Extended General Variational Inequalities and General Wiener–Hopf Equations

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Abstract In this paper, we show the extended general variational inequality problems are equivalent to solving the general Wiener–Hopf equations. By using the equivalence, we establish a general iterative algorithm for finding the solution of extended general variational inequalities. We also discuss the convergence criteria for the algorithm. Our results extend and improve the corresponding results announced by many others.

Keywords Variational inequalities · Wiener–Hopf equations · Iterative algorithm

1 Introduction

Variational inequality theory describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics and engineering sciences [1–11]. Projection methods and their variant forms including the Wiener–Hopf equations are being used to develop various numerical methods for solving variational inequalities. It has been shown that the Wiener–Hopf equations are more flexible and general than the projection methods. Noor [1–7] and Qin [10] have used the Wiener–Hopf equations technique to study the sensitivity analysis, dynamical systems as well as to suggest and analyze several iterative methods for solving variational inequalities. A new class of variational inequalities involving three nonlinear operators, which is called the extended general variational inequalities, is introduced and studied by Noor [9].

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Motivated and inspired by the above research, we establish the equivalence between extended general variational inequalities and general Wiener–Hopf equations in this paper. This alternative formulation is used to propose and analyze a new iterative algorithm for computing approximate solutions of extended general variational inequalities. We also study the conditions under which the approximate solution obtained from the iterative algorithms converges to the exact solution of the general variational inequalities. Results proved in this paper may be viewed as significant and improvement of previously known results.

2 Problem Statement and Preliminaries

Let H be a real Hilbert space whose inner product norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H. For given nonlinear operators $T, g, h : H \to H$, we consider the problem of finding $u \in H : h(u) \in K$ such that

$$\langle Tu, g(v) - h(u)\rangle \geq 0, \ \forall v \in H : g(v) \in K. \tag{1}$$

The inequality of the type (1) is called the extended general variational inequality, which was introduced by Noor in [9]. We would like to emphasize that problem (1) is equivalent to that of finding $u \in H : h(u) \in K$ such that

$$\langle Tu + h(u) - g(u), g(v) - h(u)\rangle \geq 0, \ \forall v \in H : g(v) \in K. \tag{2}$$

This equivalent formulation is also useful from the applications point of view.

We now list some special cases of the extended general variational inequalities.

(I) If $g = h$, then problem (1) is equivalent to that of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \ \forall v \in H : g(v) \in K,$$

which is known as general variational inequality, introduced and studied by Noor [3].

(II) For $g = I$, the identity operator, the extended general variational inequality (1) collapses to: Find $u \in H : h(u) \in K$ such that

$$\langle Tu, v - h(u)\rangle \geq 0, \ \forall v \in K,$$

which is also called the general variational inequality; see Noor [6].

(III) For $h = I$, the identity operator, then problem (1) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \ \forall v \in H : g(v) \in K,$$

which is also called the general variational inequality, see Noor [8].

(IV) For $g = h = I$, the identity operator, the extended general variational inequality (1) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \ \forall v \in K,$$

which is known as the classical variational inequality.

(V) If $K^* = \{u \in H; \langle u, v \rangle \geq 0, \forall v \in K\}$ is a polar (dual) convex cone of a closed convex cone K in H, then problem (1) is equivalent to that of finding $u \in K$ such that

$$g(u) \in K, \ \ Tu \in K^*, \ \ \langle g(u), Tu \rangle = 0$$

which is known as the general complementarity problem, which includes many previously known complementarity problems as special cases; see [2, 3, 6].

From the above discussion, it is clear that the extended general variational inequality (1) is most general and includes several known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences.

Related to the variational inequalities, we have the problems of solving the Wiener–Hopf equations. Now let

$$Q_K = I - gh^{-1}P_K,$$

where P_K is the projection of H onto K, I, is the identity operator. If g^{-1}, h^{-1} exists, then we consider the problem of finding $z \in H$ such that

$$\rho^{-1}Q_K z + Th^{-1}P_K z = 0, \tag{3}$$

where $\rho > 0$ is a constant. Equations of the type (3) are called general Wiener–Hopf equations. Note that, for $g = h$, we obtain the original Wiener–Hopf equation, introduced by Shi [11]. It is well known that the variational inequalities and Wiener–Hopf equations are equivalent. This equivalent has played a fundamental and basic role in developing some efficient and robust methods for solving variational inequalities and related optimization problems.

Recall the following definitions:

Definition 2.1 An operator $T : H \to H$ is said to be:

(I) Strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2, \forall u, v \in H.$$

(II) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|u - v\|, \forall u, v \in H.$$

(III) μ-coercive if there exists a constant $\mu > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \mu\|Tu - Tv\|^2, \forall u, v \in H.$$

Clearly, every μ-coercive operator is $1/\mu$-Lipschitz continuous.

(IV) Relaxed η-coercive if there exists a constant $\eta > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq (-\eta)\|Tu - Tv\|^2, \quad \forall u, v \in H$$

(V) Relaxed (ω, t)-coercive if there exist two constants $\omega, t > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq (-\omega)\|Tu - Tv\|^2 + t\|u - v\|^2, \quad \forall u, v \in H.$$

For $\omega = 0$, T is strongly monotone. This class of mappings is more general that the class of strongly monotone mappings.

We also need the following well-known result.

Lemma 2.1 *Let K be a closed convex subset of H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

if and only if $u = P_K z$, where P_K is the projection of H onto K.

It is well known that the projection operator P_K is a nonexpansive operator.

3 Main Results

First of all, using the technique of Noor [2], we prove the following result.

Theorem 3.1 *The extended general variational inequality* (1) *has a solution $u \in H : h(u) \in K$ if and only if $z \in H$ satisfies the general Wiener–Hopf equation* (3)*, where*

$$z = g(u) - \rho Tu, h(u) = P_K z,$$

where P_K is the projection of H onto K and $\rho > 0$ is a constant.

Proof Let $u \in H : h(u) \in K$ be a solution of the extended general variational inequality (1). Then, from (2), we have

$$\langle h(u) - (g(u) - \rho Tu), g(v) - h(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K$$

which implies, using Lemma 2.1, that

$$h(u) = P_K(g(u) - \rho Tu).$$

Using $Q_K = I - gh^{-1}P_K$, we have

$$(I - gh^{-1}P_K)(g(u) - \rho Tu) = g(u) - \rho Tu - gh^{-1}P_K(g(u) - \rho Tu)$$
$$= g(u) - \rho Tu - gh^{-1}h(u) = -\rho Tu$$
$$= \rho Th^{-1}P_K(g(u) - \rho Tu).$$

It follows that

$$\rho^{-1}Q_K z + Th^{-1}P_K z = 0,$$

where $z = g(u) - \rho Tu$.

Conversely, let $z \in H$ be a solution of the general Wiener–Hopf equation (3). Then, we have

$$\rho Th^{-1}P_K z = -Q_K z = (gh^{-1}P_K - I)z = gh^{-1}P_K z - z. \quad (4)$$

It follows from (4) and Lemma 2.1 that

$$0 \leq \langle gh^{-1}P_K z - z, g(v) - gh^{-1}P_K z\rangle = \langle \rho Th^{-1}P_K z, g(v) - gh^{-1}P_K z\rangle$$

for all $v \in H : g(v) \in K$. It follows that $u = h^{-1}P_K z$, that is, $h(u) = P_K z$ is a solution of (1) and $g(u) = gh^{-1}P_K Z$. Using (4), we have

$$z = g(u) - \rho Tu.$$

This completes the proof.

From the above Theorem 3.1, one can easily see that extended general variational inequalities and general Wiener–Hopf equations are equivalent. This equivalent is very useful from the numerical point of view. Using this equivalence and by an appropriate rearrangement, we suggest and analyze the following iterative algorithms for solving the extended general variational inequalities (1).

The general Wiener–Hopf equation (3) can be rewritten as

$$Q_K z = -\rho Th^{-1}P_K z,$$

which implies that

$$z - gh^{-1}P_K z = -\rho Th^{-1}P_K z,$$

Thus

$$z = gh^{-1}P_K z - \rho Th^{-1}P_K z = g(u) - \rho Tu.$$

Using the equality $z = (1-\alpha_n)z + \alpha_n z$, we obtain

$$z = (1-\alpha_n)z + \alpha_n(g(u) - \rho T u).$$

This formulation enables us to suggest the following iterative algorithm for solving the extended general variational inequalities (1).

Algorithm 3.1 For any $z_0 \in H$, compute the sequence $\{z_n\}$ by the iterative processes

$$h(u_n) = P_K z_n, z_n = g(u_n) - \rho T u_n, z_{n+1} = (1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho T u_n). \quad (5)$$

In order to prove our next main result, we need the following lemma.

Lemma 3.1 ([10]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in [0,1] with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, then

$$\lim_{n\to\infty} a_n = 0.$$

Theorem 3.2 *Let K be a closed convex subset of a real Hilbert space H. Let $g: H \to H$ be a relaxed (ω_1, t_1)-coercive and μ_1-Lipschitz continuous mapping, $h: H \to H$ be a μ_1-Lipschitz continuous mapping and let $T: H \to H$ be a relaxed (ω_2, t_2)-coercive and μ_2-Lipschitz continuous mapping. Let $\{z_n\}$, $\{u_n\}$ and $\{h(u_n)\}$ be sequences generated by Algorithm 3.1, $\{\alpha_n\}$ is a sequence in [0, 1]. Assume that the following conditions are satisfied:*

$$2\theta_1 + \theta_2 < 1, \quad \text{(C1)}$$

where $\theta_1 = \sqrt{1+\mu_1^2 - 2t_1 + 2\omega_1\mu_1^2}$, $\theta_2 = \sqrt{1+\rho^2\mu_2^2 - 2\rho t_2 + 2\rho\omega_2\mu_2^2}$.

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \quad \text{(C2)}$$

Then the sequence $\{z_n\}$, $\{u_n\}$ and $\{h(u_n)\}$ converge strongly to z^*, u^* and $h(u^*)$, respectively, where $z^* \in H$ is a solution of the general Wiener–Hopf equation (3), $u^* \in H : h(u^*) \in K$ is a solution of the extended general variational inequality (1).

Proof Letting $z^* \in H$ be a solution of the general Wiener–Hopf equation (3), we have

$$h(u^*) = P_K z^*, \quad z^* = g(u^*) - \rho T u^*, \quad z^* = (1-\alpha_n)z^* + \alpha_n(g(u^*) - \rho T u^*),$$

where $u^* \in H : h(u^*) \in K$ is a solution of the extended general variational inequality (1). Observing (5), we obtain

$$\begin{aligned}
\|z_{n+1} - z^*\| &= \|(1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho T u_n) - z^*\| \\
&= \|(1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho T u_n) - (1-\alpha_n)z^* + \alpha_n(g(u^*) - \rho T u^*)\| \\
&\le (1-\alpha_n)\|z_n - z^*\| + \alpha_n\|g(u_n) - g(u^*) - \rho(T u_n - T u^*)\|. \qquad (6)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\|g(u_n) - g(u^*) - \rho(T u_n - T u^*)\| \\
&\quad = \|u_n - u^* - (u_n - u^*) + g(u_n) - g(u^*) - \rho(T u_n - T u^*)\| \\
&\quad \le \|u_n - u^* - (g(u_n) - g(u^*))\| + \|u_n - u^* - \rho(T u_n - T u^*)\|. \qquad (7)
\end{aligned}$$

Now, we shall estimate the first term of right side of (7)

$$\begin{aligned}
&\|u_n - u^* - g(u_n) - g(u^*)\| \\
&\quad = \|u_n - u^*\|^2 - 2\langle g(u_n) - g(u^*), u_n - u^*\rangle + \|(g(u_n) - g(u^*))\|^2 \\
&\quad \le \|u_n - u^*\|^2 + 2\omega_1\|g(u)_n - g(u)^*\|^2 - 2t_1\|u_n - u^*\|^2 + \|g(u_n) - g(u^*)\|^2 \\
&\quad \le \|u_n - u^*\|^2 + 2\mu_1^2\omega_1\|u_n - u^*\|^2 - 2t_1\|u_n - u^*\|^2 + \mu_1^2\|u_n - u^*\|^2 \\
&\quad = (1 + 2\mu_1^2\omega_1 - 2t_1 + \mu_1^2)\|u_n - u^*\|^2 = \theta_1^2\|u_n - u^*\|^2, \qquad (8)
\end{aligned}$$

where $\theta_1 = \sqrt{1 + 2\mu_1^2\omega_1 - 2t_1 + \mu_1^2}$.

Next, we shall estimate the second term of right side of (7)

$$\begin{aligned}
&\|u_n - u^* - \rho(T u_n - T u^*)\| \\
&\le \|u_n - u^*\|^2 - 2\rho\langle T u_n - T u^*, u_n - u^*\rangle + \rho^2\|T u_n - T u^*\|^2 \\
&\le \|u_n - u^*\|^2 + 2\rho\omega_2\|T u_n - T u^*\|^2 - 2\rho t_2\|u_n - u^*\|^2 + \rho^2\|T u_n - T u^*\|^2 \\
&\le \|u_n - u^*\|^2 + 2\rho\omega_2\|T u_n - T u^*\|^2 - 2\rho t_2\|u_n - u^*\|^2 + \rho^2\|T u_n - T u^*\|^2 \\
&= (1 + 2\rho\mu_2^2\omega_2 - 2\rho t_2 + \rho^2\mu_2^2)\|u_n - u^*\|^2 = \theta_2^2\|u_n - u^*\|^2, \qquad (9)
\end{aligned}$$

where $\theta_2 = \sqrt{1 + 2\rho\mu_2^2\omega_2 - 2\rho t_2 + \rho^2\mu_2^2}$.
Substitute (8) and (9) into (7) yields that

$$\|g(u_n) - g(u^*) - \rho(T u_n - T u^*)\| \le (\theta_1 + \theta_2)\|u_n - u^*\|. \qquad (10)$$

Substituting (10) into (6), we arrive at

$$\|z_{n+1} - z^*\| \le (1-\alpha_n)\|z_{n+1} - z^*\| + \alpha_n(\theta_1 + \theta_2)\|u_n - u^*\|. \qquad (11)$$

Observe that

$$\|u_n - u^*\| = \|u_n - u^* - (h(u_n) - h(u^*)) + (P_K z_n - P_K z^*)\| \le \theta_1\|u_n - u^*\| + \|z_{n+1} - z^*,$$

which implies that

$$\|u_n - u^*\| \leq \frac{1}{1-\theta_1}\|z_{n+1} - z^*\|. \tag{12}$$

Now, substituting (12) into (11), we have that

$$\|z_{n+1} - z^*\| \leq (1 - \alpha_n(1 - \frac{\theta_1 + \theta_2}{1-\theta_1}))\|z_{n+1} - z^*\|,$$

From condition (C1), (C2) and Lemma 3.1, we have

$$\lim_{n\to\infty} \|z_{n+1} - z^*\| = 0.$$

From (12), we have

$$\lim_{n\to\infty} \|u_n - u^*\| = 0.$$

On the other hand, we have

$$\|h(u_n) - h(u^*)\| \leq \mu_1\|u_n - u^*\|.$$

It follows that

$$\lim_{n\to\infty} \|h(u_n) - h(u^*)\| = 0.$$

This completes the proof.

4 Conclusion

In this paper, we show that the extended general variational inequalities are equivalent to the general Wiener–Hopf equations. A general iterative algorithm for finding the solution of extended general variational inequalities is established by the equivalence. We also discuss the convergence criteria for the algorithm.

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