

# Efficiently Listing Bounded Length $st$ -Paths

Romeo Rizzi<sup>1</sup>, Gustavo Sacomoto<sup>2,3(✉)</sup>, and Marie-France Sagot<sup>2,3,4</sup>

<sup>1</sup> Dipartimento di Informatica, Università di Verona, Verona, Italy

<sup>2</sup> Université de Lyon, 69000 Lyon, France

sacomoto@gmail.com

<sup>3</sup> CNRS, UMR5558, Laboratoire de Biométrie et Biologie Évolutive,  
Université Lyon 1, 69622 Villeurbanne, France

<sup>4</sup> INRIA Grenoble Rhône-Alpes, Montbonnot-saint-martin, France

**Abstract.** The problem of listing the  $K$  shortest simple (loopless)  $st$ -paths in a graph has been studied since the early 1960s. For a non-negatively weighted graph with  $n$  vertices and  $m$  edges, the most efficient solution is an  $O(K(mn + n^2 \log n))$  algorithm for directed graphs by Yen and Lawler [Management Science, 1971 and 1972], and an  $O(K(m + n \log n))$  algorithm for the undirected version by Katoh *et al.* [Networks, 1982], both using  $O(Kn + m)$  space. In this work, we consider a different parameterization for this problem: instead of bounding the number of  $st$ -paths output, we bound their length. For the bounded length parameterization, we propose new non-trivial algorithms matching the time complexity of the classic algorithms but using only  $O(m+n)$  space. Moreover, we provide a unified framework such that the solutions to both parameterizations – the classic  $K$ -shortest and the new length-bounded paths – can be seen as two different traversals of a same tree, a Dijkstra-like and a DFS-like traversal, respectively.

## 1 Introduction

The  $K$ -shortest simple paths problem has been studied for more than 50 years (see the references in [6]). The first efficient algorithm for this problem in directed graphs with non-negative weights only appeared 10 years later independently by Yen [18] and Lawler [12]. Given a non-negatively weighted directed graph  $G = (V, E)$  with  $n = |V|$  vertices and  $m = |E|$  edges, using modern data structures [1], their algorithm lists the  $K$  distinct shortest *simple*  $st$ -paths by non-decreasing order of their lengths in  $O(K(mn + n^2 \log n))$  time. For undirected graphs, Katoh *et al.* [11] gave an improved  $O(K(m + n \log n))$  algorithm. Both algorithms use  $O(Kn + m)$  memory.

The best known algorithm for directed *unweighted* graphs is an  $\tilde{O}(Km\sqrt{n})$  randomized algorithm [16], where  $\tilde{O}(f(n))$  is a shorthand for  $O(f(n) \log^k n)$ . In a different direction, Roditty [15] noticed that the  $K$ -shortest simple paths can

---

GS and MFS were partially supported by the ERC programme FP7/2007-2013/ERC grant agreement no. [247073]10, and the French project ANR-12-BS02-0008 (Colib’read).

be efficiently approximated. Building upon his work, Bernstein [2] presented an  $\tilde{O}(Km/\epsilon)$  time algorithm for a  $(1 + \epsilon)$ -approximation. Moreover, Eppstein [7] showed that if the paths are allowed to repeat vertices, *i.e.* they are not *simple*, then the problem can be solved in  $O(K + m + n \log n)$  time. However, when the paths are *simple* and to be computed exactly, no improvement has been made on Yen and Lawler's for directed graphs or Katoh's algorithm for undirected graphs. The main bottleneck of these algorithms is their memory consumption.

Here, we consider the problem of listing all  $st$ -paths with length at most  $\alpha$ . This is a different parameterization of the  $K$ -shortest path problem, where we impose an upper-bound on the length of the output paths instead of their number. This is a natural variant of the  $K$ -shortest path problem. There are situations where it is necessary to consider all paths that are a given percentage of the optimal (*e.g.* [4]). Moreover, the bounded length problem is *almost* a particular case of the  $K$ -shortest path problem. Given any solution to the  $K$ -shortest path problem, such that the  $st$ -paths are generated one at a time in non-decreasing length order, we can use the following simple approach to solve the  $\alpha$ -bounded length variant: choose a sufficiently large  $K$  and halt the enumeration when the length of the paths is larger than  $\alpha$ . The main disadvantage of this algorithm is its space complexity which is proportional to the number of paths output hence, in the worst case, exponential in the size of the graph.

Our first and main contribution are new polynomial delay algorithms to list  $st$ -paths with length at most  $\alpha$  matching the time complexity (per path) of Yen and Lawler's algorithm for directed graphs (Sect. 3) and Katoh's for undirected graphs (Sect. 4), but using only  $O(n + m)$  internal memory. This represents an exponential improvement in memory consumption.

The main differences between the classic solutions to the  $K$ -shortest paths problem and our solutions to the  $\alpha$ -bounded paths problem are the order in which the solutions are output and the memory complexity of the algorithms.

Our second contribution is thus a unified framework where both problems can be represented in such a way that those differences arise in a natural manner (Sect. 3). Intuitively, we show that both families of algorithmic solutions correspond to two different traversals of a *same* rooted tree: a Dijkstra-like traversal for the  $K$ -shortest and a DFS-like traversal for the  $\alpha$ -bounded paths.

## 2 Preliminaries

Given a directed graph  $G = (V, E)$  with  $n = |V|$  vertices and  $m = |E|$  arcs, the in and out-neighborhoods of  $v \in V$  are denoted by  $N^-(v)$  and  $N^+(v)$ , respectively. Given a (directed or undirected) graph  $G$  with weights  $w : E \mapsto \mathbb{Q}$ , the weight, or *length*, of a path  $\pi$  is  $\sum_{(u,v) \in \pi} w(u,v)$  and is denoted by  $w(\pi)$ . We say that a path  $\pi$  is  $\alpha$ -bounded if its *length* satisfies  $w(\pi) \leq \alpha$  and  $\alpha \in \mathbb{Q}$ ; in the particular case of unit weights (*i.e.* of unweighted graphs), we say that  $\pi$  is  $k$ -bounded if  $w(\pi) \leq k$  with  $k \in \mathbb{Z}_{\geq 0}$ . A listing algorithm is *polynomial delay* if it generates the solutions, one after the other in some order, and the time elapsed until the first is output, and thereafter the time elapsed (delay) between any two consecutive

solutions, is bounded by a polynomial in the input size [9]. The general problem which we are concerned in this work is listing  $\alpha$ -bounded  $st$ -paths in  $G$ .

*Problem 1 (Listing  $\alpha$ -bounded  $st$ -paths).* Given a weighted directed graph  $G = (V, E)$ , two vertices  $s, t \in V$ , and an upper bound  $\alpha \in \mathbb{Q}$ , output all  $\alpha$ -bounded  $st$ -paths.

Clearly, any solution to the  $K$ -shortest path problem is also a solution to Problem 1, with the same (total/delay) time and space complexities. Thus Problem 1 is no harder than the classic  $K$ -shortest path problem.

We assume all directed graphs are weakly connected and all undirected graphs are connected, hence  $m \geq n - 1$ . Moreover, we assume hereafter the weights are non-negative. We remark however that a weaker assumption suffices to the applicability of our algorithms. Indeed, it is a well known fact that, when the graph  $G$  and the weights  $w : E \mapsto \mathbb{Q}$  are such that no cycle is negative, then, using Johnson's reweighting strategy [10], we can compute non-negative weights  $w'$  such that, for some constant  $C$ , we have that  $w'(\pi) = w(\pi) + C$  for any  $st$ -path  $\pi$ . This reweighting can be done in  $O(mn)$  preprocessing steps.

### 3 An $O(mn + n^2 \log n)$ -Delay Algorithm

In this section, we present an  $O(mn + n^2 \log n)$ -delay algorithm to list all  $st$ -paths with length at most  $\alpha$  in a weighted directed graph  $G$ . Thus matching the time complexity (per path) of Yen and Lawler's algorithm, while using only space linear in the *input* size.

The new algorithm, inspired by the binary partition method [3, 14], recursively partitions the solution space at every call until the considered subspace is a singleton (contains only one solution) and in that case outputs the corresponding solution. In order to have an efficient algorithm is important to explore only non-empty partitions. Moreover, it should be stressed that the order in which the solutions are output is fixed, but arbitrary.

Let us describe the partition scheme. Let  $\mathcal{P}_\alpha(s, t, G)$  be the set of all  $\alpha$ -bounded paths from  $s$  to  $t$  in  $G$ , and  $(x, s) \cdot \mathcal{P}_\alpha(s, t, G)$  denote the concatenation of  $(x, s)$  to each path of  $\mathcal{P}_\alpha(s, t, G)$ . Assuming  $s \neq t$ , we have that

$$\mathcal{P}_\alpha(s, t, G) = \bigcup_{v \in N^+(s)} (s, v) \cdot \mathcal{P}_{\alpha'}(v, t, G - s), \quad (1)$$

where  $\alpha' = \alpha - w(s, v)$ . In words, the set of paths from  $s$  to  $t$  can be *partitioned* into the disjoint union of  $(s, v) \cdot \mathcal{P}_{\alpha'}(v, t, G - s)$ , the sets of paths beginning with an arc  $(s, v)$ , for each  $v \in N^+(s)$ . Indeed, since  $s \neq t$ , every path in  $\mathcal{P}_\alpha(s, t, G)$  necessarily begins with an arc  $(s, v)$ , where  $v \in N^+(s)$ .

Algorithm 1 implements this recursive partition strategy. The solutions are only output in the leaves of the recursion tree (line 2), where the partition is always a singleton. Moreover, in order to guarantee that every leaf in the recursion tree outputs one solution, we have to test if  $\mathcal{P}_{\alpha'}(v, t, G - u)$ , where

$\alpha' = \alpha - w(u, v)$ , is not empty before the recursive call (line 7). This set is not empty if and only if the weight of the shortest path from  $v$  to  $t$  in  $G - u$  is at most  $\alpha'$ , *i.e.*  $d_{G-u}(v, t) \leq \alpha' = \alpha - w(u, v)$ . Hence, to perform this test it is enough to compute all the distances from  $t$  in the graph  $G^R - u$ , where  $G^R$  is the graph  $G$  with all arcs reversed.

Consider a generic execution of Algorithm 1 for a graph  $G$ , vertices  $s, t \in V$  and an upper bound  $\alpha$ . We can represent this execution by a rooted tree  $\mathcal{T}$ , *i.e.* the recursion tree, where each node corresponds to a call with arguments  $\langle u, t, \alpha, \pi_{su}, G' \rangle$ . The children of a given node (call) in  $\mathcal{T}$  are the recursive calls with arguments  $\langle v, t, \alpha', \pi_{su}(u, v), G' - u \rangle$  of line 8. This tree plays an important role in the unified framework of Sect. 5.

**Lemma 1.** *The recursion tree  $\mathcal{T}$  has the following properties:*

1. *The leaves of  $\mathcal{T}$  are in one-to-one correspondence with the paths in  $\mathcal{P}_\alpha(s, t, G)$ .*
2. *The leaves in the subtree rooted on a node  $\langle u, t, \alpha, \pi_{su}, G' \rangle$  correspond to the paths in  $\pi_{su} \cdot \mathcal{P}_{\alpha'}(u, t, G')$ .*
3. *The height of  $\mathcal{T}$  is bounded by  $n$ .*

---

**Algorithm 1.** `list_paths`( $u, t, \alpha, \pi_{su}, G$ )

---

```

1 if  $u = t$  then
2   |   output( $\pi_{su}$ )
3   |   return
4 end
5 compute the distances from  $t$  in  $G^R - u$ 
6 for  $v \in N^+(u)$  do
7   |   if  $d(v, t) \leq \alpha - w(u, v)$  then
8   |   |   list_paths( $v, t, \alpha - w(u, v), \pi_{su} \cdot (u, v), G - u$ )
9   |   end
10 end
```

---

The correctness of Algorithm 1 follows directly from the relation given in Eq. 1 and the correctness of the tests of line 7.

Let us now analyze its running time. The cost of a node in  $\mathcal{T}$  is the time spent by the operations inside the corresponding call, without including its recursive calls. This cost is dominated by the tests of line 7. They are performed in  $O(1)$  time by pre-computing the distances from  $t$  to all vertices in the reverse graph  $G^R - u$  (line 5). This takes  $O(t(n, m))$  time, where  $t(n, m)$  is the cost of a single source shortest path computation. By Lemma 1 the height of  $\mathcal{T}$  is bounded by  $n$ , so the path between any two leaves (solutions) in the recursion tree has at most  $2n$  nodes. Thus, the time elapsed between two solutions being output is  $O(nt(n, m))$ . Moreover, the algorithm uses  $O(m)$  space, since each recursive call has to store only the difference with the its parent graph. Recall that each solution is immediately output (line 2), not stored by the algorithm.

**Theorem 1.** *Algorithm 1 has delay  $O(nt(n, m))$ , where  $t(n, m)$  is the cost of a single source shortest path computation, and uses  $O(m)$  space.*

For unweighted (directed and undirected) graphs, the single source shortest paths can be computed using breadth-first search (BFS) running in  $O(m)$  time, so Theorem 1 guarantees an  $O(km)$  delay to list all  $k$ -bounded  $st$ -paths, since the height of the recursion tree is bounded by  $k$  instead of  $n$ . More generally, the single source shortest paths can be computed using Dijkstra’s algorithm in  $O(m + n \log n)$  time (we are assuming non-negative weights), resulting in an  $O(nm + n^2 \log n)$  delay.

## 4 An Improved Algorithm for Undirected Graphs

The total time complexity of Algorithm 1 is equal to the delay times the number of solutions, *i.e.*  $O(nt(n, m)\gamma)$ , where  $\gamma = |\mathcal{P}_\alpha(s, t, G)|$  is the number of  $\alpha$ -bounded  $st$ -paths. We now improve its total time complexity from  $O(nt(n, m)\gamma)$  to  $O((m + t(n, m))\gamma)$  in the case of weighted *undirected* graphs. On average the algorithm spends  $O(m + t(n, m))$  per solution (amortized delay), thus matching the time complexity (per path) of Katoh’s algorithm. The (worst-case) delay, however, remains the same as Algorithm 1.

The main idea to improve the complexity of Algorithm 1 is to explore the structure of the set of paths  $\mathcal{P}_\alpha(s, t, G)$  to reduce the number of nodes in the recursion tree. We avoid redundant partition steps by guaranteeing that every node in the recursion tree has at least two children. More precisely, at every call, we identify the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$ , *i.e.* the longest (considering the number of edges) path  $\pi_{ss'}$  such that  $\mathcal{P}_\alpha(s, t, G) = \pi_{ss'} \cdot \mathcal{P}_\alpha(s', t, G)$ , and append it to the current path prefix being considered in the recursive call. The intuition here is that by doing so we identify and “merge” all the consecutive single-child nodes in the recursion tree, thus guaranteeing that the remaining nodes have at least two children.

The pseudocode for this algorithm is very similar to Algorithm 1 and, for the sake of completeness, is given in Algorithm 2. We postpone the description of the  $\text{1cp}(u, t, \alpha, G)$  function to the next section, along with a discussion about the difficulties to extend it to directed graphs.

The correctness of Algorithm 2 follows directly from the correctness of Algorithm 1. The space used is the same of Algorithm 1, provided that  $\text{1cp}(u, t, \alpha, G)$  uses linear space, which, as we show in the next section, is indeed the case (Theorem 3).

Let us now analyze the total complexity of Algorithm 2 as a function of the input size and of  $\gamma$ , the number of  $\alpha$ -bounded  $st$ -paths. Let  $R$  be the recursion tree of Algorithm 2 and  $T(r)$  the cost of a given node  $r \in R$ . The total cost of the algorithm can be split in two parts, which we later bound individually, in the following way:

$$\sum_{r \in R} T(r) = \sum_{r: \text{internal}} T(r) + \sum_{r: \text{leaf}} T(r). \quad (2)$$

---

**Algorithm 2.** `list_paths`( $u, t, \alpha, \pi_{su}, G$ )
 

---

```

1  $\pi_{uu'} = \text{lcp}(u, t, \alpha, G)$ 
2 if  $u' = t$  then
3   |   output( $\pi_{su}\pi_{uu'}$ )
4   |   return
5 else
6   |   compute a shortest path tree  $T'_t$  from  $t$  in  $G^R - \pi_{uu'}$ 
7   |   for  $v \in N(u')$  do
8   |     |   if  $d(v, t) + w(u, v) \leq \alpha$  then
9   |       |   list_paths( $v, t, \alpha - w(\pi_{uu'}) - w(u', v), \pi_{su} \cdot \pi_{uu'} \cdot (u', v), G - \pi_{uu'}$ )
10  |     |   end
11  |     end
12 end
    
```

---

We have that  $\sum_{r:\text{leaf}} T(r) = O((m + t(m, n))\gamma)$ , since leaves and solutions are in one-to-one correspondence and the cost for each leaf is dominated by the cost of  $\text{lcp}(u, t, \alpha, G)$ , that is  $O(m + t(m, n))$  (Theorem 3). Now, we have that every internal node of the recursion has at least two children, otherwise  $\pi_{uu'}$  would not be the longest common prefix of  $\mathcal{P}_\alpha(u, t, G)$ . Thus,  $\sum_{r:\text{internal}} T(r) = O((m + t(m, n))\gamma)$  since in any tree the number of branching nodes is at most the number of leaves, and the cost of each internal node is dominated by the  $O(m + t(m, n))$  cost of the longest prefix computation. Therefore, the total complexity of Algorithm 2 is  $O((m + t(n, m))\gamma)$ . This completes the proof of Theorem 2.

**Theorem 2.** *Algorithm 2 outputs all  $\alpha$ -bounded  $st$ -paths in  $O((m + t(n, m))\gamma)$  time using  $O(m)$  space.*

This means that for unweighted graphs, it is possible to list all  $k$ -bounded  $st$ -paths in  $O(m)$  time per path. In addition, for weighted graphs, it is possible to list all  $\alpha$ -bounded  $st$ -paths in  $O(m + n \log n)$  time per path.

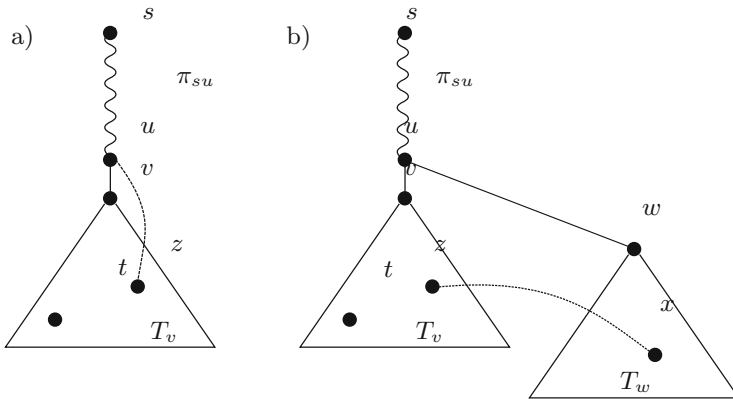
#### 4.1 Computing the Longest Common Prefix of $\mathcal{P}_\alpha(s, t, G)$

The problem of computing the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$  can be seen as a special case of the *replacement paths problem* [8]. Let  $\pi$  be a shortest  $st$ -path in  $G$ . In this problem we want to compute, for each edge  $e$  on  $\pi$ , the shortest  $st$ -path that avoids  $e$ . Given a solution to the replacement path problem we can compute the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$  using the following procedure. For each edge  $e$  along the path  $\pi$ , check whether the shortest  $st$ -path avoiding  $e$  is shorter than  $\alpha$ . There is an  $O(m + n \log n)$  algorithm to compute the replacement path in undirected graphs [13], but for directed graphs the best solution is a trivial  $O(nm + n^2 \log n)$  algorithm.

In this section, we present an alternative, arguably simpler, algorithm to compute the longest common prefix of the set of  $\alpha$ -paths from  $s$  to  $t$ , completing the description of Algorithm 2. The naive algorithm for this problem runs in

$O(nt(n, m))$  time, so that using it in Algorithm 2 would not improve the total complexity compared to Algorithm 1. Basically, the naive algorithm computes a shortest path  $\pi_{st}$  and then for each prefix in increasing order of length tests if there are at least two distinct extensions each with total weight less than  $\alpha$ . In order to test the extensions, for each prefix  $\pi_{su}$ , we recompute the distances from  $t$  in the graph  $G - \pi_{su}$ , thus performing  $n$  shortest path tree computations ( $k$  computations in the unweighted case) in the worst case.

Algorithm 3 improves the naive algorithm by avoiding those recomputations. However, before entering the description of Algorithm 3, we need a better characterization of the structure of the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$ . Lemma 2 gives this. It does so by considering a shortest path tree rooted at  $s$ , denoted by  $T_s$ . Recall that  $T_s$  is a subgraph of  $G$  and induces a partition of the edges of  $G$  into tree edges and non-tree edges. In this tree, the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$  is a prefix of the tree path from the root  $s$  to  $t$ . Additionally, any  $st$ -path in  $G$ , excluding the tree path, necessarily passes through at least one non-tree edge. The lemma characterizes the longest common prefix in terms of the non-tree edges from the subtrees rooted at siblings of the vertices in the tree path from  $s$  to  $t$ . For instance, in Fig. 1(b) the common prefix  $\pi_{su}$  can be extended to  $\pi_{su} \cdot (u, v)$  only if there is no  $\alpha$ -bounded path that passes through the subtree  $T_w$  and a non-tree edge  $(x, z)$ , where  $v$  belongs to tree path from  $s$  to  $t$  and  $w$  is one of its siblings.



**Fig. 1.** The common prefix  $\pi_{su}$  of  $\mathcal{P}_\alpha(s, t, G)$  can always be extended into an  $st$ -path using the tree path of  $T_s$  from  $u$  to  $t$ . The path  $\pi_{su}$  is the longest common prefix if and only if it can also be extended with a path containing a non-tree edge  $(x, z)$  such that  $z \in T_v$  and (a)  $x = u$  or (b)  $x \in T_w$  and  $w$  is sibling of  $v$ ; and  $d_{G'}(s, x) + w(x, z) + d_{G'}(z, t) \leq \alpha$ , where  $G' = G - (u, v)$ .

**Lemma 2.** Let  $\pi_{su} = (s = v_0, v_1), \dots, (v_{l-1}, v_l = u)$  be a common prefix of all paths in  $\mathcal{P}_\alpha(s, t, G) \neq \emptyset$  and  $T_s$  a shortest path tree rooted at  $s$ . Then,

1. the path  $\pi_{su}(u, v)$  is a common prefix of  $\mathcal{P}_\alpha(s, t, G)$ , if there is no edge  $(x, z)$  such that  $d_{G'}(s, x) + w(x, z) + d_{G'}(z, t) \leq \alpha$ , where  $G' = G - (u, v)$ ,  $z \in T_v$ , and (a)  $x = u$  or (b)  $x \in T_w$  with  $w$  a sibling of  $v$  (see Fig. 1);
2.  $\pi_{su}$  is the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$ , otherwise.

In order to use the characterization of Lemma 2 for the longest prefix of  $\mathcal{P}_\alpha(s, t, G)$ , we need to efficiently test the weight condition given in item 1, namely  $d_{G'}(s, x) + w(x, z) + d_{G'}(z, t) \leq \alpha$ , where  $G' = G - (u, v)$  and  $(u, v)$  belongs to the tree path from  $s$  to  $t$ . We have that  $d_{G'}(s, x) = d_G(s, x)$ , since  $x$  does not belong to the subtree of  $v$  in the shortest path tree  $T_s$ . Indeed, only the distances of vertices in the subtree  $T_v$  can possibly change after the removal of the tree edge  $(u, v)$ . However, in principle we have no guarantee that  $d_{G'}(z, t)$  also remains unchanged: recall that to maintain the distances from  $t$  we need a tree rooted at  $t$  not at  $s$ . Clearly, we cannot compute the shortest path tree from  $t$  for each  $G'$ ; in the worst case, this would imply the computation of  $n$  shortest path trees. For this reason, we need Lemma 3. It states that, in the specific case of the vertices  $z$  we need to compute the distance to  $t$  in  $G'$ , we have that  $d_{G'}(z, t) = d_G(z, t)$ .

**Lemma 3.** *Let  $T_s$  be a shortest path tree rooted at  $s$  and  $t$  a vertex of  $G$ . Then, for any edge  $(u, v)$ , with  $v$  closer to  $t$ , in the shortest path  $\pi_{st}$  in the tree  $T_s$ , we have that  $d_G(z, t) = d_{G'}(z, t)$ , where  $z \in T_v$  and  $G' = G - (u, v)$ .*

It is not hard to verify that Lemma 2 is also valid for directed graphs. However, the non-negative hypothesis for the weights is necessary; more specifically, we need the monotonicity property for path weights which states that for any path the weight of any subpath is not greater than the weight of the full path. Now, in Lemma 3 both the path monotonicity property and the fact that the graph is undirected are necessary. Since these two lemmas are the basis for the efficiency of Algorithm 3, it seems difficult to extend it to directed graphs.

Algorithm 3 implements the strategy suggested by Lemma 2. Given a shortest path tree  $T_s$  of  $G$  rooted at  $s$ , the algorithm traverses each vertex  $v_i$  in the tree path  $s = v_0 \dots v_n = t$  from the root  $s$  to  $t$ , and at every step finds all non-tree edges  $(x, z)$  entering the subtree rooted at  $v_{i+1}$  from a sibling subtree, i.e. a subtree rooted at  $w \in N^+(v_i) \setminus \{v_{i+1}\}$ . For each non-tree  $(x, z)$  linking the sibling subtrees found, it checks if it satisfies the weight condition  $d_{G'}(s, x) + w(x, z) + d_{G'}(z, t) \leq \alpha$ , where  $G' = G - (v_i, v_{i+1})$ . Item 2 of the same lemma implies that the first time an edge  $(x, z)$  satisfies the weight condition, the tree path traversed so far is the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$ . In order to test the weight conditions, as stated previously, we have that  $d_{G'}(s, x) = d_G(s, x)$ , since  $x$  does not belong to the subtree of  $v$  in  $T_s$ . In addition, Lemma 3 guarantees that  $d_{G'}(z, t) = d_G(z, t)$ . Thus, it is sufficient for the algorithm to compute only the shortest path trees from  $t$  and from  $s$  in  $G$ .

**Theorem 3.** *Algorithm 3 finds the longest common prefix of  $\mathcal{P}_\alpha(s, t, G)$  in  $O(m + t(n, m))$  time using  $O(m)$  space.*



**Algorithm 3.**  $\text{lcp}(s, t, \alpha, G)$ 


---

```

1 compute  $T_s$ , a shortest path tree from  $s$  in  $G$ 
2 compute  $T_t$ , a shortest path tree from  $t$  in  $G$ 
3 let  $\pi_{st} = (s = v_0, v_1) \dots (v_{n-1}, v_n = t)$  be the shortest path in  $T_s$ 
4 for  $v_i \in \{v_1, \dots, v_n\}$  do
5   for  $w \in N^+(v_i) \setminus \{v_{i+1}\}$  do
6     let  $T_w$  be the subtree of  $T_s$  rooted at  $w$ 
7     for  $(x, z) \in G$  s.t.  $x \in T_w$  or  $x = v_i$  do
8       if  $z \in T_{v_{i+1}}$  and  $d_G(s, x) + w(x, z) + d_G(z, t) \leq \alpha$  then
9         break
10      end
11    end
12  end
13 end
14 return  $\pi_{sv_{i-1}}$ 

```

---

*Proof.* The cost of the algorithm can be divided in two parts: the cost to compute the shortest path trees  $T_s$  and  $T_t$ , and the cost of the loop in line 4. The first part is bounded by  $O(t(n, m))$ . Let us now prove that the second part is bounded by  $O(m + n)$ . The cost of each execution of line 8 is  $O(1)$ , since we only need distances from  $s$  and  $t$  and the shortest path trees from  $s$  and  $t$  are already computed, and we pre-process the tree to decide in  $O(1)$  if a vertex belongs to a subtree. Hence, the cost of the loop is bounded by the number of times line 8 is executed. The neighborhood of each vertex  $x \in T_w$  is visited exactly once, since for each  $w \in N^+(v_i) \setminus \{v_{i+1}\}$  and  $w' \in N^+(v_j) \setminus \{v_{j+1}\}$  the subtrees  $T_w$  and  $T_{w'}$  are disjoint, where  $v_i$  and  $v_j$  belong to the tree path from  $s$  to  $t$ .  $\square$

## 5 $K$ -Shortest and $\alpha$ -Bounded Paths: A Unified View

The two main differences between the solutions to the  $K$ -shortest and  $\alpha$ -bounded paths problems are: (i) the order in which the paths are output and (ii) the space complexity of the algorithms. In this section, we show that both problems can be placed in a unified framework such that those differences arise in a natural way. More precisely, we show that their solutions correspond to two different traversals of the *same* rooted tree: a Dijkstra-like traversal for the  $K$ -shortest and a DFS-like traversal for the  $\alpha$ -bounded paths. This tree is a weighted version of the recursion tree of Algorithm 1, so the height is bounded by  $n$  and each leaf corresponds to an  $\alpha$ -bounded  $st$ -path (see Lemma 1).

The space complexity of the algorithms then follows from the fact that, in addition to the memory to store the tree, Dijkstra's algorithm uses memory proportional to the number of nodes, whereas the DFS uses memory proportional to the height of the tree. In addition, the order in which the solutions are output is precisely the order in which the leaves of the tree are visited, a Dijkstra-like traversal visits the leaves in increasing order of their distance from the root, whereas a DFS-like traversal visits them in an arbitrary but fixed order.

We first modify Algorithm 1 to obtain an iterative *generic* variant. The pseudocode is shown in Algorithm 4. Observe that each node in the recursion tree of Algorithm 1 corresponds to some tuple  $\langle u, t, \pi_{ut}, G' \rangle$  in line 3 of Algorithm 4. By generic we mean that the container  $Q$  is not specified in the pseudocode, the only requirement is the support for two operations: *push*, to insert a new element in  $Q$ ; and *pop*, to remove and return an element of  $Q$ . It should be clear now that depending on the container, the algorithm will perform a different traversal in the underlying recursion tree of Algorithm 1.

---

**Algorithm 4.** `list_paths_iterative`( $u, t, \alpha, \pi_{su}, G'$ )

---

```

1 push  $\langle s, t, \emptyset, G \rangle$  in  $Q$ 
2 while  $Q$  is not empty do
3    $\langle u, t, \pi_{su}, G' \rangle = Q.pop()$ 
4   if  $u = t$  then
5     output( $\pi_{su}$ )
6   else
7     compute a shortest path tree  $T_t$  from  $t$  in  $G^R - u$ 
8     for  $v \in N^+(u)$  do
9       if  $d(v, t) \leq \alpha - w(u, v)$  then
10        push  $\langle v, t, \alpha - w(u, v), \pi_{su} \cdot (u, v), G' - u \rangle$  in  $Q$ 
11       end
12     end
13   end
14 end

```

---

Algorithm 4 uses the same strategy to partition the solution space (Eq. 1). Of course, the order in which the partitions are explored depends on the type of container used for  $Q$ . We show that if  $Q$  is a stack, then the solutions are output in the reverse order of Algorithm 1 and the maximum size of the stack is linear in the size of the input. If on the other hand,  $Q$  is a priority queue, using a suitable key, the solutions are output in increasing order of their lengths, but in this case the maximum size of the priority queue is linear in the number of solutions, which is not polynomial in the size of the input.

Let  $\mathcal{T}$  be the recursion tree of Algorithm 1 (see Lemma 1). In Algorithm 4, each element  $\langle u, t, \pi_{su}, G' \rangle$  corresponds to the arguments of a call of Algorithm 1, *i.e.* a node of  $\mathcal{T}$ . For any container  $Q$  supporting push and pop operations, Algorithm 4 visits each node of  $\mathcal{T}$  exactly once, since at every iteration a node from  $Q$  is deleted and its children are inserted in  $Q$ , and  $\mathcal{T}$  is a tree. In particular, this guarantees that every leaf of  $\mathcal{T}$  is visited exactly once, thus proving the following lemma.

**Lemma 4.** *Algorithm 4 outputs all  $\alpha$ -bounded  $st$ -paths.*

Let us consider the case where  $Q$  is a stack. It is not hard to prove that Algorithm 1 is a DFS traversal of  $\mathcal{T}$  starting from the root, while Algorithm 4 is an *iterative* DFS [17] traversal of  $\mathcal{T}$  also starting from the root. Basically, an iterative DFS keeps the vertices of the fringe of the non-visited subgraph in a stack, at each iteration the next vertex to be explored is popped from the stack,

and recursive calls are replaced by pushing vertices in the stack. Now, for a fixed permutation of the children of each node in  $\mathcal{T}$ , the nodes visited in an iterative DFS traversal are in the reverse order of the nodes visited in a recursive DFS traversal, thus proving Lemma 5.

**Lemma 5.** *If  $Q$  is a stack, then Algorithm 4 outputs the  $\alpha$ -bounded  $st$ -path in the reverse order of Algorithm 1.*

For any rooted tree, at any moment during an iterative DFS traversal, the number of nodes in the stack is bounded by the sum of the degrees of the root-to-leaf path currently being explored. Recall that every leaf in  $\mathcal{T}$  corresponds to a path in  $\mathcal{P}_\alpha(s, t, G)$ . Actually, there is a one-to-one correspondence between the nodes of a root-to-leaf path  $P$  in  $\mathcal{T}$  and the vertices of the  $\alpha$ -bounded  $st$ -path  $\pi$  associated to that leaf. Hence, the sum of the degrees of the nodes of  $P$  in  $\mathcal{T}$  is equal to the sum of the degrees of the vertices  $\pi$  in  $G$ , which is bounded by  $m$ , thus proving Lemma 6.

**Lemma 6.** *The maximum number of elements in the stack of Algorithm 4 over all iterations is bounded by  $m$ .*

Let us consider now the case where  $Q$  is a priority queue. There is a one-to-many correspondence between arcs in  $G$  and arcs in  $\mathcal{T}$ , i.e. if  $\mathcal{P}_{\alpha''}(v, t, G'')$  is a child of  $\mathcal{P}_{\alpha'}(u, t, G')$  in  $\mathcal{T}$  then  $(u, v)$  is an arc of  $G$ . For every arc of  $\mathcal{T}$ , we give the weight of the corresponding arc in  $G$ . Now, Algorithm 4 using a priority queue with  $w(\pi_{su}) + d_G(u, t)$  as keys performs a Dijkstra-like traversal in this weighted version of  $\mathcal{T}$  starting from the root. Indeed, for a node  $\langle u, t, \pi_{su}, G \rangle$  the distance from the root is  $w(\pi_{su})$ , and  $d_G(u, t)$  is a (precise) estimation of the distance from  $\langle u, t, \pi_{su}, G \rangle$  to the closest leaf of  $\mathcal{T}$ . In other words, it is an  $A^*$  traversal [5] in the weighted rooted tree  $\mathcal{T}$ , using the (optimal) heuristic  $d_G(u, t)$ . As such, Algorithm 4 explores first the nodes of  $\mathcal{T}$  leading to the cheapest non-visited leaf. This is formally stated in Lemma 7.

**Lemma 7.** *If  $Q$  is a priority queue with  $w(\pi_{su}) + d_{G'}(u, t)$  as the priority key of  $\langle u, t, \pi_{su}, G' \rangle$ , then Algorithm 4 outputs the  $\alpha$ -bounded  $st$ -paths in increasing order of their lengths.*

For any choice of the container  $Q$ , each node of  $\mathcal{T}$  is visited exactly once, that is, each node of  $\mathcal{T}$  is pushed at most once in  $Q$ . This proves Lemma 8.

**Lemma 8.** *The maximum number of elements in a priority queue of Algorithm 4 over all iterations is bounded by  $\gamma$ .*

Algorithm 4 uses  $O(m\gamma)$  space since for every node inserted in the priority queue, we also have to store the corresponding graph. Moreover, using a binary heap as a priority queue, the push and pop operations can be performed in  $O(\log \gamma)$  each, where  $\gamma$  is the maximum size of the heap. Therefore, combining this with Lemma 7, we obtain the following theorem.

**Theorem 4.** *Algorithm 4 using a binary heap outputs all  $\alpha$ -bounded  $st$ -paths in increasing order of their lengths in  $O((nt(n, m) + \log \gamma)\gamma)$  total time, using  $O(m\gamma)$  space.*

## References

1. Ahuja, R.K., Mehlhorn, K., Orlin, J.B., Tarjan, R.E.: Faster algorithms for the shortest path problem. *J. ACM* **37**, 213–223 (1990)
2. Bernstein, A.: A nearly optimal algorithm for approximating replacement paths and  $k$  shortest simple paths in general graphs. In: Proceedings of the 20th ACM-SIAM Symposium on Discrete Algorithms (SODA), pp. 742–755 (2010)
3. Birmelé, E., Ferreira, R., Grossi, R., Marino, A., Pisanti, N., Rizzi, R., Sacomoto, G.: Optimal listing of cycles and  $st$ -paths in undirected graphs. In: Proceedings of the 24th Symposium on Discrete Algorithms (SODA), pp. 1884–1896 (2013)
4. Böhmová, K., Mihalák, M., Pröger, T., Sránek, R., Widmayer, P.: Robust routing in urban public transportation: how to find reliable journeys based on past observations. In: 13th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS), pp. 27–41 (2013)
5. Dechter, R., Pearl, J.: Generalized best-first search strategies and the optimality of  $A^*$ . *J. ACM* **32**(3), 505–536 (1985)
6. Dreyfus, S.E.: An appraisal of some shortest-path algorithms. *Oper. Res.* **17**(3), 395–412 (1969)
7. Eppstein, D.: Finding the  $k$  shortest paths. *SIAM J. Comput.* **28**(2), 652–673 (1999)
8. Hershberger, J., Suri, S.: Vickrey prices and shortest paths: what is an edge worth? In: Proceedings of the 42nd Symposium on Foundations of Computer Science (FOCS), pp. 252–259. IEEE Computer Society (2001)
9. Johnson, D.S., Papadimitriou, C.H., Yannakakis, M.: On generating all maximal independent sets. *Inf. Process. Lett.* **27**(3), 119–123 (1988)
10. Johnson, D.B.: Efficient algorithms for shortest paths in sparse networks. *J. ACM* **24**(1), 1–13 (1977)
11. Katoh, N., Ibaraki, T., Mine, H.: An efficient algorithm for  $K$  shortest simple paths. *Networks* **12**(4), 411–427 (1982)
12. Lawler, E.L.: A procedure for computing the  $K$  best solutions to discrete optimization problems and its application to the shortest path problem. *Manage. Sci.* **18**, 401–405 (1972)
13. Malik, K., Mittal, A.K., Gupta, S.K.: The  $k$  most vital arcs in the shortest path problem. *Oper. Res. Lett.* **8**(4), 223–227 (1989)
14. Ferreira, R., Grossi, R., Rizzi, R., Sacomoto, G., Sagot, M.-F.: Amortized  $\tilde{O}(|V|)$ -delay algorithm for listing chordless cycles in undirected graphs. In: Schulz, A.S., Wagner, D. (eds.) *ESA 2014*. LNCS, vol. 8737, pp. 418–429. Springer, Heidelberg (2014)
15. Roditty, L.: On the  $k$ -simple shortest paths problem in weighted directed graphs. In: Proceedings of the 18th ACM-SIAM Symposium on Discrete Algorithms (SODA). SIAM (2007)
16. Roditty, L., Zwick, U.: Replacement paths and  $k$  simple shortest paths in unweighted directed graphs. In: Caires, L., Italiano, G.F., Monteiro, L., Palamidessi, C., Yung, M. (eds.) *ICALP 2005*. LNCS, vol. 3580, pp. 249–260. Springer, Heidelberg (2005)
17. Sedgwick, R.: *Algorithms in C, Part 5: Graph Algorithms*, 3rd edn. Addison-Wesley Professional, Reading (2001)
18. Yen, J.Y.: Finding the  $K$  shortest loopless paths in a network. *Manage. Sci.* **17**, 712–716 (1971)