A Tentative Approach for the Wadge-Wagner Hierarchy of Regular Tree Languages of Index $[0, 2]$

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Abstract. We provide a hierarchy of tree languages recognised by nondeterministic parity tree automata with priorities in $\{0, 1, 2\}$, whose length exceeds the first fixed point of the ε operation (that itself enumerates the fixed points of $x \mapsto \omega^x$). We conjecture that, up to Wadge equivalence, it exhibits all regular tree languages of index $[0, 2]$.

1 Introduction

This paper contributes to the close investigation of regular tree languages of index $[0, 2]$. Our tool to measure and compare those languages is given by descriptive set theory through the notion of *topological complexity*. It is well known that deterministic parity tree automata recognize only languages in the $\mathbf{\Pi}^1_1$ class (coanalytic sets), whereas nondeterministic automata recognize languages that are neither analytic, nor coanalytic. The expressive power of nondeterministic automata is nonetheless bounded by the second level of the projective hierarchy, and, by Rabin's complementation result [7], all nondeterministic languages are in fact in the $\mathbf{\Delta}^1_2$ class. A more discriminating topological complexity measure than the Baire and the projective hierarchy is therefore needed: the Wadge hierarchy, which relies on the notion of reductions by continuous functions (Wadge-reducibility). Complexity classes, called Wadge degrees, consist of sets Wadge-reducible to each other, and constitute a hierarchy whose levels, called ranks, can be enumerated with ordinals. We describe a series of operations on automata that preserve the index and lift the Wadge degrees of the recognized languages[1]. These operations help us generate a hierarchy of regular tree languages of higher and higher topological complexity, one level higher than the first fixed point of the ordinal function[2] $x \mapsto \varepsilon_x$ which itself enumerates the fixed points of the exponentiation $x \mapsto \omega^x$.

[1] We emphasize that this is done without any determinacy principle. In particular, we do not require $\mathbf{\Delta}^1_2$-determinacy.

[2] Not to be mistaken with an ε-move.

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2 Preliminaries

2.1 The Wadge Hierarchy and the Wadge Game

The Wadge theory is in essence the theory of *pointclasses*[3] (see [1]). For Γ a pointclass, we denote by $\check{\Gamma}$ its *dual* class containing all the subsets whose complements are in Γ, and by $\Delta(\Gamma)$ the ambiguous class $\Gamma \cap \check{\Gamma}$. If $\Gamma = \check{\Gamma}$, we say that Γ is *self-dual.*

Given any topological space X, the *Wadge preorder* $\leq_W$ on $\mathscr{P}(X)$ is defined for $A, B \subseteq X$ by $A \leq_W B$ if and only if there exists $f\colon X \longrightarrow X$ continuous such that $f^{-1}(B) = A$. It is merely by definition a preorder which induces an equivalence relation $\equiv_W$ whose equivalence classes – denoted by $[A]_W$ – are called the *Wadge degrees.* A set $A \subseteq X$ is *self-dual* if $[A]_W = [A^{\complement}]_W$, and *non-self-dual* otherwise. We use the same terminology for the Wadge degrees. We have a direct correspondence between $(\mathscr{P}(X), \leq_W)$ restricted to Γ and the pointclasses included in Γ with inclusion: the pointclasses are exactly the initial segments of the Wadge preorder. In particular, the Wadge hierarchy tremendously refines the Borel and the projective hierarchies.

The space T_Σ equipped with the standard Cantor topology is a Polish space, and is in fact homeomorphic to the Cantor space [2]. Let $L, M \subseteq T_\Sigma$, the Wadge game $W(L, M)$ is a two-player infinite game that provides a very useful characterization for the Wadge preorder. In this game, each player builds a tree, say t_{I} and t_{II}. At every round, player I plays first, and both players add a finite number of children to the terminal nodes of their tree. Player II is allowed to skip her turn, but has to produce a tree in T_Σ throughout a game. Player II wins the game if and only if $t_I \in L \Leftrightarrow t_{II} \in M$.

Lemma 1 ([9]). *Let $L, M \subseteq T_\Sigma$. Then $L \leq_W M$ if and only if player* II *has a winning strategy in the game $W(L, M)$.*

We write $A <_W B$ when II has a winning strategy in $W(A, B)$ *and* I has a winning strategy in $W(B, A)$[4]. Given a pointclass Γ of T_Σ with suitable closure properties, the assumption of the determinacy of Γ is sufficient to prove that Γ is semi-linearly ordered by $\leq_W$, denoted SLO(Γ), i.e., that for all $L, M \in \Gamma$,

$$L \leq_W M \qquad \text{or} \qquad M \leq_W L^{\complement},$$

and that $\leq_W$ is well founded when restricted to sets in Γ [1,8]. Under these conditions, the Wadge degrees of sets in Γ with the induced order is thus a hierarchy called the *Wadge hierarchy.* Therefore, there exists a unique ordinal, called the *height* of the Γ-Wadge hierarchy, and a mapping d_W^Γ from the Γ-Wadge hierarchy onto its height, called the *Wadge rank*, such that, for every L, M non-self-dual in Γ, $d_W^\Gamma(L) < d_W^\Gamma(M)$ if and only if $L <_W M$ and $d_W^\Gamma(L) = d_W^\Gamma(M)$

[3] A pointclass is a collections of subsets of a topological space that is closed under continuous preimages.

[4] This is in general stronger than the usual $A <_W B$ if and only if $A \leq_W B$ and $B \not\leq_W A$, but the two definitions coincide when the classes considered are determined.

if and only if $L \equiv_W M$ or $L \equiv_W M^{\complement}$. The wellfoundedness of the Γ-Wadge hierarchy ensures that the Wadge rank can be defined by induction as follows:

- $d_W^{\Gamma}(\emptyset) = d_W^{\Gamma}(\emptyset^{\complement}) = 1$.
- $d_W^{\Gamma}(L) = \sup \left\{ d_W^{\Gamma}(M) + 1 : M \text{ is non-self-dual}, M <_W L \right\}$ for $L >_W \emptyset$.

Note that given two pointclasses Γ and Γ', for every $L \in \Gamma \cap \Gamma'$, we have $d_W^{\Gamma}(L) = d_W^{\Gamma'}(L)$. Under sufficient determinacy assumptions, we can therefore safely speak of *the* Wadge rank of a tree language, denoted by d_W, as its Wadge rank with respect to any topological class including it. However the main result of this article does not provide any Wadge rank for the canonical languages that are constructed, because we do not make use of any determinacy principle.

2.2 The Conciliatory Hierarchy

A *conciliatory* binary tree over a finite set Σ is a partial function $t : \{0,1\}^* \to \Sigma$ with a prefix-closed domain. Such trees can have both infinite and finite branches. A tree is called *full* if $\text{dom}(t) = \{0,1\}^*$. Let $\mathcal{T}_{\Sigma}^{\leq\omega}$ and T_{Σ} denote, respectively, the set of all conciliatory binary trees and the set of full binary trees over Σ. Given $x \in \text{dom}(t)$, we denote by t_x the subtree of t rooted at x. Let $\{0,1\}^n$ denote the set of words over $\{0,1\}$ of length n, and let t be a conciliatory tree over Σ. We denote by $t[n]$ the finite initial binary tree of height $n+1$ given by the restriction of t to $\bigcup_{0 \leq i \leq n} \{0,1\}^i$.

For conciliatory languages L, M we define the *conciliatory* version of the Wadge game: $C(L, M)$ [4,5]. The rules are similar, except for the fact that both players are now allowed to skip and to produce trees with finite branches – or even finite trees. For conciliatory languages L, M we use the notation $L \leq_c M$ if and only if II has a winning strategy in the game $C(L, M)$. If $L \leq_c M$ and $M \leq_c L$, we will write $L \equiv_c M$. The conciliatory hierarchy is thus the partial order induced by $\leq_c$ on the equivalence classes given by $\equiv_c$. We write $A <_c B$ when II has a winning strategy in $C(A, B)$ *and* I has a winning strategy in $C(B, A)$.

From a conciliatory language L over Σ, one defines the corresponding language L^b of full trees over $\Sigma \cup \{b\}$ by

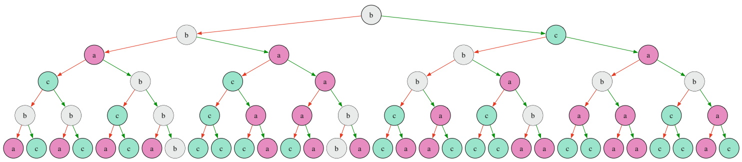
$$L^b = \left\{ t \in T_{\Sigma \cup \{b\}} : t_{[\ /b]} \in L \right\},$$

where b is an extra symbol that stands for "blank", and $t_{[\ /b]}$, the *undressing* of t, is informally the conciliatory tree over Σ obtained once all the occurrences of b have been removed in a top-down manner. More precisely, if there is a node v such that $t(v) = b$, we ignore this node and replace it with $v0$. If, for each integer n, $t(v0^n) = b$, then $v \notin \text{dom}(t_{[\ /b]})$. This process is illustrated by Fig. 1.

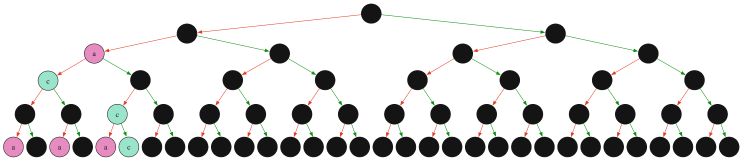
If Γ is a pointclass of full trees, we say that a conciliatory language L is in Γ if and only if L^b is in Γ.

Lemma 2. *Let L and M be conciliatory languages. Then*

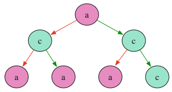
$$L \leq_c M \text{ if and only if } L^b \leq_W M^b.$$



(a) A tree t with blanks



(b) The blanks are deleted in a top-down manner.



(c) The resulting tree $t_{[\ /b]}$.

Fig. 1. The undressing process.

The mapping $L \mapsto L^b$ gives thus a natural embedding of the preorder $\leq_c$ restricted to conciliatory sets in Γ into the Γ-Wadge hierarchy. Hence, for Γ with suitable closure and determinacy properties, the conciliatory degrees of sets in Γ with the induced order constitute a hierarchy called the *conciliatory hierarchy*. We define, by induction, the corresponding *conciliatory rank* of a language:

- $d_c^\Gamma(\emptyset) = d_c^\Gamma(\emptyset^\complement) = 1$.
- $d_c^\Gamma(L) = \sup\{d_c^\Gamma(M) + 1 : M <_c L\}$ for $L >_c \emptyset$.

Similarly to the Wadge case, given two pointclasses Γ and Γ', for every conciliatory $L \in \Gamma \cap \Gamma'$, we have $d_c^\Gamma(L) = d_c^{\Gamma'}(L)$. Under sufficient determinacy assumptions, we can therefore speak safely of *the* conciliatory rank of a conciliatory tree language, denoted by d_c, as its conciliatory rank with respect to any topological class including it. Observe that the conciliatory hierarchy does not contain self-dual languages: a strategy for I in $C(L, L^\complement)$ is to skip in the first round, and then copy moves of II.

2.3 Automata and Conciliatory Trees

A *nondeterministic parity tree automaton* $\mathcal{A} = \langle \Sigma, Q, I, \delta, \mathrm{r}\rangle$ consists of a finite input alphabet Σ, a finite set Q of states, a set of initial states $I \subseteq Q$, a transition relation $\delta \subseteq Q \times \Sigma \times Q \times Q$ and a priority function $\mathrm{r} : Q \to \omega$. A run of automaton $\mathcal{A}$ on a binary conciliatory input tree $t \in \mathcal{T}_\Sigma^{\leq\omega}$ is a conciliatory tree $\rho_t \in \mathcal{T}_Q^{\leq\omega}$ with $\mathrm{dom}(\rho_t) = \{\varepsilon\} \cup \{va : v \in \mathrm{dom}(t) \wedge a \in \{0, 1\}\}$ such that the root of this tree is labeled with a state $q \in I$, and for each $v \in \mathrm{dom}(t)$, transition $(\rho_t(v), t(v), \rho_t(v_1), \rho_t(v_1)) \in \delta$. The run ρ_t is *accepting* if parity condition is satisfied on each infinite branch of ρ_t, i.e., if the highest rank of a state occurring infinitely often on the branch is even, and if the rank of each leaf node in ρ_t is even. We say that a parity tree automaton A *accepts* a conciliatory tree t if it has an accepting run on t. The language *recognized* by A, denoted $L(A)$ is the set of trees accepted by A. We let $L^\omega(A)$ denote the set of full trees recognized by A, i.e., $L^\omega(A) = L(A) \cap T_\Sigma$. Notice that as the set of states is finite, the priority function is bounded. Moreover, shifting all ranks by an even number does not change the language recognized by a parity tree automaton. It is thus sufficient to consider parity tree automata whose priorities are restricted to intervals $[\iota, \kappa]$, for $\iota \in \{0, 1\}$. We say that an automaton is of index $[\iota, \kappa]$ if its priorities are restricted to intervals $[\iota, \kappa]$. A language is of index $[\iota, \kappa]$ if there is an automaton of index $[\iota, \kappa]$ that recognises it. This gives rise to the Mostowski-Rabin hierarchy [3]. Let $W_{[0,2]}$ be the game tree language of index $[0, 2]$. One can prove that $L \leq_W W_{[0,2]}$ holds for any regular tree language L of index $[0, 2]$, but fails for $L = W^{\complement}_{[0,2]}$.

Corollary 1. *The mapping $L \mapsto L^b$ embeds the conciliatory hierarchy for $\mathbf{\Delta}^1_2$-sets restricted to languages of index* $[0, 2]$ *into the $\mathbf{\Delta}^1_2$-Wadge hierarchy restricted to languages of index* $[0, 2]$.

We use the following conventions in the diagrams. Nodes represent states of the automaton. Node labels correspond to state ranks. A red edge shows the state that is assigned to the left successor node of a transition, and a green edge goes to the right successor node. In order to lighten the notation, transitions that are not depicted on a diagram lead to some all-accepting state. Given automata $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \leq_c \mathcal{B}$ for $L(\mathcal{A}) \leq_c L(\mathcal{B})$, and same with $<_c, \leq_W, <_W$.

3 Operations on Languages and Their Automatic Counterparts

We present operations on conciliatory tree languages, which we then use to construct more and more complex languages. W.l.o.g. we assume the alphabet to be $\Sigma = \{a, c\}$.

3.1 The Sum

For $L, M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, we define $L \oplus M$ (the *sum* of L and M) as the language formed of all those trees $t \in \mathcal{T}_\Sigma^{\leq\omega}$ such that one of the following conditions holds:

- $t(10^n) = a$ for each integer n and $t_0 \in M$;
- the node 10^n is the first on the path 10^* labeled with c and either $t(10^n0) = a$ and $t_{10^n00} \in L$, or $t(10^n0) = c$ and $t_{10^n00} \in L^{\complement}$.

This operation behaves well regarding the conciliatory hierarchy.

Facts 1 ([4,5]). Given L, M, and M' any conciliatory tree languages over Σ,

1. $(L \oplus M)^{\complement} \equiv_c L \oplus M^{\complement}$.
2. The operation $\oplus$ preserves the conciliatory ordering: if $M' \leq_c M$, then

$$L \oplus M' \leq_c L \oplus M.$$

3. Assuming enough determinacy:

$$d_c(L \oplus M) = d_c(L) + d_c(M).$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two automata that recognize, respectively, the conciliatory languages M and L. Then the automaton $\mathcal{B}+\mathcal{A}$ depicted in Fig. 2 recognizes the sum of L and M. In this picture, $\mathcal{C}$ is any automaton of index $[0, 2]$ that recognizes a language equivalent to $L^{\complement}$, and the parity i and j are defined as follows:

- $i = 0$ if and only if the empty tree is accepted by $\mathcal{A}$;
- $j = 1$ if and only if $L(\mathcal{A})$ is equivalent to $L(\mathcal{A}) \to \ominus$, where $\ominus$ denotes any automaton that rejects all trees.[5]

Notice that if $\mathcal{A}$ and $\mathcal{B}$ are parity tree automata of index $[0, 2]$ such that $L(\mathcal{B})^{\complement}$ can be recognized by an automaton of index $[0, 2]$, then $\mathcal{B}+\mathcal{A}$ is a parity tree automata of index $[0, 2]$.

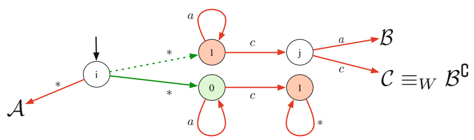


Fig. 2. The automaton $\mathcal{B}+\mathcal{A}$ that recognizes $L(\mathcal{B}) \oplus L(\mathcal{A})$. The values of i and j depend on properties of $\mathcal{A}$.

Lemma 3. *Let L, L', M and M' be conciliatory languages such that $L <_c L'$ and $M \leq_c M'$. Then the following hold.*

1. $M \oplus L <_c M' \oplus L'$;
2. $M <_c M \oplus L$.

3.2 Multiplication by a Countable Ordinal

In order to define the multiplication of a language by a countable ordinal, we first introduce the operation $\sup_{n<\omega}$. Let $(L_n)_{n\in\omega} \subseteq \mathcal{T}_{\Sigma}^{\leq\omega}$ be a countable family of conciliatory languages. Define $\sup_{n<\omega} L_n$ as the conciliatory tree language containing all of those trees $t \in \mathcal{T}_{\Sigma}^{\leq\omega}$ such that one of the following conditions holds:

[5] A player in charge of $L(\mathcal{A}) \to \ominus$ in a conciliatory game is like a player in charge of $L(\mathcal{A})$, but with the extra possibility at any moment of the play to reach a definitively rejecting position.

– $t(1^n) = a$ for all integer n;
– the node 1^n is the first on the path 1^* labeled with c and $t_{1^n0} \in L_n$.

The multiplication by a countable ordinal is now defined as an iterated sum. Let $L \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, $L \odot 1 = L$, $L \odot (\alpha + 1) = (L \odot \alpha) \oplus L$, and $L \odot \lambda = \sup_{\alpha<\lambda} L \odot \alpha$, for λ limit.

Let $\mathcal{A}$ be an automaton that recognizes the conciliatory languages L. Then the automaton $\mathcal{A}\bullet\omega$ depicted in Fig. 3(a) recognizes a language equivalent to $L \odot \omega$. In this picture, $\mathcal{C}$ is any automaton that recognizes a language equivalent to $L^\complement$. The automaton $\mathcal{A}\overline{\bullet}\omega$ that recognizes the complement of $L(\mathcal{A}\bullet\omega)$, and thus a language equivalent to the complement of $L \odot \omega$, is depicted in Fig. 3b. Notice that if $\mathcal{A}$ is of index $[0, 2]$, and if there exists an automaton that recognizes $L(\mathcal{A})^\complement$ of index $[0, 2]$, then both $\mathcal{A}\bullet\omega$ and $\mathcal{A}\overline{\bullet}\omega$ are parity tree automata of index $[0, 2]$. Hence, for every ordinal $0 < \alpha < \omega^\omega$ and for every automaton $\mathcal{A}$, there exists an automaton $\mathcal{A}\bullet\alpha$ that recognizes $L(\mathcal{A}) \odot \alpha$. Moreover, if $\mathcal{A}$ is of index $[0, 2]$, and if there exists an automaton that recognizes $L(\mathcal{A})^\complement$ of index $[0, 2]$, then $\mathcal{A}\bullet\alpha$ is a parity tree automaton of index $[0, 2]$.

(a) The automaton $\mathcal{A}\bullet\omega$. (b) The automaton $\mathcal{A}\overline{\bullet}\omega$.

Fig. 3. Automata that recognize respectively a language equivalent to $L \odot \omega$ and a language equivalent to its complement.

As a corollary of Lemma 3 and Facts 1, the multiplication by a countable ordinal behaves well regarding the conciliatory hierarchy.

Corollary 2. *Let L and M be conciliatory languages such that $L <_c M$. Then for every countable ordinals $0 < \alpha < \beta < \omega^\omega$:*

1. $L \odot \alpha <_c L \odot \beta$;
2. $L \odot \alpha <_c M \odot \alpha$.

3.3 The Pseudo-Exponentiation

Let $P \subseteq \mathcal{T}_\Sigma^{\leq\omega}$ be a conciliatory tree language. For $t \in \mathcal{T}_\Sigma^{\leq\omega}$, let:

$$i^P(t)(a_1, a_2, \ldots, a_n) = \begin{cases} t(a_1, 0, a_2, 0, \ldots, 0, a_n, 0), & \text{if } t_{a_1,0,a_2,0,\ldots,0,a_n,1} \in P; \\ b, & \text{otherwise.} \end{cases}$$

This process is illustrated in Fig. 4. The nodes in blue are called the *main run*. The blue arrows denote the dependency of a node of the main run on a subtree of auxiliary moves. If the auxiliary subtree of a main run node is not in P, then we say that the node is *killed*.

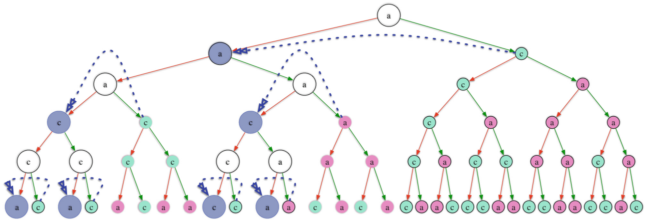


Fig. 4. Main run and auxiliary moves.

Let $L \subseteq \mathcal{T}_{\Sigma}^{\leq\omega}$, we define the *action* of P on L, in symbols (P, L), by

$$\left\{t \in \mathcal{T}_{\Sigma}^{\leq\omega} : i^P(t)_{[\ /b]} \in L\right\}.$$

Let $P_{\mathbf{\Pi}_1^0}$ be the complete closed set of all full trees over Σ with all nodes on the leftmost branch 0^* labelled by a. For $L \subseteq \mathcal{T}_{\Sigma}^{\leq\omega}$, we denote by $(\mathbf{\Pi}_1^0, L)$ the action of $P_{\mathbf{\Pi}_1^0}$ on L. This operation $(\mathbf{\Pi}_1^0, \cdot)$ behaves well regarding the conciliatory hierarchy.

Facts 2 ([4,5]). Let L and M be conciliatory tree languages over Σ. Then the following hold.

1. $(\mathbf{\Pi}_1^0, L)^{\complement} \equiv_c (\mathbf{\Pi}_1^0, L^{\complement})$.
2. If $L \leq_c M$, then $(\mathbf{\Pi}_1^0, L) \leq_c (\mathbf{\Pi}_1^0, M)$.
3. If $L <_c M$, then $(\mathbf{\Pi}_1^0, L) <_c (\mathbf{\Pi}_1^0, M)$.
4. Assuming enough determinacy, $d_c((\mathbf{\Pi}_1^0, L)) = \omega_1^{d_c(L)+\varepsilon}$, for [6] $\varepsilon \in \{-1, 0, 1\}$.

Without assuming any determinacy hypothesis, we can nonetheless prove the following Proposition that links $(\mathbf{\Pi}_1^0, \cdot)$ to $\oplus$.

Proposition 1. *Let L, L' and M be conciliatory languages such that $L <_c (\mathbf{\Pi}_1^0, M)$ and $L' <_c (\mathbf{\Pi}_1^0, M)$. Then*

1. $L \oplus L' <_c (\mathbf{\Pi}_1^0, M)$;
2. $L \odot \alpha <_c (\mathbf{\Pi}_1^0, M)$, *for any* $\alpha < \omega^\omega$.

Given any automaton $\mathcal{A}$ recognizing $L \subseteq \mathcal{T}_{\Sigma}^{\leq\omega}$, the conciliatory language $(\mathbf{\Pi}_1^0, L)$ is recognized by the automaton $(\omega^\omega)^{\mathcal{A}}$ defined from $\mathcal{A}$ by replacing each state of $\mathcal{A}$ by a "gadget", as depicted in Fig. 5. By replacing a state by the gadget we mean that all transitions ending in this state should now end in the initial state of the gadget, and that all the transitions leaving this state should now start from the final state of the gadget. This sort of gadget first appeared in [5]. Notice that if $L \subseteq \mathcal{T}_{\Sigma}^{\leq\omega}$ is of index $[0, 2]$, then $(\mathbf{\Pi}_1^0, L)$ is also of index $[0, 2]$. Observe also that the game language $W_{[0,2]}$ is a fixed point for pseudo-exponentiation, i.e.,

$$(\mathbf{\Pi}_1^0, W_{[0,2]})^b \equiv_W W_{[0,2]}.$$

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[6] $\varepsilon = \begin{cases} -1 & \text{if } d_c(L) < \omega; \\ 0 & \text{if } d_c(L) = \beta + n \text{ and } \operatorname{cof}(\beta) = \omega_1;. \\ 1 & \text{if } d_c(L) = \beta + n \text{ and } \operatorname{cof}(\beta) = \omega. \end{cases}$

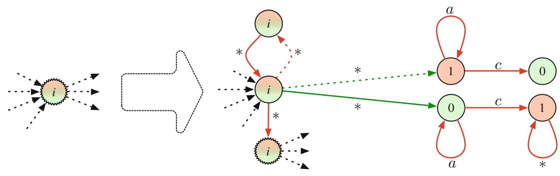


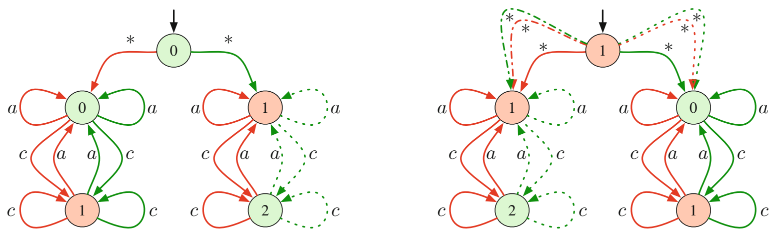
Fig. 5. The gadget to replace a state in $\mathcal{A}$.

4 Difference of Co-analytic Sets

The operations defined in Sect. 3 are *Borel* in the sense that when we apply them to Borel languages, the resulting language is still Borel. In order to describe the most of the Wadge hierarchy of languages recognized by parity tree automata of index $[0, 2]$ we need to climb higher.

4.1 The Operation $(D_2(\boldsymbol{\Pi}_1^1), \cdot)$

We define a conciliatory language of index $[0, 2]$ that is $D_2(\boldsymbol{\Pi}_1^1)$-complete (Fig. 6a) and such that its complement (Fig. 6b) is also of index $[0, 2]$, via the automata that recognize each of them. We denote by $A_{D_2(\boldsymbol{\Pi}_1^1)}$ and $A_{\check{D}_2(\boldsymbol{\Pi}_1^1)}$ the conciliatory languages recognized respectively by $\mathcal{A}_{D_2(\boldsymbol{\Pi}_1^1)}$ and $\mathcal{A}_{\check{D}_2(\boldsymbol{\Pi}_1^1)}$.



(a) $\mathcal{A}_{D_2(\boldsymbol{\Pi}_1^1)}$ (b) $\mathcal{A}_{\check{D}_2(\boldsymbol{\Pi}_1^1)}$

Fig. 6. Automata that recognize respectively a $D_2(\boldsymbol{\Pi}_1^1)$-complete and a $\check{D}_2(\boldsymbol{\Pi}_1^1)$-complete language.

For $M \subseteq \mathcal{T}_{\Sigma}^{\leq\omega}$, we denote by $(D_2(\boldsymbol{\Pi}_1^1), M)$ the action of $L(\mathcal{A}_{D_2(\boldsymbol{\Pi}_1^1)})$ on M. Observe that this operation is highly non-Borel, since if we apply it to a $\boldsymbol{\Sigma}_1^0$-complete conciliatory language, the resulting language will be complete for the pointclass of all the countable unions of $D_2(\boldsymbol{\Pi}_1^1)$ languages. The operation $(D_2(\boldsymbol{\Pi}_1^1), \cdot)$ behaves well with respect to $\leq_c$.

Theorem 1. *Let $M, M' \subseteq \mathcal{T}_\Sigma^{\leq\omega}$. If $M \leq_c M'$, then*

1. $(D_2(\mathbf{\Pi}_1^1), M)^\complement \equiv_c (D_2(\mathbf{\Pi}_1^1), M^\complement)$;
2. $(D_2(\mathbf{\Pi}_1^1), M) \leq_c (D_2(\mathbf{\Pi}_1^1), M')$.

A winning strategy for I in $C(M, M')$ can also be "remote controlled" to a winning strategy for I in $C((D_2(\mathbf{\Pi}_1^1), M), (D_2(\mathbf{\Pi}_1^1), M'))$, so that the following holds.

Corollary 3. *Let M and M' be conciliatory languages such that $M <_c M'$. Then*

$$(D_2(\mathbf{\Pi}_1^1), M) <_c (D_2(\mathbf{\Pi}_1^1), M')$$

The operation $(D_2(\mathbf{\Pi}_1^1), \cdot)$ is much stronger than $(\mathbf{\Pi}_1^0, \cdot)$, and is in fact a fixed point of it.

Proposition 2. *Let $M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$. Then*

$$\left(\mathbf{\Pi}_1^0, (D_2(\mathbf{\Pi}_1^1), M)\right) \equiv_c (D_2(\mathbf{\Pi}_1^1), M).$$

Let $\mathcal{A}$ be an automaton that recognizes $M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$. Then the conciliatory tree language $(D_2(\mathbf{\Pi}_1^1), M)$ is recognized by the automaton $\varepsilon_\mathcal{A}$ defined from $\mathcal{A}$ by replacing each state of $\mathcal{A}$ by a "gadget", as depicted in Fig. 7. As in the pseudo-exponentiation case, by replacing a state by the gadget we mean that all transitions ending in this state should now end in the initial state of the gadget, and that all the transitions starting from this state should now start from the final state of the gadget. Notice that if $M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$ is of index $[0, 2]$, then $(D_2(\mathbf{\Pi}_1^1), M)$ is also of index $[0, 2]$, and that $W_{[0,2]}$ is a fixed point of this operation. In particular the game language $W_{[0,2]}$ is above all the differences of coanalytic sets, which is a strengthening of a result obtained by Finkel and Simonnet [6].

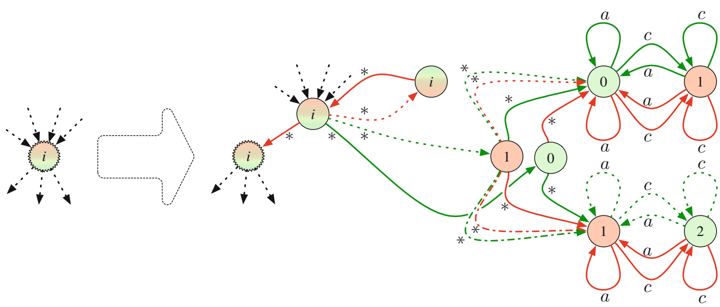


Fig. 7. The gadget to replace a state in $\mathcal{A}$.

5 A Fragment of the Wadge Hierarchy

Let $\varphi_2(0)$ denote the first fixed point[7] of the ordinal epsilon function, namely the one that enumerates the fixed points of the exponentiation of base ω:

$$\varepsilon_0 = \sup_{n<\omega} \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega^0}}}}}_{n} \; ; \; \varepsilon_{\alpha+1} = \sup_{n<\omega} \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega^{(\varepsilon_\alpha+1)}}}}}}_{n} \; ; \; \varepsilon_\lambda = \sup_{\alpha<\lambda} \varepsilon_\alpha, \text{for } \lambda \text{ limit.}$$

Finally: $\varphi_2(0) = \sup_{n<\omega} \overbrace{\varepsilon_{\cdot_{\cdot_{\cdot_{\varepsilon_0}}}}}^{n}$.

We recall that every ordinal $\alpha > 0$ admits a unique Cantor normal form of base ω^ω (CNF) which is an expression of the form $\alpha = (\omega^\omega)^{\alpha_k} \cdot \nu_k + \cdots + (\omega^\omega)^{\alpha_0} \cdot \nu_0$ where $k < \omega$, $0 < \nu_i < \omega^\omega$ for any $i \leq k$, and $\alpha_0 < \cdots < \alpha_k < \alpha$.

For every ordinal $0 < \alpha < \varphi_2(0)$, we inductively define a pair of automata $(\mathcal{A}_\alpha, \bar{\mathcal{A}}_\alpha)$ whose languages are incomparable through the conciliatory ordering. If the CNF of α is $\alpha = (\omega^\omega)^{\alpha_k} \cdot \nu_k + \cdots + (\omega^\omega)^{\alpha_0} \cdot \nu_0$ we set

$$\mathcal{A}_\alpha = \mathcal{A}_{(\omega^\omega)^{\alpha_k}} \bullet \nu_k + \cdots + \mathcal{A}_{(\omega^\omega)^{\alpha_0}} \bullet \nu_0, \qquad \bar{\mathcal{A}}_\alpha = \mathcal{A}_{(\omega^\omega)^{\alpha_k}} \bullet \nu_k + \cdots + \bar{\mathcal{A}}_{(\omega^\omega)^{\alpha_0}} \bullet \nu_0,$$

where $\mathcal{A}_{(\omega^\omega)^{\alpha_i}}$ and $\bar{\mathcal{A}}_{(\omega^\omega)^{\alpha_i}}$ are respectively

- $\ominus$ and $\oplus$ if $\alpha_i = 0$;
- $(\boldsymbol{\omega^\omega})^{\mathcal{A}_{\alpha_i}}$ and $(\boldsymbol{\omega^\omega})^{\bar{\mathcal{A}}_{\alpha_i}}$ if $\alpha_i < (\omega^\omega)^{\alpha_i}$;
- $\boldsymbol{\varepsilon}_{\mathcal{A}_{2+\beta}}$ and $\boldsymbol{\varepsilon}_{\bar{\mathcal{A}}_{2+\beta}}$ if $\alpha_i = (\omega^\omega)^{\alpha_i}$ holds[8] and $\alpha_i = \varepsilon_\beta$ for some $\beta < \alpha_i$.

Lemma 4. *For* $0 < \alpha < \beta < \varphi_2(0)$, *we have*

1. $\mathcal{A}_\alpha \not\leq_c \bar{\mathcal{A}}_\alpha$ *and* $\bar{\mathcal{A}}_\alpha \not\leq_c \mathcal{A}_\alpha$.
2. $\mathcal{A}_\alpha <_c \mathcal{A}_\beta$; $\bar{\mathcal{A}}_\alpha <_c \mathcal{A}_\beta$; $\mathcal{A}_\alpha <_c \bar{\mathcal{A}}_\beta$ *and* $\bar{\mathcal{A}}_\alpha <_c \bar{\mathcal{A}}_\beta$.

Applying the embedding $L \mapsto L^b$, we have thus generated a family $\left(\mathcal{A}_\alpha{}^b\right)_{\alpha<\varphi_2(0)}$ of parity tree automata of index $[0,2]$ that respects the strict Wadge ordering: $\alpha < \beta$ if and only if $\mathcal{A}_\alpha{}^b <_W \mathcal{A}_\beta{}^b$. Hence the main result follows.

Theorem 2. *There exists a family* $\left(\mathcal{A}_\alpha{}^b\right)_{\alpha<\varphi_2(0)}$ *of parity tree automata of index* $[0,2]$ *such that*

1. *they recognize languages of full trees over the alphabet* $\{a,b,c\}$;
2. $\alpha < \beta$ *holds if and only if* $\mathcal{A}_\alpha{}^b <_W \mathcal{A}_\beta{}^b$ *holds as well.*

Let $\mathcal{A}_{\varphi_2(0)}{}^b$ be an automaton of index $[0,2]$ over the alphabet $\{a,b,c\}$ that recognizes a language equivalent to $W_{[0,2]}$. We formulate the following conjecture.

Conjecture. Let L be a regular non-self-dual full language of index $[0,2]$. Then either $L \equiv_W W_{[0,2]}$, or there exists $\alpha < \varphi_2(0)$ such that $L \equiv_W L(\mathcal{A}_\alpha{}^b)$ or $L^{\complement} \equiv_W L(\mathcal{A}_\alpha{}^b)$.

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[7] Another way to characterise $\varphi_2(0)$ is to remember that an ordinal is the set of its predecessors and notice that a nonzero ordinal is of the form respectively ω^α iff it is closed under addition and ε_α iff it is closed under $x \longmapsto \omega^x$. Then $\varphi_2(0)$ is the first non null ordinal closed under $x \longmapsto \varepsilon_x$ as well as $x \longmapsto \omega^x$ and $x, y \longmapsto x + y$.

[8] Notice that we have $\alpha_i = (\omega^\omega)^{\alpha_i} \iff \alpha_i = \omega^{\alpha_i}$.

6 Conclusion

In this paper, we have produced a very long chain of parity tree automata of index $[0, 2]$ but of different Wadge degrees. Its length is $\varphi_2(0)+1$, where $\varphi_2(0)$ is the first fixed point of the ordinal function that itself enumerates all fixed points of the ordinal exponentiation $x \mapsto \omega^x$. We conjecture that every regular non-self-dual language of index $[0, 2]$ is, up to Wadge equivalence, recognized by an automaton in $\left(\mathcal{A}_\alpha{}^b\right)_{\alpha<\varphi_2(0)+1}$. Since degrees of self-dual languages of index $[0, 2]$ are always immediately above and below two non-self-dual degrees of languages of index $[0, 2]$, this conjecture would imply that the height of the Wadge hierarchy of regular languages of index $[0, 2]$ is exactly $\varphi_2(0) + 1$.

The whole construction is effective, meaning that the mapping $\alpha \mapsto \mathcal{A}_\alpha{}^b$ (for $0 < \alpha < \varphi_2(0)+1$) is recursive. It also means that, for any $0 < \alpha < \beta < \varphi_2(0)+1$, the relation $\mathcal{A}_\alpha{}^b <_W \mathcal{A}_\beta{}^b$ which stipulates that there exist two strategies – one that is winning for player II in the game $W(\mathcal{A}_\alpha{}^b, \mathcal{A}_\beta{}^b)$ and another one that is winning for I in the game $W(\mathcal{A}_\beta{}^b, \mathcal{A}_\alpha{}^b)$ – can be established by recursively providing such strategies. However, we did not consider any decidability issue. It thus remains open whether one can decide, given any automaton $\mathcal{B}$ and any ordinal $0 < \alpha < \varphi_2(0) + 1$, whether $\mathcal{B} <_W \mathcal{A}_\alpha{}^b$ holds or not.

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