Complex Fuzzy Set-Valued Complex Fuzzy Integral and Its Convergence Theorem

Sheng-quan Ma and Sheng-gang Li

Abstract This paper is devoted to propose the convergence problem of complex fuzzy set-valued complex fuzzy integral base on the complex fuzzy sets values complex fuzzy measure. We introduces the concepts of the complex fuzzy set-valued complex fuzzy measure in [1], the complex fuzzy set-valued measurable function in [2], and the complex fuzzy set-valued complex fuzzy integral in [3]. And then, we focuses on convergence problem of complex fuzzy set-valued complex fuzzy integral, obtained some convergence theorems.

Keywords Complex fuzzy set-valued measure · Complex fuzzy set-valued measurable function · Complex fuzzy set-valued complex fuzzy integral · Convergence theorem

1 Introductions

In 1998, fuzzy measure range is extended to the fuzzy real number field by Wu et al. [4] etc., which give the definition of Sugeno integral base on fuzzy number fuzzy measure, Guo et al. [5] etc. Also give the definition of (G) integral on fuzzy measure of fuzzy valued functions, which will be generalized the Sugeno integral to fuzzy sets [6]. In 1989 Buckley [7] proposed the concepts of fuzzy complex number, including people need to consider the measure and integration problems of fuzzy complex numbers, introduction fuzzy distance by Zhang [8–12], which discussed the fuzzy real valued measure problem of fuzzy sets, and give the fuzzy real valued

S. Ma (✉) · S. Li
College of Mathematics and Information Science, Shanxi Normal University,
Xi'an 710062, China
e-mail: mashengquan@163.com

S. Li
e-mail: shengganglinew@126.com

S. Ma
School of Information and Technology, Hainan Normal University, Haikou 571158, China

B.-Y. Cao et al. (eds.), *Fuzzy Systems & Operations Research and Management*, Advances in Intelligent Systems and Computing 367,
DOI 10.1007/978-3-319-19105-8_14

fuzzy integral; in 1996, fuzzy measure and measurable function concept is extended to fuzzy complex sets by Qiu et al. [13], which given the concept of complex fuzzy measure, complex fuzzy measurable function and complex fuzzy integral, Wang and Li [14] etc. in 1999 based on the concepts of fuzzy number of Buckley, gives the concept of fuzzy complex valued measures and fuzzy complex valued integral, obtain some important results. The [15–17] study measurable function and its integral of complex fuzzy number set, especially Sugeno and Choquet type fuzzy complex numerical integral and its properties, and its application in classification technique. In this paper, base on the research work on the basis of [1–3], gives the convergence theorem of complex fuzzy set-valued complex fuzzy integral, lays a foundation for the complex fuzzy set-valued complex fuzzy integral theory.

2 Complex Fuzzy Set-Valued Complex Fuzzy Measure

Definition 1 ([18]) Suppose (X, F) is a classic measurable space, $E \to F$, mapping $f : X \to [-\infty, +\infty]$, f is called real valued measurable function of (X, F) on $\tilde{E}$, if and only if $\forall \alpha \in (-\infty, +\infty)$, $\tilde{E} \cap \chi_{F_\alpha} \in \mathcal{F}$, $\widetilde{E} \cap x^c_{F_\alpha} \in \mathcal{F}$, where $F_\alpha = \{x : f(x) \geq \alpha\}$, mapping $\tilde{f} : X \to F^*(R)$, f is called real valued measurable function of $(X, \mathcal{F})$ on $\tilde{E}$, if and only if $\forall \lambda \in (0, 1]$, $f_\lambda{}^-(x)$, $f_\lambda{}^+(x)$ is a real valued measurable function, where

$$\tilde{f}(x) = \bigcup_{\lambda \in [0,1]} \lambda[(f(x))^-_\lambda, (f(x))^+_\lambda] \triangleq \bigcup_{\lambda \in [0,1]} \lambda[f^-_\lambda(x), f^+_\lambda(x)].$$

Definition 2 ([1]) Suppose Z is a non-empty complex numbers set, $F(Z)$ is set kinds on Z that consisting of by all complex fuzzy set $\tilde{\rho}$ is fuzzy complex valued distance that defined in $F(Z)$, set function

$$\tilde{\mu} : F(Z) \to F^*_+(K) = \{\tilde{A} + i\tilde{B} : \tilde{A}, \tilde{B} \in F^*_+(R), i = \sqrt{-1}\},$$

$$\tilde{A} + i\tilde{B} \mapsto \tilde{\mu}(\tilde{A} + i\tilde{B}) \in F^*_+(K)$$

called complex fuzzy set-valued complex fuzzy measure on $(Z, F(Z))$ if and only if

1. $\tilde{\mu}(\phi) = \tilde{0}, \tilde{0} = (\tilde{0}, \tilde{0}), \tilde{0} \in F^*(K)$,
2. $\forall \tilde{A}, \tilde{B} \in F(Z), \tilde{A} \subset \tilde{B} \Rightarrow \tilde{\mu}(\tilde{A}) \leq \tilde{\mu}(\tilde{B})$, where $\mathrm{Re}\tilde{\mu}(\tilde{A}) \leq \mathrm{Re}\tilde{\mu}(\tilde{B})$, $\mathrm{Im}\tilde{\mu}(\tilde{A}) \leq \mathrm{Im}\tilde{\mu}(\tilde{B})$,
3. $\{\tilde{A}_n\} \subset F(Z)\, \tilde{A}_n \subset \tilde{A}_{n+1}\, (n = 1, 2 \ldots) \Rightarrow \tilde{\rho} \lim_{n\to\infty} \tilde{\mu}\tilde{A}_n = \tilde{\mu}\left(\bigcup_{n=1}^{\infty} \tilde{A}_n\right)$,
4. $\left\{\tilde{A}_n\right\} \subset F(Z)\, \tilde{A}_n \supset \tilde{A}_{n+1}\, (n = 1, 2 \ldots)$,

note as $\tilde{\mu} = \mathrm{Re}\tilde{\mu} + i\mathrm{Im}\tilde{\mu} \triangleq \tilde{\mu}_R + i\tilde{\mu}_I$, $(i = \sqrt{-1})$.

Definition 3 ([2]) Suppose $Z \subset K$ is a non-empty set of complex numbers, $(Z, \mathcal{F}(Z))$ is a classical complex measurable space, $\tilde{E} \in \mathcal{F}(Z)$, mapping $f : Z \to K$, called f is a complex valued measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z))$, if and only if $\forall a + ib \in K$, $\tilde{E} \cap \chi_{F_{a,b}} \in \mathcal{F}(Z)$, and $\tilde{E} \cap {\chi^c}_{F_{a,b}} \in \mathcal{F}(Z)$, where

$$F_{a,b} = \{z \in K \,|\mathrm{Re}\,[f(z)] \geq a, \mathrm{Im}\,[f(z)] \geq b\}.$$

Definition 4 ([2]) Suppose Z is a non-empty complex numbers set, $\tilde{E} \in \mathcal{F}(Z)$, mapping $\tilde{f} : Z \to F_0(K)$, $z \mapsto \tilde{f}(z) = \mathrm{Re}\tilde{f}(z) + i\mathrm{Im}\tilde{f}(z) \in F_0(K)$, $i = \sqrt{-1}$,

$$\tilde{f}(z) = \bigcup_{\lambda \in [0,1]} \lambda \left[\left(\mathrm{Re}\tilde{f}(z)\right)\right]_\lambda + i \bigcup_{\lambda \in [0,1]} \lambda \left[\mathrm{Im}\tilde{f}(z)\right]_\lambda$$

$$\triangleq \bigcup_{\lambda \in [0,1]} \lambda \mathrm{Re}\tilde{f}_\lambda(z) + i \bigcup_{\lambda \in [0,1]} \lambda \mathrm{Im}\tilde{f}_\lambda(z)$$

$$\triangleq \bigcup_{\lambda \in [0,1]} \lambda \left[\mathrm{Re}\tilde{f}_\lambda^-(z), \mathrm{Re}\tilde{f}_\lambda^+(z)\right] + i \bigcup_{\lambda \in [0,1]} \lambda \left[\mathrm{Im}\tilde{f}_\lambda^-(z), \mathrm{Im}\tilde{f}_\lambda^+(z)\right],$$

then $(Z, \mathcal{F}(Z), \tilde{\mu})$ is complex fuzzy valued fuzzy measure space, called $\tilde{f}$ is complex fuzzy valued complex fuzzy measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z), \tilde{\mu})$ if and only if $\forall \lambda \in [0, 1]$, $\mathrm{Re}\tilde{f}_\lambda(z)$, $\mathrm{Im}\tilde{f}_\lambda(z)$, which are complex valued measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z))$

record

$$F_{(\alpha,\beta),\lambda} \triangleq \left\{z = \alpha + i\beta : \mathrm{Re} f_\lambda^\pm(z) \geq \alpha, \mathrm{Im} f_\lambda^\pm(z) \geq \beta\right\},$$

where

$$\mathrm{Re} f_\lambda^\pm(z) \geq \alpha,$$

express

$$\mathrm{Re} f_\lambda^+(z) \geq \alpha \text{ and } \mathrm{Re} f_\lambda^-(z) \geq \alpha, \mathrm{Im} f_\lambda^\pm(z) \geq \beta,$$

express

$$\mathrm{Im} f_\lambda^+(z) \geq \beta \text{ and } \mathrm{Im} f_\lambda^-(z) \geq \beta, \forall \alpha, \beta \in [0, \infty),$$

thus, $\tilde{f}$ is complex fuzzy valued complex fuzzy measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z), \tilde{\mu})$ if and only if $\forall \lambda \in [0, 1]$, $\tilde{E} \cap \chi_{F_{(\alpha,\beta))\lambda}} \in \mathcal{F}(Z)$, $\tilde{E} \cap {\chi^c}_{F_{(\alpha,\beta),\lambda}} \in \mathcal{F}(Z)$ $\tilde{M}\left(\tilde{E}\right)$ express all of the complex fuzzy set-valued measurable function on $\tilde{E}$.

3 Complex Fuzzy Set-valued Complex Fuzzy Integral and Its Properties

Definition 5 ([3]) Suppose $(Z, \mathcal{F}(Z), \tilde{\mu})$ is complex fuzzy set-valued fuzzy measure space, $\tilde{E} \in \mathcal{F}(Z)$, $\tilde{f} : Z \to F_0(K)$, define $\tilde{f}$ is complex fuzzy set-valued complex fuzzy integral on $\tilde{E}$ about $\tilde{\mu}$,

$$\int_{\tilde{E}} \tilde{f} d\tilde{\mu} \triangleq \left(\int_{\tilde{E}} \mathrm{Re} \tilde{f} d\tilde{\mu}_R, \int_{\tilde{E}} \mathrm{Im} \tilde{f} d\tilde{\mu} \right)_I,$$

where

$$\int_{\tilde{E}} \mathrm{Re} \tilde{f} d\tilde{\mu}_R = \bigcup_{\lambda \in [0,1]} \lambda \left[\int_{\tilde{E}} \mathrm{Re} f_\lambda{}^- d\tilde{\mu}_R, \int_{\tilde{E}} \mathrm{Re} f_\lambda{}^+ d\tilde{\mu}_R \right]$$

$$= \bigcup_{\lambda \in [0,1]} \lambda \left[\sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Re}\tilde{\mu}_\lambda^- \left(\tilde{A} \cap \chi_{F_{\lambda,\alpha,1}{}^-} \right), \sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Re}\tilde{\mu}_\lambda^+ \left(\tilde{A} \cap \chi_{F_{\lambda,\alpha,1}{}^+} \right) \right]$$

$$\int_{\tilde{E}} \mathrm{Im} \tilde{f} d\tilde{\mu}_I = \bigcup_{\lambda \in [0,1]} \lambda \left[\int_{\tilde{E}} \mathrm{Im} f_\lambda{}^- d\tilde{\mu}_I, \int_{\tilde{E}} \mathrm{Im} f_\lambda{}^+ d\tilde{\mu}_I \right]$$

$$= \bigcup_{\lambda \in [0,1]} \lambda \left[\sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Im}\tilde{\mu}_\lambda^- (\tilde{A} \cap \chi_{F_{\lambda,\alpha,2}{}^-}, \sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Im}\tilde{\mu}_\lambda^+ (\tilde{A} \cap \chi_{F_{\lambda,\alpha,2}{}^+} \right],$$

where

$$F_{\lambda,\alpha,1}{}^- = \left\{ z | \mathrm{Re} f_\lambda{}^- (z) \geq \alpha \right\},$$

$$F_{\lambda,\alpha,1}{}^+ = \left\{ z | \mathrm{Re} f_\lambda{}^+ (z) \geq \alpha \right\},$$

$$F_{\lambda,\alpha,2}{}^- = \left\{ z | \mathrm{Im} f_\lambda{}^- (z) \geq \alpha \right\},$$

$$F_{\lambda,\alpha,2}{}^+ = \left\{ z | \mathrm{Im} f_\lambda{}^+ (z) \geq \alpha \right\},$$

now called $\tilde{f}$ complex fuzzy set-value complex fuzzy integrable in $\tilde{E}$ about $\tilde{\mu}$.

Complex fuzzy set-valued complex fuzzy integral has the following important properties:

Theorem 1 *([3]) Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space:*

1. *$\forall \tilde{E} \in \mathcal{F}(Z)$, $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, then*

$$\int_{\tilde{E}} \tilde{f} d\tilde{\mu} \in F^{*}(K),$$

2. *Suppose $\tilde{E} \in \mathcal{F}(Z)$, if $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, $\chi_{\tilde{E}}$ is the characteristic function of $\tilde{E}$, then $\int_{\tilde{E}} \tilde{f} d\tilde{\mu} = \int \tilde{f} \chi_{\tilde{E}} d\tilde{\mu}$,*
3. *Let $\tilde{E} \in \mathcal{F}(Z)$. If $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, if $\tilde{\mu}\left(\tilde{E}\right) = \tilde{0}$, then*

$$\int_{\tilde{E}} \tilde{f} d\tilde{\mu} = \tilde{0},$$

4. *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$. If $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, if $\tilde{A} \subset \tilde{B}$, then*

$$\int_{\tilde{A}} \tilde{f} d\tilde{\mu} \subseteq \int_{\tilde{B}} \tilde{f} d\tilde{\mu},$$

5. *Let $\tilde{A} \in \mathcal{F}(Z)$. If $\tilde{f}_1, \tilde{f}_2 \in \tilde{M}\left(\tilde{E}\right)$, if $\tilde{f}_1 \subseteq \tilde{f}_2$ in $\tilde{A}$, then*

$$\int_{\tilde{A}} \tilde{f}_1 d\tilde{\mu} \subseteq \int_{\tilde{A}} \tilde{f}_2 d\tilde{\mu},$$

these properties are demonstrated in [18]. Here ignore.

4 Complex Fuzzy Set-Value Complex Fuzzy Integral and Its Convergence Theorem

Theorem 2 *Suppose $(Z, \mathcal{F}(Z), \tilde{\mu})$ is complex fuzzy set-valued fuzzy measure space, $\left\{\tilde{f}_n\right\}$ is non-negative complex fuzzy set-valued complex fuzzy integrable function sequence in $(Z, \mathcal{F}(Z), \tilde{\mu})$, $A \in \mathcal{F}(Z)$, if in A, $\left\{\tilde{f}_n\right\}$ monotone convergence in $\tilde{f}$ incrementing, then $\lim\limits_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$.*

Proof Suppose $A = X$, because $\tilde{f}_n \leq \tilde{f}$, $(n = 1, 2, \ldots)$, so by the generalized complex fuzzy set-valued complex fuzzy integral properties have the following result

$$\lim_{n\to\infty} \int_X \tilde{f}_n d\tilde{\mu} \leq \int_X \tilde{f} d\tilde{\mu},$$

now to proof the opposite inequality.

Let $I = \int_X \tilde{f} d\tilde{\mu}$. Then

1. if $I = 0$, conclusion obvious,
2. if $0 < I < \infty + i\infty$ then

$$I = \sup_{\mathrm{Re}\alpha \in [0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}\left(\tilde{f}\right)_\alpha\right) + i \sup_{\mathrm{Im}\alpha \in [0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_\alpha\right)\right)$$

to know the exist $\alpha_k > 0$ makes

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > ReI - \frac{1}{2k}, S\left(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > ImI - \frac{1}{2k},$$
$$(k = 1, 2, \ldots), \quad another \quad \tilde{f}_n \quad \uparrow \tilde{}\, f \quad have$$

$$\left(\tilde{f}_n\right)_{\alpha_k} \uparrow \left(\tilde{f}\right)_{\alpha_k},$$

then using the properties of generalized triangle norm, know exist n_k, such that, when $n \geq n_k$,

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right)$$

$$> ReI - \frac{1}{2k}, S\left(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right), > ImI - \frac{1}{2k}, (k = 1, 2, \ldots),$$

when $n \geq n_k$,

$$\int_X \tilde{f}_n d\tilde{\mu} > I - \frac{1}{k}, (k = 1, 2, \ldots),$$

by the arbitrariness of k have

$$\int_X \tilde{f} d\tilde{\mu} \leq \lim_{n \to \infty} \int_X \tilde{f}_n d\tilde{\mu}.$$

3. if $I = \infty + i\infty$, then $\alpha_k > 0$ makes

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\left(\tilde{f}_{\alpha_k}\right)\right)\right) > \mathrm{k},$$

$$S(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}(\tilde{f}_{\alpha_k})) > \mathrm{k}, (k = 1, 2, \ldots),$$

exit n_k, such that when $n \geq n_k$,

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > \mathrm{k},$$

$$S\left(\text{Im}\alpha_k, \text{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > \text{k},$$

then

$$\lim_{n\to\infty}\int_X \tilde{f}_n d\tilde{\mu} \geq \int_X \tilde{f} d\tilde{\mu} \geq S\left(\text{Re}\alpha_k, \text{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right)$$

$$+iS\left(\text{Im}\alpha_k, \text{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > k + ik\,(k = 1, 2, \ldots),$$

that is $\lim_{n\to\infty}\int_X \tilde{f}_n d\tilde{\mu} \geq \int_X \tilde{f} d\tilde{\mu}$.

Theorem 3 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space,* $\left\{\tilde{f}_n\right\}$ *is complex fuzzy set value complex fuzzy integrable function sequence in* $(Z, \mathcal{F}(Z), \tilde{\mu})$*,* $A \in \mathcal{F}(Z)$*, if in A,* $\left\{\tilde{f}_n\right\}$ *decrease monotonically converges to* $\tilde{f}$*, and for arbitrarily* $\varepsilon_i > 0$, $(i = 1, 2)$*, where* $\varepsilon = \varepsilon_1 + i\varepsilon_2$ *exit* n_0 *makes*

$$\tilde{\mu}\left(\left\{x|\tilde{f}_{n_0} > \int_A \tilde{f} d\tilde{\mu} + \varepsilon\right\} \cap A\right) < \infty + i\infty,$$

then

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}.$$

Proof $\tilde{f}_1 \geq \tilde{f}_2 \geq \cdots$, so $\int_A \tilde{f}_1 d\tilde{\mu} \geq \int_A \tilde{f}_2 d\tilde{\mu} \geq \cdots$, then

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu},$$

there

$$\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} = \mathop{\wedge}_{n=1}^{\infty}\text{Re}\int_A \tilde{f}_n d\tilde{\mu} + i\mathop{\wedge}_{n=1}^{\infty}\text{Im}\int_A \tilde{f}_n d\tilde{\mu},$$

due to $\forall n$, $\tilde{f}_n \geq \tilde{f}$, so

$$\int_A \tilde{f}_n d\tilde{\mu} \geq \int_A \tilde{f} d\tilde{\mu}$$

have

$$\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} \geq \int_A \tilde{f} d\tilde{\mu}$$

if $\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} > \int_A \tilde{f} d\tilde{\mu}$, then $\int_A \tilde{f} d\tilde{\mu} = \lambda < \infty + i\infty$ where $\lambda = \lambda_1 + i\lambda_2$, and there is $\gamma_i \in (0, \infty)$ where $\gamma = \gamma_1 + i\gamma_2$, $(i = 1, 2)$ makes

$$\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} > \gamma > \lambda,$$

$$\Rightarrow \forall n, \sup_{\mathrm{Re}\alpha \in [0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}[\left(\tilde{f}_n\right)_\alpha \cap A]\right)$$

$$+i \sup_{\mathrm{Im}\alpha \in [0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left(\left(\tilde{f}_n\right)_\alpha \cap A]\right)\right) > \gamma,$$

$$\Rightarrow \forall n, S\left(\gamma, \tilde{\mu}\left(\left(\tilde{f}_n\right)_\gamma \cap A\right)\right) = S\left(\mathrm{Re}\gamma, \mathrm{Re}\tilde{\mu}\left(\left(\tilde{f}_n\right)_\gamma \cap A\right)\right)$$

$$+iS\left(\mathrm{Im}\gamma, \mathrm{Im}\tilde{\mu}\left(\left(\tilde{f}_n\right)_\gamma \cap A\right)\right) > \gamma,$$

take $\varepsilon_l = \frac{\gamma_l + \lambda_l}{2}$ $(l = 1, 2)$ exit n_0,

$$\tilde{\mu}\left(\left\{x|\tilde{f}_{n_0} > \int_A \tilde{f}d\tilde{\mu} + \varepsilon\right\} \cap A\right) < \infty + i\infty, \gamma_l = \lambda_l + 2\varepsilon_l > \lambda_l + \varepsilon_l$$

$$\Rightarrow \left\{x|\tilde{f}_{n_0} \geq \gamma\right\} \subseteq \left\{x|\tilde{f}_{n_0} > \lambda + \varepsilon\right\},$$

$$\Rightarrow \tilde{\mu}\left(A \cap \left(\tilde{f}_{n_0}\right)_\gamma\right) \leq \tilde{\mu}\left(\left\{x|\tilde{f}_{n_0} > \lambda + \varepsilon\right\} \cap A\right) < \infty + i\infty.$$

By the continuity of $\tilde{\mu}$, $A \cap \left(\tilde{f}_{n_1}\right)_\gamma \supseteq A \cap \left(\tilde{f}_{n_2}\right)_\gamma \supseteq \ldots,$

$$\tilde{\mu}\left(A \cap \left(\tilde{f}\right)_\gamma\right)$$

$$= \tilde{\mu}\left(\bigcap_{n=1}^{\infty}\left[A \cap \left(\tilde{f}_n\right)_\gamma\right]\right) = \lim_{n\to\infty} \tilde{\mu}\left(A \cap \left(\tilde{f}_n\right)_\gamma\right) \geq \gamma \geq \lambda$$

$$\int_A \tilde{f}d\tilde{\mu} \triangleq \sup_{\mathrm{Re}\alpha \in [0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right)$$

$$+i \sup_{\mathrm{Im}\alpha \in [0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right)$$

$$\geq S\left(\mathrm{Re}\gamma, \mathrm{Re}\tilde{\mu}\left[\tilde{f}_\gamma \cap A\right]\right) + iS\left(\mathrm{Im}\gamma, \mathrm{Im}\tilde{\mu}\left[\tilde{f}_\gamma \cap A\right]\right) > \lambda = \lambda_1 + i\lambda_2,$$

conflicting with $\int_A \tilde{f}d\tilde{\mu} = \lambda$, so $\bigwedge_{n=1}^{\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f}d\tilde{\mu}$, that is

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}.$$

Theorem 4 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space,* $\left\{\tilde{f}_n\right\}$ *is complex fuzzy set value complex fuzzy integrable function sequence in* $(Z, \mathcal{F}(Z), \tilde{\mu})$ $A \in \mathcal{F}(Z)$*, if in* A*,* $\left\{\tilde{f}_n\right\}$ *convergence in* $\tilde{f}$*, and for arbitrary* $\varepsilon_k > 0$, $(k = 1, 2)$*, where,* $\varepsilon = \varepsilon_1 + i\varepsilon_2$ *exit* n_0 *makes*

$$\tilde{\mu}\left(\left\{x \mid \sup_{n\geq n_0} \tilde{f}_n > \int_A \tilde{f} d\tilde{\mu} + \varepsilon\right\} \cap A\right) < \infty + i\infty.$$

then $\lim\limits_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$*.*

Proof Let $\tilde{h}_n = \bigvee\limits_{k=1}^{n} \tilde{f}_k$, $\tilde{g}_k = \bigwedge\limits_{k=1}^{n} \tilde{f}_k$. Then $\left\{\tilde{h}_n\right\} \downarrow \tilde{f}$ and $\{\tilde{g}_n\} \uparrow \tilde{f}$, $\tilde{g}_n \leq \tilde{f}_n \leq \tilde{h}_n$, so

$$\int_A \tilde{g}_n d\tilde{\mu} \leq \int_A \tilde{f}_n d\tilde{\mu} \leq \int_A \tilde{h}_n d\tilde{\mu}$$

by theory 3,

$$\lim_{n\to\infty}\int_A \tilde{g}_n d\tilde{\mu} = \lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu}$$

$$= \lim_{n\to\infty}\int_A \tilde{h}_n d\tilde{\mu} = \lim_{n\to\infty}\int_A \tilde{f} d\tilde{\mu}.$$

Definition 6 ([3]) Given a fuzzy complex value fuzzy measure space $\left(C, \tilde{F}, \tilde{\mu}\right)$, let $\tilde{f}_n$ $(n = 1, 2, \ldots)$ and $\tilde{f} : C \to F^*(C)$ are fuzzy complex value fuzzy measurable function, $\tilde{A} \in \tilde{F}$ then

(1) $\left\{\tilde{f}_n\right\}$ almost everywhere converges to $\tilde{f}$ on $\tilde{A}$, if $\tilde{\mu}\left(\tilde{E}\right) = \tilde{0}$ for $\tilde{E} \in \tilde{F}$ and makes $\left\{\tilde{f}_n\right\}$ converges to $\tilde{f}$ point by point on $\tilde{A} - \tilde{E}$, note that $\tilde{f}_n \overset{a.e}{\to} \tilde{f}$;

(2) $\left\{\tilde{f}_n\right\}$ almost uniform converges to $\tilde{f}$ on $\tilde{A}$, if $\exists \tilde{E} \in \tilde{F}$ for $\varepsilon > 0$, $\left|\tilde{\mu}_\alpha\left(\tilde{E}\right)\right| < \varepsilon$ and makes $\left\{\tilde{f}_n\right\}$ uniform converges to $\tilde{f}$ point by point, note that $\tilde{f}_n \overset{a.u}{\to} \tilde{f}$;

(3) $\left\{\tilde{f}_n\right\}$ pseudo almost everywhere converges to $\tilde{f}$, if $\hat{\mu}\left(\tilde{A} - \tilde{E}\right) = \tilde{\mu}\left(\tilde{E}\right)$ for $\tilde{E} \in \tilde{F}$ and makes $\left\{\tilde{f}_n\right\}$ converges to $\tilde{f}$ point by point on $\tilde{A} - \tilde{E}$, note that $\tilde{f}_n \overset{p.a.e}{\to} \tilde{f}$;

(4) $\left\{\tilde{f}_n\right\}$ pseudo almost uniform converges to $\tilde{f}$, if $\lim\limits_{n\to\infty} \tilde{\mu}\left(\tilde{A} - \tilde{E}_k\right) = \tilde{\mu}\left(\tilde{A}\right)$ for $\left\{\tilde{E}_k\right\} \subset \tilde{F}$ and makes $\left\{\tilde{f}_n\right\}$ uniform converges to $\tilde{f}$ point by point on $\tilde{A} - \tilde{E}$ for any fixed point, $k = 1, 2, 3 \ldots$ note that $\tilde{f}_n \overset{p.a.u}{\to} \tilde{f}$;

(5) $\left\{\tilde{f}_n\right\}$ converges in measure to $\tilde{f}$, if

$$\lim_{n\to\infty}\tilde{\mu}\left(\left\{x|\left\{\tilde{f}_n(x)-\tilde{f}(x)\right\}>\varepsilon\right\}\cap\tilde{A}\right)=0$$

for any $\varepsilon>0$, note that $\tilde{f}_n\overset{\tilde{\mu}}{\to}\tilde{f}$;

(6) $\left\{\tilde{f}_n\right\}$ pseudo converges in measure to $\tilde{f}$, if

$$\lim_{n\to\infty}\tilde{\mu}\left(\left\{x|\left\{\tilde{f}_n(x)-\tilde{f}(x)\right\}>\varepsilon\right\}\cap\tilde{A}\right)=\tilde{\mu}\left(\tilde{A}\right),$$

note that $\tilde{f}_n\overset{p.\tilde{\mu}}{\to}\tilde{f}$.

Theorem 5 *Suppose* $(Z,\mathcal{F}(Z),\tilde{\mu})$ *is Complex fuzzy set-valued fuzzy measure space* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *is complex fuzzy set-valued complex fuzzy measurable function in* $(Z,\mathcal{F}(Z),\tilde{\mu})$, $A\in\mathcal{F}(Z)$, *if* $\tilde{f}_n\overset{a.e}{\to}\tilde{f}$, $\tilde{\mu}$ *is zero-additive, and for arbitrarily* $\varepsilon_k>0$, $(k=1,2)$, *where* $\varepsilon=\varepsilon_1+i\varepsilon_2$ *exit* n_0 *makes*

$$\tilde{\mu}\left(\left\{x|\sup_{n\geq n_0}\tilde{f}_n\rangle\right\}\int_A\tilde{f}d\tilde{\mu}+\varepsilon\cap A\right)<\infty+i\infty,$$

then $\lim_{n\to\infty}\int_A\tilde{f}_nd\tilde{\mu}=\int_A\tilde{f}d\tilde{\mu}$.

Proof Because $\tilde{f}_n\overset{a.e}{\to}\tilde{f}$ in A, then exit $B\in\mathcal{F}(Z)$, $\tilde{\mu}(B)=0$, $\tilde{\mu}$ is zero-additive, $\int_{A\backslash B}\tilde{f}d\tilde{\mu}\overset{\Delta}{=}$

$$\sup_{\mathrm{Re}\alpha\in[0,\infty)}S\left(\mathrm{Re}\alpha,\mathrm{Re}\tilde{\mu}\left[\tilde{f}_\alpha\cap(A\backslash B)\right]\right)+i\sup_{\mathrm{Im}\alpha\in[0,\infty)}S\left(\mathrm{Im}\alpha,\mathrm{Im}\tilde{\mu}\left[\tilde{f}_\alpha\cap(A\backslash B)\right]\right)$$
$$=\sup_{\mathrm{Re}\alpha\in[0,\infty)}S\left(\mathrm{Re}\alpha,\mathrm{Re}\tilde{\mu}\left[\left(A\cap\tilde{f}_\alpha\right)\backslash B\right]\right)+i\sup_{\mathrm{Im}\alpha\in[0,\infty)}S\left(\mathrm{Im}\alpha,\mathrm{Im}\tilde{\mu}\left[\left(A\cap\tilde{f}_\alpha\right)\backslash B\right]\right)$$
$$=\sup_{\mathrm{Re}\alpha\in[0,\infty)}S\left(\mathrm{Re}\alpha,\mathrm{Re}\tilde{\mu}\left(A\cap\tilde{f}_\alpha\right)\right)+i\sup_{\mathrm{Re}\alpha\in[0,\infty)}S\left(\mathrm{Im}\alpha,\mathrm{Im}\tilde{\mu}\left(A\cap\tilde{f}_\alpha\right)\right)$$
$$=\int_A\tilde{f}d\tilde{\mu}.$$

Similarly

$$\int_A\tilde{f}_nd\tilde{\mu}=\int_{A\backslash B}\tilde{f}_nd\tilde{\mu},$$

and because

$$\tilde{\mu}\left(\left\{x|\sup_{n\geq n_0}\tilde{f}_n>\int_A\tilde{f}d\tilde{\mu}+\varepsilon\right\}\cap A\right)<\infty+i\infty,$$

and from the Theorem 2, obtained $\lim_{n\to\infty}\int_{A\backslash B}\tilde{f}_nd\tilde{\mu}=\int_{A\backslash B}\tilde{f}d\tilde{\mu}$, so $\lim_{n\to\infty}\int_A\tilde{f}_nd\tilde{\mu}=\int_A\tilde{f}d\tilde{\mu}$.

Theorem 6 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *is complex fuzzy set-valued complex fuzzy measurable function in* $(Z, \mathcal{F}(Z), \tilde{\mu})$, $A \in \mathcal{F}(Z)$, *if* $\left\{\tilde{f}_n\right\}$ *uniform convergence in* $\tilde{f}$ *in A, then*

$$\lim_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}.$$

Proof
(1) If $\int_A \tilde{f} d\tilde{\mu} = \infty + i\infty$, let $\tilde{g}_n = \overset{n}{\underset{k=1}{\wedge}} \tilde{f}_k$ then

$$\lim_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} \geq \int_A \tilde{g}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu} = \infty + i\infty.$$

(2) If

$$\int_A \tilde{f} d\tilde{\mu} = \sup_{\mathrm{Re}\alpha\in[0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right) + i \sup_{\mathrm{Im}\alpha\in[0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right)$$

$= \lambda < \infty + i\infty$, then,

$$\forall\alpha \leq \lambda, \tilde{\mu}\left[\tilde{f}_\alpha \cap A\right] \geq \lambda;$$

$$\forall\alpha > \lambda, \tilde{\mu}\left[\tilde{f}_\alpha \cap A\right] \leq \lambda;$$

α_n monotone decreasing trend to λ, then $\tilde{f}_{\alpha_1} \subseteq \tilde{f}_{\alpha_2} \subseteq \cdots$ and $\overset{\infty}{\underset{n=1}{\cap}} \tilde{f}_{\alpha_n} = \tilde{f}_{\dot{\lambda}}$ by under continuous of $\tilde{\mu}$, $\tilde{\mu}\left(\tilde{f}_{\dot{\lambda}} \cap A\right) = \lim_{n\to\infty} \tilde{\mu}\left(\tilde{f}_{\alpha_n} \cap A\right) \leq \lambda$, uniform convergence in $\tilde{f}$ in A, arbitrarily $\varepsilon'_k > 0$, $(k = 1, 2)$ where $\varepsilon' = \varepsilon'_1 + i\varepsilon'_2$ exit n_0, $\forall x \in A$,

$$\tilde{f}_n(x) \leq \tilde{f}(x) + \varepsilon \Rightarrow \sup_{n\geq n_0} \tilde{f}_n(x) \leq \tilde{f}(x) + \varepsilon$$

$$\Rightarrow \left\{x \mid \sup_{n\geq n_0} \tilde{f}_n(x) \geq \lambda + \varepsilon\right\} \cap A \subseteq \left\{x \mid \tilde{f}(x) \geq \lambda\right\} \cap A = \tilde{f}_{\dot{\lambda}} \cap A$$

$$\Rightarrow \tilde{\mu}\left(\left\{x \mid \sup_{n\geq n_0} \tilde{f}_n(x) \geq \lambda + \varepsilon\right\} \cap A\right)$$

$$\leq \lambda < \infty + i\infty$$

by Theorem 3 has $\lim_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$.

Theorem 7 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space,* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *are complex fuzzy set-valued complex fuzzy measurable function on* $(Z, \mathcal{F}(Z), \tilde{\mu})$*,* $A \in \mathcal{F}(Z)$*, if* $\tilde{f}_n \overset{a.e.u}{\to} \tilde{f}$ *on A, and* $\tilde{\mu}$ *is zero-additive, then* $\lim\limits_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$.

Proof Similar to the proof method of Theorem 6.

Theorem 8 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *are complex fuzzy set-valued complex fuzzy measurable function on* $(Z, \mathcal{F}(Z), \tilde{\mu})$*,* $A \in \mathcal{F}(Z)$*, if* $\tilde{f}_n \overset{a.u}{\to} \tilde{f}$ *in A, and* $\tilde{\mu}$ *is zero-additive, and exit*

$$\{B_k\} \subseteq \mathcal{F}(Z),\, B_1 \supseteq B_2 \supseteq \cdots,\, \tilde{\mu}(B_k) \to 0$$

makes

$$\lim_{n\to\infty} \int_{A\backslash B_k} \tilde{f}_n d\tilde{\mu} = \int_{A\backslash B_k} \tilde{f} d\tilde{\mu}.$$

Proof If $\tilde{f}_n \overset{a.u}{\to} \tilde{f}$ in A, then exit $\{E_k\} \subseteq \mathcal{F}(Z)$, $\tilde{\mu}(E_k) \to 0$, $\tilde{f}_n \overset{u}{\to} \tilde{f}$ in $A\backslash E_k$, let

$$B_k = \bigcap_{i=1}^{k} E_i \subseteq E_k.$$

Then

$$A\backslash B_k = \bigcap_{i=1}^{k} (A\backslash E_i),$$

$\forall k$, $\tilde{f}_n \overset{u}{\to} \tilde{f}$ in $A\backslash E_k$, $\tilde{f}_n \overset{u}{\to} \tilde{f}$ in $A\backslash B_k$, by Theorem 6 know the conclusion is right.

5 Conclusion

In this paper, the fuzzy measure concepts was extended from general classical set to ordinary complex fuzzy set, fuzzy, research complex fuzzy set-valued complex fuzzy measure and its properties, and measurable function in complex fuzzy set value complex fuzzy measure space and its properties was studied, of the of extension of the scope of classical measure theory, generalization of the corresponding conclusion of classical measure theory; research Integral theory problem of complex fuzzy set-valued function base on complex fuzzy set-valued measure, establish complex fuzzy set-valued complex fuzzy Integral theory, which is important work in fuzzy complex analysis. This work extends the fuzzy measure and fuzzy integral theory, to lay a solid foundation for our future research on complex fuzzy set-valued complex fuzzy integral application problem.

Acknowledgments Thanks to the support by International Science & Technology Cooperation Program of China (No. 2012DFA11270).

References

1. Ma, S., Li, S.: Complex fuzzy set-valued complex fuzzy measures and their properties. Sci. World J. **2014**, Article ID 493703, 7 paper
2. Ma, S., Zhao, H., Li, S.: Measurable function on complex fuzzy set-valued complex fuzzy measure space and its properties. Fuzzy Syst. Math. (2014) (has been finalized)
3. Ma, S., Li, S., Zhao, H., Zhou, M.: Complex fuzzy set-value complex fuzzy integrals and their properties. J. Lanzhou Univ. (Nat. Sci. Ed.) (has been finalized)
4. Wu, C.X., Zhang, D.L., Guo, C.M.: Fuzzy number fuzzy measure and fuzzy integrals (I): fuzzy integrals of functions with respect to fuzzy number fuzzy measure. Fuzzy Sets Syst. **98**(3), 355–360 (1998)
5. Guo, C.M., Zhang, D.L., Wu, C.X.: Fuzzy-valued fuzzy measures and generalized fuzzy integrals. Fuzzy Sets Syst. **97**(2), 255–260 (1998)
6. Wu, C.X., Zhang, D.L., Zhang, B., et al.: Fuzzy number fuzzy measure and fuzzy integrals (I): fuzzy-valued functions with respect to fuzzy number fuzzy measure on fuzzy sets. Fuzzy Sets Syst. **107**(2), 219–226 (1999)
7. Buckley, J.J.: Fuzzy complex numbers. Fuzzy Sets Syst. **33**(3), 333–345 (1989)
8. Zhang, G.: Fuzzy number-valued fuzzy measure and fuzzy number-valued fuzzy Integral on the fuzzy set. Fuzzy Sets Syst. **49**(3), 357–376 (1992)
9. Zhang, G.: The structural characteristics of the fuzzy number-valued fuzzy measure on the fuzzy algebra and their applications. Fuzzy Sets Syst. **52**(1), 69–81 (1992)
10. Zhang, G.: The convergence for a sequence of fuzzy integrals of fuzzy number-valued function on the fuzzy set. Fuzzy Sets Syst. **59**(1), 43–57 (1993)
11. Zhang, G.: On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals on the fuzzy set. Fuzzy Sets Syst. **45**(2), 227–237 (1992)
12. Zhang, G.: On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals on the fuzzy set. Fuzzy Sets Syst. **48**(2), 257–265 (1992)
13. Qiu, J., Li, F., Su, L.: Complex fuzzy measure and complex fuzzy integral. J. Lanzhou Univ. (Nat. Sci. Ed.) **32**, 155–158 (1996)
14. Wang, G., Li, X.: Lebesugue integral with respect to the fuzzy complex-valued measure. J. Harbin Normal Univ. (Nat. Sci. Ed.) **15**(2), 20–26 (1999)
15. Ma, S., Chen, F., Wang, Q., Zhao, Z.: The design of fuzzy classifier base on Sugeno type fuzzy complex-value integral. In: Proceedings of 2011 Seventh International Conferen on Computational Intelligence and Security (published by CPS, IEEE computer society), pp. 338–342 (2011)
16. Ma, S., Chen, F., Zhao, Z.: Choquet type fuzzy complex-valued integral and its application in classification. Fuzzy Engineering and Operations Research, AISC, vol. 147, pp. 229–237. Springer (2012)
17. Ma, S., Chen, F., Wang, Q., Zhao, Z.: Sugeno type fuzzy complex-value integral and its application in classification. Procedia Eng. ELSEVIER **29**, 4140–4151 (2012)
18. Yao, O., Jun, L.: Note on the definition of fuzzy number-valued fuzzy measurable functions. J. Southeast Univ. (Nat. Sci. Ed.) **33**(6), 801–803 (2003)