Unreliable Queueing System with Cold Redundancy

Valentina Klimenok[1](✉) and Vladimir Vishnevsky[2]

[1] Department of Applied Mathematics and Computer Science, Belarusian State University, 220030 Minsk, Belarus
{klimenok,dudin@}bsu.by

[2] Institute of Control Sciences of Russian Academy of Sciences and Closed Corporation "Information and Networking Technologies", Moscow, Russia
vishn@inbox.ru

Abstract. In this paper, we analyze a queueing system with so called "cold" redundancy. The system consists of an infinite buffer, the main unreliable server (server 1) and the absolutely reliable reserve server (server 2). The input flow is a *BMAP* (Batch Markovian Arrival Process). Breakdowns arrive to the server 1 according to a *MAP* (Markovian Arrival Process). If the server 1 is fault-free, it serves a customer, if any. After breakdown occurrence the server 1 fails and the repair period starts immediately. The customer, whose service is interrupted by a breakdown, goes to the second server, where its service is restarted. When the repair period ends, the customer whose service on the server 2 has not yet completed goes back to the server 1 and its service begins anew. We assume that the switching from one server to another takes time. Switching times as well as service times and repair time have *PH* (Phase type) distribution. The queue under consideration can be applied for modeling of a hybrid communication system consisting of the FSO – Free Space Optics channel (server 1) and the radio-wave channel (server 2). We derive a condition for stable operation of the system, calculate its stationary distribution and base performance measures and derive an expression for the Laplace-Stieltjes transform of the sojourn time distribution.

Keywords: Unreliable queueing system · Batch markovian arrival process · Phase-type distribution · Stationary state distribution · Sojourn time distribution

1 Introduction

In recent years, the FSO – Free Space Optics technologies have become widespread due to their undoubted advantages. The main advantages of atmospheric optical (laser) communication link are high capacity and quality of communication. However, optical communication systems have also disadvantages, the main of which is the dependence of the communication channel on the weather condition. The unfavorable weather conditions such as rain, snow, fog, aerosols, smog

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can significantly reduce visibility and thus significantly reduce the effectiveness of atmospheric optical communication link.

As it is mentioned in [1], one of the main directions of creating the ultra-high speed (up to 10 Gbit/s) and reliable wireless means of communication is the development of hybrid communication systems based on laser and radio-wave technologies. Unlike the FSO channel, radio-wave IEEE802.11n channel is not sensitive to weather conditions and can be considered as absolutely reliable. However, it has a lower transmission speed compared with the FSO-channel. In hybrid communication system consisting of the FSO channel and the radio-wave IEEE802.11n channel the latter can be considered as as a backup communication channel. Because of the high practical need for hybrid communication systems, a considerable amount of studies of this class of systems have appeared recently. Some results of these studies are presented in [2–4].

Papers from [2] are mainly focused on the study of stationary reliability characteristics, methods and algorithms for optimal channel switching in hybrid systems by means of simulation. The paper [3] deals with hybrid communication channel with so called "hot" redundancy, where the backup IEEE 802.11n channel continuously transmits data along with the FSO channel, but, unlike the latter, at low speed. In the paper [4], the hybrid communication system with "cold" redundancy is considered, where the radio-wave link is assumed to be absolutely reliable and backs up the atmospheric optical communication link only in cases when the latter interrupts its functioning because of the unfavorable weather conditions. The paper [1] is devoted to the study of a hybrid communication system where the millimeter-wave radio channel is used as a backup one. To model this system, the authors consider two-channel queueing system with unreliable heterogeneous servers which fail alternately.

In the present paper, we consider queueing system suitable to model a hybrid communication channel with "cold" reserve under more general, in comparison with the papers cited above, assumptions about the pattern of arrival processes of customers and breakdowns, distributions of service and repair times. Besides, we assume that the switching from one server to another takes time. Switching times as well as service times and repair time have PH (Phase type) distributions. The queue under consideration can be applied for modeling of hybrid communication system with "cold" redundancy where the radio-wave link is assumed to be absolutely reliable (its work does not depend on the weather conditions) and backs up the atmospheric optical communication link only in cases when the latter interrupts its functioning because of the unfavorable weather conditions. Upon the occurrence of favorable weather conditions the data packets begin to be transmitted over the FSO channel.

2 Mathematical Model

We consider a queueing system consisting of two heterogeneous servers and infinite waiting room. One of the servers (main server, server 1) is unreliable and the other one (reserve server, server 2) is absolutely reliable. The latter is in the so-called cold standby and connects to the service of a customer only in the case when the main server is under repair.

Customers arrive into the system in accordance with a Batch Markovian Arrival Process (*BMAP*). The *BMAP* is defined by the underlying process ν_t, $t \geq 0$, which is an irreducible continuous-time Markov chain with the finite state space $\{0, \dots, W\}$, and the matrix generating function $D(z) = \sum\limits_{k=0}^{\infty} D_k z^k$, $|z| \leq 1$. The batches of customers enter the system only at the epochs of the chain ν_t, $t \geq 0$, transitions. The $(W+1) \times (W+1)$ matrices D_k, $k \geq 1$, (non-diagonal entries of the matrix D_0) define the intensities of the process ν_t, $t \geq 0$, transitions which are accompanied by generating the k-size batch of customers. The matrix $D(1)$ is an infinitesimal generator of the process ν_t, $t \geq 0$. The intensity (fundamental rate) of the *BMAP* is defined as $\lambda = \boldsymbol{\theta} D'(1)\mathbf{e}$ where $\boldsymbol{\theta}$ is the unique solution of the system $\boldsymbol{\theta} D(1) = \mathbf{0}$, $\boldsymbol{\theta}\mathbf{e} = 1$, and the intensity of batch arrivals is defined as $\lambda_b = \boldsymbol{\theta}(-D_0)\mathbf{e}$. Here and in the sequel $\mathbf{e}(\mathbf{0})$ is a column (row) vector of appropriate size consisting of 1's (0's). For more information about the *BMAP* see, e.g., [5].

If there is no customers in the system and the server 1 is fault-free at an arrival epoch, it immediately starts the service of an arriving customer. If the server 1 is under repair or serves a customer at an arrival epoch, an arriving customer is placed at the end of the queue in the buffer and is picked-up for service later on, according the FIFO discipline.

If a breakdown arrive to the server 1 during service of a customer, the repair of the server 1 and the switching to the server 2 begin immediately. After the switching time has expired, the customer goes to the server 2 where it starts its service anew. However, if during the switching time the server 1 becomes fault-free, the customer restarts its service on this server.

If the repair period on the server 1 ended but the service of a customer by the server 2 does not complete, then the switching to the server 1 begins. It is assumed that during the switching time the server 1 can not serve customers from the queue. After the switching time has expired, the customer goes to the server 1 where it starts its service anew. If at the end of the switching time the server 1 is under repair, the customer restarts its service on the server 2.

Breakdowns arrive to the server 1 according to a *MAP* which is defined by the $(V+1) \times (V+1)$ matrices H_0 and H_1. The breakdowns fundamental rate is calculated as $h = \boldsymbol{\gamma} H_1 \mathbf{e}$ where the row vector $\boldsymbol{\gamma}$ is the unique solution of the system $\boldsymbol{\gamma}(H_0 + H_1) = \mathbf{0}$, $\boldsymbol{\gamma}\mathbf{e} = 1$.

The service time of a customer by the k-th server, $k = 1, 2$, has *PH* type distribution with an irreducible representation $(\boldsymbol{\beta}^{(k)}, \mathbf{S}^{(k)})$. The service process on the k-th server is directed by the Markov chain $m_t^{(k)}$, $t \geq 0$, with the state space $\{1, \dots, M^{(k)}, M^{(k)}+1\}$ where $M^{(k)}+1$ is an absorbing state. The intensities of transitions into the absorbing state are defined by the vector $\mathbf{S}_0^{(k)} = -S^{(k)}\mathbf{e}$. The service rates are calculated as $\mu^{(k)} = -[\boldsymbol{\beta}^{(k)}(S^{(k)})^{-1}\mathbf{e}]^{-1}$, $k = 1, 2$.

The switching time from the server 2 to the server 1 has *PH* type distribution with an irreducible representation $(\boldsymbol{\alpha}^{(1)}, A^{(1)})$. The switching time from the server 1 to the server 2 has PH type distribution with an irreducible representation $(\boldsymbol{\alpha}^{(2)}, A^{(2)})$. The switching process on the k-th server is directed by the Markov chain $l_t^{(k)},\ t \geq 0$, with the state space $\{1, \ldots, L^{(k)}, L^{(k)}+1\}$ where $L^{(k)}+1$ is an absorbing state, $k = 1, 2$. The intensities of transitions into the absorbing state are defined by the vector $\mathbf{A}_0^{(k)} = -A^{(k)}\mathbf{e}$. The switching rates are defined as $\kappa_k = -[\boldsymbol{\alpha}^{(k)}(A^{(k)})^{-1}\mathbf{e}]^{-1}$, $k = 1, 2$.

The repair period has PH type distribution with an irreducible representation $(\boldsymbol{\tau}, T)$. The repair process is directed by the Markov chain $\vartheta_t,\ t \geq 0$, with the state space $\{1, \ldots, R, R+1\}$ where $R+1$ is an absorbing state. The intensities of transitions into the absorbing state are defined by the vector $\mathbf{T}_0 = -T\mathbf{e}$. The repair rate is $\phi = -(\boldsymbol{\tau}T^{-1}\mathbf{e})^{-1}$.

3 Process of the System States

Let at the moment t:

- i_t be the number of customers in the system, $i_t \geq 0$;
- $n_t = 0$ if the server 1 is fault-free and $n_t = 1$ if the server 1 is under repair;
- $r_t = 0$ if one of the servers serves a customer and $r_t = 1$ if the switching period takes place;
- $m_t^{(k)}$ be the state of the directing process of the service at the k-th busy server, $m_t^{(k)} = \overline{1, M^{(k)}}$, $k = 1, 2$;
- $l_t^{(1)}$ be the state of the directing process of the switching time from the server 2 to the server 1, $l_t^{(1)} = \overline{1, L^{(1)}}$;
- $l_t^{(2)}$ be the state of the directing process of the switching time from the server 1 to the server 2, $l_t^{(2)} = \overline{1, L^{(2)}}$;
- ϑ_t be the state of the directing process of the repair time at the server 1, $\vartheta_t = \overline{1, R}$;
- ν_t and η_t be the states of the directing process of the $BMAP$ and the MAP correspondingly, $\nu_t = \overline{0, W}$, $\eta_t = \overline{0, V}$.

The process of the system states is described by the regular irreducible continuous time Markov chain, $\xi_t, t \geq 0$, with the state space

$$X = \{(i, n, \nu, \eta),\ i = 0, n = 0, \nu = \overline{0, W}, \eta = \overline{0, V}\} \bigcup$$

$$\{(i, n, \nu, \eta, \vartheta),\ i = 0, n = 1, \nu = \overline{0, W}, \eta = \overline{0, V}, \vartheta = \overline{0, R}\} \bigcup$$

$$\{(i, n, r, \nu, \eta, m^{(1)}),\ i > 0, n = 0,\ r = 0, \nu = \overline{0, W}, \eta = \overline{0, V}, m^{(1)} = \overline{1, M^{(1)}}\} \bigcup$$

$$\{(i, n, r, \nu, \eta, l^{(1)}),\ i > 0, n = 0,\ r = 1, \nu = \overline{0, W}, \eta = \overline{0, V}, l^{(1)} = \overline{1, L^{(1)}}\} \bigcup$$

$$\{(i, n, r, \nu, \eta, m^{(2)}, \vartheta),\ i > 0, n = 1,\ r = 0, \nu = \overline{0, W}, \eta = \overline{0, V}, m^{(2)} = \overline{1, M^{(2)}},$$

$$\vartheta = \overline{1, R}\} \bigcup \{(i, n, r, \nu, \eta, \vartheta, l^{(2)}),\ i > 0, n = 1,\ r = 1, \nu = \overline{0, W},$$

$$\eta = \overline{0,V}, \vartheta = \overline{1,R}, l^{(2)} = \overline{1,L^{(2)}}\}.$$

In the following, we will assume that the states of the chain $\xi_t, t \geq 0$, are ordered as follows. Within the indicated above subsets of the set X the states of the chain are enumerated in the lexicographic order. Denote the obtained ranked sets as $X(0,0), X(0,1), X(i,n,r), i \geq 1, n, r = 1,2, r = 1,2$, and arrange these sets in the lexicographic order. Let Q_{ij}, $i,j \geq 0$, be the matrices formed by intensities of the chain transition from the state corresponding to the value i of the component i_n to the state corresponding to the value j of this component and $Q = (Q_{ij})_{i,j \geq 0}$, be the generator of the chain. For further use in the sequel, we also introduce the following notation:

- $\otimes$, $\oplus$ are the symbols of Kronecker's product and sum of matrices;
- $\bar{W} = W + 1$, $\bar{V} = V + 1$, $a = \bar{W}\bar{V}$;
- $\text{diag}\{B_i, i = \overline{1,n}\}$ is the n-size diagonal matrix with diagonal blocks B_i.

Lemma 1. *Infinitesimal generator Q of the Markov chain ξ_t, $t \geq 0$, has the following block structure*

$$Q = \begin{pmatrix} \tilde{Q}_0 & \tilde{Q}_1 & \tilde{Q}_2 & \tilde{Q}_3 & \cdots \\ \hat{Q}_0 & Q_1 & Q_2 & Q_3 & \cdots \\ O & Q_0 & Q_1 & Q_2 & \cdots \\ O & O & Q_0 & Q_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the non-zero blocks are of the form

$$\tilde{Q}_0 = \begin{pmatrix} D_0 \oplus H_0 & I_{\bar{W}} \otimes H_1 \otimes \boldsymbol{\tau} \\ I_a \otimes T_0 & D_0 \oplus H \oplus T \end{pmatrix}, \quad \hat{Q}_0 = \begin{pmatrix} I_a \otimes \boldsymbol{S}_0^{(1)} & O \\ O_{aL^{(1)} \times a} & O \\ O & I_{aR} \otimes \boldsymbol{S}_0^{(2)} \\ O_{aRL^{(2)} \times a} & O \end{pmatrix},$$

$$\tilde{Q}_k = \begin{pmatrix} D_k \otimes I_{\bar{V}} \otimes \boldsymbol{\beta}^{(1)} & O_{a \times aL^{(1)}} & O & O_{a \times aRL^{(2)}} \\ O & O & D_k \otimes I_{\bar{V}} \otimes T \otimes \boldsymbol{\beta}^{(2)} & O \end{pmatrix}, \ k \geq 1,$$

$$Q_0 = \begin{pmatrix} I_a \otimes \boldsymbol{S}_0^{(1)} \boldsymbol{\beta}^{(1)} & O & O & O \\ O & O_{aL^{(1)}} & O & O \\ O & O & I_{aR} \otimes \boldsymbol{S}_0^{(2)} \boldsymbol{\beta}^{(2)} & O \\ O & O & O & O_{aRL^{(2)}} \end{pmatrix},$$

$$Q_k = \text{diag}\{D_{k-1} \otimes I_{\bar{V}} \otimes I_{M^{(1)}}, D_{k-1} \otimes I_{\bar{V}} \otimes I_{L^{(1)}},$$
$$D_{k-1} \otimes I_{\bar{V}} \otimes I_R \otimes I_{M^{(2)}}, D_{k-1} \otimes I_{\bar{V}} \otimes I_R \otimes I_{L^{(2)}}\}, \ k \geq 2,$$

$$Q_1 = \begin{pmatrix} D_0 \oplus \mathcal{Q}_{1,1} & O & O & I_{\bar{W}} \otimes \mathcal{Q}_{1,4} \\ I_{\bar{W}} \otimes \mathcal{Q}_{2,1} & D_0 \oplus \mathcal{Q}_{2,2} & I_{\bar{W}} \otimes \mathcal{Q}_{2,3} & O \\ O & I_{\bar{W}} \otimes \mathcal{Q}_{3,2} & D_0 \oplus \mathcal{Q}_{3,3} & O \\ I_{\bar{W}} \otimes \mathcal{Q}_{4,1} & O & I_{\bar{W}} \otimes \mathcal{Q}_{4,3} & D_0 \oplus \mathcal{Q}_{4,4} \end{pmatrix},$$

where

$$\mathcal{Q}_{1,1} = H_0 \oplus S^{(1)},\ \mathcal{Q}_{1,4} = H_1 \otimes \boldsymbol{e}_{M^{(1)}} \otimes \boldsymbol{\tau} \otimes \boldsymbol{\alpha}^{(2)},$$

$$\mathcal{Q}_{2,1} = I_{\bar{V}} \otimes A_0^{(1)} \otimes \boldsymbol{\beta}^{(1)},\ \mathcal{Q}_{2,2} = D_0 \oplus H_0 \oplus A^{(1)},\ \mathcal{Q}_{2,3} = H_1 \otimes \mathbf{e}_{L^{(1)}} \otimes \boldsymbol{\tau} \otimes \boldsymbol{\beta}^{(2)},$$

$$\mathcal{Q}_{3,2} = I_{\bar{V}} \otimes \boldsymbol{T}_0 \otimes \mathbf{e}_{M^{(2)}} \otimes \boldsymbol{\alpha}^{(1)},\ \mathcal{Q}_{3,3} = (H_0 + H_1) \oplus T \oplus S^{(2)},$$

$$\mathcal{Q}_{4,1} = I_{\bar{V}} \otimes \boldsymbol{T}_0 \otimes \mathbf{e}_{L^{(2)}} \otimes \boldsymbol{\beta}^{(1)},\ \mathcal{Q}_{4,3} = I_{\bar{V}} \otimes I_R \otimes \boldsymbol{A}_0^{(2)} \otimes \boldsymbol{\beta}^{(2)},\ \mathcal{Q}_{4,4} = H \oplus T \oplus A^{(2)}.$$

Corollary 1. *The Markov chain $\xi_t, t \geq 0$, belongs to the class of continuous time quasi-Toeplitz Markov chains (QTMC), see [6].*

The proof follows from the form of the generator given by Lemma 1 and the definition of $QTMC$ given in [6].

In the following it will be useful to have expressions for the generating functions $\tilde{Q}(z) = \sum\limits_{k=1}^{\infty} \tilde{Q}_k z^k,\ Q(z) = \sum\limits_{k=0}^{\infty} Q_k z^k,\ |z| \leq 1.$

Corollary 2. *The matrix generating functions $\tilde{Q}(z),\ Q(z)$ are of the form*

$$\tilde{Q}(z) = \begin{pmatrix} (D(z) - D_0) \otimes I_{\bar{V}} \otimes \boldsymbol{\beta}^{(1)} & O & O & O \\ O & O & (D(z) - D_0) \otimes I_{\bar{V}} \otimes T \otimes \boldsymbol{\beta}^{(2)} & O \end{pmatrix}, \quad (1)$$

$$\begin{aligned} Q(z) = &Q_0 + \mathcal{Q}z + z\,\mathrm{diag}\,\{D(z) \otimes I_{\bar{V}} \otimes I_{M^{(1)}},\ D(z) \otimes I_{\bar{V}} \otimes I_{L^{(1)}}, \\ &D(z) \otimes I_{\bar{V}} \otimes I_R \otimes I_{M^{(2)}},\ D(z) \otimes I_{\bar{V}} \otimes I_R \otimes I_{L^{(2)}}\}, \end{aligned} \quad (2)$$

where the matrix $\mathcal{Q}$ has the following block form:

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_{1,1} & O & O & \mathcal{Q}_{1,4} \\ \mathcal{Q}_{2,1} & \mathcal{Q}_{2,2} & \mathcal{Q}_{2,3} & O \\ O & \mathcal{Q}_{3,2} & \mathcal{Q}_{3,3} & O \\ \mathcal{Q}_{4,1} & O & \mathcal{Q}_{4,3} & \mathcal{Q}_{4,4} \end{pmatrix}, \quad (3)$$

and blocks $\mathcal{Q}_{i,j}$ are defined in the Lemma 1.

4 Stationary Distribution Performance Measures

Theorem 1. *The necessary and sufficient condition for existence of the stationary distribution of the Markov chain $\xi_t,\ t \geq 0$, is the fulfillment of the inequality*

$$\lambda < \boldsymbol{q}_1 \mathbf{S}_0^{(1)} + \boldsymbol{q}_3 \mathbf{S}_0^{(2)}, \quad (4)$$

where the vectors $\boldsymbol{q}_1,\ \boldsymbol{q}_3$ are calculated as $\boldsymbol{q}_1 = \mathbf{x}_1(\mathbf{e}_{\bar{V}} \otimes I_{M^{(1)}}),\ \boldsymbol{q}_3 = \mathbf{x}_3(\mathbf{e}_{\bar{V}R} \otimes I_{M^{(2)}})$, and the vectors $\mathbf{x}_1,\ \mathbf{x}_3$ are sub-vectors of the vector $\mathbf{x} = (\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4)$, which is the unique solution of the system of linear algebraic equations

$$\mathbf{x}\left[\mathcal{Q} + \mathrm{diag}\left\{I_a \otimes \boldsymbol{S}_0^{(1)} \boldsymbol{\beta}^{(1)},\ O_{aL^{(1)}},\ I_{aR} \otimes \boldsymbol{S}_0^{(2)} \boldsymbol{\beta}^{(2)},\ O_{aRL^{(2)}}\right\}\right] = 0,\ \ \mathbf{xe} = 1. \quad (5)$$

Here the vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4$ have the dimensions $a, aL^{(1)}, aRM^{(2)}, aRL^{(2)}$ correspondingly.

Proof. It follows from [6], that a necessary and sufficient condition for existence of the stationary distribution of the chain ξ_t, $t \geq 0$, can be formulated in terms of the matrix generating function $Q(z)$ and has the form of the inequality

$$\mathbf{y}Q^{'}(1)\mathbf{e} < 0, \tag{6}$$

where the vector $\mathbf{y}$ is the unique solution of the system of linear algebraic equations

$$\mathbf{y}Q(1) = \mathbf{0}, \tag{7}$$

$$\mathbf{ye} = 1.$$

Let $\mathbf{x}$ be a stochastic vector separated into parts as $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$. Represent the vector $\mathbf{y}$ in the form $\mathbf{y} = (\boldsymbol{\theta} \otimes \mathbf{x_1}, \boldsymbol{\theta} \otimes \mathbf{x_2}, \boldsymbol{\theta} \otimes \mathbf{x_3}, \boldsymbol{\theta} \otimes \mathbf{x_4})$.

Then, using (2) and taking into account that $\boldsymbol{\theta} \sum\limits_{k=0}^{\infty} D_k = 0$, system (7) is reduced to the form

$$\mathbf{x}\left[\text{diag}\left\{I_{\bar{V}} \otimes \boldsymbol{S}_0^{(1)}\boldsymbol{\beta}^{(1)}, O_{\bar{V}L^{(1)}}, I_{\bar{V}R} \otimes \boldsymbol{S}_0^{(2)}\boldsymbol{\beta}^{(2)}, O_{\bar{V}RL^{(2)}}\right\} + \mathcal{Q}\right] = 0. \tag{8}$$

Adding to system (8) the normalization condition, we obtain system (5).

Similarly, we conclude that inequality (4) is reduced to the following inequality:

$$\lambda + \mathbf{x}\mathcal{Q}\mathbf{e} < 0. \tag{9}$$

Taking into account the structure of the matrix $\mathcal{Q}$ given by (3), we reduce inequality (9) to the form

$$\lambda < \mathbf{x}_1\left(\mathbf{e}_{\bar{V}} \otimes S_0^{(1)}\right) + \mathbf{x}_3\left(\mathbf{e}_{\bar{V}R} \otimes S_0^{(2)}\right). \tag{10}$$

Applying mixed product rule, we derive from (10) ergodicity condition (4).

Remark 1. Intuitive explanation of stability condition (4) is as follows. The left hand side of inequality (4) is the rate of customers arriving into the system. The right hand side of the inequality is a rate of customers leaving the system after service under overload condition. It is obvious that in steady state the former rate must be less that the latter one.

Corollary 3. *In the case of stationary Poisson flow of breakdowns and exponential distribution of service and repair times, ergodicity condition (4)–(5) is reduced to the following inequality:*

$$\lambda < q_1\mu_1 + q_3\mu_2,$$

where

$$q_1 = \frac{\phi + \kappa^{(2)}}{h}\left[\frac{\phi + \kappa^{(2)}}{h} + \frac{\kappa^{(2)}}{\kappa^{(1)}} + \frac{\kappa^{(2)}(\kappa^{(1)} + h)}{\kappa^{(1)}\phi} + 1\right]^{-1}, \; q_3 = q_1\frac{h\kappa^{(2)}(\kappa^{(1)} + h)}{\kappa^{(1)}\phi(\phi + \kappa^{(2)})}.$$

In what follows we assume inequality (4) be fulfilled. Let us enumerate the steady state probabilities in accordance with the introduced above order of the states of the chain and form the row vectors $\boldsymbol{p}_i$ of steady state probabilities corresponding the value i of the first component of the Markov chain, $i \geq 0$.

To calculate the vectors $\boldsymbol{p}_i$, $i \geq 0$, we use the numerically stable algorithm, see [6], which has been elaborated for calculating the stationary distribution of multi-dimensional continuous time quasi-Toeplitz Markov chains.

Having the stationary distribution $\boldsymbol{p}_i$, $i \geq 0$, been calculated we can find a number of stationary performance measures of the system. When calculating the performance measures, the following result will be useful, especially in the case when the distribution $\boldsymbol{p}_i$, $i \geq 0$, is heavy tailed.

Lemma 2. *The vector generating function* $\boldsymbol{P}(z) = \sum\limits_{i=1}^{\infty} \boldsymbol{p}_i z^i$, $|z| \leq 1$, *satisfies the following equation:*

$$\boldsymbol{P}(z)Q(z) = z\left[\boldsymbol{p}_1 Q_0 - \boldsymbol{p}_0 \tilde{Q}(z)\right]. \tag{11}$$

In particular, formula (11) can be used to calculate the value of the generating function $\mathbf{P}(z)$ and its derivatives at the point $z = 1$ without the calculation of infinite sums. Having these derivatives been calculated, we will able to find a number of performance measures of the system. The problem of calculating the value of the $\boldsymbol{P}(z)$ and its derivatives at the point $z = 1$ from Eq. (11) is non-trivial because the matrix $Q(z)$ is singular at the point $z = 1$.

Let us denote $f^{(n)}(z)$ the n-th derivative of the function $f(z)$, $n \geq 1$, and $f^{(0)}(z) = f(z)$.

Corollary 4. *The* m*-th,* $m \geq 0$, *derivatives of the vector generating function* $\boldsymbol{P}(z)$ *at the point* $z = 1$ *are recursively calculated from the system of linear algebraic equations*

$$\begin{cases} \boldsymbol{P}^{(m)}(1)Q(1) = \Gamma^{(m)}(1) - \sum\limits_{l=0}^{m-1} C_m^l \boldsymbol{P}^{(l)}(1)Q^{(m-l)}(1), \\ \boldsymbol{P}^{(m)}(1)Q'(1)\mathbf{e} = \frac{1}{m+1}\left[\Gamma^{(m+1)}(1) - \sum\limits_{l=0}^{m-1} C_{m+1}^l \boldsymbol{P}^{(l)}(1)Q^{(m+1-l)}(1)\right]\mathbf{e}. \end{cases} \tag{12}$$

where

$$\Gamma^{(m)}(1) = \begin{cases} \boldsymbol{p}_1 Q_0 - \boldsymbol{p}_0 \tilde{Q}(1), & m = 0, \\ \boldsymbol{p}_1 Q_0 - \boldsymbol{p}_0 \tilde{Q}(1) - \boldsymbol{p}_0 \tilde{Q}'(1), & m = 1, \\ -\boldsymbol{p}_0\left[m\tilde{Q}^{(m-1)}(1) + Q^{(m)}(1)\right], & m > 1, \end{cases}$$

and the derivatives $Q^{(m)}(1)$, $\tilde{Q}^{(m)}(1)$ *are calculated using formulas (1)–(2).*

The proof of the corollary is parallel to the one outlined in [7] and is omitted here.

Now we are able to calculate a number of performance measures of the system under consideration.

- Throughput of the system $\varrho = \boldsymbol{q}_1\mathbf{S}_0^{(1)} + \boldsymbol{q}_3\mathbf{S}_0^{(2)}$.
- Mean number of customers in the system $L = \boldsymbol{P}^{(1)}(1)\mathbf{e}$.
- Variance of the number of customers in the system $V = \boldsymbol{P}^{(2)}(1)\mathbf{e} + L - L^2$.
- Probability that a system is empty and the server 1 is fault-free (is faulty)

$$P_0^{(0)} = \boldsymbol{p}_0^{(0)}\mathbf{e},\ P_0^{(1)} = \boldsymbol{p}_0^{(1)}\mathbf{e}.$$

- Probability that the server 1 serves a customer (the server 1 is faulty and the server 2 serves a customer)

$$P_0^{(0,0)} = \boldsymbol{P}(1)\,\mathrm{diag}\left\{I_{aM^{(1)}},\, 0_{L^{(1)}+aR(M^{(2)}+L^{(2)})}\right\}\mathbf{e}.$$

$$P_0^{(1,0)} = \boldsymbol{P}(1)\,\mathrm{diag}\left\{0_{a(M^{(1)}+L^{(1)})},\, I_{aRM^{(2)}},\, 0_{aRL^{(2)}}\right\}\mathbf{e}.$$

- Probability that the switching period from the server 2 to the server 1 (from the server 1 to the server 2) takes place

$$P_0^{(0,1)} = \boldsymbol{P}(1)\,\mathrm{diag}\left\{0_{aM^{(1)}}, I_{aL^{(1)}},\, 0_{aR(M^{(2)}+L^{(2)})}\right\}\mathbf{e}.$$

$$P_0^{(1,1)} = \boldsymbol{P}(1)\,\mathrm{diag}\left\{0_{a(M^{(1)}+L^{(1)}+RM^{(2)})},\, I_{aRL^{(2)})}\right\}\mathbf{e}.$$

5 Sojourn Time Distribution

Let $V(x)$ be the stationary distribution function of the sojourn time of an arbitrary customer in the system, $v(s) = \int\limits_0^\infty e^{-sx}dV(x)$, $Re(s) \geq 0$, be the Laplace-Stieltjes transform of this function.

Theorem 2. *The Laplace-Stieltjes transform of the sojourn time stationary distribution is calculated as*

$$v(s) = \lambda^{-1}\left\{\boldsymbol{p}_0\sum_{k=1}^{\infty}\tilde{Q}_k\sum_{l=1}^{k}\Phi^l(s) + \sum_{i=1}^{\infty}\boldsymbol{p}_i\sum_{k=2}^{\infty}Q_k\sum_{l=1}^{k-1}\Phi^{i+l}(s)\right\}\mathbf{e} \tag{13}$$

where

$$\Phi(s) = (sI - \bar{Q})^{-1}Q_0,\ \ \bar{Q} = Q(1) - Q_0.$$

Proof. The proof is based on the probabilistic interpretation of the Laplace-Stieltjes transform. We assume that, independently on the system operation, the stationary Poisson input of so called catastrophes arrives. Let s, $s > 0$, be the rate of this flow. Then, the Laplace-Stieltjes transform $v(s)$ is interpreted as the probability of no catastrophe arrival during the sojourn time of a customer. This allows to derive the expression for $v(s)$ by means of probabilistic reasonings.

Let us assume that at the moment of the beginning of a customer service the initial phases of service time at the servers are already determined. Then the matrix of probabilities of no catastrophes arrival during the service time of the customer and corresponding transitions of the finite components of the Markov chain ξ_t, $t \geq, 0$ is calculated as $\hat{\Phi}(s) = \int\limits_0^\infty e^{(-sI+\bar{Q})t}\hat{Q}_0 dt = (sI - \bar{Q})^{-1}\hat{Q}_0$, if at the departure epoch there are no customers in the queue, and $\Phi(s) = \int\limits_0^\infty e^{(-sI+\bar{Q})t}Q_0 dt = (sI - \bar{Q})^{-1}Q_0$, if at the departure epoch there are customers in the queue. Note, that $\hat{\Phi}(s)\mathbf{e} = \Phi(s)\mathbf{e}$ because $\hat{Q}_0\mathbf{e} = Q_0\mathbf{e}$.

Assuming that an arbitrary customer arriving in a group of size k is placed on the j-th position with probability $1/k$ and using the law of total probability, we obtain the following expression

$$v(s) = \boldsymbol{p}_0 \sum_{k=1}^{\infty} \frac{k}{\lambda}\tilde{Q}_k \sum_{l=1}^{k} \frac{1}{k}\Phi^l(s)\mathbf{e} + \sum_{i=1}^{\infty}\boldsymbol{p}_i \sum_{k=2}^{\infty}\frac{k-1}{\lambda}Q_k \sum_{l=1}^{k-1}\frac{1}{k-1}\Phi^{i+l}(s)\mathbf{e}. \quad (14)$$

Formula (13) immediately follows from formula (14). The theorem is proved.

Corollary 5. *Mean sojourn time, $\bar{v}$, of an arbitrary customer in the system is calculated as*

$$\bar{v} = -\lambda^{-1}\left[\boldsymbol{p}_0 \sum_{k=1}^{\infty}\tilde{Q}_k \sum_{l=1}^{k}\sum_{m=0}^{l-1}\Phi^m(0) + \sum_{i=1}^{\infty}\boldsymbol{p}_i \sum_{k=2}^{\infty} Q_k \sum_{l=1}^{k-1}\sum_{m=0}^{i+l-1}\Phi^m(0)\right]\Phi'(0)\mathbf{e} \quad (15)$$

where $\Phi'(0) = -(\bar{Q})^{-2}Q_0.$

Proof. To obtain formula (15), we used the relation $\bar{v} = -v'(0)$ and the fact that the matrix $\Phi(0)$ is a stochastic one.

6 Conclusion

In this paper, we study a single-server queueing system with $BMAP$ input and cold redundancy. The queue under consideration can be applied for modeling of a hybrid communication system consisting of the FSO – Free Space Optics channel and the radio wave channel. The system is studied in steady state using matrix-analytic methods. We derive a condition for stable operation of the system, calculate its stationary distribution and key performance measures and derive an expression for the Laplace-Stieltjes transform of the sojourn time distribution. This research was carried out in the framework of the applied project. Further studies suggest the computer realization of proposed algorithms and the implementation of numerical experiments to investigate the qualitative nature of the system under study.

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