$(\in, \in\vee q)$-Fuzzy Filter Theory in FI-algebras

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Abstract The notions of $(\in, \in \vee q)$-fuzzy filters and $(\in, \in \vee q)$-fuzzy implicative (positive implicative, commutative, involution) filters in FI-algebras are introduced and their properties are investigated. Equivalent characterizations of various $(\in, \in \vee q)$-fuzzy filters are obtained. Relations among various $(\in, \in\vee q)$-fuzzy filters are discussed and it is proved that a fuzzy set is $(\in, \in\vee q)$-fuzzy positive implicative filter iff it is both $(\in, \in\vee q)$-fuzzy implicative and $(\in, \in\vee q)$-fuzzy commutative (or involution) filter.

Keywords Fuzzy logic · FI-algebra · $(\in, \in \vee q)$-fuzzy filter · $(\in, \in \vee q)$-fuzzy implicative (positive implicative, commutative, involution) filter

1 Introduction

Non-classical logic, extension and development of classical logic, has become a formal tool for computer science and artificial intelligence to deal with uncertainty information. To satisfy needs of fuzzy reasoning, many kinds of fuzzy logic algebras have been established. Among them, fuzzy implication algebras (in short, FI-algebras) posed in Ref. [1] are more extensive. Precisely, MV-algebras, bounded BCK-algebras, BL-algebras, lattice implication algebras and R_0-algebras are all FI-algebras and implications in FI-algebras generalize implication operators in various logic algebras. Thus, it is meaningful to deeply study properties of FI-algebras. Fuzzy sets introduced firstly by Zadeh in Ref. [2]. At present, some types of generalized fuzzy filters were introduced in various logical algebras by using the ideas of fuzzy sets [3–5].

In this paper, we apply the ideas and methods of fuzzy sets to FI-algebras. We introduce various concepts of $(\in, \in\vee q)$-fuzzy filters and discuss their properties. Some significative results are obtained.

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2 Preliminaries

By an fuzzy implication algebra (in short, FI-algebra), we mean an algebra $(X, \to, 0)$ such that the following conditions hold: for all $x, y, z \in X$,

(I_1) $x \to (y \to z) = y \to (x \to z)$;
(I_2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;
(I_3) $x \to x = 1$;
(I_4) $x \to y = y \to x = 1 \Rightarrow x = y$;
(I_5) $0 \to x = 1$.

In any FI-algebra X, The following statements are true, for all $x, y, z \in X$,

(I_6) $1 \to x = x, x \to 1 = 1$;
(I_7) $((x \to y) \to y) \to y = x \to y$;
(I_8) $(y \to z) \to ((x \to y) \to (x \to z)) = 1$;
(I_9) $x \leq y \Leftrightarrow x \to y = 1$;
(I_{10}) $x \leq y \Rightarrow (z \to x \leq z \to y$ and $y \to z \leq x \to z)$;
(I_{11}) $0' = 1, 1' = 0, x \leq x'', x''' = x'$;
(I_{12}) $x \leq y \Rightarrow y' \leq x'$,

where $1 = 0 \to 0$ and $x' = x \to 0$.

A fuzzy set on X is a mapping $f : X \to [0, 1]$. A fuzzy set on X with the form

$$f(y) = \begin{cases} t \in (0, 1], & y = x \\ 0, & y \neq x \end{cases} \tag{1}$$

is said to be a fuzzy point with support x and value t witch is denoted by x_t. For a fuzzy point x_t and a fuzzy set f we have

(1) x_t is said to belong to f, written as $x_t \in f$, if and only if $f(x) \geqslant t$;
(2) x_t is said to be quasi-coincident with f, written as $x_t q f$, iff $f(x) + t > 1$;
(3) If $x_t \in f$ or $x_t q f$, then we write $x_t \in\vee q f$;
(4) The symbol $\overline{\in\vee q}$ means that $\in\vee q$ does not hold.

3 $(\in, \in\vee q)$-Fuzzy Filters in FI-algebras

Definition 3.1 Let X be an FI-algebra. A fuzzy set f on X is called a $(\in, \in\vee q)$-fuzzy filter in X, if it satisfies the following conditions:

(F1) $(\forall t \in (0, 1])(\forall x \in X)(x_t \in f \Rightarrow 1_t \in\vee q f)$;
(F2) $(\forall t, r \in (0, 1])(\forall x, y \in X)((x_t \in f$ and $(x \to y)_r \in f) \Rightarrow y_{\min\{t,r\}} \in\vee q f)$.

The set of all $(\in, \in\vee q)$-fuzzy filters in X is denoted by $\mathbf{FF}(X)$.

Theorem 3.1 *The conditions (F1) and (F2) in Definition 3.1 are equivalent to the following conditions (F3) and (F4), respectively:*

(F3) $(\forall x \in X)(f(1) \geqslant \min\{f(x), 0.5\})$;
(F4) $(\forall x, y \in X)(f(y) \geqslant \min\{f(x), f(x \to y), 0.5\})$.

Proof It is similar to the proof of Theorem 1 and Theorem 2 in Ref. [5].

Remark 3.1 Let X be an FI-algebra and f a fuzzy set on X.

(1) Theorem 3.1 shows that $f \in \mathbf{FF}(X)$ iff f satisfies conditions (F3) and (F4).
(2) Let $f \in \mathbf{FF}(X)$, then it is easy to see by (1) of this remark that f satisfies the following condition:

(F5) $(\forall x, y \in X)(x \leq y \Rightarrow f(y) \geqslant \min\{f(x), 0.5\})$.

Theorem 3.2 *Let X be an FI-algebra and f a fuzzy set on X. Then $f \in \mathbf{FF}(X)$ iff f satisfies the following condition:*

(F6) $(\forall x, y, z \in X)(x \to (y \to z) = 1 \Rightarrow f(z) \geqslant \min\{f(x), f(y), 0.5\})$.

Proof Suppose $f \in \mathbf{FF}(X)$. Let $x, y, z \in X$ such that $x \to (y \to z) = 1$. Then $x \leq y \to z$, and so we have $f(y \to z) \geqslant \min\{f(x), 0.5\}$ by (F5). It follows from (F4) that $f(z) \geqslant \min\{f(y), f(y \to z), 0.5\} \geqslant \min\{f(y), f(x), 0.5\}$. Conversely, since for all $x \in X$, $x \to (x \to 1) = 1$, we have $f(1) \geqslant \min\{f(x), f(x), 0.5\} = \min\{f(x), 0.5\}$, i.e., (F3) holds. By ($I_3$) we have $(x \to y) \to (x \to y) = 1$, thus $f(y) \geqslant \min\{f(x), f(x \to y), 0.5\}$, i.e., (F4) holds. So, $f \in \mathbf{FF}(X)$ by Theorem 3.1.

4 Various $(\in, \in\vee q)$-Fuzzy Filters

4.1 $(\in, \in\vee q)$-Fuzzy Implicative Filters

Definition 4.1 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy implicative filter, if it satisfies the following condition:

(F7) $(\forall x, y, z \in X)(f(x \to z) \geqslant \min\{f(x \to (y \to z)), f(x \to y), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy implicative filters in X is denoted by $\mathbf{FIF}(X)$.

Example 4.1 Let $X = \{0, a, b, 1\}$. The operator $\to_1$ on X is defined in Table 1.
Then $(X, \to_1, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.3$, $f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FIF}(X)$.

Table 1 Definition of $\to_1$ on X

$\to_1$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Theorem 4.1 *Let X be an FI-algebra, $f \in \boldsymbol{FF}(X)$. Consider the conditions:*

(F8) $(\forall x, y, z \in X)(f(y \to z) \geqslant \min\{f(x), f(x \to (y \to (y \to z))), 0.5\})$;
(F9) $(\forall x, y \in X)(f(x \to y) \geqslant \min\{f(x \to (x \to y)), 0.5\})$;
(F10) $(\forall x, y, z \in X)(f((x \to y) \to (x \to z)) \geqslant \min\{f(x \to (y \to z)), 0.5\})$.

Then $f \in \boldsymbol{FIF}(X) \Leftrightarrow$ *(F8)*$\Leftrightarrow$ *(F9)*$\Leftrightarrow$ *(F10).*

Proof $f \in \mathbf{FIF}(X) \Rightarrow$(F8): Let $f \in \mathbf{FIF}(X)$. Take $x = y$ in (F7) first, then by (F3) and (F4) we have that $f(y \to z) \geqslant \min\{f(y \to (y \to z)), f(y \to y), 0.5\} = \min\{f(y \to (y \to z)), f(1), 0.5\} = \min\{f(y \to (y \to z)), 0.5\} \geqslant \min\{f(x), f(x \to (y \to (y \to z))), 0.5\}$. That is, (F8) holds.

(F8) $\Rightarrow$ (F9): Assume $f \in \mathbf{FF}(X)$ and f satisfies (F8). Then for all $x, y \in X$ we can obtain by (F3) that $f(x \to y) \geqslant \min\{f(1), f(1 \to (x \to (x \to y))), 0.5\} = \min\{f(x \to (x \to y)), 0.5\}$. So, f satisfies (F9).

(F9) $\Rightarrow$ (F10): Assume (F9) holds. Since for all $x, y, z \in X$ by (I_1) and (I_2) we have $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)) = x \to ((x \to y) \to (x \to z))$. By ($I_1$), (F9) and (F5) we further have that $f((x \to y) \to (x \to z)) = f(x \to ((x \to y) \to z)) \geqslant \min\{f(x \to (x \to ((x \to y) \to z))), 0.5\} = \min\{f(x \to ((x \to y) \to (x \to z))), 0.5\} \geqslant \min\{\min\{f(x \to (y \to z)), 0.5\}, 0.5\} = \min\{f(x \to (y \to z)), 0.5\}$. That is (F10) holds.

(F10)$\Rightarrow$ $f \in \mathbf{FIF}(X)$: Assume $f((x \to y) \to (x \to z)) \geqslant \min\{f(x \to (y \to z)), 0.5\}$ for all $x, y \in X$. By $((x \to y) \to (x \to z)) \to ((x \to y) \to (x \to z)) = 1$ and Theorem 3.2 we have that $f(x \to z) \geqslant \min\{f((x \to y) \to (x \to z)), f(x \to y), 0.5\}$. So, by (F10) we have that $f(x \to z) \geqslant \min\{f(x \to (y \to z)), f(x \to y), 0.5\}$. That is (F7) holds, thus, $f \in \mathbf{FIF}(X)$.

4.2 $(\in, \in\vee q)$-Fuzzy Positive Implicative Filters

Definition 4.2 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy positive implicative filter, if it satisfies the following condition:

(F11) $(\forall x, y, z \in X)(f(y) \geqslant \min\{f(x), f(x \to ((y \to z) \to y)), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy positive implicative filters in X is denoted by $\mathbf{FPIF}(X)$.

Example 4.2 Let $X = \{0, a, b, 1\}$. The operator $\to_2$ on X is defined in Table 2.

Then $(X, \to_2, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = 0.3$, $f(a) = f(b) = f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FPIF}(X)$.

Table 2 Definition of $\to_2$ on X

$\to_2$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Theorem 4.2 *Let X be an FI-algebra, $f \in \boldsymbol{FF}(X)$. Consider the conditions:*

(F12) $(\forall x, y \in X)(f(x) \geqslant \min\{f((x \to y) \to x), 0.5\})$*;*
(F13) $(\forall x \in X)(f(x) \geqslant \min\{f(x' \to x), 0.5\})$*;*
(F14) $(\forall x, y, z \in X)(f((x' \to x) \to x) \geqslant \min\{f(1), 0.5\})$.

Then $f \in \boldsymbol{FPIF}(X)$ ⇔(F12)⇔(F13)⇔ (F14).

Proof $f \in \mathbf{FPIF}(X) \Rightarrow$(F12): Assume that $f \in \mathbf{FPIF}(X)$. Then by using (F11) and (F3) we have that $f(x) \geqslant \min\{f(1), f(1 \to ((x \to y) \to x)), 0.5\} = \min\{f(1), f((x \to y) \to x), 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. So, (F12) holds.

(F12)$\Rightarrow$ $f \in \mathbf{FPIF}(X)$: Assume $f \in \mathbf{FF}(X)$ and (F12) holds. Then for all $x, y, z \in X$, by (F4) we have that $f(y) \geqslant \min\{f((y \to z) \to y), 0.5\} \geqslant \min\{f(x), f(x \to ((y \to z) \to y)), 0.5\}$. That is (F11) holds and so $f \in \mathbf{FPIF}(X)$.

(F12)$\Rightarrow$(F13): Putting $y = 0$ in (F12), one gets (F13).

(F13)$\Rightarrow$(F12): By (I_{10}) we have $x' = x \to 0 \leq x \to y$. By ($\mathrm{I}_{10}$) again we get $(x \to y) \to x \leq x' \to x$. By (F13) and (F5) we further have $f(x) \geqslant \min\{f(\neg x \to x), 0.5\} \geqslant \min\{f((x \to y) \to x), 0.5\}$, i.e., (F12) holds.

(F14)$\Rightarrow$(F12): Assume $f \in \mathbf{FF}(X)$ and (F14) holds. Since by (I_{10}) we have $x \to 0 \leq x \to y$ and $(x \to y) \to x \leq (x \to 0) \to x = x' \to x$. By (F5) we get $f(x' \to x) \geqslant \min\{f((x \to y) \to x), 0.5\}$. Thus by (F4) and (F14), we have $f(x) \geqslant \min\{f(\neg x \to x), f((x' \to x) \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, \min\{f(1), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), f(1), 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. So, (F12) holds.

(F13)$\Rightarrow$(F14): Assume (F13) holds. Then by (I_1), (I_2) and (F5) we obtain that $f((x' \to x) \to x) = f(\alpha) \geqslant \min\{f((\alpha \to 0) \to \alpha), 0.5\} = \min\{f((((x' \to x) \to x) \to 0) \to ((x' \to x) \to x)), 0.5\} = \min\{f((x' \to x) \to ((((x' \to x) \to x) \to 0) \to x)), 0.5\} \geqslant \min\{f((((x' \to x) \to x) \to 0) \to x'), 0.5\} = \min\{f((((x' \to x) \to x) \to 0) \to (x \to 0)), 0.5\} \geqslant \min\{f(x \to ((x' \to x) \to x)), 0.5\} = \min\{f((x' \to x) \to (x \to x)), 0.5\} = \min\{f(1), 0.5\}$. So (F14) holds and the proof is thus finished.

Theorem 4.3 *Let X be an FI-algebra. Then $\boldsymbol{FPIF}(X) \subseteq \boldsymbol{FIF}(X)$.*

Proof Assume $f \in \mathbf{FPIF}(X)$. Since $z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to (z \to (z \to x))$ by (I_8), we have by (F5) that $f((z \to y) \to (z \to (z \to x))) \geqslant \min\{f(z \to (y \to x)), 0.5\}$. By (F11), ($\mathrm{I}_1$) and ($\mathrm{I}_7$) we have further that $f(z \to x) \geqslant \min\{f(1), f(1 \to (((z \to x) \to x) \to (z \to x))), 0.5\} = \min\{f(((z \to x) \to x) \to (z \to x)), 0.5\} = \min\{f(z \to (((z \to x) \to x) \to x)), 0.5\} = \min\{f(z \to (z \to x)), 0.5\} \geqslant \min\{f(z \to y), f((z \to y) \to (z \to (z \to x))), 0.5\} \geqslant \min\{f(z \to y), \min\{f(z \to (y \to x)), 0.5\}, 0.5\} = \min\{f(z \to y), f(z \to (y \to x)), 0.5\}$, i.e., (F7) holds. Thus $f \in \mathbf{FIF}(X)$ and thus $\mathbf{FPIF}(X) \subseteq \mathbf{FIF}(X)$.

Remark 4.1 The converse of Theorem 4.3 may not be true. Consider the FI-algebra $(X, \to_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.7$, it is easy to verify that $f \in \mathbf{FIF}(X)$. But $f \notin \mathbf{FPIF}(X)$, since $f((a' \to a) \to a) = f((0 \to a) \to a) = f(a) = 0.4 \ngeqslant 0.5 = \min\{f(1), 0.5\}$.

Theorem 4.4 *Let X be an FI-algebra. Then* $\boldsymbol{FIF}(X) \subseteq \boldsymbol{FPIF}(X)$ *if and only if the following condition holds for all* $f \in \boldsymbol{FIF}(X)$*:*

(F15) $(\forall x, y \in X)(f((x \to y) \to y) \geqslant \min\{f((y \to x) \to x), 0.5\})$.

Proof Assume $\mathbf{FIF}(X) \subseteq \mathbf{FPIF}(X)$. Then for all $f \in \mathbf{FIF}(X)$, we have that $f \in \mathbf{FPIF}(X)$. Since $y \leq (x \to y) \to y$, we have $((x \to y) \to y) \to x \leq y \to x$. By ($I_1$) and ($I_2$), we have further that $(((x \to y) \to y) \to x) \to ((x \to y) \to y) \geq (y \to x) \to ((x \to y) \to y) = (x \to y) \to ((y \to x) \to y) \geq (y \to x) \to x$. Hence by (F11) and (F5) we have that $f((x \to y) \to y) \geqslant \min\{f(1), f(1 \to (((x \to y) \to y) \to x) \to ((x \to y) \to y))), 0.5\} = \min\{f((((x \to y) \to y) \to x) \to ((x \to y) \to y)), 0.5\} \geqslant \min\{\min\{f((y \to x) \to x), 0.5\}, 0.5\} = \min\{f((y \to x) \to x), 0.5\}$, that is f satisfies (F15).

Conversely, let $f \in \mathbf{FIF}(X)$ and for any $x, y \in X$, it satisfies that $f((x \to y) \to y) \geqslant \min\{f((y \to x) \to x), 0.5\}$. It follows from ($I_2$) that $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$. Thus by (F5) and (F9), we have that $f((y \to x) \to x) \geqslant \min\{f((x \to y) \to y), 0.5\} \geqslant \min\{\min\{f((x \to y) \to ((x \to y) \to y)), 0.5\}, 0.5\} = \min\{f((x \to y) \to ((x \to y) \to y)), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. Since $y \leq x \to y$, we have $(x \to y) \to x \leq y \to x$ and by (F5) $f(y \to x) \geqslant \min\{f((x \to y) \to x), 0.5\}$. So, by (F4) we can obtain that $f(x) \geqslant \min\{f(y \to x), f((y \to x) \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, \min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. Hence $f \in \mathbf{FPIF}(X)$ by (F12), and so $\mathbf{FIF}(X) \subseteq \mathbf{FPIF}(X)$.

4.3 $(\in, \in\vee q)$-Fuzzy Commutative Filters

Definition 4.3 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy commutative filter, if it satisfies the following condition:

(F16) $(\forall x, y, z \in X)(f(((x \to y) \to y) \to x) \geqslant \min\{f(z), f(z \to (y \to x)), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy commutative filters in X is denoted by $\mathbf{FCF}(X)$.

Example 4.3 Let $X = \{0, a, b, 1\}$. The operator $\to_3$ on X is defined in Table 3.

Then $(X, \to_3, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.3$, $f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FCF}(X)$.

Table 3 Definition of $\to_3$ on X

$\to_3$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Theorem 4.5 *Let X be an FI-algebra, $f \in$ **FF**(X). Then $f \in$ **FCF**(X) if and only if f satisfies the following condition:*

(F17) $(\forall x, y \in X)(f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\})$.

Proof Assume $f \in \mathbf{FCF}(X)$. Then taking $z = 1$ in (F16) we can obtain that $f(((x \to y) \to y) \to x) \geqslant \min\{f(1), f(1 \to (y \to x)), 0.5\} = \min\{f(1), f(y \to x), 0.5\} = \min\{f(y \to x), 0.5\}$. That is (F17) holds.

Conversely, assume (F17) holds, i.e., for all $x, y \in X$, $f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\}$. Then for all $x, y, z \in X$, since $f \in \mathbf{FF}(X)$, by (F4), we have $f(y \to x) \geqslant \min\{f(z), f(z \to (y \to x)), 0.5\}$. So, $f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\} \geqslant \min\{\min\{f(z), f(z \to (y \to x)), 0.5\}, 0.5\} = \min\{f(z), f(z \to (y \to x)), 0.5\}$, i.e., (F16) holds. Thus $f \in \mathbf{FCF}(X)$.

Theorem 4.6 *Let X be an FI-algebra. Then **FPIF**$(X) \subseteq$ **FCF**(X).*

Proof Assume $f \in \mathbf{FPIF}(X)$. Since $x \leq ((x \to y) \to y) \to x$, by (I_{10}) we have that $(((x \to y) \to y) \to x) \to y \leq x \to y$. Letting $\alpha = ((x \to y) \to y) \to x$, by (I_1) and (I_{10}), we have that $(\alpha \to y) \to \alpha = ((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$. So, $f((\alpha \to y) \to \alpha) \geqslant \min\{f(y \to x), 0.5\}$ by (F5). Then by (F12) we have that $f(((x \to y) \to y) \to x) = f(\alpha) \geqslant \min\{f((\alpha \to y) \to \alpha), 0.5\} \geqslant \min\{\min\{f(y \to x), 0.5\}, 0.5\} = \min\{f(y \to x), 0.5\}$, that is, f satisfies (F17). Thus $f \in \mathbf{FCF}(X)$ by Theorem 4.5.

Remark 4.2 The converse of Theorem 4.6 may not be true. Consider the FI-algebra $(X, \to_3, 0)$ in Example 4.3, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.7$, it is easy to verify that $f \in \mathbf{FCF}(X)$. But $f \notin \mathbf{FPIF}(X)$, since $f(b) = 0.4 \not\geqslant 0.5 = \min\{f((b \to a) \to b), 0.5\}$.

Theorem 4.7 *Let X be an FI-algebra. Then **FPIF**$(X) =$ **FIF**$(X) \cap$ **FCF**(X).*

Proof By Theorems 4.3 and 4.6 we have that $\mathbf{FPIF}(X) \subseteq \mathbf{FIF}(X) \cap \mathbf{FCF}(X)$.

To show that $\mathbf{FIF}(X) \cap \mathbf{FCF}(X) \subseteq \mathbf{FPIF}(X)$, assume $f \in \mathbf{FIF}(X) \cap \mathbf{FCF}(X)$. Then for all $x, y \in X$, since $x \leq (x \to y) \to y$, we have $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$ by (I_{10}). Thus by (F9) and (F5) we obtain that $f((x \to y) \to y) \geqslant \min\{f((x \to y) \to ((x \to y) \to y)), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. Since $y \leq x \to y$, we have $(x \to y) \to x \leq y \to x$ by (I_8). Thus by (F17) and (F5) we obtain further that $f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. So, by (F4) we have that $f(x) \geqslant \min\{f((x \to y) \to y), f(((x \to y) \to y) \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, \min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. It follows from Theorem 4.2 that $f \in \mathbf{FPIF}(X)$. Hence $\mathbf{FIF}(X) \cap \mathbf{FCF}(X) \subseteq \mathbf{FPIF}(X)$.

Remark 4.3 Let X be an FI-algebra. We claim that $\mathbf{FIF}(X) \nsubseteq \mathbf{FCF}(X)$ and $\mathbf{FCF}(X) \nsubseteq \mathbf{FIF}(X)$. In fact, on the one hand, consider the FI-algebra $(X, \rightarrow_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.6$, then $f \in \mathbf{FIF}(X)$, but $f \notin \mathbf{FCF}(X)$, since $f(((b \rightarrow a) \rightarrow a) \rightarrow b) = f(b) = 0.4 < 0.5 = \min\{f(a \rightarrow b), 0.5\}$. On the other hand, consider the FI-algebra $(X, \rightarrow_3, 0)$ in Example 4.3, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.6$, then $f \in \mathbf{FCF}(X)$, but $f \notin \mathbf{FIF}(X)$, since $f(b \rightarrow a) = f(b) = 0.4 < 0.5 = \min\{f(b \rightarrow (b \rightarrow a)), f(b \rightarrow b), 0.5\}$.

4.4 $(\in, \in\vee q)$-Fuzzy Involution Filters

Definition 4.4 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy involution filter, if it satisfies the following condition:

(F18) $(\forall x, y \in X)(f(y) \geqslant \min\{f(x), f((x \rightarrow y)''), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy involution filters in X is denoted by $\mathbf{FINF}(X)$.

Example 4.4 Let $X = \{0, a, b, 1\}$. The operator $\rightarrow_4$ on X is defined in Table 4.

Then $(X, \rightarrow_4, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = 0.3$, $f(b) = f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FINF}(X)$.

Remark 4.4 Let X be an FI-algebra. Then $\mathbf{FF}(X) \subseteq \mathbf{FINF}(X)$ general does not hold. In fact, consider the FI-algebra $(X, \rightarrow_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = 0.4$, $f(b) = f(1) = 0.6$, then $f \in \mathbf{FF}(X)$, but $f \notin \mathbf{FINF}(X)$, since $f(a) = 0.4 < 0.5 = \min\{f(b), f((b \rightarrow a)''), 0.5\}$.

Theorem 4.8 *Let X be an FI-algebra, $f \in \boldsymbol{FF}(X)$. Then $f \in \boldsymbol{FINF}(X)$ if and only if f satisfies the following condition:*

(F19) $(\forall x \in X)(f(x) \geqslant \min\{f(x''), 0.5\})$.

Proof Assume $f \in \mathbf{FINF}(X)$. Then for all $x \in L$, by (F18) and (F3) we can obtain that $f(x) \geqslant \min\{f(1), f((1 \rightarrow x)''), 0.5\} = \min\{f(1), f(x''), 0.5\} = \min\{f(x''), 0.5\}$. That is (F19) holds.

Conversely, assume (F19) holds, i.e., for all $x \in X$, $f(x) \geqslant \min\{f(x''), 0.5\}$. Then for all $x, y \in X$, since $f \in \mathbf{FF}(X)$, by (F4), we have $f(y) \geqslant \min\{f(x), f(x \rightarrow y), 0.5\} \geqslant \min\{f(x), \min\{f((x \rightarrow y)''), 0.5\}, 0.5\} = \min\{f(x), f((x \rightarrow y)''), 0.5\}$. i.e., (F18) holds. Thus $f \in \mathbf{FINF}(X)$.

Table 4 Definition of $\rightarrow_4$ on X

$\rightarrow_4$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Remark 4.5 Let X be an FI-algebra. We claim that $\mathbf{FINF}(X) \nsubseteq \mathbf{FIF}(X)$ and $\mathbf{FIF}(X) \nsubseteq \mathbf{FINF}(X)$. In fact, on the one hand, consider the FI-algebra $(X, \rightarrow_4, 0)$ in Example 4.4, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.2$, $f(1) = 0.9$, then $f \in \mathbf{FINF}(X)$, but $f \notin \mathbf{FIF}(X)$, since $f(a \rightarrow 0) = f(b) = 0.2 < 0.5 = \min\{f(a \rightarrow a), f(a \rightarrow (a \rightarrow 0)), 0.5\}$. On the other hand, consider the FI-algebra $(X, \rightarrow_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = 0.2$, $f(b) = f(1) = 0.9$, then $f \in \mathbf{FIF}(X)$, but $f \notin \mathbf{FINF}(X)$, since $f(a) = 0.2 < 0.5 = \min\{f(b), f((b \rightarrow a)''), 0.5\}$.

Theorem 4.9 *Let X be an FI-algebra. Then $\boldsymbol{FCF}(X) \subseteq \boldsymbol{FINF}(X)$.*

Proof Assume $f \in \mathbf{FCF}(X)$. Since $0 \rightarrow (x \rightarrow y) = 1$, by (F17), we have that $f((x \rightarrow y)'' \rightarrow (x \rightarrow y)) = f((((x \rightarrow y) \rightarrow 0) \rightarrow 0) \rightarrow (x \rightarrow y)) \geqslant \min\{f(0 \rightarrow (x \rightarrow y)), 0.5\} = \min\{f(1), 0.5\}$. Thus by (F4) we can obtain that $f(y) \geqslant \min\{f(x), f(x \rightarrow y), 0.5\} \geqslant \min\{f(x), \min\{f((x \rightarrow y)''), f((x \rightarrow y)'' \rightarrow (x \rightarrow y)), 0.5\}, 0.5\} \geqslant \min\{f(x), \min\{f((x \rightarrow y)''), \min\{f(1), 0.5\}, 0.5\}, 0.5\} = \min\{f(x), f((x \rightarrow y)''), f(1), 0.5\} = \min\{f(x), f((x \rightarrow y)''), 0.5\}$, that is, f satisfies (F18). Thus $f \in \mathbf{FINF}(X)$ by Definition 4.4.

Corollary 4.1 *Let X be an FI-algebra. Then $\boldsymbol{FPIF}(X) \subseteq \boldsymbol{FINF}(X)$.*

Remark 4.6 The converse of Theorem 4.9 and Corollary 4.1 may not be true. Let $X = \{0, a, b, c, 1\}$. The operator $\rightarrow_5$ on X is defined in Table 5.

Then $(X, \rightarrow_5, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.3$, $f(c) = f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FINF}(X)$. But $f \notin \mathbf{FCF}(X)$, and thus $f \notin \mathbf{FPIF}(X)$, since $f(((b \rightarrow a) \rightarrow a) \rightarrow b) = f((a \rightarrow a) \rightarrow b) = f(1 \rightarrow b) = f(b) = 0.3 < 0.5 = \min\{f(a \rightarrow b), 0.5\}$.

Theorem 4.10 *Let X be an FI-algebra. Then $\boldsymbol{FPIF}(X) = \boldsymbol{FIF}(X) \cap \boldsymbol{FINF}(X)$.*

Proof By Theorem 4.3 and Corollary 4.1 we have that $\mathbf{FPIF}(X) \subseteq \mathbf{FIF}(X) \cap \mathbf{FINF}(X)$. To show that $\mathbf{FIF}(X) \cap \mathbf{FINF}(X) \subseteq \mathbf{FPIF}(X)$, assume $f \in \mathbf{FIF}(X) \cap \mathbf{FINF}(X)$, Then for all $x \in X$, since $x \leqslant x''$, we have $x' \rightarrow x \leqslant x' \rightarrow x''$ by (I_{10}). Thus by (F5) we have that $f(x' \rightarrow x'') \geqslant \min\{f(x' \rightarrow x), 0.5\}$, and thus by $f \in \mathbf{FIF}(X)$ and (F9) we have $f(x'') = f(x' \rightarrow 0) \geqslant \min\{f(x' \rightarrow (x' \rightarrow 0)), 0.5\} = \min\{f(x' \rightarrow x''), 0.5\}$. This together with $f \in \mathbf{FINF}(X)$ and (F19) we can obtain that $f(x) \geqslant \min\{f(x''), 0.5\} \geqslant \min\{f(x' \rightarrow x''), 0.5\} \geqslant \min\{f(x' \rightarrow x), 0.5\}$, i.e., f satisfies (F13). It follows from Theorem 4.2 that $f \in \mathbf{FPIF}(X)$. Hence $\mathbf{FIF}(X) \cap \mathbf{FINF}(X) \subseteq \mathbf{FPIF}(X)$.

Table 5 Definition of $\rightarrow_5$ on X

$\rightarrow_5$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

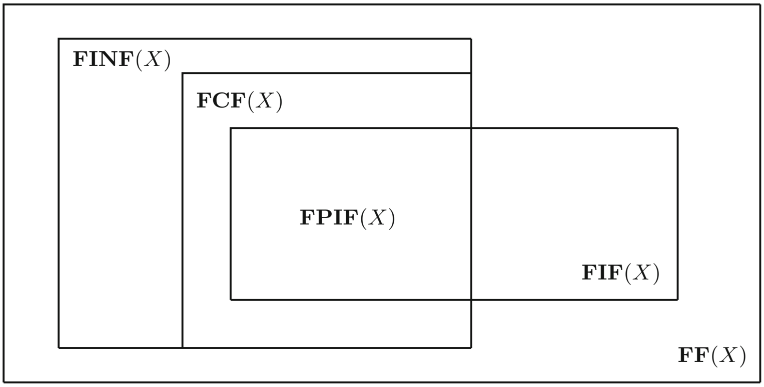

Fig. 1 Relations among sets of various $(\in, \in\vee q)$-fuzzy filters in FI-algebras

5 Conclusion

In this paper, we introduced the notions of $(\in, \in\vee q)$-fuzzy filters and $(\in, \in\vee q)$-fuzzy implicative (resp., positive implicative, commutative, involution) filters in FI-algebras and investigated their properties and relationships (as Fig. 1). Some significant characterizations of the mentioned kinds of filters are obtained. Since FI-algebras are more extensive, the results obtained in this paper can be applied to some other logical algebras such as MV-algebras, BL-algebras, R_0-algebras, MTL-algebras and lattice implication algebras and so on. So, it is hoped that more research topics of Non-classical Mathematical Logic will arise with this work.

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