Comparative Approach to the Multi-Valued Logic Construction for Preferences

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Abstract. This paper is aimed at the giving of a comparative approach to the preferences modelling. This approach is conceived to grasp the fuzzy nature of preferences what determines the choice two fuzzy logic formalisms for their representation discussed by P. Hajek and L. Godo. These two (appropriately modified) formalism are used to propose two formalism for preferences: *Fuzzy Modal Preferential Logic* (FMPL) and *Comparative Possibilistic Multi-Modal Propositional Logic* (CPMPL). We also justify some metalogical properties of both systems such as their completeness and we discuss a satisfiability problem for them. In result, we propose a short juxtaposition of the properties of the considered systems.

Keywords: Fuzzy logic for preferences · Multi-valued logic for preferences · Preferences modelling

1 Introduction

Different aspects of *preferences modelling* are considered in temporal knowledge representation. On one one hand, preferences and their representation are discussed in the context of the temporal planning issue as its specific constraints type – see for example: [8], [10], sometimes in term of STPP and STPPU— like in [11]. On the other hand, other approaches—such as: [15], [13] expose an epistemic or a game-theoretic nature of preferences. It is usually discussed there in terms of different variants of epistemic logic or logic of action. All the approaches refer, however, to the *fuzzy nature* of preferences partially and indirectly. The recent attempt for modelling user's preferences on attributes from [3] seems to be too algebraic and relatively narrow approach.

The lack of the exact multi-valued or fuzzy logic system for preferences representation and–as result–lack of a knowledge about its properties forms a main motivation of this paper. Therefore, we propose a new formalism of multi-valued logic and fuzzy logic for preferences modeling in this paper. For this purpose, we

L. Rutkowski et al. (Eds.): ICAISC 2015, Part I, LNAI 9119, pp. 172–183, 2015.
DOI: 10.1007/978-3-319-19324-3_16

especially adopt a formalism of systems involved in a newly interpreted modal necessity proposed and developed by L.Godo and F. Esteva in [4], in [6] and [7]– retrospectively discussed also in [5]. This approach allows us to apply a multi-valued modal systems with a decreasing or increasing degree of necessity of the modal operators to the preferences. It has been shown in [4] that if a set S of such degrees taken from a fuzzy set [0,1] is finite, then systems of possibilistic logic with modal operators with necessity (possibility) degrees from S remain complete.

1.1 Goal and a Novelty of the Paper

The first aim of this paper is a putting forward proposal of a new *Fuzzy-Modal Preferential Logic* FMPL as suitable for preferences representation. In order to expose the advantages of this newly proposed system we intend to compare FMPL with some alternative fuzzy system being an appropriate modification of the Hajek's *comparative possibilistic multi-modal propositional logic* CPMPL — considered in [5]— but without reference to preferences. For this purpose we justify a handful of properties of both systems such as completeness, model checking problem and its complexity. We show these properties in terms of the newly proposed interval semantics. In other words, this paper proposes an answer of the following questions:

- How to construct a multi-valued logic for preferences?
- Which (useful) properties have this system?
- Why a multi-valued system seems to be more comfortable than an alternative fuzzy comparative one?

1.2 Terminological Background and Methodological Remarks

All of the considered frames in our interval-based semantics will be based on an order $\langle S, \leq \rangle$ which:

- is strongly discrete, i.e. there are only finitely many points between any two points,
- the order contains the least element,
- for any *a, b, c* $\in$ S, if $a \leq c$ and $b \leq c$, then either $a \leq b$ or $b \leq a$.

We will consider a finite set $\mathcal{A} = \{1, \ldots, m\}$ of agents such that each agent is endowed with a set of a local states L_i and $i \in \mathcal{A}$. If e is an environment of an agent the set of local states is denoted by L_e. Each agent is equipped with a set of local actions and protocolar functions that produce a transition relation t. In such a framework we adopt the following definitions.

Definition 1. *Let $\mathcal{L}$ be a given language. An interval-based interpreted system* IBIS *is a tuple (S, s_0, t, Lab) such that*

S is a set of global states (which can be points or intervals) reachable from the initial state s_0,

t is a standard transition relation between states, and
Lab is a labeling function, which for an interval $I = s_1, s_2, \ldots, s_k$ is defined: $Lab(I) = Lab(s_1, s_2, \ldots, s_k) = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_k)$, where terms $\hat{s}_i$ represents the point s_i in $\mathcal{L}$, for $i < k$. In particular pairs $(a, b) \in S^2$ are associated to formulas pair $(\hat{a}, \hat{b})$.

Definition 2. *An interval is a finite path in* IBIS*, or as a sequence $I = s_1 s_s \ldots s_k$ such that $s_i t s_{i+1}$ for $1 \leq i \leq k-1$ and a transition t.*

Definition 3. *A generalized Kripke frame is a tuple $M = (S, s_0, t, Lab)$, where s_0, t and Lab are as above and $S \subseteq L_e \times L_1 \times \ldots L_m$ is a set of reachable (in any way) global states. It easy to see that an unravelling generalized Kripke frame is an IBIS.* [1]

Definition 4. *(Model checking). Given a logic $\mathcal{L}$, a generalized Kripke frame M, an interval $I = s_0 \ldots s_k$ and a formula $\phi \in \mathcal{L}$, the model checking problem for $\mathcal{L}$ amounts to checking whether or not it holds $M, I \models \phi$.*

In a model checking procedure for a newly constructed FMPL we make use of the reduction method to the Boolean quantified satisfiability problem recognized as a PSpace-problem from [9]. We use a standard logical notation.

1.3 Paper's Organization

The paper is organized as follows. In Section 2 we introduce a multi-valued logic for preferences (FMPL) representation by means of the necessity definition proposed by Godo and Hajek. We check completeness of this system and perform its model checking for the interval-based semantics. In Section 3 we consider comparative fuzzy preferential logic CPMPL as the alternative system for preferences and we will discuss its difficulties. We expose the conditional nature of the completeness proof, the difficulties with an interval-based semantics construction for CPMPL and undefinability of a CPMPL-satisfiability problem in this semantics type. In section 4 we give a short juxtaposition of properties of both systems. In the last section of the paper we formulate conclusions and we sketch a promising research direction in this area.

2 Fuzzy Modal Preferential Logic

It has just been mentioned that preferences are often expressed in terms of modal-epistemic logic and represented by relations being some types of orders– especially in many game-theoretic approaches where they are interpreted in infinite game trees. In this paragraph we intend to understand preferences in a 'broader' sense and represent them by partial orders rejecting a linearity condition for them. We adopt such a way of their understanding because such a

[1] For clarity, we omit the detailed definition of unravelling.

generality ensures more flexibility with respect to the incomparable worlds. This *manouvre* is also dictated by a purely meta-logical properties of the formalism that we use for preferences representation. Indeed, partial orders represent the modal system S4 which shows to have much more comfortable properties such as completeness (in a multi-modal version as $S4_m$, i.e. S4 with m-modalities) than S4.3 – suitable for linear orders.

We also intend to expose a fuzzy nature of preferences — often formulated with a different degree of a certainty in many practical situations as the one described below.

Example 1. We can prefer to go to the theater today afternoon with a preference $\frac{3}{10}$ *and whenever, but not today-with a preference* $\frac{7}{10}$ *and only in our birthday we are strongly motivated with a preference = 1 (modal necessity) to go to the theater.*

It is clear that a modal representation of preferences seems to be too sharp in this matter because—as based on two-valued reasoning. However, it seems to be reasonable to restrict a *spectrum* of possible fuzzy values associated with our preferences to their finite set $S \subseteq [0, 1]$. This *manouvre* will turn out reasonable from the technical point of view, as well. In essence, it ensures completeness of the considered system.

2.1 Fuzzy Modal Preferential Logic FMPL — Syntax and Semantics

It has already been said that preferences will be represented by us in an interval-based semantics as transitive and reflexive orders. Moreover, we decided to interpret their in a fuzzy manner. It will be reflected in types of operators that we introduce. In essence, we want to consider operators of a type $[\mathrm{Pref}]_i^\alpha \phi$ read: "an agent i (strongly) prefers ϕ with a degree $\alpha \in [0, 1]$", and $\langle \mathrm{Pref} \rangle_i^\alpha \phi$ "an agent i weakly prefers ϕ with a degree α"

A language $\mathcal{L}$ of FMPL is given by a grammar:

$$\phi \;:=\; p \,|\, \neg\phi \,|\, \phi \wedge \psi \,|\, [\mathrm{Pref}]_i^\alpha \phi \,|\, \langle \mathrm{Pref} \rangle_i^\alpha \phi$$

Axioms: The axioms of FMPL are: the axioms of Boolean propositional calculus, and

$[\mathrm{Pref}]_i^\alpha(\phi \to \chi) \to ([\mathrm{Pref}]_i^\alpha \phi \to [\mathrm{Pref}]_i^\alpha \chi)$ (axiom K)
$[\mathrm{Pref}]_i^\alpha \phi \to [\mathrm{Pref}]_i^\alpha [\mathrm{Pref}]_i^\alpha \phi$ (axiom 4)
$[\mathrm{Pref}]_i^\alpha \phi \to \phi$ (axiom T)

As inference rules we adopt *Modus Ponens* and the inference rule for $[\mathrm{Pref}]_i^\alpha$ operator: from $\phi \to \psi$ infer $[\mathrm{Pref}]_i^\alpha \phi \to [\mathrm{Pref}]_i^\alpha \phi$ for some $\alpha \in G$, with G being a finite subset of $[0, 1]$. The finiteness of G is important for the satisfiability of FMPL, and it enables the completeness for the point-wise semantics [4,5]. This will be discussed in the context of interval-based semantics.

As usual, we define FMPL as the smallest theory in a language $\mathcal{L}$(FMPL) which contains the above axioms and closed on the above inference rules.

FMPL in An Interval-Based Semantics and Satisfiability Problem. We intend to interpret this FMPL in an interval-based semantics. It will be based on an inductive definition of satisfaction for formulas of $\mathcal{L}$(FMPL) that we will introduce below. For this purpose assume as given an IBIS with a labeling function Lab defined earlier. Recall that if an interval $I = s_1 \ldots s_k$, than $Lab(I) = Lab(s_1 \ldots s_k) = (\widehat{s_1} \ldots, \widehat{s_1})$, where $\widehat{s_i}$ represents a point s_i in a given language. Assume also that $\sim_i$ is an accessibility relation between intervals (for some agent i). For two given intervals, say $I = s_1 \ldots s_k$ and $I' = s_1^{'} \ldots s_l^{'}$ we only assume wrt $\sim_i^{\alpha}$ the following: $I \sim_i^{\alpha} I'$ means that $k \leq l$ and j-local states of I and I' are identical (for $j \leq k$).

Naturally, the necessity degrees α — associated to modal formulas of FMPL — should be reflected in the accessibility relation $\sim_i$ interpreting these formulas. We will write: $I \sim^{\alpha} I'$ iff the relation $\sim (I, I')$ is associated to a value at least α (symb: $\alpha \leq \| \sim (I, I')\|$. In terms of a convention of Hajek we can say that intervals I and I' are at least α-similar [5]. It is easy to observe that $\sim_i^{\alpha}$ is a partial order, hence it can be an appropriate representation for preferences in the light of earlier arrangements.

Definition 5. *(**Satisfaction**) Given a formula $\phi \in \mathcal{L}$(FMPL), an IBIS and an interval I we define inductively the fact that ϕ is satisfied in IBIS and in an interval I (symb.$I \models \phi$) as follows:*

IBIS, $I \models p$ iff $p \in Lab(I)$ for all $p \in Var$.

IBIS, $I \models \neg\phi$ iff if its not such a case that IBIS, $I \models \phi$.

IBIS, $I \models \phi \wedge \psi$ iff there is such a case that IBIS, $I \models \phi$ and IBIS, $I \models \psi$.

IBIS, $I \models [\mathrm{Pref}]_i^{\alpha}\phi$, where $i \in A$, iff for all $I \sim_i^{\alpha} I'$ it holds IBIS, $I' \models \phi$ for a fixed α.

IBIS, $I \models \langle\mathrm{Pref}\rangle_i^{\alpha}\phi$, where $i \in A$, iff there is such I' that $I \sim_i^{\alpha}\ I'$ and it holds IBIS, $I' \models \phi$ for a fixed α.

The key clause of the above definition is this one referring to the modal operators $[\mathrm{Pref}]_i^{\alpha}\phi$ and $\langle\mathrm{Pref}\rangle_i^{\alpha}\phi$. These conditions assert that such modal formulas are satisfied in an interval I and model IBIS iff the same formula ϕ holds in all intervals accessible from this I in the sense of the accessible relation $\sim_i$ by an agent i.

Example 2. Consider a formula $[\mathrm{Pref}]_i^{\alpha}\phi$ and intervals I^k of the form: $s_1 s_2 \ldots s_k$ (for $k = 1, 2, \ldots 5$) that are labeled by a formula ϕ, and an interval $s_1 \ldots s_6$, which is not labeled by ϕ. Assume that $I \sim_i I^k$ and $I \sim_i s_1 \ldots s_6$ for some accessibility relation $\sim_i$. It is easy to observe that a formula $[\mathrm{Pref}]_i^{\alpha}\phi$ is not satisfied in I because not for all intervals accessible from, say I^{access}, it holds $I^{\mathrm{access}} \models \phi$. In fact, $s_1 \ldots s_6 \not\models \phi$ (due to above satisfaction definition).

We intend to justify the PSpace completeness of satisfiability problem for FMPL. The proof uses a reduction of a satisfiability problem for FMPL to the satisfiability problem for $S4_m$ that is known as PSpace-complete, [2]. In the proof we adopt the proof ideas from [9]. This proof idea requires, however, some additional justification. In fact, we said that accessibility relations in IBIS are associated to one of a values from a finite $G \in [0.1]$ in our case. Hence we will consider a slightly modified key relationships between accessibility relations in Kripke frames of the form $\mathrm{K} = \langle W, S_1 \ldots, S_k \rangle$ and in IBIS in the form: $I_1 S_i I_2$ in K $\iff I_1 \sim_i^\alpha I_2$ in IBIS, for $\alpha \in G$ and $i \leq k$. It means that we will consider only intervals et least α-similar to the initial ones in satisfaction condition for formulas of FMPL.

Theorem 1. *(Satisfiability + Completeness) FMPL is complete wrt the interval IBIS-based semantics. The satisfiability problem for FMPL is PSpace-complete.*

Proof. (Outline) This proof uses a reduction of a satisfiability problem for FMPL to the satisfiability problem for $S4_m$ what can be deduced as PSpace-complete because of the satisfiability problem for S4 — known as PSpace-complete, [2]. In the proof we adopt the proof ideas from [9].

In order to reduce the case of $\mathrm{S4}_m$ to the case of FMPL, it suffices to show that $IBIS, I \models_{FMPL} \phi \iff M \models_{S4_m} f(\phi)$ for some model M and a bijection f that transforms the formulas of *FMPL* into formulas of $\mathrm{S4}_m$ replacing each box-operator $[\ \ ]_j \phi$ of $\mathrm{S4}_m$ by $[\mathrm{Pref}]_i^\alpha \phi$, where $j \in \{1, \ldots, m\}$ and $\mathrm{card}\{G\} = m$. $(\rightarrow)$ Assume first that ϕ is satisfied in a model $\mathrm{IBIS} = (S, s_0, \sim_i^\alpha, t, Lab)$ and in an interval I. We aim at giving of such a Kripke model $\mathcal{M}$ that $f(\phi)$ is satisfied in it. This situation is, however, easy because it is enough to assume that a domain $|M|$ of $\mathcal{M}$ consists of the intervals from IBIS such that it holds for all i: $I_1 S_i I_2 in \mathcal{M} \iff I_1 \sim_i^\alpha I_2 in$IBIS for a fixed $\alpha \in G$. One can inductively check that $f(\phi)$ is satisfied in the Kripke model $\mathcal{M}$.

$(\leftarrow)$ Assume now that $(M, \omega) \models f(\phi)$ for $\mathcal{M} = (W, \omega_0, S_1, \ldots, S_m)$ and that M contains only the relations in ϕ. We can construct a model $\mathrm{IBIS} = \langle W \bigcup \{s_0\}, s_0, t, Lab \rangle$ with a new point s_0 as an initial one. A transition relation t is not empty because it contains (s_0, t) for some t-due to its definition. We finally define $Lab(s_0, \omega) = \pi(\omega)$. It remains only to justify by induction that ϕ is satisfied in the interval $s_0 \omega$ in IBIS.

We analyze only the case of the modal box-operator. For this purpose we assume inductively that $\mathrm{IBIS}, s_0 \omega \models \phi \iff M, \omega \models f(\phi)$. We want to prove that IBIS, $s_0 \omega \models [\mathrm{Pref}]_i^\alpha \phi$, or that $\forall s' w' (s_0 \omega \sim^\alpha s' \omega' \wedge s' \omega' \models \phi)$ for some fixed α. Assume also that $M, \omega \models [\mathrm{Pref}]_i^\alpha f(\phi)$, which is equivalent to $\forall \omega' (\omega S \omega' \wedge \omega' \models f(\phi))$. Meanwhile $\omega S \omega' \iff (s' = s_0$ and $s_0 \omega \sim^\alpha s' \omega')$ for all ω' and the relation S. Therefore we have $\forall \omega' (s' = s_0$ and $s_0 \omega \sim^\alpha s' \omega' \wedge \omega' \models f(\phi))$. The induction assumption allows us to deduce that $\forall s' \omega' (s' = s_0 \wedge s_0 \omega \sim^\alpha s' \omega' \wedge s' \omega' \models \phi)$ what proves this case.

The above procedure has been carried out for the fixed parameter $\alpha \in G \subseteq [0, 1]$. The same reasoning can be used for each parameter $\beta \in G$. The finiteness of G ensures that the complexity of the satisfiability problem for FMPL remains PSpace-complete. □

Note that an assuming of a linearity condition for preferences changes a complexity of the satisfiability problem. In fact, it holds the following theorem:

Theorem 2. *The satisfiability problem for for FMPL with a linearity condition for preferences is NP-complete.*

Proof. In essence, such a system can form a kind of a multi-modal $S4.3_m$ for some finite m. It has been shown in [2] that a satisfiability problem for S4.3 — even for the Horn clauses — is NP-complete. □

One can raise a question whether a finiteness condition imposed on a set of necessity degree can ensure a completeness of FMPL wrt the interval based semantics with discrete finite intervals, like in a pointwise semantics in [4]. This is not obvious, since it requires at least a density condition for the space of intervals. How inappropriate is the condition of a finite discreteness imposed on intervals, can be seen from the fact that S4remains complete wrt the Cantor set. Recall that this set can be defined as a set of all real numbers represented as $\sum_{i=1}^{\infty} a_i/3^i$ for $a_i \in \{0, 2\}$.

Theorem 3. *Assume that FMPL is such that a set of necessity degrees G has one fixed value α. Then it is complete wrt the Cantor space.*

Proof. It immediately follows from the fact that S4 is complete wrt to the Cantor space [14] and from the fact that such a restricted FMPL can be translated to S4. □

3 Comparative Multi-modal Preferential Logic and Its Difficulties

As it has been mentioned before, the preferences have a comparative nature. It means that having a preference, say $Pref_1$, with respect to a subject or activity A, we can have a possibility to compare $Pref_1$ with some other, say $Pref_2$, wrt to the same preference object or activity. In essence, we are usually willing to formulate our preferences *against* the other alternative ones.

Example 3. We put a preference P_1: 'going to the cinema today' with a value $\frac{3}{10}$ against a preference P_2 'goint to the cinema tomorrow" with a values $\frac{4}{25}$.

In the light of the above statements it seems to be reasonable to consider a new alternative approach to the preferences representation that will be more suitable for such a comparative nature of preferences. In fact: some A can be more preferable than B and B more than C *etc.* It remains to chose only the appropriate formal representation for such a property. The so-called *comparative possibilistic modal propositional logic* CPMPL with a one binary modality ∇–interpreted as $\|\phi \nabla \chi\| = 1$ iff possibility of ϕ is less than or equal to the possibility of χ–described in [5] can be suitable for such a task from the purely theoretical point of view.

In this section we introduce a slightly modified version of CPMPL as suitable for preferences representation. Secondly, we comparatively examine both FMPL and CPMPL in further subsections. This presentation will be, however, aimed at the exposing of the advantages of the earlier multi-valued preferential approach because–as we will argue later–this new comparative fuzzy approach is not so convenient as one can think. On the other hand, we observe an interesting fact that a rejecting of a linearity condition is not required to preserve a completeness in a case of CPMPL-formalism.

3.1 Comparative Possibilistic Multi-modal Preferential Logic-Syntax and Semantics

As this alternative fuzzy approach to the preferences representation we adopt a multi-modal system CPMPL proposed in [6]–in a slight modified version. CPMPL is built in a language given by a grammar:

$\phi ::= \phi \mid \neg\phi \mid \phi \wedge \chi \mid \phi \triangle \chi$.

Because a comparison between preferences is both a transitive relation (if A is weaker preferable than B and B weaker than C, than than A is weaker preferable than C) and a linear one (either A is more preferable than B, or B is more preferable than A), we adopt the following ***axioms system*:**

- axioms of propositional calculus,
- $(\phi\triangle\chi) \rightarrow ((\chi\triangle\psi) \rightarrow (\phi\triangle\psi)$, (transitivity)
- $(\phi\triangle\chi) \vee (\chi\triangle\phi)$, (linearity)
- $(\phi\triangle\chi) \rightarrow ((\phi \vee \chi)\triangle(\chi \vee \psi))$ (disjunction)
- $(\phi\triangle\chi) \rightarrow [\,](\phi\triangle\chi)$ (boxing 1).

As inference rules we adopt *Modus Ponens* and necessitation rule of the form: from $\phi \rightarrow \chi$ deduce $\phi\triangle\chi$.

CPMPL will be interpreted by us in a Kripke frames $\mathcal{K} = (W, \mu, e)$, where $W \neq \emptyset$ is a set of states, μ is a probability measure and $e : \mathcal{L}(CPMPL) \mapsto [0, 1]$ is a standard fuzzy evaluation function. We adopt the following interpretation of formulas in a model $\mathcal{K}$: $\|\phi\triangle\chi\|_M = \mu(\phi) \leq \mu(\chi)$ (probability of χ is greater than probability of ϕ) for $w = \|\phi\|_K$ and $v = \|\chi\|_K$.

Example 4. Assume that ϕ_C = 'It is preferable to visit a cinema today' and χ_T = It is preferable to visit a theather today' and $\|\phi_C\| = \frac{45}{100}$ and $\|\phi_T\| = \frac{21}{100}$. Than $\|\phi_T\| = \mu$(visiting a cinema) $= \frac{21}{100} \leq \frac{45}{100} = \mu$(visiting a theater) and $\|\phi_T\triangle\chi_C\|$ is satisfied in some model K.

3.2 Completeness of CPMPL

It has been shown in [6](see also: [5]) that CPMPL is complete wrt to the proposed semantics. This proof leads by an reinterpretation of this system in some fuzzy tense system, called MTL in [6]. The proof idea bases on a translation * of the language $\mathcal{L}$(CPMPL) into the language $\mathcal{L}$(MTL) such that: $p^* = p$, $(\phi\triangle\chi)^* = [\,](\phi^* \rightarrow F\chi^*)$, where F is a temporal operator 'always in the future' of LTL and [ ] is a normal modal box-type operator. We omit the details of the proof that can be easily found in [6], pp. 16-17. Meanwhile we will propose a short outline of the new completeness proof of this system.

Idea of our proof will be based on the alternative translation of formulas of CPMPL into propositional variables of so-called Rational Pavelka Logic RPL. The idea of translation is easy: bimodal formula, say $\odot(k, j)$, is represented by double-indexed propositional variable $p_{k,j}$; unary modal operator, say $\odot j$,–by a singularly indexed propositional variable p_j.

Our proof-similar to the ideas of of completeness proof for Rational Pavelka logic from [5] will be essentially based on the definition of so-called *truth degree* and *provability degree* defined as: $\|\phi\|_T = inf\{\|\phi\|_M : M$ is a model of $T\}$ and $|\phi|_T = sup\{r : T \vdash (\widehat{r} \rightarrow \phi)\}$ (*resp.*) for an arbitrary formula $\phi \in \mathcal{L}(RPL)$. In such a terminological framework, the Pavelka style of completeness proofs consists of a proving of a unique equality:

$$\|\phi|_T = |\phi\|_T \; {}^{2}.$$

In order to prove the CPMPL-completeness we also adopt the following lemmas (without their proofs).

Lemma 1. *([5], p. 130) If $T \nvdash \widehat{r} \rightarrow \phi$ then the theory $T \cup \{\phi \rightarrow \widehat{r}\}$ is consistent.*

Lemma 2. *([5], p. 81) Let T be consistent and complete. Then the evaluation $e(p_i) = |p_i|$ is a model for T.*

Theorem 4. ***Completeness*** *CPMPL is complete wrt to the proposed semantics. More precisely, for each theory T in a language $\mathcal{L}(CPMPL)$ and a formula ϕ of this language it holds: $|\phi|_T = \|\phi\|_T$.*

Proof. (Short outline). For a use of the proof we should make a simple translation of the bi-modal CPMPL-formulas $\phi \bigtriangleup \chi$ into the propositional variables $p_{\phi,\chi}$– similarly as it has been made by a translation of the formulas of fuzzy probability system FP discussed in [5], p. 235.

In order to justify the theorem we have to justify that both $|\phi|_T \leq \|\phi\|_T$ and $\|\phi\|_T \leq |\phi|_T$. Meanwhile, the proof for $|\phi|_T \leq \|\phi\|_T$ (soundness) is routine. In order to prove the converse we must show that for each rational $r < \|\phi\|, T \vdash \widehat{r} \rightarrow \phi$, where $\widehat{r}$ is a representation of r in a language of CPMPL. By a transposition it is enough to show that if T does *not* prove $\widehat{r} \rightarrow \phi$, than $\|\phi\| \leq r$. Assume therefore that $T \nvdash \widehat{r} \rightarrow \phi$ what means that $T \cup \{\phi \rightarrow \widehat{r}\}$ is consistent and it must have a complete Lindenbaum extension, say $T^{'}$. Then-by the above lemma– an evaluation $e(p_i) = |p_i|$ and $e(\phi \rightarrow \widehat{r}) = 1$, thus $e(\phi) \leq r$. In particular, it means that $e(p_{\phi,\ chi}) = |p_{\phi,\chi}|$ and $e((\phi \triangle \chi) \rightarrow \widehat{r}) = 1$, thus $e(\phi \triangle \chi) \leq r$ what finishes our proof. □

Remark 1. Note that the translation of the formulas can be also carried out in two steps. Firstly, we can represent a CPMPL-formulas $\phi \triangle \chi$ by an unary modal operator $P(\phi|\chi)$ for a conditional probability-read as: 'probability of χ under a condition ϕ' in the appropriate language. In the second step, we can represent $P(\phi|\chi)$ by a single propositional variable $p_{\phi,\chi}$.

3.3 Further Properties of CPMPL and Its Difficulties....

We have just shown how to arrive at the completeness theorem *via* method of formulas translation. It is not difficult to observe that the above proof is essentially based on the earlier lemmas and the same translation between formulas of CPMPL and formulas of Pavelka's system. This situation seems to correspond well with

[2] It exactly means that a formula ϕ is equally provable in a theory T as satisfiable in its models.

the situation exposed in [6] of the similarly conditioned proving of completeness of CPMPL *via* referring to a fuzzy tense logic. Hence, one can state that a completeness proof for CPMPL cannot be so direct like in a case of FMPL where we make use of the direct translation to better-known completeness of $S4.3_m$. Moreover, the CPMPL-completeness does not solve the model checking problem. This problem will be shortly discussed in this subsection together with further properties of CPMPL and its difficulties such as interval-based semantics construction.

Interval-Based Semantics Construction. The pointwise Kripke fuzzy semantics for CPMPL has already been introduced in outline. The interval-based semantics construction seems to be, however, more problematic one for the following reasons–if it is possible at all.

• Assuming that formulas ϕ and χ (from $\phi\triangle\chi$) will be interpreted in intervals, say I_1 and I_2 (*resp.*), it is not clear how to define relationships between I_1 and I_2. Fuzzy logic semantics-in a contrast with a modal logic one-is not necessary based on the accessibility relations between states or intervals.

• It is not clear whether the relations of intervals for CPMPL-formulas have/or should have more temporal or spatial nature.

• Even if the interval-based semantics construction is possible, it is not clear which of the spatial-temporal interval relations (for example of Allen's sort) can be suitable for an interpretation of $\phi\triangle\chi$. It appears that this formula requires more a quantitative interpretation approach than the qualitative one. Indeed, $\phi\triangle\chi$ can be much better semantically interpreted by a fact that a *measure* of an interval I_1 (associated with ϕ) is *smaller* than a *measure* of I_2 associated with χ than by the temporal-spatial relationships between I_1 and I_2 such as: *meet, overlap, begin etc.*

Satisfiability Problem. In order to consider a satisfiability problem for CPMPL return to the idea of the representation of a formula $\phi\triangle\chi$ by $[\,](\phi \rightarrow F\chi)$ for F being a temporal 'future' operator of linear temporal logic LTL. Such a representation justifies that a CPMPL-satisfiability problem is not only more complicated than a model checking for LTL, but also for our FMPL that does not contain such 'mixed' modalities. Meanwhile, it is known that the model checking for LTL is in PSpace and–even–PSpace-hard. This first fact was solved via reduction to the formulas satisfiability problem(see: [12], p. 15); the second one-by a reduction to the tiling problem (see: [12], p. 18-19). It allows us to formulate the following

Corollary 1. *The satisfiability problem for CPMPL is (at least) in PSpace and is even (at least) PSpace-hard.*

The direct justification of this fact should be, however, (at least) slightly more complicated that the satisfiability proof for FMPL and systems with separate, non-combined modalities.

All the above remarks on the CPMPL-satisfiability problem refer to the pointwise semantics. Meanwhile–in the light of the exposed difficulties with the interval-based semantics construction–the CPMPL-satisfiability problem for this semantics type cannot be even formulated.

4 FMPL *versus* Comparative Fuzzy Preferential Logic

We venture to formulate a handful of comparative remarks on both systems for preferences representation. There are handful similarities between them, but also a handful of differences. Both FMPL and CPMPL are complete and the completeness can be obtained by a referring to the other systems earlier recognized as complete ones and *via* an appropriate formulas translation. CPMPL grasps the comparative nature of preferences, but it has worst technical and meta-logical properties. The detailed comparison is given in the table below.

Properties	**Fuzzy-Modal Preferential Logic**	**Comparative Multi-M. Preferential Logic**
•*completeness*	yes	yes
•*proof of completeness*	*via* formulas translation, useful in a model checking problem solving	*via* formulas translation
•*nature of logic*	multi-valued	fuzzy
•*comparative nature of preferences*	non-reflected	reflected
•*semantics type*	pointwise, interval-based	only pointwise
•*nature of semantics*	Kripke modal	Kripke fuzzy
•*satisfiability problem*	PSPACE	(at least) PSPACE
•*satisfiability problem for interval-based semantics*	in PSPACE, PSPACE-hard	non exists
•*ability for a combination with other system (for actions, temporal relations etc.)*	high	restricted, non direct

5 Further Implementations and Concluding Remarks

In this paper we put forward a comparative analysis of two 'candidatures' for a role of the appropriate multi-valued/fuzzy logic for preferences representation. In order to single out the best logic, we considered a handful of their properties such as completeness, satisfiability problems and a problem of the interval-based semantics construction. We concluded that–although a comparative fuzzy preferential logic better grasp the comparative nature of preferences than the alternative system–it shows worst meta-logical properties. It turns out that *Fuzzy-Modal Preferential Logic* shows a better ability for a combination with other systems for actions, temporal relations *etc.*

This last property–a system's ability to be combined–is especially important from the point of view of a long term goal of our research. In future we just intend to develop hybrid models for temporal reasoning with preferences and other constraints. FMPL seems to be also more able to be extended to a new system with preferences for two agents groups that can express the mutually inconsistent preferences. It appears that the Allen's detecting inconsistency algorithm from [1] can be easily implemented for such a system. This issue requires, however, a more detailed analysis.

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