbb{Z}(p^{\infty})$ and its endomorphism ring J_p. The annihilator ideals are $p^n J_p$, and $\mathbb{Z}(p^n)(\leq \mathbb{Z}(p^{\infty}))$ are the kernel subgroups.
(b) Let $A \cong \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$, an n-dimensional $\mathbb{Q}$-vector space. Then $\mathsf{E} = \operatorname{End} A$ is isomorphic to the $n \times n$-matrix ring over $\mathbb{Q}$. Every subspace is a kernel subgroup and every left ideal an annihilator.

Finite Topology Matrix representation of linear transformations is an important issue in linear algebra. For groups it is possible to establish a similar, though much less informative, representation of endomorphisms of direct sums. Since we do not wish to restrict our study to finite direct decompositions, we need to introduce a natural topology in endomorphism rings.

Actually, endomorphism rings admit various topologies defined in terms of the underlying groups. They play an increasingly important role in the study of groups (also modules, rings, etc.). We will discuss the most significant topology, the so-called **finite topology** that was introduced into the theory of abelian groups by Szele [8].

The terminology comes from the fact that the open neighborhoods are defined in terms of finite subsets of the group: every finite subset X in A defines an open set about $\alpha \in \operatorname{End} A$, viz.

$$U_X(\alpha) = \{\eta \in \operatorname{End} A \mid \eta x = \alpha x \ \ \forall x \in X\}.$$

It is clear that $U_X(\alpha) = \cap_{x \in X} U_x(\alpha)$; also, $U_X(\alpha) = \alpha + U_X(0)$ for each $\alpha \in \operatorname{End} A$. Thus, the finite topology can more conveniently be defined with the aid of a subbase of neighborhoods of 0, by the open sets $U_X = \{\eta \in \operatorname{End} A \mid \eta X = 0\}$, taken for all finite subsets $X \leq A$. The finite topology is evidently Hausdorff, and moreover, it is linear: the open sets U_X are left ideals in $\operatorname{End} A$. As a consequence, the continuity of the addition in $\operatorname{End} A$ is immediate. Moreover, we can state:

Theorem 1.11. *The endomorphism ring* End$A$ *of a group A is a complete topological ring in the finite topology.*

Proof. To prove that ring multiplication is continuous, let $\alpha, \beta \in \operatorname{End} A$, and let $\alpha\beta + U_X$ be an open neighborhood of $\alpha\beta$. Since U_X is a left ideal and since $U_{\beta X}\beta \leq U_X$, the desired continuity follows from

$$(\alpha + U_{\beta X})(\beta + U_X) \subseteq \alpha\beta + U_{\beta X}\beta + U_X \subseteq \alpha\beta + U_X.$$

Therefore, End$A$ is a topological ring.

To verify completeness, let $\{\alpha_X\}_{X \in I}$ denote a Cauchy net where the index set I (the set of finite subsets of A) is partially ordered by inclusion. Reminder: the Cauchy net satisfies: given $X \in I$, $\alpha_Y - \alpha_Z \in U_X$ holds for all $Y, Z \in I$ containing some $X_0 \in I$; here $X \subseteq X_0$ may be assumed, so that $\alpha_Y x = \alpha_Z x$ $(\forall x \in X)$ for large Y, Z. Therefore, the common value of the $\alpha_Y x$ for large Y defines an element $x' \in A$, and it is readily seen that the assignment $\alpha : x \mapsto x'$ is an endomorphism of A. This is the limit of our Cauchy net, as is evident from $\alpha - \alpha_Y \in U_X$ for all $Y \supset X_0$. □

Observe that the factor group $(\operatorname{End} A)/U_X$ is isomorphic to a subgroup of a direct sum of copies of A, because U_X is the kernel of the homomorphism $\operatorname{End} A \to A^X$ defined as $\theta \mapsto (\theta x_1, \dots, \theta x_n)$, where $X = \{x_1, \dots, x_n\}$.

The finite topology raises several questions. An immediate one is whether or not the finite topology can be discrete for an infinite group.

Theorem 1.12 (Arnold–Murley [1]). *The endomorphism ring of a group A is discrete in the finite topology only if A is self-small. The converse holds for countable* End$A$.

Proof. Suppose A is not self-small, so there exists a map $\phi : A \to \oplus_{i<\omega} A_i$ where $A_i \cong A$ and $\pi_i\phi \neq 0$ for all i; here π_i denotes the ith projection. Define $G_n = \{a \in A \mid \pi_i\phi(a) = 0 \ \forall i \geq n\}$, an increasing chain of proper subgroups with union A. Then $\{\eta \in \operatorname{End} A \mid \eta(G_n) = 0\} \neq 0$ for all $n \leq \omega$. Any finite subset X of A is contained in some G_n, so $U_X \neq 0$. Thus no open set in the finite topology is 0, and End$A$ is not discrete in the finite topology.

Conversely, suppose that End$A$ is not discrete in the finite topology. Let $U_0 > \dots > U_n > \dots$ be a strictly descending chain of neighborhoods in End$A$; if End$A$ is countable, then there is no harm in assuming that $\cap_{n<\omega} U_n = 0$ as well. Choose $\eta_n \in U_n$ such that $\eta_n \notin U_{n+1}$. Define a homomorphism $\phi : A \to \oplus_{i<\omega} A_i$ with $A_i \cong A$ by setting $\phi = \sum_{n<\omega} \eta_n$. This means that A is not self-small. □

Example 1.13.

(a) The finite topology on the endomorphism ring of A is discrete if A is finite, or torsion-free of finite rank, or a rigid group, but not discrete if A is an infinite p-group.
(b) J_p is self-small, but its endomorphism ring is not discrete in the finite topology. The same holds for torsion-complete p-groups with finite UK-invariants.

When the Finite Topology is Compact Compactness being always of particular interest, let us turn our attention to the question as to when End$A$ is compact in the finite topology.

Before stating the relevant result, consider the fully invariant subgroup $O_x = \{\theta x \mid \theta \in \operatorname{End} A\}$ of A which we will call the **orbit of** $x \in A$. The evaluation map $\theta \mapsto \theta x$ is a group homomorphism of $\operatorname{End} G$ onto O_x whose kernel is U_x. This leads to the group isomorphism $(\operatorname{End} A)/U_x \cong O_x$ (which is moreover an $\operatorname{End} A$-module map).

Proposition 1.14. *The endomorphism ring* $\operatorname{End} A$ *of a group* A *is compact in the finite topology if and only if* A *is a torsion group whose* p*-components are finitely cogenerated.*

Proof. Knowing that $\operatorname{End} A$ is complete in the finite topology, for the compactness of $\operatorname{End} A$ it is necessary and sufficient that all neighborhoods U_x be of finite indices. This is the case exactly if the orbits are finite.

First, suppose $\operatorname{End} A$ is compact. As $\langle x \rangle$ is a subgroup of O_x, A ought to be a torsion group. By Corollary 2.3 in Chapter 5, every non-zero p-component A_p of A contains a cocyclic summand; let C_p be one of minimal order in A_p. The orbit of its cogenerator is the socle of A_p, so it must be finite. Thus A_p is finitely cogenerated (hence a finite direct sum of cocyclic groups).

Conversely, if $A = \oplus_p A_p$ with finitely cogenerated A_p, then for every $n \in \mathbb{N}$, $A[n]$ is finite. But $x \in A[n]$ implies $O_x \leq A[n]$, thus all U_x are of finite indices in $\operatorname{End} A$ (and the same holds for all neighborhoods U_X). Consequently, $\operatorname{End} A$ is compact in the finite topology. $\square$

Though the endomorphism ring of a p-group is rarely algebraically compact as a ring, we can still claim that its endomorphism group is algebraically compact. In fact, from Theorem 2.1 in Chapter 7 we know that $\operatorname{Hom}(A, *)$ is algebraically compact whenever A is a p-group. The invariants can be computed by making use of theorems by Pierce [1].

Matrix Representations of Endomorphisms Once we have a ring topology in $\operatorname{End} A$, it makes sense to form convergent infinite sums. An infinite sum $\sum_{i \in I} \alpha_i$ with $\alpha_i \in \operatorname{End} A$ is **convergent** if, for each $x \in A$, almost all $\alpha_i x = 0$, in which case $\alpha \in \operatorname{End} A$ is its limit where $\alpha x = \sum_i \alpha_i x$ for $x \in A$. A matrix $\|\alpha_{ji}\|$ with entries in $\operatorname{End} A$ is said to be **column-convergent** if for each column i, the sum $\sum_j \alpha_{ji}$ is convergent.

Suppose $A = \oplus_{i \in I} A_i$ is a (finite or infinite) direct sum, and ϵ_i are the associated projections, viewed as mutually orthogonal idempotents in $\operatorname{End} A$. Every $a \in A$ can be written as $a = \sum_i \epsilon_i a$ where almost all terms vanish. For $\alpha \in \operatorname{End} A$, we then have $\alpha a = \sum_i \alpha \epsilon_i a = \sum_{i,j} (\epsilon_j \alpha \epsilon_i) a$. In this way, with every $\alpha \in \operatorname{End} A$ we associate an $I \times I$-matrix $\|\alpha_{ji}\|$ where $\alpha_{ji} = \epsilon_j \alpha \epsilon_i$. If $\|\beta_{ji}\|$ with $\beta_{ji} = \epsilon_j \beta \epsilon_i$ is the matrix associated with $\beta \in \operatorname{End} A$, then the matrices associated with $\alpha - \beta$ and $\alpha\beta$ are precisely the difference matrix $\|\alpha_{ji} - \beta_{ji}\|$ and the product matrix $\| \sum_k \alpha_{jk} \beta_{ki} \|$, respectively. We conclude that $\phi \colon \alpha \to \|\alpha_{ji}\|$ is a ring homomorphism whose kernel is evidently 0.

For a fixed index i, $\alpha \epsilon_i a = \sum_j \alpha_{ji} a$ converges for every $a \in A$, indicating that the matrix $\|\alpha_{ji}\|$ is column-convergent. Conversely, if a matrix with entries $\alpha_{ji} \in \epsilon_j (\operatorname{End} A) \epsilon_i$ is column-convergent, then it must come from an $\alpha \in \operatorname{End} A$, namely, from $\alpha = \sum_{i,j} \alpha_{ji}$. This is a convergent sum, because for each $a \in A$, the sum $\sum_{i,j} \alpha_{ji} a$ is finite. If $\epsilon_j(\operatorname{End} A)\epsilon_i$ is identified with $\operatorname{Hom}(A_i, A_j)$, then we can state our findings as follows.

Proposition 1.15. *Let $A = \oplus_{i \in I} A_i$ be a direct sum decomposition of A. Then* $\operatorname{End} A$ *is isomorphic to the ring of all column-convergent $I \times I$-matrices $\|\alpha_{ji}\|$ where $\alpha_{ji} \in$* $\operatorname{Hom}(A_i, A_j)$. □

Needless to say, if the A_i are small objects, then every column contains but a finite number of non-zero entries, and if the group is a finite direct sum, then its endomorphism ring will be the full matrix ring with the indicated entries.

Example 1.16. Let $A = \oplus_{i \in I} \langle a_i \rangle$ be a free group. In the matrix representation of $\operatorname{End} A$, the entries are integers, and in each column, almost all entries are 0.

Example 1.17. If $D = \oplus_{i \in I} D_i$ is a torsion-free divisible group with $D_i \cong \mathbb{Q}$, then we are in the same situation as in the preceding example, the only difference is that the entries of the matrix can be arbitrary rational numbers.

Example 1.18. Let $A = A_0 \oplus (\oplus_p A_p)$, where A_0 is torsion-free, while the A_p are p-groups belonging to different primes. Then the matrices representing the endomorphisms are $\omega \times \omega$-matrices of the form

$$\begin{pmatrix} \alpha_{00} & 0 & 0 & \dots & 0 & \dots \\ \alpha_{20} & \alpha_{22} & 0 & \dots & 0 & \dots \\ \alpha_{30} & 0 & \alpha_{33} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{p0} & 0 & 0 & \dots & \alpha_{pp} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where $\alpha_{00} \in \operatorname{End} A_0$, $\alpha_{p0} \in \operatorname{Hom}(A_0, A_p)$, and $\alpha_{pp} \in \operatorname{End} A_p$.

★ **Notes.** There is an extensive literature on endomorphism rings; this is one of the most investigated areas in the theory. Readers interested in the subject are referred to Krylov–Mikhalev–Tuganbaev [KMT] where a large amount of material on the endomorphism rings of abelian groups is presented in a systematic manner with full proofs. Important results are also available on endomorphism rings of groups (or modules) with distinguished submodules; these are instrumental in deriving results on endomorphism rings.

The finite topology provides additional information about the relation between the group and its endomorphism ring. It is a rare possibility of defining the finite topology intrinsically (i.e., solely in the endomorphism ring, without referring to the group), but it seems it is a most relevant feature in cases when the endomorphism ring determines the group, like for torsion groups, certain completely decomposable groups, etc. Recently, May [7] discusses the use of finite topology.

There are a few ideals in the endomorphism ring that are of interest from the group theoretical point of view. The most widely studied ideal is the Jacobson radical $\mathsf{J}(A)$. In general, there is no satisfactory characterization for $\mathsf{J}(A)$ in terms of the group A, only in special cases. Another notable ideal is Ines A, the set of inessential endomorphisms. For a study of right ideals, see Faticoni [2]. No detailed discussion will be given here to the ideals of $\operatorname{End} A$.

Let us point out some results by Mishina on endomorphism rings. In her paper [3], she characterizes the groups A for which every endomorphism of each subgroup extends to an endomorphism of A: these groups are either divisible, or torsion with homogeneous p-components. In another paper [4], she shows that all the endomorphisms of all factor groups A/C lift to A if and only if A is either free, or torsion with homogeneous p-components, or else the direct sum of a divisible torsion group and a finitely generated free group. (Similar results are proved for automorphisms, see Sect. 1, Exercise 8 in Chapter 17.)

Recently, several publications deal with the so-called *algebraic entropy* which was recently introduced in abelian groups. The paper Dikranjan–Goldsmith–Salce–Zanardo [1] contains lots of interesting results on the entropy of endomorphisms.

Exercises

(1) If $|A| = p^n$, then $|\operatorname{End} A| \leq p^{n^2}$.
(2) (a) If $G \leq A$, then $\operatorname{Hom}(A, G)$ is a right ideal in $\operatorname{End} A$.
(b) If G is fully invariant, then $\operatorname{Hom}(A, G)$ is a two-sided ideal.
(3) (Lawver) All endomorphic images of A are fully invariant exactly if, for every $a \in A$ and for all $\eta, \xi \in \operatorname{End} A$, there is a $b \in A$ such that $(\eta\xi)a = \xi b$.
(4) Show that the finite topology of $\operatorname{End} A$ for a separable torsion-free group A can be defined intrinsically (i.e. without reference to A).
(5) Describe the finite topology of J_p as an endomorphism ring of $\mathbb{Z}(p^\infty)$ and as that of J_p.
(6) (a) The direct sum and the direct product of elementary p-groups T_p for different primes p have isomorphic endomorphism rings.
(b) However, these endomorphism rings are not isomorphic as topological rings (equipped with the finite topology).
(7) For an infinite group A, $\operatorname{End} A$ is always infinite. Give examples where $|A| < |\operatorname{End} A|$, and where $|A| > |\operatorname{End} A|$.
(8) (a) Let $\{G_i\}_{i\in I}$ be a system of subgroups of A which is directed upwards under inclusion such that $\cup_{i\in I} G_i = A$. Define a topology in $\operatorname{End} A$ by declaring the set of left ideals $\mathsf{L}_i = \{\theta \in \operatorname{End} A \mid \theta G_i = 0\}$ as a base of neighborhoods about 0. Show that $\operatorname{End} A$ is a complete group in this topology, and if the G_i are fully invariant in A, then $\operatorname{End} A$ is a topological ring.
(b) (Pierce) Let A be a p-group and $G_i = A[p^i]$ $(i < \omega)$. Then $\operatorname{End} A$ is a complete topological ring in this topology.
(9) A self-small group is not the direct sum of infinitely many non-zero groups, but it may be decomposed into the direct sum of any finite number of non-zero summands. [Hint: torsion-complete with standard basic.]
(10) A group is self-small if all of its endomorphisms $\neq 0$ are monic.
(11) (Dlab) Let D be a divisible p-group of countable rank. Representing endomorphisms by matrices over J_p, show that in each column almost all entries are divisible by p^k for every $k > 0$.
(12) Suppose $A = \oplus_{i\in I} A_i$ with countable summands A_i. In the matrix representations of endomorphisms, every column contains at most countably many non-zero entries.
(13) The set $\operatorname{End}_{\mathsf{E}} A$ of E-endomorphisms of A is the center of the ring $\mathsf{E} = \operatorname{End} A$.
(14) For any group A, the center of the ring $\operatorname{End}(A \oplus \mathbb{Z})$ is $\cong \mathbb{Z}$.

2 Endomorphism Rings of p-Groups

It is a rather intriguing question to find general properties shared by the endomorphism rings of p-groups. Fortunately, substantial information is available, and we wish to discuss some relevant results.

Role of Basic Subgroups We start with the observation that underscores the relevance of basic subgroups also from the point of view of endomorphisms: *any endomorphism of a reduced p-group is completely determined by its action on a basic subgroup.* Actually, a stronger statement holds: any homomorphism of a p-group A into a reduced group C is determined by its restriction to a basic subgroup B of A. The exact sequence $0 \to B \to A \to A/B \to 0$ induces the exact sequence $0 = \mathrm{Hom}(A/B, C) \to \mathrm{Hom}(A, C) \to \mathrm{Hom}(B, C)$ which justifies our claim.

The Finite Topology It should be pointed out that in case of reduced p-groups, the finite topology of the endomorphism ring can be defined intrinsically, without reference to the underlying group. If $x \in A$, then there are a projection $\epsilon : A \to \langle c \rangle$ onto a cyclic summand and a $\theta \in \mathsf{E}$ such that $x = \theta c$. Manifestly, the neighborhood U_x (annihilating x) is nothing else than the left annihilator ideal $(\theta\epsilon)^\perp$.

Proposition 2.1. *Let A be a separable p-group.*

(i) *The finite topology of its endomorphism ring E can be defined by taking the left annihilators of the primitive idempotents.*
(ii) *In the finite topology of E, the left ideal E_0 of E generated by the primitive idempotents is dense, and its completion is E.*

Proof. As $U_x \le U_{px}$ and every element is contained in a finite summand, the U_ϵ for primitive idempotents form a subbase.

For (ii) we show that for every $\theta \in \mathsf{E}$ and for every neighborhood U_x, the coset $\theta + U_x$ intersects E_0. Now $U_x = (p^k\epsilon)^\perp$ (if $\langle x \rangle = p^k\epsilon A$) is a left ideal, so $1-\epsilon \in U_x$ implies $-\theta(1-\epsilon) \in U_\epsilon$ whence $\theta\epsilon \in \theta + U_\epsilon$ follows. $\square$

Structure of End for p-Groups We are in the fortunate situation that a lot is known about the endomorphism rings of p-groups. As a matter of fact, Liebert [3] gave a complete characterization in the separable case. A much less informative, but perhaps more attractive information is recorded in the next theorem.

Before stating the theorem, we recall a definition: a ring E is a **split extension** of a subring R by an ideal L of E if there exists a ring homomorphism $\rho : \mathsf{E} \to \mathsf{R}$ that is the identity on R, and $\mathsf{L} = \mathrm{Ker}\rho$. We write $\mathsf{E} = \mathsf{R} \oplus \mathsf{L}$ (direct sum in the group sense).

Theorem 2.2 (Pierce [1]). *For a p-group A,* End A *is a split extension*

$$\mathrm{End}\, A \cong \mathsf{R} \oplus \mathrm{End}_s\, A,$$

where R is a ring whose additive group is the completion of a free p-adic module, and $\mathrm{End}_s A$ *is the ideal of small endomorphisms of A.*

Proof. This is an immediate consequence of Theorem 3.3 in Chapter 7. $\square$

Baer–Kaplansky Theorem The endomorphism ring of an abelian group may contain more or less information about the group itself. There are arbitrarily large torsion-free groups whose endomorphism rings are just $\mathbb{Z}$, or a subring of $\mathbb{Q}$, so their endomorphism rings reveal no more about their structure than indecomposability. But, on the other hand, if A is cocyclic, then $\operatorname{End} A$ (now $\cong \mathbb{Z}/p^k\mathbb{Z}$ or $\cong J_p$) completely characterizes A among the torsion groups. Indeed, in this case $\operatorname{End} A$ has only two idempotents: 0 and 1, so any torsion group C with $\operatorname{End} C \cong \operatorname{End} A$ must be indecomposable, and hence cocyclic. It is a trivial exercise to check that only $C \cong \mathbb{Z}(p^k)$ ($k \in \mathbb{N}$ or ∞) is a possibility.

Example 2.3. It can very well happen that a torsion group and a torsion-free group have isomorphic endomorphism rings. Example: J_p is the endomorphism ring of both $\mathbb{Z}(p^\infty)$ and J_p. Exercise 6 in Sect. 1 provides an example of the same situation for a torsion and a mixed group.

It is natural to wonder in which cases the group can entirely be recaptured from its endomorphism ring, or, more accurately, when the isomorphy of endomorphism rings implies that the groups themselves are isomorphic. We will show that this is always the case for torsion groups. The proof relies on a distinguished feature of torsion groups: they have lots of summands, even indecomposable summands, so an adequate supply of idempotent endomorphisms is at our disposal.

We start with a preparatory lemma.

Lemma 2.4 (Richman–Walker [1]). *Let* $\mathsf{E} = \operatorname{End} A$ *where* A *is a* p*-group.*

(i) *If* A *is a bounded or a divisible group, then* $A \cong \mathsf{E}\epsilon$ *for an idempotent* $\epsilon \in \mathsf{E}$.
(ii) *If* A *has an unbounded basic subgroup, then there are idempotent endomorphisms* π_i ($i \in \mathbb{N}$) *such that*

$$A \cong \varinjlim \mathsf{E}\pi_i$$

with connecting maps $\gamma_i\colon\ \eta\pi_i \mapsto \eta\pi_i\pi_{i+1}$ ($\eta \in \mathsf{E}, i \in \mathbb{N}$).

Proof. Let A be a bounded p-group, and $\epsilon \in \mathsf{E}$ a primitive idempotent of maximal order p^k. Then $\epsilon A = \langle c\rangle$ is a summand of A. We claim that $\alpha : \mathsf{E}\epsilon \to A$ is a group isomorphism where $\alpha : \eta\epsilon \mapsto \eta\epsilon(c) = \eta(c)$ ($\eta \in \mathsf{E}$). It is clearly a homomorphism, and it is surjective, since c can be mapped onto every element of A by a suitable η. Moreover, it is monic, because if we have $\eta(c) = 0$ for some $\eta \in \mathsf{E}$, then also $\eta\epsilon = 0$. Similar argument applies if A is divisible.

Next, let A have an unbounded basic subgroup. Then there exist primitive idempotents $\epsilon_i \in \mathsf{E}$ of orders p^{n_i} with $n_1 < \cdots < n_i < \ldots$ such that $A = C_1 \oplus \cdots \oplus C_i \oplus A_i$ with $C_j < A_i$ for all $j > i$, where $C_i = \epsilon_i A$ is cyclic of order p^{n_i}. Choose endomorphisms $\phi_i : A \to C_i$ satisfying $\phi_i \restriction C_i = \mathbf{1}_{C_i}$, $\phi_i(C_{i+1}) = C_i$, while $\phi_i(C_j) = 0$ for $j \neq i, i+1$ and $\phi_i(A_{i+1}) = 0$. Then for every $i \in \mathbb{N}$, $\pi_i = \epsilon_i + \phi_i\epsilon_{i+1}$ is an idempotent endomorphism with $\operatorname{Im} \pi_i = C_i$. Evidently, we can select generators c_i of C_i inductively so as to satisfy $\phi_i(c_{i+1}) = c_i$ for all i.

Consider the E-modules $\mathsf{E}\pi_i$ ($i \in \mathbb{N}$). The E-map $\gamma_i : \mathsf{E}\pi_i \to \mathsf{E}\pi_{i+1}$ defined by $\gamma_i(\pi_i) = \pi_i\pi_{i+1}$ is monic; indeed, if $\eta\pi_i\pi_{i+1} = 0$ ($\eta \in \mathsf{E}$), then $\eta\pi_i(c_i) = \eta\pi_i\pi_{i+1}(c_{i+1}) = 0$ implies $\eta\pi_i = 0$ (since π_i is 0 everywhere else). We thus have a direct system $\mathsf{E}\pi_i$ ($i \in \mathbb{N}$) of left E-modules with connecting maps γ_i. As above, we can show that the map $\alpha_i : \eta\pi_i \mapsto \eta(c_i)$ of $\mathsf{E}\pi_i$ to $A[p^{n_i}]$ is an E-isomorphism. Moreover, the diagram

$$\begin{tikzcd}
\mathsf{E}\pi_i \arrow[r, "\alpha_i"] \arrow[d, "\gamma_i"] & A[p^{n_i}] \arrow[d, "\beta_i"] \\
\mathsf{E}\pi_{i+1} \arrow[r, "\alpha_{i+1}"] & A[p^{n_{i+1}}]
\end{tikzcd}$$

(where β_i is the inclusion map) commutes for each i, because $\beta_i\alpha_i(\eta\pi_i) = \eta(c_i)$ and $\alpha_{i+1}\gamma_i(\eta\pi_i) = \alpha_{i+1}(\eta\pi_i\pi_{i+1}) = \eta\pi_i(c_{i+1}) = \eta(\epsilon_i + \phi_i\epsilon_{i+1})(c_{i+1}) = \eta\phi_i(c_{i+1}) = \eta(c_i)$. Since $A = \varinjlim A[p^{n_i}]$, we obtain $A \cong \varinjlim \mathsf{E}\pi_i$. $\square$

Theorem 2.5 (Baer [9], Kaplansky [1]). *If A and C are torsion groups whose endomorphism rings are isomorphic, then $A \cong C$.*

Moreover, every ring isomorphism ψ : $\operatorname{End} A \to \operatorname{End} C$ is induced by a group isomorphism $\phi: A \to C$; i.e. $\psi : \eta \mapsto \phi\eta\phi^{-1}$.

Proof. The proof can at once be reduced to p-groups. So suppose A, C are p-groups, and $\psi : \operatorname{End} A \to \operatorname{End} C$ is a ring isomorphism. To simplify notation, write $\mathsf{E} = \operatorname{End} A$.

If A is bounded or divisible, then by Lemma 2.4(i) we have $A \cong \mathsf{E}\epsilon$ for an idempotent $\epsilon \in \mathsf{E}$. Similarly, $C \cong \psi(\mathsf{E})\psi(\epsilon)$ whence the existence of an isomorphism $\phi: A \to C$ is immediate.

If $A = B \oplus D$ where B is bounded and D is divisible, then $\operatorname{End} A$ has a torsion-free part $\cong \operatorname{End} D$ that is an ideal, whose two-sided annihilator is $\cong \operatorname{End} B$. These ideals are carried by ψ to the corresponding ideals of $\operatorname{End} C$. The settled cases imply that A and C have isomorphic bounded and divisible subgroups.

If A has unbounded basic subgroup, then in view of Lemma 2.4(ii) there are idempotents $\pi_i \in \mathsf{E}$ such that $A \cong \varinjlim \mathsf{E}\pi_i$, and clearly $C \cong \varinjlim \psi(\mathsf{E})\psi(\pi_i)$ with corresponding connecting maps. Thus again $A \cong C$ follows.

We proceed to the second claim. For simplicity we identify A with $\mathsf{E}\epsilon$ or with $\varinjlim \mathsf{E}\pi_i$, as the case may be, and similarly for C. Let $\phi : A \to C$ be the isomorphism induced by ψ from $\mathsf{E}\epsilon$ to $\psi(\mathsf{E})\psi(\epsilon)$, or by ψ between the direct limits. Then the endomorphism $\psi(\eta)$ ($\eta \in \mathsf{E}$) acting on C can also be obtained by using ϕ^{-1}, applying η and followed by ϕ, i.e. $\psi(\eta) = \phi\eta\phi^{-1}$, as claimed. $\square$

Since all the primitive idempotents are contained in the ideal $\operatorname{End}_s A$ of small endomorphisms, from the foregoing proof we conclude:

Corollary 2.6 (Pierce [1]). *The torsion groups A and C are isomorphic exactly if the rings $\operatorname{End}_s A$ and $\operatorname{End}_s C$ are isomorphic.* $\square$

An immediate corollary to Theorem 2.5 is the following remarkable fact:

Corollary 2.7 (Baer [9]). *Every automorphism of the endomorphism ring of a torsion group is inner.*

Proof. Let α be an automorphism of $\operatorname{End} A$, where A is a torsion group. Theorem 2.5 asserts that it must act as $\alpha : \eta \mapsto \phi\eta\phi^{-1}$ for some automorphism ϕ of A ($\eta \in \operatorname{End} A$). Here ϕ is viewed as a unit in $\operatorname{End} A$. □

Liebert's Theorem The following characterization of the endomorphism rings of separable p-groups is an important document, though it seems difficult to use it. We state it without proof.

Theorem 2.8 (Liebert [3]). *For a ring* E*, there exists a separable p-group A such that* $\operatorname{End} A \cong \mathsf{E}$ *if and only if the following conditions are satisfied:*

(i) *the sum* E_0 *of all minimal non-nil left ideals is a ring whose additive group is a p-group, and whose left annihilator in* E *is* 0*;*
(ii) *if π, ρ are primitive idempotents in* E*, then the additive group of* $\pi\mathsf{E}\rho$ *is cyclic;*
(iii) *if π, ρ are primitive idempotents in* E *such that* $o(\pi) \leq o(\rho)$*, then the left annihilator of* $\mathsf{E}\rho$ *is contained in the left annihilator of* $\mathsf{E}\pi$*, and* $\mathsf{E}\pi\mathsf{E}\rho = \mathsf{E}\rho[o(\pi)]$*;*
(iv) *a proper right ideal of* E_0 *whose left annihilator in* E_0 *is nilpotent is not a summand in* E_0*;*
(v) E *is complete in its finite topology.* □

Center of the Endomorphism Ring We proceed to identify the center of the endomorphism ring of a torsion group. The general case immediately reduces to primary groups, and it is more elegant to formulate the result for p-groups.

Theorem 2.9 (Charles [1], Kaplansky [K]). *The center of the endomorphism ring of a p-group consists of multiplications by p-adic integers or by elements of the residue class ring $\mathbb{Z}/p^k\mathbb{Z}$ according as the group is unbounded or bounded by p^k (with minimal k).*

Proof. Multiplication by a rational or a p-adic integer is an endomorphism of any p-group A; it evidently commutes with every $\eta \in \operatorname{End} A$.

Conversely, γ in the center of $\operatorname{End} A$ means that the map $\gamma : A \to A$ is an $\operatorname{End} A$-module homomorphism. We now appeal to Lemma 6.1(i) below to infer that γ must act as multiplication by a $\rho \in J_p$. □

The Jacobson Radical The Jacobson radical $\mathsf{J}(A)$ of the endomorphism ring E of a p-group A has been extensively studied, but so far there is no satisfactory characterization. Pierce [1] compared it to the set

$$\mathsf{H}(A) = \{\eta \in \mathsf{E} \mid \eta(p^n A[p]) \leq p^{n+1} A[p]\},$$

which is an ideal of E located between $p\mathsf{E}$ and E (also called **Pierce radical**); it consists of all endomorphisms that strictly increase the heights of elements of finite heights in the socle. From the point of view of endomorphism rings of p-groups,

$\mathsf{H}(A)$ seems to be more tractable than the Jacobson radical. We now prove results that involve $\mathsf{H}(A)$.

Lemma 2.10 (Pierce [1]).

(i) *For every reduced p-group A,* $\mathsf{H}(A)$ *contains the Jacobson radical* $\mathsf{J}(A)$ *of* End A.
(ii) *For a separable p-group A,* $\mathsf{J}(A) = \mathsf{H}(A)$ *if and only if, for all* $a \in A[p]$ *and all* $\eta \in \mathsf{H}(A)$*, the infinite sum* $\sum_{n<\omega} \eta^n(a)$ *converges* (*in the p-adic topology of A*).
(iii) *If A is torsion-complete, then* $\mathsf{J}(A) = \mathsf{H}(A)$.

Proof.

(i) Assume $\eta \in \mathsf{J}(A)$, but $\eta \notin \mathsf{H}(A)$. Thus for some $n < \omega$, there is an $a \in A[p]$ of finite height n such that also $h_p(\eta a) = n$. Then a is in the socle of a summand $\langle b \rangle$, and ηa is in the socle of a summand $\langle c \rangle$, both of order p^{n+1}. Evidently, there is a $\xi \in \operatorname{End} A$ mapping $\langle c \rangle$ onto $\langle b \rangle$ such that $\xi(\eta a) = a$. Now $\eta \in \mathsf{J}(A)$ implies $\xi\eta \in \mathsf{J}(A)$, so $1-\xi\eta$ is an automorphism of A. However, $(1-\xi\eta)a = 0$, a contradiction.

(ii) If $\eta \in \mathsf{H}(A)$ and $\mathsf{H}(A) = \mathsf{J}(A)$, then $1-\eta$ is an automorphism of A. Thus, given $a \in A[p]$, there is a $b \in A[p]$ such that $(1 - \eta)b = a$. It follows that the partial sums $b_n = a + \eta(a) + \cdots + \eta^n(a) = (1 - \eta^{n+1})b$ satisfy $b - b_n = \eta^{n+1}(b)$. Since $\eta \in \mathsf{H}(A)$ guarantees that $h_p(\eta^{n+1}(b)) > n$, the sequence b_n converges to b.

Conversely, if we know that all the infinite sums of the stated kind converge, then we can show that $1 - \eta$ is an automorphism of A for all $\eta \in \mathsf{H}(A)$. For each $a \in A[p]$, we have $h_p((1 - \eta)a) = h_p(a)$, and therefore $\operatorname{Ker}(1 - \eta) = 0$. To see that $\operatorname{Im}(1 - \eta) = A$, the proof goes by induction on the order of $a \in A$, to verify the existence of a $c \in A$ such that $(1-\eta)c = a$. If $o(a) = p$, then for c we choose $\sum_{n<\omega} \eta^n(a)$. If $o(a) = p^{k+1}$, then $b' = \sum_{n<\omega} \eta^n(p^k a)$ must belong to $p^k A$, because all the partial sums belong to this (closed) subgroup. Hence $b' = p^k u$ for some $u \in A$, and we have $(1 - \eta)p^k u = p^k a$. Now $a - (1 - \eta)u$ is of order $\leq p^k$, so by the induction hypothesis there is a $v \in A$ satisfying $(1 - \eta)v = a - (1 - \eta)u$. Then $c = u + v$ is mapped by $1 - \eta$ upon a.

(iii) If A is a torsion-complete p-group, then the sums in (ii) always converge, and therefore, $\mathsf{J}(A) = \mathsf{H}(A)$. □

Example 2.11. Let $A = \oplus_{n<\omega} \langle a_n \rangle$ where $o(a_n) = p^n$. Then $\mathsf{J}(\operatorname{End} A) \neq \mathsf{H}(\operatorname{End} A)$. Indeed, the correspondence $\eta : a_n \mapsto pa_{n+1}$ defines an endomorphism in $\mathsf{H}(A)$, and the sequence $(1 + \eta + \cdots + \eta^k)a_1$ $(k < \omega)$ does not converge.

Proposition 2.12 (Pierce [1]).

(i) *For a reduced p-group A, there is a ring embedding*

$$\psi\colon\ (\operatorname{End} A)/\mathsf{H}(A) \to \prod_{n<\omega} \mathsf{M}_{f_n(A)} \tag{16.2}$$

where $\mathsf{M}_{f_n(A)}$ denotes a matrix ring over the prime field F_p whose dimension is the nth UK-invariant $f_n(A)$ of A. Im ψ *contains the direct sum of the matrix rings.*

(ii) *For a separable p-group A, ψ is an isomorphism if and only if A is torsion-complete.*

Proof.

(i) Every $\eta \in \operatorname{End} A$ induces a linear transformation η_n of the $f_n(A)$-dimensional vector space $p^n A[p]/p^{n+1}A[p]$. Thus $\eta \mapsto (\eta_n)_{n<\omega}$ induces a ring homomorphism from End A to the right-hand side of (16.2), whose kernel is exactly $\mathsf{H}(A)$.

If we write the basic subgroup of A as $B = \oplus_{n<\omega} B_n$ where B_n is the direct sum of cyclic groups of order p^{n+1}, then $p^n A[p]/p^{n+1}A[p] \cong B_n[p]$. It is clear that any linear transformation η_n on $B_n[p]$ extends to an endomorphism of B_n, and then to an endomorphism η of A. This is always true for a simultaneous extension of a finite number of η_ns, and also for infinitely many provided A is torsion-complete.

(ii) That ψ is an isomorphism only if the separable p-group A is torsion-complete can be seen from the representation of elements in separable p-groups as $\sum_{n<\omega} b_n$ with $b_n \in B_n$. If an arbitrarily chosen collection $\eta_n \in \operatorname{End} B_n$ $(n < \omega)$ extends to $\eta \in \operatorname{End} A$, then all $\sum_{n<\omega} \eta(b_n)$ must be contained in A, thus $\overline{B} \leq A$. □

The proof shows that Im ψ is a subdirect product of the matrix rings.

★ **Notes.** The study of the endomorphism rings of p-groups A was initiated by Pierce [1]; in this important paper, he proved several relevant results. In a sequel [3] to [1], he characterizes End A within a class of rings when A is separable with a prescribed basic subgroup. A more satisfactory realization theorem is due to Liebert [1, 3] who characterized the endomorphism rings as rings, first for bounded, and later for separable p-groups. (Though it is an important contribution, it still falls short of the true significance, since the conditions are not illuminating.) See also Liebert [4]. Goldsmith [2] examines endomorphism rings of non-separable p-groups.

Generalizing Corner [5], Dugas–Göbel [3] show that for every reduced ring R over J_p, whose additive group is torsion-free and algebraically compact, there exists a separable p-group A such that End A is a split extension of R and $\operatorname{End}_s A$. They also prove, for every cardinal κ, the existence of κ separable p-groups, all with such a fixed R, so that all homomorphisms between them are small.

Ideals in endomorphism rings have been discussed by Liebert [2] and Monk [1]. According to Hausen [4] and Ivanov [1], the sum of nilpotent ideals is the collection of all $\eta \in \operatorname{End} A$, for which there is a finite chain $0 = A_0 < A_1 < \cdots < A_n = A$ of fully invariant subgroups such that $\eta A_{i+1} \leq A_i$. Hausen [5] generalizes the ideal $\mathsf{H}(A)$ by defining I_A as the collection of all $\eta \in \operatorname{End} A$ for which there is a finite sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = \tau$ of ordinals such that $\eta(p^{\sigma_i}A[p]) \leq p^{\sigma_{i+1}}A[p]$ for $i = 0, 1, \ldots, n-1$, where τ denotes the length of A. Her theorems are concerned with totally projective groups.

Several authors discussed a modified version of the Baer–Kaplansky theorem: when the isomorphism of the Jacobson radicals of the endomorphism rings implies the isomorphism of the groups. See, e.g., Flagg [1], Hausen–Johnson [1], Hausen–Praeger–Schultz [1], Schultz [2]. Puusemp [1] showed that the conclusion of Theorem 2.5 remains true if only the isomorphy of the multiplicative endomorphism semigroups is assumed. There are theorems similar to Theorem 2.5 on certain mixed groups. For the endomorphism semigroup, see also Sebel'din [3]. In several papers, May–Toubassi (see, e.g., [1]) showed that two mixed local groups of torsion-free rank 1 with totally projective torsion subgroups are isomorphic if their endomorphism rings are isomorphic; for a survey, see May [4]. Files–Wickless [1] extended the Baer–Kaplansky theorem to a class of global mixed groups. Files [1] proved the isomorphy of reduced local Warfield groups with isomorphic endomorphism rings.

Nunke [6] has a nice generalization of Theorem 2.8: If A, C are unbounded p-groups, then $\operatorname{End} A$ and $\operatorname{End} C$ embed in $\operatorname{End}(\operatorname{Tor}(A, C))$ in such a way that they are centralizers of each other, and their intersection is precisely the center of $\operatorname{End}(\operatorname{Tor}(A, C))$. The special case $C \cong \mathbb{Z}(p^\infty)$ yields Theorem 2.9.

A fairly large literature deals with the problem as to when $\operatorname{End} A$ equals the subring generated by its units (i.e., by $\operatorname{Aut} A$). Castagna [1] gives an example where this subring is a proper subring. Hill [6] shows that if $p > 2$, then every endomorphism of a totally projective p-group is the sum of two automorphisms. Hill–Megibben–Ullery [1] prove the same for local Warfield groups. See also Goldsmith–Meehan–Wallutis [1] where the unit sum numbers (number of units needed to be added to get the endomorphisms) are investigated.

Bunina–Mikhalëv [1] investigate when the endomorphism rings of two p-groups are elementarily equivalent.

Exercises

(1) Give more examples to show that a torsion-free group and a torsion group may have isomorphic endomorphism rings.

(2) (Levi) Find the endomorphisms which map every subgroup into itself. In particular, for p-groups.

(3) If A is a torsion group, then the $\mathbb{Z}$-adic topology of $\operatorname{End} A$ is finer than its finite topology. [Hint: $n(\operatorname{End} A) \leq U_x$ if $nx = 0$.]

(4) Let A be a separable p-group with basic subgroup B. Then $\operatorname{End} A$ is a closed subring in $\operatorname{End} \overline{B}$ (in the finite topology).

(5) (Szele–Szendrei) A torsion group A has commutative endomorphism ring exactly if $A \leq \mathbb{Q}/\mathbb{Z}$.

(6) Find two non-isomorphic p-groups with isomorphic endomorphism groups.

(7) Verify the analogue of the Baer–Kaplansky theorem for adjusted cotorsion groups.

(8) For a divisible p-group D, the Jacobson radical of $\operatorname{End} D$ is equal to $p \operatorname{End} D$.

(9) (a) Assume A is a torsion group. The endomorphisms of A with finitely cogenerated images form an ideal $\mathsf{V}(A)$ in $\operatorname{End} A$. It is the ideal generated by the primitive idempotents.

(b) Follow the proof of the Baer–Kaplansky theorem to conclude that a ring isomorphism $\mathsf{V}(A) \cong \mathsf{V}(C)$ implies $A \cong C$ provided that C, too, is torsion.

(10) Let A be a torsion-complete p-group, and $\eta \in \operatorname{End} A$ with $\operatorname{Ker} \eta = 0$. If η maps a basic subgroup into a basic subgroup, then $\eta \in \operatorname{Aut} A$.

(11) (D'Este [1]) A group C is said to be an *E-dual* of A if $\operatorname{End} A$ and $\operatorname{End} C$ are anti-isomorphic rings. A reduced p-group has an E-dual if and only if it is torsion-complete with finite UK-invariants. [Hint: summands B_n in a basic subgroup $B = \oplus_n B_n$ are finite; A is separable and $\overline{B} \leq A$.]

3 Endomorphism Rings of Torsion-Free Groups

In contrast to torsion groups, non-isomorphic torsion-free groups may very well have isomorphic endomorphism rings. Another major difference in the behavior of endomorphism rings between torsion and torsion-free groups lies in the fact that only minor restrictive conditions hold for the torsion-free case. We will see in Theorem 7.1 that only slight restriction on the ring (to be cotorsion-free) is enough to guarantee that it is an endomorphism ring of a torsion-free group.

Example 3.1 (Corner [2]). A ring whose additive group is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$ cannot be the endomorphism ring of an abelian group. For, such a group must be torsion-free divisible, and it cannot be of rank 1, neither of rank ≥ 2, because then the rank of its endomorphism ring is 1, and ≥ 4, respectively. (Similar argument holds for $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$.)

Example 3.2 (Sąsiada [4], Corner [2]). There exist torsion-free groups of finite rank whose endomorphism groups are isomorphic, but their endomorphism rings are not. Indeed, by Theorem 3.3 below, there exist such groups with endomorphism rings isomorphic to the ring $\mathbb{Z} \oplus \mathbb{Z}$ and to the ring of the Gaussian integers $\mathbb{Z} + \mathbb{Z}i$.

Corner's Theorem We enter our study of torsion-free endomorphism rings with the following striking theorem. Though it is a corollary to Theorem 7.1 that is by far more general, we offer a full proof of this historically important result that opened new prospects in the theory; this proof is needed, because we will give no detailed proof for Theorem 7.1. The idea of localization in the proof is due to Orsatti [2], so Corner's method will be needed merely in the more tractable local case.

Theorem 3.3 (Corner [2]). *Every countable reduced torsion-free ring is isomorphic to the endomorphism ring of some countable reduced torsion-free group.*

Proof. Let R be a countable, p-local reduced torsion-free ring. It is a $\mathbb{Z}_{(p)}$-algebra, Hausdorff in its p-adic topology, so it is a pure subring in its p-adic completion $\tilde{\mathsf{R}}$, which is a torsion-free J_p-algebra with the same identity. Choose a maximal set $\{z_n \ (n < \omega)\}$ in R that is linearly independent over J_p. Thus, for every $a \in \mathsf{R}$, there is a dependence relation

$$p^k a = \pi_1 z_1 + \cdots + \pi_m z_m \qquad (\pi_i \in J_p),$$

for some k and m, where the coefficients π_i are uniquely determined up to factors p^j. We denote by S the pure subring of J_p generated by $\mathbb{Z}_{(p)}$ and the π_i for all $a \in \mathsf{R}$. Clearly, S is still countable. We proceed with the construction which requires several steps.

(a) Suppose $\pi_1, \ldots, \pi_n \in J_p$ are linearly independent over S. We claim: $\pi_1 r_1 + \cdots + \pi_n r_n = 0$ ($r_i \in \mathsf{R}$) implies that all $r_i = 0$. In fact, for sufficiently large $\ell \in \mathbb{N}$, there are relations $p^\ell r_i = \rho_{i1} z_1 + \cdots + \rho_{im} z_m$ with $\rho_{ij} \in \mathsf{S}$, thus $\sum_i \sum_j \pi_i \rho_{ij} z_j = 0$. By the independence of the z_j, we have $\sum_i \pi_i \rho_{ij} = 0$ ($j = 1, \ldots, m$) whence, by hypothesis on the π_i, all $\rho_{ij} = 0$, and $r_i = 0$.

(b) For each non-zero $a \in \mathsf{R}$ choose a pair ρ_a, σ_a of p-adic integers such that the set $\{\rho_a, \sigma_a \mid 0 \neq a \in \mathsf{R}\}$ is algebraically independent over S. This is possible, for S is countable, and the transcendence degree of J_p over S is the continuum. Set

$$e_a = \rho_a 1 + \sigma_a a \in \tilde{\mathsf{R}}, \tag{16.3}$$

and define the group A as the pure subgroup

$$A = \langle \mathsf{R}, \mathsf{R}e_a \ \forall a \in \mathsf{R}\rangle_* \leq \tilde{\mathsf{R}}. \tag{16.4}$$

Obviously, A is countable, reduced, and torsion-free.

(c) It is evident from the definition that A is a left R-module. It is faithful, as different elements of R act differently on $1 \in A$. Consequently, R is isomorphic to a subring of $\operatorname{End} A$.

(d) In order to show that it is not a proper subring, select an $\eta \in \operatorname{End} A$. Since R is pure and dense in A (which is pure and dense in $\tilde{\mathsf{R}}$), it follows that $\tilde{A} = \tilde{\mathsf{R}}$. By Proposition 2.10 in Chapter 6, η extends uniquely to a J_p-endomorphism $\tilde{\eta}$ of $\tilde{\mathsf{R}}$. Then

$$\eta e_a = \tilde{\eta}(\rho_a 1 + \sigma_a a) = \rho_a(\tilde{\eta}1) + \sigma_a(\tilde{\eta}a) = \rho_a(\eta 1) + \sigma_a(\eta a)$$

for any $a \in A$. More explicitly, write

$$p^k(\eta e_a) = b_0 + \sum_{i=1}^{n} b_i e_{a_i},\ p^k(\eta 1) = c_0 + \sum_{i=1}^{n} c_i e_{a_i},\ p^k(\eta a) = d_0 + \sum_{i=1}^{n} d_i e_{a_i}$$

for $a_i, b_i, c_i, d_i \in \mathsf{R}$, and for some $k, n \in \mathbb{N}$. Substitution yields

$$\begin{aligned} b_0 + &\sum_{i=1}^{n} b_i(\rho_{a_i} 1 + \sigma_{a_i} a_i) \\ &= \rho_a[c_0 + \sum_{i=1}^{n} c_i(\rho_{a_i} 1 + \sigma_{a_i} a_i)] + \sigma_a\left[d_0 + \sum_{i=1}^{n} d_i(\rho_{a_i} 1 + \sigma_{a_i} a_i)\right] \end{aligned}$$

where we may assume that $a_1 = a$. Comparing the corresponding coefficients on both sides, we use algebraic independence to argue that $b_1 = c_0$, $b_1 a = d_0$, while all other b_i, c_i, d_i vanish. This means $p^k(\eta 1) = c_0$, $p^k(\eta a) = c_0 a$ which thus holds for all $a \in \mathsf{R}$. Therefore, with the notation $\eta 1 = c \in \mathsf{R}$, we have $\eta a = ca$ for all $a \in \mathsf{R}$, showing that η acts on R as left multiplication by $c \in \mathsf{R}$. The same holds for $\tilde{\eta}$ and for $\eta = \tilde{\eta} \restriction A$. This completes the proof of the local case.

(e) Moving to the global case, suppose R is as stated in the theorem. We get $\tilde{\mathsf{R}} = \prod_p \tilde{\mathsf{R}}_{(p)}$ where $\tilde{\mathsf{R}}_{(p)}$ is the p-adic completion of the reduced part of the localization $\mathsf{R}_{(p)} = \mathbb{Z}_{(p)} \otimes \mathsf{R}$. For $a \in \mathsf{R}$ we write $a = (\dots, a_p, \dots)$ with $a_p \in \mathsf{R}_{(p)}$.

Just as in (b), for each $a \in \mathsf{R}$ choose ρ_a, σ_a now as $\rho_a = (\dots, \rho_{pa}, \dots)$, $\sigma_a = (\dots, \sigma_{pa}, \dots)$ with $\rho_{pa}, \sigma_{pa} \in J_p$ algebraically independent over $\mathsf{S}_{(p)}$ for each p; note that if $\mathsf{R}_{(p)} = 0$, then $\rho_{pa} = \sigma_{pa} = 0$ can be chosen. Defining e_a as in (16.3) and A as in (16.4), A becomes a countable subgroup of $\tilde{\mathsf{R}}$. As in (c), we argue that R is isomorphic to a subring of $\operatorname{End} A$. Every $\eta \in \operatorname{End} A$ extends uniquely to $\tilde{\eta} \in \operatorname{End}(\tilde{\mathsf{R}}^+)$ which must act coordinate-wise in each $\tilde{\mathsf{R}}_{(p)}$, because these are fully invariant subrings in $\tilde{\mathsf{R}}$. By the local case, $\tilde{\eta}$ is left multiplication by the $\mathsf{R}_{(p)}$-component of $\eta 1 = c \in \mathsf{R}$, thus $\tilde{\eta}$ must agree with the left multiplication by c on all of $\tilde{\mathsf{R}}$, in particular on A. This establishes the claim that $\operatorname{End} A \cong \mathsf{R}$. □

Since there is a set of cardinality $2^{\aleph_0}$ of pairwise disjoint finite subsets of algebraically independent elements of J_p, and since these define non-isomorphic torsion-free groups in the above construction, it is clear that there are $2^{\aleph_0}$ non-isomorphic solutions in Theorem 3.3.

The Topological Version Another point of interest emerges if the endomorphism rings are equipped with the finite topology. Then all endomorphism rings of countable reduced torsion-free groups can be characterized, even if they are uncountable. Note that the necessity of the condition stated in the next theorem is immediate: each left ideal L in Theorem 3.4 is defined to consist of all $\eta \in \operatorname{End} A$ that annihilate a fixed $a \in A$. However, the proof of sufficiency involves more ring theory than we care to get into, and therefore we state the theorem without proof.

Theorem 3.4 (Corner [4]). *A topological ring* R *is isomorphic to* $\operatorname{End} A$ *for a countable reduced torsion-free group A if and only if it is complete in the topology with a base of neighborhoods of* 0 *consisting of left ideals* L_n $(n < \omega)$ *such that the factor groups* R/L_n *are countable, reduced and torsion-free.* □

Quasi-Endomorphism Ring This is a most useful tool in dealing with finite rank torsion-free groups. The set of quasi-endomorphisms of a torsion-free A (see Sect. 9 in Chapter 12) is a $\mathbb{Q}$-algebra

$$\mathbb{Q} \operatorname{End} A = \mathbb{Q} \otimes \operatorname{End} A.$$

The fully invariant pure subgroups of A form a lattice $\mathfrak{F}$ where $G \cap H$ and $(G+H)_*$ are the lattice operations $(G, H \in \mathfrak{F})$.

Lemma 3.5 (Reid [3]). *Let A be a torsion-free group. The correspondences*

$$G \mapsto \mathbb{Q} \otimes G \qquad \textit{and} \qquad M \mapsto M \cap A$$

are inverse to each other between the lattice $\mathfrak{F}$ of fully invariant pure subgroups G and the $\mathbb{Q}\,\mathrm{End}\,A$*-submodules M of* $\mathbb{Q} \otimes A$.

Proof. Straightforward. $\square$

Isomorphic Endomorphism Rings It seems that it does not make much sense to pose the question as to when the isomorphy of endomorphism rings of torsion-free groups implies the isomorphy of the groups themselves. Surprisingly it has an answer, albeit in a very special case, by the following theorem.

Theorem 3.6 (Sebel'din [1]). *Suppose that* $\mathrm{End}\,A \cong \mathrm{End}\,C$ *where A and C are direct sums of rational groups each of which is p-divisible for almost all primes p. Then* $A \cong C$.

Proof. If A has κ summands of type $\mathbb{Z}_{(p)}$ for a prime p, then $\mathrm{End}\,A$ has κ orthogonal primitive idempotents of this type. Hence from $\mathrm{End}\,A \cong \mathrm{End}\,C$ it follows that A and C must have the same numbers of summands of types $\mathbb{Z}_{(p)}$. Factor out the ideals of elements of types $\mathbb{Z}_{(p)}$ for all p, and repeat the same argument for types $\mathbb{Z}_{(p,q)}$ for different primes p, q (i.e., for rational groups r-divisible for all primes $r \neq p, q$). We can then conclude in the same way the equality of the numbers of summands of these types. If we keep doing this, including more and more primes, then the claim follows. $\square$

Sebel'din points out that this is a sharp result: the hypothesis on A in the preceding theorem cannot be weakened: if A is of rank 1 and if its type $\mathbf{t}$ is finite at infinitely many primes, then there are non-isomorphic C of rank one with isomorphic endomorphism ring.

In an another special case, more can be stated:

Theorem 3.7 (Wolfson [2]). *Let* A, C *be homogeneous separable torsion-free groups of type* $\mathbf{t}(\mathbb{Z})$. *If* $\psi : \mathrm{End}\,A \to \mathrm{End}\,C$ *is a ring isomorphism, then there exists an isomorphism* $\phi : A \to C$ *such that* $\psi(\eta) = \phi\eta\phi^{-1}$ *for all* $\eta \in \mathrm{End}\,A$.

Proof. If $\epsilon \in \mathrm{End}\,A$ is a primitive idempotent, then $A \cong (\mathrm{End}\,A)\epsilon$, and the same holds for C. Hence the existence of ϕ is immediate. The rest follows as in the proof of Theorem 2.5. $\square$

★ **Notes.** Theorem 3.3 is one of the most significant theorems on endomorphism rings. It has been generalized, see Theorem 7.2. Corner [2] also proves that for finite rank groups, Theorem 3.3 has a noteworthy improvement: a reduced torsion-free ring of rank n is isomorphic to the endomorphism ring of a torsion-free group of rank $\leq 2n$. This is the sharpest result in general. Zassenhaus [1] found conditions for a ring of rank n to be the endomorphism ring of a group of the same rank. For a generalization of this result, see Dugas–Göbel [6].

Theorem 3.3 has been generalized by several authors, see Sect. 7. It was Göbel [1] who observed that Corner's theorem should be valid for uncountable groups under suitable assumptions. The first generalization to arbitrary cardinalities was given by Dugas–Göbel [2] for cotorsion-free rings under the hypothesis V = L.

There are numerous results on the endomorphism rings of a few selected classes of torsion-free groups. For example, Dugas–Thomé [2] discuss the Butler version, while the endomorphism rings of separable torsion-free groups were characterized by Metelli–Salce [1] in the homogeneous case, by Webb [1] for homogeneously decomposable groups, and by Bazzoni–Metelli [1] in the general case. The conditions are similar to those given by Liebert [3] for p-groups. Blagoveshchenskaya [1] investigates the case of countable torsion-free groups.

Faticoni [Fat] observes and demonstrates that for torsion-free finite rank groups, the endomorphism ring modulo the nilradical is more tractable than the ring itself. Chekhlov [3] considers groups whose idempotent endomorphisms are central, including the separable and cotorsion cases. For a reduced torsion-free A, Krylov [5] defines the ideal $\mathsf{H}(A)$ of $\mathsf{E} = \operatorname{End} A$ as the set of all $\eta \in \mathsf{E}$ for which $h_p(\eta x) > h_p(x)$ for all $x \in A$ if the latter height is finite (this being the torsion-free analogue of the ideal denoted by the same symbol in the torsion case). If A is of finite rank, then the Jacobson radical J of E contains $\mathsf{H}(A)$ and is nilpotent mod it. For more on End, see also Krylov [6].

Exercises

(1) If A is torsion-free, then $p \operatorname{End} A = \operatorname{End} A$ exactly if $pA = A$.

(2) Every reduced torsion-free ring of rank one is the endomorphism ring of a torsion-free group of rank $n \in \mathbb{N}$.

(3) (Corner) If $\operatorname{End} A$ is countable, reduced and torsion-free, then A must be reduced and torsion-free.

(4) (Corner) A ring whose additive group is $\cong J_p \oplus J_p$ cannot be the endomorphism ring of any group.

(5) For an arbitrarily large cardinal κ, there exist torsion-free groups of cardinality κ whose endomorphism rings have cardinality 2^κ.

(6) For a finite rank torsion-free A, $\operatorname{End} A$ has no divisors of zero if and only if all endomorphisms are monic if and only if $\mathbb{Q} \operatorname{End} A$ is a division ring.

(7) The center of the endomorphism ring of a homogeneous separable torsion-free group is a subring of $\mathbb{Q}$.

(8) (Hauptfleisch) Suppose that $\operatorname{End} A \cong \operatorname{End} C$ where A and C are homogeneous separable torsion-free groups of types $\mathbf{t}$ and $\mathbf{s}$, respectively.

   (a) If $\mathbf{t} = \mathbf{s}$, then every isomorphism between the endomorphism rings is induced by an isomorphism between the groups.

   (b) If $\mathbf{t} \neq \mathbf{s}$, then $A \otimes S \cong C \otimes T$ where T, S are rational groups of type $\mathbf{t}$ and $\mathbf{s}$, respectively.

(9) Is Theorem 3.7 true for any type $\mathbf{t}$?

(10) (J. Reid, Orsatti) Call a ring R **subcommutative** if for all $r, s \in \mathsf{R}$ there is $t \in \mathsf{R}$ such that $rt = sr$.

   (a) The ring of the integral quaternions $q = a + bi + cj + dk$ with $a, b, c, d \in \mathbb{Z}$ is subcommutative.

   (b) If $\operatorname{End} A$ is subcommutative, then endomorphic images of A are fully invariant subgroups.

   (c) Conclude that the full invariance of endomorphic images does not imply the commutativity of the endomorphism ring.

4 Endomorphism Rings of Special Groups

Our next concern is with the endomorphism rings of some important types of groups, like projective, injective, etc. groups.

Endomorphism Rings of Free Groups We start with free groups. It might be helpful to note in advance that if F is a free group, then the image of an endomorphism η is also a free subgroup, and therefore $\operatorname{Ker}\eta$ is always a summand of F.

At this point, we need a definition: a ring R is called **Baer ring** if the left (or, equivalently, the right) annihilator of a non-empty subset of R is generated by an idempotent.

Theorem 4.1 (Wolfson [1], Rangaswamy [2], Tsukerman [1]). *The endomorphism ring of a free group is a Baer ring if and only if the group is countable.*

Proof. We show that, for a free group F, $\operatorname{End} F$ is a Baer ring if and only if F enjoys the strong summand intersection property, i.e. intersections of any number of summands are summands. Then the claim will follow from Proposition 7.12 in Chapter 3.

Assume F has the strong summand intersection property. Consider the right annihilator $\mathsf{N} \leq \operatorname{End} F = \mathsf{E}$ of a subset $X \subset \mathsf{E}$, and let $K = \cap_{\xi \in X} \operatorname{Ker}\xi$. Then K is a summand of F by hypothesis, thus $K = \epsilon F$ for an idempotent $\epsilon \in \mathsf{E}$. Hence $\xi\epsilon = 0$ for all $\xi \in X$, moreover, $\xi\alpha = 0$ for an $\alpha \in \mathsf{E}$ if and only if $\alpha F \leq \epsilon F$—which holds if and only if $\epsilon\alpha = \alpha$, i.e. $\alpha \in \epsilon\mathsf{E}$. Thus $\mathsf{N} = \epsilon\mathsf{E}$, and $\operatorname{End} F$ is a Baer ring.

Conversely, suppose $\operatorname{End} F = \mathsf{E}$ is a Baer ring, and $K = \cap_{\xi \in X} \operatorname{Ker}\xi$ for some set $X \subset \mathsf{E}$, where the summands are written in the form $\operatorname{Ker}\xi$. Define N as the right annihilator ideal of X, thus $\mathsf{N} = \epsilon\mathsf{E}$ with an idempotent ϵ. Then $K = \epsilon F$ is a summand of F. □

Endomorphism Rings of Algebraically Compact Groups Next, we consider algebraically compact groups. We know from Theorem 2.11 in Chapter 7 that their endomorphism groups are also algebraically compact, but we can now prove much more.

Theorem 4.2.

(a) *The endomorphism ring of an algebraically compact group is a right algebraically compact ring.*
(b) *The same holds if the group is torsion-complete.*

Proof.

(a) We show: if A is algebraically compact, then $\mathsf{E} = \operatorname{Hom}(A, A)$ is an algebraically compact right E-module (the first A being viewed as a left E-module). Let N be a pure submodule in a right E-module M. It suffices to prove that the group homomorphism

$$\operatorname{Hom}_{\mathsf{E}}(M, \operatorname{Hom}(A, A)) \to \operatorname{Hom}_{\mathsf{E}}(N, \operatorname{Hom}(A, A))$$

induced by the inclusion map $N \to M$ is surjective. In view of a well-known natural isomorphism, this can be rewritten as

$$\mathrm{Hom}(M \otimes_{\mathsf{E}} A, A) \to \mathrm{Hom}(N \otimes_{\mathsf{E}} A, A).$$

Since $N \to M$ is a pure inclusion as right E-modules, $N \otimes_{\mathsf{E}} A \to M \otimes_{\mathsf{E}} A$ is a pure inclusion group-theoretically. Therefore, by the pure-injectivity of A, the map in question is in fact surjective.

(b) A reduced torsion group A has the same endomorphism ring as its cotorsion hull $A^\bullet$. For a torsion-complete group, the cotorsion and pure-injective hulls are identical. □

Endomorphism Rings of Divisible Groups We now turn to injective groups; their endomorphism rings are very special, indeed.

Theorem 4.3. *The endomorphism ring* E *of a divisible group* D *is a right algebraically compact ring whose Jacobson radical* J *consists of those* $\eta \in \mathsf{E}$ *for which* $\mathrm{Ker}\,\eta$ *is an essential subgroup in* D.

Idempotents mod J *lift, and* E/J *is a von Neumann regular ring.*

Proof. By Theorem 4.2, E is an algebraically compact ring. Let $\chi \in \mathsf{E}$ be such that $\mathrm{Ker}\,\chi$ is essential in D. Then $\mathrm{Ker}\,\eta\chi$ is also essential in D for every $\eta \in \mathsf{E}$. Since $\mathrm{Ker}\,\eta\chi \cap \mathrm{Ker}(1-\eta\chi) = 0$, we obtain that $\mathrm{Ker}(1-\eta\chi) = 0$, so $1-\eta\chi$ is a monomorphism. Evidently, $\mathrm{Ker}\,\eta\chi \le \mathrm{Im}(1-\eta\chi)$, therefore $\mathrm{Im}(1-\eta\chi)$ is an essential divisible subgroup, whence it follows that, for all η, $1-\eta\chi$ is an automorphism of D, i.e. an invertible element in E. Thus $\chi \in \mathsf{J}$. Conversely, if $\chi \in \mathsf{J}$, then let K be a subgroup in D with $K \cap \mathrm{Ker}\,\chi = 0$. By the injectivity of D, there is a $\xi : D \to D$ such that $\xi\chi x = x$ for all $x \in K$. This means that $K \le \mathrm{Ker}(1-\xi\chi)$ where $1-\xi\chi \in \mathrm{Aut}\,D$ (as $\chi \in \mathsf{J}$). Thus $K = 0$, and $\mathrm{Ker}\chi$ is essential in D.

If $\chi \in \mathsf{E}$ satisfies $\chi^2 - \chi \in \mathsf{J}$, then $N = \mathrm{Ker}(\chi^2-\chi)$ is essential in D. Now $\chi N \cap \mathrm{Ker}\,\chi = 0$, for if χa ($a \in N$) is annihilated by χ, then $0 = \chi(\chi a) = \chi a$. The subgroup $\chi N \oplus \mathrm{Ker}\,\chi$ is essential in D, since it contains N: every $a \in N$ can be written in the form $a = \chi a + (1-\chi)a \in \chi N \oplus \mathrm{Ker}\,\chi$. Thus $D = B \oplus C$ where $\chi N \le B, \mathrm{Ker}\,\chi \le C$. If $\epsilon : D \to B$ denotes the projection with $\epsilon C = 0$, then $\chi - \epsilon$ annihilates every $a = \chi a + (1-\chi)a \in N$. Therefore, $\chi - \epsilon \in \mathsf{J}$, showing that χ lifts to ϵ mod J.

Assume now that $\chi \in \mathsf{E}$, and $K \le D$ is maximal with respect to the property $K \cap \mathrm{Ker}\,\chi = 0$. Then $K + \mathrm{Ker}\,\chi \le \mathrm{Ker}(\chi - \chi\xi\chi)$ where $\xi \in \mathsf{E}$ is as above. Hence $\mathrm{Ker}(\chi - \chi\xi\chi)$ is essential in D, and so $\chi - \chi\xi\chi \in \mathsf{J}$. We conclude that E/J is a von Neumann regular ring. □

Although we won't do it here, with some extra effort we can even prove that the factor ring E/J is self-injective; see Notes.

Endomorphism Rings of Cotorsion Groups Finally, we prove something concerning endomorphism groups of cotorsion groups that should not be surprising.

Theorem 4.4. *The endomorphism group of a cotorsion group is cotorsion.*

Proof. If C is an adjusted cotorsion group, then $\operatorname{End} C \cong \operatorname{End} tC$ by Proposition 1.7(ii). As tC is torsion, $\operatorname{End} tC$ is algebraically compact.

In general, let $G = C \oplus A \oplus D$ where C is adjusted cotorsion, A is reduced torsion-free algebraically compact, and D is divisible. Since homomorphism groups into algebraically compact groups are algebraically compact, $\operatorname{Hom}(G, A \oplus D)$ is algebraically compact (Theorem 2.11 in Chapter 7). To find the structure of $\operatorname{Hom}(G, C)$, observe that $\operatorname{Hom}(D, C) = 0$, thus it remains to check $\operatorname{Hom}(A, C)$. If B is a basic subgroup of the complete group A, then we have an exact sequence $0 \to B \to A \to \oplus \mathbb{Q} \to 0$. This implies the exactness of $0 = \operatorname{Hom}(\oplus \mathbb{Q}, C) \to \operatorname{Hom}(A, C) \to \operatorname{Hom}(B, C) \to \operatorname{Ext}(\oplus \mathbb{Q}, C) = 0$, thus the middle terms are isomorphic. As B is the direct sum of copies of $\mathbb{Z}_{(p)}$ for various primes p, $\operatorname{Hom}(B, C)$ will be the direct product of copies of $C_{(p)}$ for various primes p, so cotorsion. □

★ **Notes.** In general, endomorphism rings of cotorsion groups need not be cotorsion rings, but they are in the torsion-free case (Theorem 4.2). For the endomorphism ring of an algebraically compact group, a result similar to the divisible case can be established: idempotents mod its Jacobson radical J lift, and E/J is a von Neumann regular self-injective ring. This follows from a ring-theoretical result by Zimmermann–Zimmermann-Huisgen [Math. Z. **161**, 81–93 (1978)] stating that the radical of a pure-injective ring has this property.

Exercises

(1) (Szélpál) (a) $\operatorname{End} A$ is a torsion ring if and only if A is a bounded group.

(b) $\operatorname{End} A$ is torsion-free if and only if the reduced part of A is torsion-free, and $pA \neq A$ implies $A[p] = 0$.

(2) The endomorphism ring of a cotorsion group need not be algebraically compact as a group.

(3) The endomorphism ring of the additive group of any injective module is algebraically compact.

(4) (Rangaswamy) For a torsion group A, $\operatorname{End} A$ is a Baer ring if and only if, for every prime p, A_p is either divisible or elementary.

5 Special Endomorphism Rings

So far our investigations relating a group and its endomorphism ring have mainly been concerned with the question as to how the group structure is reflected in the endomorphism ring. In this section we approach this problem from the opposite direction, and the dominant theme will be the kind of influence properties of endomorphism rings have on the underlying group. The results in this section would seem to suggest that specific endomorphism rings are indeed special. Needless to say, we are primarily interested in conventional ring properties.

It seems sensible at this point to stress that one should not have high expectations of the interrelations between interesting group and important ring properties. We keep emphasizing that significant group and ring properties seldom match: groups whose endomorphism rings are of special interest in ring theory are few and far between.

We are going to survey several cases of interest.

Elementary Properties We start assembling a few simple observations that will simplify subsequent arguments. We will abbreviate $\mathsf{E} = \operatorname{End} A$.

(a) *Suppose* $\alpha \in \mathsf{E}$. *If* $n|\alpha$, *then* $\alpha A \leq nA$, *and if* $n\alpha = 0$, *then* $\alpha A \leq A[n]$ *for* $n \in \mathbb{N}$. Indeed, if $\beta \in \mathsf{E}$ satisfies $n\beta = \alpha$, then $\alpha A = n\beta A \leq nA$, and if $n\alpha = 0$, then $n\alpha A = 0$, so $\alpha A \leq A[n]$.

(b) *If* $\operatorname{End} A = \mathsf{R}_1 \oplus \mathsf{R}_2$ *is a ring direct sum, then* $A = B_1 \oplus B_2$ *where* $\mathsf{R}_i \cong \operatorname{End} B_i$ *and* $\operatorname{Hom}(B_i, B_j) = 0$ *for* $i \neq j$. This is a simple consequence of Sect. 1(D).

(c) *If* A *has a sequence of direct decompositions*

$$A = A_0 \oplus \cdots \oplus A_n \oplus C_n \qquad \textit{with} \quad C_n = A_{n+1} \oplus C_{n+1} \quad (n < \omega)$$

where $A_n \neq 0$ $(n < \omega)$, *then neither the left nor the right* (*principal*) *ideals of* E *satisfy the minimum condition.* To verify this, let $\pi_n : A \to C_n$ denote the obvious projection. Then $\pi_n\pi_{n+1} = \pi_{n+1}$, but no $\alpha \in \mathsf{E}$ exists satisfying $\pi_{n+1}\alpha = \pi_n$, because $\operatorname{Im} \pi_{n+1}\alpha \leq \operatorname{Im} \pi_{n+1} < \operatorname{Im} \pi_n$. This establishes the proper inclusion $\pi_{n+1}\mathsf{E} < \pi_n\mathsf{E}$ for every n. Also, $\pi_{n+1}\pi_n = \pi_{n+1}$, and there is no $\beta \in \mathsf{E}$ with $\beta\pi_{n+1} = \pi_n$, since clearly $\operatorname{Ker} \pi_n < \operatorname{Ker} \pi_{n+1} \leq \operatorname{Ker} \beta\pi_{n+1}$. This shows that $\mathsf{E}\pi_{n+1} < \mathsf{E}\pi_n$.

(d) *If* γ *is in the center of* E, *then* $\operatorname{Im} \gamma$ *and* $\operatorname{Ker} \gamma$ *are fully invariant subgroups of* A. For each $\eta \in \mathsf{E}$, we have $\eta\gamma(A) = \gamma\eta(A) \leq \operatorname{Im} \gamma$, and if $a \in \operatorname{Ker} \gamma$, then $\gamma\eta(a) = \eta(\gamma(a)) = 0$ means $\eta(a) \in \operatorname{Ker} \gamma$.

(e) *In a torsion ring* (*with or without identity*), *the divisible subgroup is contained in the annihilator of the ring.* Thus $1 \in \mathsf{R}$ implies R is reduced.

Recall that an element $\alpha \in \mathsf{R}$ is **regular** (in the sense of von Neumann) if $\alpha\beta\alpha = \alpha$ for some $\beta \in \mathsf{R}$, and π**-regular** if α^m is regular for some $m \in \mathbb{N}$.

Lemma 5.1 (Rangaswamy [1]). *An endomorphism* α *of* A *is a* (*von Neumann*) *regular element in* $\operatorname{End} A$ *if and only if both* $\operatorname{Im} \alpha$ *and* $\operatorname{Ker} \alpha$ *are summands of* A.

Proof. Assume $\alpha \in \mathsf{E} = \operatorname{End} A$ satisfies $\alpha\beta\alpha = \alpha$ for some $\beta \in \mathsf{E}$. Since $\alpha\beta$ and $\beta\alpha$ are idempotents, they are projections of A, so their images and kernels are summands of A. The inclusions $\operatorname{Im} \alpha\beta\alpha \leq \operatorname{Im} \alpha\beta \leq \operatorname{Im} \alpha$ and $\operatorname{Ker} \alpha \leq \operatorname{Ker} \beta\alpha \leq \operatorname{Ker} \alpha\beta\alpha$ imply $\operatorname{Im} \alpha = \operatorname{Im} \alpha\beta$ and $\operatorname{Ker} \alpha = \operatorname{Ker} \beta\alpha$, so $\operatorname{Im} \alpha, \operatorname{Ker} \alpha$ are summands of A.

Conversely, suppose $\operatorname{Im} \alpha = B$ and $\operatorname{Ker} \alpha = K$ are summands of A, say, $A = B \oplus C = K \oplus H$ for some $C, H \leq A$. Because of $K \cap H = 0$, α maps H isomorphically into, and evidently onto, B. This means, there is a $\beta \in \operatorname{End} A$ that annihilates C and is inverse to $\alpha \restriction H$ on B. Writing $a = g + h \in K \oplus H$, we get $(\alpha\beta\alpha)a = (\alpha\beta)\alpha h = \alpha h = \alpha a$ for every $a \in A$, whence $\alpha\beta\alpha = \alpha$ follows. □

Simple Endomorphism Rings Our survey of special endomorphism rings starts with division rings.

Proposition 5.2 (Szele [2]). *The endomorphism ring of A is a division ring if and only if $A \cong \mathbb{Q}$ or $A \cong \mathbb{Z}(p)$ for some prime p.*

A division ring is the endomorphism ring of a group if and only if it is the additive group of a prime field.

Proof. If $\operatorname{End} A$ is a division ring, then every non-zero $\alpha \in \operatorname{End} A$ is an automorphism, which entails, for every prime p, either $pA = A$ or $pA = 0$. It also follows that $0, 1$ are the only idempotents in $\operatorname{End} A$. After ruling out $\mathbb{Z}(p^\infty)$, it is clear that the only possibility is either $A \cong \mathbb{Z}(p)$ for some p, or else $A \cong \mathbb{Q}$. □

By a **simple ring** is meant a ring R whose only ideals are 0 and R.

Theorem 5.3. $\operatorname{End} A$ *is a simple ring if and only if, for some integer n, either $A \cong \oplus_n \mathbb{Q}$ or $A \cong \oplus_n \mathbb{Z}(p)$ for a prime p.*

A simple ring is the endomorphism ring of a group if and only if it is a complete matrix ring of finite order over a prime field.

Proof. Sufficiency is obvious from the matrix representation of direct sums. Conversely, if $\mathsf{E} = \operatorname{End} A$ is a simple ring, then for every prime p, either $p\mathsf{E} = 0$ or $p\mathsf{E} = \mathsf{E}$. In the first alternative, (a) above implies that $A \leq A[p]$, i.e. A is an elementary p-group. If $p\mathsf{E} = \mathsf{E}$ for every p, then (a) applied to $\alpha = 1$ shows that A is divisible. In this case, the socle of E, if not zero, would be a non-trivial ideal, therefore E must be torsion-free, and division by every p is an automorphism of A. Consequently, A is likewise torsion-free divisible. Our conclusion is that either $A \cong \oplus \mathbb{Z}(p)$ or $A \cong \oplus \mathbb{Q}$. The endomorphisms mapping A onto a finite rank subgroup form an ideal in E, by simplicity this must be all of E. Hence A is a finite direct sum, and we are done, since the second assertion is pretty clear from our argument. □

Artinian Endomorphism Rings The preceding theorem shows that simple rings can be endomorphism rings only if they are left and right artinian. It is not difficult to identify all artinian endomorphism rings.

Theorem 5.4.

(i) $\operatorname{End} A$ *is left* (*right*) *artinian if and only if*

$$A = B \oplus D$$

where B is finite and D is torsion-free divisible of finite rank.

(ii) (Szász [1]) *The same holds for left* (*right*) *perfect rings.*

Proof.

(i) If $\mathsf{E} = \operatorname{End} A$ is left (right) artinian, then in the set of ideals $\{n\mathsf{E} \mid n \in \mathbb{N}\}$ there is a minimal one, say, $m\mathsf{E}$; clearly, it must be divisible. (a) above shows that $nm|m\mathbf{1}_A$ implies $mA \le nmA$ for every $n \in \mathbb{N}$, i.e. mA is divisible. Consequently, $A = B \oplus D$ where $mB = 0$ and D is divisible. From (c) we derive that both B and D are of finite rank; in particular, B is a finite group. The presence of $\mathbb{Z}(p^\infty)$ is ruled out, since a ring with $\mathbf{1}$ cannot have a non-zero annihilator.

For the converse, assume A has the stated decomposition. Then the two summands are fully invariant, so $\operatorname{End} A = \operatorname{End} B \oplus \operatorname{End} D$. Hence $\operatorname{End} B$ is finite and $\operatorname{End} D$ is a complete matrix ring of finite order over $\mathbb{Q}$, so $\operatorname{End} A$ is artinian.

(ii) The same proof goes through under the milder hypothesis that the principal left (right) ideals satisfy the minimum condition, i.e. $\operatorname{End} A$ is a left (right) perfect ring. $\square$

PID Endomorphism Rings We have seen that the endomorphism rings of Procházka-Murley groups were PID. Clearly, a group with PID endomorphism ring is indecomposable. More specific results can be stated about the behavior of summands with PID endomorphism rings. A typical result is recorded in the next proposition. We recall that quasi-isomorphic groups whose endomorphism rings are PID have isomorphic endomorphism rings (cf. Lemma 9.8 in Chapter 12).

Proposition 5.5. *Let $A = A_1 \oplus \cdots \oplus A_n$ where the summands are quasi-isomorphic finite rank torsion-free groups and have endomorphism rings isomorphic to the same principal ideal domain* D.

(i) *For every endomorphism η of A,* $\operatorname{Ker}\eta$ *is a summand of A. It is the direct sum of* D*-modules quasi-isomorphic to the A_i.*

(ii) *A pure subgroup of A that is quasi-isomorphic to one* (*and hence to each*) *of A_i, is a summand of A.*

Proof.

(i) An endomorphism η of A can be represented as an $n \times n$ matrix $\mathbb{M} = \|\eta_{ij}\|$ with entries in the field Q of quotients of D. For the proof, we multiply $\mathbb{M}$ by a suitable $0 \ne \gamma \in \mathsf{D}$ so as to have all the entries contained in D. By hypothesis, D is a PID, so there exist invertible matrices $\mathbb{A} = \|\alpha_{ki}\|, \mathbb{B} = \|\beta_{j\ell}\| \in \mathsf{M}_n(\mathsf{D})$ (the full matrix ring of order n over D) such that $\mathbb{AMB} = \|\delta_{k\ell}\|$ is a diagonal matrix: $\delta_{k\ell} = 0$ for $k \ne \ell$ (and $\delta_{kk}|\delta_{\ell\ell}$ if $k < \ell$). An easy calculation shows that its diagonal entries are

$$\delta_{kk} = \sum_{i,j} \alpha_{ki}(\gamma\eta_{ij})\beta_{jk} \in \gamma\mathsf{D} \quad (k \le n),$$

where the indicated inclusion $\delta_{kk} \in \gamma\mathsf{D}$ is justified by the fact (as we shall see in the next paragraph) that the diagonal elements represent endomorphisms of quasi-isomorphic modules, so by Lemma 9.8 in Chapter 12 they have to be elements of $\operatorname{End} A_i = \mathsf{D}$. (One can also argue that $\gamma\mathbb{AMB}$ represents a map

$A \to \rho A$ which must evidently be γ times of a map $A \to A$.) Consequently, the matrix of η admits a diagonal form with entries in Q.

This fact translated to maps asserts that there are automorphisms ρ, σ of A (corresponding to $\mathbb{A}, \mathbb{B}$) such that $\chi = \rho\eta\sigma$ is represented by a diagonal matrix. The zero columns in this matrix correspond to the kernel of χ; consequently, the kernel is of the form $\operatorname{Ker}\chi = B_1 \oplus \cdots \oplus B_m$ with $m \leq n$. Each component B_j is obtained as a sum of isomorphic copies of some of the components A_i, so it is evidently quasi-isomorphic to the A_i. Hence for $\operatorname{Ker}\eta = \operatorname{Ker}\rho\eta = \operatorname{Ker}\chi\sigma^{-1} = \sigma(\operatorname{Ker}\chi)$ we have

$$\operatorname{Ker}\eta = \sigma(B_1 \oplus \cdots \oplus B_m) = \sigma B_1 \oplus \cdots \oplus \sigma B_m.$$

(ii) Let $\pi_i : A \to A_i$ denote the projections in the given direct decomposition, and let us fix quasi-isomorphisms $\alpha_i : A_i \to C$ $(i = 1, \ldots, n)$ where C is a pure subgroup in A. Evidently, $\alpha_i\pi_i \restriction C$ is an endomorphism of C, say, it is equal to some $\epsilon_i \in \mathsf{D}$ (Lemma 9.8 in Chapter 12). Let $\epsilon = \gcd\{\epsilon_1, \ldots, \epsilon_n\}$ be calculated in D; thus, we have $\epsilon = \tau_1\epsilon_1 + \cdots + \tau_n\epsilon_n$ for suitable $\tau_i \in \mathsf{D}$. Then $\eta = \sum_i \tau_i\alpha_i\pi_i : A \to C$ is an endomorphism of A whose restriction to C acts like ϵ.

We use induction on n to complete the proof. If $n = 1$, then $C \leq A_1$, and by purity equality holds. Next suppose $n > 1$. Evidently, $\xi = \eta - \epsilon$ is a non-zero endomorphism of A that annihilates C. Owing to (i), $\operatorname{Ker}\xi$ is a summand of A, and also a direct sum of groups quasi-isomorphic to the A_i. Since $\xi \neq 0$, this summand has less than n components, so induction hypothesis applies. The claim that C is a summand follows at once. □

Noetherian Endomorphism Rings It is a trivial observation that if $\operatorname{End} A$ is noetherian, then A decomposes into the direct sum of a finite number of indecomposable groups. It seems difficult to say much more in general about groups with noetherian endomorphism rings: just consider arbitrarily large torsion-free groups whose endomorphism rings are $\cong \mathbb{Z}$. The only hope is to put aside torsion-free groups, and concentrate on torsion groups. Luckily, we have a complete description in this case.

Theorem 5.6. *Suppose A is a torsion group.* $\operatorname{End} A$ *is left* (*or right*) *noetherian exactly if A is finitely cogenerated.*

Proof. If $\operatorname{End} A$ is noetherian, then A is a finite direct sum of indecomposable (i.e. cocylic) groups, so it is finitely cogenerated. Conversely, if A is finitely cogenerated, then by Theorem 5.3 in Chapter 4, $A = B \oplus D$ with finite B and finite rank divisible D. In the additive decomposition $\operatorname{End} A = \operatorname{Hom}(B, D) \oplus \operatorname{End} D$, the first summand is finite, while the second summand is a finite direct sum of complete matrix rings over the p-adic integers, for various primes. Thus $\operatorname{End} A$ is a finite extension of a two-sided noetherian ring. □

Notice a kind of duality: a torsion group A has the maximum (minimum) condition of subgroups if and only if $\operatorname{End} A$ has the minimum (maximum) condition on left (or right) ideals.

Regular Endomorphism Rings Our next project in this section is concerned with (von Neumann) regular rings.

Example 5.7.

(a) The endomorphism ring of an elementary group is regular. This is obvious from the fact that every subgroup is a summand; see Lemma 5.1.
(b) A direct product $\prod_p A_p$ of elementary p-groups A_p, for different primes p, has also regular endomorphism ring. This follows from the isomorphism $\mathrm{End}(\prod_p A_p) \cong \mathrm{End}(\oplus_p A_p)$.

Example 5.8. Torsion-free divisible groups have regular endomorphism rings.

Theorem 5.9. *If* $\mathrm{End}\, A$ *is a von Neumann regular ring, then* $A = D \oplus G$ *where*

(i) *D is torsion-free divisible; it is* 0 *if G is not torsion;*
(ii) *G is a pure subgroup between $\oplus_{p\in P} T_p$ and $\prod_{p\in P} T_p$, where T_p is an elementary p-group, and P is a set of primes.*

Proof. If $\mathrm{End}\, A$ is a regular ring, then to every $\alpha \in \mathrm{End}\, A$ there is a $\beta \in \mathrm{End}\, A$ such that $\alpha\beta\alpha = \alpha$. Specifically, if α is multiplication by p, then $p\beta p = p$, which implies that A contains no elements of order p^2, so A_p, the p-component of A, is an elementary group. Since also $p = p\beta p\beta p = p\beta p\beta p\beta p = \dots$, it is clear that, for every $a \in A$, $h_p(pa) = \infty$. Therefore, pG is p-divisible. Write $A = D \oplus G$ with D divisible and G reduced; at this point we already know that D is torsion-free, and $T = tG$ is an elementary group. We have $G = T_p \oplus G(p)$ with p-divisible $G(p)$, and it follows that $G(p) = pG$. As G/T_p is p-divisible, G/T is divisible.

The intersection $\cap_p pG$ is clearly a torsion-free subgroup of G. For every $a \in G$, pa belongs to every G_q (for every prime q), and is divisible in G_q by every p^k $(k \geq 2)$ uniquely. This means $\cap_p pG$ is divisible, and hence 0. Thus the intersection of the kernels of the projections $\pi_p : G \to T_p$ is 0, and consequently, G is isomorphic to a subgroup of the direct product $\prod_{p\in P} T_p$, containing $T = \oplus_{p\in P} T_p$ with divisible G/T.

If G is not torsion and $D \neq 0$, then A has an endomorphism mapping $G \to D$ non-trivially whose kernel contains T, but is not a summand (G has no proper summand containing T). This is in violation to Lemma 5.1. $\square$

Semi-Local Endomorphism Rings Finally, we consider local and semi-local endomorphism rings. Recall that a ring R is said to be **semi-local** if R/J is a semi-simple artinian ring; as usual, here J denotes the Jacobson radical of R. R is **local** if R/J is a division ring.

Lemma 5.10 (Orsatti [1]). *A torsion group has local endomorphism ring if and only if it is cocyclic.*

Proof. A local ring has only two idempotents: 0 and 1, therefore a group with local endomorphism ring is indecomposable. A torsion group is indecomposable if and only if it is cocyclic. Since $\mathrm{End}\, \mathbb{Z}(p^n) \cong \mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{End}\, \mathbb{Z}(p^\infty) \cong J_p$ are local rings, the claim is evident. $\square$

For the semi-local case we prove:

Theorem 5.11 (Călugăreanu [1]). *A group A has semi-local endomorphism ring if and only if $A = T \oplus G$ where T is a finitely cogenerated group, and G is torsion-free with semi-local* End G.

In case G has finite rank, End G *is semi-local exactly if $pG = G$ for almost all primes p.*

Proof. Assume End A is semi-local. A semi-local ring has but a finite number of orthogonal idempotents, hence A is a finite direct sum of indecomposable groups. This must be true for $tA = T$ as well, whence $A = T \oplus G$ where T and G are as stated (because summands inherit semi-local endomorphism rings). If $pG \neq G$, then also $p\,\mathrm{End}\,G \neq \mathrm{End}\,G$, and there is at least one maximal left ideal L_p between $p\,\mathrm{End}\,G$ and $\mathrm{End}\,G$. Since $\mathsf{L}_p \neq \mathsf{L}_q$ if $p \neq q$ are primes, and since in a semi-local ring the Jacobson radical is the intersection of finitely many maximal left ideals, $pG < G$ can hold only for a finite number of primes.

Conversely, if A is as stated, then both T and G have semi-local endomorphism rings, and it is easy to check that the same is true for A (the radical of End A is then the direct sum of the radicals of End T and End G). If G is of finite rank, then so is End A, and because $\mathrm{End}\,A/p\,\mathrm{End}\,A$ is not 0 but for finitely many primes p, and then it is finite, the same must hold for $(\mathrm{End}\,A)/\mathsf{J}$. Hence J is a finite intersection of maximal left ideals, and $(\mathrm{End}\,A)/\mathsf{J}$ is a finite direct sum of simple artinian rings. □

★ **Notes.** It was Szele who initiated a systematic study of groups whose endomorphism rings belong to a class of rings of interest, and since then this has been a recurrent theme in the literature. While the theorems above convince us that some familiar ring properties of the endomorphism ring might impose severe restrictions on the groups, it is unlikely that classes of groups whose endomorphism rings share some other prominent ring properties, like commutativity, hereditariness etc. will admit a satisfactory description in the near future, since these do not impose much restriction on the groups. There exists an extensive literature on special endomorphism rings of restricted classes of groups, many of these results are technical or involve more advanced ring theory.

For more results on cases when End A is regular, or when its principal right ideals are projective, see Glaz–Wickless [1]. Karpenko–Misyakov [1] describe the groups whose endomorphism rings have regular center. Mader [5] investigates the maximal regular ideal in endomorphism rings. Groups with π-regular endomorphism rings were studied in Fuchs–Rangaswamy [1]; the results are similar to Example 5.7. See F. Kasch–A. Mader, *Regularity and substructures of Hom* (2009) where regular homomorphisms were studied.

Misyakova [1] examines the case when the endomorphism ring is semi-prime (i.e., the intersection of prime ideals is 0). Salce–Menegazzo [1] investigate groups with linearly compact endomorphism rings. Several papers are devoted to self-injective endomorphism rings, see Albrecht [2], Ivanov [6], Rangaswamy [3].

Exercises

(1) The endomorphism ring of a separable torsion-free group A is left (or right) noetherian if and only if A is completely decomposable of finite rank.

(2) Find groups in which every endomorphism is either an automorphism or nilpotent.
(3) (Szele–Szendrei)

  (a) If $\operatorname{End} A$ is commutative, then the p-components A_p are cocyclic, and $A/t(A)$ is p-divisible for every p with $A_p \neq 0$.
  (b) A splitting mixed group A has commutative endomorphism ring exactly if both $\operatorname{End} t(A)$ and $\operatorname{End}(A/t(A))$ are commutative, and A satisfies the condition in (a).

(4) The endomorphism ring of $A = \oplus_p \mathbb{Z}(p) \oplus \prod_p \mathbb{Z}(p)$ is not regular.
(5) (Rangaswamy) Kernels and images of all the endomorphisms of A are pure subgroups in A if and only if $t(A)$ is elementary and $A/t(A)$ is divisible.
(6) (Rangaswamy) If the kernel of an $\eta \in \operatorname{End} A$ is a summand of A, then the right annihilator of η is a projective right ideal of $\operatorname{End} A$. [Hint: if $\epsilon : A \to \operatorname{Ker} \eta$ is projection, then $\operatorname{Ann} \eta = \epsilon \operatorname{End} A$.]
(7) (Călugăreanu) A divisible group has semi-local endomorphism ring exactly if it is of finite rank.
(8) If A is a p-group and $\operatorname{End} A$ is Dedekind-finite (i.e. $\alpha\beta = \mathbf{1}$ $(\alpha, \beta \in \operatorname{End} A)$ implies $\beta\alpha = \mathbf{1}$), then the UK-invariants of A are finite.

6 Groups as Modules Over Their Endomorphism Rings

Since every group A is a left module over its own endomorphism ring $\mathsf{E} = \operatorname{End} A$, it is tempting to strive for a better understanding as to how A behaves as an E-module. A large number of special cases have already been investigated (mostly restricted to subcategories of $\mathcal{A}b$), but so far no systematic study is available. Here we collect a few special cases of interest in order to give a flavor of this interesting topic.

Preliminaries A few trivial facts to keep in mind:

(A) A group A is a faithful E-module (i.e. no non-zero element of A is annihilated by E).
(B) The E-submodules are exactly the fully invariant subgroups.
(C) If $\phi : A \to N$ is an E-homomorphism into an E-module N, then $\operatorname{Ker} \phi$ is a fully invariant subgroup of A.

Our starting point is a brief analysis of the E-homomorphisms into and from A. The basic facts for the primary case are summarized in the following lemma.

Lemma 6.1. *Let A be a p-group with a fully invariant subgroup H.*

(i) *An endomorphism is an E-map if and only if it is multiplication by a p-adic integer.*
(ii) *If A is either bounded or has an unbounded basic subgroup, then a homomorphism $\alpha : A \to A/H$ is an E-map if and only if it is the canonical homomorphism followed by multiplication by a p-adic integer.*

Proof. The 'if' parts of both claims are obvious, so we proceed to prove necessity. Let $A = C \oplus U$ be a decomposition of A where C is a cocyclic group. If C is cyclic of order p^n, then let $c \in C$ denote a generator, and if $C \cong \mathbb{Z}(p^\infty)$, then pick any non-zero element $c \in C$, say, of order p^n. Choose an $\eta \in \mathsf{E}$ so as to satisfy:

$$\eta\colon\ c \mapsto c + x, \quad u \mapsto tu \quad (\forall\, u \in U) \tag{16.5}$$

where t is any integer; $x \in A[p^n]$ is chosen arbitrarily if C is cyclic, while if C is quasi-cyclic, then x can be any element of order $\leq p^n$ in the divisible subgroup of A.

(i) We need a brief calculation. Suppose $\theta \in \mathsf{E}$ is an E-map, and write $\theta(c) = kc + v \in A$ with $k \in \mathbb{Z}, v \in U$. Then

$$\eta\theta(c) = kc + kx + tv \quad \text{and} \quad \theta\eta(c) = kc + v + \theta(x).$$

The equality $\eta\theta = \theta\eta$ holds for all possible choices of t and x; this yields $v = 0$ and $\theta(x) = kx$. Hence we conclude that if C is cyclic, then θ acts on $A[p^n]$ as multiplication by k. If C is quasi-cyclic, then the same can be said only about the action of θ in the divisible subgroup of H. The integer k depends on the order of c, but for different elements the numbers must match, so it follows by usual arguments that there is a $\rho \in J_p$ such that θ acts as multiplication by ρ. In case A is bounded by p^n, then only multiplications mod p^n need to be considered.

(ii) Assume A is as stated above in (ii), so it has a cyclic summand $C = \langle c \rangle$ of order p^n with maximal n or with arbitrarily large n. If $A = C \oplus U$, then clearly, $A/H = C/(H \cap C) \oplus U/(H \cap U)$. For an E-map $\alpha : A \to A/H$, we set $\alpha c = k\bar{c} + \bar{v}$ with $k \in \mathbb{Z}, v \in U$ where bars indicate cosets mod H. If η is as above in (16.5), then $\eta\bar{c} = \bar{c} + \bar{x}$ and $\eta\bar{u} = t\bar{u}$. We then have

$$\eta\alpha(c) = \eta(k\bar{c} + \bar{v}) = k\bar{c} + k\bar{x} + t\bar{v},\ \alpha\eta(c) = \alpha(c + x) = k\bar{c} + \bar{v} + \alpha x.$$

These have to be equal for all permissible choices of x and t, whence $\bar{v} = 0$, $\alpha c = k\bar{c}$, and $\alpha x = k\bar{x}$. Hence α acts as multiplication by the integer k on $A[p^n]$ mod H, in particular, on $\langle\bar{c}\rangle \cong \langle c\rangle/(H \cap \langle c\rangle)$. As in (i), we argue that these numbers k define a $\rho \in J_p$ such that $\alpha x = \rho\bar{x}$ for all $x \in A$. Consequently, $\alpha = \phi\rho$ where $\phi : A \to A/H$ is the canonical map.

(From Theorem 6.11 below it will follow that (ii) fails for p-groups of the form $B \oplus D$ with bounded B and divisible $D \neq 0$.) □

From the preceding lemma it is obvious that the ring of E-endomorphisms of a p-group is isomorphic to the center of E.

To formulate the precise statements in the following theorems, we will need new definitions. To simplify, following [KMT], we adopt an easy terminology: a group will be called **endo-P** if it has property P as a left E-module.

Endo-Finitely-Generated Groups First, we exhibit a few examples for endo-finitely generated groups. Some are even endo-cyclic.

Example 6.2. An immediate example for an endo-finitely-generated group is any bounded group B. If $b \in B$ is an element of maximal order, then B is generated by b over $\mathsf{E} = \operatorname{End} B$. Thus $B = \mathsf{E}b$ is endo-cyclic.

Example 6.3. A torsion-free divisible group is also endo-cyclic. The direct sum of a torsion-free divisible group and a bounded group is endo-finitely generated.

Example 6.4. An E-group A (see Sect. 6 in Chapter 18 for definition) is endo-cyclic: if $a \in A$ corresponds to $1 \in \mathsf{E}$, then $A = \mathsf{E}a$.

Endo-finitely generated torsion groups admit a full characterization.

Proposition 6.5. *A torsion group is endo-finitely-generated if and only if it is bounded.*

Proof. As pointed out in Example 6.2, a bounded group is endo-cyclic. On the other hand, if A is unbounded torsion, then a finite set can generate over $\operatorname{End} A$ only a bounded subgroup of A. □

Endo-Artinian and Endo-Noetherian Groups It is not difficult to characterize the groups that are artinian over their endomorphism rings; however, only little has been established for the noetherian case.

Theorem 6.6 (Krylov–Mikhalev–Tuganbaev[KMT]).

(i) *A group A is endo-artinian if and only if $A = B \oplus D$ where B is a bounded group, and D is a divisible group with a finite number of non-zero p-components.*
(ii) *A is endo-noetherian exactly if $A = B \oplus C$ where B is a bounded group, and C is torsion-free endo-noetherian.*

Proof.

(i) Assume A is endo-artinian. Then there is a minimal subgroup among the subgroups of the form nA ($n \in \mathbb{N}$), say, mA is minimal. This mA is divisible, so $A = B \oplus mA$ with $mB = 0$. Clearly, mA can have but a finite number of p-components $\neq 0$, because these are E-submodules.

Conversely, if A is of the form stated in the theorem, then B has only a finite number of fully invariant subgroups. The torsion-free part of D has no proper fully invariant subgroup $\neq 0$, while the fully invariant proper subgroups of a p-component D_p of D are of the form $D_p[p^k]$, so they satisfy the minimum condition.
(ii) Supposing A is endo-noetherian, there is a maximal subgroup in the set of subgroups of the form $A[n]$ ($n \in \mathbb{N}$). This must coincide with the torsion subgroup tA of A, thus $A = A[m] \oplus C$ for some $m \in \mathbb{N}$ and torsion-free C. Such a C must be endo-noetherian, because the fully invariant subgroups of A containing $A[m]$ correspond to those of C.

For the converse, note that all fully invariant subgroups of $A = B \oplus C$ are among the direct sums of fully invariant subgroups of B and C. □

Endo-noetherian torsion-free groups are abundant, and it seems impossible to characterize them reasonably.

Example 6.7. The strongly irreducible torsion-free groups of finite rank are endo-noetherian, since their non-zero fully invariant subgroups have finite indices.

Endo-Projective Groups As far as endo-projective groups are concerned, only the torsion groups have a satisfactory characterization, the general case seems to be out of reach at this time. It is an easy exercise to see that a torsion group is endo-projective exactly if its p-components are, thus it suffices to deal with p-groups.

Theorem 6.8 (Richman–Walker [1]). *A p-group is endo-projective if and only if it is bounded.*

Proof. Suppose the p-group A is E-projective. If A is bounded, then by Lemma 2.4(i), $A \cong \mathsf{E}\epsilon$ with an idempotent ϵ, thus A is a summand of the free module E.

If A is unbounded, then we concentrate on the group homomorphism

$$\chi\colon \operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) \otimes_{\mathsf{E}} A \to \operatorname{Hom}_{\mathsf{E}}(A, A)$$

acting as $(\theta \otimes a)(a') \mapsto \theta(a')(a)$, where $\theta \in \operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E})$ and $a, a' \in A$. Since A is a p-group, $\operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) \otimes_{\mathsf{E}} A$ is also a p-group, while $\operatorname{Hom}_{\mathsf{E}}(A, A)$ is the center of E that is isomorphic to J_p (cf. Theorem 2.9). Thus $\chi = 0$. It is clear that $\theta(a')(a) = 0$ for all $a, a' \in A$ only if $\operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) = 0$. But if A is E-projective, then A is a summand of a direct sum of copies of E, so $\operatorname{Hom}_{\mathsf{E}}(A, \mathsf{E}) \neq 0$. Therefore, no unbounded A can be endo-projective. □

Example 6.9. Examples of endo-projective torsion-free groups are $\mathbb{Z}$ and $\mathbb{Q}$. Also, $\mathbb{Q} \oplus \mathbb{Q}$ is endo-projective as it is isomorphic to $\mathsf{E}\epsilon$ for an idempotent ϵ.

Endo-Projective Dimension It is natural to inquire about the endo-projective dimension of groups. We will denote by p.d.A the E-projective dimension of A.

Theorem 6.10 (Douglas–Farahat [1]).

(i) *A torsion group is either endo-projective or has endo-projective dimension* 1.
(ii) *The same holds for a divisible group.*

Proof.

(i) It suffices to deal with p-groups A. Let ϵ_n denote the projection of A onto a summand $C_n \cong \mathbb{Z}(p^n)$. Then $A[p^n]$ is E-projective, being isomorphic to $\operatorname{Hom}(C_n, A) \cong \mathsf{E}\epsilon_n$. Thus p.d.$A = 0$ if A is bounded.

Next suppose A has an unbounded basic subgroup. Then A is the union of a countable ascending chain of E-projective subgroups $A[p^n]$ for integers n for which A has cyclic summands of order p^n. A well-known lemma of Auslander states that if the groups in such a chain have projective dimensions $\leq m$, then the projective dimension of the union is $\leq m + 1$; in our case, $m = 0$. Thus p.d.$A \leq 1$ in this case.

Finally, let $A = B \oplus D$ where B is p^n-bounded, and $D \neq 0$ is divisible; we may assume $B \neq 0$, since the divisible case will be settled in (ii). Consider the submodules $A_k = A[p^{n+k}]$ for $k < \omega$ whose union is A. For a cyclic summand C

of order p^n, we have $A_0 \cong \mathrm{Hom}(C, A)$ as E-modules, so p.d.$A_0 = 0$. If we prove that p.d.$(A_k/A_{k-1}) = 1$, then another form of Auslander's lemma on projective dimensions will assure that p.d.A cannot exceed 1. It is immediately seen that the factor group A_k/A_{k-1} is generated over E by any element $d \in D$ of order p^{n+k}, so it is isomorphic to E/L where $\mathsf{L} = \{\eta \in \mathsf{E} \mid \eta d \in A_{k-1}\}$ is a left ideal of E. Write $A = D' \oplus A'$ where $d \in D' \cong \mathbb{Z}(p^\infty)$, and define $\chi \in \mathsf{E}$ as multiplication by p on D' and the identity on A'. It follows that $\mathsf{L} = \mathsf{E}\chi \cong \mathsf{E}$, so L is E-projective. Hence p.d.$(A_k/A_{k-1}) =$ p.d.$(\mathsf{E}/\mathsf{L}) = 1$, and p.d.$A \le 1$.

(ii) If D is a divisible group that is not torsion, then $D \cong \mathsf{E}\pi$ where π denotes the projection onto a summand $\cong \mathbb{Q}$, since $D = \mathsf{E}d$ for any $d \in D$ of infinite order. In this case, D is evidently endo-projective.

If D is a divisible p-group, then form a projective resolution of $\mathbb{Z}(p^\infty)$ as a J_p-module: $0 \to H \to F \to \mathbb{Z}(p^\infty) \to 0$ where F, H are free J_p-modules (submodules of free are free as J_p is a PID). Now $\mathsf{E} = \mathrm{End}\, D$ is a (torsion-free, hence) flat J_p-module, so the tensored sequence

$$0 \to \mathsf{E} \otimes_{J_p} H \to \mathsf{E} \otimes_{J_p} F \to \mathsf{E} \otimes_{J_p} \mathbb{Z}(p^\infty) = D \to 0$$

is exact. Moreover, it is exact even as an E-sequence. The first two tensor products are free E-modules, whence p.d.$D \le 1$ is the consequence of a Kaplansky inequality for projective dimensions in an exact sequence. □

In contrast, the endo-projective dimension of a torsion-free group can be any integer or ∞; see the Notes.

Endo-Quasi-Projective Groups Turning our attention to the quasi-projective case, we can prove the following characterization for p-groups.

Theorem 6.11 (Fuchs [19]). *A p-group is endo-quasi-projective if and only if it is bounded or has an unbounded basic subgroup.*

Proof. Evidently, a group A is quasi-projective over E if and only if, for each fully invariant subgroup H of A, every E-homomorphism $\alpha : A \to A/H$ factors as $\alpha = \phi\theta$ where $\phi : A \to A/H$ is the canonical map, and $\theta : A \to A$ is a suitable E-map, i.e. multiplication by some $\rho \in J_p$.

To prove necessity, we have to rule out the case when $A = B \oplus D$ where $p^m B = 0$ for some $m \in \mathbb{N}$ and $D \neq 0$ is divisible. Let $\phi : A \to A/H$ be the canonical map with $H = A[p^m]$. Choose $\alpha : A \to A/H$ so as to satisfy $\alpha B = 0$ and $\alpha \restriction D : D \to D/(D[p^m])$ an isomorphism; clearly, α is an E-map. There is no $\rho \in J_p$ such that $\alpha = \phi\rho$, since $\alpha D[p^m] \neq 0$, but $\phi\rho D[p^m] = 0$ for all $\rho \in J_p$. Consequently, A must be as stated.

For sufficiency, note that a bounded group is by Theorem 6.8 endo-projective, while for groups with unbounded basic subgroups, an appeal to Lemma 6.1(ii) completes the proof. □

Endo-Flat Groups In order to describe the endo-flat torsion groups, we may again restrict our consideration to p-groups.

Theorem 6.12 (Richman–Walker [1]). *A p-group is endo-flat if and only if it is bounded or has an unbounded basic subgroup.*

Proof. If A is bounded, then it is E-projective, so a fortiori E-flat. It is shown in Theorem 2.4 that if A is a p-group with an unbounded basic subgroup, then $A[p^n]$ is E-projective (for any $n \in \mathbb{N}$ if there is a primitive idempotent of order p^n). Hence A is E-flat as the direct limit of E-projective groups.

It remains to show that $A = B \oplus D$ is not endo-flat if B is bounded and $D \neq 0$ is divisible. If $p^n B = 0$ (with smallest n), then $\mathsf{E} = \operatorname{End} A$ is additively the direct sum of a p^n-bounded group and a torsion-free group, and the same must hold for E-projective modules. Direct limits of groups that are direct sums of p^n-bounded and torsion-free groups are again of the same kind. Therefore, if $D \neq 0$, then A cannot be flat. □

We find it rather surprising that a torsion group is endo-flat if and only if it is endo-quasi-projective.

Example 6.13. Let A be a torsion-free group whose endomorphism ring is a PID (e.g. any rigid group). Then A is endo-flat, because it is torsion-free as an E-module. For more examples see Faticoni–Goeters [1].

Endo-Injective Groups We do not wish to discuss endo-injectivity in general, we only wish to prove that finite groups share this property. We will then state the general result without proof (in order to avoid the search for additional structural information).

Proposition 6.14. *Finite groups are endo-injective.*

Proof. We prove that a finite p-group F is endo-injective. For convenience, let us abbreviate $\operatorname{Hom}(F, \mathbb{Z}(p^\infty)) = F^\circ$. If F is viewed as a left E-module, then module theory tells us that F° becomes a right E-module, and $F^{\circ\circ}$ a left E-module. For finite F, we have the canonical E-isomorphism $F \to F^{\circ\circ}$. Since the action of E on F on the left is the same as the action of E on F° on the right, from Theorem 6.8 we infer that F° is a projective right E-module. We appeal to the natural isomorphism

$$\operatorname{Ext}^1_{\mathsf{E}}(C, \operatorname{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}^{\mathsf{E}}_1(B, C), \mathbb{Q}/\mathbb{Z})$$

that holds if C is a left and B is a right E-module (see Cartan–Eilenberg [CE]). Choose $B = F^\circ$, then its projectivity implies that the right-hand side is 0, so the same holds for the Ext on the left for each C. Hence $\operatorname{Hom}_{\mathbb{Z}}(F^\circ, \mathbb{Q}/\mathbb{Z})) = F^{\circ\circ} \cong F$ is an injective left E-module. □

The main result on endo-injectivity is the following theorem.

Proposition 6.15 (Richman–Walker [3]). *A group A is endo-injective if and only if it is of the form*

$$A = \prod\nolimits_p A_p \oplus D$$

where each A_p is a finite p-group, D is a divisible group of finite rank, and either

(i) $D = 0$*; or*
(ii) *D is not a mixed group, and almost all* $A_p = 0$. □

★ **Notes.** There is an extensive literature on groups (and more generally, on modules) that are of special kind over their own endomorphism rings. Significant advances have been made, and the proofs are getting more and more involved in ring theory. This area is certainly a gold mine for research problems. More on this subject may be found in the monograph Krylov–Mikhalev–Tuganbaev [KMT].

Some sample results from the rich selection:

(i) Reid [5, 6] has an in-depth study of the endo-finitely generated case.
(ii) Niedzwecki–Reid [1] described cyclic endo-projective groups A. Their result states that $A = \mathsf{R}^+ \oplus M$ where R is an E-ring and M is an E-module, i.e. an R-module with $\mathrm{Hom}_{\mathsf{R}}(\mathsf{R}, M) = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}, M)$.
(iii) See Arnold–Pierce–Reid–Vinsonhaler–Wickless [1] for more on endo-projectivity. They show that a torsion-free group is endo-projective whenever it is endo-quasi-projective, and an endo-projective group is 2-generated.
(iv) For arbitrary torsion-free groups, endo-quasi-projectivity was investigated by Vinsonhaler–Wickless [1], Vinsonhaler [1], Bowman–Rangaswamy [1].
(v) A is endo-flat exactly if for every finitely generated left ideal L of $\mathrm{End}\,A$, the canonical map $\mathsf{L} \otimes A \to \mathsf{L}A$ is an isomorphism (Goeters–Reid [1]). Cf. also Albrecht–Faticoni [1], Albrecht–Goeters–Wickless [1].
(vi) A completely decomposable group is endo-flat exactly if in the set of types of summands, two types with upper bound have also a lower bound (Richman–Walker [6]). This result is generalized by Rangaswamy [5] to separable groups.
(vii) Detachable p-groups are endo-quasi-injective (Richman [4]). Here ‘detachable’ means that every element a in the socle is contained in a summand C such that $p^\sigma C[p] = \langle a \rangle$ for $\sigma = h_p(a)$. (Separable and totally projective p-groups are detachable.) For more endo-quasi-injective groups, see Poole–Reid [1].
(viii) Endo-torsion and other module properties were discussed by Faticoni [3].
(ix) A torsion-free A is endo-noetherian if and only if it is quasi-isomorphic to a finite direct sum of strongly irreducible groups (Paras [1]).
(x) Endo-uniserial groups (the fully invariant subgroups form a chain) have been investigated by Hausen [7], [8], and Dugas–Hausen [1].
(xi) Endomorphism rings with chain conditions were studied by Albrecht [1].
(xii) The question when a group is endo-slender was considered by Huber [2], Mader [4], Eda [5]. Eda proved that a necessary and sufficient condition for a reduced vector group $A = \prod_{i \in I} R_i$ to be endo-slender is that for each infinite subset J of I, there exists an index $i \in I$ such that the set $\{j \in J \mid \mathbf{t}(R_j) \leq \mathbf{t}(R_i)\}$ is infinite.
(xiii) Self-injective endomorphism rings are subjects of several of papers, see e.g. Rangaswamy [3], Ivanov [6], Albrecht [2].
(xiv) Goldsmith–Vámos [1] deal with the so-called clean endomorphism rings.

After it has been observed that injective modules over any ring have algebraically compact additive groups, Richman–Walker [3] characterized the algebraically compact groups that are injective over the endomorphism ring. Vinsonhaler–Wickless [2] consider the endo-injective hull of separable p-groups: they are reduced algebraically compact groups, whose p-basic subgroups can be calculated.

Projective dimensions over the endomorphism ring were studied by Douglas–Farahat [1]. Inter alia they prove that, for any group A and for its powers A^I, the endo-projective dimensions are equal. That the endo-projective dimension of a finite rank torsion-free group can be any integer or ∞ was shown by Bobylev [1] for countable rank torsion-free groups. In the finite rank case, the same was proved for finite dimensions by Angad-Gaur [1], employing Corner type constructions.

Vinsonhaler–Wickless [3] discussed the flat dimension of completely decomposable torsion-free groups, and proved that it can be any integer.

Wickless [2] studied the torsion subgroup as an endo-submodule. Properties of $\mathrm{Hom}(C, A), \mathrm{Ext}(C, A), A \otimes C, \mathrm{Tor}(A, C)$, viewed as modules over $\mathrm{End}\,A$ or $\mathrm{End}\,C$, are popular subjects in the Russian literature. This is an interesting area of research which already produced several noteworthy results.

Exercises

(1) If A is endo-finite, and H is a fully invariant subgroup of A, then A/H is also endo-finite. [Hint: cosets of an E-generating set for A.]
(2) Subgroups and finite direct sums of endo-artinian groups are likewise endo-artinian.
(3) Finite direct sums and summands of endo-noetherian groups are again endo-noetherian.
(4) Find the groups that are both endo-artinian and endo-noetherian.
(5) For any group C, the group $A = \mathbb{Z} \oplus C$ is endo-cyclic and endo-projective.
(6) (Niedzwecki–Reid) If A is generated by n elements over $\mathrm{End}\,A$, then A^n is cyclic over $\mathrm{End}(A^n)$.
(7) There are torsion-free groups A of arbitrarily large cardinality κ such that—as $\mathrm{End}\,A$-modules—they cannot be generated by less than κ elements.
(8) (Wu–Jans) If A is endo-quasi-projective, then A/H is also endo-quasi-projective for any fully invariant subgroup H.
(9) (Poole–Reid) Let A be a direct sum of fully invariant subgroups A_i. Then A is endo-quasi-injective if and only if each A_i is endo-quasi-injective.
(10) (a) (Brameret, Feigelstock) A p-group is endo-uniserial if and only if it is divisible or a direct sum of copies of $\mathbb{Z}(p^n)$ and $\mathbb{Z}(p^{n+1})$ for some $n \in \mathbb{N}$.
(b) (Hausen) A torsion-free endo-uniserial group of finite rank is p-local for some p, and every non-zero fully invariant subgroup is of finite index.
(11) Find groups, other than finite groups, that are both endo-projective and endo-injective.

7 Groups with Prescribed Endomorphism Rings

It is natural to ask the question: if we are given a ring, under what conditions is it an endomorphism ring? And if it is an endomorphism ring, how can we find groups to fit? These questions have been answered for separable p-groups partially by Pierce [1], and completely by Liebert [3]; the conditions are quite complicated (see Theorem 2.8), and so are the constructions.

We now consider the questions in general. For torsion-free groups, we will have a mild sufficient condition on the ring to satisfy to be an endomorphism ring, but the real challenge is to construct groups once the target ring has been selected. Torsion-free groups of totally different, even arbitrarily large cardinalities may have isomorphic endomorphism rings. We will return to torsion groups, but prefer to think of the endomorphism rings in the form of Theorem 2.2, and our aim will be to find groups once the ring R is prescribed. Unfortunately, for mixed groups, only very little can be said.

The Torsion-Free Case Corner's Theorem 3.3 provides a satisfactory condition for countable rings. Of course, one prefers conditions that are applicable in arbitrary cardinalities. It turns out that the study of this problem in full generality requires sophisticated machinery that would go beyond the scope of this volume. Therefore, we just state the theorems without elaborating the proofs. Before doing so, let us recall that the endomorphism ring a group of cardinality κ has cardinality at most 2^κ.

For cotorsion-free rings the result is most satisfactory.

Theorem 7.1 (Dugas–Göbel [2], Shelah [4]). *Let* R *denote a cotorsion-free ring, and* κ *a cardinal satisfying* $\kappa^{\aleph_0} = \kappa$ *and* $\kappa^+ \geq |\mathsf{R}|$. *There exists a group* A *of cardinality* κ^+ *such that*

$$\operatorname{End} A \cong \mathsf{R}.$$

Moreover, there is a rigid system of 2^κ *groups with this property.* $\square$

There are various proofs of this theorem, all rely on some set-theoretical principle, like the Black Box or the Shelah elevator. The group A is constructed as a sandwich group between the direct sum B of copies of R^+ and its $\mathbb{Z}$-adic completion $\tilde{B}$. In this way, we could get too many endomorphisms, and the essence of the construction is to find out how to get rid of unwanted maps. This is a difficult and tiring technical process—and additional principles are needed to accomplish this task. For details, we refer to the excellent expositions Eklof–Mekler [EM], Göbel–Trlifaj [GT], or the original papers.

In the publications, slightly different conditions are stated for the cardinals involved. The main issue in the preceding theorem is that, under fairly general conditions, there exist arbitrarily large groups having the given ring as their endomorphism ring. Interestingly, for small cardinals a different approach seems to be necessary.

We should mention a topological version that is actually stronger than Theorem 7.1 inasmuch as it gives an explicit necessary and sufficient condition (recall that all endomorphism rings are complete in the finite topology). It extends Theorem 3.4 to arbitrary cardinalities.

Theorem 7.2 (Dugas–Göbel [5]). *A topological ring* R *is isomorphic to the finitely topologized endomorphism ring of a cotorsion-free group if and only if* R *is complete and Hausdorff in a topology that admits a basis of neighborhoods of* 0 *consisting of right ideals* N *with cotorsion-free factor groups* R/N. $\square$

The Torsion Case The problem for torsion groups is a different ball game. The endomorphism rings are extremely restricted: they are built on algebraically compact groups, and admit unavoidable small endomorphisms. The big problem is that these small endomorphisms form an ideal that already totally characterizes the group itself. Thus the only hope is to have some control on the endomorphism rings modulo small endomorphisms. We know from Theorem 2.2 that $\operatorname{End} A$ is a split extension of the ideal $\operatorname{End}_s A$ by an algebraically compact ring that faithfully reflects direct decompositions with unbounded summands. We have something to say about this ring.

Theorem 7.3 (Corner [5], Dugas–Göbel [3]). *Let* R *denote a ring whose additive group is the p-adic completion of a free group. There exists a separable p-group A such that*

$$\operatorname{End} A \cong \mathsf{R} \oplus \operatorname{End}_s A. \qquad \square$$

Moreover, if the ring R is given as stated, then for every infinite cardinal κ, there exists a family $\{A_\sigma \mid \sigma < \kappa\}$ of separable p-groups such that for each σ, $\operatorname{End} A_\sigma \cong \mathsf{R} \oplus \operatorname{End}_s A_\sigma$, and $\operatorname{Hom}(A_\rho, A_\sigma) = \operatorname{Hom}_s(A_\rho, A_\sigma)$ whenever $\rho \neq \sigma$.

Reduced Mixed Groups Corner–Göbel [1] gave a unified treatment of constructing groups from prescribed endomorphism rings that included mixed groups as well. On a different vein, Franzen–Goldsmith constructed a functor from the category of reduced torsion-free groups to the category of reduced mixed group to transfer results from one category to the other. They proved:

Theorem 7.4 (Franzen–Goldsmith [1]). *For every reduced countable torsion-free ring* R *there exists a reduced countable mixed group A such that A/tA is divisible, and*

$$\operatorname{End} A \cong \mathsf{R} \oplus \operatorname{Hom}(A, tA). \qquad \square$$

★ **Notes.** Starting with Corner's epoch making Theorem 3.3, a significant amount of work has been done on constructing torsion-free groups, even for arbitrarily large cardinalities, with prescribed endomorphism rings. Using a consequence of V = L, Dugas–Göbel [2] proved that every cotorsion-free ring is an endomorphism ring. For the general version, see Shelah [4]. The condition of cotorsion-freeness appears in almost all of the publications.

The consequences of Theorem 7.2 are legion, especially in constructing fascinating examples. It seems easier to construct a ring with required properties than large groups satisfying prescribed conditions. We have already taken advantage of this procedure.

May [5] considers cases when the rings are not necessarily cotorsion-free. Arnold–Vinsonhaler [3] proved that every reduced ring R whose additive group is a finite rank Butler group is the endomorphism ring of a finite rank Butler group provided that $p\mathsf{R} \neq \mathsf{R}$ holds for at least five primes p. Another kind of realization theorem is due to Dugas–Irwin–Khabbaz [1] who considered subrings of the Baer-Specker ring $\mathsf{P} = \mathbb{Z}^{\aleph_0}$ as split extensions of the ideal of finite rank endomorphisms as endomorphism rings of pure subgroups of P.

Exercises

(1) Prove that there exists no group whose endomorphism ring is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$ or to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ or to $J_p \oplus J_p$ for any prime p.
(2) Prove the existence of a mixed group with divisible torsion-free part such that its endomorphism ring has a homomorphic image of a ring with countable free additive group. [Hint: Theorem 7.4.]

Problems to Chapter 16

PROBLEM 16.1. Is the anti-isomorphic ring of an endomorphism ring also an endomorphism ring?

PROBLEM 16.2. Find the groups such that every endomorphism is the sum of (two) monomorphisms.

There are several publications on the unit sum number, see Notes to Sect. 2.

PROBLEM 16.3 (Megibben). Investigate the endomorphism rings of $\aleph_1$-free $\aleph_1$-separable torsion-free groups.

PROBLEM 16.4. Which groups have endo-projective cover?

PROBLEM 16.5. Relate the endo-flat cover of a group to the group.

PROBLEM 16.6. Discuss pure-projective and pure-injective dimensions over endomorphism rings.

PROBLEM 16.7. Find the stable range of endomorphism rings.

PROBLEM 16.8. Let S be a ring containing $\operatorname{End} A$ as a subring, sharing the same identity. When does a group G exist that contains A and satisfies $\operatorname{End} G \cong \mathsf{S}$?

PROBLEM 16.9. The endo-Goldie-dimension of a torsion-free group can be arbitrarily large. Given the endomorphism ring, can we say something about the minimum?

PROBLEM 16.10. Which groups A have the property that every automorphism of $\operatorname{Aut} A$ extends to an automorphism of $\operatorname{End} A$?

PROBLEM 16.11. For a subgroup C of A define $\dot{C} = \cap_{\eta \in S} \operatorname{Ker} \eta$ where $S = \{\eta \in \operatorname{End} A \mid \eta C = 0\}$. Which subgroups satisfy $\dot{C} = C$?

PROBLEM 16.12. Any polynomial identity (besides commutativity) in endomorphism rings?

Chapter 17
Automorphism Groups

Abstract Needless to say, automorphism groups contain in general much less information about the group structure than the endomorphism rings. There is an enormous contrast between torsion and torsion-free groups, perhaps even larger than from the point of view of endomorphism rings. Here again, the case of p-groups is more favorable (at least for $p > 2$) inasmuch as their automorphism groups determine the groups up to isomorphism. The torsion-free case seems uncontrollable, but a close examination shows that, interestingly, only a handful of finite groups may occur as automorphism groups of torsion-free groups of finite rank.

We remark at the outset that we will not give complete proofs of the two most important theorems on automorphism groups: that $\operatorname{Aut} A$ determines A if A is a p-group ($p > 2$), and the full description of finite automorphism groups of finite rank torsion-free groups. The proofs are too long and require lot of arguments on non-commutative groups.

1 Automorphism Groups

Automorphisms The automorphisms of a group A form a group under composition (written multiplicatively): the product $\alpha\beta$ of two automorphisms α, β of A is defined as

$$(\alpha\beta)a = \alpha(\beta a) \qquad \text{for all } a \in A.$$

This is, in general a non-commutative group, the **automorphism group** of A; denoted $\operatorname{Aut} A$. It is exactly the group of units in the ring $\operatorname{End} A$.

The groups of order 1 and 2 have no automorphisms other than the identity. But every group of order > 2 has other automorphisms as well: $a \mapsto -a$ is an automorphism different from $\mathbf{1}_A$ except when A is an elementary 2-group. If A is an elementary 2-group of order ≥ 4, then permutations of the elements of a basis give rise to automorphisms $\neq \mathbf{1}_A$.

The automorphism groups of some important groups are listed next.

Example 1.1. $\operatorname{Aut} \mathbb{Z} \cong \mathbb{Z}(2)$. Besides $\mathbf{1}_{\mathbb{Z}}$, multiplication by -1 is the only automorphism.

Example 1.2. An automorphism of $\mathbb{Z}(n)$ carries a generator a into another generator b, and clearly $b = ka$ with $\gcd(k, n) = 1$. Conversely, any such k gives rise to a well-defined automorphism of $\mathbb{Z}(n)$ mapping a to ka. It follows that $\operatorname{Aut} \mathbb{Z}(n)$ is isomorphic to the multiplicative group of the residue classes of integers coprime to n. Thus $\operatorname{Aut} \mathbb{Z}(n)$ is commutative, and its order is given by Euler's totient function $\varphi(n)$.

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Example 1.3. As End $\mathbb{Z}(p^\infty) \cong J_p$, it follows that Aut $\mathbb{Z}(p^\infty)$ is isomorphic to the multiplicative group of p-adic units. (This group will be explicitly described in Sect. 8 in Chapter 18.) The same holds for Aut J_p.

Example 1.4. Let R be a rational group of type $\mathbf{t} = (k_1, \dots, k_n, \dots)$. Then $a \mapsto p_n a$ ($a \in R$) is an automorphism of R if and only if $k_n = \infty$. Since each endomorphism is a multiplication by a non-zero rational number, it follows that Aut R must be isomorphic to the multiplicative group of all rational numbers whose nominators and denominators are divisible by primes p_n only for which $k_n = \infty$. Thus Aut R is isomorphic to the additive group which is the direct sum of $\mathbb{Z}(2)$ and as many copies of $\mathbb{Z}$ as the number of $k_n = \infty$ in $\mathbf{t}$.

Example 1.5. Let A be an elementary p-group of order p^m. Then A is an m-dimensional vector space over the prime field of characteristic p. Since automorphisms of A are linear transformations of the vector space, Aut A is isomorphic to the general linear group GL(m, p) (non-commutative).

Elementary Facts We list below some facts that we have found useful to keep in mind.

(a) *If C is a characteristic subgroup of A and $\alpha \in \operatorname{Aut} A$, then $\alpha \restriction C$ is an automorphism of C, and $a + C \mapsto \alpha a + C$ $(a \in A)$ is an automorphism of A/C.*
(b) *An isomorphism $\phi : A \to C$ between groups induces an isomorphism ϕ^* : $\operatorname{Aut} A \to \operatorname{Aut} C$ via $\phi^*(\alpha) = \phi\alpha\phi^{-1}$.*
(c) *If A is a direct sum, and if we represent the endomorphisms by matrices, then the automorphisms correspond to the invertible matrices.*
(d) *If $A = B \oplus C$, then* $\operatorname{Aut} B$ *may be regarded as a subgroup of* $\operatorname{Aut} A$, *identified with the set of all $\alpha \in \operatorname{Aut} A$ satisfying $\alpha \restriction C = \mathbf{1}_C$.* (This identification depends on the choice of the complement C.)
(e) *If $A = \oplus_{i \in I} A_i$, then under the identification indicated in* (d), *each of* $\operatorname{Aut} A_i$ *may be considered as a subgroup of* $\operatorname{Aut} A$. *The cartesian product $\prod_{i \in I} \operatorname{Aut} A_i$ is a subgroup of* $\operatorname{Aut} A$, it consists of those $\alpha \in \operatorname{Aut} A$ that carry each A_i into itself. If all the A_i are characteristic subgroups in A, then $\operatorname{Aut} A \cong \prod_{i \in I} \operatorname{Aut} A_i$. Hence:
(f) *If A is a torsion group, then* $\operatorname{Aut} A \cong \prod_p \operatorname{Aut} A_p$.
(g) *Every automorphism of $p^n A$ extends to an automorphism of A.* It suffices to prove this for $n = 1$, and when A has no summand of order p. Let γ be an automorphism of pA. If $\{b_i\}_{i \in I}$ is a p-basis of A, then write $a \in A$ in the form $a = \sum_{i=1}^m k_i b_i + pc$ with $k_i \in \mathbb{Z} \setminus p\mathbb{Z}$ and $c \in A$. Note that the terms $k_i b_i$ and pc are uniquely determined by a. For each $i \in I$, choose $a_i \in A$ such that $pa_i = \gamma(pb_i) \in pA$, and define $\theta : A \to A$ by

$$\theta(a) = \sum_{i=1}^m k_i a_i + \gamma(pc).$$

It is clear that θ is an endomorphism of A such that $\theta \restriction pA = \gamma$. To see that θ is monic, assume $\theta(a) = 0$ for some $a \in A$. Then $\gamma(pa) = 0$, so $pa = 0$. Hypothesis on A implies $a \in pA$, therefore, $\theta(a) = \gamma(a) = 0$ is impossible. Finally, to show that θ is epic, for a given $x \in A$ pick a $y \in A$ such that $\gamma(py) = px$. Then $x - \theta(y)$ is of order p, thus $x - \theta(y) = pc$ for some $c \in A$. If $z \in A$ satisfies $pz = \gamma(pc)$, then $x = \theta(y + z)$. Consequently, $\theta \in \operatorname{Aut} A$.

(h) (Baer [3]) *If* C *is a characteristic subgroup of* A*, then*

$$\text{Fix } C = \{\alpha \in \text{Aut}\, A \mid \alpha c = c \ \forall c \in C\}$$

is a normal subgroup of $\text{Aut}\, A$*. Conversely, if* Δ *is a normal subgroup of* $\text{Aut}\, A$*, then*

$$\text{Fix } \Delta = \{a \in A \mid \delta a = a \ \forall \delta \in \Delta\}$$

is a characteristic subgroup of A*.* This is a Galois-type correspondence.

Stabilizer Let $0 = C_0 < C_1 < \cdots < C_n = A$ be a chain of subgroups of A. By the **stabilizer** of this chain we mean the subgroup consisting of all $\alpha \in \text{Aut}\, A$ such that α induces the identity on all of the factor groups C_i/C_{i-1}; in other words, $\alpha c_i - c_i \in C_{i-1}$ ($\forall\, c_i \in C_i$) for all $i \geq 1$. By the stabilizer of a subgroup C is meant the stabilizer of the chain $0 < C < A$.

Lemma 1.6. *The stabilizer* Σ *of a subgroup* C *of* A *is isomorphic to* $\text{Hom}(A/C, C)$*, so it is commutative. It is a normal subgroup of* $\text{Aut}\, A$ *whenever* C *is a characteristic subgroup of* A*.*

Proof. If $\alpha, \beta \in \Sigma$, then for $a \in A$ we have $\alpha a = a + c$ and $\beta a = a + d$ for some $c, d \in C$. Hence $(\alpha\beta)a = \alpha(a + d) = a + c + d = (\beta\alpha)a$. The mappings $\overline{\alpha} : a + C \mapsto (\alpha - 1)a = c$ and $\overline{\beta} : a + C \mapsto (\beta - 1)a = d$ are well-defined homomorphisms $A/C \to C$ such that $\overline{\alpha\beta} : a + C \mapsto c + d$. Therefore, $\alpha \mapsto \overline{\alpha}$ is a homomorphism $\Sigma \to \text{Hom}(A/C, C)$ with trivial kernel. It is onto, since the map $\alpha : a \mapsto \eta(a + C)$ belongs to Σ for each $\eta \in \text{Hom}(A/C, C)$.

The second assertion is straightforward to check. ☐

Example 1.7. Let Σ denote the stabilizer of the infinite chain

$$0 < A[p] < A[p^2] < \cdots < A[p^n] < \cdots < A$$

for an unbounded p-group A. Define

$$\Sigma_n = \{\alpha \in \Sigma \mid \alpha(a) = a \ \forall a \in A[p^n]\} \qquad (n < \omega).$$

Each Σ_n is a normal subgroup of $\text{Aut}\, A$, and in the descending normal chain $\Sigma = \Sigma_0 \rhd \Sigma_1 \rhd \cdots \rhd \Sigma_n \rhd \cdots \rhd \{1\}$ the factor group Σ_n/Σ_{n+1} is a subgroup of the stabilizer of $A[p^n]$ in $A[p^{n+1}]$, so it is an elementary p-group as it is clear from Lemma 1.6. A group admitting a normal chain with elementary p-groups as factor groups is often called (in non-abelian group theory) *generalized p-nilpotent*.

Of special interest is the case when A is a reduced p-group (of length τ), and the chain is

$$A[p] \geq pA[p] \geq \cdots \geq p^\sigma A[p] \geq \cdots \geq p^\tau A[p] = 0, \qquad (17.1)$$

subgroups in the socle of A. The dimensions of the factor groups $S_\sigma = p^\sigma A[p]/p^{\sigma+1}A[p]$ are given by the UK-invariants of A, so $\text{Aut}\, S_\sigma$ is the general linear group $\mathrm{GL}(f_\sigma(A), p)$. As the groups in the chain are fully invariant, every $\alpha \in \mathrm{Aut}\, A$ induces an automorphism

$$\alpha_\sigma : a + p^{\sigma+1}A[p] \mapsto \alpha a + p^{\sigma+1}A[p] \qquad (a \in p^\sigma A[p])$$

of S_σ. In this way, we obtain a homomorphism $\psi : \alpha \mapsto (\alpha_0, \dots, \alpha_\sigma, \dots)$ of $\mathrm{Aut}\, A$ into the cartesian product $\prod_{\sigma<\tau} \mathrm{Aut}\, S_\sigma$ whose kernel is the stabilizer Σ of the chain (17.1). As every automorphism of S_n $(n < \omega)$ extends to an automorphism of A, $\mathrm{Im}\, \psi$ will contain all of $\mathrm{Aut}\, S_n$ $(n < \omega)$. Consequently, for a separable p-group, $\mathrm{Im}\, \psi$ is a subdirect product of the groups $\mathrm{Aut}\, S_n$ $(n < \omega)$.

Proposition 1.8 (Freedman [1], Hill [10]). *The homomorphism*

$$\psi : \mathrm{Aut}\, A \to \prod_{\sigma<\tau} \mathrm{Aut}\, S_\sigma$$

is surjective if the p-group A is Σ-cyclic, torsion-complete, or totally projective.

Proof. The claim is trivial for Σ-cyclic groups. For any p-group A, a collection of automorphisms $\alpha_n \in \mathrm{Aut}\, S_n$ $(n < \omega)$ lifts to an automorphism of a basic subgroup B, and every such map can be extended to A whenever A is torsion-complete Theorem 3.11 in Chapter 17.

If A is totally projective, then we apply transfinite induction on the length τ of A. We are done if $\tau \le \omega$. Suppose A has length $\sigma + 1$, and we select an arbitrary sequence $(\alpha_0, \dots, \alpha_\sigma)$ of automorphisms of S_ρ $(\rho \le \sigma)$. By induction, $A/p^\sigma A$ admits an automorphism α' inducing the given α_ρ on S_ρ $(\rho < \sigma)$. We apply Lemma 4.5 in Chapter 11, choosing both groups as $A/p^{\sigma+1}A = A$ and both nice subgroups as $S_\sigma = p^\sigma A/p^{\sigma+1}A$. The height-preserving isomorphism ϕ we now use between the two copies of S_σ is the given automorphism α_σ of S_σ, which is now extended by the theorem to an automorphism of A such that all the monic maps between the Hill invariants are induced by the given α_ρ $(\rho < \sigma)$ (see the proof of Lemma 4.5 in Chapter 11). This yields an $\alpha \in \mathrm{Aut}\, A$ inducing the prescribed set of the α_ρ. Applying an element in the stabilizer of (17.1), we can even guarantee that α induces α' on $A/p^\sigma A$. This will allow us to build matching automorphisms on the factor groups $A/p^\rho A$ leading to a desired automorphism of A. □

Involutions Recall that direct decompositions are recognizable in terms of idempotent endomorphisms. There is a similar, though not so effective, tool for automorphisms: the **involutions**, i.e. automorphisms θ such that $\theta^2 = \mathbf{1}$. As we shall see, they do not provide as much information as the projections, and, in addition, to make it work and to avoid unpleasant nuisance, one has to assume that multiplication by 2 is an automorphism of the group.

Throughout (j)-(ℓ), we will tacitly assume that A *is a group such that* $a \mapsto 2a$ *is an automorphism of* A. (In particular, A has no 2-component.)

(j) *An involution θ defines two subgroups*

$$A_\theta^- = \{a \in A \mid \theta a = -a\} \qquad \textit{and} \qquad A_\theta^+ = \{a \in A \mid \theta a = a\}$$

such that $A = A_\theta^- \oplus A_\theta^+$. Thus an involution $\theta \neq \pm 1$ gives rise to a non-trivial direct decomposition. The projections associated with this direct sum are

$$\epsilon = \tfrac{1}{2}(1-\theta) \qquad \text{and} \qquad 1-\epsilon = \tfrac{1}{2}(1+\theta).$$

Conversely, the involutions corresponding to the projections $\epsilon, 1-\epsilon$ are $1-2\epsilon, 2\epsilon-1$.

Example 1.9. That without the hypothesis $2A = A$ the decomposition (j) may fail is shown e.g. by the torsion-free group $A = \langle p^{-\infty}a, q^{-\infty}b, \frac{1}{2}(a+b)\rangle$, defined for different odd primes p, q. The correspondence $a \mapsto -a,\ b \mapsto b$ induces an involution, but A is indecomposable.

(k) *Two involutions θ, ζ of A commute if and only if*

$$A_\theta^- = (A_\theta^- \cap A_\zeta^-) \oplus (A_\theta^- \cap A_\zeta^+) \ \ \textit{and} \ \ A_\theta^+ = (A_\theta^+ \cap A_\zeta^-) \oplus (A_\theta^+ \cap A_\zeta^+),$$

and similarly for A_ζ^-, A_ζ^+. Note that if these equalities hold, then the two involutions commute on all four intersections, so also on A.

(ℓ) *A direct decomposition*

$$A = C_1 \oplus \cdots \oplus C_n \qquad (C_i \neq 0) \tag{17.2}$$

defines a set $\{\theta_1, \ldots, \theta_n\}$ of commuting involutions such that $\theta_i \restriction C_j = \delta_{ij}\mathbf{1}_{C_j}$ where δ_{ij} is the Kronecker delta. Conversely, a set $\{\theta_1, \ldots, \theta_n\}$ of pairwise commuting involutions determines a unique decomposition (17.2).

★ **Notes.** Our knowledge about the automorphism groups of abelian groups is rather fragmentary. Several examples of groups are known that can never be automorphism groups of abelian groups, and it is hard to visualize what sort of condition will make a group into an automorphism group. (In contrast, e.g. for commutative rings, every group can be an automorphism group.) No systematic analysis of all automorphism groups of abelian groups is available as yet. Whatever the outcome of such a study will be, it is clear that the normal structure of automorphism groups will be a major factor in understanding these groups. This justifies the widespread interest in search for normal subgroups in Aut A.

A troublesome aspect in the study of automorphism groups is that the methods of group theory, albeit extremely powerful and sharp, seem to be inapt in this case. We feel that automorphism groups of abelian groups deserve much more attention by non-commutative group theorists than they have received in the past: it is a fertile area that might also have interesting repercussions to general group theory.

Exercises

(1) A summand that is a characteristic subgroup is necessarily fully invariant.

(2) (a) Every automorphism of A extends to an automorphism of its injective hull. The extension is unique if A is torsion-free.

(b) Assume $A^1 = 0$. Every automorphism of A extends uniquely to an automorphism of its $\mathbb{Z}$-adic completion $\hat{A}$.

(c) Every automorphism of a reduced group can be extended uniquely to its cotorsion hull.

(3) The elements of the stabilizer subgroup of a chain carry each link into itself.

(4) (Leptin) $\alpha \in \operatorname{Aut} A$ induces the identity on A/nA and leaves the elements $A[n]$ fixed if and only if it is of the form $\alpha = 1 - n\eta$ for some $\eta \in \operatorname{End} A$. [Hint: for $n = p^k$, $(1-\alpha)A \leq p^k A$, $(1-\alpha)A[p^k] = 0$, consider p-basic in A.]

(5) The group ring $\mathbb{Z}(\mathfrak{G})$ of $\mathfrak{G} = \operatorname{Aut} A$ with integral coefficients is mapped into $\operatorname{End} A$ in the obvious way. Show that the image of this ring-homomorphism is the subring of $\operatorname{End} A$ generated by $\mathfrak{G}$.

(6) The automorphisms of A and the automorphisms of C induce commuting automorphisms on the groups $\operatorname{Hom}(C,A), \operatorname{Ext}(C,A), C \otimes A, \operatorname{Tor}(C,A)$.

(7) Let $\langle a_i \rangle$ be cyclic of prime order p_i, and $T = \oplus_{i<\omega} \langle a_i \rangle$ for different p_i.

(a) Every automorphism of T extends uniquely to $G = \prod_{i<\omega} \langle a_i \rangle$.

(b) $|\operatorname{Aut} G| = 2^{\aleph_0}$.

(c) Two pure subgroups of G are isomorphic if and only if there exists an $\alpha \in \operatorname{Aut} G$ carrying one onto the other.

(8) (Mishina) A group A has the property that every automorphism of every subgroup extends to an automorphism of A if and only if A is one of the following groups: (a) divisible group; (b) torsion group with homogeneous p-components; (c) direct sum of a divisible torsion group and a rank 1 torsion-free group. [Hint: consider Σ-cyclic subgroups.]

(9) To give an example of a characteristic subgroup that fails to be fully invariant, choose a ring R of algebraic integers in an extension of degree 2 of $\mathbb{Q}$ such that ± 1 are the only units in R (e.g., $\mathsf{R} = \mathbb{Z}[\sqrt{-5}]$). Pick a torsion-free group A of rank 4 with $\operatorname{End} A \cong \mathsf{R}$, and show that (a) all subgroups of A are characteristic; (b) all endomorphisms of A are monic; and (c) A contains no fully invariant subgroup of rank 1.

2 Automorphism Groups of p-Groups

With the exception of groups of order ≤ 2, all p-groups have non-trivial automorphisms; moreover, if the group is infinite, then it has at least $2^{\aleph_0}$ automorphisms (cf. Exercise 3).

The Center For torsion groups A, our first concern is to identify the center of $\operatorname{Aut} A$. We may restrict our considerations to p-groups.

Theorem 2.1 (Baer [5]). *If $p \neq 2$, then the center of the automorphism group of a p-group A consists of*

(i) *multiplications by p-adic units, if A is unbounded, and*
(ii) *multiplications by integers k with $1 \leq k < p^n$ coprime to p, if A is bounded, and p^n is the smallest bound.*

Proof. It follows from Theorem 2.9 in Chapter 16 that the multiplications listed in (i) or (ii) belong to the center. It is more difficult to prove that there are no other elements in the center. Fortunately, the arguments in Lemma 6.1 in Chapter 16 can be applied with $\theta \in \operatorname{Aut} A$, because η can be chosen as an automorphism by requiring that $\gcd(t, p) = 1$. (In the present situation, the hypothesis $p \neq 2$ is needed in the proof to conclude that $v = 0$.) $\square$

There is a single special case:

$$A = \mathbb{Z}(2^m) \oplus B \oplus \mathbb{Z}(2^\infty) \quad \text{with } \ 2^{m-1}B = 0 \ (m \geq 1) \tag{17.3}$$

that is responsible for making the prime $p = 2$ exceptional in the preceding theorem. For this group, the center of $\operatorname{Aut} A$ is the direct product of the multiplicative group of the 2-adic units and $\mathbb{Z}(2) = \langle \alpha \rangle$, where α is the identity everywhere except on the generators c of the largest cyclic summand $\mathbb{Z}(2^m)$; in this situation, $\alpha(c) = c + v$, where v denotes the unique element of order 2 and of infinite height.

To see this, we take a closer look at the proof of Lemma 6.1(i) in Chapter 16. Let A be a 2-group that we write as $A = \langle c \rangle \oplus B \oplus D$, where $o(c) = 2^m$, D is the divisible part, and so far no assumption on B. Now $\theta(c) = kc + v$ $(v \in B \oplus D)$, and suppose $v \neq 0$. Everything works in the proof (in particular, θ has to be multiplication by a dyadic unit ρ in D), except that v could be of order 2, and that k must be odd to have $\theta \in \operatorname{Aut} A$. Replacing θ by an odd multiple, $k = 1$ may be assumed. If $\chi : \langle c \rangle \to B \oplus D$ is any homomorphism, then θ has to commute with the automorphism β defined by $\beta(c) = c + \chi(c)$, $\beta(x) = x$ $(x \in B \oplus D)$, therefore

$$c + \chi(c) + v = \beta(c + v) = \beta\theta(c) = \theta\beta(c) = \theta(c + \chi(c)) = c + v + \theta\chi(c).$$

Thus θ fixes all homomorphic images of c in $B \oplus D$ which is $(B \oplus D)[2^m]$. This conclusion applied to a cyclic summand of order $\geq 2^m$ in B would imply $v = 0$, so $2^{m-1}B = 0$ follows. If v is contained in a cyclic summand $\langle b \rangle < B$, then θ commutes with the automorphism γ that is the identity on the complement of $\langle b \rangle$ and satisfies $\gamma(b) = \xi(b) + b$ with a monomorphism $\xi : \langle b \rangle \to \langle c \rangle$. This leads to $c + v = \theta(c) = \theta\gamma(c) = \gamma\theta(c) = \gamma(c + v) = c + \xi(v) + v$, whence $\xi(v) = 0$, a contradiction. Therefore, $v \in D$. If there is another $w \in D$ of order 2, then there is an automorphism of D (and hence of A) that switches v and w; this would not commute with θ. Consequently, $D \cong \mathbb{Z}(2^\infty)$, A has to be of form (17.3), and the automorphisms have to be as stated.

Accordingly, from Theorem 8.6 in Chapter 18 it follows that for the center $\mathfrak{z}(\operatorname{Aut} A)$ of the automorphism group of a p-group A we have: if $p > 2$, then

$$\mathfrak{z}(\operatorname{Aut} A) \cong \mathbb{Z}(p-1) \oplus J_p \quad \text{or} \quad \cong \mathbb{Z}(p-1) \oplus \mathbb{Z}(p^{m-1})$$

according as A is unbounded or bounded by p^m, while if $p = 2$, then

$$\mathfrak{z}(\operatorname{Aut} A) \cong J_2 \quad \text{or} \quad \cong \mathbb{Z}(2^{m-1}) \quad \text{or} \quad \cong \mathbb{Z}(2) \oplus J_2$$

according as A is unbounded, bounded by 2^m, or of exceptional type.

Since a group is abelian exactly if it coincides with its center, from the preceding theorem it is immediate that *the automorphism group of a p-group is commutative if and only if it is cocyclic or isomorphic to* $\mathbb{Z}(2^\infty) \oplus \mathbb{Z}(2)$.

Involutions of p-Groups It appears that involutions enter the picture naturally. Involutions contain a lot of information about p-groups, but we have to learn how to read them. We now go on to collect some relevant observations about involutions in p-groups.

If $S \subseteq \operatorname{Aut} A$,

$$\mathfrak{c}(S) = \{\alpha \in \operatorname{Aut} A \mid \alpha\phi = \phi\alpha \ \forall \phi \in S\}$$

will denote the **centralizer** of S in $\operatorname{Aut} A$. Of course, $\mathfrak{c}(\operatorname{Aut} A) = \mathfrak{z}(\operatorname{Aut} A)$. An involution θ is said to be **extremal** if either A_θ^+ or A_θ^- is indecomposable $\neq 0$. In (a)–(e) *we keep assuming that $p \neq 2$.*

(a) *The p-group A is bounded, and p^n is the l.u.b. for the orders of its elements if and only if* $\mathfrak{z}(\operatorname{Aut} A) \cong \mathbb{Z}(p^{n-1}(p-1))$.

(b) *If A is a p-group, and if $\eta \in \operatorname{End} A$ is such that* $\operatorname{Ker} \eta = 0$ *and* $\eta(p^n A[p]) = p^n A[p]$ *for all $n < \omega$, then η is an automorphism.* Evidently, η is monic and $A[p] \leq \operatorname{Im} \eta$. To prove by induction on n that $A[p^n] \leq \operatorname{Im} \eta$, suppose $a \in A$ has order p^n ($n > 1$). By induction hypothesis, there are a $b \in A[p^n]$ such that $\eta(p^{n-1}b) = p^{n-1}a$, and a $c \in A$ with $\eta(c) = a - \eta(b)$. Hence $a = \eta(b + c)$, and η is epic as well.

(c) *Assume $A = C_1 \oplus \cdots \oplus C_k$ ($C_i \neq 0$) is a direct decomposition of the p-group A. Let θ_i denote the involutions that belong to this decomposition. Then the centralizer*

$$\mathfrak{c}\{\theta_1, \ldots, \theta_k\} = \operatorname{Aut} C_1 \times \cdots \times \operatorname{Aut} C_k,$$

and if $k \geq 3$, then the center of this centralizer contains exactly 2^k *involutions.* That under the identification stated in Sect. 1 (d), all $\operatorname{Aut} C_i$ belong to the centralizer of the set of the θ_i is evident. Conversely, if $\alpha \in \operatorname{Aut} A$ commutes with each of θ_i, then $\alpha C_i = \alpha A_{\theta_i}^- \leq A_{\theta_i}^- = C_i$ for every i, whence $\alpha C_i = C_i$, so $\alpha \in \times \operatorname{Aut} C_i$. The final claim is evident.

(d) *If an involution θ of A is extremal, then for every commuting involution ζ, in the direct decomposition stated in Sect. 1 (h) there are at most* 3 *non-zero summands.*

Example 2.2 (Leptin [2]). It can happen that, for different primes p, q, a p-group and a q-group have isomorphic automorphism groups, though this is a very rare phenomenon. This is the case if and only if the groups are cyclic: $\mathbb{Z}(p^k)$ and $\mathbb{Z}(q)$ such that $3 \leq p < q$ and $q - 1 = p^{k-1}(p-1)$. (E.g., q is a Fermat prime, and $p = 2$.)

The main result on p-groups is the following theorem which we state without proof. It is an important contribution to the theory of p-groups.

Theorem 2.3 (Leptin [2], Liebert [5]). *Assume $p > 2$ and A, C are p-groups. If* $\operatorname{Aut} A \cong \operatorname{Aut} C$*, then $A \cong C$.* □

★ **Notes.** Automorphism groups of finite abelian groups have been investigated by several authors, starting with Shoda [1], and as a result, a lot is known about them. A further step was taken by Baer [3] who studied the normal structure of $\operatorname{Aut} A$ for infinite p-groups A; he also investigated the correspondence between characteristic subgroups of A and normal subgroups of $\operatorname{Aut} A$. That the results gave special status to the prime number 2 should come as no surprise. Later, the study of the normal structure of $\operatorname{Aut} A$ has advanced considerably, see Faltings [1], Freedman [1], Leptin [3], Mader [1, 2], Hill [10], and above all Hausen [1, 2, 3].

The amount of open questions as to how the normal structure can be described is staggering. Several papers were devoted to the question of characteristic subgroups whose automorphisms extend to automorphisms of the containing group, as well as to pairs of subgroups whose isomorphism extends to an automorphism of the entire group. Solvable automorphism groups were studied by Shlyafer [1] and Brandl [1]: for a p-group A, $\operatorname{Aut} A$ is solvable only if A has the minimum condition on subgroups. See also papers by Hausen listed in the Bibliography.

The multiplicative analogue of the Baer–Kaplansky theorem is more subtle; it has not been completely settled: the case $p = 2$ is still open. The work started with Freedman [1] who proved that for $p \geq 5$ two countable reduced p-groups A, C are isomorphic whenever $p^{\omega}A \cong p^{\omega}C$ and $\operatorname{Aut} A \cong \operatorname{Aut} C$. It was a big step forward when Leptin [2] succeeded in relaxing the requirements of countability and the isomorphy of the first Ulm subgroups. In cases $p = 2, 3$, involutions created serious problems. Only 25 years later was able Liebert [5] to find a proof, using completely different methods, that included also the prime $p = 3$. Unfortunately, the proofs do not provide a method to encode relevant structural information about the group from the automorphism group. Schultz [3] claimed a proof for $p = 2$, but his proof relied on a lemma that held only for $p \geq 3$. The case $p = 2$ seems awfully difficult, and the Leptin-Liebert theorem might not extend to this special case. For $p \geq 3$, Liebert [5] describes the possible isomorphisms between $\operatorname{Aut} A$ and $\operatorname{Aut} A'$ for isomorphic p-groups A, A'; there exist some not induced by any group isomorphism.

Leptin [2] has an interesting in-depth study of commuting involutions. For an illuminating survey on results concerning automorphism groups up to the year 1999, we refer to Schultz [4] (the claim for $p = 2$ should be ignored).

The elementary equivalence of automorphism groups of p-groups ($p \geq 3$) were discussed by Bunina–Roǐzner [1].

Exercises

(1) (a) An elementary group of order p^m has $(p^m - 1)(p^m - p) \cdots (p^m - p^{m-1})$ automorphisms.

(b) $|\operatorname{Aut} A| = 2^{\kappa}$ if A is an elementary p-group of cardinality κ.

(2) If A is a p-group of order p^m, then $|\operatorname{Aut} A|$ is divisible by p^{m-1}.
(3) (Boyer, E. Walker, Khabbaz, Mader, Hill)

(a) If A is an infinite reduced p-group, then $|\operatorname{Aut} A| = 2^{|B|}$, where B is a basic subgroup of A.
(b) If A is a non-reduced p-group, then $|\operatorname{Aut} A| = 2^{|A|}$.

(4) (Baer) The subgroup of elements in a p-group A that are left fixed under all automorphisms of A is not $\{0\}$ if and only if $p = 2$ and A has a unique element g of order 2 of maximal height (in which case this subgroup is $\langle g \rangle$).
(5) (Baer) Let A be a separable p-group and $p \geq 3$. The set of elements of A left fixed under all automorphisms that fix a subgroup C elementwise is C^-, the closure of C in the p-adic topology. (If $p = 2$, the exceptional subgroup $\langle g \rangle$ of the preceding example, if exists, must be adjoined to C^-.)
(6) (Hill) Let A be a Σ-cyclic p-group. For $n \geq 1$, every automorphism of $A[p^n]$ that preserves heights (computed in A) extends to an automorphism of A. [Hint: extend it to $A[p^{n+1}]$.]
(7) (Megibben) Every automorphism of a large subgroup of a p-group A that preserves heights (computed in A) extends to an automorphism of A.
(8) If A is a p-group, and $\alpha \in \operatorname{Aut} A$ induces the identity on $p^\sigma A/p^{\sigma+1}A$ for some ordinal σ, then it also induces the identity on $p^{\sigma+n}A/p^{\sigma+n+1}A$ for all integers $n \geq 1$.
(9) (Mader) Let A be a divisible p-group, and Σ_n the normal subgroup of $\operatorname{Aut} A$ that consists of those $\alpha \in \operatorname{Aut} A$ which leave $A[p^n]$ element-wise fixed. Show that $(\operatorname{Aut} A)/\Sigma_n \cong \operatorname{Aut} A[p^n]$.
(10) (Hausen) The automorphism group of a finitely cogenerated group is residually finite (i.e., every element $\neq 1$ is contained in a normal subgroup of finite index). [Hint: consider $\alpha \in \operatorname{Aut} A$ fixing $A[p^n]$ element-wise.]
(11) (Tarwater) Let A be a homogeneous Σ-cyclic p-group. It has an automorphism α carrying the subgroup G into the subgroup H if and only if $G \cong H$ and $A[p]/G[p] \cong A[p]/H[p]$.
(12) (Baer) For a p-group A with $p \geq 3$, all the torsion subgroups of $\operatorname{Aut} A$ are finite exactly if A is finitely cogenerated. [Hint: an infinity of independent summands yields an infinite torsion group in the centralizer of the system of involutions; conversely, reduce to divisible groups and consider matrix representation.]
(13) Suppose $A = B \oplus C$, and B is fully invariant in A. Then $\operatorname{Aut} A$ is the semidirect product of the stabilizer of the chain $0 < B < A$ by the subgroup $\operatorname{Aut} B \times \operatorname{Aut} C$. [Hint: map $\operatorname{Aut} A$ onto $\operatorname{Aut} B \times \operatorname{Aut} C$.]

3 Automorphism Groups of Torsion-Free Groups

The most relevant results we know concerning automorphism groups of torsion-free groups are about the finite automorphism groups. Before tackling this special situation, let us consider a few facts without putting any restrictive condition on the automorphism groups.

(A) *If a group* $\mathfrak{G}$ *is isomorphic to* $\mathrm{Aut}\,A$ *for a torsion-free group* A*, then* $\mathfrak{G}$ *is isomorphic to the group of units in a subring* R *of a* $\mathbb{Q}$*-algebra* S*. The converse holds if* $\mathfrak{G}$ *is finite.*

Indeed, in the torsion-free case, $\mathrm{Aut}\,A$ is the unit group in $\mathrm{End}\,A$ that is a subring of the $\mathbb{Q}$-algebra $\mathbb{Q} \otimes \mathrm{End}\,A$. If $\mathfrak{G}$ is a finite subgroup in the unit group of a $\mathbb{Q}$-algebra R, then it is the group of units in the subring $\mathsf{R} \subseteq \mathbb{Z}[\mathfrak{G}]$ generated by $\mathfrak{G}$. In view of Theorem 3.3 in Chapter 16, there exists a group A with $\mathrm{End}\,A \cong \mathsf{R}$.

(B) *Monic endomorphisms of a finite rank torsion-free group are automorphisms.* More generally, *if the additive group of a ring* R *is a finite rank torsion-free group, then the one-sided units of* R *are two-sided.*

Assume $r, s \in \mathsf{R}$ satisfy $rs = 1$, but $sr \neq 1$. Then $r(sr - 1) = 0$, so the right annihilator $\mathrm{Ann}\,r \neq 0$. Therefore, the right annihilator $\mathrm{Ann}\,r \leq \mathsf{R}$ contains a pure subgroup of rank ≥ 1. Then an obvious argument with ranks shows that $rs = 1$ is impossible.

The automorphism group of a torsion-free group rarely contains any relevant information about the structure of the group itself—this is evident from several examples. Not even the rank 1 groups can be recaptured from their automorphism groups, as is illustrated by the following example.

Example 3.1. Let A be a rational group of type $(k_1, \dots, k_n, \dots)$. If all the k_n are finite, then $\mathrm{Aut}\,A \cong \mathbb{Z}(2)$. In all other cases, $\mathrm{Aut}\,A$ is a direct product of $\mathbb{Z}(2)$ and a free abelian group whose rank is equal to the number of infinite k_n's.

Finite Automorphism Groups Our knowledge of (non-commutative) groups that can be realized as automorphism groups of torsion-free groups does not go much further than the finite case (and sporadic examples). In fact, up to now, the most important result concerning the automorphism groups of torsion-free groups is the classification of finite automorphism groups. The full discussion involves more non-commutative group theory than we care to get into; we refer to the excellent presentation by Corner [10]. However, we can extract essential facts from the sole hypothesis that the automorphism groups are torsion—we will see that even this milder condition alone means severe restriction.

In the following statements we use the notations: $\mathfrak{G} = \mathrm{Aut}\,A$ and $\mathsf{E} = \mathrm{End}\,A$ where A is a reduced torsion-free group. In the arguments, we take advantage of the fact that $\mathfrak{G}$ is the unit group in E.

(a) *If* $\mathfrak{G}$ *is a torsion group, then* E *contains no nilpotent elements* $\neq 0$.
For the proof we may suppose that the nilpotent $\eta \in \mathsf{E}$ satisfies $\eta^2 = 0$. Then the endomorphisms $1+\eta$ and $1-\eta$ are inverse to each other, so they belong to $\mathfrak{G}$. But $(1 + \eta)^n = 1 + n\eta \neq 1$ for $n \in \mathbb{N}$ unless $\eta = 0$ (due to the torsion-freeness of E).

(b) *If $\mathfrak{G}$ is torsion, then every involution $\theta \in \mathfrak{G}$ is in the center of $\mathfrak{G}$.*
For every $\alpha \in \mathfrak{G}$, the endomorphisms $(1+\theta)\alpha(1-\theta)$, $(1-\theta)\alpha(1+\theta)$ are nilpotent, so they are 0 by (a). Therefore $2(\theta\alpha - \alpha\theta) = (1+\theta)\alpha(1-\theta) - (1-\theta)\alpha(1+\theta) = 0$, whence $\theta\alpha = \alpha\theta$ follows.

(c) *If $\mathfrak{G}$ is torsion, and $\alpha \in \mathfrak{G}$ is of odd order $m > 1$, then $m = 3$.*
Manifestly, the proof can be confined to the case $m = p^k$ for a prime $p \geq 3$. The endomorphism $\beta = 1 - \alpha + \alpha^2 - + \cdots + \alpha^{m-3}$ has an inverse:

$$\beta^{-1} = \begin{cases} \alpha^2 - \alpha^4 + - \cdots + \alpha^{m+1} & \text{if } m \equiv 1 \bmod 4, \\ \alpha^3 - \alpha^5 + - \cdots + \alpha^m & \text{if } m \equiv -1 \bmod 4; \end{cases}$$

thus $\beta \in \mathfrak{G}$. Since $\alpha^m - 1$ annihilates A, there must exist a minimal polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\alpha)$ annihilates A. Clearly, $f(x)$ divides $x^{p^k} - 1$, but not $x^{p^{k-1}} - 1$, whence from the irreducibility (over $\mathbb{Z}$) of the p^kth cyclotomic polynomial $\phi_{p^k}(x) = (x^{p^k} - 1)/(x^{p^{k-1}} - 1)$, we deduce the divisibility relation $\phi_{p^k}(x)|f(x)$. Consequently, the mapping $\alpha \mapsto \zeta$ (where ζ is a primitive p^kth complex root of unity) extends to a ring homomorphism $\chi \colon \mathbb{Z}[\alpha] \to \mathbb{Q}[\zeta]$ of the subring of E generated by α into the algebraic extension of $\mathbb{Q}$ by ζ. The image of β under χ is the complex number $1 - \zeta + \zeta^2 - + \cdots + \zeta^{m-3} = (1 + \zeta^{p^k-2})/(1+\zeta)$ whose absolute value is $\neq 1$ unless ζ^{p^k-2} equals ζ or its complex conjugate $\bar{\zeta} = 1/\zeta$. In the first alternative we have $p^k = 3$, while the second alternative: $\zeta^{p^k-1} = 1$ is impossible. Since $\chi(\beta)$ must be a root of unity, the only possibility is that α is of order 3.

(d) *If $\mathfrak{G}$ is a torsion group, then it does not contain any element of order* 8.
In fact, if $\alpha \in \mathfrak{G}$ is of order 8, then

$$\beta = 1 + (1-\alpha^4)(1+\alpha-\alpha^3) \quad \text{and} \quad \gamma = 1 + (1-\alpha^4)(1-\alpha+\alpha^3)$$

are inverse to each other in E. As in (c), we can show that the subring $\mathbb{Z}[\alpha]$ of E generated by α maps into $\mathbb{Q}[\zeta]$ *via* $\alpha \mapsto \zeta = (1+i)/\sqrt{2}$ = a primitive 8th root of unity. This homomorphism sends β into $3 + 2\sqrt{2}$ which is a unit of infinite order in $\mathbb{Q}[\zeta]$; thus β cannot be of finite order.

At this point, from (c) and (d) we deduce that $\mathfrak{G}$, if torsion, contains only elements whose orders are divisors of 12.

(e) *If $\mathfrak{G}$ is torsion, then the involution -1 is not contained in any cyclic subgroup of order* 12.
We show that the assumption that $\alpha \in \mathfrak{G}$ of order 12 satisfies $\alpha^6 = -1$ is contradictory. Because then

$$\beta = 1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 \quad \text{and} \quad \gamma = 1 - \alpha^2 - \alpha^3 - \alpha^4 - \alpha^5$$

are inverse automorphisms, and—as we wish to prove—of infinite order. We again examine the polynomial $g(x) \in \mathbb{Z}[x]$ of minimal degree such that $g(\alpha)A = 0$. Now $x^6 + 1 = (x^2+1)(x^4 - x^2 + 1)$ with irreducible factors.

As $\alpha^2 \neq -1$ is obvious, it follows that the cyclotomic polynomial $\phi_{12}(x) = x^4 - x^2 + 1$ is a divisor of $g(x)$. As above, we get a ring homomorphism carrying β into $1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 = (1 + \zeta^5)/(1 + \zeta)$, where ζ denotes a primitive 12th root of unity. But $|1 + \zeta^5| \neq |1 + \zeta|$, so β must be of infinite order.

Primordinal Groups We stop here with the listing of properties of finite (torsion) automorphism groups $\mathfrak{G}$, but will formulate the precise result. The theorem below refers to six groups, which are called **primordinal groups**. The following is the list of the primordinal groups (the non-cyclic groups are given by generators and defining relations):

$$\textit{cyclic groups: } \mathbb{Z}(2),\ \mathbb{Z}(4),\ \mathbb{Z}(6),$$

$$\textit{quaternion group: } Q_8 = \{\alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^2\},$$

$$\textit{dicyclic group: } DC_{12} = \{\alpha, \beta \mid \alpha^3 = \beta^2 = (\alpha\beta)^2\},$$

$$\textit{binary tetrahedral group: } BT_{24} = \{\alpha, \beta \mid \alpha^3 = \beta^3 = (\alpha\beta)^2\},$$

where the indices indicate the group orders. For the proof of the following theorem we refer to Corner [10]. (B-blocks are groups constructed from BT_{24}.)

Theorem 3.2 (Hallett–Hirsch [1], Corner [10]). *A finite group $\mathfrak{G}$ is the automorphism group of some torsion-free group if and only if it is a subdirect product of primordinal groups such that either*

(i) $\mathfrak{G}$ *has a direct factor of order* 2*; or*
(ii) *there exists a direct decomposition*

$$\mathfrak{G} = \mathfrak{G}_0 \times \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_r$$

where $\mathfrak{G}_1, \dots, \mathfrak{G}_r$ are B-blocks, and $\mathfrak{G}_0$ is a subdirect product of copies of the groups $\mathbb{Z}(4), \mathrm{Q}_8, \mathrm{DC}_{12}$ *in such a way that it contains the diagonal involution* -1. □

Examples for Primordinal Automorphism Groups The following examples exhibit torsion-free groups with primordinal automorphism groups.

Example 3.3. $\mathbb{Z}(2)$ *as an automorphism group.* We have numerous examples for groups with $\mathbb{Z}(2)$ as automorphism groups, the simplest is $\mathbb{Z}$. All the examples are indecomposable.

Example 3.4. $\mathbb{Z}(4)$ *as an automorphism group.* As the unit group of the Gaussian integers is cyclic of order 4, the existence of an example is guaranteed by Theorem 3.3 in Chapter 16. An explicit example is as follows.

Let $p_1, \dots, p_i, \dots$ be different primes $\equiv 1 \bmod 4$. Then -1 is a quadratic residue mod p_i, so we can find $k_i \in \mathbb{N}$ such that $k_i^2 \equiv -1 \bmod p_i$ for each i. Define

$$A = \langle a, b, p_i^{-1}(a + k_i b) \ \ (i \in \mathbb{N})\rangle. \tag{17.4}$$

Then $a \mapsto -b,\ b \mapsto a$ induces an automorphism α of order 4. We show that A has no automorphisms other than the powers of α. Every $\beta \in \operatorname{Aut} A$ acts as $\beta a = ra + sb, \beta b = ta + ub$ with integers r, s, t, u satisfying $ru - st = \pm 1$. As $\beta(a + k_i b) = (r + k_i t)a + (s + k_i u)b$ ought to be divisible by p_i, it must be an integral linear combination of $p_i a, p_i b, a + k_i b$. This leads to the relation $s + k_i u \equiv k_i(r + k_i t) \mod p_i$ whence $s + t \equiv k_i(r - u) \bmod p_i$, and so $(s+t)^2 \equiv -(r-u)^2 \bmod p_i$. This holds for all i, so congruence becomes equality, whence $t = -s, r = u$ follows. This, in view of $ru - st = \pm 1$ leaves only four possibilities: $r = \pm 1, s = 0$ and $r = 0, s = \pm 1$. These correspond to the powers of α, in particular, the choice $r = 0, s = -1$ to α.

Example 3.5. $\mathbb{Z}(6)$ *as an automorphism group.* To give an explicit example, consider an infinite set $p_1, \dots, p_i, \dots$ of primes $\equiv 1 \bmod 6$. Then there exist integers k_i such that $k_i^2 + k_i \equiv -1 \bmod p_i$. Consider the group A as in (17.4) with the new set of primes. The correspondence $a \mapsto b, b \mapsto -a + b$ induces an automorphism α of order 6. For $\beta \in \operatorname{Aut} A$, write as before $\beta a = ra + sb, \beta b = ta + ub$ with $r, s, t, u \in \mathbb{Z}$ satisfying $ru - st = \pm 1$. Then $p_i | \beta(a + k_i b) = (r + k_i t)a + (s + k_i u)b$ implies $k_i(r + k_i t) \equiv s + k_i u \bmod p_i$. Replacing s by $-sk_i - sk_i^2$ and canceling by k_i, we obtain $r + k_i t \equiv -(1 + k_i)s + u \bmod p_i$, or $r + s - u \equiv -k_i(s + t) \bmod p_i$. Eliminating k_i, we get

$$(r + s - u)^2 - (r + s - u)(s + t) + (s + t)^2 \equiv 0 \mod p_i.$$

Again, this holds for all i, thus this congruence becomes equality. As the equation $x^2 - xy + y^2 = 0$ has only the trivial solution in real numbers x, y, it follows that $u = r + s, t = -s$. Therefore, $r^2 + rs + s^2 = ru - st = \pm 1$, whence the only possibilities are $r = \pm 1, s = 0; r = 1, s = -1$, and those obtained by switching r and s. Consequently, A has at most six automorphisms.

Example 3.6. The quaternion group Q_8 *as an automorphism group.* We select two disjoint infinite sets of primes, $p_1, \dots, p_i, \dots$ and $q_1, \dots, q_i, \dots$, all $\equiv -1 \bmod 4$, and $k_i, \ell_i \in \mathbb{Z}$ such that $k_i^2 \equiv -1 \bmod p_i$, $\ell_i^2 \equiv -1 \bmod q_i$. Our group will now be

$$A = \langle a, b, c, d, p_i^{-1}(a + k_i b), p_i^{-1}(d + k_i c), q_i^{-1}(a + \ell_i c), q_i^{-1}(b + \ell_i d) \ \ (i \in \mathbb{N}) \rangle.$$

Using the same argument as in previous examples, from the divisibility relations it follows that $\gamma \in \operatorname{Aut} A$ must act like this:

$$\gamma a = ra + sb + tc + ud, \quad \gamma b = -sa + rb + uc - td,$$
$$\gamma c = -ta - ub + rc + sd, \quad \gamma d = -ua + tb - sc + rd$$

for some $r, s, t, u \in \mathbb{Z}$, such that the determinant of the coefficient matrix—which is equal to $(r^2 + s^2 + t^2 + u^2)^2$—is a unit in $\mathbb{Z}$, thus it is equal to 1. Evidently, the equation $r^2 + s^2 + t^2 + u^2 = 1$ has exactly 8 solutions in integers, and the correspondences

$$\alpha\colon a \mapsto b, b \mapsto -a, c \mapsto d, d \mapsto -c, \quad \beta\colon a \mapsto c, b \mapsto -d, c \mapsto -a, d \mapsto b$$

(with the choices $s = 1, r = t = u = 0$, resp. $t = 1, r = s = u = 0$) induce automorphisms such that $\alpha^2 = \beta^2 = (\alpha\beta)^2 = -1$. Thus $\operatorname{Aut} A \cong Q_8$.

Example 3.7. The dicyclic group DC_{12} *as an automorphism group.* Again, we start with two disjoint infinite sets p_i and q_i of primes such that $p_i \equiv 1 \bmod 4$ and $q_i \equiv 1 \bmod 6$ for all $i \in \mathbb{N}$. Choose integers k_i, ℓ_i so as to satisfy $k_i^2 \equiv -1 \bmod p_i$ and $\ell_i^2 + \ell_i \equiv -1 \bmod q_i$, and define

$$A = \langle a, b, c, d, p_i^{-1}(a + k_i b), p_i^{-1}(c + k_i d), q_i^{-1}(a + \ell_i c), q_i^{-1}(d + \ell_i b) \ \ (i \in \mathbb{N}) \rangle.$$

It is easy to check that the correspondences

$$\alpha : a \mapsto c, b \mapsto d, c \mapsto -a + c, d \mapsto -b + d, \quad \beta : a \mapsto d, b \mapsto -c, c \mapsto b, d \mapsto -a$$

give rise to automorphisms of A such that $\alpha^3 = \beta^2 = (\alpha\beta)^2 = -1$. The same technique with divisibility as before leads to the conclusion that, for every $\gamma \in \operatorname{Aut} A$ we must have

$$\gamma a = ra + sb + tc + ud, \qquad \gamma b = -sa + rb - uc - td,$$

$$\gamma c = -ta + (u+s)b + (r+t)c - sd, \qquad \gamma d = -(u+s)a - tb + sc + (r+t)d$$

with suitable $r, s, t, u \in \mathbb{Z}$. The determinant of the coefficient matrix must be a unit in $\mathbb{Z}$, thus $(r^2 + rt + t^2 + s^2 + su + u^2)^2 = 1$. This equation has 12 integral solutions: $r = \pm 1, s = t = u = 0$; $r = -t = 1, s = u = 0$, as well as those obtained from these two by permissible permutations.

Example 3.8. The binary tetrahedral group BT_{24} *as an automorphism group.* Let $p_i, q_i, k_i.\ell_i$ have the same meaning as in the preceding example. This time, A is defined as

$$A = \langle a, b, c, d, p_i^{-1}(a + k_i b), p_i^{-1}(a - c + k_i d), q_i^{-1}(a + \ell_i d), q_i^{-1}(c + \ell_i b) \ \ (i \in \mathbb{N})\rangle.$$

The maps

$$\alpha : \ a \mapsto d,\ b \mapsto -a + c,\ c \mapsto -a - b + c + d,\ d \mapsto -a + d,$$

$$\beta : \ a \mapsto c,\ b \mapsto b - d,\ c \mapsto -a + c,\ d \mapsto b$$

induce automorphisms of A satisfying $\alpha^3 = \beta^3 = (\alpha\beta)^2 = -1$. It is a tiring but straightforward calculation to show that α, β generate a subgroup of order 24, and there are no other automorphisms.

Direct Products of Primordinal Groups Finally, we show how to find a torsion-free group whose automorphism group is a finite direct product of primordinal groups.

We start off with the observation that our Examples 3.3–3.8 can easily be modified by replacing the group A by the tensor product $A \otimes R$, where R is a rational group whose type $(k_1, \ldots, k_n, \ldots)$ contains only integers, and $k_n = 0$ whenever the nth prime occurs among the primes used in the example. Needless to say, only the case of infinitely many $k_n \neq 0$ bears any interest.

Now let $\mathfrak{G} = \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_n$ where each $\mathfrak{G}_j$ is isomorphic to one of the primordinal groups. For each $\mathfrak{G}_j$ select a torsion-free group A_j with $\operatorname{Aut} A_j \cong \mathfrak{G}_j$. The primes $p \not\equiv 1 \bmod 4$ (or mod 6) are infinite in number, so we can select a rigid system $R_1, \ldots, R_n$ of rational groups of the types mentioned above such that $k_n > 0$ only if $p_n \not\equiv 1 \bmod 4$ and mod 6. It is readily seen that then also the groups $B_j = A_j \otimes R_j$ form a rigid system. Hence

$$\operatorname{Aut}(B_1 \oplus \cdots \oplus B_n) = \operatorname{Aut} B_1 \times \cdots \times \operatorname{Aut} B_n \cong \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_n.$$

Orderable Groups as Automorphism Groups It was Corner who pointed out that groups admitting total orders are candidates for examples that can be realized as automorphism groups of torsion-free groups modulo the ubiquitous $\mathbb{Z}(2)$. The basic observation is that for such a group $\mathfrak{G}$, the group ring $\mathbb{Z}[\mathfrak{G}]$ with integer coefficients has no units other than the group elements and their negatives. The reason is that the product of $x = \sum_{i=1}^{k} n_i g_i$ and $y = \sum_{j=1}^{\ell} m_j h_j$ (where $n_i, m_j \in \mathbb{Z},\ g_i, h_j \in \mathfrak{G}$) can never be 1 unless each of x and y is $\pm$ a group element. In fact, if $n_i \neq 0 \neq m_j$

for all i, j, and if $g_1 < \cdots < g_k, h_1 < \cdots < h_\ell$ under an arbitrarily selected total order, then in the product xy the terms $n_1 m_1 g_1 h_1$ and $n_k m_\ell g_k h_\ell$ never cancel, and are different unless $k = 1 = \ell$. As a result, for an orderable group $\mathfrak{G}$, the unit group of the group ring $\mathbb{Z}[\mathfrak{G}]$ is isomorphic to $\mathfrak{G} \times \mathbb{Z}(2)$.

Theorem 3.9. *For every orderable group $\mathfrak{G}$, there exist torsion-free groups whose automorphism groups are isomorphic to $\mathfrak{G} \times \mathbb{Z}(2)$.*

Proof. We appeal to Theorem 7.1 in Chapter 16 to derive the existence of torsion-free groups whose endomorphism rings are isomorphic to $\mathbb{Z}[\mathfrak{G}]$. Each of these groups has the desired property. □

Example 3.10. An abelian group is orderable if and only if it is torsion-free. All non-commutative free groups are orderable, and by a theorem of Malcev, every locally nilpotent torsion-free group admits a total order.

★ **Notes.** The first relevant results on automorphism groups of torsion-free groups are in the memoir Beaumont–Pierce [2] on rank 2 groups. They show how complicated these groups can be even in this extremely special case. See also Król [2].

It was Hallett–Hirsch [1] who started the systematic investigations of finite automorphism groups of torsion-free groups. It is really surprising that these groups are made up of only a handful groups by using subdirect products. Their results were corrected and substantially improved by Corner in a monumental paper [10], published posthumously. Robust arguments were needed to verify the precise conditions on the admissible subdirect products. It might be helpful to call attention to the semidirect product representations $DC_{12} \cong \mathbb{Z}(3) \rtimes \mathbb{Z}(4)$ and $BT_{24} \cong Q_8 \rtimes \mathbb{Z}(3)$.

Hallett–Hirsch [1] also point out that if a torsion-free group H, commutative or not, has finite automorphism group, then it ought to be abelian. Because if $\operatorname{Aut} H$ is finite, then $\mathfrak{z}(H)$ is of finite index in H. Schur's theorem tells us that then the commutator subgroup H' is finite, so it must be trivial in a torsion-free group.

May [6] has more results on commutative automorphism groups of torsion-free groups; he succeeded in describing those of countable rank.

p-adic modules display a simpler behavior: Corner–Goldsmith [1] establish the isomorphy of reduced torsion-free modules over J_p ($p \neq 2$) provided their automorphism groups are isomorphic.

We have had several occasions to observe that there exist torsion-free groups of arbitrarily large cardinalities with automorphism groups $\cong \mathbb{Z}(2)$—which is clearly the smallest possible group that can be the automorphism group of a torsion-free group $\neq 0$. The question concerning the existence of indecomposable torsion-free groups of cardinality κ whose automorphism groups are of largest possible cardinality, viz. of size 2^κ, was answered for arbitrarily large κ in the affirmative in Fuchs [22].

The study of automorphism groups is certainly an attractive ground for research, there are lots of unanswered questions.

Bekker has several papers on the holomorph of abelian groups; Bekker–Nedov [1] show that in some special cases the holomorph characterizes the group.

Exercises

(1) The stabilizer of any chain of subgroups of a torsion-free group is a torsion-free group.

(2) Let $\mathfrak{G}$ be an elementary 2-group of order 2^n. There is a finite rank indecomposable torsion-free group A such that $\operatorname{Aut} A \cong \mathfrak{G}$.

(3) For any cardinal κ, the elementary 2-group of cardinality 2^κ is the automorphism group of some torsion-free group of rank κ.
(4) (Hallett–Hirsch) A finite abelian group F is isomorphic to $\operatorname{Aut} A$ for a torsion-free A if and only if (i) $|F|$ is even; (ii) $\alpha^{12} = 1$ for all $\alpha \in F$; and (iii) not every $\alpha \in F$ of order 2 is contained in a cyclic group of order 12.
(5) (de Vries–de Miranda) Let p_1, p_2, q_1, q_2 be different primes. The automorphism group of

$$A = \langle p_1^{-\infty} a, p_1^{-\infty} b, p_2^{-\infty} c, p_2^{-\infty} d,$$

$$q_1^{-\infty}(a+c), q_1^{-\infty}(b+d), q_2^{-\infty}(a+d), q_2^{-\infty}(b-c+d)\rangle$$

is $\cong \mathbb{Z}(6)$. [Hint: $a \mapsto b, b \mapsto -a+b, c \mapsto d, d \mapsto -c+d$ is a generator.]
(6) (Hallett–Hirsch) Suppose A is a torsion-free group with $\operatorname{Aut} A \cong \mathrm{Q}_8 = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^2 \rangle$.
   (a) For every $0 \neq a \in A$, the elements $a, \alpha(a), \beta(a), \alpha\beta(a)$ are independent.
   (b) If $\operatorname{rk} A$ is finite, then it is a multiple of 4.

Problems to Chapter 17

PROBLEM 17.1. Study the normal structure of automorphism groups of p-groups.

PROBLEM 17.2. For a p-group A, what are the connections between properties of $\operatorname{Aut} A$ and the UK-invariants of A?

PROBLEM 17.3. For which finite rank torsion-free groups A are all the automorphisms of $\operatorname{End} A$ inner?

PROBLEM 17.4. If $\operatorname{Aut} A$ (the unit group of $\operatorname{End} A$) does not generate $\operatorname{End} A$, are there then characteristic subgroups in A that are not fully invariant?

PROBLEM 17.5. Characterize $\operatorname{Aut} A$ for separable p-groups A, or for simply presented p-groups.

PROBLEM 17.6. Find a simple proof for the Leptin–Liebert theorem. Is Theorem 2.3 true also for $p = 2$?

Chapter 18
Groups in Rings and in Fields

Abstract The most frequent occurrence of abelian groups, apart from vector spaces, is undoubtedly the groups found in rings and fields. This chapter is devoted to their study.

While in general we deal exclusively with associative rings with 1, in the first two sections of this chapter we also include rings without identity as well as not necessarily associative rings (called 'narings' for short) in order to make the discussion smoother. As a matter of fact, the collection of narings on a group A displays more pleasant features than the set of associative rings, as demonstrated by the group Mult A. This group, suggested by R. Baer, crystallizes the idea of building narings on a group (thus from the additive point of view, associativity in rings seems less natural).

The paper devoted to the additive groups of rings, published by Rédei–Szele [1] on the special case of torsion-free rings of rank 1, was the beginning of Szele's ambitious program on the systematic study of additive groups. Our current knowledge on the additive groups of rings, apart from artinian and regular rings, is still more fragmentary than systematic, though a large amount of material is available in the literature. The inherent problem is that interesting ring properties rarely correspond to familiar group properties. Due to limitation of space, we shall not pursue this matter here; we refer the reader to Feigelstock's two-volume treatise [Fe]. The problem of rings isomorphic to the endomorphism ring of their additive group (called E-rings) attracted much attention; we present a few miscellaneous results on them.

Our final topic concerns multiplicative groups: groups of units in commutative rings and multiplicative groups of fields. While the theory for rings has not reached maturity, there are several essential results in the case of fields, mostly due to W. May. Unfortunately, we cannot discuss them here, because they require more advanced results on fields.

1 Additive Groups of Rings and Modules

In this section we collect a few fundamental facts on the additive groups of rings. But first of all, conventions on the terminology are in order.

Rings and Their Additive Groups In this chapter, by a **ring** we shall mean an associative ring (with or without identity), and by a **naring** a not necessarily associative ring (there are many of those of importance, like Lie rings, alternative rings, etc.). As customary, we will attribute to the ring properties of its underlying additive group in cases when no confusion may arise. Consequently, terms like p-ring, torsion or torsion-free ring, divisible and reduced ring, pure ideal, etc. will make perfectly good sense without any further comment. If desirable, distinction will be made between the ring R and its additive group R^+; however, extreme caution is necessary in the context of direct decompositions. A ring R whose

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additive group is isomorphic to the group A will be called a **ring on** A; we also say R **is supported by** A.

Lemma 1.1. *For all* $a, b \in \mathsf{R}$ *and* $m, n \in \mathbb{N}$*, we have:*

(a) *If* $ma = 0$ *and* $nb = 0$*, then* $dab = 0$ *where* $d = \gcd\{m, n\}$.
(b) *If* $m|a$ *and* $mb = 0$*, then* $ab = 0$.
(c) $m|a$ *and* $n|b$ *imply* $mn|ab$.

Proof. Follow the proof of Lemma 1.2 in Chapter 8. □

Some immediate corollaries of this lemma are listed below.

(A) In every ring R, the following are (two-sided) ideals:

  (i) $n\mathsf{R}$ and $\mathsf{R}[n]$ for every $n \in \mathbb{N}$;
  (ii) the torsion part $t(\mathsf{R})$, and its p-components;
  (iii) the Ulm subgroups, the subgroup $p^{\sigma}\mathsf{R}$ for every ordinal σ, including the p-divisible part $p^{\infty}\mathsf{R}$ and the divisible subgroup of R;
  (iv) the fully invariant subgroups of the additive group R^{+};
  (v) if R is torsion-free, then $\mathsf{R}(\mathbf{t})$ is an ideal for every type $\mathbf{t}$.

  Similar statements hold if R is replaced by a left or right ideal. (iv) follows from the fact that in any ring R, multiplication by an element from the left (or from the right) is an endomorphism of R^{+}.

(B) If R is a torsion ring, then there is a ring direct decomposition $\mathsf{R} = \oplus_p \mathsf{R}_p$ into p-components.

(C) If $p|e$ for an idempotent e, then $p^k|e$ for all $k \in \mathbb{N}$. If the ring R has an identity $\mathbf{1}$, then $m|\mathbf{1}$ $(m \in \mathbb{N})$ implies $m^k|a$ for all $a \in \mathsf{R}$ and $k \in \mathbb{N}$.

(D) From (C) we obtain that the p-heights satisfy

$$h_p(ab) \geq h_p(a) + h_p(b) \quad \text{for all } a, b \in \mathsf{R}.$$

(E) Consequently, if R is a torsion-free ring, then the characters obey the rule $\chi(ab) \geq \chi(a)\chi(b)$ for all $a, b \in \mathsf{R}$.

(F) Elements in the torsion subgroup of a ring are annihilated by elements in the first Ulm subgroup, in particular, by the elements in the divisible part of the additive group. The additive group of a p-ring is therefore separable whenever it has an identity; actually, it must then be a bounded group.

Example 1.2. Let R be a torsion-free ring. If $a \in \mathsf{R}$ is not nilpotent, then either $\mathbf{t}(a)$ is idempotent or all the types of a^n $(n \in \mathbb{N})$ are different.

Extending Ring to a Larger Group A more relevant problem is concerned with the extension of the ring structure on a group to a larger group. The first question which comes to mind is whether or not a ring can be extended to a ring on the injective hull of its additive group. From (F) it is obvious that in the divisible hull of a torsion group all products ought to be 0. Thus the real interest lies in the torsion-free case, which can be dealt with satisfactorily.

Theorem 1.3. *The ring structure on a torsion-free group can be extended uniquely to a ring on the divisible hull of its additive group.*

Proof. There is only one way to extend the multiplication $\cdot$ from a ring on a torsion-free A to its divisible hull D: if $x, y \in D$ and $m, n \in \mathbb{N}$ such that $mx, my \in A$, then the extension μ (if exists) must be

$$\mu(x, y) = (mn)^{-1} mx \cdot ny \in D.$$

This is independent of the choice of m, n, and makes D into a ring—this is easy to check. $\square$

Notice that the ring on D inherits commutativity (or associativity), and preserves identity if the ring on A had one.

Example 1.4. Let R be an integral domain such that R^+ is torsion-free of finite rank n. Then the unique extension ring on its divisible hull D is a field. In fact, given $r \neq 0, s$ in R, the equation $rx = s \in \mathsf{R}$ is solvable in D, since $r\mathsf{R}$ is of rank n, so for some $x \in \mathsf{R}$, s must be equal to rx.

There are two more noteworthy embeddings of a group that should be examined: one into its $\mathbb{Z}$-adic completion, and one into its cotorsion hull. Before entering into the analysis of when and in how many ways a ring structure can be extended in these cases, we prove an auxiliary lemma.

Lemma 1.5. *Let A be a reduced group. A ring structure on a pure dense subgroup of A can be extended at most in one way to a ring on A.*

Proof. From a pure-exact sequence $0 \to C \to A \to A/C \to 0$, a repeated application of Theorem 3.1 in Chapter 8 yields the pure-exact sequences $0 \to C \otimes C \to A \otimes C \to (A/C) \otimes C \to 0$ and $0 \to A \otimes C \to A \otimes A \to A \otimes (A/C) \to 0$, leading to the exact sequence

$$0 \to C \otimes C \to A \otimes A \to [(A/C] \otimes C] \oplus [A \otimes (A/C)] \to 0. \tag{18.1}$$

Clearly, the terms involving the divisible group A/C are divisible, thus the induced map $\mathrm{Hom}(A \otimes A, A) \to \mathrm{Hom}(C \otimes C, A)$ is monic. This means that any $C \otimes C \to C$ can have at most one extension $A \otimes A \to A$. $\square$

We are now prepared to look at the embeddings in completions and in cotorsion hulls.

Theorem 1.6. *Let G denote the pure-injective or the cotorsion hull of A. A ring structure on A can be extended to a ring on G, which is unique whenever A is reduced.*

Proof. First, let G be the completion of A. As above, from the pure-exact sequence $0 \to A \to G \to G/A \to 0$ we derive the conclusion that $0 \to A \otimes A \to G \otimes G$ is pure-exact, and by pure-injectivity, that the induced map $\mathrm{Hom}(G \otimes G, G) \to \mathrm{Hom}(A \otimes A, G)$ is surjective. Hence the extensibility is manifest. If A is reduced, the preceding lemma takes care of uniqueness.

If G is the cotorsion hull of A, then in the exact sequence $0 \to A \otimes A \to G \otimes G \to [(G/A) \otimes A] \oplus [G \otimes (G/A)] \to 0$ (that is like (18.1)) the direct sum is torsion-free and divisible. A reference to Theorem 9.2 in Chapter 9 shows that the map $\mathrm{Hom}(G \otimes G, G) \to \mathrm{Hom}(A \otimes A, G)$ is surjective, completing the proof. □

★ **Notes.** The pioneering papers: Beaumont [1], Szele [1], Rédei–Szele [1], Beaumont–Zuckerman [1] (all published around 1950) provide only a superficial analysis of the relation between the structure of a ring and its additive group, in very special cases. They stimulated interest in the additive groups of rings, and in the next decade several more substantial papers were published, including Beaumont–Pierce [1] which initiated a more systematic study of rings on finite rank torsion-free groups.

Exercises

(1) Describe all (associative) rings on the following groups: $\mathbb{Z}(p^n), \mathbb{Z}(p^\infty), \mathbb{Z}$.

(2) (a) In a torsion-free ring R with identity 1, $\chi(1) \leq \chi(a)$ for each $a \in \mathsf{R}$.
  (b) An unbounded p-ring cannot have identity.

(3) If U, V are fully invariant subgroups (ideals) of R, then their product is contained in their intersection $U \cap V$.

(4) (Steinfeld) Let R be a ring without divisors of zero. Then either $p\mathsf{R} = 0$ for some prime p, or R is torsion-free whose typeset is directed upward.

(5) In a torsion-free ring R, the nil radical is a pure ideal.

(6) Let M denote a maximal (left) ideal of the torsion-free ring R. Then either $p\mathsf{R} \leq \mathsf{M}$ for some prime p, or R/M is torsion-free divisible.

(7) The ring constructed in Theorem 1.3 has the property that every torsion-free divisible ring containing a subring isomorphic to the ring on A also contains a ring isomorphic to the one on D.

(8) Let R be a ring with additive group A such that $A^1 = 0$. Then the unique extension ring on the completion $\tilde{A}$ inherits polynomial identities (in particular, associativity, commutativity) from R, and is a $\tilde{\mathbb{Z}}$-algebra. [Hint: $\tilde{A}$ is an inverse limit.]

(9) Let A and D be as in Theorem 1.3.

  (a) Establish a bijection between the pure left ideals of A and D.
  (b) This correspondence preserves primeness.
  (c) D can have an identity even if A has none.
  (d) If D has an identity, then every left ideal of D is pure.
  (e) D has a non-zero nilpotent ideal exactly if A has got one.

(10) A ring on a splitting mixed group need not be a direct sum of a torsion and a torsion-free ring. [Hint: on $\langle a \rangle \oplus \langle b \rangle \cong \mathbb{Z} \oplus \mathbb{Z}(p)$ define $a^2 = b, ab = ba = b^2 = 0$.]

(11) For a ring R with identity, by a **universal R-module** on the group A is meant an R-module U along with a group homomorphism $\mu : A \to U$ such that for every group homomorphism $\alpha : A \to M$ from A into an R-module M, there is a unique R-homomorphism $\psi : U \to M$ such that $\psi\mu = \alpha$. Verify the existence and uniqueness of universal R-modules. [Hint: $\mathsf{R} \otimes A$.]

2 Rings on Groups

In the preceding section we discussed additive groups of rings, in this section we reverse our approach, and will focus on ring structures on given groups. Here it seems reasonable to consider not only the associative rings, but to include also narings in the discussion.

Every group A can be furnished with a ring structure in a trivial way by defining all products to be 0; such a ring is called the **zero-ring on** A. In general, there exist several other rings on A.

Rings on a Given Group By **multiplication** on a group A, we mean a function

$$\mu : A \times A \to A$$

that is additive in both arguments. The multiplications in A are in a bijective correspondence with the naring structures on A. Observe that left multiplications by $a \in A$ are endomorphisms of the group, thus the multiplication μ in R defines a homomorphism $\eta : A \to \operatorname{End} A$ such that $\mu(a, x) = \eta(a)(x)$ $(a, x \in A)$. *Vice versa,* every such η defines a naring on A. In this way, the naring structures on A can be identified with the elements of

$$\operatorname{Mult} A = \operatorname{Hom}(A, \operatorname{End} A) \cong \operatorname{Hom}(A \otimes A, A).$$

$\operatorname{Mult} A$ is called the **group of multiplications on** A (R. Baer).

Example 2.1. (a) Let $A = \langle a \rangle$ be a cyclic group. Then $\operatorname{Mult} A \cong A$ generated by $\eta : a \mapsto \mathbf{1}_A$ $(\eta \in \operatorname{Hom}(A, \operatorname{End} A))$. It is a rare phenomenon that all the multiplications on a group are associative and commutative.

(b) Let $A = \langle a \rangle \oplus \langle b \rangle$, both cyclic groups of the same order. The multiplication defined by the following map is not associative: $\eta(a) : a \mapsto b, b \mapsto b$; $\eta(b) : a \mapsto a, b \mapsto a$ (the rest follows by distributivity).

Example 2.2. Let A be a torsion-free group $\neq \mathbb{Z}$ with $\operatorname{End} A \cong \mathbb{Z}$. Then $\operatorname{Mult} A = 0$, A is a nil group (see below).

We call a group N a **nil group** if there is no naring on N except for the zero-ring, i.e. $\operatorname{Mult} N = 0$, and call N a **quasi-nil** group if it admits but a finite number of non-isomorphic rings. In the torsion case, both the nil, and the quasi-nil groups can be described without difficulty; moreover, with a little extra effort, even the case of mixed nil groups can be settled.

Proposition 2.3 (Szele [1]). *A torsion group is nil if and only if it is divisible. There exists no genuine mixed group that is nil.*

Proof. A summand of a nil group is nil, and since cyclic groups of orders p^k ($k \in \mathbb{N}$) are not nil, it is clear that the torsion part of a nil group must be divisible. On the other hand, for a torsion divisible group D we have $D \otimes D = 0$, so $\operatorname{Mult} D = 0$.

For a mixed nil group N, tN is divisible, and if $N \neq tN \neq 0$, then there is a non-trivial map $0 \neq N \otimes N \to tN$, so $\operatorname{Mult} N \neq 0$. $\square$

Proposition 2.4 (Fuchs [7]). *A torsion group N is quasi-nil exactly if $N = B \oplus D$ where B is finite and D is divisible.*

Proof. The basic subgroup B of a quasi-nil torsion group N must be finite. Otherwise, it is easy to construct rings R on N such that the squares R^2 have different finite cardinalities. Thus N must have the indicated structure.

Conversely, assume $N = B \oplus D$, where $B = \langle b_1 \rangle \oplus \cdots \oplus \langle b_k \rangle$ is a finite and D is a divisible group. The narings on N are completely determined by the products $b_i b_j$ of basis elements; the products might have non-zero D-coordinates, but all of them are contained in a finite rank summand of D, so as far as isomorphism is concerned, this summand may be regarded the same for all narings. This means that there are but a finite number of choices for the products $b_i b_j$, and therefore N is quasi-nil, indeed. $\square$

Example 2.5. (a) A rank 1 torsion-free group is nil if and only if its type is not idempotent. A completely decomposable torsion-free group that is homogeneous of non-idempotent type is a nil group.

(b) (Corner) There are homogeneous torsion-free nil groups of type $(0, \dots, 0, \dots)$. [Hint: Sect. 4, Exercise 5 in Chapter 12.]

Proposition 2.6 (Wickless [1]). *All rings on a group A are nilpotent if and only if $A = D \oplus C$, where D is torsion divisible, and C is reduced torsion-free supporting only nilpotent rings.*

Proof. If all rings on A are nilpotent, then A cannot have any finite cyclic summand, or contain a copy of $\mathbb{Q}$, since these support non-nilpotent rings. Hence $A = D \oplus C$, where D is torsion divisible and C is reduced torsion-free. It is clear that only nilpotent rings can exist on C. Conversely, if A has such a decomposition, then D is an ideal in any ring R on A, and $\bar{\mathsf{R}} = \mathsf{R}/D$ is nilpotent. Then $(\bar{\mathsf{R}})^m = 0$ (the mth power of the ring) for an $m \in \mathbb{N}$, thus $(\mathsf{R})^{2m} \leq (D)^2 = 0$. $\square$

Rings on Finite Rank Torsion-Free Groups Substantial results were obtained by Beaumont–Pierce concerning rings on finite rank torsion-free groups. The highlight is a generalization of the famous Wedderburn principal theorem which is stated next without detailed proof. We write $\mathsf{A}^* = \mathbb{Q} \otimes \mathsf{A}$, and identify $1 \otimes a$ with $a \in \mathsf{A}$; A^* is a $\mathbb{Q}$-algebra if A is a ring.

Theorem 2.7 (Beaumont–Pierce [1]). *Assume A is a torsion-free ring of finite rank, and let $\mathsf{A}^* = \mathsf{S}^* \oplus \mathsf{N}^*$ be the Wedderburn decomposition of the $\mathbb{Q}$-algebra A^*, where S^* is a semi-simple subalgebra and N^* is the radical of A^*. Then $\mathsf{S} = \mathsf{A} \cap \mathsf{S}^*$*

is a subring, and $\mathsf{N} = \mathsf{A} \cap \mathsf{N}^*$ *is the largest nilpotent ideal of* A*; they satisfy* $\mathbb{Q} \otimes \mathsf{S} = \mathsf{S}^*$, $\mathbb{Q} \otimes \mathsf{N} = \mathsf{N}^*$.

Furthermore, $\mathsf{S} \oplus \mathsf{N}$ *is a subring of finite index in* A.

Proof. Everything follows without difficulty from the classical Wedderburn theorem, except for the last statement. That $\mathsf{S} \oplus \mathsf{N}$ is a finite index subring requires a long proof. (We can add that the additive group of S is quotient-divisible.) □

Role of Basic Subgroups We turn our attention to an interesting phenomenon underlining the relevance of basic subgroups.

Proposition 2.8 (Fuchs [5]). *Let A be a p-group, and $B = \oplus_{i\in I}\langle b_i\rangle$ a basic subgroup of A. Any multiplication $\mu \in$ Mult A is completely determined by the set of the $\mu(b_i, b_j)$ for all $i, j \in I$.*

Conversely, every choice of the elements $\mu(b_i, b_j)$ for all $i, j \in I$ (subject to the necessary condition that $o(\mu(b_i, b_j)) \le \min\{o(b_i), o(b_j)\}$) gives rise to a naring on A.

Proof. There are several proofs to choose from, the following is most elementary and direct. Needless to say, that if μ is given for the basis elements of B (of course, the values need not be in B), then it extends linearly to all of B. To find $\mu(a, a')$ for all $a, a \in A$, let $o(a') = p^k$, and write accordingly $a = b + p^k x$ with $b \in B, x \in A$, to obtain

$$\mu(a, a') = \mu(b, a') + \mu(p^k x, a') = \mu(b, a') + \mu(x, p^k a') = \mu(b, a').$$

In a similar fashion, $\mu(b, a') = \mu(b, b')$ for a suitable $b' \in B$, establishing the first part of our claim.

For the second part, note that, every $b \in B$ being a unique linear combination of the b_i, the ring postulates for $\mu(b, b')$ for all $b, b' \in B$ are readily checked. If μ is extended to the whole of A (in the way shown in the preceding paragraph), then just a routine verification is needed to show that the extended μ is well defined and satisfies the requisite postulates. □

Example 2.9. Let $\overline{B}$ be the torsion-completion of the direct sum $B = \oplus_{i\in I}\langle b_i\rangle$ of cyclic p-groups. Rings can be defined on $\overline{B}$ in various ways, we need to specify the products $b_i b_j$.

(a) Putting $b_i b_j = b_i$ or b_j according as $o(b_i) \le o(b_j)$ or not.
(b) Order the index set I, and in case $o(b_i) = o(b_j)$, set $b_i b_j = b_i$ or $= b_j$ according as $i \le j$ or $i > j$.
(c) Of course, $b_i b_j$ can be chosen arbitrarily, e.g. $b_i b_j = b_k$ at random (subject to the necessary order condition) or a linear combination.

In (a)–(c), B is a subring of the ring on $\overline{B}$ such that all the products belong to B. In case (b)–(c), associativity is in doubt.

Example 2.10. Let A be a p-group with basic subgroup B. Then Mult$A \cong \mathrm{Hom}(B \otimes B, A)$; this follows from Proposition 2.8.

Absolute Properties When dealing with rings on a given group A, an inevitable problem is to find those subgroups of A that enjoy certain property P, like being an ideal, or an annihilator, in *every* ring supported by A. We shall call them **absolute P** on A.

We only consider **absolute ideals**, i.e. subgroups that are definitely (one- or two-sided) ideals in every ring on A; evidently, problems for left-, right-, or two-sided ideals are identical, because the anti-isomorphic ring is on the same group. We observed above that the fully invariant subgroups of A enjoy this absolute property, but we will see in a moment that not only these subgroups. To identify the absolute ideals, we define an ideal of $\operatorname{End} A$ which plays a decisive role in the answer. This ideal was introduced by Fried [1], so we shall refer to it as the **Fried ideal** F of $\operatorname{End} A$. It is defined as the trace of A in $\operatorname{End} A$:

$$\mathsf{F} = \langle \operatorname{Im} \phi \mid \phi \in \operatorname{Hom}(A, \operatorname{End} A)\rangle.$$

To show that F is in fact an ideal in $\operatorname{End} A$, notice that for $\eta \in \operatorname{End} A$, the mappings $a \mapsto \eta(\phi a)$ and $a \mapsto (\phi a)\eta$ are homomorphisms $A \to \operatorname{End} A$, as is readily checked. Thus $\eta(\phi a)$ and $(\phi a)\eta$ are among the generators of F for all $a \in A$, so F is in fact an ideal of $\operatorname{End} A$.

Example 2.11. (a) (Fried [1]) Let A be a reduced p-group, so $\operatorname{End} A$ is a reduced algebraically compact group. In this case, F is the torsion subgroup of $\operatorname{End} A$.
(b) Let $A = \mathbb{Z}(p^\infty)$, so $\operatorname{End} A \cong J_p$. Thus $\mathsf{F} = 0$.
(c) If A is a torsion-free group of rank one, then $\mathsf{F} = \operatorname{End} A$ or 0 according as $\mathbf{t}(A)$ is idempotent or not.

Theorem 2.12 (Fried [1]). *A subgroup C of a group A is an absolute ideal if it is invariant under the Fried ideal F of* $\operatorname{End} A$*, i.e. $\mathsf{F}C \leq C$. The converse holds if we admit narings on A as well.*

Proof. Let R be a (na)ring on A. We associate with $a \in \mathsf{R}$ the left multiplication $\lambda_a : x \mapsto ax$ $(x \in A)$. The correspondence $\phi : a \mapsto \lambda_a$ is a homomorphism of A into $\operatorname{End} A$, and hence into F. A subgroup C of A is a left ideal in R exactly if every λ_a carries C into itself. The same argument applies to the right multiplications. The conclusion is that if $\mathsf{F}C \leq C$, then C is an absolute ideal of A.

Conversely, every $\phi \in \operatorname{Hom}(A, \operatorname{End} A)$ gives rise to a ring multiplication by defining the product of $a, c \in A$ as $ac = (\phi a)c$ (which may be non-associative). Therefore, if C is an ideal in every naring on A, then $(\phi a)c \in C$ for $c \in C$, i.e. $\mathsf{F}C \leq C$. □

Example 2.13. (a) $\mathsf{F} = 0$ means that all subgroups are absolute ideals. This is the case, e.g., for J_p, $\mathbb{Z}(p^\infty)$, and for rational groups of non-idempotent types.
(b) In a p-group A, all subgroups of A^1 are absolute ideals.

★ **Notes.** Borho [1] investigates the number n of non-isomorphic associative rings (identity not required) on finite rank torsion-free groups. He shows that only $n = 1, 2, 3$, and ∞ are possible. $n = 1$ holds for nil groups, $n = 2$ only for $\mathbb{Q}$, and $n = 3$ for $\mathbb{Q} \oplus R$ where R is a rational group of non-idempotent type. For more information about rings on torsion-free groups of finite rank, see Freedman [2], Niedzwecki [1], and on mixed groups, Jackett [2].

Arnold [4] proves that if R^+ is a finite rank Butler group, then N^+ is Butler and $\mathsf{R}^+/\mathsf{N}^+$ is almost completely decomposable (N denotes the nil radical). Feigelstock [1] considers rings R that are T-nilpotent (i.e., for any sequence $a_i \in \mathsf{R}$ $(i < \omega)$, there is $n < \omega$ such that $a_1 a_2 \cdots a_n = 0$), and shows that a torsion-free group cannot have the property that all rings on it are T-nilpotent.

Beaumont–Lawver [1] study groups supporting semi-simple rings, Haimo [2] investigates groups that support radical (non-zero) rings. For rings on completely decomposable groups, see Ree–Wisner [1] and Gardner [1]. Kompantseva [1] studies Mult on reduced algebraically compact groups.

Absolute properties of subgroups were investigated in several papers. In particular, absolute Jacobson radicals were described in special cases. Eklof–Mez [1] show that on a reduced p-adic algebraically compact group A, the subgroup pA is the absolute Jacobson radical. For absolute nil-ideals, see Kompantseva [2].

Bergman [1] and Hill [14] investigate the additive groups of rings generated by idempotents. Interestingly, they are always Σ-cyclic.

Exercises

(1) Mult A is a bimodule over End A.

(2) (a) The group Mult D is torsion-free and divisible for divisible D.
  (b) All multiplications on a torsion-free divisible D are associative if and only if rk $D \leq 1$.

(3) (a) Mult $J_p \cong J_p$ for every prime p (cf. Example 1.10 in Chapter 8).
  (b) All rings (identity not required) on J_p are isomorphic to one of the following rings: $p^k J_p$ ($k < \omega$) and the zero ring on J_p.

(4) (a) A homomorphism $\eta : A \to \operatorname{End} A$ defines an associative ring structure on the group A if and only if it satisfies $\eta(x)\eta(y) = \eta(\eta(x)(y))$ for all $x, y \in A$.
  (b) In order to get an associative ring on A, $\operatorname{Im} \eta$ ought to be a subring in End A. If η is injective, then this condition is sufficient as well.

(5) (Fried) For a subgroup B of A, define $B^* = \{a \in A \mid \mathsf{F}a \leq B\}$.

  (a) Show that B^* is a subgroup of A.
  (b) B is an ideal in every naring over A if and only if $B \leq B^*$ if and only if there is a fully invariant subgroup C of A such that $C \leq B \leq C^*$.

(6) The absolute annihilator of a torsion group A contains A^1, and there is an associative and commutative ring on A whose annihilator is exactly A^1.

(7) (a) Every element in the Frattini ideal $p\mathsf{R}$ of a p-ring R generates a nilpotent ideal of R, but the ideal $p\mathsf{R}$ itself need not be nilpotent.
  (b) (Jackett) There is an associative and commutative ring on a p-group A such that pA is exactly the set of its nilpotent element. (Thus pA is the nil radical.)
  (c) The absolute Jacobson radical of a p-group A contains pA.

(8) Let A be a p-group such that there are at most countably many non-isomorphic rings on A. Prove that A is a nil group. [Hint: argue with the basic subgroup.]

(9) Let R be a p-ring, and B a basic subgroup of R^+. If every element of B is nilpotent, then the same holds for R. [Hint: for $a \in \mathsf{R}$ there is a $b \in B$ with $a^k = b^k$.]

(10) (Feigelstock) A p-group A has the property that every associative ring on A is commutative if and only if the reduced part of A is cyclic.

3 Additive Groups of Noetherian Rings

We proceed to investigate the additive structures of some important types of rings. Our study opens with the class of noetherian rings.

Nilpotent Noetherian Rings We begin with the nilpotent case.

Lemma 3.1 (Szele [6]). *A group is the additive group of a nilpotent left or right noetherian ring if and only if it is finitely generated.*

Proof. Let R be a noetherian ring with $(\mathsf{R})^k = 0$ (kth power) for some $k \in \mathbb{N}$. The subgroups between $(\mathsf{R})^{i+1}$ and $(\mathsf{R})^i$ ($i = 1, \dots, k-1$) are ideals of R, thus they satisfy the maximum condition. Equivalently, $(\mathsf{R})^i/(\mathsf{R})^{i+1}$ is finitely generated as a group. Therefore, also R^+ is finitely generated. Conversely, the zero-ring over a finitely generated group is nilpotent noetherian. □

Mixed Noetherian Rings Split We cannot tell much about the additive structure of a general noetherian ring R, besides its torsion subgroup T. The subgroups $\mathsf{R}[n]$ are ideals for all $n \in \mathbb{N}$, thus there is a maximal one, say, $\mathsf{R}[m]$, among them. Then necessarily $\mathsf{T} = \mathsf{R}[m]$, $m\mathsf{T} = 0$, thus $\mathsf{R}^+ = \mathsf{T}^+ \oplus C$ for some torsion-free subgroup C. This C is isomorphic to the additive group of the noetherian ring R/T (but it need not be a ring as products of its elements need not belong to C). Hence

Lemma 3.2. *A group A is the additive group of a left or right noetherian ring if and only if $A = T \oplus C$, where T is a bounded group, and C is torsion-free that can carry a noetherian ring structure.*

Proof. The converse will follow from Lemma 4.3. □

In view of this lemma, the study of additive groups of noetherian rings may be restricted to the torsion-free case. A further reduction to the reduced case is immediate as soon as we notice that there are noetherian rings, even fields, on every torsion-free divisible group.

Example 3.3. The additive group of an algebraic number field of degree n gives an example of a field with additive group $\cong \mathbb{Q}^{(n)}$, while for an infinite cardinal κ, the quotient field of a polynomial ring over $\mathbb{Q}$ with κ indeterminates yields an example with additive group $\mathbb{Q}^{(\kappa)}$.

Our present knowledge of noetherian rings on reduced torsion-free groups does not go beyond a few elementary remarks.

(A) *For every left ideal L of a noetherian ring R, we have $n\mathsf{L}_* \leq \mathsf{L}$ for some $n \in \mathbb{N}$: L_*/L is bounded for the purification L_* of L.*
This follows from the ascending chain condition on L_*/L.

(B) *The types of elements in R satisfy the minimum condition.*
In fact, to a properly descending chain $\mathbf{t}_1 > \cdots > \mathbf{t}_n > \dots$ of types of elements in R, there corresponds a properly ascending chain $\mathsf{R}(\mathbf{t}_1) < \cdots < \mathsf{R}(\mathbf{t}_n) < \dots$ of ideals of R.

Noetherian Rings on Free Groups The next theorem shows that arbitrarily large free groups support noetherian domains.

Theorem 3.4 (O'Neill [1]). *For every cardinal κ, the free group on κ generators supports a commutative noetherian domain.*

Proof. If $\kappa = n$ is an integer, then a ring of algebraic integers of degree n is a Dedekind domain whose additive group is a free group on n generators.

So suppose κ is infinite, and x_σ ($\sigma < \kappa$) are indeterminates. Let P denote the polynomial ring in these indeterminates with coefficients in $\mathbb{Z}$. The primitive polynomials (i.e., with 1 as the gcd of coefficients) form a multiplicative semigroup S in P, and define the ring R as the localization of P at S. That is, R consists of all fractions f/g where $f, g \in \mathsf{P}$ and g is primitive. Since every polynomial in R is the product of an integer and primitive polynomials, from the definition it should be evident that the only ideals of R are $n\mathsf{R}$ for integers $n \geq 0$. This shows that R is a (commutative) noetherian integral domain.

It remains to show that R^+ is a free group. That the additive group P^+ is free follows from the unique representation of polynomials as sums of different monomials. We break the proof into several steps.

Step 1. *Subrings R_σ and subgroups G_σ.* For every ordinal $0 < \sigma < \kappa$, let R_σ be the set of all fractions $f/g \in \mathsf{R}$ where f, g contain explicitly only indeterminates x_ρ with $\rho < \sigma$. In addition, let $\mathsf{P}_\sigma = \mathsf{P} \cap \mathsf{R}_\sigma$. Furthermore, let G_σ be the subgroup of $\mathsf{R}_{\sigma+1}$ which is generated by the fractions f/g subject to the conditions:

(i) $f, g \in \mathsf{P}_{\sigma+1}$;
(ii) f, g have no common factors except ± 1;
(iii) either g contains a term with x_σ, or else f is a monomial with x_σ as a factor.

Then every fraction $f/g \in G_\sigma$ satisfies (iii) even if f is not monomial, because if the common denominator of a finite sum of generators contains no x_σ, then all the numerators in this sum must contain x_σ as a factor.

Step 2. *We claim:* $\mathsf{R}^+ = \oplus_{\sigma<\kappa} G_\sigma$. That the G_σ generate R^+ requires no comment, only the direct sum property needs a verification. To this end, we prove that $G_\sigma \cap \sum_{\rho<\sigma} G_\rho = 0$. Assume $f/g = u/v \neq 0$ where either f or g contains x_σ, but u, v have only indeterminates x_ρ with $\rho < \sigma$. It is well known that P is a unique factorization domain, so we may assume that f, g have no common factor $\neq \pm 1$. Then $fv = gu$ shows that the factors of f are factors of u, so f cannot contain x_σ, and neither can g, a contradiction. Thus the sum is direct. This argument also shows that $\mathsf{R}^+_{\sigma+1} = \mathsf{R}^+_\sigma \oplus G_\sigma$.

Step 3. *G_σ is free.* In a finite rank subgroup H of R, let $g \in \mathsf{P}$ be a common denominator of elements in a maximal independent set. Then gH is a finite rank subgroup in the free group P^+, so it is finitely generated free. Here g can be cancelled to argue that H itself is finitely generated free. Owing to Pontryagin's criterion, this allows us to conclude that G_σ is free for countable ordinals σ.

We also observe that for $\sigma \geq \omega$, $G_\sigma \cong G_{\sigma+1}$ is obvious, since they are generated by using equipotent sets of indeterminates, so freeness is inherited when moving to the next ordinal. If σ is a limit ordinal, then $\mathsf{R}_\sigma = \cup_{\rho<\sigma}\mathsf{R}_\rho$ along with $\mathsf{R}^+_{\rho+1} = \mathsf{R}^+_\rho \oplus G_\rho$ guarantees that $\mathsf{R}_\sigma = \oplus_{\rho<\sigma} G_\sigma$. This completes the proof. □

★ **Notes.** There are several minor results on the additive groups of noetherian rings in the literature, but so far no relevant information has been found. Theorem 3.4 is an exception, since it shows that no serious restriction can be expected.

Exercises

(1) Let R be a torsion-free ring, and D the ring on the divisible hull of R^+. D is noetherian if so is R, but it can be noetherian even if R is not.
(2) A homogeneous completely decomposable torsion-free group supports a noetherian domain if and only if its type is idempotent.
(3) A finitely generated free group admits countably many pairwise non-isomorphic rings; all noetherian.
(4) For every subgroup G of a finitely generated free group F, there is a ring R on F such that the additive group of $(\mathsf{R})^2$ is G.

4 Additive Groups of Artinian Rings

Next, we consider left **artinian rings** (with or without identity), i.e. rings in which the left ideals satisfy the minimum condition. We will see that the minimum condition has a profound impact on the additive structure—this should not be surprising, since the artinian condition is very restrictive. Our main result will establish a necessary and sufficient condition for a group to be the additive group of an artinian ring.

Nilpotent Artinian Rings To begin with, we consider nilpotent artinian rings R, that is, $(\mathsf{R})^k = 0$ for some $k \in \mathbb{N}$ (called the exponent of R). Their additive groups are easily characterized.

Proposition 4.1 (Szele [6]). *A group A is the additive group of a nilpotent artinian ring if and only if A satisfies the minimum condition on subgroups. Thus A is a torsion group characterized in Theorem 5.3 in Chapter 4.*

Proof. In a nilpotent ring R of exponent k, every subgroup C satisfying $(\mathsf{R})^{i+1} \leq C \leq (\mathsf{R})^i$ $(i = 1, \ldots, k-1)$ is an ideal of R. Hence if R is artinian, the minimum condition is satisfied by the subgroups of $(\mathsf{R})^i/(\mathsf{R})^{i+1}$ for all i, and hence by those of R. Conversely, the zero-ring on a group with minimum condition on subgroups is artinian (and nilpotent). □

Preliminary Lemmas Removing the hypothesis of nilpotency, we turn our attention to artinian rings in general. Their study starts with two preliminary lemmas.

Lemma 4.2. *The additive group of a field is a torsion-free divisible group or an elementary p-group according as the characteristic of the field is* 0 *or p. Every such group is the additive group of a field.*

Proof. The fields are vector spaces over $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$, depending on their characteristic. The existence of finite algebraic extensions of degree n and extensions of transcendence degree κ of the prime fields (for any cardinal κ) prove the second assertion. (See also Example 3.3.) □

The following lemma is less obvious.

Lemma 4.3. *Let $A \cong \oplus_\kappa \mathbb{Z}(p^k)$ with an infinite cardinal κ, for a fixed $k \in \mathbb{N}$. There exists a commutative ring R with identity on A such that $\mathsf{R}, p\mathsf{R}, \dots, p^k\mathsf{R} = 0$ are the only ideals of R.*

Proof. Let G be a totally ordered abelian group with $|G| = \kappa$. Define R as the formal Laurent series ring over the ring $\mathbb{Z}/p^k\mathbb{Z}$ for G. That is, R consists of all formal power series $f = \sum_{\gamma\in G} n_\gamma x^\gamma$ with coefficients $n_\gamma \in \mathbb{Z}/p^k\mathbb{Z}$, subject to the condition that the set of exponents γ of the non-zero terms is well-ordered in the total order of G. Of course, the operations in R are the same as for power series. It is well known that these formal Laurent series form a field provided that the coefficients are taken from a field. Using this property for $\mathbb{Z}/p\mathbb{Z}$, it is clear that there is no ideal between $p\mathsf{R}$ and R, and hence it follows that the only ideals in R are the obvious $\mathsf{R}, p\mathsf{R}, \dots, p^k\mathsf{R} = 0$.

A somewhat simpler argument is as follows. Let I be an index set of cardinality κ, T the polynomial ring over $\mathbb{Z}/p^k\mathbb{Z}$ in indeterminates x_i with $i \in I$, and R the ring T localized at the semigroup of all primitive polynomials; here 'primitive' simply means that not all the coefficients are divisible by p. It is immediate that the additive group of T, and hence of R, is a direct sum of as many copies of $\mathbb{Z}(p^k)$ as $|I|$, and it is also easy to verify that the only ideals of R are $\mathsf{R}, p\mathsf{R}, \dots, p^k\mathsf{R} = 0$. □

Characterization of the Additive Structure The next result gives us full insight into the additive structure of artinian rings.

Theorem 4.4 (Szele–Fuchs [1]). *A group A is the additive group of a* (*left*) *artinian ring* (*commutative artinian ring*) *if and only if it is a direct sum $A = B \oplus C \oplus D$ where*

(i) *B is a bounded group;*
(ii) *C is a finite direct sum of quasi-cyclic groups $\mathbb{Z}(p^\infty)$ for equal or different primes p;*
(iii) *$D = \oplus_\kappa \mathbb{Q}$ is a κ-dimensional $\mathbb{Q}$-vector space, κ any cardinal.*

Proof. Suppose R is an artinian ring. In view of the minimum condition, the set of ideals $n\mathsf{R}$ ($n = 1, 2, \dots$) contains a minimal one, say, $m\mathsf{R} = \mathsf{L}$. This must satisfy $p\mathsf{L} = \mathsf{L}$ for each prime p, showing that L is divisible as a group. As such, it is, group-theoretically, a summand: $\mathsf{R}^+ = \mathsf{L}^+ \oplus B$ holds for some subgroup B. Then $\mathsf{L}^+ = m\mathsf{R}^+ = m\mathsf{L}^+ \oplus mB$, whence $mB = 0$ follows. We conclude that R^+ is a direct sum of a divisible group L^+ and a bounded group B. Write $\mathsf{L}^+ = C \oplus D$, where C is torsion divisible and D is torsion-free divisible. To complete the proof of necessity, we need only show that C is of finite rank. But this is an immediate consequence of Sect. 1 (F), since C is annihilated by L and B, so all subgroups in C are ideals, thus C must satisfy the minimum condition.

Conversely, assume A has the indicated structure. We then write $A = B_1 \oplus \cdots \oplus B_n \oplus C \oplus D$, where C, D are as in (ii)–(iii), while each B_i is a homogeneous Σ-cyclic group. To build a ring over A, we invoke Lemma 4.3 to get rings with finitely many ideals on each B_i, and define the zero-ring on C, and a field on D. This yields a commutative artinian ring on A (which has an identity exactly if $C = 0$). □

Note that the summand C is contained in the annihilator ideal of any artinian ring on A. From the proof it is clear that the one-sided ideals of a (left) artinian ring have the same kind of additive structure as the ring.

We can derive some not so obvious ring-theoretical consequences from the last theorem.

Theorem 4.5 (Szele–Fuchs [1], Szász [2]). *Every left artinian ring* R *is the ring-theoretical direct sum of a torsion-free left artinian ring* S*, and a finite number of left artinian p-rings* T_p*, belonging to different primes p:*

$$\mathsf{R} = \mathsf{S} \oplus \mathsf{T}_{p_1} \oplus \cdots \oplus \mathsf{T}_{p_m}.$$

Every torsion-free left artinian ring, in particular, S*, has a left identity.*

Proof. We start with the proof of the second claim. Since a torsion-free artinian ring S cannot be nilpotent, ring theory tells us that it must contain an idempotent $e \neq 0$ such that $e + \mathsf{N}$ is the identity element in the semi-simple artinian ring S/N (N denotes the nilpotent radical of S). By Theorem 4.4, the additive group of every left ideal of S is divisible, so for every $a \in \mathsf{S}$, the left ideal generated by $a' = a - ea$ is divisible. Therefore $\frac{1}{2}a' = na' + ba'$ for some $n \in \mathbb{Z}, b \in \mathsf{S}$, whence $(2n-1)a' = -2ba'$ and $a' = ca'$ with $c = -2(2n-1)^{-1}b \in \mathsf{S}$. We obtain $e(ca') = ea' = 0$, which implies $a' = (c - ec)a' = \cdots = (c - ec)^k a'$ for every $k > 1$. Since $c - ec \in \mathsf{N}$ and N is nilpotent, we get $(c - ec)^k = 0$ for some $k \in \mathbb{N}$, and so $a' = 0$. This means that $a = ea$ for every $a \in \mathsf{S}$, i.e. e is a left identity for S.

To prove the first claim, note that the torsion part T of a left artinian ring R is an ideal in R, and is the direct sum of its p-components T_p, where in view of Theorem 4.4 only a finite number of them can be different from 0. Thus $\mathsf{R}^+ = D \oplus \mathsf{T}^+$ where D is a torsion-free divisible subgroup. By what has been proved, the factor ring R/T has a left identity $e + \mathsf{T}$ where $e \in D$ may be assumed (so that $e\mathsf{T} = 0 = \mathsf{T}e$). Setting $\mathsf{S} = e\mathsf{R}$, we have the Peirce-decomposition $a = ea + (a - ea)$ where $ea \in \mathsf{S}, a - ea \in \mathsf{T}$; thus $\mathsf{S}^+ \cong D$ as both are complements of T. Now S is a subring—this is clear from $(\mathsf{S})^2 = (e\mathsf{R})^2 \leq e\mathsf{R} = \mathsf{S}$. The direct sum $\mathsf{R} = \mathsf{S} \oplus \mathsf{T}$ holds also in the ring-theoretical sense, since the torsion and torsion-free parts of an artinian ring are subrings annihilating each other. □

Thus every left artinian ring is the direct sum of a finite number of (indecomposable) left artinian rings, each of which is either torsion-free or a p-ring. An immediate consequence is that the structure theory of artinian rings can be reduced to those of torsion-free and p-rings. Since quasi-cyclic groups contribute next to nothing to the structure of artinian rings, among artinian p-rings only the bounded ones are of real interest.

Embedding in Artinian Ring with Identity We turn our attention to two basically ring-theoretical questions concerning artinian rings: *when can an artinian ring be embedded in an artinian ring with identity?* And, *when is an artinian ring noetherian?* Though these are actually questions that should be treated within ring theory, their discussion in this volume is justified by the unusual situation that not only their full solutions rely heavily upon group-theoretic reasoning on the additive structure of the ring, but even the actual conditions are in group-theoretic terms.

It is a well-known classical result that every ring without identity can be embedded in a ring with identity (extending it by $\mathbb{Z}$). However, in most cases such an embedding changes the ring structure to a great extent (it certainly will not be artinian). A different method helps to get some extended rings also artinian, however, it turns out that it is not possible to embed every artinian ring in an artinian ring with identity. The precise result on embeddability is as follows.

Theorem 4.6 (Szele–Fuchs [1]). *A left artinian ring can be embedded in a left artinian ring with* (*one- or two-sided*) *identity if and only if it does not contain any quasi-cyclic subgroup.*

Proof. To prove the 'only if' part, suppose R is artinian with an identity. From Theorem 4.4 we conclude that the quasi-cyclic subgroups of R are always contained in the annihilator of the ring. The annihilator ideal is evidently 0 whenever a left or right identity is present. Thus the stated condition is necessary.

Conversely, suppose R is artinian without quasi-cyclic subgroups. By Theorem 4.5, R is then the direct sum of a torsion-free divisible artinian ring S and a finite number of artinian rings T_p whose additive groups are direct sums of cyclic groups of the same prime power order. If S or T_p fails to contain an identity, then we can adjoin one as follows. The set $\mathsf{S}^* = (\mathbb{Q}, \mathsf{S})$ under the coordinate-wise addition and under the usual multiplication rule

$$(m, r)(n, s) = (mn, ms + nr + rs) \qquad (r, s \in \mathsf{S}, m, n \in \mathbb{Q})$$

is a ring that is artinian with identity $(1, 0)$. Similarly, for a p^k-bounded T_p, the set $(\mathbb{Z}/p^k\mathbb{Z}, \mathsf{T}_p)$ under similar operations is artinian with identity $(1 + p^k\mathbb{Z}, 0)$. The direct sum of these rings with identity provides the desired embedding. □

When Artinian Implies Noetherian Passing to the second question we raised above, the answer is provided by the following theorem.

Theorem 4.7 (Szele–Fuchs [1], Fuchs [8]). *A left artinian ring is left noetherian if and only if it contains no quasi-cyclic subgroup. Equivalently, if and only if its annihilator is finite.*

Proof. Each quasi-cyclic subgroup in an artinian ring R is contained in the annihilator of the ring, and every subgroup of the annihilator is an ideal of R. Since quasi-cyclic groups violate the maximum condition on subgroups, a left artinian ring that is left noetherian cannot contain any quasi-cyclic subgroup.

Next suppose R is left artinian and has no quasi-cyclic subgroup. In order to prove that it is left noetherian, we show that R as a left R-module is of finite length (it has a finite composition series). Let N denote the nilpotent radical of R; hence, $(\mathsf{N})^k = 0$ for a minimal $k \in \mathbb{N}$. If $\mathsf{N} = \mathsf{R}$, then R is nilpotent, so Proposition 4.1 implies the minimum condition on subgroups. From the absence of quasi-cyclic subgroups we conclude that R is finite in this case. Our attention now narrows to the case when

$$\mathsf{R} = \mathsf{N}^0 > \mathsf{N} > (\mathsf{N})^2 > \cdots > (\mathsf{N})^{k-1} > (\mathsf{N})^k = 0$$

is a properly descending chain of ideals. The factor modules $M_i = (\mathsf{N})^{i-1}/(\mathsf{N})^i$ ($i = 1, \dots, k$) are annihilated by N, they can therefore be viewed as left modules over the semi-simple artinian ring R/N which we shall denote by R_0. It is a classical result that a module over a semi-simple artinian ring decomposes into the direct sum of simple (unital) submodules and a submodule annihilated by the ring. This applies to the M_i where the submodules correspond to left ideals of R. Thus every M_i is a direct sum of a finite number of simple R_0-modules and a finite submodule. Hence it is obvious that R is of finite length, so it is definitely noetherian.

The annihilator ideal of an artinian-noetherian ring must satisfy both the minimum and the maximum conditions on subgroups, so it must be finite. Conversely, if the annihilator is finite in an artinian ring, then it cannot contain any quasi-cyclic subgroup (that are annihilators in artinian rings), so the ring is noetherian by the preceding arguments. □

An obvious corollary is a frequently cited theorem of C. Hopkins: *a left artinian ring with a left or right-sided identity is noetherian.*

★ **Notes.** In the theory of additive structures of rings, the most complete results are related to artinian rings. The results shed more light also on some of their ring theoretical properties. It is unusual that important ring properties can be derived from the additive structure.

The ring direct decomposition in Theorem 4.5 was extended to perfect rings by Huynh [1]. (A ring is left perfect if its principal left ideals satisfy the minimum condition.)

Exercises

(1) If the ring is not artinian, then its quasi-cyclic subgroups need not belong to the annihilator of the ring.

(2) (Szász)

(a) In a perfect ring, quasi-cyclic subgroups belong to the annihilator of the ring.

(b) A group A is the additive group of a perfect ring exactly if it is the direct sum of a divisible torsion-free group and a torsion group.

(3) If the torsion-free ring S in Theorem 4.5 is of finite rank, then it has a two-sided identity.
(4) A nilpotent artinian ring is embeddable in an artinian ring with identity if and only if it is finite.
(5) Let M be a (not necessarily unital) left module over a ring R such that the submodules of M satisfy the minimum condition.

    (a) The group of M^+ is the direct sum of a divisible and a bounded group.
    (b) If R is artinian, then M has the maximum condition on submodules exactly if it contains no quasi-cyclic subgroup.

(6) (Fuchs) Let R be a ring with restricted minimum condition on left ideals (i.e., minimum condition holds modulo every non-zero left ideal).

    (a) If R contains elements $\neq 0$ of finite order, then its additive structure is the same as in Theorem 4.4. [Hint: $\mathsf{R}[p]$.]
    (b) If R is torsion-free, then it is homogeneous of idempotent type. [Hint: ideals of the form $\cap_i p_i^{k_i}\mathsf{R}$.]

5 Additive Groups of Regular Rings

Our program for this section is to study the additive structures of (von Neumann) regular and generalized regular rings (associativity will be assumed, but not identity). Significant information can be obtained in these cases about the additive groups. To begin with, we recall the relevant definitions.

Structure of the Additive Group An element a of a ring R is said to be **regular** if $axa = a$ for some $x \in \mathsf{R}$, and m**-regular** for $m \in \mathbb{N}$ if a^m is a regular element. A ring R is **regular** or m**-regular** if each of its elements has this property. R is π**-regular** if every element of R is m-regular for some $m \in \mathbb{N}$ (depending on the element).

Theorem 5.1 (Fuchs [5]). *The additive group of a regular ring is the direct sum of a torsion-free divisible group and a reduced group C sandwiched between the direct sum and the direct product of its p-components:*

$$\oplus_p \mathsf{T}_p \leq C \leq \prod_p \mathsf{T}_p.$$

T_p^+ are elementary p-groups, and C/T^+ is torsion-free divisible.

There is a bijection between the regular subrings of $\prod_p \mathsf{T}_p / \oplus_p \mathsf{T}_p$ and the regular rings sandwiched between the direct sum and direct product of the p-components T_p.

Proof. Let R be a regular ring. If $p^k | a \in \mathsf{R}$, then $p^{2k} | axa = a$. Hence an element a in a regular ring is either not divisible by the prime p or divisible by every power of p. In the latter alternative, its order cannot be a power of p, since ax is then annihilated by a. Hence the p-component T_p of R must be an elementary p-group, and as such it is a summand: $\mathsf{R} = \mathsf{T}_p \oplus \mathsf{R}_p$. This is, moreover, a ring-direct sum, since R_p is p-divisible. This is so, since for every $b \in \mathsf{R}_p$, the element pb is divisible by every power of p, and as R_p contains no elements of order p, b itself must be divisible by p. It also follows that R_p is a ring, and that a torsion-free regular ring is divisible.

Clearly, if q is a prime $\neq p$, then the q-component T_q is contained in R_p, and a repeated application of the last direct sum decomposition yields the ring-direct sum decomposition

$$\mathsf{R} = \mathsf{T}_{p_1} \oplus \cdots \oplus \mathsf{T}_{p_k} \oplus \mathsf{R}_0, \tag{18.2}$$

where $p_1, \ldots, p_k$ are different primes, and multiplications by these primes are automorphisms of the complement $\mathsf{R}_0 = \mathsf{R}_{p_1} \cap \cdots \cap \mathsf{R}_{p_k}$.

Next consider $\mathsf{D} = \cap_p \mathsf{R}_p$ with p running over all primes; it is obviously a torsion-free ideal in R. From what has been said before, it is clear that D is divisible: each $d \in \mathsf{D}$ is divisible by p, and thus p-divisible. Write $\mathsf{R}^+ = \mathsf{D}^+ \oplus C$ where C denotes a reduced group containing $\mathsf{T} = \oplus_p \mathsf{T}_p$. For each p, (18.2) defines a projection ρ_p : $\mathsf{R} \to \mathsf{T}_p$ such that $\cap_p \operatorname{Ker} \rho_p = \cap_p \mathsf{R}_p = \mathsf{D}$. We conclude that R/D is isomorphic to a ring-theoretical subdirect sum of the rings $\operatorname{Im} \rho_p = \mathsf{T}_p$. Now R/T—as a torsion-free regular ring—is divisible, and so is its pure subgroup C/T. Thus we arrive at the first part of the claim.

From the above discussion it is clear that every regular ring R between the direct sum and the direct product defines a regular ring in R/T. Conversely, the complete inverse image of such a ring is a regular ring between the direct sum and the direct product, as a ring that contains a regular ring as an ideal with regular factor ring is itself a regular ring. $\square$

Example 5.2. If $C = \oplus_p \mathsf{T}_p$ or $C = \prod_p \mathsf{T}_p$, then there is a regular ring with additive group C: just choose for T_p any field of characteristic p. (See also the ring M *infra*.)

Generalized Regular Rings In sharp contrast to regular rings, the additive groups of m-regular ($m \geq 2$) and π-regular rings can be arbitrary, since zero-rings share all these properties. However, the situation changes drastically if the rings are supposed to have an identity, or are just (left) ideals in m-regular, etc. rings with identity. Here we will restrict ourselves to the discussion of π-regularity.

Theorem 5.3 (Fuchs–Rangaswamy [1]). *If* L *is a left ideal in a* π*-regular ring* R *with identity, then*

(i) *for every prime* p*, the* p*-component* L_p *of* L *is bounded, and* L *is a ring-direct sum* $\mathsf{L} = \mathsf{L}_p \oplus p^m\mathsf{L}$ *For some* $m \in \mathbb{N}$*;*
(ii) *for the torsion subgroup* T *of* L, L/T *is a divisible torsion-free* π*-regular ring.*

Proof. (i) The π-regularity of R implies $(p{\cdot}1)^m x (p{\cdot}1)^m = (p{\cdot}1)^m$, i.e. $p^{2m}x = p^m{\cdot}1$ holds for some $x \in \mathsf{R}$ and $m \in \mathbb{N}$. Hence, $p^{2m}xa = p^m a$ for every $a \in \mathsf{L}$, showing that every $p^m a$ is divisible by p^{2m} in the principal left ideal $\mathsf{R}a$. We derive that $p^{2m}\mathsf{L} = p^m\mathsf{L}$, and $p^m\mathsf{L}$ is p-divisible. Hence if $a \in \mathsf{L}_p$ is of order p^k, then $h_p(p^m a) \geq \omega$, thus for some $y \in \mathsf{R}$, $p^m a = p^k(ya) = y(p^k a) = 0$, whence $p^m\mathsf{L}_p = 0$ is obtained. Hence L_p is bounded, so $\mathsf{L}^+ = \mathsf{L}_p^+ \oplus C$ for some subgroup C of L. Notice that $p^m\mathsf{L} = p^m C$ is p-divisible, and division by p in C is unique. Consequently, $C = p^m C$, and we obtain that $\mathsf{L} = \mathsf{L}_p \oplus p^m\mathsf{L}$ is a ring-theoretical direct sum. This completes the proof of (i).

(ii) All that we have to do is to observe that L/T is an epic image of the p-divisible rings $p^m\mathsf{L}$, and π-regularity is inherited by surjective images. □

Note that the integer m in the preceding theorem depends only on prime p, and is the same for all left ideals of the π-regular ring.

Embedding in Regular Ring with Identity We wish to solve the problem of embedding of regular rings in rings with identity preserving regularity. We wish to show that there is always such an embedding. Oddly enough, this is another purely ring-theoretical question that is answered by taking full advantage of our knowledge of the additive groups.

To start with, we construct a commutative regular ring M with identity as follows. For every prime p, take the prime field $\mathsf{F}_p = \mathbb{Z}/p\mathbb{Z}$ of characteristic p; let ϵ_p denote its identity element. Evidently, $\mathsf{F} = \prod_p \mathsf{F}_p$ is a commutative regular ring with identity $\epsilon = (\ldots, \epsilon_p, \ldots)$. The quotient $\mathsf{F}/\oplus_p \mathsf{F}_p$ is a torsion-free divisible regular ring in which the coset of ϵ generates a pure subgroup $\mathsf{M}/\oplus_p \mathsf{F}_p$ isomorphic to $\mathbb{Q}$ as a ring as well. This M is a regular ring: it contains the regular ring $\oplus_p\mathsf{F}_p$ as an ideal modulo which it is regular.

It might be of interest to point out that the ring F is the completion of M, and its ring structure is completely determined by M.

Proposition 5.4 (Fuchs–Halperin [1]). *Every regular ring is a unital algebra over the commutative regular ring* M.

Proof. Let R be a regular ring. To define the action of $x \in \mathsf{M}$ on $a \in \mathsf{R}$, we write $x = (\ldots, x_p, \ldots)$ with $x_p \in \mathsf{F}_p$, and notice that, by construction, there is a rational number mn^{-1} $(m, n \in \mathbb{N})$ such that $nx_p \equiv m \bmod p$ for almost all primes p. Select a finite set $\{p_1, \ldots, p_k\}$ of primes which includes all prime divisors of m, n as well as the primes for which the last congruence fails to hold. With such a set of primes, we make a ring-decomposition

$$\mathsf{M} = \mathsf{F}_{p_1} \oplus \cdots \oplus \mathsf{F}_{p_k} \oplus \mathsf{M}_0, \tag{18.3}$$

where M_0 is an ideal of M such that multiplication by each p_i $(i = 1, \ldots, k)$ is an automorphism of M_0. Accordingly, we have

$$x = x_{p_1} + \cdots + x_{p_k} + x_0 \qquad (x_{p_i} \in \mathsf{F}_{p_i}, x_0 \in \mathsf{M}_0).$$

Now R has a decomposition $\mathsf{R} = \mathsf{T}_{p_1} \oplus \cdots \oplus \mathsf{T}_{p_k} \oplus \mathsf{R}_0$, corresponding to (18.3). If we write accordingly $a = a_{p_1} + \cdots + a_{p_k} + a_0$ with $a_{p_i} \in \mathsf{T}_{p_i}, a_0 \in \mathsf{R}_0$, then we can define

$$xa = x_{p_1}a_{p_1} + \cdots + x_{p_k}a_{p_k} + mn^{-1}a_0. \tag{18.4}$$

Take into account that T_p is an F_p-vector space, so that $x_p a_p$ makes sense for every p, and so does $mn^{-1}a_0$ by virtue of the choice of the primes (multiplication by n is an automorphism on R_0). We still have to convince ourselves that xa does not change if we select a larger set of primes or use a different form for mn^{-1}. But this follows right away from our selection of primes which guarantees that each x_p with $p \notin \{p_1, \ldots, p_k\}$ acts on F_p as a multiplication by mn^{-1}. We leave it to the reader to check the algebra postulates to conclude that R is an M-algebra, indeed. $\square$

By taking full advantage of the last result, it becomes easy to verify the following theorem.

Theorem 5.5 (Fuchs–Halperin [1]). *Every regular ring can be embedded as an ideal in a regular ring with identity.*

Proof. Given a regular ring R (without identity), we form the set of pairs $(x, r) \in \mathsf{M} \times \mathsf{R}$, and define operations as usual to make it into a ring R^*: addition: $(x, r) + (y, s) = (x + y, r + s)$, and multiplication:

$$(x, r) \cdot (y, s) = (xy, xs + yr + rs) \quad \text{with} \quad x, y \in \mathsf{M}, r, s \in \mathsf{R}.$$

Then $(\epsilon, 0)$ is the identity in R^*, and $r \mapsto (0, r)$ $(r \in \mathsf{R})$ embeds R in R^* as an ideal. As both the ideal and the factor ring are regular, so is R^*. $\square$

★ **Notes.** We refer to additional literature to the embedding problem by various authors, see, e.g., Jackett [1] where more information is given about the embedding between the direct sum and the direct product. Proposition 5.4 and Theorem 5.5 are excellent examples to demonstrate the relevance of the additive group in rings. No method of proof is known to me that avoids the additive structure.

Results on π-regular rings, similar to those in Proposition 5.4 and Theorem 5.5, were proved by Fuchs–Rangaswamy [1], provided the additive group satisfies a necessary condition. The methods were similar. For additional results, consult Feigelstock [Fe], where more additive structures are examined. Feigelstock [2] deals with the additive groups of self-injective rings.

Exercises

(1) A group is the additive group of a boolean ring (every element is idempotent) if and only if it is an elementary 2-group.

(2) A group is isomorphic to the additive group of a noetherian regular ring exactly if it is a finite direct sum of a torsion-free divisible group and elementary p-groups.
(3) Show that the torsion subgroup of a Baer ring is an elementary group (for definition, see Sect. 4 in Chapter 16).
(4) The ring M above is the best choice in the following sense. If M' is any commutative regular ring R with identity such that every regular ring is a unital M'-algebra, then there is an identity-preserving surjective homomorphism $\phi : \mathsf{M}' \to \mathsf{M}$ with $x'a = \phi(x')a$ for all $x' \in \mathsf{M}', a \in \mathsf{R}$.
(5) Given $n \in \mathbb{N}$, construct a regular ring between $\oplus_p \mathsf{F}_p$ and $\prod_p \mathsf{F}_p$ whose torsion-free rank is $n \in \mathbb{N}$ (notation as before Proposition 5.4). [Hint: vectors with solutions of the congruence $x^n \equiv 1 \bmod p$ as coordinates.]
(6) A torsion-free divisible m-regular ring without identity is an ideal in an m-regular ring with identity.

6 *E*-Rings

Every ring with identity is a subring of the endomorphism ring of its additive group. In [AG], one of the problems asked for the characterization of rings R for which $\operatorname{End} \mathsf{R}^+ \cong \mathsf{R}$. In his dissertation, Schultz [1] made a good progress towards the solution of the problem.

Properties of *E*-Ring Following Schultz, we say that R is an ***E*-ring** if every endomorphism of R^+ is a left multiplication $\dot{r}$ by an element $r \in \mathsf{R}$. Accordingly, a group A is called an ***E*-group** if it supports an E-ring; this ring is uniquely determined by A as it is isomorphic to $\operatorname{End} A$.

We observe right away that every E-ring has an identity; furthermore, for an E-ring R, the correspondence $r \mapsto \dot{r}$ is a ring isomorphism $\mathsf{R} \to \operatorname{End} \mathsf{R}^+$, and $\eta \mapsto \eta(1)$ ($\eta \in \operatorname{End} \mathsf{R}^+$) is its inverse. Actually, the identity has a distinguished role: R is an E-ring if and only if the only endomorphism of R^+ that carries $1 \in \mathsf{R}^+$ to 0 is the zero endomorphism, see Theorem 6.3(ii).

Example 6.1. (a) The following are easy examples of E-groups: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Q} \oplus \mathbb{Z}/n\mathbb{Z}$ for every $n \in \mathbb{N}$. Also, the rational groups of idempotent types are E-groups.
(b) Subrings of a finite algebraic extension of $\mathbb{Q}$ (containing 1) are E-rings. (Proof is omitted.)
(c) Examples of E-rings include all subrings of $\tilde{\mathbb{Z}}$ with 1, in particular, J_p for any prime p.

Example 6.2 (Douglas–Farahat). We can define a finite rank torsion-free E-ring on the group $A = \langle p_1^{-\infty} a_1, \dots, p_n^{-\infty} a_n, 2^{-1}(a_1 + \cdots + a_n) \rangle$ (with different odd primes p_i) for any $n \geq 2$, by setting $a_i^2 = 2a_i$ and $a_i a_j = 0$ for all $i \neq j$. The ring identity is the element $2^{-1}(a_1 + \cdots + a_n)$.

As a starting point, we prove the following lemma that gives several necessary and sufficient conditions on a ring to be an E-ring.

Theorem 6.3 (Bowshell–Schultz [1]). *These are equivalent for a ring* R *with* 1*:*

(i) R *is an E-ring;*
(ii) *If* $\eta \in \mathrm{End}(\mathsf{R}^+)$ *satisfies* $\eta(1) = 0$*, then* $\eta = 0$*;*
(iii) $\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}, \mathsf{R}) = \mathrm{Hom}_{\mathsf{R}}(\mathsf{R}, \mathsf{R})$ *where* R *is viewed as a right module over itself;*
(iv) $\mathrm{End}(\mathsf{R}^+)$ *is commutative.*

Proof. (i) $\Rightarrow$ (ii) By (i), $\eta \in \mathrm{End}(\mathsf{R}^+)$ is multiplication by some $s \in \mathsf{R}$. Thus $0 = \eta(1) = s \cdot 1$ in R^+, also in R. Thus $s = 0$, which means $\eta = 0$.

(ii) $\Rightarrow$ (iii) Let $\eta \in \mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}, \mathsf{R})$. If $\eta(1) = s \in \mathsf{R}$, then $(\eta - s)(1) = 0$, thus (ii) implies $\eta = \dot{s}$. Hence $\eta(rt) = srt = \eta(r)t$ for all $r, t \in \mathsf{R}$.

(iii) $\Rightarrow$ (iv) First we prove that R is commutative: for $r, s \in \mathsf{R}$ we have by (iii) $rs = \dot{s}(r) = \dot{s}(1 \cdot r) = \dot{s}(1)r = sr$ (dots denote now right multiplication). To show that $\eta, \xi \in \mathrm{End}(\mathsf{R}^+)$ commute, note that $\eta\xi(r) = \eta(\xi(1)r) = \eta(1)\xi(1)r$ which is by the commutativity of R equal to $\xi\eta(r)$.

(iv) $\Rightarrow$ (i) For $r \in \mathsf{R}, \eta \in \mathrm{End}(\mathsf{R}^+)$, we have $\eta(r) = \eta(\dot{r}(1)) = (\eta\dot{r})(1))$ which is—by commutativity—equal to $\dot{r}\eta(1) = \eta(1)r$. Hence η is left multiplication by $\eta(1) \in \mathsf{R}$, and R is an E-ring. $\square$

We continue with listing a few pertinent properties of E-rings.

(A) *Let* R *be an E-ring* (*so commutative*), *and* $A = \mathsf{R}^+$. *There is a bijection between the elements of A and rings* (*with or without units*) *on A.* For $0 \neq a \in A$, define a new multiplication $r \circ s = ras$ (last product computed in R). This gives rise to a ring structure on A with $1 \circ 1 = a$, so all these rings are different. To see that there is no other ring structure S on A, suppose $\star$ is a ring operation in S. Then for a fixed $r \in \mathsf{R}$, $s \mapsto r \star s$ is an endomorphism of A, thus it is multiplication by some $\bar{r} \in \mathsf{R}$. Also, the correspondence $r \mapsto \bar{r}$ is an endomorphism of A, so $\bar{r} = ra$ for some $a \in \mathsf{R}$. We conclude that $r \star s = \bar{r}s = ras$, as claimed.

(B) *If A is an E-group, and if* $\mu \in \mathrm{Mult}\,A$ *defines an E-ring on A, then* $\nu_a(x, y) = \mu(a, \mu(x, y))$ *for all* $a \in A$ *yields all the elements of* $\mathrm{Mult}\,A$. This follows from (A).

Observe that (B) shows that if A is an E-group, then every element of $\mathrm{Mult}\,A$ represents an associative and commutative ring structure on A.

(C) *For an E-group A, the Fried ideal is all of* $\mathrm{End}\,A$. Thus a subgroup in an E-group is an absolute ideal if and only if it is fully invariant.

(D) *Endomorphic images of an E-group are fully invariant subgroups.* This is a simple consequence of the commutativity of the endomorphism ring.

(E) *An endomorphic image of the additive group of an E-ring* R *is the additive group of a principal ideal of* R*, it is itself an E-group.* Every endomorphism is a multiplication by a ring element $a \in \mathsf{R}$, so it is of the form $r \mapsto ra$. If we define multiplication on $\mathsf{R}a$ via $ra \circ sa = ras$ $(= rsa)$ $(r, s \in \mathsf{R})$, then we get a ring (with identity a) on $\mathsf{R}a$.

(F) *Every endomorphism of the additive group of a principal ideal* $\mathsf{R}a$ *extends to all of the E-group* R^+. By (E), $\mathsf{R}a$ is an *E*-ring, so each endomorphism of its additive group is a multiplication by some $ra \in \mathsf{R}a$. The multiplication by ra in R is a desired extension.

(G) *Every direct decomposition of the additive group of an E-ring* R *is also a ring-direct decomposition.* By (D), the summands are fully invariant. Products of elements from different summands must be 0 by the full invariance.

(H) *The direct sum* $\mathsf{R} \oplus \mathsf{S}$ *of two rings is an E-ring exactly if both* R *and* S *are E-rings and* $\mathrm{Hom}(\mathsf{R}^+, \mathsf{S}^+) = 0 = \mathrm{Hom}(\mathsf{S}^+, \mathsf{R}^+)$. One way, the implication follows from (D) and (F). For the reverse implication observe that if R and S are as stated, then the endomorphism ring of the direct sum is just the direct sum of the endomorphism rings.

(J) *The additive group of an E-ring cannot contain any pure subgroups of the form* $\mathbb{Q} \oplus \mathbb{Q}$, $\mathbb{Z}(p^m) \oplus \mathbb{Z}(p^n)$ *with* $m, n \in \mathbb{N}$ *or* $J_p \oplus J_p$ *for any prime p.* These subgroups would then be ring-direct summands and *E*-rings (see (D), (G)), but their endomorphism rings are not commutative.

(K) *If* S *is a ring with identity such that* $\mathsf{S}^+ \sim \mathsf{R}^+$ *for a finite rank torsion-free E-ring* R*, then* S *is likewise an E-ring.* Since $\mathbb{Q} \otimes \mathrm{End}\, \mathsf{S}^+ \cong \mathbb{Q} \otimes \mathrm{End}\, \mathsf{R}^+$, both endomorphism rings are simultaneously commutative Theorem 6.3.

Special Cases In a few cases, *E*-rings with particular kinds of additive groups can be completely classified.

Lemma 6.4 (Schultz [1]). *Suppose* R *is an E-ring.*

(i) *If* R *is torsion, then* $\mathsf{R} \cong \mathbb{Z}/n\mathbb{Z}$ *for some* $n \in \mathbb{N}$.
(ii) *If* R *is not reduced, then* $\mathsf{R} \cong \mathbb{Q} \oplus \mathbb{Z}/n\mathbb{Z}$ *for some* $n \in \mathbb{N}$.
(iii) *If* R^+ *is finitely generated, then* $\mathsf{R} \cong \mathbb{Z}$ *or* $\cong \mathbb{Z}/n\mathbb{Z}$ *with* $n \in \mathbb{N}$.

Proof. (i)–(ii) If R is an *E*-ring, then by (H) all of its summands are *E*-rings. This rules out quasi-cyclic groups $\mathbb{Z}(p^\infty)$, and by (J) also more than one copy of $\mathbb{Q}$. Again by (H), if $\mathbb{Q}$ is a summand, then its complement must be a reduced torsion group. Such a torsion group cannot contain two cyclic p-groups as summands for the same p (see (J)), and so it must be a subgroup of $\mathbb{Q}/\mathbb{Z}$. The existence of identity shows that it must be cyclic. R^+ in (i) (in (ii)) has commutative endomorphism ring, so R is an *E*-ring.

(iii) In case R^+ is finitely generated, we argue that no direct decomposition may contain two isomorphic copies of $\mathbb{Z}$, or more than one cyclic p-group for the same prime. The possibility $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ is ruled out by calculating its endomorphism ring, so R^+ must be as stated. □

We mention the following characterization of torsion-free *E*-rings that are of finite rank.

Theorem 6.5 (Bowshell–Schultz [1]). *A finite rank torsion-free E-ring* R *is quasi-isomorphic to a ring-direct sum*

$$\mathsf{R} \sim \mathsf{R}_1 \oplus \cdots \oplus \mathsf{R}_n$$

of E-rings R_i *such that* $\mathbb{Q}\mathsf{R}_i$ *is an algebraic number field, and, for all* $i \neq j$, $\mathrm{Hom}(\mathsf{R}_i, \mathsf{R}_j) = 0$.

Proof. Sufficiency is clear in view of Example 6.1 and (H).

From Theorem 2.7 it follows that the direct decomposition of the $\mathbb{Q}$-algebra $\mathsf{R}^* = \mathsf{R} \otimes \mathbb{Q} = \mathsf{S}^* \oplus \mathsf{N}^*$ yields a subring $\mathsf{A} = \mathsf{S} \oplus \mathsf{N}$ of finite index in R, where S is a subring such that S^* is a semi-simple $\mathbb{Q}$-algebra, and N is the nil-radical of A. If R is an E-ring, then by (K), so is A, and then by (G), so are S and N. As N cannot have identity, it must be 0. The commutativity of R implies that S^* is a ring-direct sum of finite rank fields F_i $(i = 1, \dots, n)$. Such fields ought to be algebraic number fields, so setting $\mathsf{R}_i = \mathsf{F}_i \cap \mathsf{R}$, the proof is complete. □

Satisfactory result exists also for cotorsion E-rings. Note that the next theorem implies that the cardinality of such a ring cannot exceed $2^{\aleph_0}$.

Theorem 6.6 (Bowshell–Schultz [1]). *A reduced cotorsion E-ring* R *is a direct sum of two cotorsion E-rings:*

(a) $\mathsf{R}_1 = \prod_{p \in X} \mathbb{Z}(p^{k_p})$ *with* $k_p \in \mathbb{N}$, *and*
(b) $\mathsf{R}_2 = \prod_{p \in Y} J_p$

where X, Y *are disjoint sets of primes. Conversely, a direct sum of rings in* (a) *and* (b) *with disjoint* X, Y *is an E-ring.*

Proof. Assume R is a cotorsion E-ring. By Theorem 7.3 in Chapter 9 and (G), R is the direct sum of an adjusted cotorsion ring R_1, and a torsion-free ring R_2 on an algebraically compact group. If T denotes the torsion subgroup of R_1, then by (J) the p-components of T are cyclic, and therefore its cotorsion hull is as stated in (a). Again by (J), R_2 cannot have any summand of the form $J_p \oplus J_p$ for any prime p, whence (b) is clear.

For the converse, it is obvious that the rings in (a) and (b) are E-rings.

Since an E-ring cannot contain a direct sum $J_p \oplus \mathbb{Z}(p^k)$ as a summand (it has non-commutative endomorphism ring), the index sets X, Y must be disjoint. On the other hand, if they are disjoint, the direct sum $\mathsf{R}_1 \oplus \mathsf{R}_2$ is obviously an E-ring. □

Example 6.7 (Bowshell–Schultz [1]). Let R_i $(i \in I)$ be a set of E-rings such that $\mathrm{Hom}(\prod_{j \neq i \in I} \mathsf{R}_i, \mathsf{R}_j) = 0$ for each $j \in I$. In addition, suppose that there exist a prime p, and surjective ring maps $\phi_i : \mathsf{R}_i \to \mathbb{Z}/p\mathbb{Z}$ for each i. Then the subgroup of $\prod_{i \in I} \mathsf{R}_i$ consisting of all $(\dots, r_i, \dots)$ $(r_i \in \mathsf{R}_i)$ with $\phi_i(r_i) = \phi_j(r_j)$ for all $i, j \in I$ is an E-ring R. This ring is indecomposable if and only if all the R_i are indecomposable. Sufficiency follows from the easily established fact that R contains no idempotents except for 0, 1.

★ **Notes.** Pierce–Vinsonhaler [1] has an important collection of results on finite rank torsion-free E-rings. See Vinsonhaler [2] for an informative survey of E-rings and E-modules.

The question of existence of arbitrarily large E-rings was for a while a major problem; it has been answered by Dugas–Mader–Vinsonhaler [1] in the affirmative by using Shelah's Black Box. Dugas [4] proves that there exist arbitrarily large E-modules. Pierce [4] has numerous results on E-modules. A theorem by Faticoni [1] shows that being an E-ring is not as restrictive as one might think: every countable reduced torsion-free commutative ring is contained as a pure subring in an E-ring.

Hausen–Johnson [2] define the *E-ring core* of a ring R by a transfinite process as follows. Let $C_0 = \mathsf{R}$. If C_ρ has been defined for an ordinal ρ, then let $C_{\rho+1}$ be the ring of those $r \in C_\rho$ for which multiplication by r is in the center of $\mathrm{End}(C_\rho)$. For a limit ordinal σ, set $C_\sigma = \bigcap_{\rho<\sigma} C_\rho$. There is an ordinal λ with $C_\lambda = C_{\lambda+1}$, and this subring (evidently an E-ring) is called the **E-ring core** of R.

Absolute E-rings (i.e., remain E-rings in any generic extension of the universe) were investigated by Göbel–Herden–Shelah [1]. Herden–Shelah [1] proved that absolute E-rings of cardinality κ exist if and only if κ is less than the first Erdős cardinal.

Generalized E-rings were studied by Feigelstock–Hausen–Raphael [1]. A slight generalization is when the isomorphy $A \cong (\mathrm{End}\, A)^+$ is not required to be induced by multiplication. Further generalization is when only the existence of an epimorphism $A \to (\mathrm{End}\, A)^+$ is assumed. It turns out that this is not a proper generalization in case A is torsion-free of finite rank: A is then an E-ring. Göbel–Shelah–Strüngmann [1] construct a group A, that is not an E-group, but satisfies $A \cong (\mathrm{End}\, A)^+$ with a non-commutative endomorphism ring—quite an interesting phenomenon.

Dugas–Feigelstock [1] define **A-ring** as a ring R with 1 having the property that every automorphism of its additive group is left multiplication by a unit of R. They show that A-rings retain some of the features of E-rings, e.g. their unit groups $U(\mathsf{R})$ are abelian.

Exercises

(1) The torsion subgroup of an E-ring must be separable.
(2) Cotorsion E-groups are algebraically compact.
(3) (Schultz) Let A be a completely decomposable torsion-free E-group of finite rank. Then the rank 1 summands are of incomparable idempotent types.
(4) Show that R_1 in Lemma 6.4 contains pure subrings that are not cotorsion, but are still E-rings.
(5) (Goeters) Let R be a torsion-free ring with $\mathrm{rk}^p\, \mathsf{R} = 1$. If it is p-reduced, then it is an E-ring. [Hint: R^+ embeds as a p-pure subgroup in J_p, $\eta \in \mathrm{End}\, \mathsf{R}^+$ is multiplication by $\pi \in J_p$.]
(6) (Faticoni) Let E_p be a collection of p-local E-rings for different primes p. Then the ring $\mathsf{E} = \prod_p \mathsf{E}_p$ is again an E-ring.

7 Groups of Units in Commutative Rings

In this section, all rings are associative and commutative with identity.

Unit Groups In every ring R with identity 1, the elements which have a (multiplicative) inverse with respect to 1 form a group under multiplication, called the **unit group** of R. It will be denoted by $U(\mathsf{R})$. Thus 1 is the identity of the group of units, and the symbol $\times$ will stand for the direct product. We will consider unit groups sometimes as a multiplicative, some other time as being converted into an additive group. From the context it will always be clear which operation is meant.

The correspondence $F : \mathsf{R} \mapsto U(\mathsf{R})$ is functorial. In fact, if $\phi : \mathsf{R} \to \mathsf{S}$ is a ring homomorphism preserving identities, then ϕ induces a group homomorphism $\bar{\phi} : U(\mathsf{R}) \to U(\mathsf{S})$: it is just the restriction map. It is remarkable that F has a left adjoint functor G which is defined by letting the group ring over the integers correspond to a group, i.e. $G\colon A \mapsto \mathbb{Z}[A]$. Indeed, for each group A and each ring R, there is a natural bijection between the set of homomorphisms $A \to U(\mathsf{R})$ and the set of identity preserving ring maps $\mathbb{Z}[A] \to \mathsf{R}$.

Example 7.1. (a) The units in $\mathbb{Z}/p^k\mathbb{Z}$, for a prime power p^k, are the cosets coprime to p. Number theory tells us that this is a cyclic group of order $\varphi(p^k) = p^k - p^{k-1}$ generated by any primitive root mod p^k (φ is the Euler totient function).

(b) The units of the ring $\mathbb{Z}/n\mathbb{Z}$ are represented by the residue classes that are coprime to n. The unit group is a direct product of cyclic groups, its order is $\varphi(n)$. It is cyclic if and only if $n = p^k, 2p^k$ for odd primes p, or else $n = 2, 4$.

Example 7.2. The unit group $U(J_p)$ is isomorphic to $\mathbb{Z}(p-1) \oplus J_p$ for every odd prime p. This will follow from Theorem 8.6.

Example 7.3. If R is the ring of integers in a finite algebraic extension of $\mathbb{Q}$, then by Dirichlet's theorem on units, $U(\mathsf{R})$ is the direct product of a finite cyclic group of even order and a finitely generated free group.

Before listing some informative facts about unit groups, we repeat: all rings to be considered are commutative with 1, even if this is not stated explicitly.

(A) The unit group of a cartesian product of rings is the cartesian product of the unit groups.

(B) The polynomial ring $\mathsf{S} = \mathsf{R}[x_i]_{i\in I}$ over a domain R with any number of indeterminates x_i satisfies $U(\mathsf{S}) \cong U(\mathsf{R})$. In fact, only a constant polynomial may be a unit.

(C) The situation for formal power series rings is completely different. Let $\mathsf{S} = \mathsf{R}[[x]]$ be the power series ring over a domain R in a single indeterminate x. Then we have a direct product $U(\mathsf{S}) \cong U(\mathsf{R}) \times E$ where E is the multiplicative group of all power series with constant term 1. This follows from the fact that every unit of S is the product of a unit of R and an element of E.

(D) We now consider the localization R_S of R at a semigroup S; suppose S contains only non-zero divisors, and $1 \in S$. Then $U(\mathsf{R}_S) \cong U(\mathsf{R})G$ where G denotes the group of quotients of S.

(E) Let M be an R-module. The 'idealization' $\mathsf{R}(M)$ is a ring with additive group $\mathsf{R} \oplus M$ with the following rules of operations:

$$(r,a)+(s,b) = (r+s, a+b), \ (r,a)(s,b) = (rs, rb+sa) \quad (r,s \in \mathsf{R}, a,b \in M).$$

Then $(1,0)$ is the identity of $\mathsf{R}(M)$, and (r,a) is a unit if and only if $r \in U(\mathsf{R})$. As a result, we have $U(\mathsf{R}(M)) \cong U(\mathsf{R}) \times M$.

An immediate consequence of (E) (with $\mathsf{R} = \mathbb{Z}$) is the following theorem.

Theorem 7.4. *For every group A, there exists a ring with unit group isomorphic to the additive group $\mathbb{Z}(2) \oplus A$.* □

(F) Of special interest is the role the Jacobson radical J of R plays in connection with $U(\mathsf{R})$. For all $r \in \mathsf{J}$, $1 + r \in U(\mathsf{R})$, and if J is regarded as a group under the 'circle' operation $r \circ s = r + s + rs$, then the correspondence $r \mapsto 1 + r$ is an isomorphism of $(\mathsf{J}, \circ)$ with a subgroup of $U(\mathsf{R})$.

(G) The element -1 is always a unit of order 2, except when the additive group of the ring is a 2-group. It is very seldom that $U(\mathsf{R})$ has odd order (see Ditor [1]). Indeed, in this case R^+ is a 2-group, and the elements $1 + x$ with $x \in \mathsf{J}$ form a subgroup whose order is the order of the Jacobson radical J; hence $\mathsf{J} = 0$. This means that R is a semi-simple artinian ring on an elementary 2-group. Therefore, R is a direct sum of a finite number of Galois-fields, and $U(\mathsf{R})$ is a finite direct product of cyclic groups of orders $2^n - 1$ for various $n \geq 1$.

Groups that Can Be Unit Groups We shall now proceed to the problem of describing those groups that can occur as unit groups in a commutative ring, and give a brief account of the main results available in the literature.

It is almost trivial to prove (we do not need the not-so-obvious Theorem 7.4) that every group A (now we have to use the multiplicative notation) embeds as a subgroup in the unit group of some ring: just form the group ring $\mathbb{Z}[A]$ of A over the integers (or over any ring with 1). The elements $a \in A$ are evidently units in this ring. Notably, the canonical embedding $\chi : A \to U(\mathbb{Z}[A])$ (where $\chi(a) = a$ for all $a \in A$) has the 'universal' property: if R is any ring with 1, and if $\alpha : A \to U(\mathsf{R})$ is any homomorphism, then there exists an identity preserving ring homomorphism $\bar{\alpha} : \mathbb{Z}[A] \to \mathsf{R}$ such that $\alpha = \bar{\alpha}\chi$. For the proof, it suffices to point out that there is one and only one way of extending α to $\mathbb{Z}[A]$, *viz.* $\bar{\alpha} : \sum n_i a_i \mapsto \sum n_i \alpha(a_i)$ $(n_i \in \mathbb{Z})$.

It is a delicate, difficult unsolved problem to characterize the groups that are the unit groups in rings. In the domain case, more can be said.

Lemma 7.5. *The torsion part of the unit group of an integral domain* D *is locally cyclic. Conversely, every locally cyclic torsion group with an element of order* 2 *is the torsion subgroup of the unit group of some domain.*

Proof. Let u be a unit of order $n \in \mathbb{N}$ in D. It is a root of an equation $x^n = 1$ of degree n. Just as in a field, also in a domain an equation of degree n can have at most n roots, whence it follows that the p-socle of $U(\mathsf{D})$ contains at most p elements. This forces a torsion group to be locally cyclic, i.e. isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$.

Conversely, let T be a torsion locally cyclic group with an element of order 2. If T is infinite, then it is the union of cyclic groups of orders $2\,m_i$ (with increasing m_i) which we can represent as the union of complex roots of unity of degrees $2\,m_i$. Correspondingly, we consider the tower of splitting fields F_i of polynomials $x^{2\,m_i} - 1$, and define D as the set union of algebraic integers in these fields. Then D will contain exactly the desired roots of unity, so it is a domain as desired. □

We have more specific information on unit groups of rings of algebraic integers in finite algebraic extensions K of $\mathbb{Q}$. By Dirichlet's theorem, it is a direct product of a finite cyclic group of even order and $r_1 + r_2 - 1$ infinite cyclic groups. Here r_1 denotes the number of real, and r_2 the number of pairs of conjugate complex roots of the defining polynomial of the field K.

Example 7.6. (a) The unit group of the ring of Gaussian integers $\mathbb{Z} + i\mathbb{Z}$ is cyclic of order 4: it consists of the complex numbers $\pm 1, \pm i$.

(b) The unit group of the ring of Eisenstein integers $\mathbb{Z} + \rho\mathbb{Z}$ (where $\rho^2 + \rho + 1 = 0$) is isomorphic to $\mathbb{Z}(6) \oplus \mathbb{Z}$.

We shall require a lemma, generalizing a result by Cohn [2].

Lemma 7.7. *Let U be the unit group of an integral domain, and assume A is a multiplicative group containing U such that A/U is torsion-free. Then A, too, is the unit group of some domain.*

Proof. Let R be a domain with $U(\mathsf{R}) = U$. We view A as an extension of U by A/U, and select a representative $a_x \in A$ in each coset x modulo U. We obtain a factor set $u_{x,y} \in U$ as defined by

$$a_x a_y = u_{x,y} a_{xy} \quad \text{for all } x, y \in A/U. \tag{18.5}$$

Define S as an algebra over R with the set $\{a_x\}_{x \in A/U}$ as basis such that the basis elements multiply according to the rule (18.5). Since the $u_{x,y}$ are coming from A, they satisfy the obligatory associativity conditions to guarantee that S will become an R-algebra. It is manifest that the multiples ua_x of the basis elements a_x (where $u \in U$) form a subgroup of $U(\mathsf{S})$ canonically isomorphic to A.

It still remains to show that S is a domain, and $U(\mathsf{S})$ contains no elements other than those of the form ua_x. Every torsion-free abelian group admits a linear order compatible with the group operation. Choose an arbitrary, but fixed linear order on A/U, and write a non-zero element $\xi \in \mathsf{S}$ in the form $\xi = \sum_{i=1}^m r_i a_{x_i}$ with $0 \neq r_i \in \mathsf{R}$, and a_{x_i} in the above set of representatives such that $x_1 < \cdots < x_m$. If $\zeta = \sum_{j=1}^n s_j a_{y_j}$ with $0 \neq s_j \in \mathsf{R}$ and $y_1 < \cdots < y_n$ is another element of S, then in the product $\xi\zeta$, $a_{x_1 y_1}$ will be the smallest basis element with coefficient $r_1 s_1 u_{x_1,y_1} \neq 0$. Thus S has no zero-divisors. Since the largest basis element a_{x_m,y_n} in the product $\xi\zeta$ will also have a non-vanishing coefficient, it follows that $\xi\zeta = 1$ only if $m = 1 = n$, and in addition, r_1, s_1 are units in R. □

Group Rings For group rings, the main difficulty lies in the emergence of the so-called non-trivial units: units in $\mathsf{R}[A]$ that are not products of units in R with elements of A. There is an extensive literature on the unit groups of groups rings of a group over a commutative ring. In this volume, we do not discuss group rings.

Example 7.8. Let $A = \langle a \rangle$ be cyclic of order 5, and let $\mathbb{Q}^{(3)}$ denote the ring of rationals whose denominators are powers of 3. Then $\mathbb{Q}^{(3)}[A]$ has non-trivial units. To prove this, consider $\gamma = 1 + a + a^2 + a^3 + a^4 \in \mathbb{Q}^{(3)}[A]$. Then $a\gamma = \gamma$, whence $\gamma^2 = 5\gamma$ follows. An easy calculation shows that $1 - 2\gamma$ is a unit with inverse $1 - \frac{2\gamma}{9}$.

★ **Notes.** Rings with cyclic groups of units were described by Gilmer [1] for finite rings, and by Pearson–Schneider [1] for infinite rings.

It was G. Higman [Proc. London Math. Soc. **46**, 231–248 (1938/39)] who started a systematic study of unit groups in group rings; he investigated the units of group rings over finite algebraic extensions of $\mathbb{Z}$. In general, the study of the unit groups in (commutative) group rings is an interesting area of research. Several significant results are available. The main contributors include S. Berman, P. Danchev, G. Karpilovsky, W. May, T. Mollov, N.A. Nachev, W. Ullery. From the point of view of abelian group theory, the results by Danchev are especially interesting: how the structure of the unit group of the group ring over a p-group depends on the structure of the p-group (e.g., if the group is totally projective).

To characterize the unit groups of commutative rings seems to be quite a difficult problem. Recently several publications dealt with very special cases, even these turned out to be far from easy. The relation between the unit group and the additive structure of the Jacobson radical is certainly one of the crucial points to be investigated.

Exercises

(1) If R is an integral domain and A is torsion-free, then $U(\mathsf{R}[A]) \cong U(\mathsf{R}) \times A$.
(2) If U, V are unit groups of some rings, then so is their direct product $U \times V$.
(3) The canonical embedding $\chi_A : A \to \mathbb{Z}[A]$ is functorial: every group homomorphism $\alpha : A \to B$ induces a ring homomorphism $\bar{\alpha} : \mathbb{Z}[A] \to \mathbb{Z}[B]$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\chi_A} & \mathbb{Z}[A] \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\bar{\alpha}} \\ B & \xrightarrow{\chi_B} & \mathbb{Z}[B] \end{array}$$

(4) The list of prime numbers that can be orders of unit groups in rings is: 2 and $p = 2^n - 1$.
(5) Let R be a domain of characteristic 0 whose unit group is torsion. Then $U(\mathsf{R}) \cong \mathbb{Z}(2)$ or $\mathbb{Z}(4)$.
(6) Find the unit group in the following regular rings: F and M (for the notation, see Sect. 5).
(7) What is the unit group of the bounded artinian ring in Lemma 4.3?

8 Multiplicative Groups of Fields

Our study of the multiplicative groups of units continues with the most important special case when the ring is a field. Intuitively, it is expected that the characteristic of the field will play a decisive role in the structure of its multiplicative group.

If K denotes a field, then the symbol $\mathsf{K}^\times$ is used for its multiplicative group. For obvious reason, we have to adhere to the multiplicative notation in $\mathsf{K}^\times$.

Torsion Groups in Fields It is easy to describe the torsion subgroups of fields.

Proposition 8.1. *A torsion group is isomorphic to the torsion subgroup of* $\mathsf{K}^\times$ *for some field* K *of characteristic* 0 *if and only if it is locally cyclic with non-trivial* 2*-component.*

Proof. See the proof of Lemma 7.5. □

Turning to the case of finite characteristics, let us point out that if char $\mathsf{K} = p$, then 1 is the only root of the polynomial $x^p - 1 = (x-1)^p$. Hence the p-component of $\mathsf{K}^\times$ is trivial. This is just an indication that the positive characteristic case must be handled differently.

Special Fields In the following, we examine the multiplicative groups of the most important fields.

1. **Prime fields:** $\mathbb{Q}$ and $\mathsf{F}_p = \mathbb{Z}/p\mathbb{Z}$ for each prime p. The fundamental theorem of arithmetic states that every rational number $\neq 0$ can be written uniquely in the form $\pm p_1^{k_1} \dots p_n^{k_n}$ where p_i are different primes and $k_i \neq 0$ are integers. Hence $\mathbb{Q}^\times$ is the direct product of $\langle -1\rangle$ and the cyclic groups $\langle p\rangle$ for all primes p. We can now state that $\mathbb{Q}^\times$ is the direct product of a cyclic group of order 2 and a countably generated free group.

   For prime characteristics, the existence of primitive roots mod p implies that the multiplicative group is cyclic, so $\mathsf{F}_p^\times \cong \mathbb{Z}(p-1)$.

2. **Finite algebraic extensions of prime fields.** Recall the fundamental theorem of ideal theory from algebraic number theory: in the ring R of algebraic integers in a finite algebraic extension of $\mathbb{Q}$, a fractional ideal $\mathsf{L} \neq 0$ is a uniquely determined product $\mathsf{L} = \mathsf{P}_1^{k_1} \cdots \mathsf{P}_n^{k_n}$ with $0 \neq k_i \in \mathbb{Z}$, where P_i are different prime ideals. In other words, the non-zero ideals form a multiplicative group that is a countably generated free group.

Theorem 8.2 (Skolem [1]). *The multiplicative group of a finite algebraic extension* K *of a prime field has the following form:*

(i) *If* char $\mathsf{K} = 0$*, then* $\mathsf{K}^\times \cong \mathbb{Z}(2m) \times F$*, where* $m \in \mathbb{N}$ *and* F *is a countably generated free group;*
(ii) *If* char $\mathsf{K} = p$*, then* $\mathsf{K}^\times \cong \mathbb{Z}(p^n - 1)$ *with* $n \in \mathbb{N}$.

Conversely, every group of the form (i) *or* (ii) *can be realized as the multiplicative group of a finite algebraic extension of a prime field.*

Proof. (i) Evidently, $a, b \in \mathsf{K}^\times$ generate the same (proper or fractional) ideal if and only if ab^{-1} is a unit in the ring R of algebraic integers in K. Hence $\phi : a \mapsto \mathsf{R}a$ is a multiplicative homomorphism of $\mathsf{K}^\times$ into the group of ideals such that $\operatorname{Ker}\phi$ is the group U of units in R. Consequently, $\mathsf{K}^\times$ is an extension of U by a subgroup of a (countable) free group, so it is of the indicated form.

(ii) The situation is totally different for prime characteristics, since a finite algebraic extension of degree n of F_p is a Galois field of p^n elements. The multiplicative group of a finite field is cyclic, so $\mathsf{K}^\times \cong \mathbb{Z}(p^n - 1)$, as stated.

To verify the second part, note that the existence theorem on Galois fields settles case (ii) at once. For case (i), it suffices to point out that from the proof of Lemma 7.5 it is clear that if ζ is a complex, $2m$th primitive root of unity, then the torsion subgroup of $\mathbb{Q}(\zeta)^\times$ is cyclic of order $2m$. □

3. **Algebraically closed fields.** In an algebraically closed field A, for every $a \in \mathsf{A}$ and for every prime p, the equation $x^p = a$ has p roots in A, except when char $\mathsf{A} = p$. Therefore, $\mathsf{A}^\times$ must be a divisible group, and hence a direct product of groups isomorphic to $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$. The precise result is as follows.

Theorem 8.3. *A group is isomorphic to the multiplicative group of an algebraically closed field* A *if and only if it is of the form*

$$\mathsf{A}^\times \cong \begin{cases} \mathbb{Q}/\mathbb{Z} \oplus D & \text{if char } \mathsf{A} = 0; \\ \oplus_{q \neq p} \mathbb{Z}(q^\infty) \oplus D & \text{if char } \mathsf{A} = p; \end{cases}$$

where D is a torsion-free divisible group. D can be of any infinite cardinality, or, in the second alternative only, also $D = 0$.

Proof. If char $\mathsf{A} = 0$, then A must contain n roots of unity for every n, so it contains a subgroup $\cong \mathbb{Q}/\mathbb{Z}$. On the other hand, if char $\mathsf{A} = p$, then—as noticed above—the p-component of $\mathsf{A}^\times$ is trivial. But otherwise, the same argument applies, with the exception that $\mathsf{A}^\times$ may be torsion (for algebraic extensions of the prime field). As far as $|\mathsf{A}|$ is concerned, observe that the algebraic closure of a field of infinite cardinality has exactly the same cardinality. □

4. **Real-closed fields.** Recall the definition: a real-closed field C is a field that can be linearly ordered, but no proper algebraic extension admits linear order; here, the order is meant to be compatible with the field operations: sums and products of positive elements are positive. The only possibility is char $\mathsf{C} = 0$. From the theory of real-closed fields it is known that $\mathsf{A} = \mathsf{C}(\sqrt{-1})$, a quadratic extension, is an algebraically closed field. As a consequence, all polynomials of odd degrees must have a factor of degree 1 in $\mathsf{C}[x]$; in particular, extraction of pth roots for odd primes p is possible throughout in C. But these pth roots are unique, since ± 1 are the only roots of unity in C as it is shown by the following argument: if $1 \neq \zeta \in \mathsf{A}$ is a pth root of 1, then $1 + \zeta^2 + \cdots + \zeta^{2(p-1)} = (\zeta^{2p} - 1)/(\zeta^2 - 1) = 0$ shows that -1 is a sum of squares if ζ is in the field, thus $\zeta \notin \mathsf{C}$. As far as square roots in general are concerned, we know that in a linear order of C, the positive elements are complete squares, i.e. for every $c \in \mathsf{C}$, one of $c, -c$ has a square root in C. The following theorem summarizes the facts.

Theorem 8.4. *A group is isomorphic to the multiplicative group of a real-closed field exactly if it is of the form*

$$\mathbb{Z}(2) \oplus D$$

where D is a torsion-free divisible group of cardinality $\kappa \geq \aleph_0$. Every infinite cardinal κ can occur. □

5. ***p*-adic number fields.** Recalling that every p-adic number $\neq 0$ can be written uniquely as $p^k \pi$ with $k \in \mathbb{Z}$ and π a p-adic unit, it is clear that the multiplicative

group of the p-adic number field is a direct product $\mathbb{Z}\times U_p$ where U_p is the group of units in J_p. The units of the form $\pi = 1 + s_1p + s_2p^2 + \dots$ $(0 \le s_i < p)$ form a subgroup E of U_p, and every unit is uniquely the product of an element in the reduced residue class mod p and an element $\pi \in E$. Thus $U_p \cong V \times E$ where $V \cong \mathbb{Z}(p-1)$ is the group of residue classes mod p prime to p.

We concentrate on E, and prove an auxiliary lemma. It is based on an ingenious idea that goes back to K. Hensel, representing elements of E by their 'logarithm' values in J_p.

Lemma 8.5. *Let $p > 2$ and $e = g^{p-1}$, where g is a primitive root* mod p^n *for all $n \in \mathbb{N}$. Furthermore, let σ_k denote the kth partial sum of the p-adic integer $\sigma = s_1p + \cdots + s_np^n + \dots$ $(0 \le s_n < p)$. The sequence e^{σ_k} $(k < \omega)$ converges to an element $\pi \in E$, denoted e^σ, and the correspondence $e^\sigma \mapsto \pi$ is an isomorphism $pJ_p \to E$.*

Proof. By number theory, there exists a primitive root g mod p that is also primitive root modulo all powers of p (if g is an arbitrary primitive root for p, then either g or $g+p$ works for all p^n). Our choice guarantees that $e = g^{p-1}$ satisfies $e^m \equiv 1 \bmod p^k$ for some $m \in \mathbb{N}$ exactly if $p^{k-1}|m$. For a p-adic integer $\sigma = s_1p + \cdots + s_np^n + \cdots \in pJ_p$, consider the sequence $e^0, e^{s_1p}, \dots, e^{s_1p+\cdots+s_kp^k} = e^{\sigma_k}, \dots$. It converges to a p-adic integer π, since $e^{\sigma_{k-1}} \equiv e^{\sigma_k} \bmod p^k$ for all $k \in \mathbb{N}$. As $\pi \equiv e^{\sigma_{k-1}} \bmod p^k$, we have $\pi \in E$. We now show that the correspondence $\sigma \mapsto \pi$ is a bijection.

For every $\pi \in E$ and for every $k \ge 1$, there is a unique integer σ_{k-1} such that $0 \le \sigma_{k-1} < p^{k-1}$ and $e^{\sigma_{k-1}} \equiv \pi \bmod p^k$ (with $\sigma_0 = 0$). Then uniqueness implies $\sigma_{k-1} \equiv \sigma_k \bmod p^{k-1}$, thus the integers σ_k converge to a p-adic integer σ such that $\sigma \equiv \sigma_{k-1} \bmod p^k$ for all $k \in \mathbb{N}$. It is routine to check that the correspondence $\eta : \pi \mapsto e^\sigma$ is a homomorphism from the multiplicative group E into the additive group pJ_p. □

In case $p = 2$, the number $e = 5$ has the property that $e^m \equiv 1 \bmod 2^k$ is tantamount to the divisibility relation $2^{k-2}|m$. We can establish the same 'logarithmic' correspondence, the only difference is that it will be between E and $4J_2$.

The preceding lemma and the remarks provide a proof for the following theorem.

Theorem 8.6 (K. Hensel). *The multiplicative group of the p-adic number field $\mathbb{Q}_p^*$ is isomorphic to the additive group*

$$\mathsf{K}^\times \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}(p-1) \oplus J_p & \textit{if } p \neq 2, \\ \mathbb{Z} \oplus \mathbb{Z}(2) \oplus J_2 & \textit{if } p = 2. \end{cases}$$

★ **Notes.** The problem of finding the multiplicative structure of fields is as old as algebraic number theory. Dirichlet's theorem on units was the first remarkable result, followed by Galois fields, and by the multiplicative groups p-adic number fields by Hensel. Skolem's theorem is of much later origin, presumably in a less precise form it was known before (as all the ingredients were available long before).

Naturally, the big question is to survey all groups that do occur as multiplicative groups of fields. So far no satisfactory characterization has been given in terms of what algebraists would regard satisfactory, namely, by a set of sentences of the first-order language of group theory. As a matter of fact, S.R. Kogalowski [Dokl. Akad. Nauk SSSR **140**, 1005–1007 (1961)] has shown that the class of multiplicative groups of fields is not arithmetically closed in the sense of A.I. Malcev, and so not axiomatizable; cf. also Sabbagh [1].

There are numerous remarkable results on the multiplicative structure of fields. To discuss any of these results in detail would take us too far afield: the proofs are intricate arguments based on theorems in field theory. A serious attempt to clarify the multiplicative structures of fields was undertaken by W. May who proved several substantial theorems, of which the most significant is perhaps the one that describes the multiplicative group up to a free factor: Given a group A whose torsion subgroup is a subgroup of $\mathbb{Q}/\mathbb{Z}$ and has a non-trivial 2-component, there exists a field K of characteristic 0 such that $\mathsf{K}^\times \cong A \times F$ with a free group F. Regretfully, we cannot go more deeply into the subject, we cannot even sketch here the proof of this result; see May [1]. Another result of his tells us about the change of unit groups under certain field extensions [3].

Exercises

(1) The additive and the multiplicative groups of a field are never isomorphic.

(2) (a) A finite group is isomorphic to the multiplicative group of a field exactly if it is cyclic of order $p^n - 1$ for some prime p and integer $n \geq 1$.

(b) A finite group is the torsion subgroup of the multiplicative group of a field exactly if it is cyclic of even order or of the form indicated in (a).

(3) (Schenkman) Let N be the field generated by all algebraic numbers of degree $\leq n$, for a fixed $n \geq 2$. Then $\mathsf{N}^\times$ is a direct product of cyclic groups. [Hint: N does not contain mth root of 1 for $m > 4n!$; use Pontryagin's theorem.]

(4) A field K of characteristic p is called *perfect* if irreducible polynomials in $\mathsf{K}[x]$ have only simple roots. Show that to be perfect it is necessary and sufficient that $\mathsf{K}^\times$ be p-divisible.

(5) (Sabbagh) The group $\mathbb{Q} \oplus \mathbb{Z}(2)$ is not isomorphic to the multiplicative group of any field.

Problems to Chapter 18

PROBLEM 18.1. If a torsion-free group can support a left noetherian ring, can it also support a two-sided (or a commutative) noetherian ring?

PROBLEM 18.2 (Niedzwecki–Reid [1]). Study the additive structure of a ring modulo the pure subgroup generated by 1.

PROBLEM 18.3. Given A, define groups A_n $(n < \omega)$ by the rule: $A_0 = A$, and $A_{n+1} = (\operatorname{End} A_n)^+$. How long can this sequence be before it becomes stationary?

It becomes stationary at 0 if A is an E-group, and at 1 if A is a rigid group.

PROBLEM 18.4. Study the Picard groups of E-rings.

PROBLEM 18.5. Characterize the unit groups of finite commutative rings.

See Pearson–Schneider [1].

PROBLEM 18.6. Characterize the unit groups of commutative von Neumann regular and noetherian rings.

PROBLEM 18.7. Study the change of the unit groups under ring (field) extensions.

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