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Bing-Yuan Cao
Zeng-Liang Liu
Yu-Bin Zhong
Hong-Hai Mi *Editors*

Fuzzy Systems & Operations Research and Management

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Bing-Yuan Cao · Zeng-Liang Liu
Yu-Bin Zhong · Hong-Hai Mi
Editors

Fuzzy Systems & Operations Research and Management

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China

Zeng-Liang Liu
Foreign Affairs Office
National Defense University
Beijing
China

Yu-Bin Zhong
Higher Education Mega Center
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Guangzhou
China

Hong-Hai Mi
Zhuhai Campus
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Zhuhai
China

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Foreword

The volume put forward by Profs. Cao, Liu, Zhong, and Mi constitutes a collection of papers presented at the 7th International Conference on Fuzzy Information and Engineering (ICFIE'2014) and the 1st International Conference of Operations Research and Management (ICORM'2014) held during November 7–11, 2014 in Zhuhai, China.

The areas of Fuzzy Engineering and Fuzzy Information have been recently enjoying a vivid growth with a significant progress and numerous and innovative applications in a variety of areas. Theory, methodology, algorithms, and development guidelines and reports on the best practices come hand in hand. Fuzzy sets with their generalizations, conceptual enhancements, and new operations, transformations, and processing mechanisms call for a systematic and coherent treatment of the fundamentals to move them to the next level of sophistication. The methodology of fuzzy sets has been the cornerstone of successful and efficient algorithms. There are several visible directions including a hybridization of fuzzy sets and other technologies of Soft Computing (Computational Intelligence). The ensuing algorithms fully benefit from the prudently orchestrated synergy of the fuzzy sets, neurocomputing and population-based optimization.

Operations Research and Management are undoubtedly those domains in which the role of information granules and fuzzy sets, in particular, assume a central position. In this regard, two tendencies are apparently visible. In the first one, we witness augmenting the existing approaches, sometimes quite well established, by bringing into the picture fuzzy sets. The intention is to endow the existing methods with the new quality, new insights, and functionalities including raising support for enhanced human-centricity of the ensuing systems. The second one, which becomes more dominant nowadays, looks at the original treatment of the problem, its quite substantial reformulation and engagement of fuzzy sets in a fully innovative, non-traditional fashion not being biased of how the problem was handled so far.

This volume offers a truly diverse collection of contributions in which the authors report on the recent accomplishments in the area. The papers can be split into several general categories, which reflect upon the main research directions, namely Fuzzy Systems and Applications, Fuzzy Mathematics and Applications, Fuzzy Information and Computing, and finally Operations Research and Management and Applications.

The Editors deserve our congratulations on professionally assembling such a diverse and representative collection of research studies covering a spectrum of fundamental and applied facets of fuzzy set technology.

Prof. Witold Pedrycz
University of Alberta
Edmonton, Canada

Preface

This book is a monograph from submissions by the 7th International Conference on Fuzzy Information and Engineering (ICFIE'2014) and the 1st International Conference of Operations Research and Management (ICORM'2014) during November 7–11, 2014 in Zhuhai, China. The monograph is published by *Advances in Intelligent and Soft Computing* (AISC), Springer, ISSN: 1867-5662.

This year, we received more than 100 submissions. Each paper underwent a rigorous review process. Only high-quality papers are included in the book.

The book, containing papers, is divided into five main parts:

In Part I, subjects on "Fuzzy Systems and Its Applications."
In Part II, themes on "Fuzzy Mathematics and Its Applications."
In Part III, topics discussed on "Fuzzy Information and Computer."
In Part IV, ideas circling around "Operations Research and Management and Its Applications."
In Part V, dissertations on "Others."

We appreciate the organizations sponsored by International Fuzzy Information and Engineering Association (chips); Fuzzy Information and Engineering Branch of ORSC; Operations Research Society of Guangdong Province; China Guangdong, Hong Kong and Macao Operations Research Society and undertaken by Beijing Normal University, Zhuhai, Guangdong, China.

Heartfelt thanks to the Guangdong Provincial Science and Technology Association, Guangzhou University, China, for fund for co-sponsorships.

We are grateful to Mazandaran University, Iran; Fuzzy Information and Engineering Branch of International Institute of General Systems Studies in China (IIGSS-GB) for support.

We appreciate the Editorial Committee, reviewers, in particular, Prof. Witold Pedrycz in Canada's chief scientist intelligent computing. We are thankful to our students: Postdoctoral: Xue-gang Zhou, Ph.D.: Xiao-peng Yang; Master: Ze-jian Qin and Geng-tao Zhang, who have contributed a lot to the development of this issue. We wish to express our heartfelt appreciation to all the authors and participants for their great contributions that made these conferences possible and all the

hard work worthwhile. Meanwhile, we are thankful to China Education and Research Foundation; China Science and Education Publishing House, and China Charity Press Publishing and its president Mr. Dong-cai Lai for sponsoring.

Finally, we thank the publisher, Springer, for publishing the AISC (Notes: Our series of conference proceedings by Springer, like *Advances in Soft Computing* (ASC), AISC, (ASC 40, ASC 54, AISC 62, AISC 78, AISC 82 and AISC 147, included into EI and indexed by Thomson Reuters Conference Proceedings Citation Index (ISTP); AISC 211 and AISC 254 awaiting EI index), and thank the supports coming from international magazine *Fuzzy Information and Engineering* by Elsevier, and *Operations Research Management and Fuzzy Mathematics* by China Science and Education Press (Hong Kong).

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Part I
Fuzzy Systems and Its Applications

Analytic Representation Theorem of Fuzzy-Valued Function Based on Methods of Fuzzy Structured Element

Si-Cong Guo, Ying Zhao and Hua-Dong Wang

Abstract The paper introduces the representation method of fuzzy structured element in fuzzy-valued function analytics systematically. It includes the concept of the fuzzy structured element, operations of fuzzy numbers, the analytic expression of fuzzy-valued functions and its differential and integral, they are all based on the fuzzy structured element. Theorems of the fuzzy structured element not only provide methods for analytic representation of fuzzy analysis and operations, but also start a new way for studying on the theory and application of fuzzy analysis.

Keywords Fuzzy structured element · Fuzzy numbers · Fuzzy-valued functions · Analytic representation theorems · Differential and integral calculus

1 Introduction

The concept of fuzzy numbers and fuzzy-valued functions was firstly proposed by Chang and Zadeh [3]. And then, followed by Dubois and Prade [4, 5] who defined operations of fuzzy numbers and differential of fuzzy-valued functions by the extension principle [15]. Many papers have shown researches on fuzzy numbers and fuzzy-valued functions, such as [1, 2, 6, 7, 13, 14, 17].

S.-C. Guo · Y. Zhao (✉)
Institute of Intelligence Engineering and Mathematics, Liaoning Technical University, Fuxin 123000, China
e-mail: zhaoying198916@163.com

Y. Zhao
Institute of Information Science and Engineering, Northeastern University, Shenyang 110000, China

H.-D. Wang
Research Center on Fictitious Economy and Data Science, Chinese Academy of Sciences, Beijing 100190, China

H.-D. Wang
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100190, China

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From the point of uncertain mathematic developing system, it is natural and significant to make fuzzy analytics as an extension of interval analytics. Based on this idea, the theoretical framework of fuzzy number calculations, fuzzy-valued functional differential and its integral in the sense of Riemann have a strong background objectively [4, 5]. Theories of interval analysis has been developed quite mature in past several decades [16]. The connection between fuzzy analytics and interval analytics is the expansion principle [12] in fuzzy mathematics. However, the expansion principle has crisp arguments on its expressing, and then the operations and transformations of fuzzy numbers and fuzzy-valued functions cannot be operated practically, these are also barriers affecting the practical application of the fuzzy analysis method. In order to solve problems in the real world effectively with fuzzy analytics like the classic analytics, some breakthroughs should be made in the problems, such as the quick expression of fuzzy number calculations, the analytic expressions of elementary fuzzy-valued functions, differential and integral calculus of fuzzy-valued functions, discriminate for convergence and summation of fuzzy series, analytic expressions of fuzzy-valued functions, and so on.

2 Fuzzy Structured Element and Fuzzy Numbers

Convexity and normality are two constant properties of the fuzzy numbers [3, 4, 13]. Let A be a convex and normal fuzzy set on the real numbers field R, f is a monotonic function on R, by the expansion principle of Zadeh, it is easy to know that the fuzzy set $f(A)$ must be convex and normal. This implies that convexity and normality are inherent structured properties of the fuzzy numbers for monotonic transformations. Therefore, we defined a fuzzy set on R with convexity and normality, and called it a fuzzy structured element, then all fuzzy numbers can be obtained by the monotonic transformations of the fuzzy structured element. This is the basic idea of proposing and building the fuzzy structured element [8].

Definition 2.1 ([8]) Let E be a fuzzy set on R, $E(x)$ is the membership function of $E, \forall x \in R$, then, E is called a fuzzy structured element, if $E(x)$ satisfies the following properties:

(1) $E(0) = 1, E(1 + 0) = E(-1 - 0) = 0$;

(2) $E(x)$ is a function of monotonic increasing and right continuous on $[-1, 0]$, monotonic decreasing and left continuous on $(0, 1]$;

(3) $E(x) = 0(-\infty < x < -1 \text{ or } 1 < x < +\infty)$.

E is called a normal fuzzy structured element, if (1) $\forall x \in (-1, 1), E(x) > 0$; (2) $E(x)$ is increasing and continuous on $[-1, 0]$, strictly monotonic decreasing and continuous on $(0, 1]$.

E is called a symmetrical fuzzy structured element, if $E(x) = E(-x)$.

Theorem 2.1 ([10]) *Let E be any fuzzy structured element on R, and $E(x)$ is its membership function, the function $f(x)$ is continuous and monotonic on* $[-1, 1]$, $\hat{f}(x)$ *is* *a set-valued function extensional from* $f(x)$. *Then* $\hat{f}(E)$ *is a bounded closed fuzzy number on R, and the membership function of* $\hat{f}(E)$ *is* $E(f^{-1}(x))$*, where* $f^{-1}(x)$ *is a rotational symmetric function for x and y (If* $f(x)$ *is a strictly monotonic function, then* $f^{-1}(x)$ *is the inverse function of* $f(x)$*).*

Theorem 2.2 ([10]) *For a given normal fuzzy structured element E and any finite fuzzy number A, there always exists a monotonic bounded function f on* $[-1, 1]$*, such that* $A = f(E)$*.*

Theorem 2.3 ([10]) *Suppose that E is a fuzzy structured element on R, f is a monotonic bounded function on* $[-1, 1]$*, E is a given fuzzy structured element on R, a fuzzy number A. If f is monotonic increasing, then the* $\lambda-$*level set of A is a closed interval on R, and it can be denoted as* $A_\lambda = [f(E)]_\lambda = f[e_\lambda^-, e_\lambda^+] = [f(e_\lambda^-), f(e_\lambda^+)]$*. If* $A_\lambda = [a_\lambda^-, a_\lambda^+]$*,* $B_\lambda = [b_\lambda^-, b_\lambda^+]$ *is monotonic decreasing, then* $A_\lambda = [f(e_\lambda^+), f(e_\lambda^-)]$*.*

Further contents about fuzzy structured element can be found in [9–11].

3 Same Sequential Transformations of Monotonic Functions and Operations of Fuzzy Numbers

It is easy to know that if $f(x)$ is a monotonic function on the symmetrical internal $[-1, 1]$, then both $f(-x)$ and $-f(x)$ are monotonic functions. If $f(x)$ satisfies $f(x) > 0$ or $f(x) < 0$, then $\frac{1}{f(x)}$ is also a monotonic function, and it has opposite monotonicity with $f(-x)$. Therefore, we can obtain that $-f(-x)$, $\frac{1}{f(-x)}$ and $-\frac{1}{f(x)}$ have same sequential monotonicity with $f(-x)$ on $[-1, 1]$. Denote the family of all functions which are bounded and have same monotonicity on $[-1, 1]$ by $D[-1, 1]$. Give $f \in D[-1, 1]$, same sequential monotonic transformations of f are defined as following:

$$\begin{aligned}
\tau_0 &: f \to \tau_0(f) = f^{\tau_0} = f,\\
\tau_1 &: f \to \tau_1(f) = f^{\tau_1} = -f(-x),\\
\tau_2 &: f \to \tau_2(f) = f^{\tau_2} = \frac{1}{f(-x)}, (f(-x) \neq 0),\\
\tau_3 &: f \to \tau_3(f) = f^{\tau_3} = -\frac{1}{f(x)}, (f(x) \neq 0).
\end{aligned}$$

Let A and B be two fuzzy numbers, $\otimes$ be a binary operation on the real field. If $f(A, B) = A \otimes B$, by the multivariate expansion principle, we have

$$A \otimes B = \bigcup_{\lambda \in [0,1]} \lambda \wedge (A_\lambda \otimes B_\lambda), \forall \lambda \in [0, 1],$$

where $A_\lambda = [a_\lambda^-, a_\lambda^+]$, $B_\lambda = [b_\lambda^-, b_\lambda^+]$ are the $\lambda-$level sets of A and B, respectively.

Theorem 3.1 *Suppose that E is a given symmetrical fuzzy structured element, g is a monotonic and bounded function on* $[-1, 1]$*, a fuzzy number* $B = g(E)$*, then we have*

$$(1)\ -B = g^{\tau_1}(E), \quad (2)\ 1/B = g^{\tau_2}(E), \quad (3)\ 1/B = g^{\tau_1\tau_2}(E) = g^{\tau_3}(E).$$

Proof Suppose that g is a monotonic increasing function, $E_\lambda = [e_\lambda^-, e_\lambda^+]$ denotes the $\lambda-$level set of E.

(1) For all $\lambda \in [0, 1]$, $B_\lambda = [g(E)]_\lambda = [g(e_\lambda^-), g(e_\lambda^+)]$, by the definition of interval numbers, we have $-B_\lambda = [g(-e_\lambda^-), g(-e_\lambda^+)]$. Since E is a symmetrical fuzzy structured element, then $-B_\lambda = [-g(-e_\lambda^-), -g(-e_\lambda^+)] = [g^{\tau_1}(E_\lambda)] = [g^{\tau_1}(E)]_\lambda$. By expansion principle, $-B_\lambda = \bigcup_{\lambda\in[0,1]} \lambda \wedge (-B_\lambda) = \bigcup_{\lambda\in[0,1]} \lambda \wedge [g^{\tau_1}(E)]_\lambda = g^{\tau_1}(E)$.

(2) For all $\lambda \in [0, 1]$, $B_\lambda = [g(E)]_\lambda = [g(e_\lambda^-), g(e_\lambda^+)]$, by the definition of interval numbers, $\left(\frac{1}{B}\right)_\lambda = \frac{1}{B_\lambda} = [\frac{1}{g(e_\lambda^+)}, \frac{1}{g(e_\lambda^-)}] = [\frac{1}{g(-e_\lambda^-)}, \frac{1}{g(-e_\lambda^+)}] = g^{\tau_2}(E_\lambda) = [g^{\tau_2}(E)]_\lambda$.

According to the expansion principle, the conclusion is proved.

(3) Since $\frac{1}{B_\lambda} = \left[\frac{1}{g(-e_\lambda^-)}, \frac{1}{g(-e_\lambda^+)}\right]$, then we have

$$-\frac{1}{B_\lambda} = \left[-\frac{1}{g\left(-e_\lambda^+\right)}, -\frac{1}{g\left(-e_\lambda^-\right)}\right] = \left[-\frac{1}{g\left(e_\lambda^-\right)}, -\frac{1}{g\left(e_\lambda^+\right)}\right] = g^{\tau_3}\left(E_\lambda\right) = \left[g^{\tau_3}\left(E\right)\right]_\lambda.$$

Theorem 3.2 *Let E be a symmetrical fuzzy structured element. Then*

$$f(e_\lambda^-) = -f^{\tau_1}(e_\lambda^+),\ f(e_\lambda^+) = -f^{\tau_1}(e_\lambda^-).$$

Proof On the one hand, since $f^{\tau_1}(x) = -f(-x)$, then $f^{\tau_1}(e_\lambda^-) = -f(-e_\lambda^-)$. On the other hand, since E is a symmetrical fuzzy structured element, then $e_\lambda^- = -e_\lambda^+$, so we obtain that $f(e_\lambda^-) = -f^{\tau_1}(e_\lambda^+)$. Similarly, we can prove that $f(e_\lambda^+) = -f^{\tau_1}(e_\lambda^-)$.

Theorem 3.3 *Suppose that E is a symmetrical fuzzy structured element, $f, g \in D[-1, 1]$ (suppose that they are all monotonic increasing functions), fuzzy numbers $A = f(E)$ and $B = g(E)$, $\forall\lambda \in [0, 1]$, then $f_\lambda(E) = [f(e_\lambda^-), f(e_\lambda^+)]$, $f^{\tau_k}(x)$ are same order monotonic transformations of $f(x)$, then*

(1) If A and B are any bounded fuzzy numbers, then $A + B = (f + g)(E)$, with the membership function $\mu_{A+B}(x) = E((f + g)^{-1}(x))$.

(2) If A and B are any bounded fuzzy numbers, then $A - B = (f + g^{\tau_1})(E)$, with the membership function $\mu_{A-B}(x) = E((f + g^{\tau_1})^{-1}(x))$.

(3) If A and B are positive fuzzy numbers, then $A \cdot B = (f \cdot g)(E)$, with the membership function $\mu_{A\cdot B}(x) = E((f \cdot g)(x))$.

(4) If A and B are negative fuzzy numbers, then $A \cdot B = (f^{\tau_1} \cdot g^{\tau_1})(E)$, with the membership function $\mu_{A\cdot B}(x) = E((f^{\tau_1} \cdot g^{\tau_1})^{-1}(x)) = E(-(f \cdot g)^{-1}(x))$.

(5) If A is a negative fuzzy number, B is a positive fuzzy number, then $A \cdot B = (f^{\tau_1} \cdot g)^{\tau_1}(E) = (-f \cdot g^{\tau_1})(E)$, with the membership function

$$\mu_{A\cdot B}(x) = E((f \cdot g^{\tau_1})^{-1}(x)).$$

(6) If A and B are positive fuzzy numbers, and 0 is not included in the support of B, that is $0 \notin \operatorname{supp} B$, then $A \div B = A \cdot \frac{1}{B} = f(E) \cdot g^{\tau_2}(E) = (f \cdot g^{\tau_2})(E)$, with the membership function $\mu_{A\div B}(x) = E((f \cdot g^{\tau_2})^{-1}(x))$.

(7) If A and B are negative fuzzy numbers, and 0 *is not included in the support of B, that is* $0 \notin \text{supp } B$*, then* $A \div B = A \cdot \frac{1}{B} = f^{\tau_1}(E) \cdot g^{\tau_3}(E) = (f^{\tau_1} \cdot g^{\tau_3})(E)$*, with the membership function* $\mu_{A \div B}(x) = E((f^{\tau_1} \cdot g^{\tau_3})^{-1}(x))$*.*

(8) If A is a negative fuzzy number, B is a positive fuzzy number, and 0 *is not included in the support of B, then* $A \div B = (-f \cdot g^{\tau_3})(E)$*, with the membership function* $\mu_{A \div B}(x) = E((-f \cdot g^{\tau_3})^{-1}(x))$*.*

Proof Suppose that $f, g \in D[-1, 1]$ and they are all monotonic increasing functions, for any $\lambda \in [0, 1]$, we have $[f(E)]_\lambda = [f(e_\lambda^-), f(e_\lambda^+)]$, $[g(E)]_\lambda = [g(e_\lambda^-), g(e_\lambda^+)]$.

(1) Let $H = A + B$, by the definition of interval numbers, $[H]_\lambda = [f(E)]_\lambda + [g(E)]_\lambda$. Then $A + B = (f + g)(E)$. Since $f + g$ is a monotonic and bounded function on $[-1, 1]$, therefore, $\mu_{A+B}(x) = E((f + g)^{-1}(x))$.

(2) Let $H = A - B \cdot y$ the definition of interval number, we have

$$[H]_\lambda = [f(E)]_\lambda - [g(E)]_\lambda = [f(e_\lambda^-) - g(e_\lambda^+), f(e_\lambda^+) - g(e_\lambda^-)].$$

Since E is a symmetrical fuzzy structured element, according to Theorem 3.1, we obtain that

$$\begin{aligned}[H]_\lambda &= [f(e_\lambda^-) - g(e_\lambda^+), f(e_\lambda^+) - g(e_\lambda^-)] \\ &= [f(e_\lambda^-) + g^{\tau_1}(e_\lambda^-), f(e_\lambda^+) + g^{\tau_1}(e_\lambda^+)] = [f(E)]_\lambda + [g(E)]_\lambda,\end{aligned}$$

then $A - B = (f + g^{\tau_1})(E)$. Since $f + g$ is a monotonic and bounded function on $[-1, 1]$, therefore, $\mu_{A-B}(x) = E((f + g^{\tau_1})^{-1}(x))$.

(3) Let $H = A \cdot B$, since A and B are positive fuzzy numbers. Then $0 \le f(e_\lambda^-) \le f(e_\lambda^+)$, $0 \le g(e_\lambda^-) \le g(e_\lambda^+)$. By the definition of interval number, we have

$$\begin{aligned}A_\lambda \cdot B_\lambda &= [f(E)]_\lambda \cdot [g(E)]_\lambda = [f(e_\lambda^-) \cdot g(e_\lambda^-), f(e_\lambda^+) \cdot g(e_\lambda^+)] \\ &= [(fg)(e_\lambda^-), (fg)(e_\lambda^+)] = [(fg)(E)]_\lambda = [h(E)]_\lambda,\end{aligned}$$

where $h = f \cdot g$. Therefore, $H = h(E) = (f \cdot g)(E)$, from Theorem 2.1, we have $\mu_H(x) = \mu_{A \cdot B}(x) = [E(f \cdot g)^{-1}(x)]$.

(6) Since A and B are positive fuzzy numbers, f and g are monotonic increasing functions, then $0 \le f(e_\lambda^-) \le f(e_\lambda^+)$, $0 \le g(e_\lambda^-) \le g(e_\lambda^+)$. By the definition of interval number, we have

$$\begin{aligned}A_\lambda \div B_\lambda &= [f(E)]_\lambda \div [g(E)]_\lambda \\ &= \left[f(e_\lambda^-), f(e_\lambda^+)\right] \div \left[g(e_\lambda^-), g(e_\lambda^+)\right] \\ &= \left[\frac{f(e_\lambda^-)}{g(e_\lambda^+)}, \frac{f(e_\lambda^+)}{g(e_\lambda^-)}\right] \left[\frac{f(e_\lambda^-)}{g(e_\lambda^-)}, \frac{f(e_\lambda^+)}{g(e_\lambda^+)}\right] \\ &= \left[f(e_\lambda^-) \cdot g^{\tau_2}(e_\lambda^-), f(e_\lambda^+) \cdot g^{\tau_2}(e_\lambda^+)\right] \\ &= f(E_\lambda) \cdot g^{\tau_2}(E_\lambda) = \left[(f \cdot g^{\tau_2})(E)\right]_\lambda\end{aligned}$$

then $A \div B = (f \cdot g^{\tau_2})(E)$, therefore, $\mu_{A \div B}(x) = E((f \cdot g^{\tau_2})^{-1}(x))$.

The conclusions of (4), (5), (7) and (8) can be proved similarly.

In many situations of fuzzy number operations, the monotonic functions f and x are easily to be founded. Therefore, through the way in Theorem 3.3, the choice of the membership function after the fuzzy number operation can be made easily.

4 Order of Fuzzy Numbers

The property that the fuzzy number space and the family of standard monotonic functions on $[-1, 1]$ is homeomorphic implies that a bounded real fuzzy number and a standard monotonic function $B_{[-1,1]}$ on $[-1, 1]$ is one to one [9]. Therefore, the order between fuzzy numbers is equivalent to the order relation between two corresponding monotonic functions, which provides a way to simplify the calculation of ranking fuzzy numbers. Denote the class of all bounded fuzzy numbers is denoted by $N(R)$.

Definition 4.1 Let $A, B \in N(R)$, for any $\lambda \in (0, 1]$, $A_\lambda = \left[a_\lambda^-, a_\lambda^+\right]$, $B_\lambda = \left[b_\lambda^-, b_\lambda^+\right]$. If $a_\lambda^- \geq b_\lambda^-$, $a_\lambda^+ \geq b_\lambda^+$, then we call $A \geq B$, and the order "$\geq$" is named as the natural order of fuzzy numbers.

According to the express theorem of the fuzzy structured element, suppose that E is any given fuzzy structured element, the fuzzy numbers A and B are denoted as $A = f(E)$, $B = g(E)$, then $A \geq B$ is equal to that $f(x) \geq g(x)$ with respect to all $x \in [-1, 1]$. Since $(N(R), \geq)$ is a partial order set, sometimes there exists the order "$\geq$" between two fuzzy numbers A and B, sometimes there exists not. So the order relation is not appropriate for ranking fuzzy numbers.

Definition 4.2 Let $A \in N(R)$, and $A = f(E)$, we define

$$O(A) = \int_{-1}^{1} f(x)dx$$

as the adjoint number of fuzzy number A.

Definition 4.3 Let $A, B \in N(R)$. If $O(A) > O(B)$, then we define that $A \succ B$. If $O(A) = O(B)$, we define that $A \sim B$. (Similarly, we can define the dual order "$\prec$"). If $A \succ B$ and $A \sim B$, we define that $A \succeq B$ (Similarly, we can define the dual order "$\preceq$").

For any $A, B \in N(R)$, one of the relations $A \succ B$, $A \prec B$ and $A \sim B$ must be true, and it satisfies the properties as follows:

(1) If $A \succeq B$, $B \succeq C$, then $A \succeq C$;

(2) If $A \succ B$, $B \succ A$, then $A \sim B$. That is, $(N(R), \succeq)$ is a full ordered set, and if $A \geq B$, then $A \succeq B$ must be true. Conversely, it might not be true. We call the order "$\succeq$" as structured order.

5 Analytical Representations of Fuzzy-Valued Functions

A function is a mapping from a set of numbers onto another set of numbers, we define the mapping from a set of real numbers onto a set of fuzzy numbers as a fuzzy-valued function. Let X and Y be two real sets. $D \subseteq X$, f is a mapping from D to $N(R)$. For any $x \in D$, there exists a unique fuzzy number $y \in N(R)$, such that $y = f(x)$, we call that f is a fuzzy-valued function on D.

Theorem 5.1 *Suppose that E is a given normal fuzzy structured element on Y, $\tilde{f}(x)$ is any bounded fuzzy-valued function on X, there always exists a binary function $g(x, y)$, such that for any given $x \in X$, $g(x, y)$ is a monotonic bounded function with respect to y on $[-1, 1]$, and satisfies $\tilde{f}(x) = g(x, E)$.*

Proof Since E is a given normal fuzzy structured element, $\tilde{f}(x)$ is a bounded fuzzy-valued function on X. According to the definition of fuzzy-valued functions, for all given $x \in X$, $\tilde{f}(x)$ is a bounded and closed fuzzy number. From Theorem 2.3, for a given x, there exists a monotonic and bounded function $g_x(y)$ with respect to y on Y, such that $\tilde{f}(x) = g_x(E)$. Since $x \in X$, then $g_x(E)$ is a function with respect to x, we denote it as $g_x(E) = g(x, E) = g(x, y)|_{y=E}$.

Example 5.1 Let E be a given normal fuzzy structured element on Y. Then $\tilde{f}_1(x) = h(x) + p(x)E$, where $h(x)$ is a bounded function, $p(x) \geq 0$, $\tilde{f}_2(x) = h(x)$, where $h(x) \neq 0$ are fuzzy-valued functions on X. From Theorem 2.2, the membership functions of these fuzzy-valued functions can be denoted by fuzzy structured element E as

$$\mu_{\tilde{f}_1(x)}(y) = E\left(\frac{y - h(x)}{p(x)}\right) \mu_{\tilde{f}_2(x)}(y) = E\left(\frac{1}{a} \ln \frac{y}{h(x)}\right), \forall x \in X, y \in Y.$$

6 The Ordered Cone and the Local Ordered Cone

Definition 6.1 Let K be a real cone, and $0 \in K$ is its vertice. We define the relation of sequence as $x > y \Leftrightarrow x - y \in K$. If K_0 is a subset of K, for any $x, y \in K_0$, $x > y$ or $y > x$, then K_0 is called an ordered cone.

Considering a class of binary functions $FD = \{f(x, y)|x \in R, y \in [-1, 1]\}$. For any $x \in X$, $f(x, y)$ is a monotonic function with respect to y on $[-1, 1]$, $f(x, y)$ can be regarded as a family of monotonic functions with a parameter x, we define that $f_x(y) = f(x, y), x \in X$.

Theorem 6.1 *Suppose that $f_x(y) = f(x, y) \in FD$, for any $x \in X$, if there exists a ε−neighborhood of x denoted as $(x - \varepsilon, x + \varepsilon)$, then for any $x + \Delta x \in (x - \varepsilon, x + \varepsilon)$, $f_{x+\Delta x}(y) - f_x(y)$ is still a monotonic function on $[-1, 1]$, then for any $-1 \leq y_1 \leq y_2 \leq 1$, we have*

$$f_{x+\Delta x}(y_2) - f_{x+\Delta x}(y_1) \geq f_x(y_2) - f_x(y_1)$$

or

$$f_{x+\Delta x}(y_2) - f_{x+\Delta x}(y_1) \leq f_x(y_2) - f_x(y_1).$$

Proof Suppose that $g(y) = f_{x+\Delta x}(y) - f_x(y)$ is a monotonic increasing function with respect to y, then $g(y_1) \leq g(y_2)$. We have $f_{x+\Delta x}(y_1) - f_x(y_1) \leq f_{x+\Delta x}(y_2) - f_x(y_2)$, that is $f_{x+\Delta x}(y_2) - f_{x+\Delta x}(y_1) \geq f_x(y_2) - f_x(y_1)$. If $g(y) = f_{x+\Delta x}(y) - f_x(y)$ is a monotonic increasing function with respect to y, then we can obtain similarly $f_{x+\Delta x}(y_2) - f_{x+\Delta x}(y_1) \leq f_x(y_2) - f_x(y_1)$.

Theorem 6.2 *Let $f_x(y) = f(x, y)$ be a monotonic function with respect to y on $[-1, 1]$, which satisfies Theorem 6.1 and it is differentiable at the point $x \in X$. Then $f'_x(y) = \frac{\partial f(x,y)}{\partial x}$ is still monotonic with respect to y on $[-1, 1]$.*

Proof Suppose that $g(y) = f_{x+\Delta x}(y) - f_x(y)$ is a monotonic increasing function with respect to y, according to Theorem 6.1, for any $-1 \leq y_1 \leq y_2 \leq 1$, we have $f_{x+\Delta x}(y_1) - f_x(y_1) \leq f_{x+\Delta x}(y_2) - f_x(y_2)$, then

$$f'_x(y_1) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y_1) - f(x, y_1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f_{x+\Delta x}(y_1) - f_x(y_1)}{\Delta x}$$

$$\leq \lim_{\Delta x \to 0} \frac{f_{x+\Delta x}(y_2) - f_x(y_2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y_2) - f(x, y_2)}{\Delta x} = f'_x(y_2).$$

Therefore, for any $-1 \leq y_1 \leq y_2 \leq 1$, we have then $f'_x(y)$ is a monotonic with respect to y on $[-1, 1]$. If $g(y) = f_{x+\Delta x}(y) - f_x(y)$ is a monotonic decreasing function with respect to y, we can prove it similarly.

Definition 6.2 Let $f(x, y)$ be a monotonic function with respect to y on $[-1, 1]$. For any $x \in D$, there exists a $\varepsilon-$neighborhood at x (the size of ε is related to x), composes an ordered cone for variable x in this $\varepsilon-$neighborhood, then $f_x(y) = f(x, y)$ is called a local ordered cone on D.

7 General Forms of Fuzzy-Valued Functional Differential and Riemann Integral

In fuzzy analytics, differential and integral of fuzzy valued functions have some different definitions, the following two definitions are depending on the definition of the derivative on interval-valued functions and the expansion principle, and they are two of the most common definitions.

Definition 7.1 ([5]) Suppose that $\tilde{f}(x)$ is a fuzzy-valued function on $D \subseteq X$, $f_\lambda(x) = [f_{1(\lambda)}(x), f_{2(\lambda)}(x)]$ is the $\lambda-$level set of $\tilde{f}(x)$, the derivative of $f_\lambda(x)$ is

$f'_\lambda(x) = [\min\{f'_{1(\lambda)}(x), f'_{2(\lambda)}(x)\}, \max\{f'_{1(\lambda)}(x), f'_{2(\lambda)}(x)\}]$. If for any $\lambda \in (0, 1]$, $f_\lambda(x)$ is derivable, then fuzzy-valued function $\tilde{f}(x)$ is derivable on D, and

$$\tilde{f}'(x) = \underset{\lambda\in(0,1]}{\cup} \lambda \wedge f'_\lambda(x).$$

Definition 7.2 ([12]) Let $f_\lambda(x) = [f_{1(\lambda)}(x), f_{2(\lambda)}(x)]$ be the $\lambda-$level set of $\tilde{f}(x)$. For any $\lambda \in (0, 1]$, the interval-valued functions $f_{1(\lambda)}(x)$ and $f_{2(\lambda)}(x)$ are integrable on D (in the sense of Riemann), then $\tilde{f}(x)$ is integrable on D, and

$$\int_D \tilde{f}(x)dx = \underset{\lambda\in(0,1]}{\cup} \lambda \wedge \int_D f_\lambda(x)dx,$$

$$\int_D f_\lambda(x)dx = [\int_D f_{1(\lambda)}(x)dx, \int_D f_{2(\lambda)}(x)dx].$$

The structure of the fuzzy-valued functional derivative in Definition 7.2 is meaningful in the view that fuzzy-valued functions are generated from interval-valued functions.

Definition 7.3 This will give a definition of fuzzy-valued functions based on the fuzzy structured element, and show that it is equal to Definition 7.2 and the derivative of fuzzy-valued functions can be transformed to the derivative of general functions.

Definition 7.4 Suppose that $\tilde{f}(x) = g(x, E)$ is a fuzzy-valued function generated by the fuzzy structured element E, $g(x, y)$ is a monotonic bounded function for y on $[-1, 1]$ and spans an ordered cone or a local ordered cone, then $\tilde{f}(x)$ is derivable on D, and $\frac{d\tilde{f}(x)}{dx} = \frac{\partial g(x,y)}{\partial x}\big|_{y=E}$.

The following theorem proves that the forms of derivative in both Definitions 7.2 and 7.3 are equivalent.

Theorem 7.1 *A necessary and sufficient condition for that the fuzzy-valued function $\tilde{f}(x) = g(x, E)$ is derivable under the conditions of Definition 7.3 is that for any $\lambda \in (0, 1]$, the $\lambda-$level set $f_\lambda(x)$ of $\tilde{f}(x)$ is derivable on D, and its derivative $\tilde{f}(x)$ is a fuzzy-valued function on D.*

Proof From Definition 7.3, $\forall y_0 \in [-1, 1]$, if $\tilde{f}(x)$ is derivable on D, then $\tilde{f}(x)$ is derivable with respect to x on $D \subseteq X$. Since for any $\lambda \in (0, 1]$, $f_\lambda(x) = [g(x, e_\lambda^-), g(x, e_\lambda^+)]$, $e_\lambda^- \in [-1, 1], e_\lambda^+ \in [-1, 1]$. We obtain that $g(x, e_\lambda^-)$ and $g(x, e_\lambda^+)$ are derivable for x on D. That is, for any $\lambda \in (0, 1]$, the $\lambda-$level set $f_\lambda(x)$ of $\tilde{f}(x)$ is derivable on D; On the contrary, for any $\lambda \in (0, 1]$, the $\lambda-$level set $f_\lambda(x)$ of $\tilde{f}(x)$ is derivable on D, then $f_\lambda(x) = [g(x, e_\lambda^-), g(x, e_\lambda^+)]$ is derivable for x on $D \subseteq X$, that is $g(x, e_\lambda^-)$ and $g(x, e_\lambda^+)$ are derivable for x on D. Since the membership function $E(x)$ of the fuzzy structured element E is monotonic on $[-1, 0]$ and $[0, 1]$, when λ is taken over $(0, 1]$, e_λ^- is taken over $[-1, 0]$, e_λ^+ is taken over $[0, 1]$, thus for any $y_0 \in [-1, 1]$, $g(x, y_0)$ is derivable for x on $D \subseteq X$, let $y = y_0$, that is $g(x, y)$ is derivable on $[-1, 1]$ for x.

Since $g(x, y)$ spans an ordered cone on X, then the derivative $g'(x, y)$ is still a monotonic bounded function for y on $[-1, 1]$, this ensures the derivative $g'(x, E) = \frac{\partial g(x,y)}{\partial x}|_{y=E}$ is also a fuzzy-valued function on D, the theorem is proved.

Theorem 7.2 *Let $\tilde{f}(x) = g(x, E)$ be a fuzzy-valued function generated by the fuzzy structured element E. If $g(x, y)$ is integrable with respect to x on $D \subseteq X$ (in the sense of Riemann), then the fuzzy-valued function $\tilde{f}(x)$ is integrable on D, and $\int_D \tilde{f}(x)dx = \int_D g(x, y)dx \,|_{y=E}$.*

Proof Suppose that $g(x, y)$ is monotonic increasing function for y on $[-1, 1]$, $\lambda \in (0, 1]$, $E_\lambda = [e_\lambda^-, e_\lambda^+]$, by Definition 7.2, we obtain that

$$\int_D \tilde{f}(x)dx = \bigcup_{\lambda\in(0,1]} \lambda \wedge \int_D f_\lambda(x)dx = \bigcup_{\lambda\in(0,1]} \lambda \wedge [\int_D g(x, e_\lambda^-)dx, \int_D g(x, e_\lambda^+)dx].$$

Let $J(y) = \int_D g(x, y)dx$. Since $g(x, y)$ is monotonic increasing function with respect to y on $[-1, 1]$, then $J(y)$ is a monotonic increasing function for y on $[-1, 1]$. Thus

$$\begin{aligned}\int_D \tilde{f}(x)dx &= \bigcup_{\lambda\in(0,1]} \lambda \wedge [J(e_\lambda^-), J(e_\lambda^+)] = \bigcup_{\lambda\in(0,1]} \lambda \wedge J(E_\lambda) = \bigcup_{\lambda\in(0,1]} \lambda \wedge J_\lambda(E) \\ &= J(E) = \int_D g(x, y)dx \,|_{y=E}.\end{aligned}$$

8 Conclusions

This paper introduced the method of the fuzzy structured element, transformed the operations and the order of fuzzy numbers into operations and sequence of functions from $B_{[-1,1]}$ by the fuzzy structured element, respectively. We have obtained many important theorems of fuzzy-valued functions, including the representation theorems and general forms of differential and Riemann integral.

References

1. Anastassiou, G.A.: On H-fuzzy differentiation. Math. Balkanica **16**, 155–193 (2002)
2. Bedregal, B.C.: On interval fuzzy negations. Fuzzy Sets Syst. **161**(17), 2290–2313 (2010)
3. Chang, S.L., Zadeh, L.A.: On fuzzy mapping and control. Trans. Syst. Man Cybern. **2**(1), 30–34 (1972)
4. Dubois, D., Prade, H.: Operations on fuzzy numbers. Int. J. Syst. Sci. **9**(6), 613–626 (1978)
5. Dubois, D., Prade, H.: Toward fuzzy differential calculus: part 3, differentiation. Fuzzy Sets Syst. **8**(3), 225–233 (1982)

6. Dubois, D., Prade, H.: Random sets and fuzzy interval analysis. Fuzzy Sets Syst. **42**(1), 87–101 (1991)
7. Gehrke, C.M., Walker, E.: Some comments on interval valued fuzzy sets. Int. J. Intell. Syst. **11**, 751–759 (1996)
8. Guo, S.C.: Methods of the structured element in fuzzy analysis (I), (II). J. Liaoning Tech. Univ. (Nat. Sci.) **21**(5), 670–673, 808–810 (2002)
9. Guo, S.C.: Homeomorphic properties between fuzzy number space and the family of standard bounded monotonic functions. Adv. Nat. Sci. **14**(11), 1318–1321 (2004)
10. Guo, S.C.: Principle of mathematical analysis based on structured element. Northeast University Press, Shenyang (2004)
11. Guo, S.C.: Analytic express methods and calculus of fuzzy-valued function based on structured element. Sci. Technol. Eng. **15**(5), 513–517 (2005)
12. Luo, C.Z.: Introduction of Fuzzy Sets. Introduction of Fuzzy Sets. Beijing Normal University Press, Beijing (1994)
13. Mizumoto, M., Tanaka, K.: Algebraic Properties of Fuzzy Numbers. International Conference on Cybernetics and Society, Washington (1976)
14. Moore, R.E., Lodwick, W.: Interval analysis and fuzzy set theory. Fuzzy Sets Syst. **135**(1), 5–9 (2003)
15. Puri, M., Ralescu, D.: Differentials of fuzzy functions. Math. Anal. Appl. **91**, 552–558 (1983)
16. Shen, Z.H.: Methods and applications for interval analysis. Appl. Math. Numer. Math. **2**, 1–28 (1982)
17. Wu, C.Z., Gong, Z.T.: On Henstock integral of fuzzy-number-valued functions I. Fuzzy Sets Syst. **120**(3), 523–532 (2001)

Posynomial Geometric Programming with Intuitionistic Fuzzy Coefficients

Zeinab Kheiri and Bing-yuan Cao

Abstract In this paper, we introduce posynomial geometric programming problems with intuitionistic fuzzy numbers, it is formulated in intuitionistic fuzzy environment introducing intuitionistic fuzzinees in objective and constraint coefficients. This paper presents an approach based on (α, β)-cuts of intuitionistic fuzzy numbers to solve posynomial geometric programming problems with the data as triangular and trapezoidal intuitionistic fuzzy numbers.

Keywords Intuitionistic fuzzy set · Triangular and trapezoidal intuitionistic fuzzy numbers · Posynomial geometric programming · (α, β)-cuts and interval-valued function

1 Introduction

Geometric programming (GP) provides a power tool for solving a variety of optimization problems. In the real world, many applications of geometric programming (GP) are engineering design problems in which some of the problem parameters are estimating of actual values. Posynomial geometric programming many application oriented mathematical models deal with real numbers. In our daily life, we frequently deal with vague or imprecise information. The intuitionistic fuzzy sets were first introduced by Atanassov [1] which is a generalization of the concept of fuzzy set. The generalized concept of intuitionistic fuzzy number (IFN) introduced by Grzegrorzewski [5] in 2003 receives high attention and different definitions of IFNs have been proposed. Wang et al. [14] gave the definition of intuitionistic trapezoidal

Z. Kheiri · B. Cao
School of Mathematics and Information Science, Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong, Higher Education Institutes, Guangzhou University, Guangzhou 510006, Guangdong, China
e-mail: caobingy@163.com

B. Cao (✉)
Guangzhou University, Guangzhou 510550, China

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fuzzy number and interval intuitionistic trapezoidal fuzzy number. Recent years have witnessed a growing interest in the study of decision making problems with intuitionistic fuzzy sets/numbers, see, [7, 9, 15]. The approaches to modeling uncertainty of linear programming by the intuitionistic fuzzy methodology are also described in [4, 10]. Dubey and Mehra [4] solved linear programming with triangular intuitionistic fuzzy numbers in which triangular intuitionistic fuzzy numbers are converted to crisp set and solved. In this paper, we consider Posynomial Geometric Programming with intuitionistic fuzzy coefficient, and by using (α, β)-cuts, the intuitionistic posynomial geometric programming transformed into interval optimization problem. This interval optimization problems by parametric functional form of an interval number reduce to crisp Posynomial Geometric Programming and then solve the problem by dual problem of the Geometric Programming. This paper is organized as follows: Sect. 2 the concepts of triangular, trapezoidal intuitionistic fuzzy numbers and cut sets as well as arithmetical operations are introduced. In Sect. 3 a preliminary of Geometric Programming is presented. Formulation of intuitionistic fuzzy Posynomial Geometric Programming (IFPGP) with intuitionistic fuzzy coefficients and the conversion of IFPGP into classical posynomial geometric programming have been done in Sect. 4. Numerical examples and a comparison analysis are given in Sect. 5. The paper is concluded in Sect. 6.

2 Preliminaries

We quote several different definitions of triangular and trapezoidal intuitionistic fuzzy numbers.

Definition 1 ([1]) An intuitionistic fuzzy set (IFS) $\tilde{A}^I$ in X is given by

$$\tilde{A}^I = \{\langle x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)\rangle : x \in X\},$$

where the functions $\mu_{\tilde{A}^I}(x) : X \to [0, 1]$ and $\nu_{\tilde{A}^I}(x) : X \to [0, 1]$, with the condition $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1$, define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set $\tilde{A}^I$ which is a subset of X. For each $\tilde{A}^I$ in X, we can compute the intuitionistic index of the element in x to the set $\tilde{A}^I$, which is defined as follows:

$$\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x),$$

where $\pi_{\tilde{A}^I}(x)$ is also called a hesitancy degree of x to $\tilde{A}^I$. it is obvious that $x \in X, 0 \leq \pi_{\tilde{A}^I}(x) \leq 1$.

Definition 2 An intuitionistic fuzzy subset $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in R\}$ of the real line is called an intuitionistic fuzzy number (IFN) if:
(i) A is IF-normal, if there exist at least two points $x_0, x_1 \in X$ such that $\mu_{\tilde{A}^I}(x_0) = 1$, and $\nu_{\tilde{A}^I}(x_1) = 0$, it is easily seen that given intuitionistic fuzzy set $\tilde{A}^I$ is IF-normal if

there is at least one point that surely belongs to A and at least one point which does not belong to $\tilde{A}^I$,
(ii) $\tilde{A}^I$is IF-convex, an Intuitionistic Fuzzy Set $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in R\}$ of the real line is called IF-convex, if $\forall x_1, x_2 \in R, \forall \lambda \in [0, 1]$

$$\mu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}^I}(x_1), \mu_{\tilde{A}^I}(x_2)\},$$

$$\nu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\nu_{\tilde{A}^I}(x_1), \nu_{\tilde{A}^I}(x_2)\},$$

Thus A is IF convex if its membership function is fuzzy convex and its non membership function is fuzzy concave,
(iii) $\mu_{\tilde{A}^I}(x)$ is upper semicontinuous and $\nu_{\tilde{A}^I}(x)$ is lower semicontinuous,
(iv) $\tilde{A}^I = \{(x \in R\}|\nu_{\tilde{A}^I}(x) < 1\}$ is bounded.

Definition 3 ([13]) A generalized triangular intuitionistic fuzzy number (GTIFN) $\tilde{a}^I = \langle(a, l_\mu, r_\mu, \omega_a), (a, l_\nu, r_\nu, \upsilon_a)\rangle$ is a special intuitionistic fuzzy set on a real number set R, whose membership function and non-membership function are defined as follows

$$\mu_{\tilde{a}^I}(x) = \begin{cases} \frac{x-a+l_\mu}{l_\mu}\omega_a, & a - l_\mu \leq x \leq a, \\ \omega_a, & x = a, \\ \frac{a+r_\mu-x}{r_\mu}\omega_a, & a \leq x \leq a + r_\mu, \\ 0, & otherwise. \end{cases}$$

$$\nu_{\tilde{a}^I}(x) = \begin{cases} \frac{(a-x)+\upsilon_a(x-a+l_\nu)}{l_\nu}, & a - l_\nu \leq x \leq a, \\ \upsilon_a, & x = a, \\ \frac{(x-a)+\upsilon_a(a+r_\nu-x)}{r_\nu}, & a \leq x \leq a + r_\mu, \\ 1, & otherwise, \end{cases}$$

where l_μ, r_μ, and l_ν, r_ν are called the spreads of membership and non-membership functions, respectively, and a is called mean value. ω_a and υ_a represent the maximum degree of membership and minimum degree of non-membership, respectively, such that they satisfy the conditions $0 \leq \omega_a \leq 1, 0 \leq \upsilon_a \leq 1$ and $0 \leq \omega_a + \upsilon_a \leq 1$.

If $a - l_\nu \geq 0$, then generalized triangular intuitionistic fuzzy number (GTIFN) $\tilde{a}^I$ is called positive GTIFN and if $a + r_\nu \leq 0$ is called negative GTIFN.

When $\omega_a = 1, \upsilon_a = 0$ is called normal intuitionistic fuzzy number, namely traditional fuzzy number.

Definition 4 An intuitionistic fuzzy number $\tilde{a}^I = \langle(a_1, a_2, \gamma_\mu, \tau_\mu, \omega_a), (a_1, a_2, \gamma_\nu, \tau_\nu, \upsilon_a)\rangle$ is said to be a generalized trapezoidal intuitionistic fuzzy number (GTrIFN) if its membership and non-membership functions, are respectively, given by

$$\mu_{\tilde{a}^I}(x) = \begin{cases} \frac{x-a_1+\gamma_\mu}{\gamma_\mu}\omega_a, & a_1-\gamma_\mu \le x \le a_1, \\ \omega_a, & a_1 \le x \le a_2, \\ \frac{a_2+\tau_\mu-x}{\tau_\mu}\omega_a, & a_2 \le x \le a_2+\tau_\mu, \\ 0, & otherwise. \end{cases}$$

$$\nu_{\tilde{a}^I}(x) = \begin{cases} \frac{(a_1-x)+\upsilon_a(x-a_1+\gamma_\nu)}{\gamma_\nu}, & a_1-\gamma_\nu \le x \le a_1, \\ \upsilon_a, & a_1 \le x \le a_2, \\ \frac{(x-a_2)+\upsilon_a(a_2+\tau_\nu-x)}{\tau_\nu}, & a_2 \le x \le a_2+\tau_\nu, \\ 1, & otherwise, \end{cases}$$

where $a_1 \le a_2, \gamma_\mu, \tau_\mu \ge 0$. γ_μ, τ_μ, and γ_ν, τ_ν are called the spreads of membership and non-membership functions, respectively, such that $\gamma_\mu \le \gamma_\nu$ and $\tau_\mu \le \tau_\nu$.

ω_a and υ_a represent the maximum degree of membership and minimum degree of non-membership, respectively, satisfying $0 \le \omega_a \le 1, 0 \le \upsilon_a \le 1$ and $0 \le \omega_a + \upsilon_a \le 1$. When $\omega_a = 1, \upsilon_a = 0$ is called normal intuitionistic fuzzy number, namely traditional fuzzy number. Generally, there is $(a_1, a_2, \gamma_\mu, \tau_\mu) = (a_1, a_2, \gamma_\nu, \tau_\nu)$ intuitionistic trapezoidal fuzzy number $\tilde{a}^I$, here, denoted as $\tilde{a}^I = \langle(a_1, a_2, \gamma_\mu, \tau_\mu); \omega_a, \upsilon_a\rangle$. When $a_1 = a_2$, the intuitionistic trapezoidal fuzzy number becomes intuitionistic triangular fuzzy number. A generalized trapezoidal intuitionistic fuzzy number is called positive if $a_1 - \gamma_\nu \ge 0$.

Definition 5 ([8]) Scalar multiplication:
(i) If $\tilde{a}^I = \langle(a, l_\mu, r_\mu, \omega_a), (a, l_\nu, r_\nu, \upsilon_a)\rangle$ is a GTIFN, then

$$K\tilde{a^I} = \begin{cases} \langle(ka, kl_\mu, kr_\mu, \omega_a), (ka, kl_\nu, kr_\nu, \upsilon_a)\rangle, & for\ \ k > 0, \\ \langle(ka, -kr_\mu, -kl_\mu, \omega_a), (ka, -kr_\nu, -kl_\nu, \upsilon_a)\rangle, & for\ \ k < 0, \end{cases}$$

(ii) If $\tilde{a}^I = \langle(a_1, a_2, \gamma_\mu, \tau_\mu, \omega_a), (a_1, a_2, \gamma_\nu, \tau_\nu, \upsilon_a)\rangle$ be GTrIFN, then

$$k\tilde{a}^I = \begin{cases} \langle(ka_1, ka_2, k\gamma_\mu, k\tau_\mu, \omega_a), (ka_1, ka_2, k\gamma_\nu, k\tau_\nu, \upsilon_a)\rangle, & for\ \ k > 0, \\ \langle(ka_1, ka_2, -k\tau_\mu, -k\gamma_\mu, \omega_a), (ka_1, ka_2, -k\tau_\nu, -k\gamma_\nu, \upsilon_a)\rangle, & for\ \ k < 0. \end{cases}$$

Definition 6 A (α, β)-cut set of GTIFN $\tilde{a}^I = \langle(a, l_\mu, r_\mu, \omega_a), (a, l_\nu, r_\nu, \upsilon_a)\rangle$ is defined as

$$\tilde{a}^I_{\alpha,\beta} = \{x : \mu_{\tilde{a}(x)} \ge \alpha, \nu_{\tilde{a}}(x) \le \beta\},$$

where $0 \le \alpha \le \omega_a, \upsilon_a \le \beta \le 1$.

A α-cut set of GTIFN $\tilde{a}^I$ is a crisp subset of R, which is defined as

$$\hat{a}_\alpha = [a_L(\alpha), a_R(\alpha)] = [(a - l_\mu) + \frac{l_\mu\alpha}{\omega_a}, (a + r_\nu) - \frac{r_\nu\alpha}{\omega_a}].$$

According to Definitions 3 and 7, by using membership function, $\mu_{\tilde{a}}(x) \geq \alpha$, we have

$$\frac{x - a + l_\mu}{l_\mu}\omega_a \geq \alpha \rightarrow x \geq (a - l_\mu) + \frac{l_\mu \alpha}{\omega_a},$$

and

$$\frac{a + r_\mu - x}{r_\mu}\omega_a \geq \alpha \rightarrow x \leq (a + r_\nu) - \frac{r_\nu \alpha}{\omega_a},$$

we gain

$$[a_L(\alpha), a_R(\alpha)] = [(a - l_\mu) + \frac{l_\mu \alpha}{\omega_a}, (a + r_\nu) - \frac{r_\nu \alpha}{\omega_a}].$$

Similarly a β-cut of GTIFN $\tilde{a}^I$ is defined as

$$\hat{a}_\beta = [a_L(\beta), a_R(\beta)] = [(a - l_\nu) + \frac{(1-\beta)l_\nu}{1 - \upsilon_a}, (a + r_\nu) - \frac{(1-\beta)r_\nu}{1 - \upsilon_a}].$$

Definition 7 The (α, β)-cut set of GTrIFN $\tilde{a}^I = \langle(a_1, a_2, \gamma_\mu, \tau_\mu, \omega_a), (a_1, a_2, \gamma_\nu, \tau_\nu, \upsilon_a)\rangle$ is defined as usually, by

$$\tilde{a}^I_{\alpha,\beta} = \{x : \mu_{\tilde{a}}(x) \geq \alpha, \nu_{\tilde{a}}(x) \leq \beta\},$$

where $0 \leq \alpha \leq \omega_a, \upsilon_a \leq \beta \leq 1$.

A α-cut set of GTrIFN $\tilde{a}^I$ is a crisp subset of R, which is defined as

$$\hat{a}_\alpha = [a_L(\alpha), a_R(\alpha)] = [(a_1 - \gamma_\mu) + \frac{\gamma_\mu \alpha}{\omega_a}, (a_2 + \tau_\mu) - \frac{\tau_\mu \alpha}{\omega_a}].$$

Similarly a β-cut of GTrIFN $\tilde{a}$ is defined as

$$\hat{a}_\beta = [a_L(\beta), a_R(\beta)] = [(a_1 - \gamma_\nu) + \frac{(1-\beta)\gamma_\nu}{1 - \upsilon_a}, (a_2 + \tau_\nu) - \frac{(1-\beta)\tau_\nu}{1 - \upsilon_a}].$$

Theorem 1 ([8]) *Let $\tilde{a}^I$ be any GTIFN or GTrIFN. For any $\alpha \in [0, \omega_a]$ and $\beta \in [\upsilon_a, 1]$, where $0 \leq \alpha + \beta \leq 1$ the following equality is valid:*

$$\tilde{a}^I_{\alpha,\beta} = \hat{a}_\alpha \cap \hat{a}_\beta. \tag{1}$$

Proof See [1].

According Theorem 1 and definition of the intersection between $\hat{a}_\alpha$ and $\hat{a}_\beta$, we have following result

$$\tilde{a}_{\alpha,\beta} = [a_L, a_R], \tag{2}$$

where

$$a_L = \max\{a_L(\alpha), a_L(\beta)\}, \tag{3}$$

and

$$a_R = \min\{a_R(\alpha), a_R(\beta)\}. \tag{4}$$

Theorem 2 ([4]) *For each $a > 0$, the exponential function $f(x) = a^x$, is continuous.*

Note 1: Multiplication of two continuous functions is continuous. Ishihashi and Tanaka [6] defined three definitions to rank intervals. In this paper, according to our approach, we just introduce their definition for order relation is determined by left and right limits of an interval.

Definition 8 Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ are two closed intervals, then order relation between two closed intervals as

$$A \leq B, \quad \textit{iff} \quad a_L \leq b_L \text{ and } a_R \leq b_R. \tag{5}$$

Definition 9 ([10]) (Interval-valued function) Let $a > 0$, $b > 0$ and consider the interval $[a, b]$. From a mathematical point of view, any real number can be represented on a line. Similarly, we can represent an interval by a function. If the interval is of the form $[a, b]$, the interval-valued function is taken as

$$h(\rho) = a^{(1-\rho)} b^{\rho} \textit{ for } \rho \in [0, 1]. \tag{6}$$

The choice of the parameter ρ reflects some attitude on the part of the decision maker.

Lemma 1 *For given $[a, b]$, $a > 0, b > 0$, then $h(\rho) = a^{(1-\rho)} b^{\rho}$ for $\rho \in [0, 1]$ is a strictly monotone increasing continuous function.*

Proof According to Theorem 2 and Note 1, $h(\rho)$ is continuous.

Since $0 \leq \rho \leq 1$, then

$$\frac{d(h(\rho))}{d\rho} = \rho(1-\rho)\frac{1}{a^{\rho} b^{(1-\rho)}} \geq 0,$$

then $h(\rho)$ is monotone increasing and the proof is complete.

Lemma 2 *Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ are two closed intervals, if $A \leq B$ then for $\rho \in [0, 1]$*

$$h_A(\rho) \leq h_B(\rho).$$

Proof From Definition 8, $a_L \leq b_L \ \ and \ \ a_R \leq b_R$, since $\rho \in [0, 1]$, we obtain two following inequalities

$$a_L^{(1-\rho)} \leq b_L^{(1-\rho)}, \tag{7}$$

$$a_R^{\rho} \leq b_R^{\rho}. \tag{8}$$

Then we have $a_L^{(1-\rho)} a_R^{\rho} \leq b_L^{(1-\rho)} b_R^{\rho}$, hence, $h_A(\rho) \leq h_B(\rho)$.

3 Posynomial Geometric Programming

The general primal problem of posynomial geometric programming (PGP) [2, 12] is to

$$\begin{array}{l} \min \quad g_0(x) \\ s.t. \quad g_i(x) \leq 1, (1 \leq i \leq p), \\ \quad x > 0, \end{array} \tag{9}$$

where the functions g_i, $0 < i < p$ are posynomials, i.e.,

$$g_i(x) = \sum_{k=1}^{J_i} c_{ik} \Pi_{l=1}^{m} x_l^{\gamma_{ikl}} (1 \leq k \leq J_i, 1 \leq i \leq p, 1 \leq l \leq m). \tag{10}$$

The exponents γ_{ikl} are arbitrary real numbers, the coefficients c_{ik} are assumed to be positive constants and the decision variables $x = (x_1, \ldots, x_n)^T \in R^n$ are required to be strictly positive. In trade-off analysis we vary the constraints and see the effect on the optimal value of the problem. Starting from the geometric programming (1) that is a standard PGP, we form a perturbed PGP [12], by replacing the number one with parameter b_i which are all positive constants, on the right-hands side of each constraint.

$$\begin{array}{l} \min \quad g_0(x) \\ s.t. \quad g_i(x) \leq b_i, (1 \leq i \leq p), \\ \qquad x > 0. \end{array} \tag{11}$$

When $b_i = 1$, this reduces to the standard PGP, Eq. (9), otherwise, the constraints need some amendment to be standard PGP, Eq. (9). To solve the standard PGP can use the dual problem of the PGP [12]. The corresponding posynomial GP dual problem is to

$$
\begin{aligned}
&\max d(w) = \prod_{k=1}^{J_0}(\tfrac{c_{0k}}{w_{0k}})^{w_{0k}} \prod_{i=1}^{p}\prod_{k=1}^{J_p}(\tfrac{c_{ik}\lambda_i}{w_{ik}})^{w_{ik}}\\
&s.t.\quad \lambda_i = \sum_{k=1}^{J_i} w_{ik},\\
&\quad \lambda_0 = 1,\\
&\quad \sum_{i=1}^{p}\sum_{k=1}^{J_i}\gamma_{ikl}w_{ik} = 0 \quad 1 \le l \le m,\\
&\quad w_{ik} > 0 \quad 0 \le i \le p,\ \ 1 \le k \le J_p.
\end{aligned}
\tag{12}
$$

The quantity $J-(m+1)$ is termed a degree of difficulty in geometric programming. In the case of a constrained geometric programming problem, J denotes the total number of terms in all the posynomials and m represents the number of primal variables. The posynomial function contains J_0 terms in the objective function and J_p terms in the inequality constrains where $J = J_0 + J_1 + \cdots + J_p$.

4 Posynomial Geometric Programming with Intuitionistic Fuzzy Coefficient

In this section, Posynomial Geometric Programming with intuitionistic fuzzy coefficient and its solution producer are described. Let $\tilde{c}^I_{ik}$, $(0 \le i \le p)$ and $\tilde{b}^I_i$, $(1 \le i \le p)$ denote the intuitionistic fuzzy number that can be GTrIFN, GTIFN or STrIFN. The posynomial geometric programming problem with intuitionistic fuzzy number coefficients is of the following form:

$$
\begin{aligned}
&\min\quad \tilde{g}_0(x) = \sum_{k=1}^{J_0}\tilde{c}^I_{0k}\Pi_{l=1}^{m}x_l^{\gamma_{0kl}}\\
&s.t.\quad \tilde{g}_i(x) = \sum_{k=1}^{J_i}\tilde{c}^I_{ik}\Pi_{l=1}^{m}x_l^{\gamma_{ikl}} \le \tilde{b}^I_i,\ (1 \le i \le p),\\
&\quad x > 0,
\end{aligned}
\tag{13}
$$

by using (α, β)-cut of the intuitionistic fuzzy coefficients and parameter b_i and according to Eq. (1), the model is reduced to

$$
\begin{aligned}
&\min\quad \sum_{k=1}^{J_0}[c_{L_{0k}}, c_{R_{0k}}]\Pi_{l=1}^{m}x_l^{\gamma_{0kl}}\\
&s.t.\quad \sum_{k=1}^{J_i}[c_{L_{ik}}, c_{R_{ik}}]\Pi_{l=1}^{m}x_l^{\gamma_{ikl}} \le [b_{L_i}, b_{R_i}],\ (1 \le i \le p),\\
&\quad x > 0,
\end{aligned}
\tag{14}
$$

where

$$c_{L_{ik}} = \max\{c_{L_{ik}}(\alpha), c_{L_{ik}}(\beta)\},\ (0 \le i \le p), \tag{15}$$

$$c_{R_{ik}} = \min\{c_{R_{ik}}(\alpha), c_{R_{ik}}(\beta)\},\ (0 \le i \le p), \tag{16}$$

$$b_{L_i} = \max\{b_{L_i}(\alpha), b_{L_i}(\beta)\},\ (1 \le i \le p), \tag{17}$$

$$b_{R_i} = \min\{b_{R_i}(\alpha), b_{R_i}(\beta)\},\ (1 \le i \le p). \tag{18}$$

In model (14), we denote $g_0(x) = \sum_{k=1}^{J_0}[c_{L_{0k}}, c_{R_{0k}}]\Pi_{l=1}^{m} x_l^{\gamma_{0kl}}(x)$, and

$$g_i(x) = \sum_{k=1}^{J_i}[c_{L_{ik}}, c_{R_{ik}}]\Pi_{l=1}^{m} x_l^{\gamma_{ikl}}.$$

The model (14) is the posynomial geometric programming problem with interval coefficients. This model (14) can be transformed into the following parametric form:

$$\begin{aligned} \min \quad & g_0(x;\rho) = \textstyle\sum_{k=1}^{J_0}(c_{L_{0k}})^{(1-\rho)}(c_{R_{0k}})^{\rho}\Pi_{l=1}^{m} x_l^{\gamma_{0kl}} \\ s.t. \quad & g_i(x;\rho) = \textstyle\sum_{k=1}^{J_i}(c_{L_{ik}})^{(1-\rho)}(c_{R_{ik}})^{\rho}\Pi_{l=1}^{m} x_l^{\gamma_{ikl}} \le (b_{L_i})^{(1-\rho)}(b_{R_i})^{\rho},\ (1 \le i \le p), \\ & x > 0. \end{aligned} \tag{19}$$

The following theorem shows that model (14) can be transformed to a parametric posynomial geometric programming that is model (19).

Theorem 3 *The interval posynomial geometric programming problem*

$$\begin{aligned} \min \quad & \textstyle\sum_{k=1}^{J_0}[c_{L_{0k}}, c_{R_{0k}}]\Pi_{l=1}^{m} x_l^{\gamma_{0kl}} \\ s.t. \quad & \textstyle\sum_{k=1}^{J_i}[c_{L_{ik}}, c_{R_{ik}}]\Pi_{l=1}^{m} x_l^{\gamma_{ikl}} \le [b_{L_i}, b_{R_i}],\ (1 \le i \le p), \\ & x > 0, \end{aligned} \tag{20}$$

is equivalent to following parametric posynomial geometric programming

$$\begin{aligned} \min \quad & g_0(x;\rho) = \textstyle\sum_{k=1}^{J_0}(c_{L_{0k}})^{(1-\rho)}(c_{R_{0k}})^{\rho}\Pi_{l=1}^{m} x_l^{\gamma_{0kl}} \\ s.t. \quad & g_i(x;\rho) = \textstyle\sum_{k=1}^{J_i}(c_{L_{ik}})^{(1-\rho)}(c_{R_{ik}})^{\rho}\Pi_{l=1}^{m} x_l^{\gamma_{ikl}} \le (b_{L_i})^{(1-\rho)}(b_{R_i})^{\rho},\ (1 \le i \le p), \\ & x > 0. \end{aligned} \tag{21}$$

Proof Let Q_1 and Q_2 be the sets of all feasible solutions to (20) and (21), respectively. Then $x \in Q_1$ if and only if:

$$\sum_{k=1}^{J_i}[c_{L_{ik}}, c_{R_{ik}}]\Pi_{l=1}^{m} x_l^{\gamma_{ikl}} \le [b_{L_i}, b_{R_i}]. \tag{22}$$

Then, for any k, we take $d_k \in [c_{L_{ik}}, c_{R_{ik}}]$ and $q \in [b_{L_i}, b_{R_i}]$, problem (22) is substituted by the following crisp problem:

$$\sum_{k=1}^{J_i} d_k \Pi_{l=1}^{m} x_l^{\gamma_{ikl}} \le q. \tag{23}$$

From Definition 9, the interval-valued functions of $C = [c_{L_{ik}}, c_{R_{ik}}]$ and $B = [b_{L_i}, b_{R_i}]$ for any fixed i, are obtained respectively as

$$h_C(\rho) = c_{L_{ik}}^{(1-\rho)} c_{R_{ik}}^{\rho} \; for \; \rho \in [0, 1],$$

$$h_B(\rho) = b_{L_i}^{(1-\rho)} b_{R_i}^{\rho} \; for \; \rho \in [0, 1].$$

According to Lemma 1, $h_C(\rho)$ and $h_B(\rho)$ are strictly monotone increasing continuous functions. We obtain $d_k \in h_C(\rho)$ and $q \in h_B$, then $for \; \rho \in [0, 1]$, problem (23) reduces to

$$\sum_{k=1}^{J_i} c_{L_{ik}}^{(1-\rho)} c_{R_{ik}}^{\rho} \Pi_{l=1}^{m} x_l^{\gamma_{ikl}} \leq b_{L_i}^{(1-\rho)} b_{R_i}^{\rho}. \tag{24}$$

Then $x \in Q_2$, hence $Q_1 = Q_2$.

Now we suppose $x_0 = (x_{01}, \ldots, x_{0n})^T$ to be an optimal feasible solution to (20), then for all $x \in Q_1$, we have:

$$g_0(x) \geq g_0(x_0)$$

$$\Leftrightarrow \sum_{k=1}^{J_0} [c_{L_{0k}}, c_{R_{0k}}] \Pi_{l=1}^{m} x_l^{\gamma_{0kl}} \geq \sum_{k=1}^{J_0} [c_{L_{0k}}, c_{R_{0k}}] \Pi_{l=1}^{m} x_{0l}^{\gamma_{0kl}}$$

We take

$$\varphi_k \in [c_{L_{0k}}, c_{R_{0k}}]$$

for any k, then above problemes turn to

$$\sum_{k=1}^{J_0} \varphi_k \Pi_{l=1}^{m} x_l^{\gamma_{0kl}} \geq \sum_{k=1}^{J_0} \varphi_k \Pi_{l=1}^{m} x_{0l}^{\gamma_{0kl}}. \tag{25}$$

From Definition 9, the interval-valued function of $D = [c_{L_{0k}}, c_{R_{0k}}]$, is obtained

$$h_D(\rho) = c_{L_{0k}}^{(1-\rho)} c_{R_{0k}}^{\rho} \; for \; \rho \in [0, 1].$$

Then $\varphi_k \in h_D(\rho)$, problem (25) reduces to

$$\sum_{k=1}^{J_0} c_{L_{0k}}^{(1-\rho)} c_{R_{0k}}^{\rho} \Pi_{l=1}^{m} x_l^{\gamma_{0kl}} \geq \sum_{k=1}^{J_0} c_{L_{0k}}^{(1-\rho)} c_{R_{0k}}^{\rho} \Pi_{l=1}^{m} x_{0l}^{\gamma_{0kl}}. \tag{26}$$

We conclude that x_0 is an optimal feasible solution to (21).

Problem (19) called perturbed PGP, that the constraints need some amendment to be standard PGP, Eq. (6). We turn the inner program of model (19) to the following standard posynomial geometric program form:

$$\begin{aligned} \min \quad & g_0(x;\rho) = \sum_{k=1}^{J_0} (c_{L_{0k}})^{(1-\rho)} (c_{R_{0k}})^{\rho} \Pi_{l=1}^{m} x_l^{\gamma_{0kl}} \\ s.t. \quad & g_i(x;\rho) = \sum_{k=1}^{J_i} \frac{(c_{L_{ik}})^{(1-\rho)} (c_{R_{ik}})^{\rho}}{(b_{L_i})^{(1-\rho)} (b_{R_i})^{\rho}} \Pi_{l=1}^{m} x_l^{\gamma_{ikl}} \le 1, (1 \le i \le p), \\ & x > 0, \end{aligned} \tag{27}$$

We drive standard posynomial geometric program problem, and can solve by the dual problem of the PGP.

For $\rho = 0$ and $\rho = 1$ the lower bound and upper bound of the interval value of the parameter is used to find the optimal solution respectively. These two values yield the lower and upper bounds of the optimal solution. Although, one can gain the intermediate optimal result by using a proper value of ρ.

5 Numerical Examples

We present few examples to depict the working of the proposed technique for solving with intuitionistic fuzzy coefficients.

Design of a Two-Bar Truss

We consider a simple mechanical design problem, the two-bar truss is subjected to a vertical load $2p$ and is to be designed for minimum weight. The members have a tubular section with mean diameter d and wall hickness t and the maximum permissible stress in each member (σ_0) is approximately equal to 60,000 psi. Determine the values of h and d using geometric programming for the following data [12]:
$p = 33{,}000$ Ib, $t = 0.1$ in., $b = 30$ in., $\sigma_0 = 60{,}000$ psi, and ρ (*density*) $= 0.3$ lb/in^3.

$$\begin{aligned} & \min\ 2\rho\pi dt\sqrt{b^2+h^2} \\ & st. \frac{p}{\pi dt}\frac{\sqrt{b^2+h^2}}{h} \le \sigma_0, \\ & d, h \ge 0. \end{aligned} \tag{28}$$

The optimal solution is $h^* = 30$, $d^* = 2.474874$ and optimal objective value is 19.74. Now we consider the intuitionistic fuzzy optimization model of the two-bar truss as following:

$$\begin{aligned} & \min\ 2\tilde{\rho}^I\pi dt\sqrt{b^2+h^2} \\ & st. \frac{\tilde{p}^I}{\pi dt}\frac{\sqrt{b^2+h^2}}{h} \le \tilde{\sigma}_0^I, \\ & d, h \ge 0. \end{aligned} \tag{29}$$

where $\tilde{p}^I = 33,\tilde{000}^I$ Ib $= \langle(33000, 10, 30; 0.5); (33000, 40, 50; 0.25)\rangle$, $t = 0.1$ in., $b = 30$ in., $\tilde{\sigma_0}^I = 60,\tilde{000}^I = \langle(60000, 20, 30; 0.75); (60000, 30, 70; 0.2)\rangle$, *psi*, and $\tilde{\rho}^I$ (*density*) $= 0.\tilde{3}^I$ lb/in^3 $= \langle(0.3, 0.04, 0.08; 0.75); (0.3, 0.06, 0.14; 0.2)\rangle$, are the TIFNs. It can be seen that the objective and constraint functions are not posynomials, due to the presence of the $\sqrt{b^2+h^2}$. The functions can be converted to posynomials by introducing a new variable y as

$$y = b^2 + h^2.$$

Thus the intuitionistic fuzzy optimization problem can be stated as:

$$\begin{aligned} &\min\ 0.628\tilde{0.3}^I dy^{1/2} \\ &\tfrac{33,\tilde{000}^I}{0.314} y^{1/2} d^{-1} h^{-1} \le 60,\tilde{000}^I, \\ &900y^{-1} + y^{-1}h^2 \le 1 \\ &\quad y, d, h \ge 0. \end{aligned} \tag{30}$$

According to Definition 5, achieve the following model:

$$\begin{aligned} &\min\ \langle(0.188, 0.025, 0.050; 0.75); (0.188, 0.037, 0.087; 0.2)\rangle dy^{1/2} \\ &s.t.\quad \langle(105095.54, 31.84, 95.54; 0.5); (105095.54, 127.38, 159.23; 0.25)\rangle y^{1/2} d^{-1} h^{-1} \\ &\qquad \le 60,\tilde{000}^I, \\ &\quad 900y^{-1} + y^{-1}h^2 \le 1, \\ &\quad y, d, h \ge 0, \end{aligned} \tag{31}$$

by using (α, β)-cut of the intuitionistic fuzzy coefficients and parameter $\tilde{\sigma_0}^I$ that are $\alpha = 0.2$, $\beta = 0.5$ and according to Eq. (1), the model (19) is reduced to

$$\begin{aligned} &\min\ (0.174, 0.220) dy^{1/2} \\ &s.t.\quad (105076.43, 105148.61) y^{1/2} d^{-1} h^{-1} \le (59988.75, 60022), \\ &\quad 900y^{-1} + y^{-1}h^2 \le 1, \\ &\quad y, d, h \ge 0. \end{aligned} \tag{32}$$

This interval optimization problem can be transformed into the parametric form (10),

$$\begin{aligned} &\min\ (0.174)^{(1-\rho)} (0.220)^{\rho} dy^{1/2} \\ &s.t.\quad (105076.43)^{1-\rho} (105148.61)^{\rho} y^{1/2} d^{-1} h^{-1} \le (59988.75)^{(1-\rho)} (60022)^{\rho}, \\ &\quad 900y^{-1} + y^{-1}h^2 \le 1, \\ &\quad y, d, h \ge 0, \\ &\quad \rho \in [0, 1]. \end{aligned} \tag{33}$$

Turn this parametric perturbed PGP to standard geometric program form:

Table 1 Numerical results

ρ	Optimal objective value	d^*	y^*	h^*
0	18.28673	2.477140	1800.000	30
0.1	18.72100	2.477173	1800.000	30
0.2	19.16559	2.477205	1800	30
0.3	19.62074	2.477238	1800	30
0.4	20.08669	2.477271	1800	30
0.5	20.56371	2.477304	1800	30
0.6	21.05206	2.477337	1800	30
0.7	21.55201	2.477370	1800	30
0.8	22.06382	2.477402	1800	30
0.9	22.58780	2.477435	1800	30
1	23.13703	2.478841	1799.999	30

$$
\begin{aligned}
&\min\ (0.174)^{(1-\rho)}(0.220)^{\rho} d y^{1/2} \\
&\tfrac{(105076.43)^{1-\rho}(105148.61)^{\rho}}{(59988.75)^{(1-\rho)}(60022)^{\rho}} y^{1/2} d^{-1} h^{-1} \leq 1, \\
&\quad 900 y^{-1} + y^{-1} h^2 \leq 1, \\
&\quad y, d, h \geq 0, \\
&\rho \in [0, 1].
\end{aligned} \tag{34}
$$

This model is parametric standard PGP, for $\rho \in [0, 1]$, numerical solutions of this problem are presented in following Table 1.

For $\rho = 0$, the lower bound of interval value of the coefficient is used to find the optimal solution. And $\rho = 1$, present the upper bound of the interval coefficients is used for the optimal solution. These two values yield the lower and upper bounds of the optimal solution. We change $33,\tilde{000}^I$ to 32990, 33030, 32960, 33050, $60,\tilde{000}^I$ to 59980, 66030, 65970, 66070, and $0.\tilde{3}^I$ to 0.26, 0.38, 0.24, 0.44, in model (19) respectively. Then finding optimal objective values, 17.13109, 25.02297, 15.85812, 28.91133. Thus from the above discussion we see that the optimal objective values for $\rho \in [0, 1]$, are between 15.85812 and 28.91133.

The Optimal Box Design Problem

A box manufacturer wants to determine the optimal dimensions for making boxes to sell to customers. the cost for productions is $C_1 = 2/\text{sqft}$ dollars and the cost for producing the top and bottom is $C_1 = 3/\text{sqft}$ dollars are more cardboard is used for the top and bottom of the boxes. The volume of the box is to be set at a limit of $V = 4\text{ft}^3$ which can be varied for difference customer specifications. If the dimensions of the box are W for the width, H for the box height, and L for the box length, what should the dimension be based on the cost values and box volume [3].

The problem is minimization the box cost for a specific box volume, the primal objective function is:

$$\begin{array}{ll} \min & COST = C_1HW + C_1HL + C_2WL \\ st. & WLH \geq V, \\ & W, L, H \geq 0. \end{array} \tag{35}$$

However, the model is not geometric programming, because in geometric programming the inequalities must be written in the form of $\leq$. Thus, the primal constraint becomes:

$$\begin{array}{ll} \min & COST = 2HW + 2HL + 3WL \\ st. & 4W^{-1}L^{-1}H^{-1} \leq 1, \\ & W, L, H \geq 0. \end{array} \tag{36}$$

The optimal solution is $L^* = 1.386723$, $W^* = 1.386723$, $H^* = 2.080084$ and optimal objective value is 17.30699. Now we consider the intuitionistic fuzzy optimization model of the optimal box design problem as following:

$$\begin{array}{ll} \min & COST = \tilde{2}^I HW + \tilde{2}^I HL + \tilde{3}^I WL \\ st. & \tilde{4}^I W^{-1}L^{-1}H^{-1} \leq 1, \\ & W, L, H \geq 0, \end{array} \tag{37}$$

where $\tilde{2}^I = \langle(1.5, 2.5, 0.5, 0.5; 0.75); (1.5, 2.5, 0.6, 0.7; 0.2)\rangle$,
$\tilde{3}^I = \langle(2.8, 3.2, 0.6, 0.8; 3/4); (2.8, 3.2, 0.5, 0.5; 0.2; 1/6)\rangle$
and $\tilde{4}^I = \langle(3.5, 4.69, 0.5, 0.6; 3/4); (3.5, 4.69, 0.25, 0.1; 1/4)\rangle$ are GTrINFs.

By using (α, β)-cut of the intuitionistic fuzzy coefficients that are $\alpha = 0.3,\ \beta = 0.6$ and according to Eq. (1), the model (26), is reduced to:

$$\begin{array}{ll} \min & COST = (1.2, 2.8)HW + (1.2, 2.8)HL + (2.54, 3.46)WL \\ st. & (3.65, 4.63)W^{-1}L^{-1}H^{-1} \leq 1, \\ & W, L, H \geq 0. \end{array} \tag{38}$$

This interval optimization problem can be transformed into the parametric form (10),

$$\begin{array}{ll} \min & COST = (1.2)^{1-\rho}(2.8)^{\rho}HW + (1.2)^{1-\rho}(2.8)^{\rho}HL + (2.54)^{1-\rho}(3.46)^{\rho}WL \\ s.t. & (3.65)^{1-\rho}(4.63)^{\rho}W^{-1}L^{-1}H^{-1} \leq 1 \\ & W, L, H \geq 0. \end{array} \tag{39}$$

This model is parametric standard PGP, for $\rho \in [0, 1]$, numerical solutions of this problem are presented in following Table 2.

We change $\tilde{2}^I$ to 1, 3, 0.9, 3.2, $\tilde{3}^I$ to 2.2, 4, 2.3, 3.7, and $\tilde{4}^I$ to 3, 4.09, 3.25, 4.79 in model (24) respectively. Then $COST^* = 8.116018, 25.33403, 8.099305, 28.63159$. Thus from the above discussion we see that the optimal objective values for $\rho \in [0, 1]$, are between 8.099305 and 28.63159.

Table 2 Numerical results for optimal box design problem

ρ	*COST*	*L**	*W**	*H**
0	10.95759	1.199168	1.199168	2.538240
0.1	11.90166	1.230542	1.230542	2.468273
0.2	12.92707	1.262840	1.262840	2.400236
0.3	14.04082	1.295933	1.295933	2.334074
0.4	15.25053	1.329892	1.329892	2.269735
0.5	16.56446	1.364742	1.364742	2.207170
0.6	17.99160	1.400505	1.400505	2.146330
0.7	19.54169	1.437205	1.437205	2.087167
0.8	21.22533	1.474867	1.474867	2.029634
0.9	23.05403	1.513516	1.513516	1.973688
1	25.04029	1.553177	1.553177	1.919283

6 Conclusion

The present paper propose a solution procedure for posynomial geometric programming with IFN, where the coefficients of objective and constraint functions and right side are GTIFNs or GTrIFNs. The new concept of the posynomial geometric programming in an intuitionistic fuzzy number environment is introduced in this paper. In this approach, we use (α, β)-cut of the intuitionistic fuzzy numbers turn into interval numbers. To solve the interval optimization problem by interval-valued function to crisp PGP. The most important advantage, nobody solve the PGP problem in intuitionistic fuzzy numbers environment.

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References

1. Atanassov, K.T.: Intuitionistic Fuzzy Sets: Theory and Applications. Springer Physica Verlag, Heidelberg (1999)
2. Boyd, S., Kim, S.J., Vandenberghe, L., Hassibi, A.: Atutorial on Geometric Programming. Springer Science, Business Media, LLC (2007)
3. Creese, R.C.: Geometric programming for desing and cost optimization (with illustrative case study problems and solutions). Synthesis Lectures On Engineering (2010)
4. Dubey, D., Mehra, A.: Linear programming with triangular intuitionistic fuzzy number. In: EUSFLAT (2011)
5. Grzegorzewski, P.: Intuitionistic fuzzy numbers. In: Accepted for the Proceedings of the IFSA 2003 World Congress
6. Ishihashi, H., Tanaka, M.: Multiobjective programming in optimization of the interval objective function. Euro. J. Oper. Res. 48, 219–225 (1990)

7. Jianqiang, W., Zhong, Z.: Aggregation operators on intuitionistic trapezoidal fuzzy number and its application to multi-criteria decision making problems. J. Syst. Eng. Electron. **20**, 321326 (2009)
8. Li, D.F.: A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems. Comput. Math. Appl. **60**, 1557–1570 (2010)
9. Li, D.F., Nan, J.X., Zhang, M.J.: A ranking method of triangular intuitionistic fuzzy numbers and application to decision making. Int. J. Comput. Intell. Syst. **3**, 522–530 (2010)
10. Mahapatra, G.S., Mandal, T.K.: Posynomial parametric geometric programming with interval valued coefficient. J. Optim. Theory Appl. **154**, 120132 (2012)
11. Nachammai, A.L., Thangaraj, Dr. P.: Solving intuitionistic fuzzy linear programming by using metric distance ranking. Researcher **5**(4), 65–70 (2013)
12. Rao, S.S.: Engineering Optimization Theory and Practice, 3rd edn. A Wiley Interscience Publication, Wiley, New York (1996)
13. Seikh, M.R., Nayak, P., Pal, M.: Generalized triangular fuzzy numbers in intuitionistic fuzzy environment. Int. J. Eng. Res. Dev. **5**, 08–13 (2012)
14. Wang, J.Q., Zhang, Z.: Aggregation operators on intuitionistic fuzzy numbers and its applications to multi-criteria decision making problems. J. Syst. Eng. Electron. **20**, 321–326 (2009)
15. Wei, G., Lin, R., Zhao, X., Wang, H.: Some aggregating operators based on the Choquet integral with fuzzy number intuitionistic fuzzy information and their applications to multiple attribute decision making. Control Cybern. **41** (2012)

Generalized Multi-fuzzy Soft Set and Its Application in Decision Making

Hai-dong Zhang, Shi-long Liao and Wei-yuan Ma

Abstract The soft set theory, proposed by Molodtsov, can be used as a general mathematical tool for dealing with uncertainty. By introducing a generalization parameter which itself is multi-fuzzy, we define generalized multi-fuzzy soft sets which are an extension to the multi-fuzzy soft sets. Some operations on a generalized multi-fuzzy soft set are investigated, such as complement operation, union and intersection operations, "AND" and "OR" operations. Finally, application of generalized multi-fuzzy soft sets in decision making problems has been shown.

Keywords Multi-fuzzy set · Multi-fuzzy soft set · Generalized multi-fuzzy soft set · Decision making

1 Introduction

Molodtsov [1] initiated a novel concept called soft sets as a new mathematical tool for dealing with uncertainties. The soft set theory is free from many difficulties that have troubled the usual theoretical approaches. It has been found that fuzzy sets, rough sets, and soft sets are closely related concepts [2]. Soft set theory has potential applications in many different fields including the smoothness of functions, game theory, operational research, Perron integration, probability theory, and measurement theory [1, 3]. Research works on soft sets are very active and progressing rapidly in these years. Maji et al. [4] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Jun [5] introduced the notion of soft BCK/BCI-algebras. Jun and Park [6] discussed the applications of soft sets in ideal theory of BCK/BCI-algebras. Feng et al. [7] applied soft set theory to the study of semirings and initiated the notion of soft semirings. Furthermore, based on [4], Ali et al. [8] introduced some new operations on soft sets and improved the notion

H. Zhang (✉) · S. Liao · W. Ma
School of Mathematics and Computer Science, Northwest University for Nationalities, Lanzhou 730030, China
e-mail: lingdianstar@163.com

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of complement of soft set. They proved that certain De Morgans laws hold in soft set theory. Qin and Hong [9] introduced the notion of soft equality and established lattice structures and soft quotient algebras of soft sets. Park et al. [10] discussed some properties of equivalence soft set relations.

The soft set model can be combined with other mathematical models. Maji et al. [11, 12] presented the notion of generalized fuzzy soft sets theory which is based on a combination of the fuzzy set and soft set models. Yang et al. [13] presented the concept of the interval-valued fuzzy soft sets by combining interval-valued fuzzy set [14–16] and soft set models. Feng et al. [17] provided a framework to combine fuzzy sets, rough sets and soft sets all together, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. Shabir [18] presented a new approach to soft rough sets by combining the rough set and soft set.

Recently, authors [19] proposed the concept of the multi-fuzzy set which is a more general fuzzy set using ordinary fuzzy sets as building blocks, and its membership function is an ordered sequence of ordinary fuzzy membership functions. The notion of multi-fuzzy sets provides a new method to represent some problems which are difficult to explain in other extensions of fuzzy set theory, such as color of pixels. By combining the multi-fuzzy set and soft set models, Yang et al. [20] presented the concept of the multi-fuzzy soft set, and provided its application in decision making under an imprecise environment. The purpose of this paper is to extend the concept of multi-fuzzy soft set by introducing a generalization parameter which itself is multi-fuzzy, from which we can obtain a new soft set model: generalized multi-fuzzy soft set theory.

The rest of this paper is organized as follows. The following section briefly reviews some background on soft set, multi-fuzzy set and multi-fuzzy soft set. In Sect. 3, the concept of generalized multi-fuzzy soft set is presented. Some operations on the generalized multi-fuzzy soft set are then defined. Also their some interesting properties have been investigated. An application of generalized multi-fuzzy soft set in decision making problem has been shown in Sect. 4. Section 5 concludes the paper.

2 Preliminaries

In this section we give few definitions regarding soft sets and multi-fuzzy sets.

Definition 1 ([1]) Let U be an initial universe set and E be a universe set of parameters. A pair (F, A) is called a soft set over U if $A \subset E$ and $F : A \rightarrow P(U)$, where $P(U)$ is the set of all subsets of U.

Definition 2 ([12]) Let U be an initial universe set and E be a universe set of parameters. A pair (F, A) is called a fuzzy soft set over U if $A \subset E$ and $F : A \rightarrow F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.

Definition 3 ([19]) Let U be an initial universe set and k be a positive integer. A multi-fuzzy set $\tilde{A}$ in U is a set of ordered sequences.

$$\tilde{A} = \{u/(\mu_1(u), \mu_2(u), \ldots, \mu_k(u)) : u \in U\},$$

where $\mu_i \in F(U), i = 1, 2, \ldots k$.

The function $\mu_{\tilde{A}} = (\mu_1, \mu_2, \ldots, \mu_k)$ is called the multi-membership function of multi-fuzzy set $\tilde{A}$, and k is called the dimension of $\tilde{A}$. The set of all multi-fuzzy sets of dimension k in U is denoted by $M^k FS(U)$.

If $\sum_{i=1}^{k} \mu_i(u) \leq 1, \forall u \in U$, then the multi-fuzzy set of dimension k is called a normalized multi-fuzzy set. If $\sum_{i=1}^{k} \mu_i(u) = l > 1$, for some $u \in U$, we redefine the multi-membership degree $(\mu_1(u), \mu_2(u), \ldots, \mu_k(u)) : u \in U\}$ as $\frac{1}{l}(\mu_1(u), \mu_2(u), \ldots, \mu_k(u)) : u \in U\}$, then the non-normalized multi-fuzzy set can be changed into a normalized multi-fuzzy set.

Example 1 Suppose a color image is approximated by a $m \times n$ matrix of pixels. Let U be the set of all pixels of the color image. For any pixel u in U, the membership values $\mu_r(u), \mu_g(u), \mu_b(u)$ are being the normalized red value, green value and blue value of the pixel u respectively. So the color image can be approximated by the collection of pixels with the multi-membership function (μ_r, μ_g, μ_b) and it can be represented as a multi-fuzzy set

$$\tilde{A} = \{u/(\mu_r(u), \mu_r(u), \mu_b(u)) : u \in U\}.$$

In a two dimensional image, color of pixels cannot be characterized by a membership function of an ordinary fuzzy set, but it can be characterized by a three dimensional membership function (μ_r, μ_g, μ_b). In fact, a multi-fuzzy set can be understood to be a more general fuzzy set using ordinary fuzzy sets as its building blocks.

Definition 4 ([19]) Let $\tilde{A} = \{u/(\mu_1(u), \mu_2(u), \ldots, \mu_k(u)) : u \in U\}$ and $\tilde{B} = \{u/(\gamma_1(u), \gamma_2(u), \ldots, \gamma_k(u)) : u \in U\}$ be two multi-fuzzy sets of dimension k in U. We define the following relations and operations

(1) $\tilde{A} \sqsubseteq \tilde{B}$ iff $\mu_i(u) \leq \gamma_i(u), \forall u \in U$, and $1 \leq i \leq k$.
(2) $\tilde{A} = \tilde{B}$ iff $\mu_i(u) = \gamma_i(u), \forall u \in U$, and $1 \leq i \leq k$.
(3) $\tilde{A} \sqcup \tilde{B} = \{u/(\mu_1(u) \vee \gamma_1(u), \mu_2(u) \vee \gamma_2(u), \ldots, \mu_k(u) \vee \gamma_k(u)) : u \in U\}$.
(4) $\tilde{A} \sqcap \tilde{B} = \{u/(\mu_1(u) \wedge \gamma_1(u), \mu_2(u) \wedge \gamma_2(u), \ldots, \mu_k(u) \wedge \gamma_k(u)) : u \in U\}$.
(5) $\tilde{A}^c = \{u/(\mu_1^c(u), \mu_2^c(u), \ldots, \mu_k^c(u)) : u \in U\}$.

Definition 5 ([20]) Let U be an initial universe set and E be a universe set of parameters. A pair $(\tilde{F}, A)$ is called a multi-fuzzy soft set of dimension k over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : A \to M^k FS(U)$.

A multi-fuzzy soft set is a mapping from parameters to $M^k FS(U)$. It is a parameterized family of multi-fuzzy subsets of U. For $e \in A$, $\tilde{F}(e)$ may be considered as the set of e-approximate elements of the multi-fuzzy soft set $(\tilde{F}, A)$.

Example 2 Suppose that $U = \{c_1, c_2, c_3, c_4\}$ is the set of color cloths under consideration, $A = \{e_1, e_2, e_3\}$ is the set of parameters, where e_1 stands for the parameter 'color' which consists of red, green and blue, e_2 stands for the parameter 'ingredient' which is made from wool, cotton and acrylic, and e_3 stands for the parameter 'price' which can be various: high, medium and low. We define a multi-fuzzy soft set of dimension 3 as follows

$\tilde{F}(e_1) = \{c_1/(0.4, 0.2, 0.3), c_2/(0.2, 0.1, 0.6), c_3/(0.1, 0.3, 0.4), c_4/(0.3, 0.1, 0.3)\}$,
$\tilde{F}(e_2) = \{c_1/(0.1, 0.2, 0.6), c_2/(0.3, 0.2, 0.4), c_3/(0.5, 0.3, 0.1), c_4/(0.6, 0.1, 0.3)\}$,
$\tilde{F}(e_3) = \{c_1/(0.3, 0.4, 0.1), c_2/(0.4, 0.1, 0.2), c_3/(0.2, 0.2, 0.5), c_4/(0.7, 0.1, 0.2)\}$.

3 Generalized Multi-fuzzy Soft Sets

3.1 Concept of Generalized Multi-fuzzy Soft Sets

In this subsection, we give a modified definition of multi-fuzzy soft set.

Definition 6 Let U be an initial universe and E be a set of parameters. The pair (U, E) is called a soft universe. Suppose that $\tilde{F} : E \to M^k FS(U)$, and $\tilde{f}$ is a multi-fuzzy subset of E, i.e. $\tilde{f} : E \to I^k = [0, 1]^k$, we say that $\tilde{F}_{\tilde{f}}$ is a generalized multi-fuzzy soft set (GMFSS, in short) over the soft universe (U, E) if and only if $\tilde{F}_{\tilde{f}}$ is a mapping given by

$$\tilde{F}_{\tilde{f}} : E \to M^k FS(U) \times I^k,$$

where $\tilde{F}_{\tilde{f}}(e) = (\tilde{F}(e), \tilde{f}(e))$, $\tilde{F}(e) \in M^k FS(U)$, and $\tilde{f}(e) \in I^k$.

For all $e \in E$, $\tilde{F}_{\tilde{f}}(e)$ is actually a generalized multi-fuzzy set of (U, E), where $u \in U$, it can be written as:

$\tilde{F}_{\tilde{f}}(e) = (\tilde{F}(e), \tilde{f}(e))$.

Here, $\tilde{F}(e) = \{u/(\mu^1_{\tilde{F}(e)}(u), \mu^2_{\tilde{F}(e)}(u) \cdots \mu^k_{\tilde{F}(e)}(u)) : u \in U\}$,
$\tilde{f}(e) = (\mu^1_{\tilde{f}(e)}(u), \mu^2_{\tilde{f}(e)}(u), \ldots, \mu^k_{\tilde{f}(e)}(u))$.

Remark 1 A generalized multi-fuzzy soft set is also a special case of a soft set because it is still a mapping from parameters to $M^k FS(U) \times I^k$,

(1) If $k = 1$, then $\tilde{F}_{\tilde{f}}$ will be degenerated to be a generalized fuzzy soft sets [11].
(2) If $k = 2$, $\mu^1_{\tilde{F}(e)}(u) + \mu^2_{\tilde{F}(e)}(u) \leq 1$ and $\mu^1_{\tilde{f}(e)}(u) + \mu^2_{\tilde{f}(e)}(u) \leq 1$, then $\tilde{F}_{\tilde{f}}$ will be degenerated to be a generalized intuitionistic fuzzy soft sets [21].

Example 3 Let U be a set of three shirts under consideration of a decision maker to purchase, which is denoted by $U = \{u_1, u_2, u_3\}$. Let $E = \{e_1, e_2, e_3\}$ be set of parameters, where e_1 stands for the parameter 'color' which consists of red, green and blue, e_2 stands for the parameter 'ingredient' which is made from wool, cotton, and acrylic, and e_3 stands for the parameter 'price' which can be various: high, medium and low. Let $\tilde{f} : E \to I^3$ be defined as follows: $\tilde{f}(e_1) = (0.3, 0.6, 0.1)$, $\tilde{f}(e_2) = (0.3, 0.5, 0.2)$, $\tilde{f}(e_3) = (0.6, 0.2, 0.1)$. We define a function $\tilde{F}_{\tilde{f}} : E \to M^3FS(U) \times I^3$, given by as follows:

$$\tilde{F}_{\tilde{f}}(e_1) = (\{u_1/(0.2, 0.6, 0.2), u_2/(0.1, 0.8, 0.1), u_3/(0.2, 0.6, 0.2)\}, (0.3, 0.6, 0.1)),$$
$$\tilde{F}_{\tilde{f}}(e_2) = (\{u_1/(0.2, 0.2, 0.5), u_2/(0.3, 0.6, 0.1), u_3/(0.7, 0.2, 0.0)\}, (0.3, 0.5, 0.2)),$$
$$\tilde{F}_{\tilde{f}}(e_3) = (\{u_1/(0.8, 0.1, 0.1), u_2/(0.3, 0.4, 0.1), u_3/(0.6, 0.2, 0.2)\}, (0.6, 0.2, 0.1)).$$

Then $\tilde{F}_{\tilde{f}}$ is a GMFSS over (U, E).

In matrix from this can be expressed as

$$\tilde{F}_{\tilde{f}} = \begin{pmatrix} (0.2, 0.6, 0.2) & (0.1, 0.8, 0.1) & (0.2, 0.6, 0.2) & (0.3, 0.6, 0.1) \\ (0.2, 0.2, 0.5) & (0.3, 0.6, 0.1) & (0.7, 0.2, 0.0) & (0.3, 0.5, 0.2) \\ (0.8, 0.1, 0.1) & (0.3, 0.4, 0.1) & (0.6, 0.2, 0.2) & (0.6, 0.2, 0.1) \end{pmatrix},$$

where the ith row vector represent $\tilde{F}_{\tilde{f}}(e_i)$, the ith column vector represent u_i, the last column represent the range of $\tilde{f}$ and it will be called membership multi-fuzzy matrix of $\tilde{F}_{\tilde{f}}$.

Definition 7 Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two GMFSSs over (U, E). Now $\tilde{F}_{\tilde{f}}$ is said to be a generalized multi-fuzzy soft subset of $\tilde{G}_{\tilde{g}}$ if and only if

(1) $\tilde{f}$ is a multi-fuzzy subset of $\tilde{g}$;
(2) $\tilde{F}(e)$ is also a multi-fuzzy subset of $\tilde{G}(e)$, $\forall e \in E$.

In this case, we write $\tilde{F}_{\tilde{f}} \sqsubseteq \tilde{G}_{\tilde{g}}$.

Example 4 Consider the GMFSS $\tilde{F}_{\tilde{f}}$ over (U, E) given in Example 3. Let $\tilde{G}_{\tilde{g}}$ be another GMFSS over (U, E) defined as follows:

$$\tilde{G}_{\tilde{g}}(e_1) = (\{u_1/(0.1, 0.5, 0.0), u_2/(0.1, 0.5, 0.1), u_3/(0.1, 0.4, 0.1)\}, (0.2, 0.5, 0.1)),$$
$$\tilde{G}_{\tilde{g}}(e_2) = (\{u_1/(0.1, 0.1, 0.4), u_2/(0.2, 0.4, 0.1), u_3/(0.6, 0.1, 0.0)\}, (0.2, 0.3, 0.1)),$$
$$\tilde{G}_{\tilde{g}}(e_3) = (\{u_1/(0.5, 0.1, 0.0), u_2/(0.1, 0.3, 0.1), u_3/(0.5, 0.1, 0.1)\}, (0.4, 0.1, 0.1)).$$

Clearly, we have $\tilde{G}_{\tilde{g}} \sqsubseteq \tilde{F}_{\tilde{f}}$.

Definition 8 Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two GMFSSs over (U, E). Now $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ are said to be a generalized multi-fuzzy soft equal if and only if

(1) $\tilde{F}_{\tilde{f}}$ is a generalized multi-fuzzy soft subset of $\tilde{G}_{\tilde{g}}$;
(2) $\tilde{G}_{\tilde{g}}$ is a generalized multi-fuzzy soft subset of $\tilde{F}_{\tilde{f}}$,

which can be denoted by $\tilde{F}_{\tilde{f}} = \tilde{G}_{\tilde{g}}$.

3.2 Operations on Generalized Multi-fuzzy Soft Sets

Definition 9 Let $\tilde{F}_{\tilde{f}}$ be a GMFSS over (U, E). Then the complement of $\tilde{F}_{\tilde{f}}$, denoted by $\tilde{F}^c_{\tilde{f}}$, is defined by $\tilde{F}^c_{\tilde{f}} = \tilde{G}_{\tilde{g}}$, where $\tilde{G}(e) = \tilde{F}^c(e)$, and $\tilde{g}(e) = \tilde{f}^c(e)$.

From the above definition, we can see that $(\tilde{F}^c_{\tilde{f}})^c = \tilde{F}_{\tilde{f}}$.

Example 5 Consider the GMFSS $\tilde{G}_{\tilde{g}}$ over (U, E) defined in Example 4. Thus, by Definition 9, we have

$$\tilde{G}^c_{\tilde{g}} = \begin{pmatrix} (0.9, 0.5, 1.0) & (0.9, 0.5, 0.9) & (0.9, 0.6, 0.9) & (0.8, 0.5, 0.9) \\ (0.9, 0.9, 0.6) & (0.8, 0.6, 0.9) & (0.4, 0.9, 1.0) & (0.8, 0.7, 0.9) \\ (0.5, 0.9, 1.0) & (0.9, 0.7, 0.9) & (0.5, 0.9, 0.9) & (0.6, 0.9, 0.9) \end{pmatrix}.$$

Definition 10 The union operation on the two GMFSSs $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$, denoted by $\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}}$, is defined by a mapping given by $\tilde{H}_{\tilde{h}} : E \to M^k FS(U) \times I^k$, such that $\tilde{H}_{\tilde{h}}(e) = (\tilde{H}(e), \tilde{h}(e))$, where $\tilde{H}(e) = \tilde{F}(e) \sqcup \tilde{G}(e)$, $\tilde{h}(e) = \tilde{f}(e) \sqcup \tilde{g}(e)$.

Definition 11 The intersection operation on the two GMFSSs $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$, denoted by $\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}}$, is defined by a mapping given by $\tilde{H}_{\tilde{h}} : E \to M^k FS(U) \times I^k$, such that $\tilde{H}_{\tilde{h}}(e) = (\tilde{H}(e), \tilde{h}(e))$, where $\tilde{H}(e) = \tilde{F}(e) \sqcap \tilde{G}(e)$, $\tilde{h}(e) = \tilde{f}(e) \sqcap \tilde{g}(e)$.

Example 6 Let us consider the GMFSS $\tilde{F}_{\tilde{f}}$ in Example 3. Let $\tilde{G}_{\tilde{g}}$ be another GMFSS over (U, E) defined as follows:

$$\tilde{G}_{\tilde{g}}(e_1) = (\{u_1/(0.5, 0.5, 0.8), u_2/(0.2, 0.4, 0.1), u_3/(0.8, 0.4, 0.2)\}, (0.4, 0.5, 0.7)),$$
$$\tilde{G}_{\tilde{g}}(e_2) = (\{u_1/(0.4, 0.1, 0.8), u_2/(0.1, 0.4, 0.7), u_3/(0.6, 0.4, 0.5)\}, (0.2, 0.3, 0.5)),$$
$$\tilde{G}_{\tilde{g}}(e_3) = (\{u_1/(0.6, 0.4, 0.3), u_2/(0.3, 0.5, 0.1), u_3/(0.6, 0.4, 0.8)\}, (0.6, 0.8, 0.3)).$$

Then

$$\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}} = \begin{pmatrix} (0.5, 0.6, 0.8) & (0.2, 0.8, 0.1) & (0.8, 0.6, 0.2) & (0.4, 0.6, 0.7) \\ (0.4, 0.2, 0.8) & (0.3, 0.6, 0.7) & (0.7, 0.4, 0.5) & (0.3, 0.5, 0.5) \\ (0.8, 0.4, 0.3) & (0.3, 0.5, 0.1) & (0.6, 0.4, 0.8) & (0.6, 0.8, 0.3) \end{pmatrix},$$

$$\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}} = \begin{pmatrix} (0.2, 0.5, 0.2) & (0.2, 0.4, 0.1) & (0.2, 0.4, 0.2) & (0.3, 0.5, 0.1) \\ (0.2, 0.1, 0.5) & (0.1, 0.4, 0.1) & (0.6, 0.2, 0.0) & (0.2, 0.3, 0.2) \\ (0.6, 0.1, 0.1) & (0.3, 0.4, 0.1) & (0.6, 0.2, 0.2) & (0.6, 0.2, 0.1) \end{pmatrix}.$$

Definition 12 A GMFSS is said to a generalized $\tilde{F}-$empty multi-fuzzy soft set, denoted by $\tilde{F}_{\tilde{0}}$, if $\tilde{F}_{\tilde{0}} : E \to M^k FS(U) \times I^k$, such that $\tilde{F}_{\tilde{0}}(e) = (\tilde{F}(e), \tilde{0}(e))$, where $\tilde{0}(e) = (0, 0, \ldots 0), \forall e \in E$.

If $\tilde{F}(e) = \{u/(0, 0, \ldots, 0) : u \in U\}$ then the generalized $\tilde{F}-$empty multi-fuzzy soft set is called a generalized empty multi-fuzzy soft set, denoted by $\tilde{\varnothing}$.

Definition 13 A GMFSS is said to a generalized $\tilde{F}-$universal multi-fuzzy soft set, denoted by $\tilde{F}_{\tilde{1}}$, if $\tilde{F}_{\tilde{1}} : E \rightarrow M^k FS(U) \times I^k$, such that $\tilde{F}_{\tilde{1}}(e) = (\tilde{F}(e), \tilde{1}(e))$, where $\tilde{1}(e) = (1, 1, \ldots, 1), \forall e \in E$.

If $\tilde{F}(e) = \{u/(1, 1, \ldots, 1) : u \in U\}$ then the generalized $F-$universal multi-fuzzy soft set is called a generalized universal multi-fuzzy soft set, denoted by $\tilde{1}$.

From Definitions 12 and 13, obviously we have

(1) $\tilde{\emptyset} \sqsubseteq F_{\tilde{0}} \sqsubseteq \tilde{F}_{\tilde{f}} \sqsubseteq \tilde{F}_{\tilde{1}} \sqsubseteq \tilde{1}$,
(2) $\tilde{\emptyset}^c = \tilde{1}$,
(3) $\tilde{1}^c = \tilde{\emptyset}$.

Theorem 1 *Let $\tilde{F}_{\tilde{f}}$ be a GMFSS over (U, E). Then the following holds:*

(1) $\tilde{F}_{\tilde{f}} \sqcup \tilde{\emptyset} = \tilde{F}_{\tilde{f}}, \tilde{F}_{\tilde{f}} \sqcap \tilde{\emptyset} = \tilde{\emptyset}$,
(2) $\tilde{F}_{\tilde{f}} \sqcup \tilde{1} = \tilde{1}, \tilde{F}_{\tilde{f}} \sqcap \tilde{1} = \tilde{F}_{\tilde{f}}$,
(3) $\tilde{F}_{\tilde{f}} \sqcup F_{\tilde{0}} = \tilde{F}_{\tilde{f}}, \tilde{F}_{\tilde{f}} \sqcap F_{\tilde{0}} = F_{\tilde{0}}$,
(4) $\tilde{F}_{\tilde{f}} \sqcup F_{\tilde{1}} = F_{\tilde{1}}, \tilde{F}_{\tilde{f}} \sqcap F_{\tilde{1}} = \tilde{F}_{\tilde{f}}$.

Proof Straightforward.

Remark 2 Let $\tilde{F}_{\tilde{f}}$ be a GMFSS over (U, E). If $\tilde{F}_{\tilde{f}} \neq \tilde{1}$ or $\tilde{F}_{\tilde{f}} \neq \tilde{\emptyset}$, then $\tilde{F}_{\tilde{f}} \sqcup \tilde{F}^c_{\tilde{f}} \neq \tilde{1}$, and $\tilde{F}_{\tilde{f}} \sqcap \tilde{F}^c_{\tilde{f}} \neq \tilde{\emptyset}$.

Theorem 2 *Let $\tilde{F}_{\tilde{f}}$, $\tilde{G}_{\tilde{g}}$ and $\tilde{H}_{\tilde{h}}$ be any three GIMFSSs over (U, E). Then the following holds:*

(1) $\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}} = \tilde{G}_{\tilde{g}} \sqcup \tilde{F}_{\tilde{f}}$,
(2) $\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}} = \tilde{G}_{\tilde{g}} \sqcap \tilde{F}_{\tilde{f}}$,
(3) $\tilde{F}_{\tilde{f}} \sqcup (\tilde{G}_{\tilde{g}} \sqcup \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}}) \sqcup \tilde{H}_{\tilde{h}}$,
(4) $\tilde{F}_{\tilde{f}} \sqcap (\tilde{G}_{\tilde{g}} \sqcap \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}}) \sqcap \tilde{H}_{\tilde{h}}$.

Proof The properties follow from the definitions.

Theorem 3 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two GMFSSs over (U, E). Then De-Morgan's laws are valid:*

(1) $(\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}})^c = \tilde{F}^c_{\tilde{f}} \sqcap \tilde{G}^c_{\tilde{g}}$,
(2) $(\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}})^c = \tilde{F}^c_{\tilde{f}} \sqcup \tilde{G}^c_{\tilde{g}}$.

Proof For all $e \in E$,

$$\begin{aligned}
(\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}})^c &= ((\tilde{F}(e) \sqcup \tilde{G}(e), \tilde{f}(e) \sqcup \tilde{g}(e)))^c \\
&= (\tilde{F}^c(e) \sqcap \tilde{G}^c(e)), (\tilde{f}^c(e) \sqcap \tilde{g}^c(e)) \\
&= (\tilde{F}^c(e), \tilde{f}^c(e)) \sqcap (\tilde{G}^c(e) \sqcap \tilde{g}^c(e)) = \tilde{F}^c_{\tilde{f}} \sqcap \tilde{G}^c_{\tilde{g}}.
\end{aligned}$$

Likewise, the proof of (2) can be made similarly.

Theorem 4 *Let $\tilde{F}_{\tilde{f}}$, $\tilde{G}_{\tilde{g}}$ and $\tilde{H}_{\tilde{h}}$ be any three GMFSSs over (U, E). Then,*

(1) $\tilde{F}_{\tilde{f}} \sqcup (\tilde{G}_{\tilde{g}} \sqcap \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcup \tilde{G}_{\tilde{g}}) \sqcap (\tilde{F}_{\tilde{f}} \sqcup \tilde{H}_{\tilde{h}})$,
(2) $\tilde{F}_{\tilde{f}} \sqcap (\tilde{G}_{\tilde{g}} \sqcup \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \sqcap \tilde{G}_{\tilde{g}}) \sqcup (\tilde{F}_{\tilde{f}} \sqcap \tilde{H}_{\tilde{h}})$.

Proof The proof follows from definition and distributive property of multi-fuzzy set.

Definition 14 Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two GMFSSs over (U, E). The "$\tilde{F}_{\tilde{f}}$ AND $\tilde{G}_{\tilde{g}}$", denoted by $\tilde{F}_{\tilde{f}} \wedge \tilde{G}_{\tilde{g}}$, is defined by $\tilde{F}_{\tilde{f}} \wedge \tilde{G}_{\tilde{g}} = \tilde{H}_{\tilde{h}}$, where $\tilde{H}_{\tilde{h}}(\alpha, \beta) = (\tilde{H}(\alpha, \beta), \tilde{h}(\alpha, \beta))$, $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \sqcap \tilde{G}(\beta)$, and $\tilde{h}(\alpha, \beta) = \tilde{f}(\alpha) \sqcap \tilde{g}(\beta)$, for all $(\alpha, \beta) \in E \times E$.

Definition 15 Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two GMFSSs over (U, E). The "$\tilde{F}_{\tilde{f}}$ OR $\tilde{G}_{\tilde{g}}$", denoted by $\tilde{F}_{\tilde{f}} \vee \tilde{G}_{\tilde{g}}$, is defined by $\tilde{F}_{\tilde{f}} \vee \tilde{G}_{\tilde{g}} = \tilde{O}_{\tilde{o}}$, where $\tilde{O}_{\tilde{o}}(\alpha, \beta) = (\tilde{O}(\alpha, \beta), \tilde{o}(\alpha, \beta))$, $\tilde{O}(\alpha, \beta) = \tilde{F}(\alpha) \sqcup \tilde{G}(\beta)$, and $\tilde{o}(\alpha, \beta) = \tilde{f}(\alpha) \sqcup \tilde{g}(\beta)$, for all $(\alpha, \beta) \in E \times E$.

Example 7 Let us consider two GMFSSs $\tilde{G}_{\tilde{g}}$ and $\tilde{F}_{\tilde{f}}$ in Example 6. Then we have $\tilde{G}_{\tilde{g}} \wedge \tilde{F}_{\tilde{f}} = \tilde{H}_{\tilde{h}}$ and $\tilde{G}_{\tilde{g}} \vee \tilde{F}_{\tilde{f}} = \tilde{O}_{\tilde{o}}$ as follows:

$$
\begin{aligned}
\tilde{H}_{\tilde{h}}(e_1, e_1) &= (\{u_1/(0.2, 0.5, 0.2), u_2/(0.1, 0.4, 0.1), u_3/(0.2, 0.4, 0.2)\}, (0.3, 0.5, 0.1)),\\
\tilde{H}_{\tilde{h}}(e_1, e_2) &= (\{u_1/(0.2, 0.2, 0.5), u_2/(0.2, 0.4, 0.1), u_3/(0.7, 0.2, 0.0)\}, (0.3, 0.5, 0.2)),\\
\tilde{H}_{\tilde{h}}(e_1, e_3) &= (\{u_1/(0.5, 0.1, 0.1), u_2/(0.2, 0.4, 0.1), u_3/(0.6, 0.2, 0.2)\}, (0.4, 0.2, 0.1)),\\
\tilde{H}_{\tilde{h}}(e_2, e_1) &= (\{u_1/(0.2, 0.1, 0.2), u_2/(0.1, 0.4, 0.1), u_3/(0.2, 0.4, 0.2)\}, (0.2, 0.3, 0.1)),\\
\tilde{H}_{\tilde{h}}(e_2, e_2) &= (\{u_1/(0.2, 0.1, 0.5), u_2/(0.1, 0.4, 0.1), u_3/(0.6, 0.2, 0.0)\}, (0.2, 0.3, 0.2)),\\
\tilde{H}_{\tilde{h}}(e_2, e_3) &= (\{u_1/(0.4, 0.1, 0.1), u_2/(0.1, 0.4, 0.1), u_3/(0.6, 0.2, 0.2)\}, (0.2, 0.2, 0.1)),\\
\tilde{H}_{\tilde{h}}(e_3, e_1) &= (\{u_1/(0.2, 0.4, 0.2), u_2/(0.1, 0.5, 0.1), u_3/(0.2, 0.4, 0.2)\}, (0.3, 0.6, 0.1)),\\
\tilde{H}_{\tilde{h}}(e_3, e_2) &= (\{u_1/(0.2, 0.2, 0.3), u_2/(0.3, 0.5, 0.1), u_3/(0.6, 0.2, 0.0)\}, (0.3, 0.5, 0.2)),\\
\tilde{H}_{\tilde{h}}(e_3, e_3) &= (\{u_1/(0.6, 0.1, 0.1), u_2/(0.3, 0.4, 0.1), u_3/(0.6, 0.2, 0.2)\}, (0.6, 0.2, 0.1))
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{O}_{\tilde{o}}(e_1, e_1) &= (\{u_1/(0.5, 0.6, 0.8), u_2/(0.2, 0.8, 0.1), u_3/(0.8, 0.6, 0.2)\}, (0.4, 0.6, 0.7)),\\
\tilde{O}_{\tilde{o}}(e_1, e_2) &= (\{u_1/(0.5, 0.5, 0.8), u_2/(0.3, 0.6, 0.1), u_3/(0.8, 0.4, 0.2)\}, (0.4, 0.5, 0.7)),\\
\tilde{O}_{\tilde{o}}(e_1, e_3) &= (\{u_1/(0.8, 0.5, 0.8), u_2/(0.3, 0.4, 0.1), u_3/(0.8, 0.4, 0.2)\}, (0.6, 0.5, 0.7)),\\
\tilde{O}_{\tilde{o}}(e_2, e_1) &= (\{u_1/(0.4, 0.6, 0.8), u_2/(0.1, 0.8, 0.7), u_3/(0.6, 0.6, 0.5)\}, (0.3, 0.6, 0.5)),\\
\tilde{O}_{\tilde{o}}(e_2, e_2) &= (\{u_1/(0.4, 0.2, 0.8), u_2/(0.3, 0.6, 0.7), u_3/(0.7, 0.4, 0.5)\}, (0.3, 0.5, 0.5)),\\
\tilde{O}_{\tilde{o}}(e_2, e_3) &= (\{u_1/(0.8, 0.1, 0.8), u_2/(0.3, 0.4, 0.7), u_3/(0.6, 0.4, 0.5)\}, (0.6, 0.3, 0.5)),\\
\tilde{O}_{\tilde{o}}(e_3, e_1) &= (\{u_1/(0.6, 0.6, 0.3), u_2/(0.3, 0.8, 0.1), u_3/(0.6, 0.6, 0.8)\}, (0.6, 0.8, 0.3)),\\
\tilde{O}_{\tilde{o}}(e_3, e_2) &= (\{u_1/(0.6, 0.4, 0.5), u_2/(0.3, 0.6, 0.1), u_3/(0.7, 0.4, 0.8)\}, (0.6, 0.8, 0.3)),\\
\tilde{O}_{\tilde{o}}(e_3, e_3) &= (\{u_1/(0.8, 0.4, 0.3), u_2/(0.3, 0.5, 0.1), u_3/(0.6, 0.4, 0.8)\}, (0.6, 0.8, 0.3)).
\end{aligned}
$$

Theorem 5 *Let $\tilde{F}_{\tilde{f}}$ and $\tilde{G}_{\tilde{g}}$ be two GMFSSs over (U, E). Then*

(1) $(\tilde{F}_{\tilde{f}} \wedge \tilde{G}_{\tilde{g}})^c = \tilde{F}^c_{\tilde{f}} \vee \tilde{G}^c_{\tilde{g}}$,
(2) $(\tilde{F}_{\tilde{f}} \vee \tilde{G}_{\tilde{g}})^c = \tilde{F}^c_{\tilde{f}} \wedge \tilde{G}^c_{\tilde{g}}$.

Proof (1) Suppose that $\tilde{F}_{\tilde{f}} \wedge \tilde{G}_{\tilde{g}} = \tilde{H}_{\tilde{h}}$, where $\tilde{H}_{\tilde{h}}(\alpha, \beta) = (\tilde{F}(\alpha) \sqcap \tilde{G}(\beta), \tilde{f}(\alpha) \sqcap \tilde{g}(\beta))$. Therefore, for all $(\alpha, \beta) \in E \times E$, we have $\tilde{H}^c_{\tilde{h}}(\alpha, \beta) = (\tilde{F}^c(\alpha) \sqcup \tilde{G}^c(\beta), \tilde{f}^c(\alpha) \sqcup \tilde{g}^c(\beta))$.

Again suppose that $\tilde{F}^c_{\tilde{f}} \vee \tilde{G}^c_{\tilde{g}} = \tilde{O}_{\tilde{o}}$, where $\tilde{O}_{\tilde{o}}(\alpha, \beta) = (\tilde{F}^c(\alpha) \sqcup \tilde{G}^c(\beta), \tilde{f}^c(\alpha) \sqcup \tilde{g}^c(\beta))$. Hence $\tilde{H}^c_{\tilde{h}} = \tilde{O}_{\tilde{o}}$.

Likewise, the proof of (2) can be made similarly.

Theorem 6 *Let $\tilde{F}_{\tilde{f}}$, $\tilde{G}_{\tilde{g}}$ and $\tilde{H}_{\tilde{h}}$ be any three GMFSSs over (U, E). Then we have*

(1) $\tilde{F}_{\tilde{f}} \wedge (\tilde{G}_{\tilde{g}} \wedge \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \wedge \tilde{G}_{\tilde{g}}) \wedge \tilde{H}_{\tilde{h}}$,
(2) $\tilde{F}_{\tilde{f}} \vee (\tilde{G}_{\tilde{g}} \vee \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \vee \tilde{G}_{\tilde{g}}) \vee \tilde{H}_{\tilde{h}}$,
(3) $\tilde{F}_{\tilde{f}} \wedge (\tilde{G}_{\tilde{g}} \vee \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \wedge \tilde{G}_{\tilde{g}}) \vee (\tilde{F}_{\tilde{f}} \wedge \tilde{H}_{\tilde{h}})$,
(4) $\tilde{F}_{\tilde{f}} \vee (\tilde{G}_{\tilde{g}} \wedge \tilde{H}_{\tilde{h}}) = (\tilde{F}_{\tilde{f}} \vee \tilde{G}_{\tilde{g}}) \wedge (\tilde{F}_{\tilde{f}} \vee \tilde{H}_{\tilde{h}})$.

Proof The proof follows from definition and distributive property of multi-fuzzy set.

4 Application of Generalized Multi-fuzzy Soft Sets

In this section, we define an aggregate multi-fuzzy set of a GMFSS. We also define GMFSS aggregation operator that produce an aggregate multi-fuzzy set from a GMFSS over (U, E).

Definition 16 Let $\tilde{F}_{\tilde{f}}$ be a GMFSS over (U, E), i.e. $\tilde{F}_{\tilde{f}} \in \text{GMFSS}(U, E)$. Then GMFSS-aggregation operator, denoted by $GMFSS_{agg}$, is defined by

$$GMFSS_{agg} : GMFSS(U, E) \to M^kFS(U),\ GMFSS_{agg}\tilde{F}_{\tilde{f}} = \tilde{H},$$

where

$$\tilde{H} = \{u/\mu_{\tilde{H}}(u) = (\mu^1_{\tilde{H}}(u), \mu^2_{\tilde{H}}(u), \ldots, \mu^k_{\tilde{H}}(u)) : u \in U\}$$

which is a multi-fuzzy set over U. The value $\tilde{H}$ is called aggregate multi-fuzzy set of $\tilde{F}_{\tilde{f}}$.

Here, the membership degree $\mu_{\tilde{H}}(u_i)$ of u_i is defined as follows

$$\mu_{\tilde{H}}(u_i) = (a^1_i, a^2_i, \ldots, a^k_i) = (inf\{1, \sum_{x \in E} \mu^1_{\tilde{f}}(x)\mu^1_{\tilde{F}(x)}(u_i)\}, inf\{1, \sum_{x \in E} \mu^2_{\tilde{f}}(x)\mu^2_{\tilde{F}(x)}(u_i)\}, \ldots,$$
$$inf\{1, \sum_{x \in E} \mu^k_{\tilde{f}}(x)\mu^k_{\tilde{F}(x)}(u_i)\}).$$

From the above definition, it is noted that the $GIVFSS_{agg}$ on the multi-fuzzy set is an operation by which several approximate functions of a GMFSS are combined to produce a single multi-fuzzy set that is the aggregate multi-fuzzy set of the GMFSS.

Once an aggregate multi-fuzzy set has been arrived at, it may be necessary to choose the best single alternative from this set. Therefore, we can construct a GMFSS-decision making method by the following algorithm.

Step 1 Construct a GMFSS over (U, E),

Step 2 Find the aggregate multi-fuzzy set $\tilde{H}$,

Step 3 Compute the score r_i of u_i such that

$$r_i = \sum_{u_j \in U} ((a_i^1 - a_j^1) + (a_i^2 - a_j^2) + \cdots + (a_i^k - a_j^k)).$$

Step 4 Find the largest value in S where $S = \max_{u_i \in U}\{r_i\}$.

Example 8 Assume that a company want to fill a position. There are four candidates who form the set of alternatives, $U = \{u_1, u_2, u_3, u_4\}$. The hiring committee consider a set of parameters, $E = \{x_1, x_2, x_3\}$, where x_1 stands for "experience", which includes three levels: rich, average, poor, x_2 stands for "computer knowledge" which includes three levels: skilled, average, poor, and x_3 stands for "young age", which includes three levels: old, middle, little.

After a serious discussion each candidate is evaluated from point of view of the goals and the constraint according to a chose subset

$$\tilde{f} = \{x_1/(0.69, 0.10, 0.21), x_2/(0.32, 0.58, 0.10), x_3/(0.23, 0.47, 0.27)\}.$$

Finally, the committee constructs the following GMFSS over (U, E).

Step 1 Let the constructed GMFSS, $\tilde{F}_{\tilde{f}}$, be as follows,

$$\tilde{F}_{\tilde{f}} = \begin{pmatrix} (0.42, 0.28, 0.20) & (0.22, 0.49, 0.18) & (0.28, 0.42, 0.20) & (0.36, 0.54, 0.10) & (0.69, 0.10, 0.21) \\ (0.22, 0.19, 0.18) & (0.15, 0.45, 0.20) & (0.68, 0.22, 0.00) & (0.25, 0.35, 0.30) & (0.32, 0.58, 0.10) \\ (0.61, 0.19, 0.10) & (0.37, 0.43, 0.10) & (0.61, 0.29, 0.10) & (0.62, 0.28, 0.10) & (0.23, 0.47, 0.27) \end{pmatrix}.$$

Step 2 The aggregate multi-fuzzy set can be found as

$\tilde{H} = \{u_1/(0.50, 0.23, 0.09), u_2/(0.28, 0.51, 0.08), u_3/(0.55, 0.31, 0.07), u_4/(0.47, 0.39, 0.08)\}$

Step 3 For all $u_i \in U$, compute the score r_i of u_i such that

$$r_i = \sum_{u_j \in U} ((a_i^1 - a_j^1) + (a_i^2 - a_j^2) + \cdots + (a_i^k - a_j^k)).$$

Thus, we have $r_1 = -0.28, r_2 = -0.08, r_3 = 0.16, r_4 = 0.20$.

Step 4 The decision is anyone of the elements in S where $S = \max_{u_i \in U}\{r_i\}$.

In our example, the candidate u_4 is the best choice because $\max_{u_i \in U}\{r_i\} = \{u_4\}$.

5 Conclusion

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainty. However, it is difficult to be used to represent the fuzziness of problem. In order to handle these types of problem parameters, some fuzzy extensions of soft set theory are presented, yielding fuzzy soft set theory. In this paper, the notion of generalized multi-fuzzy soft set theory is proposed. Our generalized multi-fuzzy soft set theory is an extension of generalized fuzzy soft set theory and generalized intuitionistic fuzzy soft set theory. Some operations are defined on generalized multi-fuzzy soft sets. The basic properties of the generalized multi-fuzzy soft sets are also presented and discussed. Finally, an application of this theory has been applied to solve a decision making problem.

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References

1. Molodtsov, D.: Soft set theory-first results. Comput. Math. Appl. **37**, 19–31 (1999)
2. Aktas, H., Cagman, N.: Soft sets and soft groups. Inf. Sci. **177**, 2726–2735 (2007)
3. Molodtsov, D.: The Theory of Soft Sets. Moscow, URSS Publishers (2004)
4. Maji, P.K., Biswas, R., Roy, A.R.: Soft set theory. Comput. Math. Appl. **45**, 555–562 (2003)
5. Jun, Y.B.: Soft BCK/BCI-algebras. Comput. Math. Appl. **56**, 1408–1413 (2008)
6. Jun, Y.B., Park, C.H.: Applications of soft sets in ideal theory of BCK/BCI-algebras. Inf. Sci. **178**, 2466–2475 (2008)
7. Feng, F., Jun, Y.B., Zhao, X.Z.: Soft semirings. Comput. Math. Appl. **56**, 2621–2628 (2008)
8. Ali, M.I., Feng, F., Liu, X., Min, W.K., Shabir, M.: On some new operations in soft set theory. Comput. Math. Appl. **57**, 1547–1553 (2009)
9. Qin, K.Y., Hong, Z.Y.: On soft equality. J. Comput. Appl. Math. **234**, 1347–1355 (2010)
10. Park, J.H., Kima, O.H., Kwun, Y.C.: Some properties of equivalence soft set relations. Comput. Math. Appl. **63**, 1079–1088 (2012)
11. Maji, P.K., Samanta, S.K.: Generalized fuzzy soft sets. Comput. Math. Appl. **59**, 1425–1432 (2010)
12. Maji, P.K., Biswas, R., Roy, A.R.: Fuzzy soft set. J. Fuzzy Math. **9**(3), 589–602 (2001)
13. Yang, X.B., Lin, T.Y., Yang, J.Y., Li, Y., Yu, D.: Combination of interval-valued fuzzy set and soft set. Comput. Math. Appl. **58**(3), 521–527 (2009)
14. Gorzalczany, M.B.: A method of inference in approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets Syst. **21**, 1–17(1987)
15. Deschrijver, G., Kerre, E.E.: On the relationship between some extensions of fuzzy set theory. Fuzzy Sets Syst. **133**, 227–235 (2003)
16. Deschrijver, G., Kerre, E.E.: Implicators based on binary aggregation operators in interval-valued fuzzy set theory. Fuzzy Sets Syst. **153**, 229–248 (2005)
17. Feng, F., Li, C.X., Davvaz, B., Ali, M.I.: Soft sets combined with fuzzy sets and rough sets: a tentative approach. Soft Comput. **14**, 899–911 (2010)
18. Shabir, M., Ali, M.I., Shaheen, T.: Another approach to soft rough sets. Knowl. Based Syst. **40**, 72–80 (2013)

19. Sebastian, S., Ramakrishnan, T.V.: Multi-fuzzy sets: an extension of fuzzy sets. Fuzzy Inf. Eng. **1**, 35–43 (2011)
20. Yang, Y., Tan, X., Meng, C.C.: The multi-fuzzy soft set and its application in decision making. Appl. Math. Model. **37**, 4915–4923 (2013)
21. Agarwal, M., Biswas, K.K., Hanmandlu, M.: Generalized intuitionistic fuzzy soft sets with applications in decision-making. Appl. Soft Comput. **13**, 3552–3566 (2013)

Distance Measures for Interval-Valued Intuitionistic Hesitant Fuzzy Sets

Ya-ru Wei, Lin-qing Gao, Chao Wang and Ming-hu Ha

Abstract In order to effectively deal with some decision-making problems on interval-valued intuitionistic hesitant fuzzy environment, some distance measures for interval-valued intuitionistic hesitant fuzzy sets are defined, and corresponding properties are given and proved. The effectiveness and practicality of the distance measures are verified finally.

Keywords Interval-valued intuitionistic hesitant fuzzy sets · Distance measures · Decision-making · Fuzzy sets

1 Introduction

In 1965, the definition of fuzzy sets was introduced by Zadeh [1] to deal with fuzzy and uncertain information problems in our life. In order to expand the fuzzy sets, the concept of intuitionistic fuzzy sets (IFS) was proposed by Atanassov [2, 3] in 1986. Intuitionistic fuzzy sets are more flexibility and practicality than the traditional fuzzy sets in dealing with ambiguity and uncertainty. While, in some practical problems, due to many affect factors such as the complexity of the objective environment, professional skills of decision makers, decision makers often fail to provide accurate preference information for decision schemes. In order to expand the intuitionistic fuzzy sets, the concept of interval-valued intuitionistic fuzzy sets was proposed by Atanassov [4] in 1989, whose membership and non-membership are represented by interval numbers. The interval-valued intuitionistic fuzzy sets (IVIFS) may be better to describe the decision-makers' preference information and reflect the nature and

Y. Wei · L. Gao
Department of Management, Hebei University,
Baoding 071002, Hebei, People's Republic of China

C. Wang · M. Ha (✉)
School of Economics and Management, Hebei University of Engineering,
Handan 056038, Hebei, People's Republic of China
e-mail: mhha@hbu.edu.cn

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characteristics of the problem. Similarly, hesitant fuzzy sets as an expansion of the fuzzy sets were proposed by Torra and Narukawa [5] and Torra [6] in 2009 and 2010, which the membership of an element is a set composed by several possible values between 0 and 1. Subsequently, the relationship [7], distance and similarity measures [8], operators [9–11] of hesitate fuzzy set were discussed.

Distance measures have a wide range of applications in approximate reasoning, decision analysis and medical diagnosis. The distance measures for the above extensions of fuzzy set theory played an important role in the decision-making information aggregation. In 2013, the definition of interval-valued intuitionistic hesitant fuzzy sets (IVIHFS) and some operations was given by Zhang [12], but the corresponding distance measures have not been proposed. In this paper, the distance measures for interval-valued intuitionistic hesitant fuzzy set will be defined and discussed.

2 Distance Measures for IVIHFS

In 2013, Zhang [12] introduced the interval-valued intuitionistic hesitant fuzzy sets (IVIHFS), which extends the hesitant fuzzy set to interval-valued intuitionistic hesitant fuzzy environments. IVIHFS are very useful tools to deal with the situation in which decision maker hesitate between several possible interval-valued intuitionistic fuzzy numbers when assess to an attribute.

Definition 2.1 (*see* [12]) Let X be a nonempty set. Then an interval-valued intuitionistic hesitant fuzzy set (IVIHFS) on X is given by

$$\tilde{E} = \{\langle x, h_{\tilde{E}}(x)\rangle | x \in X\},$$

where $h_{\tilde{E}}(x)$ is a set of some IVIFNS.

For convenience, $\tilde{h} = h_{\tilde{E}}(x)$ is denoted as an interval-valued intuitionistic hesitant fuzzy element (IVIHFE), and $\tilde{H}$ is a set of all the IVIHFEs. If $\alpha \in \tilde{h}$, then α is an IVIFN denoted by $\alpha = (\mu_\alpha, \nu_\alpha) = ([\mu_\alpha^-, \mu_\alpha^+], [\nu_\alpha^-, \nu_\alpha^+])$.

Definition 2.2 Let $\tilde{A} = \{\langle x, \tilde{h}_{\tilde{A}}(x)\rangle | x \in X\}$, $\tilde{B} = \{\langle x, \tilde{h}_{\tilde{B}}(x)\rangle | x \in X\}$ be two IVIHFS. Then $\tilde{h}_{\tilde{A}}(x)$ and $\tilde{h}_{\tilde{B}}(x)$ are IVIHFEs. The distance measure for IVIHFS as following:

$$d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}}(x_i)) = \frac{1}{4l_{x_i}} \sum_{j=1}^{l_{x_i}} d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i)), \tag{1}$$

where $h_{\tilde{A}}^{\sigma(j)}(x_i)$ and $h_{\tilde{B}}^{\sigma(j)}(x_i)$ are the jth largest values in $\tilde{h}_{\tilde{A}}(x)$ and $\tilde{h}_{\tilde{B}}(x)$; $l_{x_i} = \max(l(h_{\tilde{A}}(x_i)), l(h_{\tilde{B}}(x_i)))$, $l(h_{\tilde{A}}(x_i))$ and $l(h_{\tilde{B}}(x_i))$ are the number of $\tilde{h}_{\tilde{A}}(x)$ and $\tilde{h}_{\tilde{B}}(x)$ respectively, which will be used thereafter.

Definition 2.3 Let $\tilde{A} = \{\langle x, \tilde{h}_{\tilde{A}}(x)\rangle | x \in X\}$, $\tilde{B} = \{\langle x, \tilde{h}_{\tilde{B}}(x)\rangle | x \in X\}$ be two IVIHFS. Based on the normalized Hamming distance, we can give an interval-valued intuitionistic hesitant fuzzy normalized Hamming distance:

$$
\begin{aligned}
d_{IVIHFH}(\tilde{A}, \tilde{B}) &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}} d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i))\right] \\
&= \frac{1}{n}[\frac{1}{4l_{x_i}}\sum_{j=1}^{l_{x_i}}(|(\mu_{\tilde{A}}^{\sigma(j)}(x_i))^+ - (\mu_{\tilde{B}}^{\sigma(j)}(x_i))^+| + |(\mu_{\tilde{A}}^{\sigma(j)}(x_i))^- - (\mu_{\tilde{B}}^{\sigma(j)}(x_i))^-| \\
&\quad + |(\nu_{\tilde{A}}^{\sigma(j)}(x_i))^+ - (\nu_{\tilde{B}}^{\sigma(j)}(x_i))^+| + |(\nu_{\tilde{A}}^{\sigma(j)}(x_i))^- - (\nu_{\tilde{B}}^{\sigma(j)}(x_i))^-|)],
\end{aligned} \tag{2}
$$

where

$$
\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i) = \left([(\mu_{\tilde{A}}^{\sigma(j)}(x_i))^-, (\mu_{\tilde{A}}^{\sigma(j)}(x_i))^+], [(\nu_{\tilde{A}}^{\sigma(j)}(x_i))^-, (\nu_{\tilde{A}}^{\sigma(j)}(x_i))^+]\right),
$$

$$
\tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i) = \left([(\mu_{\tilde{B}}^{\sigma(j)}(x_i))^-, (\mu_{\tilde{B}}^{\sigma(j)}(x_i))^+], [(\nu_{\tilde{B}}^{\sigma(j)}(x_i))^-, (\nu_{\tilde{B}}^{\sigma(j)}(x_i))^+]\right).
$$

Proposition 2.1 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVIHFS on $X = \{x_1, x_2, \ldots, x_n\}$. Then the distance measure between$\tilde{A}$ and $\tilde{B}$ is defined by $d(\tilde{A}, \tilde{B})$, which satisfies the following properties:*

(1) $0 \le d(\tilde{A}, \tilde{B}) \le 1$;
(2) $d(\tilde{A}, \tilde{B}) = 0$ *if and only if* $\tilde{A} = \tilde{B}$;
(3) $d(\tilde{A}, \tilde{B}) = d(\tilde{B}, \tilde{A})$*;*
(4) $d(\tilde{A}, \tilde{B}) \le d(\tilde{A}, \tilde{C}) + d(\tilde{C}, \tilde{B})$.

It is proved that the interval-valued intuitionistic hesitant fuzzy normalized Hamming distance satisfies the above proposition.

Proof (1) By the distance of interval-valued intuitionistic fuzzy sets, we have

$$
0 \le d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}}(x_i)) \le 1.
$$

So $0 \le d_{IVIHFH}(\tilde{A}, \tilde{B}) \le 1$;

(2) When $\tilde{A} = \tilde{B}$, we can easily get $d_{IVIHFH}(\tilde{A}, \tilde{B}) = 0$.
Now we need prove $d(\tilde{A}, \tilde{B}) = 0 \Rightarrow \tilde{A} = \tilde{B}$.
Let $d_{IVIHFH}(\tilde{A}, \tilde{B}) = 0$. We can get

$$
\frac{1}{l_{x_i}} d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i)) = 0 \Rightarrow d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i)) = 0,
$$

So $\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i) = \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i)$ and $\tilde{A} = \tilde{B}$.

(3) For any two interval-valued intuitionistic fuzzy numbers $\tilde{M}$ and $\tilde{N}$, there have $d(\tilde{M}, \tilde{N})=d(\tilde{N}, \tilde{M})$. Similarly, $d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i))=d(\tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i))$;

(4) Let $d_{IVIHFH}(\tilde{A}, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{4l_{x_i}}\sum_{j=1}^{l_{x_i}} d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i))\right]$,

$$d_{IVIHFH}(\tilde{A}, \tilde{C}) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{4l_{x_i}}\sum_{j=1}^{l_{x_i}} d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{C}}^{\sigma(j)}(x_i))\right],$$

$$d_{IVIHFH}(\tilde{C}, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{4l_{x_i}}\sum_{j=1}^{l_{x_i}} d(\tilde{h}_{\tilde{C}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i))\right].$$

Then we have

$$d_{IVIHFH}(\tilde{A}, \tilde{C}) + d_{IVIHFH}(\tilde{C}, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{4l_{x_i}}\sum_{j=1}^{l_{x_i}} (d(\tilde{h}_{\tilde{A}}^{\sigma(j)}, \tilde{h}_{\tilde{C}}^{\sigma(j)}) + d(\tilde{h}_{\tilde{C}}^{\sigma(j)}, \tilde{h}_{\tilde{B}}^{\sigma(j)})\right].$$

For the three interval-valued intuitionistic fuzzy numbers $\tilde{M}$, $\tilde{N}$ and $\tilde{P}$, we have

$$d(\tilde{M}, \tilde{P}) + d(\tilde{P}, \tilde{N}) \geq d(\tilde{M}, \tilde{N}).$$

So we have

$$d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{C}}^{\sigma(j)}(x_i)) + d(\tilde{h}_{\tilde{C}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i)) \geq d(\tilde{h}_{\tilde{A}}^{\sigma(j)}(x_i), \tilde{h}_{\tilde{B}}^{\sigma(j)}(x_i)).$$

Then,

$$d_{IVIHFH}(\tilde{A}, \tilde{C}) + d_{IVIHFH}(\tilde{C}, \tilde{B}) \geq d_{IVIHFH}(\tilde{A}, \tilde{B}).$$

It is obvious that the interval-valued intuitionistic hesitant fuzzy normalized Hamming distance measure satisfies Proposition 2.1.

Definition 2.4 Let $\tilde{A} = \{\langle x, \tilde{h}_{\tilde{A}}(x)\rangle | x \in X\}$, $\tilde{B} = \{\langle x, \tilde{h}_{\tilde{B}}(x)\rangle | x \in X\}$ be two IVIHFS. Then the generalized interval-valued intuitionistic hesitant fuzzy weighted distance is given by:

$$\begin{aligned} d_{GIVIHFW}(\tilde{A}, \tilde{B}) = [\sum_{i=1}^{n} w_i (\tfrac{1}{4l_{x_i}} \sum_{j=1}^{l_{x_i}} &|(\mu_{\tilde{A}}^{\sigma(j)}(x_i))^+ - (\mu_{\tilde{B}}^{\sigma(j)}(x_i))^+|^\lambda \\ + |(\mu_{\tilde{A}}^{\sigma(j)}(x_i))^- - (\mu_{\tilde{B}}^{\sigma(j)}(x_i))^-|^\lambda &+ |(\nu_{\tilde{A}}^{\sigma(j)}(x_i))^+ - (\nu_{\tilde{B}}^{\sigma(j)}(x_i))^+|^\lambda \\ + |(\nu_{\tilde{A}}^{\sigma(j)}(x_i))^- - (\nu_{\tilde{B}}^{\sigma(j)}(x_i))^-|^\lambda)]^{\frac{1}{\lambda}}&, \end{aligned} \tag{3}$$

where $\sum_{i=1}^{n} w_i = 1$ and $\lambda > 0$.

While, the corresponding similarity measure is given by:

$$s(\tilde{A}, \tilde{B}) = 1 - d(\tilde{A}, \tilde{B}).$$

3 Example Analysis

Now an example is given to verify the effectiveness and practicality of interval-valued intuitionstic hesitant fuzzy distance measure.

Example 3.1 A factory intends to select a new site for new buildings (adopted by [13]). Three alternatives $A_i(i = 1, 2, 3)$ are available, and three criteria to decide which site to choose: P_1 (price); P_2 (location); P_3 (environment). The criteria weight vector is $\omega = (0.5, 0.3, 0.2)^T$. Assume that the characteristics of the alternative $A_i(i = 1, 2, 3)$ with respect to the criteria $P = (P_1, P_2, P_3)$ are respected by IVIHFEs. The result shows in Table 1.

The optimal alternative is got according to the following processing:

Step 1: The Standardization of interval-valued intuitionistic hesitant fuzzy matrix. Each element $\tilde{h}_{\tilde{A}_i}(x_j)(i = 1, 2, 3; j = 1, 2, 3)$ in the interval-valued intuitionistic hesitant fuzzy numbers sorted in descending order, and then get a new matrix $\ddot{\tilde{H}}$, listed in Table 2.

Step 2: Calculate the interval-valued intuitionistic hesitant fuzzy weighted Hamming distance between every alternative and the ideal site {([1,1],[0,0])}.

$$d_{H\tilde{A}_1} = 0.1375 + 0.1875 + 0.0825 = 0.4075,$$

$$d_{H\tilde{A}_2} = 0.200 + 0.1388 + 0.0850 = 0.4238,$$

$$d_{H\tilde{A}_3} = 0.2875 + 0.1088 + 0.0575 = 0.4538.$$

Table 1 The intial desicion matrix

	P_1	P_2	P_3
A_1	{([0.5,0.6],[0.3,0.4]), ([0.7,0.8],[0.1,0.2]), ([0.6,0.8],[0.1,0.2])}	{([0.2,0.4],[0.5,0.6])}	{([0.3,0.5],[0.3,0.4]), ([0.5,0.6],[0.2,0.3])}
A_2	{([0.3,0,6],[0.2,0.3])}	{([0.4,0.5],[0.4,0.5]), ([0.4,0.6],[0.3,0.4])}	{([0.3,0.4],[0.5,0.6]), ([0.5,0.6],[0.2,0.3])}
A_3	{([0.2,0.3],[0.6,0.7]), ([0.4,0.5],[0.3,0.4])}	{([0.6,0.7],[0.2,0.3]), (0.4,0.6],[0.3,0.4])]}	{([0.7,0.8],[0.1,0.2]), ([0.4,0.6],[0.2,0.3])}

Table 2 The desicion matrix added

	P_1	P_2	P_3
A_1	{([0.7,0.8],[0.1,0.2]), ([0.6,0.8],[0.1,0.2]), ([0.5,0.6],[0.3,0.4])}	{([0.2,0.4],[0.5,0.6])}	{([0.5,0.6],[0.2,0.3]), (0.3,0.5],[0.3,0.4])}
A_2	{([0.3,0.6],[0.2,0.3])}	{([0.4,0.6],[0.3,0.4]), ([0.4,0.5],[0.4,0.5])}	{([0.5,0.6],[0.2,0.3]), ([0.3,0.4],[0.2,0.3])}
A_3	{([0.4,0.5],[0.3,0.4]), ([0.2,0.3],[0.6,0.7])}	{([0.6,0.7],[0.2,0.3]), ([0.4,0.6],[0.3,0.4])}	{([0.7,0.8],[0.1,0.2]), ([0.4,0.6],[0.2,0.3])}

Table 3 Sorting distance

$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	Sorting result
0.4075	0.4238	0.4538	$\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$

Step 3: Sort the distance that calculated according to the step 3 (shown in Table 3), the alternative with the minimum distance is optimal. After summarizing, we can get as following Table 3.

4 Conclusions

In this paper, two distance measures, the interval-valued intuitionistic hesitant fuzzy normalized Hamming distance and the interval-valued intuitionistic hesitant fuzzy normalized weighted Hamming distance are discussed. The distance measures of interval-valued intuitionstic hesitant fuzzy set provide a theoretical foundation for decision-making in forms of interval-valued intuitionstic hesitant fuzzy sets. However, the distance measures with membership degree often have subjectivity, and how to reduce subjectivity will be further studied.

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References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
2. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
3. Atanassov, K.: More on intuitionistic fuzzy sets. Fuzzy Sets Syst. **33**, 37–46 (1989)
4. Atanassov, K.: Operators over interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst. **64**, 159–174 (1994)

5. Torra, V., Narukawa, Y.: On hesitant fuzzy sets and decision. In: The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, pp. 1378–1382 (2009)
6. Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**, 529–539 (2010)
7. Chen, N., Xu, Z.S., Xia, M.M.: Correlation coefficients of hesitant fuzzy sets and their application to clustering analysis. Appl. Math. Model. **37**, 2179–2211 (2013)
8. Xu, Z.S., Xia, M.M.: Distance and similarity measure for hesitant fuzzy sets. Inf. Sci. **181**, 2128–2138 (2011)
9. Xia, M.M., Xu, Z.S.: Hesitant fuzzy information aggregation in decision making. Int. J. Approx. Reason. **52**, 395–407 (2011)
10. Xia, M.M., Xu, Z.S., Chen, N.: Some hesitant fuzzy aggregation operators with their application in group decision making. Group Decis. Negot. **22**, 259–279 (2013)
11. Chen, N., Xu, Z.S., Xia, M.M.: Interval-valued hesitant preference relations and their applications to group decision making. Knowl.-Based Syst. **37**, 528–540 (2013)
12. Zhang, Z.M.: Interval-valued intuitionistic hesitant fuzzy aggregation operators and their application in group decision-making. J. Appl. Math. 1–33 (2013)
13. Zhu, B., Xu, Z.S., Xia, M.M.: Hesitant fuzzy geometric Bonferroni means. Inf. Sci. **205**, 72–85 (2012)

Fuzzy Inference Modeling Method Based on T-S Fuzzy System

Ming-zuo Jiang, Chun-ling Zhang, Xue-hai Yuan and Hong-xing Li

Abstract A kind of fuzzy inference modeling method based on T-S fuzzy system is proposed. New input-output models and state-space models are constructed respectively by applying this method to time-invariant second-order freedom movement systems modeling. The obtained differential equation models are used to simulate the second-order equations, and the results show that the models achieve a good approximation precision.

Keywords Fuzzy control · Fuzzy inference modeling method · T-S fuzzy system · Input-Output model · State-space model

1 Introduction

It is well known that the theory of automatic control is mainly based on mathematical models of systems which are often represented as differential equations. By using these models, we can, for example, design control scheme and analyze system properties such as stability, controllability and observability.

We know that fuzzy control is suitable for the control systems with fuzzy environment being hard to model. The structure of fuzzy system is one of the important problems of fuzzy system. Fuzzy system mainly consists of fuzzification, fuzzy inference and defuzzification [1–6]. In detail, fuzzification includes singleton fuzzifier and parameter singleton fuzzifier [7]; fuzzy inference involves Compositional Rules of Inference (CRI) [8] and Triple I Method [9]; defuzzification contains center of gravity method, center average method and maximum value method. The structure of fuzzy system is closely related to fuzzy inference modeling. In 2002, based on interpola-

M. Jiang · C. Zhang · H. Li · X. Yuan (✉)
Faculty of Electronic Information and Electrical Engineering,
Dalian University of Technology, Dalian 116024, China
e-mail: yuanxh@dlut.edu.cn

M. Jiang · C. Zhang · H. Li
Foundation Department, Dalian University of Technology at Panjin, Panjin 124221, China

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tion mechanism of fuzzy logic system [10], Li put forward a new kind of modeling method of fuzzy system [11]. It transforms fuzzy inference rule base describing a time-invariant system into nonlinear differential equations with variable coefficients. Furthermore, in 2009, Li realized fuzzy modeling of time-varying system [12]. In 2013, [13] gave a fuzzy inference modeling method based on fuzzy transformation.

In recent years, T-S fuzzy system has drawn broad attention of scholars. As we know, the rule consequents of this kind of systems are represented by linear or nonlinear relationship between inputs and outputs, so each rule contains more system information such that few rules are needed to express system information. It also provides a crisp output that need not to be defuzzified, which makes the technique computationally efficient for describing nonlinear systems. Moreover, T-S fuzzy models enjoy great advantages in stability analysis and controller design for nonlinear systems because of its relatively simple and fixed structure and universal function approximation capability. Therefore, T-S fuzzy system has been recognized as one of the successful tools to handle complex nonlinear systems.

In this paper, we present a kind of fuzzy inference modeling method based on T-S fuzzy system. It is applied to time-invariant second-order freedom movement systems modeling, and new input-output models and new state-space models are respectively established. In addition, some simulation examples are given to show the advantages of this method.

2 Preliminaries

In 1985, Takagi and Sugeno proposed T-S fuzzy system [1, 14]. It is essentially a nonlinear model that can effectively deal with complex multi-variable nonlinear system. Unlike Mamdani fuzzy system, the rule consequent of T-S fuzzy system is represented by a linear or nonlinear relationship between input and output so that more information can be contained in each rule. Therefore, the use of fewer rules can achieve the demanded control effect, which makes the controller more simple [1, 14–17]. The fuzzy rule base of a T-S fuzzy system comprises of a collection of fuzzy if-then rules in the following form:

If x is A_i and y is B_j, then

$$z_{ij} = c_{ij}^0 + c_{ij}^1 x + c_{ij}^2 y (i = 1, 2, \dots m; j = 1, 2, \dots n), \tag{1}$$

where $A_i (i = 1, \dots, m)$ and $B_j (j = 1, \dots, n)$ in (1) are both fuzzy sets with triangular membership functions. The coefficients c_{ij}^0, c_{ij}^1 and c_{ij}^2 in (1) can be defined as below.

For differential functions, the coefficients are determined by taking the derivatives. Let $g(x_i, y_j) = z_{ij} (i = 1, \dots, m; j = 1, \dots, n)$, where $g \in C^2([a_1, b_1] \times [a_2, b_2])$. Then we have $c_{ij}^0 = g(x_i, y_j) - c_{ij}^1 x_i - c_{ij}^2 y_j$, $c_{ij}^1 = \frac{\partial g}{\partial x}\Big|_{(x,y)=(x_i,y_j)}$, and $c_{ij}^2 = \frac{\partial g}{\partial y}\Big|_{(x,y)=(x_i,y_j)}$. For non-differential functions, numerical approximation mechanism allows us to determine the coefficients as follows:

$$\begin{aligned} z &= f(x, y) \\ &\approx f(x_i, y_j) + \tfrac{f(x_{i+1},y_j)-f(x_i,y_j)}{x_{i+1}-x_i}(x - x_i) + \tfrac{f(x_i,y_{j+1})-f(x_i,y_j)}{y_{j+1}-y_j}(y - y_j) \\ &= \tfrac{f(x_{i+1},y_j)-f(x_i,y_j)}{x_{i+1}-x_i}x + \tfrac{f(x_i,y_{j+1})-f(x_i,y_j)}{y_{j+1}-y_j}y \\ &\quad + f(x_i, y_j) - \tfrac{f(x_{i+1},y_j)-f(x_i,y_j)}{x_{i+1}-x_i}x_i - \tfrac{f(x_i,y_{j+1})-f(x_i,y_j)}{y_{j+1}-y_j}y_j. \end{aligned} \tag{2}$$

Therefore, the coefficients are got here:

$$c_{ij}^0 = f(x_i, y_j) - \frac{f(x_{i+1}, y_j) - f(x_i, y_j)}{x_{i+1} - x_i}x_i - \frac{f(x_i, y_{j+1}) - f(x_i, y_j)}{y_{j+1} - y_j}y_j,$$
$$c_{ij}^1 = \frac{f(x_{i+1}, y_j) - f(x_i, y_j)}{x_{i+1} - x_i},\ c_{ij}^2 = \frac{f(x_i, y_{j+1}) - f(x_i, y_j)}{y_{j+1} - y_j}.$$

3 Fuzzy Inference Modeling Method Based on T-S Fuzzy System

3.1 Input-Output Model of Time-Invariant System

In this section, we will discuss the problem of second-order time-invariant freedom movement system (input $u(t) = 0$) modeling. Let the universes of the input variables respectively be $Y = [a_1, b_1],\ \dot{Y} = [a_2, b_2]$ and the output variable $\ddot{Y} = [c, d]$, then $y(t) \in Y,\ \dot{y}(t) \in \dot{Y},\ \ddot{y}(t) \in \ddot{Y}$. Let $A = \{A_i(y)\}_{1\le i\le m},\ B = \{B_j(\dot{y})\}_{1\le j\le n}$ and $C = \{C_{ij}(\ddot{y})\}_{1\le i\le n, 1\le j\le m}$ be the fuzzy partitions of the universes Y, $\dot{Y}$ and $\ddot{Y}$ respectively. $y_i,\ \dot{y}_j$ and $\ddot{y}_{ij}$ with the conditions $a_1 = y_1 < y_2 < \cdots < y_m = b_1$ and $a_2 = \dot{y}_1 < \dot{y}_2 < \cdots < \dot{y}_n = b_2$ are respectively the peak points of triangular membership functions $A_i(y)$, $B_j(\dot{y})$ and $C_{ij}(\ddot{y})$. There is no demand for order to $\ddot{y}_{ij}$. For fuzzy inference rules

If $y(t)$ is A_i and $\dot{y}(t)$ is B_j, then $\ddot{y}(t)$ is C_{ij} $(i = 1, 2, \ldots m;\ j = 1, 2, \ldots n)$, we are ready to prove the following main results.

Theorem 1 *Model the second-order time-invariant freedom movement system based on T-S fuzzy system; the obtained input-output model can be expressed by the following nonlinear differential equation with variable coefficients:*

$$\begin{aligned} \ddot{y}(t) &= S(y(t), \dot{y}(t)) \\ &= a_1(y(t), \dot{y}(t))y(t) + a_2(y(t), \dot{y}(t))\dot{y}(t) \\ &\quad + a_3(y(t), \dot{y}(t))y^2(t) + a_4(y(t), \dot{y}(t))y(t)\dot{y}(t) + a_5(y(t), \dot{y}(t))\dot{y}^2(t) \\ &\quad + a_6(y(t), \dot{y}(t))y^2(t)\dot{y}(t) + a_7(y(t), \dot{y}(t))y(t)\dot{y}^2(t) + a_8(y(t), \dot{y}(t)). \end{aligned} \tag{3}$$

Proof Based on fuzzy logic interpolation mechanism, when $(y(t), \dot{y}(t)) \in [y_i, y_{i+1}] \times [\dot{y}_j, \dot{y}_{j+1}]$, the local equation on the piece (i, j) can be written as

$$\ddot{y}^{ij}(t) = A_i(y(t))B_j(\dot{y}(t))\ddot{y}_{ij} + A_{i+1}(y(t))B_j(\dot{y}(t))\ddot{y}_{i+1j} + A_i(y(t))B_{j+1}(\dot{y}(t))\ddot{y}_{ij+1} + A_{i+1}(y(t))B_{j+1}(\dot{y}(t))\ddot{y}_{i+1j+1}, \tag{4}$$

where the concrete forms of the inference antecedents are

$$A_i(y(t)) = \frac{y_{i+1} - y(t)}{y_{i+1} - y_i}, \quad A_{i+1}(y(t)) = \frac{y(t) - y_i}{y_{i+1} - y_i},$$
$$B_j(\dot{y}(t)) = \frac{\dot{y}_{j+1} - \dot{y}(t)}{\dot{y}_{j+1} - \dot{y}_j}, \quad B_{j+1}(\dot{y}(t)) = \frac{\dot{y}(t) - \dot{y}_j}{\dot{y}_{j+1} - \dot{y}_j}, \tag{5}$$

and those of the inference consequents are

$$\ddot{y}_{ij} = c^0_{ij} + c^1_{ij}y(t) + c^2_{ij}\dot{y}(t), \quad \ddot{y}_{i+1j} = c^0_{i+1,j} + c^1_{i+1,j}y(t) + c^2_{i+1,j}\dot{y}(t),$$
$$\ddot{y}_{i,j+1} = c^0_{i,j+1} + c^1_{i,j+1}y(t) + c^2_{i,j+1}\dot{y}(t), \quad \ddot{y}_{i+1,j+1} = c^0_{i+1,j+1} + c^1_{i+1,j+1}y(t) + c^2_{i+1,j+1}\dot{y}(t). \tag{6}$$

Substitute (5) and (6) into (4), after simplification, we obtain

$$\ddot{y}^{ij}(t) = a^{ij}_1 y(t) + a^{ij}_2 \dot{y}(t) + a^{ij}_3 y^2(t) + a^{ij}_4 y(t)\dot{y}(t) + a^{ij}_5 \dot{y}^2(t) + a^{ij}_6 y^2(t)\dot{y}(t) + a^{ij}_7 y(t)\dot{y}^2(t) + a^{ij}_8.$$

When $(y(t), \dot{y}(t)) \notin [y_i, y_{i+1}] \times [\dot{y}_j, \dot{y}_{j+1}]$,

$$a^{ij}_1 = a^{ij}_2 = a^{ij}_3 = a^{ij}_4 = a^{ij}_5 = a^{ij}_6 = a^{ij}_7 = a^{ij}_8 = 0.$$

When $(y(t), \dot{y}(t)) \in [y_i, y_{i+1}] \times [\dot{y}_j, \dot{y}_{j+1}]$, $h_1 = y_{i+1} - y_i$, $h_2 = \dot{y}_{j+1} - \dot{y}_j$,

$$\begin{aligned}
a^{ij}_1 &= (-\dot{y}_{j+1}c^0_{ij} + y_{i+1}\dot{y}_{j+1}c^1_{ij} + \dot{y}_{j+1}c^0_{i+1,j} - y_i\dot{y}_{j+1}c^1_{i+1,j} + \dot{y}_jc^0_{i,j+1} \\
&\quad - y_{i+1}\dot{y}_jc^1_{i,j+1} - \dot{y}_jc^0_{i+1,j+1} + y_i\dot{y}_jc^1_{i+1,j+1})/h_1h_2, \\
a^{ij}_2 &= (-y_{i+1}c^0_{ij} + y_{i+1}\dot{y}_{j+1}c^2_{ij} + y_ic^0_{i+1,j} - y_i\dot{y}_{j+1}c^2_{i+1,j} + y_{i+1}c^0_{i,j+1} \\
&\quad - y_{i+1}\dot{y}_jc^2_{i,j+1} - y_ic^0_{i+1,j+1} + y_i\dot{y}_jc^2_{i+1,j+1})/h_1h_2, \\
a^{ij}_3 &= (-\dot{y}_{j+1}c^1_{ij} + \dot{y}_{j+1}c^1_{i+1,j} + \dot{y}_jc^1_{i,j+1} - \dot{y}_jc^1_{i+1,j+1})/h_1h_2, \\
a^{ij}_4 &= (c^0_{ij} - y_{i+1}c^1_{ij} - \dot{y}_{j+1}c^2_{ij} - c^0_{i+1,j} + y_ic^1_{i+1,j} + \dot{y}_{j+1}c^2_{i+1,j} \\
&\quad - c^0_{i,j+1} + y_{i+1}c^1_{i,j+1} + \dot{y}_jc^2_{i,j+1} + c^0_{i+1,j+1} - y_ic^1_{i+1,j+1} - \dot{y}_jc^2_{i+1,j+1})/h_1h_2, \\
a^{ij}_5 &= (-y_{i+1}c^2_{ij} + y_ic^2_{i+1,j} + y_{i+1}c^2_{i,j+1} - y_ic^2_{i+1,j+1})/h_1h_2, \\
a^{ij}_6 &= (c^1_{ij} - c^1_{i+1,j} - c^1_{i,j+1} + c^1_{i+1,j+1})/h_1h_2, \\
a^{ij}_7 &= (c^2_{ij} - c^2_{i+1,j} - c^2_{i,j+1} + c^2_{i+1,j+1})/h_1h_2, \\
a^{ij}_8 &= (y_{i+1}\dot{y}_{j+1}c^0_{ij} - y_i\dot{y}_{j+1}c^0_{i+1,j} - y_{i+1}\dot{y}_jc^0_{i,j+1} + y_i\dot{y}_jc^0_{i+1,j+1})/h_1h_2.
\end{aligned}$$

By summing the coefficients of the local equations, we have

$$a_1(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_1^{ij},\quad a_2(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_2^{ij},\quad a_3(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_3^{ij},$$
$$a_4(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_4^{ij},\quad a_5(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_5^{ij},\quad a_6(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_6^{ij},$$
$$a_7(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_7^{ij},\quad a_8(y(t),\dot{y}(t))=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_8^{ij}.$$

Then we derive the overall nonlinear differential equation when $(y(t),\dot{y}(t))\in Y\times\dot{Y}$:

$$\begin{aligned}
\ddot{y}(t)&=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\ddot{y}^{ij}(t)\\
&=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}[a_1^{ij}y(t)+a_2^{ij}\dot{y}(t)+a_3^{ij}y^2(t)+a_4^{ij}y(t)\dot{y}(t)\\
&\qquad+a_5^{ij}\dot{y}^2(t)+a_6^{ij}y^2(t)\dot{y}(t)+a_7^{ij}y(t)\dot{y}^2(t)+a_8^{ij}]\\
&=\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_1^{ij}\right)y(t)+\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_2^{ij}\right)\dot{y}(t)+\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_3^{ij}\right)y^2(t)+\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_4^{ij}\right)y(t)\dot{y}(t)\\
&\quad+\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_5^{ij}\right)\dot{y}^2(t)+\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_6^{ij}\right)y^2(t)\dot{y}(t)+\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_7^{ij}\right)y(t)\dot{y}^2(t)+\left(\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}a_8^{ij}\right)\\
&=a_1(y(t),\dot{y}(t))y(t)+a_2(y(t),\dot{y}(t))\dot{y}(t)+a_3(y(t),\dot{y}(t))y^2(t)+a_4(y(t),\dot{y}(t))y(t)\dot{y}(t)\\
&\quad+a_5(y(t),\dot{y}(t))\dot{y}^2(t)+a_6(y(t),\dot{y}(t))y^2(t)\dot{y}(t)+a_7(y(t),\dot{y}(t))y(t)\dot{y}^2(t)+a_8(y(t),\dot{y}(t)).
\end{aligned}$$

Example 1 The Van der Pol equation $\ddot{y}(t)-\mu(1-y^2(t))\dot{y}(t)+y(t)=0$ with the initial condition $(y(0),\dot{y}(0))=(2,0)$ is simulated, where $\mu=1$. Let time $T=20$. The universes of Y and $\dot{Y}$ are both divided into seven rules. The simulation curves and the original curves of $y(t)$ and $\dot{y}(t)$ are respectively shown in Figs. 1 and 2, where the dotted lines show the simulation curves and the solid lines represent the original curves. More details about the simulation steps can be found in reference [2]. As can be seen from Figs. 1 and 2, the simulation curves almost coincide with the original curves, which means that the model established by fuzzy inference modeling method based on T-S fuzzy system has good approximation accuracy.

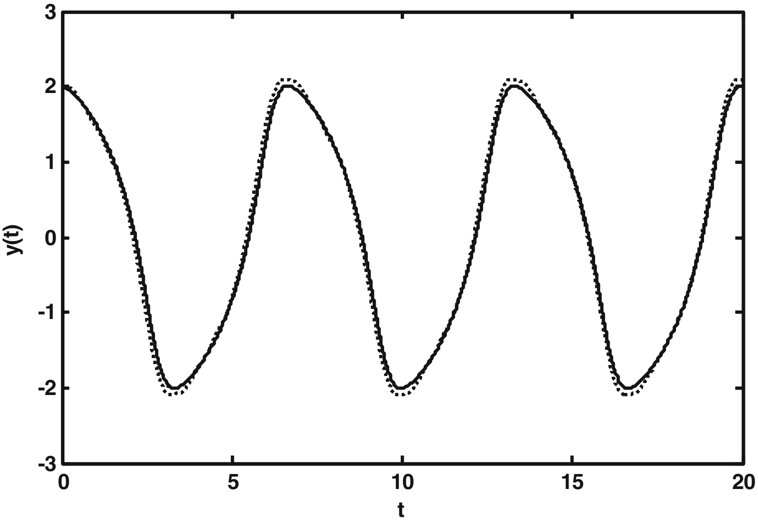

Fig. 1 Simulation curve of $y(t)$

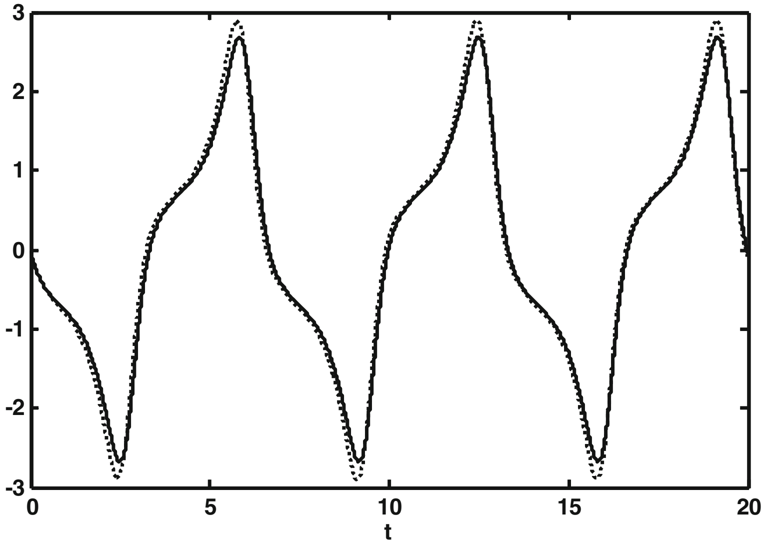

Fig. 2 Simulation curve of $\dot{y}(t)$

3.2 State-Space Models of Time-Invariant Systems

This section takes second-order time-invariant freedom movement system as an example to analyze the state-space model in detail. The universes of $x_1(t), x_2(t)$, $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are respectively denoted by $X_1 = [a_1, b_1],\ X_2 = [a_2, b_2],\ X_1' = [c_1, d_1]$ and $X_2' = [c_2, d_2]$, where the fuzzy partitions of these universes are $A = \{A_i\}_{(1\le i\le m)},\ B = \{B_j\}_{(1\le j\le n)},\ C = \{C_{ij}\}_{(1\le i\le m, 1\le j\le n)}$ and $D = \{D_{ij}\}_{(1\le i\le m, 1\le j\le n)}$ respectively. A_i, B_j, C_{ij} and D_{ij} are taken as triangular membership functions, and their peak points are respectively $x_i^{(1)},\ x_j^{(2)},\ \dot{x}_{ij}^{(1)},\ \dot{x}_{ij}^{(2)}$ with the conditions $a_1 = x_1^{(1)} < x_2^{(1)} < \cdots < x_m^{(1)} = b_1$ and $a_2 = x_1^{(2)} < x_2^{(2)} < \cdots < x_m^{(2)} = b_2$. There is no demand for order to $\dot{x}_{ij}^{(1)}$ and $\dot{x}_{ij}^{(2)}$. Then we can get the following fuzzy inference rules:

If $x_1(t)$ is $A_i,\ x_2(t)$ is B_j, then

$$\dot{x}_1(t) \text{ is } C_{ij} \text{ and } \dot{x}_2(t) \text{ is } D_{ij}.\ (i = 1, 2, \ldots m;\ j = 1, 2, \ldots n). \tag{7}$$

The fuzzy rule base (7) can be represented as a piecewise interpolation function of two variables based on T-S fuzzy system and fuzzy logic interpolation mechanism as follows:

$$(\dot{x}_1(t), \dot{x}_2(t)) = (S_1(x_1(t), x_2(t)), S_2(x_1(t), x_2(t))), \tag{8}$$

where

$$\dot{x}_1(t) = S_1(x_1(t), x_2(t)) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_i(x_1(t))B_j(x_2(t))\dot{x}_{ij}^{(1)},$$

and

$$\dot{x}_2(t) = S_2(x_1(t), x_2(t)) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_i(x_1(t))B_j(x_2(t))\dot{x}_{ij}^{(2)}.$$

Thus the fuzzy rule base can be expressed as nonlinear differential equations with variable coefficients, and then we get the following theorem.

Theorem 2 *For second-order time-invariant system modeling, based on T-S fuzzy system, if $A_i(x_1(t))$ and $B_j(x_2(t))$ are both triangular membership functions, the obtained state-space model can be constructed as the following nonlinear differential equations with variable coefficients:*

$$\begin{aligned}\dot{x}_1(t) &= S_1(x_1(t), x_2(t)) \\ &= a_{11}(x_1(t), x_2(t))x_1(t) + a_{12}(x_1(t), x_2(t))x_2(t) \\ &\quad + a_{13}(x_1(t), x_2(t))x_1^2(t) + a_{14}(x_1(t), x_2(t))x_1(t)x_2(t) \\ &\quad + a_{15}(x_1(t), x_2(t))x_2^2(t) + a_{16}(x_1(t), x_2(t))x_1^2(t)x_2(t) \\ &\quad + a_{17}(x_1(t), x_2(t))x_1(t)x_2^2(t) + a_{18}(x_1(t), x_2(t)),\end{aligned} \tag{9}$$

$$
\begin{aligned}
\dot{x}_2(t) &= S_2(x_1(t), x_2(t)) \\
&= a_{21}(x_1(t), x_2(t))x_1(t) + a_{22}(x_1(t), x_2(t))x_2(t) \\
&\quad + a_{23}(x_1(t), x_2(t))x_1^2(t) + a_{24}(x_1(t), x_2(t))x_1(t)x_2(t) \\
&\quad + a_{25}(x_1(t), x_2(t))x_2^2(t) + a_{26}(x_1(t), x_2(t))x_1^2(t)x_2(t) \\
&\quad + a_{27}(x_1(t), x_2(t))x_1(t)x_2^2(t) + a_{28}(x_1(t), x_2(t)).
\end{aligned} \tag{10}
$$

For $l = 1, 2$, we have

$$
a_{l1}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l1}^{ij}, \quad a_{l2}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l2}^{ij}, \quad a_{l3}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l3}^{ij},
$$
$$
a_{l4}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l4}^{ij}, \quad a_{l5}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l5}^{ij}, \quad a_{l6}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l6}^{ij},
$$
$$
a_{l7}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l7}^{ij}, \quad a_{l8}(x_1(t), x_2(t)) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} a_{l8}^{ij}.
$$

The local coefficients on the piece (i, j) are defined as:

When $(x_1(t), x_2(t)) \notin [x_i^{(1)}, x_{i+1}^{(1)}] \times [x_j^{(2)}, x_{j+1}^{(2)}]$,

$$
a_{l1}^{ij} = a_{l2}^{ij} = a_{l3}^{ij} = a_{l4}^{ij} = a_{l5}^{ij} = a_{l6}^{ij} = a_{l7}^{ij} = a_{l8}^{ij} = 0,
$$

and when $(x_1(t), x_2(t)) \in [x_i^{(1)}, x_{i+1}^{(1)}] \times [x_j^{(2)}, x_{j+1}^{(2)}]$, $h_1 = x_{i+1}^{(1)} - x_i^{(1)}$, $h_2 = x_{j+1}^{(2)} - x_j^{(2)}$,

$$
\begin{aligned}
a_{l1}^{ij} &= (-x_{j+1}^{(2)}c_{ij}^{0(l)} + x_{i+1}^{(1)}x_{j+1}^{(2)}c_{ij}^{1(l)} + x_{j+1}^{(2)}c_{i+1,j}^{0(l)} - x_i^{(1)}x_{j+1}^{(2)}c_{i+1,j}^{1(l)} \\
&\quad + x_j^{(2)}c_{i,j+1}^{0(l)} - x_{i+1}^{(1)}x_j^{(2)}c_{i,j+1}^{1(l)} - x_j^{(2)}c_{i+1,j+1}^{0(l)} + x_i^{(1)}x_j^{(2)}c_{i+1,j+1}^{1(l)})/h_1h_2, \\
a_{l2}^{ij} &= (-x_{i+1}^{(1)}c_{ij}^{0(l)} + x_{i+1}^{(1)}x_{j+1}^{(2)}c_{ij}^{2(l)} + x_i^{(1)}c_{i+1,j}^{0(l)} - x_i^{(1)}x_{j+1}^{(2)}c_{i+1,j}^{2(l)} \\
&\quad + x_{i+1}^{(1)}c_{i,j+1}^{0(l)} - x_{i+1}^{(1)}x_j^{(2)}c_{i,j+1}^{2(l)} - x_i^{(1)}c_{i+1,j+1}^{0(l)} + x_i^{(1)}x_j^{(2)}c_{i+1,j+1}^{2(l)})/h_1h_2, \\
a_{l3}^{ij} &= (-x_{j+1}^{(2)}c_{ij}^{1(l)} + x_{j+1}^{(2)}c_{i+1,j}^{1(l)} + x_j^{(2)}c_{i,j+1}^{1(l)} - x_j^{(2)}c_{i+1,j+1}^{1(l)})/h_1h_2, \\
a_{l4}^{ij} &= (c_{ij}^{0(l)} - x_{i+1}^{(1)}c_{ij}^{1(l)} - x_{j+1}^{(2)}c_{ij}^{2(l)} - c_{i+1,j}^{0(l)} + x_i^{(1)}c_{i+1,j}^{1(l)} + x_{j+1}^{(2)}c_{i+1,j}^{2(l)} \\
&\quad - c_{i,j+1}^{0(l)} + x_{i+1}^{(1)}c_{i,j+1}^{1(l)} + x_j^{(2)}c_{i,j+1}^{2(l)} + c_{i+1,j+1}^{0(l)} - x_i^{(1)}c_{i+1,j+1}^{1(l)} - x_j^{(2)}c_{i+1,j+1}^{2(l)})/h_1h_2, \\
a_{l5}^{ij} &= (-x_{i+1}^{(1)}c_{ij}^{2(l)} + x_i^{(1)}c_{i+1,j}^{2(l)} + x_{i+1}^{(1)}c_{i,j+1}^{2(l)} - x_i^{(1)}c_{i+1,j+1}^{2(l)})/h_1h_2, \\
a_{l6}^{ij} &= (c_{ij}^{1(l)} - c_{i+1,j}^{1(l)} - c_{i,j+1}^{1(l)} + c_{i+1,j+1}^{1(l)})/h_1h_2, \\
a_{l7}^{ij} &= (c_{ij}^{2(l)} - c_{i+1,j}^{2(l)} - c_{i,j+1}^{2(l)} + c_{i+1,j+1}^{2(l)})/h_1h_2, \\
a_{l8}^{ij} &= (x_{i+1}^{(1)}x_{j+1}^{(2)}c_{ij}^{0(l)} - x_i^{(1)}x_{j+1}^{(2)}c_{i+1,j}^{0(l)} - x_{i+1}^{(1)}x_j^{(2)}c_{i,j+1}^{0(l)} + x_i^{(1)}x_j^{(2)}c_{i+1,j+1}^{0(l)})/h_1h_2.
\end{aligned}
$$

The proof of this theorem can be completed by the method analogous to that of Theorem 1. According to this theorem, it follows that we can piecewisely solve the solutions of (9) and (10).

Example 2 In Example 1, let $x_1(t) = y(t),\ x_2(t) = \dot{y}(t)$, then the simulation curves and the original curves of $x_1(t)$ and of $x_2(t)$ are shown in Figs. 1 and 2, respectively.

4 Comparison of Different Fuzzy Inference Modeling Methods

In this section, a simulation experiment under condition of time-invariant will be provided to compare approximation performance of the models using different fuzzy modeling methods. They are fuzzy inference modeling method based on T-S fuzzy system, fuzzy inference modeling methods mentioned in references [11, 12], and fuzzy inference modeling method based on fuzzy transformation [13]. By comparing the maximum approximation errors of the input-output models and state-space models, it turns out that the proposed method is effective and superior in approximating equations.

In order to evaluate approximation capability of the proposed method, different fuzzy inference modeling methods are used to simulate the Var der Pol equation. The set of the initial values and the numbers of rules are the same as those of Example 1. We make careful comparison and overall summary of approximate effect of these models. For simplicity, the following notations are needed:

Model I represents the model using fuzzy inference modeling method based on Mamdani fuzzy system [11].

Model II represents the model using fuzzy inference modeling method based on fuzzy transformation [13].

Model III represents the model using fuzzy inference modeling method based on T-S fuzzy system.

The simulation curves of the three models of the Var der Pol equation are similar to Figs. 1 and 2. In order to directly distinguish which model has more powerful ability to approach the Var der Pol equation, the maximum approximation errors of the input-output models and the state-space models are given in the below two tables.

As can be seen in Tables 1 and 2, the max errors of the model using fuzzy inference modeling method based on T-S fuzzy system is smaller than those of the model using

Table 1 Max errors of different input-output models

Max error	Model		
	Model I	Model II	Model III
The max errors of $y(t)$	0.2650	0.5202	0.2471
The max errors of $\dot{y}(t)$	0.5745	0.7834	0.4318

Table 2 Max errors of different state-space models

Max error	Model		
	Model I	Model II	Model III
The max errors of $x_1(t)$	0.2650	0.4581	0.2471
The max errors of $x_2(t)$	0.5745	0.6642	0.4318

fuzzy inference modeling method based on Mamdani fuzzy system and the model using fuzzy inference modeling method based on fuzzy transformation. The approximation effect is better than the other two methods, so fuzzy inference modeling method based on T-S fuzzy system has the superiority to some extent.

5 Conclusion

This paper presents a fuzzy inference modeling method based on T-S fuzzy system. We model the time-varying and time-invariant second-order freedom movement systems and simulate the Var der Pol equation by the obtained differential equation model. Moreover, we compare the simulation results of this model with another two models using the fuzzy inference modeling method based on Mamdani fuzzy system and the fuzzy inference modeling method based on fuzzy transformation. It is shown that the proposed method can better approximate the original equation.

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References

1. Wang L.X.: A Course in Fuzzy Systems and Control. Tsinghua University Press, Beijing (2003)
2. Yuan, X.H., Liu, Z.L., Lee, E.S.: Center-of-gravity fuzzy systems based on normal fuzzy implications. Comput. Math. Appl. **61**(9), 2879–2898 (2011)
3. Li, Y.M., Shi, Z.K., Li, Z.H.: Approximation theory of fuzzy systems based upon genuine many-valued implications-SISO cases. Fuzzy Sets Syst. **130**(2), 147–157 (2002)
4. Zeng, X.J., Madan, G.S.: Approximation theory of fuzzy systems-SISO case. IEEE Trans. Fuzzy Syst. **2**(2), 162–176 (1994)
5. Wang, J.Y., Liu, M., Li, H.X.: Analysis of difference between control function and interpolation expression of SISO fuzzy controller. Acta Electronica Sin. **37**(2), 424–428 (2009)
6. Shan, W.W., Jin, D.M., Liang, Y.: Variable universe adaptive fuzzy logic controller CMOS analog chip implementation. Acta Electronica Sin. **37**(5), 913–917 (2009)
7. Yuan, X.H., Li, H.X., Sun, K.B.: Fuzzy Systems and their approximation capability based on parameter singleton fuzzifier methods. Acta Electronica Sin. **39**(10), 2372–2377 (2011)
8. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning I. Inform. Sci. **8**(3), 199–249 (1975)
9. Wang, G.J.: Non-Classical Logic and Approximate Reasoning. Science Press, Beijing (2000)

10. Li, H.X.: Interpolation mechanism of fuzzy control. Sci. China (Series E) **41**(3), 312–320 (1998)
11. Li, H.X., Wang, J.Y., Miao, Z.H.: Modeling on fuzzy control systems. Sci. China (Series A) **12**(45), 1506–1517 (2002)
12. Li, H.X., Song, W.Y., Yuan, X.H., Li, Y.C.: Time-varying system modeling method based on fuzzy inference. J. Syst. Sci. Math. Sci. **29**(8), 1109–1128 (2009)
13. Yuan, X.H., Li, H.X., Yang, X.: Fuzzy system and fuzzy inference modeling method based on fuzzy transformation. Acta Electronica Sin. **41**(4), 674–680 (2013)
14. Takagi, T., Sugeno, M.: Fuzzy identification of systems and its applications to modeling and control. IEEE Trans. Syst. Man Cybern. **15**(1), 116–132 (1985)
15. Ban, X.J., Gao, X.Z., Huang, X.L., Vasilakos, A.V.: Stability analysis of the simplest Takagi-sugeno fuzzy control system using circle criterion. Inform. Sci. **177**(20), 4387–4409 (2007)
16. Liu, P.Y., Li, H.X.: Hierarchical T-S fuzzy system and its universal approximation. Inform. Sci. **169**(3), 279–303 (2005)
17. Chen, X.J.: Research on TS fuzzy Model Based Simulation of Double Inverted Pendulum System. Hefei University of Technology, Hefei (2005)

Extended General Variational Inequalities and General Wiener–Hopf Equations

Xiao-Min Wang, Yan-Yan Zhang, Na Li and Xiu-Yan Fan

Abstract In this paper, we show the extended general variational inequality problems are equivalent to solving the general Wiener–Hopf equations. By using the equivalence, we establish a general iterative algorithm for finding the solution of extended general variational inequalities. We also discuss the convergence criteria for the algorithm. Our results extend and improve the corresponding results announced by many others.

Keywords Variational inequalities · Wiener–Hopf equations · Iterative algorithm

1 Introduction

Variational inequality theory describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics and engineering sciences [1–11]. Projection methods and their variant forms including the Wiener–Hopf equations are being used to develop various numerical methods for solving variational inequalities. It has been shown that the Wiener–Hopf equations are more flexible and general than the projection methods. Noor [1–7] and Qin [10] have used the Wiener–Hopf equations technique to study the sensitivity analysis, dynamical systems as well as to suggest and analyze several iterative methods for solving variational inequalities. A new class of variational inequalities involving three nonlinear operators, which is called the extended general variational inequalities, is introduced and studied by Noor [9].

X.-M. Wang (✉) · Y.-Y. Zhang
School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao 066004, China
e-mail: xmwang0823@163.com

N. Li · X.-Y. Fan
School of Science, Northeastern University, Shenyang 110004, China
e-mail: 576927605@qq.com

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Motivated and inspired by the above research, we establish the equivalence between extended general variational inequalities and general Wiener–Hopf equations in this paper. This alternative formulation is used to propose and analyze a new iterative algorithm for computing approximate solutions of extended general variational inequalities. We also study the conditions under which the approximate solution obtained from the iterative algorithms converges to the exact solution of the general variational inequalities. Results proved in this paper may be viewed as significant and improvement of previously known results.

2 Problem Statement and Preliminaries

Let H be a real Hilbert space whose inner product norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H. For given nonlinear operators $T, g, h : H \to H$, we consider the problem of finding $u \in H : h(u) \in K$ such that

$$\langle Tu, g(v) - h(u)\rangle \geq 0, \ \forall v \in H : g(v) \in K. \tag{1}$$

The inequality of the type (1) is called the extended general variational inequality, which was introduced by Noor in [9]. We would like to emphasize that problem (1) is equivalent to that of finding $u \in H : h(u) \in K$ such that

$$\langle Tu + h(u) - g(u), g(v) - h(u)\rangle \geq 0, \ \forall v \in H : g(v) \in K. \tag{2}$$

This equivalent formulation is also useful from the applications point of view.

We now list some special cases of the extended general variational inequalities.

(I) If $g = h$, then problem (1) is equivalent to that of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \ \forall v \in H : g(v) \in K,$$

which is known as general variational inequality, introduced and studied by Noor [3].

(II) For $g = I$, the identity operator, the extended general variational inequality (1) collapses to: Find $u \in H : h(u) \in K$ such that

$$\langle Tu, v - h(u)\rangle \geq 0, \ \forall v \in K,$$

which is also called the general variational inequality; see Noor [6].

(III) For $h = I$, the identity operator, then problem (1) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \ \forall v \in H : g(v) \in K,$$

which is also called the general variational inequality, see Noor [8].

(IV) For $g = h = I$, the identity operator, the extended general variational inequality (1) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \ \ \forall v \in K,$$

which is known as the classical variational inequality.

(V) If $K^* = \{u \in H; \langle u, v\rangle \geq 0, \forall v \in K\}$ is a polar (dual) convex cone of a closed convex cone K in H, then problem (1) is equivalent to that of finding $u \in K$ such that

$$g(u) \in K, \ \ Tu \in K^*, \ \ \langle g(u), Tu\rangle = 0$$

which is known as the general complementarity problem, which includes many previously known complementarity problems as special cases; see [2, 3, 6].

From the above discussion, it is clear that the extended general variational inequality (1) is most general and includes several known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences.

Related to the variational inequalities, we have the problems of solving the Wiener–Hopf equations. Now let

$$Q_K = I - gh^{-1}P_K,$$

where P_K is the projection of H onto K, I, is the identity operator. If g^{-1}, h^{-1} exists, then we consider the problem of finding $z \in H$ such that

$$\rho^{-1}Q_K z + Th^{-1}P_K z = 0, \tag{3}$$

where $\rho > 0$ is a constant. Equations of the type (3) are called general Wiener–Hopf equations. Note that, for $g = h$, we obtain the original Wiener–Hopf equation, introduced by Shi [11]. It is well known that the variational inequalities and Wiener–Hopf equations are equivalent. This equivalent has played a fundamental and basic role in developing some efficient and robust methods for solving variational inequalities and related optimization problems.

Recall the following definitions:

Definition 2.1 An operator $T : H \to H$ is said to be:

(I) Strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u - v\|^2, \forall u, v \in H.$$

(II) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u - v\|, \forall u, v \in H.$$

(III) μ-coercive if there exists a constant $\mu > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \mu\|Tu - Tv\|^2, \forall u, v \in H.$$

Clearly, every μ-coercive operator is $1/\mu$-Lipschitz continuous.

(IV) Relaxed η-coercive if there exists a constant $\eta > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq (-\eta)\|Tu - Tv\|^2, \quad \forall u, v \in H$$

(V) Relaxed (ω, t)-coercive if there exist two constants $\omega, t > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq (-\omega)\|Tu - Tv\|^2 + t\|u - v\|^2, \quad \forall u, v \in H.$$

For $\omega = 0$, T is strongly monotone. This class of mappings is more general that the class of strongly monotone mappings.

We also need the following well-known result.

Lemma 2.1 *Let K be a closed convex subset of H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

if and only if $u = P_K z$, where P_K is the projection of H onto K.

It is well known that the projection operator P_K is a nonexpansive operator.

3 Main Results

First of all, using the technique of Noor [2], we prove the following result.

Theorem 3.1 *The extended general variational inequality (1) has a solution $u \in H : h(u) \in K$ if and only if $z \in H$ satisfies the general Wiener–Hopf equation (3), where*

$$z = g(u) - \rho Tu, h(u) = P_K z,$$

where P_K is the projection of H onto K and $\rho > 0$ is a constant.

Proof Let $u \in H : h(u) \in K$ be a solution of the extended general variational inequality (1). Then, from (2), we have

$$\langle h(u) - (g(u) - \rho Tu), g(v) - h(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K$$

which implies, using Lemma 2.1, that

$$h(u) = P_K(g(u) - \rho Tu).$$

Using $Q_K = I - gh^{-1}P_K$, we have

$$\begin{aligned}(I - gh^{-1}P_K)(g(u) - \rho Tu) &= g(u) - \rho Tu - gh^{-1}P_K(g(u) - \rho Tu)\\ &= g(u) - \rho Tu - gh^{-1}h(u) = -\rho Tu\\ &= \rho Th^{-1}P_K(g(u) - \rho Tu).\end{aligned}$$

It follows that

$$\rho^{-1}Q_K z + Th^{-1}P_K z = 0,$$

where $z = g(u) - \rho Tu$.

Conversely, let $z \in H$ be a solution of the general Wiener–Hopf equation (3). Then, we have

$$\rho Th^{-1}P_K z = -Q_K z = (gh^{-1}P_K - I)z = gh^{-1}P_K z - z. \tag{4}$$

It follows from (4) and Lemma 2.1 that

$$0 \le \langle gh^{-1}P_K z - z, g(v) - gh^{-1}P_K z\rangle = \langle \rho Th^{-1}P_K z, g(v) - gh^{-1}P_K z\rangle$$

for all $v \in H : g(v) \in K$. It follows that $u = h^{-1}P_K z$, that is, $h(u) = P_K z$ is a solution of (1) and $g(u) = gh^{-1}P_K Z$. Using (4), we have

$$z = g(u) - \rho Tu.$$

This completes the proof.

From the above Theorem 3.1, one can easily see that extended general variational inequalities and general Wiener–Hopf equations are equivalent. This equivalent is very useful from the numerical point of view. Using this equivalence and by an appropriate rearrangement, we suggest and analyze the following iterative algorithms for solving the extended general variational inequalities (1).

The general Wiener–Hopf equation (3) can be rewritten as

$$Q_K z = -\rho Th^{-1}P_K z,$$

which implies that

$$z - gh^{-1}P_K z = -\rho Th^{-1}P_K z,$$

Thus

$$z = gh^{-1}P_K z - \rho Th^{-1}P_K z = g(u) - \rho Tu.$$

Using the equality $z = (1-\alpha_n)z + \alpha_n z$, we obtain

$$z = (1-\alpha_n)z + \alpha_n(g(u) - \rho Tu).$$

This formulation enables us to suggest the following iterative algorithm for solving the extended general variational inequalities (1).

Algorithm 3.1 For any $z_0 \in H$, compute the sequence $\{z_n\}$ by the iterative processes

$$h(u_n) = P_K z_n, z_n = g(u_n) - \rho Tu_n, z_{n+1} = (1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho Tu_n). \quad (5)$$

In order to prove our next main result, we need the following lemma.

Lemma 3.1 ([10]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\lambda_n)a_n + b_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in [0,1] with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, then

$$\lim_{n\to\infty} a_n = 0.$$

Theorem 3.2 *Let K be a closed convex subset of a real Hilbert space H. Let $g : H \to H$ be a relaxed (ω_1, t_1)-coercive and μ_1-Lipschitz continuous mapping, $h : H \to H$ be a μ_1-Lipschitz continuous mapping and let $T : H \to H$ be a relaxed (ω_2, t_2)-coercive and μ_2-Lipschitz continuous mapping. Let $\{z_n\}$, $\{u_n\}$ and $\{h(u_n)\}$ be sequences generated by Algorithm 3.1, $\{\alpha_n\}$ is a sequence in [0, 1]. Assume that the following conditions are satisfied:*

$$2\theta_1 + \theta_2 < 1, \quad \text{(C1)}$$

where $\theta_1 = \sqrt{1 + \mu_1^2 - 2t_1 + 2\omega_1\mu_1^2}$, $\theta_2 = \sqrt{1 + \rho^2\mu_2^2 - 2\rho t_2 + 2\rho\omega_2\mu_2^2}$.

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \quad \text{(C2)}$$

Then the sequence $\{z_n\}$, $\{u_n\}$ and $\{h(u_n)\}$ converge strongly to z^*, u^* and $h(u^*)$, respectively, where $z^* \in H$ is a solution of the general Wiener–Hopf equation (3), $u^* \in H : h(u^*) \in K$ is a solution of the extended general variational inequality (1).

Proof Letting $z^* \in H$ be a solution of the general Wiener–Hopf equation (3), we have

$$h(u^*) = P_K z^*, \quad z^* = g(u^*) - \rho Tu^*, \quad z^* = (1-\alpha_n)z^* + \alpha_n(g(u^*) - \rho Tu^*),$$

where $u^* \in H : h(u^*) \in K$ is a solution of the extended general variational inequality (1). Observing (5), we obtain

$$\begin{aligned} \|z_{n+1} - z^*\| &= \|(1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho T u_n) - z^*\| \\ &= \|(1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho T u_n) - (1-\alpha_n)z^* + \alpha_n(g(u^*) - \rho T u^*)\| \\ &\le (1-\alpha_n)\|z_n - z^*\| + \alpha_n\|g(u_n) - g(u^*) - \rho(T u_n - T u^*)\|. \end{aligned} \tag{6}$$

On the other hand, we have

$$\begin{aligned} &\|g(u_n) - g(u^*) - \rho(T u_n - T u^*)\| \\ &\quad = \|u_n - u^* - (u_n - u^*) + g(u_n) - g(u^*) - \rho(T u_n - T u^*)\| \\ &\quad \le \|u_n - u^* - (g(u_n) - g(u^*))\| + \|u_n - u^* - \rho(T u_n - T u^*)\|. \end{aligned} \tag{7}$$

Now, we shall estimate the first term of right side of (7)

$$\begin{aligned} &\|u_n - u^* - g(u_n) - g(u^*)\| \\ &\quad = \|u_n - u^*\|^2 - 2\langle g(u_n) - g(u^*), u_n - u^*\rangle + \|(g(u_n) - g(u^*))\|^2 \\ &\quad \le \|u_n - u^*\|^2 + 2\omega_1\|g(u)_n - g(u)^*\|^2 - 2t_1\|u_n - u^*\|^2 + \|g(u_n) - g(u^*)\|^2 \\ &\quad \le \|u_n - u^*\|^2 + 2\mu_1^2\omega_1\|u_n - u^*\|^2 - 2t_1\|u_n - u^*\|^2 + \mu_1^2\|u_n - u^*\|^2 \\ &\quad = (1 + 2\mu_1^2\omega_1 - 2t_1 + \mu_1^2)\|u_n - u^*\|^2 = \theta_1^2\|u_n - u^*\|^2, \end{aligned} \tag{8}$$

where $\theta_1 = \sqrt{1 + 2\mu_1^2\omega_1 - 2t_1 + \mu_1^2}$.

Next, we shall estimate the second term of right side of (7)

$$\begin{aligned} &\|u_n - u^* - \rho(T u_n - T u^*)\| \\ &\le \|u_n - u^*\|^2 - 2\rho\langle T u_n - T u^*, u_n - u^*\rangle + \rho^2\|T u_n - T u^*\|^2 \\ &\le \|u_n - u^*\|^2 + 2\rho\omega_2\|T u_n - T u^*\|^2 - 2\rho t_2\|u_n - u^*\|^2 + \rho^2\|T u_n - T u^*\|^2 \\ &\le \|u_n - u^*\|^2 + 2\rho\omega_2\|T u_n - T u^*\|^2 - 2\rho t_2\|u_n - u^*\|^2 + \rho^2\|T u_n - T u^*\|^2 \\ &= (1 + 2\rho\mu_2^2\omega_2 - 2\rho t_2 + \rho^2\mu_2^2)\|u_n - u^*\|^2 = \theta_2^2\|u_n - u^*\|^2, \end{aligned} \tag{9}$$

where $\theta_2 = \sqrt{1 + 2\rho\mu_2^2\omega_2 - 2\rho t_2 + \rho^2\mu_2^2}$.
Substitute (8) and (9) into (7) yields that

$$\|g(u_n) - g(u^*) - \rho(T u_n - T u^*)\| \le (\theta_1 + \theta_2)\|u_n - u^*\|. \tag{10}$$

Substituting (10) into (6), we arrive at

$$\|z_{n+1} - z^*\| \le (1-\alpha_n)\|z_{n+1} - z^*\| + \alpha_n(\theta_1 + \theta_2)\|u_n - u^*\|. \tag{11}$$

Observe that

$$\|u_n - u^*\| = \|u_n - u^* - (h(u_n) - h(u^*)) + (P_K z_n - P_K z^*)\| \le \theta_1\|u_n - u^*\| + \|z_{n+1} - z^*,$$

which implies that

$$\|u_n - u^*\| \leq \frac{1}{1-\theta_1}\|z_{n+1} - z^*\|. \tag{12}$$

Now, substituting (12) into (11), we have that

$$\|z_{n+1} - z^*\| \leq (1 - \alpha_n(1 - \frac{\theta_1 + \theta_2}{1-\theta_1}))\|z_{n+1} - z^*\|,$$

From condition (C1), (C2) and Lemma 3.1, we have

$$\lim_{n\to\infty} \|z_{n+1} - z^*\| = 0.$$

From (12), we have

$$\lim_{n\to\infty} \|u_n - u^*\| = 0.$$

On the other hand, we have

$$\|h(u_n) - h(u^*)\| \leq \mu_1\|u_n - u^*\|.$$

It follows that

$$\lim_{n\to\infty} \|h(u_n) - h(u^*)\| = 0.$$

This completes the proof.

4 Conclusion

In this paper, we show that the extended general variational inequalities are equivalent to the general Wiener–Hopf equations. A general iterative algorithm for finding the solution of extended general variational inequalities is established by the equivalence. We also discuss the convergence criteria for the algorithm.

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References

1. Noor, M.A., Wang, Y.J., Xiu, N.: Some new projection methods for variational inequalities. Appl. Math. Comput. **137**, 423–435 (2003)
2. Noor, M.A.: Wiener–hopf equations and variational inequalities, J. Optim.Theory Appl. **79**, 197–206 (1993)

3. Noor, M.A.: General variational inequalities. Appl. Math. Lett. **1**, 119–122 (1988)
4. Noor, M.A.: Sensitivity analysis for quasi variational inequalities. J. Optim. Theory Appl. **95**, 399–407 (1997)
5. Noor, M.A.: Wiener–hopf equations technique for quasimonotone variational inequalities. J. Optim. Theory Appl. **103**, 705–714 (1999)
6. Noor, M.A.: Some developments in general variational inequalities. Appl. Math. Comput. **152**, 199–277 (2004)
7. Noor, M.A.: General variational inequalities and nonexpansive mappings. J. Math. Anal. Appl. **331**, 810–822 (2007)
8. Noor, M.A.: Differentiable nonconvex functions and general variational inequalities. Appl. Math. Comput. **199**, 623–630 (2008)
9. Noor, M.A.: Extended general variational inequalities. Appl. Math. Lett. **22**, 182–186 (2009)
10. Qin, X., Noor M.A.: General Wiener-Hopf equation technique for nonexpansive mappings and general variational inequalities in Hilbert spaces. Appl. Math. Comput. **201**, 716–722 (2008)
11. Shi, P.: Equivalence of variational inequalities with Wiener–Hopf equations. Proc. Am. Math. Soc. **111**, 339–346 (1991)

Econometric Study on the Influencing Factors of China Life Insurance Product Demand Based on Fuzzy Variable Time Series

Xiao-yue Zhou and Kai-qi Zou

Abstract In this paper a new kind of time series named fuzzy variable time series is proposed and the econometric models of the influencing factors of China life insurance product demand based on it are constructed. This paper focuses on the importance of the variable with fuzziness and defines the discrete and continuous fuzzy variable time series. The multiple regression model and long-term equilibrium regression model of influencing factors of China life insurance product demand based on fuzzy variable time series are gotten. The gap between the econometrics and fuzzy mathematics is bridged by the concept of fuzzy variable which will lead a new development pattern for quantized model of econometrics.

Keywords Fuzzy variable · Fuzzy variable time series · Influencing factor

1 Introduction

The analysis of life insurance demand of modern life theory is based on the expected utility theory and it is established by Menahem in 1965 [1]. In the early studies, the multiple regression models are applied to study the demand factors of life insurance products [2–5]. Later, the panel data model, time series model and the autoregressive distributed lag model are applied to discuss this problem on many interrelated reference literatures [6–9]. In these researches, a kind of variable is always neglected which is the variable with fuzziness, such as the higher educational level, and so on. The model can't be scientific enough if the data are not denoted correctly [10]. For the fuzziness of these data nothing but fuzzy logic can be applied and more and more researchers realize the importance of it. In the work of [11–13], Song and Chissom presented the theory of fuzzy time series to overcome the drawback of the classical

X. Zhou (✉) · K. Zou
College of Information Engineering, Dalian University, Dalian 116622, China
e-mail: xiaoyao781028@126.com

K. Zou
e-mail: zoukq@vip.sina.com

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time series methods. They defined the universe of discourse and proposed the fuzzy set and fuzzy relationship. Later, many scholars began to focus on fuzzy time series and some achievements are gotten [14–16]. In these papers, the authors deal with the forecasting problems under a fuzzy environment and they get some fuzzy inferential rules which are different from any stochastic methods. But these fuzzy time series are impossible for the researchers to get an econometric model and the quantity relationship of variables. In order to find the variables with fuzziness for econometrics, we have to seek after new ways.

2 Fuzzy Variable Time Series

2.1 Definitions of Fuzzy Variable Time Series

In classical econometrics the variables can be classified into exogenous variable, lagged dependent variable and dummy variable. In empirical analysis some econometric data with fuzzy characteristics are not defined rationally such as the educational level of the whole population. The educational level should be expressed by fuzzy language such as 'high', 'not high', 'low' or the grade of membership. That means it should be a variable represented by fuzzy sets. But the ratio of the graduates to the whole population is usually used to represent the educational level in econometric model and the importance of people with other educational level is ignored. The method of fuzzy variable based on fuzzy theory is proposed to solve this problem in this paper. The variable with fuzzy characteristic is defined as fuzzy variable.

Definition 2.1 Let $u \in U$.

$$\mu_A : U \to [0, 1].$$

Here μ_A is a mapping from U to the interval [0, 1] and A is called a fuzzy subset in U. μ is called the membership function of A and written as μ_A. The value of $\mu_A(u)$ represents the grade of membership of u in A and it shows the grade that u belongs to A in [17].

Definition 2.2 Let $a_{u_i j}$ be the time series of a variable of period j in econometrics.

$$\mu_A : U \to [0, 1],$$

μ_A is the membership function of A. The value of $\mu_A(u_i)$ is the grade of membership of u_i to A. Let

$$z_j = \frac{\sum_{i=1}^{n} a_{u_i j} \cdot \mu_A(u_i)}{\sum_{i=1}^{n} a_{u_i j}}.$$

Here $u = 1, 2, \ldots, n$, $\sum_{i=1}^{n} a_{u_i j}$ is the sum of $a_{u_i j}$ of period j. z_j is called the discrete fuzzy variable time series of period j.

The continuous fuzzy variable time series z_x of time x are gotten and listed as:

$$z_x = \frac{\int_{x0}^{x} \mu_A(u_i) \cdot a_{u_i y} dy}{\int_{x0}^{x} a_{u_i y} dy}$$

for $x_0 \leq x \leq 1$.

Definition 2.3 Let z_j be the fuzzy variable time series of period j.

$$\overline{z_j} = E(z_j) = \frac{\sum_{j=1}^{k} z_j}{k} = \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{n} a_{u_i j} \cdot \mu_A(u_i)}{\sum_{i=1}^{n} a_{u_i j}}}{k}$$

$\overline{z_j}$ is defined as the mean of z_j and it denotes the average level of z_j.

$$D(z_j) = \frac{\sum_{j=1}^{k} (z_j - \overline{z_j})^2}{k}$$

$D(z_j)$ is defined as the variance of z_j and it measures how the data z_j is close to the mean.

In Papers [18–20] get some econometric models based on fuzzy variable time series are gotten and the results show that they can be applied in econometric study and make the models more objective and accurate. The application of fuzzy variable combined econometrics with fuzzy mathematics basically.

2.2 An Example of Fuzzy Variable Time Series

The data are always real numbers in practical study and it is very significant for the researchers to get fuzzy variable time series for the variables with fuzziness accurately. In econometrics, some variables with fuzziness are not denoted scientific enough such as the variable 'level of population ageing'. Ageing of population is a summary term for shift in the age distribution of a population toward older ages. For consciousness of the old age security, the higher the level of the population ageing is, the more the demand of life insurance product is. The 'level of population ageing' is usually defined by the ratio of the people whose age is above 60 to the whole population. But 'the level of the population ageing' is a fuzzy concept and it should be quantized by the method of fuzzy variable. Consider L.A. Zadeh's membership function of 'old people', the membership function of age u to fuzzy set A 'level of population ageing' is defined as shown:

$$\mu_A(u_i) = \begin{cases} 1 - \left[1 + (\frac{26-25}{5})^2\right]^{-1} & if\ u_i \leq 26 \\ 1 - \left[1 + (\frac{u_i-25}{5})^2\right]^{-1} & if\ 27 \leq u_i \leq 79 \\ 1 & if\ u_i \geq 80 \end{cases} \quad (1)$$

The fuzzy degree of fuzzy set A 'level of population ageing' is calculated by (1), that is

$$L(A) = \frac{2}{80}\sum_{i=1}^{80} |\mu_A(u_i) - A_{0.5}(u_i)| = 0.4096.$$

$L(A)$ states that the variable data 'level of population ageing' is so fuzzy that fuzzy variable should be used to denote it. The fuzzy variable time series of the 'level of population ageing' in China from 1995 to 2007 are gotten as below.

$z_{1995} = 0.4409$, $z_{1996} = 0.5181$, $z_{1997} = 0.4715$, $z_{1998} = 0.4837$, $z_{1999} = 0.4946$, $z_{2000} = 0.5064$, $z_{2001} = 0.5131$, $z_{2002} = 0.5267$, $z_{2003} = 0.5464$, $z_{2004} = 0.5487$, $z_{2005} = 0.5022$, $z_{2006} = 0.5227$, $z_{2007} = 0.5162$

The fuzzy variable time series of 'level of population ageing' reveal that the ageing of population is deeper and deeper and our country is a representative nation of 'getting old' as shown in Fig. 1.

What's this trend telling us? The improvement of medical conditions and the rising of living standards bring rapid ageing population which leads China to an ageing society. Not only old people but also young people need life insurance products, so it is necessary to consider the demand of whole population when we study the demand for life insurance products.

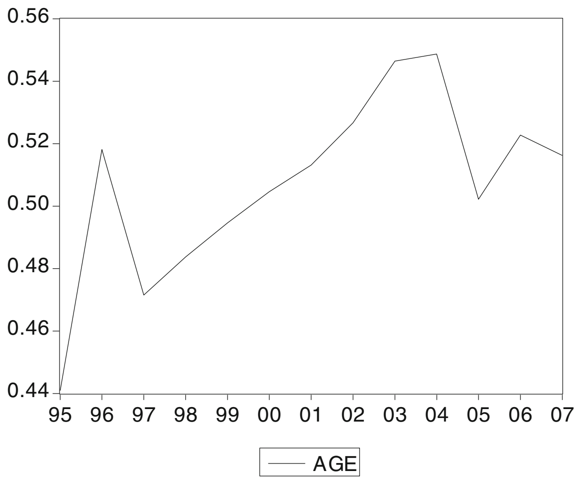

Fig. 1 Fuzzy variable time series of the level of population ageing from 1995 to 2007

3 Influencing Factors Models of China Life Insurance Product Demand

3.1 Multiple Regression Model of the Influencing Factors of China Life Insurance Product Demand

In econometrics, linear regression is an approach for modeling the relationship between a scalar dependent variable y and one or more independent variables denoted x. For more than one independent variable, the process is called a multiple linear regression. Thus the normal model takes the form

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_k x_{ki} + \mu_i, i = 1, 2\ldots, n$$

In this model, y is the dependent variable, $x_{1i}, x_{2i}, \ldots, x_{ki}$ are called independent variables, β_k is the parameters to be estimated and n is the sample size. This paper applies the multiple regression models to study the demand relationship between life insurance products and its influencing factors.

3.1.1 Dependent Variable of the Model

The amount of money to be charged for a certain amount of insurance coverage is called the premium. The more the premium is the more the demand of life insurance product is. In the model the premium of China life insurance industry is taken as the dependent variable. Because it increases rapidly, the logarithm of it is gotten into the model, noted by LNPREMIUM.

3.1.2 Independent Variables of the Model

(1) Economic development level
The gross domestic product (GDP) and per capita disposable income (INCOME) are used to represent this factor.

(2) Related goods
The substitute goods of life insurance product are the saving, stock, real estate, and so on. For the short-term nature of stock investment and the long-term nature of life insurance product, the factor of stock isn't put into the model in this paper. Because the substitute relationship between saving and life insurance product is not obvious enough, the author doesn't focus on it. The cross-elasticity of demand between life insurance product and real estate is bigger than 0 which denotes that the real estate (HOUSE) can be considered as the substitute good of life insurance product as shown in Table 1.

Table 1 Cross-elasticity of demand between life insurance product and real estate

	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008
Cross-elasticity	6.99	−32.69	4.70	15.12	15.99	6.52	0.37	0.99	1.78	1.47	1.52

(3) The factors of population

The most important factors of population are the total population of China (POPULATION), the age structure (AGE) and the degree of culture form (CULTURE). If the proportion of old people to the whole population rises the demand of life insurance product will increase. The proportion is denoted by the ageing of population and its fuzzy variable time series are gotten in 2.2 already. The awareness of insurance is enhanced with the degree of culture of the population being higher and higher.

The L-fuzzy degree of the degree of culture form is 0.4 which shows that it is so fuzzy that it should be defined according to the method of fuzzy variable. The fuzzy variable time series of the degree of culture form from 1993 to 2010 are gotten as below.

$z_{1993} = 0.2722$, $z_{1994} = 0.2689$, $z_{1995} = 0.2711$, $z_{1996} = 0.2748$, $z_{1997} = 0.2716$, $z_{1998} = 0.2753$, $z_{1999} = 0.2813$, $z_{2000} = 0.2881$, $z_{2001} = 0.2889$, $z_{2002} = 0.2826$, $z_{2003} = 0.2846$, $z_{2004} = 0.2917$, $z_{2005} = 0.3031$, $z_{2006} = 0.3205$, $z_{2007} = 0.3407$, $z_{2008} = 0.3637$, $z_{2009} = 0.3807$, $z_{2010} = 0.3957$

The degree of culture form in China is rising slowly as shown in Fig. 2.

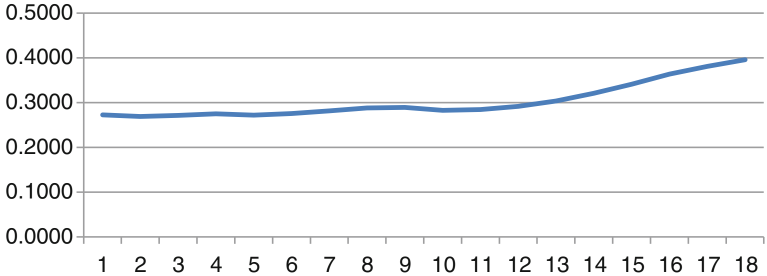

Fig. 2 Fuzzy variable time series of the degree of culture form from 1993 to 2010

3.1.3 The Multiple Regression Model of the Demand Factors of China Insurance Life Product

The multiple regression model is gotten as shown:

$$\begin{aligned}\text{LNPREMIUM} = &-806.02 + 1.34 * \text{AGE} + 6.24 * \text{LNCULTURE} - 9.16 * \text{LNGDP}+\\ &0.50 * \text{LNHOUSE} + 6.13 * \text{LNINCOME} + 73.71 * \text{LNPOPULATION}\end{aligned}$$

R-squared=0.993915, F-statistic=163.3274, Durbin-Watson stat=2.1070

By rejecting the variable POPULATION, the short-term multiple regression model of the demand factors of China life insurance industry is shown as:

$$\begin{aligned}\text{LNPREMIUM} = &-39.17 + 2.0837 * \text{AGE} + 4.7506 * \text{LNCULTURE}+\\ &3.5759 * \text{LNGDP} - 0.8056 * \text{LNHOUSE} + 0.4751 * \text{LNINCOME}\end{aligned}$$

Some conclusions are gotten from this model. Firstly, the level of economic development is very important to the demand of China life insurance product. When GDP or INCOME increases 1 %, the demand of China life insurance product will increase 3.58 or 0.474 %. The economic development will promote the demand of life insurance product and the life insurance industry grows fast in economic developed area. Secondly the degree culture form increases 1 %, the demand will increases 2.0827 %. The result shows that the CULTURE plays an important role in the life insurance industry developing and the demand of life insurance product will increase with the deepening of the aging degree in China. At last, the coefficient of real estate is negative which illustrates the substitute relationship between life insurance product and real estate. The demand of life insurance product will reduce if the house price increases continously.

3.2 The Long-Term Equilibrium Regression Model of the Influencing Factors of China Life Insurance Product Demand

The theory of cointegration is the basis of dynamic econometrics. For the cointegration relationship of dependent variable and independent variable the autoregressive distributed lag model can be changed to VEC model.

In most cases, if we combine two variables which are I(1), then their combination will also be I(1). More generally, if we combine variables with differing orders of integration, the combination with differing orders of integration, the combination will have an order of integration equal to the largest. i.e. If $X_{i,t} \sim \mathrm{I}(d_i)$ for $i = 1, 2, \ldots, k$, so we have k variables each integrated of order d_i.

Let

$$z_t = \sum_{i=1}^{k} \alpha_i X_{i,t}$$

Then $z_t \sim \mathrm{I}(\max d_i)$.

Definition 3.1 Let z_t be a $k \times 1$ vector of variables, then the components of z_t are cointegrated of order (d, b) if

(i) All components of z_t are $\mathrm{I}(d)$

(ii) There is at least one vector of coefficients α such that $\alpha z_t \sim \mathrm{I}(d{-}b)$.

The variables are cointegrated which means that a linear combination of them will be stationary. Johansen and Juselius proposed a method of testing the cointegration of variables called JJ test in 1990s. By JJ testing the lnPREMIUM, lnHOUSE, lnINCOME are cointegrated under the level of 5 % and a long-term model is gotten.

$$\text{lnPREMIUM} = \text{c} + 6.9256\text{lnINCOME} - 6.0509\text{lnHOUSE}$$

Focus on this model, the conclusions that the demand of life insurance is proportional to per capita disposable income (INCOME) and inversely proportional to the real estate price over a long period of time are gotten. The results show that the economic developing level take great influence on the demand of China life insurance industry. The rising wages and regulation of housing price will lead to a higher demand of life insurance product in the long run.

4 Conclusion

The techniques used in econometrics have been employed in a widening variety of fields, including political methodology health economics, and numerous others. Many researchers focus more attention on the applications of econometrics but the variable data with fuzziness are always neglected. The author proposes a new kind of variable named fuzzy variables according to fuzzy theory. The discrete and continuous fuzzy variable time series are gotten and they can be applied in econometric models and data mining. In this paper the short-term and long-term demand factors models of China life insurance industry are gotten by using the method of fuzzy variable time series. The results show that the economic developing level take great influence on the demand of China life insurance industry and the structure of age has the same changing direction with the demand of life insurance industry.

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References

1. Menahem, Y.: Uncertain life time, life insurance and the theory of the consumer. Rev. Econ. Stud. **32**(2), 137–150 (1965)
2. Hankansson, N.H.: Optimal investment and consumption strategies under risk for a class of utility functions. Int. Econ. Rev. **10**, 443–466 (1969)
3. Fischer, S.: A life cycle model of life insurance purchases. Int. Econ. Rev. **14**, 132–152 (1973)
4. Karni, E., Zilcha, I.: Risk aversion in the theory of life insurance: the Fisherian model. J. Risk Insur. **53**, 606–620 (1986)
5. Browne, J., Kim, K.: An international analysis of life insurance demand. J. Risk Insur. **60**(4), 642–645 (1993)
6. Beck, T., Webb, I.: Economic, demographic and institutional determinants of life insurance consumption across countries. World Bank Econ. Rev. **17**(1), 51–88 (2002)
7. Zhang, F.J.: The empirical analysis of the economic factor of China insurance industry. J. Quant. & Tech. Econ. **3**, 25–31 (2004)
8. Cai, Q.J., Hu, D.: The analysis of influencing factors of China life insurance and its mechanism. J. Insur. Prof. Coll. **6**, 1–6 (2006)
9. Guo, L.J., Xiong, Z.G.: The empirical analysis of the influencing factor of China investing life insurance. Insur. Stud. **3**, 31–36 (2007)
10. Zhou, X.Y., Zou, K.Q., Wang, Y.F.: Fuzzy variable of econometrics based on fuzzy membership function. ICIC Express Lett. **4**(5), 1799–1804 (2010)
11. Song, Q., Chissom, B.S.: New models for forecasting enrollments: fuzzy time series and neural network approaches. Annu. Meet. Am. Educ. Res. Assoc. **4**, 12–16 (1993)
12. Song, Q., Chissom, B.S.: Fuzzy time series and its models. Fuzzy Sets Syst. **54**, 269–277 (1993)
13. Song, Q., Chissom, B.S.: Enrollments forecasting with fuzzy time series-part II. Fuzzy Sets Syst. **62**, 1–9 (1994)
14. Chen, S.: Forecasting enrollments based on fuzzy time series. Fuzzy Sets Syst. **81**, 311–319 (1996)
15. Jilani, T.A., Burney, S.M.A., Ardil, C.: Fuzzy metric approach for fuzzy time series forecasting based on frequency density based partitioning. Proc. World Acad. Sci., Eng. Technol. **23**, 333–338 (2007)
16. Wu, B., Lin, Y.C.: Fuzzy time series analysis and forecasting: with an example to Taiwan weighted stock index. ACTA Math. Appl. Sin. **25**, 67–76 (2002)
17. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–358 (1965)
18. Zhou, X.Y., Zou, K.Q.: Fuzzy variable of econometrics based on fuzzy membership function. ICIC Express Lett. **4**(5), 1799–1804 (2010)
19. Zou, K.Q., Zhou, X.Y.: Multivariate regression model based on fuzzy variable in econometrics. In: 2009 International Conference on Artificial Intelligence and Computational Intelligence, pp. 111–114 (2010)
20. Zou, K.Q., Zhou, X.Y.: Co-integration analysis based on fuzzy variables in econometrics. ICIC Express Lett. **3**(4), 1393–1398 (2009)
21. Li, A.G., Zhang, Z.H.: Fuzzy Mathematics and Its Applications. Metallurgical Industry Press, Beijing (2005)
22. Zhang, X.T.: A Guide to using eViews. Beijing, China Machine Press (2007)
23. Li, Z.N., Ye, A.Z.: Higher econometrics. Beijing, Tsinghua University Press (2000)
24. http://www.stats.gov.cn/
25. http://www.circ.gov.cn/

Part II
Fuzzy Mathematics and Its Applications

Sufficient Conditions of Cut Sets on Intuitionistic Fuzzy Sets

Yiying Shi, Xuehai Yuan, Yuchun Zhang and Yuhong Zhang

Abstract In this paper, a monomorphism is established from the three-valued fuzzy sets to intuitionistic fuzzy sets, and with this monomorphism, the three-valued fuzzy sets can be embedded in intuitionistic fuzzy sets. Then the general definition of the cut sets on the intuitionistic fuzzy sets is proposed. Finally, the axiomatic descriptions for different cut sets are presented and three most intrinsic properties for each cut sets are listed.

Keywords Intuitionistic fuzzy sets · Cut sets · Sufficient condition · Monomorphism

1 Introduction

Since the concept of fuzzy sets was proposed by Zadeh in 1965, the theory has been widely applied in many fields [1]. It deals with inference that is approximate rather than exact. Compared with classical sets, in which the value of the variable is zero or one, fuzzy logical sets variable may have the truth value that ranges between 0 and 1 in degree. But in the fuzzy membership function, the degrees of certainty, negativity and uncertainty can't be expressed simultaneously. Usually a certain degree of

Y. Shi (✉) · X. Yuan
School of Control Science and Engineering, Dalian University of Technology,
Dalian 116024, People's Republic of China
e-mail: shiyiying98@163.com

Y. Shi · Y. Zhang
School of Science, Shenyang Ligong University, Shenyang 110159,
People's Republic of China

X. Yuan
School of Science, Dalian University of Technology at Panjin, Panjin 124221,
People's Republic of China

Y. Zhang
Department of Basic Education, City Institute, Dalian University of Technology,
Dalian 116600, People's Republic of China

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hesitation or uncertainty is existed in cognitive process of human beings which makes people's perception expressed as certainty, negativity and uncertainty. So in 1983, Atanassov put forward the concept of intuitionistic fuzzy sets covering membership, non-membership and hesitation degree [2, 3]. In recent years the theory is utilized in decision-making, fuzzy control, logic programming and pattern recognition, etc.

In theory of fuzzy sets and systems, cut sets are the bridges between the fuzzy sets and classical sets which play significant roles in fuzzy optimization, decision making [4, 5], fuzzy algebra [6, 7], fuzzy reasoning [8, 9], fuzzy topology [10, 11], fuzzy measure, fuzzy analysis [12–16], fuzzy logic [17], and so on. Based on cut sets, the decomposition theorems and representation theorems can be established [18]. The cut sets of intuitionistic fuzzy sets are studied in [19], and the cut sets, the decomposition theorems and representation theorems on intuitionistic fuzzy sets are studied in [20, 21]. These results play an active role in the corresponding fuzzy systems. But until now there are not axiomatic descriptions for the cut sets of the intuitionistic fuzzy sets. Therefore, in this article we present axiomatic descriptions for different cut sets. This paper is organized as follows: The preliminaries are listed in Sect. 2. The definition of the f-cut sets and the sufficient conditions are introduced in Sect. 3. And Sect. 4 is the conclusions.

2 Preliminary

Definition 1 ([1]): Let X be a set .The mapping $A : X \to [0, 1]$ is called a fuzzy subset of X.

Definition 2 ([20, 21]): Let X be a set and $\mu_A : X \to [0, 1]$, $\nu_A : X \to [0, 1]$ be two mappings. If

$$\mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$$

then we call $A = (X, \mu_A, \nu_A)$ an intuitionistic fuzzy set over X, and denote $A(x) = (\mu_A(x), \nu_A(x))$.

Let $\mathcal{IF}(X)$ denote the class of all intuitionistic fuzzy sets over X, and $A = (X, \mu_A, \nu_A) \in \mathcal{IF}(X)$, $B = (X, \mu_B, \nu_B) \in \mathcal{IF}(X)$.

(1) $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x), \forall x \in X$.
(2) $A^c = (X, \nu_A, \mu_A)$.
(3) $A \cup B = (X, \mu_{A\cup B}, \nu_{A\cup B})$, where $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x), \nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x)$.
(4) $A \cap B = (X, \mu_{A\cap B}, \nu_{A\cap B})$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x), \nu_{A\cap B}(x) = \nu_A(x) \vee \nu_B(x)$.
(5) $\tilde{X}(x) \equiv (1, 0),\ \tilde{\phi}(x) \equiv (0, 1), \forall x \in X$.

Definition 3 ([20, 21]): Let A be a subset of X and $\lambda \in [0, 1]$.

(1) If $A_\lambda, A_\lambda \in 3^X$ and

$$A_\lambda(x) = \begin{cases} 1 & \mu_A(x) \geq \lambda \\ \frac{1}{2} & \mu_A(x) < \lambda \leq 1 - \nu_A(x) \\ 0 & \lambda > 1 - \nu_A(x) \end{cases}, \quad A_{\underline{\lambda}}(x) = \begin{cases} 1 & \mu_A(x) > \lambda \\ \frac{1}{2} & \mu_A(x) \leq \lambda < 1 - \nu_A(x) \\ 0 & \lambda \geq 1 - \nu_A(x) \end{cases}$$

then we call A_λ and $A_{\underline{\lambda}}$ the λ-upper cut set and λ-strong upper cut set of A, respectively.

(2) If $A^\lambda, A^{\underline{\lambda}} \in 3^X$ and

$$A^\lambda(x) = \begin{cases} 1 & \nu_A(x) \geq \lambda \\ \frac{1}{2} & \nu_A(x) < \lambda \leq 1 - \mu_A(x) \\ 0 & \lambda > 1 - \mu_A(x) \end{cases}, \quad A^{\underline{\lambda}}(x) = \begin{cases} 1 & \nu_A(x) > \lambda \\ \frac{1}{2} & \nu_A(x) \leq \lambda < 1 - \mu_A(x) \\ 0 & \lambda \geq 1 - \mu_A(x) \end{cases}$$

then we call A^λ and $A^{\underline{\lambda}}$ the λ-lower cut set and λ-strong lower cut set of A, respectively.

(3) If $A_{[\lambda]}, A_{[\underline{\lambda}]} \in 3^X$ and

$$A_{[\lambda]}(x) = \begin{cases} 1 & \mu_A(x) + \lambda \geq 1 \\ \frac{1}{2} & \nu_A(x) \leq \lambda < 1 - \mu_A(x) \\ 0 & \lambda < \nu_A(x) \end{cases}, \quad A_{[\underline{\lambda}]}(x) = \begin{cases} 1 & \mu_A(x) + \lambda > 1 \\ \frac{1}{2} & \nu_A(x) < \lambda \leq 1 - \mu_A(x) \\ 0 & \lambda \leq \nu_A(x) \end{cases}$$

then we call $A_{[\lambda]}$ and $A_{[\underline{\lambda}]}$ the λ-upper quasi cut set and λ-strong upper quasi cut set of A, respectively.

(4) If $A^{[\lambda]}, A^{[\underline{\lambda}]} \in 3^X$ and

$$A^{[\lambda]}(x) = \begin{cases} 1 & \nu_A(x) + \lambda \geq 1 \\ \frac{1}{2} & \mu_A(x) \leq \lambda < 1 - \nu_A(x) \\ 0 & \lambda < \mu_A(x) \end{cases}, \quad A^{[\underline{\lambda}]}(x) = \begin{cases} 1 & \nu_A(x) + \lambda > 1 \\ \frac{1}{2} & \mu_A(x) < \lambda \leq 1 - \nu_A(x) \\ 0 & \lambda \leq \mu_A(x) \end{cases},$$

then we call $A^{[\lambda]}$ and $A^{[\underline{\lambda}]}$ the λ-lower quasi cut set and λ-strong lower quasi cut set of A, respectively.

Definition 4 ([20, 21]): Let $A \in 3^X$, $\lambda \in [0, 1]$,

$$f_i : [0, 1] \times 3^X \to L^X(\lambda, A) \mapsto f_i(\lambda, A), i = 1, 2, \ldots, 8$$

be the following mappings:

$$f_1(\lambda, A)(x) = \begin{cases} (0, 1) & A(x) = 0 \\ (\lambda, 1 - \lambda) & A(x) = 1 \\ (0, 1 - \lambda) & A(x) = \frac{1}{2} \end{cases}, \quad f_2(\lambda, A)(x) = \begin{cases} (\lambda, 1 - \lambda) & A(x) = 0 \\ (1, 0) & A(x) = 1 \\ (\lambda, 0) & A(x) = \frac{1}{2} \end{cases},$$

$$f_3(\lambda, A)(x) = \begin{cases} (1-\lambda, \lambda) & A(x) = 0 \\ (0, 1) & A(x) = 1 \\ (0, \lambda) & A(x) = \frac{1}{2} \end{cases}, \quad f_4(\lambda, A)(x) = \begin{cases} (1, 0) & A(x) = 0 \\ (1-\lambda, \lambda) & A(x) = 1 \\ (1-\lambda, 0) & A(x) = \frac{1}{2} \end{cases},$$

$$f_5(\lambda, A)(x) = \begin{cases} (0, 1) & A(x) = 0 \\ (1-\lambda, \lambda) & A(x) = 1 \\ (0, \lambda) & A(x) = \frac{1}{2} \end{cases}, \quad f_6(\lambda, A)(x) = \begin{cases} (1-\lambda, \lambda) & A(x) = 0 \\ (1, 0) & A(x) = 1 \\ (1-\lambda, 0) & A(x) = \frac{1}{2} \end{cases},$$

$$f_7(\lambda, A)(x) = \begin{cases} (\lambda, 1-\lambda) & A(x) = 0 \\ (0, 1) & A(x) = 1 \\ (0, 1-\lambda) & A(x) = \frac{1}{2} \end{cases}, \quad f_8(\lambda, A)(x) = \begin{cases} (1, 0) & A(x) = 0 \\ (\lambda, 1-\lambda) & A(x) = 1 \\ (\lambda, 0) & A(x) = \frac{1}{2} \end{cases},$$

then we have the following theorems:

Theorem 1 ([20, 21]): *Let $A = (X, \mu_A, \nu_A)$ be an intuitionistic fuzzy set. Then*

(1) $A = \bigcup_{\lambda \in [0,1]} f_1(\lambda, A_\lambda) = \bigcap_{\lambda \in [0,1]} f_2(\lambda, A_\lambda)$.
(2) $A = \bigcup_{\lambda \in [0,1]} f_1(\lambda, A_{\underline{\lambda}}) = \bigcap_{\lambda \in [0,1]} f_2(\lambda, A_{\underline{\lambda}})$.
(3) If mapping H: [0, 1] → 3^X satisfies

$$A_{\underline{\lambda}} \subset H(\lambda) \subset A_\lambda,$$

then

(I) $A = \bigcup_{\lambda \in [0,1]} f_1(\lambda, H(\lambda)) = \bigcap_{\lambda \in [0,1]} f_2(\lambda, H(\lambda))$.
(II) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supset H(\lambda_2)$.
(III) $A_\lambda = \bigcap_{\alpha < \lambda} H(\alpha)$, $A_{\underline{\lambda}} = \bigcup_{\alpha > \lambda} H(\alpha)$.

Theorem 2 ([13]): *Let $A = (X, \mu_A, \nu_A)$ be an intuitionistic fuzzy set. Then*

(1) $A = \bigcup_{\lambda \in [0,1]} f_3(\lambda, A^\lambda) = \bigcap_{\lambda \in [0,1]} f_4(\lambda, A^\lambda)$.
(2) $A = \bigcup_{\lambda \in [0,1]} f_3(\lambda, A^{\underline{\lambda}}) = \bigcap_{\lambda \in [0,1]} f_4(\lambda, A^{\underline{\lambda}})$.
(3) If mapping H: [0, 1] → 3^X satisfies

$$A^{\underline{\lambda}} \subset H(\lambda) \subset A^\lambda,$$

then

(I) $A = \bigcup_{\lambda \in [0,1]} f_3(\lambda, H(\lambda)) = \bigcap_{\lambda \in [0,1]} f_4(\lambda, H(\lambda))$.
(II) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supset H(\lambda_2)$.
(III) $A^\lambda = \bigcap_{\alpha < \lambda} H(\alpha)$, $A^{\underline{\lambda}} = \bigcup_{\alpha > \lambda} H(\alpha)$.

Theorem 3 ([20, 21]): *Let* $A = (X, \mu_A, \nu_A)$ *be an intuitionistic fuzzy set. Then*

(1) $A = \underset{\lambda \in [0,1]}{\cup} f_5(\lambda, A_{[\lambda]}) = \underset{\lambda \in [0,1]}{\cap} f_6(\lambda, A_{[\lambda]})$.
(2) $A = \underset{\lambda \in [0,1]}{\cup} f_5(\lambda, A_{[\underline{\lambda}]}) = \underset{\lambda \in [0,1]}{\cap} f_6(\lambda, A_{[\underline{\lambda}]})$.
(3) If mapping H: [0, 1] $\rightarrow 3^X$ *satisfies*

$$A_{[\underline{\lambda}]} \subset H(\lambda) \subset A_{[\lambda]},$$

then

(I) $A = \underset{\lambda \in [0,1]}{\cup} f_5(\lambda, H(\lambda)) = \underset{\lambda \in [0,1]}{\cap} f_6(\lambda, H(\lambda))$.
(II) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \subset H(\lambda_2)$.
(III) $A_{[\lambda]} = \underset{\alpha > \lambda}{\cap} H(\alpha),\ A_{[\underline{\lambda}]} = \underset{\alpha < \lambda}{\cup} H(\alpha)$.

Theorem 4 ([20, 21]): *Let* $A = (X, \mu_A, \nu_A)$ *be an intuitionistic fuzzy set. Then*

(1) $A = \underset{\lambda \in [0,1]}{\cup} f_7(\lambda, A^{[\lambda]}) = \underset{\lambda \in [0,1]}{\cap} f_8(\lambda, A^{[\lambda]})$.
(2) $A = \underset{\lambda \in [0,1]}{\cup} f_7(\lambda, A^{[\underline{\lambda}]}) = \underset{\lambda \in [0,1]}{\cap} f_8(\lambda, A^{[\underline{\lambda}]})$.
(3) If mapping H: [0, 1] $\rightarrow 3^X$ *satisfies*

$$A^{[\underline{\lambda}]} \subset H(\lambda) \subset A^{[\lambda]},$$

then

(I) $A = \underset{\lambda \in [0,1]}{\cup} f_7(\lambda, H(\lambda)) = \underset{\lambda \in [0,1]}{\cap} f_8(\lambda, H(\lambda))$.
(II) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \subset H(\lambda_2)$.
(III) $A^{[\lambda]} = \underset{\alpha > \lambda}{\cap} H(\alpha),\ A^{[\underline{\lambda}]} = \underset{\alpha < \lambda}{\cup} H(\alpha)$.

3 Definition of the f-cut Sets and Sufficient Conditions of the Cut Sets

Lemma 1 *Let* $\varphi : 3^X \rightarrow \mathcal{IF}(X) A \mapsto \varphi(A) = (A, A^c)$ *be a mapping. Then*

(1) φ *is an injection.*
(2) φ *can preserve the operations such as union* $(\cup)$*, intersection* $(\cap)$ *and complementary* (c)*.*

Proof
(1) It is easy to prove.

(2)

$$\begin{aligned}\varphi(A\cup B)(x) &= ((A\cup B)(x), 1-(A\cup B)(x))\\ &= (A(x)\vee B(x), 1-A(x)\vee B(x))\\ &= (A(x)\vee B(x), (1-A(x))\wedge(1-B(x)))\\ &= (A(x), (1-A(x))\vee(B(x), (1-B(x)))\\ &= (\varphi(A)\cup\varphi(B))(x).\\ \varphi(A\cap B)(x) &= ((A\cap B)(x), 1-(A\cap B)(x))\\ &= (A(x)\wedge B(x), 1-A(x)\vee B(x))\\ &= (A(x)\wedge B(x), (1-A(x))\vee(1-B(x)))\\ &= (A(x), (1-A(x))\wedge(B(x), (1-B(x)))\\ &= (\varphi(A)\cap\varphi(B))(x).\end{aligned}$$

$$\begin{aligned}\varphi(A^c)(x) &= (A^c(x), A(x))\\ &= (1-A(x), A(x))\\ &= (A(x), (1-A(x))^c\\ &= (\varphi(A)^c)(x).\end{aligned}$$

Remark By Lemma 1, it is can be concluded that A and $\varphi(A)$ constitute a monomorphism. That is to say, A has no distinctive with $\varphi(A)$. By the above lemma, the three-valued fuzzy sets can be viewed as the intuition fuzzy sets, so in the following A and $\varphi(A)$ can be indiscriminately.

Definition Let f: [0, 1] $\times\mathcal{IF}(X)\to 3^X$ be a mapping, then we call $f(\lambda$, A) is a f-cut set of the intuition fuzzy set A.

Property 1 *If the mapping f: [0, 1] $\times\mathcal{IF}(X)\to 3^X$ satisfies the following conditions:*

(1) $f(\lambda, \bigcap_{t\in T} A_t) = \bigcap_{t\in T} f(\lambda, A_t)$.

(2) When $\lambda > 0$ and A is a three-valued fuzzy set, we have $f(\lambda, A) = A$.

(3)

$$f(\lambda, f_2(\alpha, A)) = \begin{cases} X & \alpha\geq\lambda\\ f(\lambda, A) & \alpha<\lambda\end{cases}.$$

Then $\forall A\in\mathcal{IF}(X)$, $\forall\lambda\in I$, we have $f(\lambda, A) = A_\lambda$.

Proof When $\lambda > 0$,

$$\begin{aligned}f(\lambda, A) &= f(\lambda, \bigcap_{\alpha} f_2(\alpha, A_\alpha))\\ &= \bigcap_{\alpha} f(\lambda, f_2(\alpha, A_\alpha))\\ &= \bigcap_{\alpha<\lambda} f(\lambda, f_2(\alpha, A_\alpha))\\ &= \bigcap_{\alpha<\lambda} f(\lambda, A_\alpha)\\ &= \bigcap_{\alpha<\lambda} A_\alpha\\ &= A_\lambda.\end{aligned}$$

When $\lambda = 0$, we let $\alpha > 0$, then $f(\lambda, f_2(\alpha, A)) = X = A_0$.
Therefore, $\forall A\in\mathcal{IF}(X)$, $\forall\lambda\in I$, we have $f(\lambda$, A) $= A_\lambda$.

Property 2 *If the mapping f: [0, 1] $\times \mathcal{IF}(X) \to 3^X$ satisfies the following conditions:*

(1) $f(\lambda, \underset{t\in T}{\cup} A_t) = \underset{t\in T}{\cup} f(\lambda, A_t)$.
(2) When $\lambda < 1$ and A is a three-valued set, we have $f(\lambda, A) = A$.
(3)

$$f(\lambda, f_1(\alpha, A)) = \begin{cases} \phi & \alpha \le \lambda \\ f(\lambda, A) & \alpha > \lambda \end{cases}.$$

Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, A) = A_{\underline{\lambda}}$.

Proof When $\lambda < 1$,

$$\begin{aligned} f(\lambda, A) &= f(\lambda, \underset{\alpha}{\cup} f_1(\alpha, A_{\underline{\alpha}})) \\ &= \underset{\alpha}{\cup} f(\lambda, f_1(\alpha, A_{\underline{\alpha}})) \\ &= \underset{\alpha>\lambda}{\cup} f(\lambda, f_1(\alpha, A_{\underline{\alpha}})) \\ &= \underset{\alpha>\lambda}{\cup} f(\lambda, A_{\underline{\alpha}}) \\ &= \underset{\alpha>\lambda}{\cup} A_{\underline{\alpha}} \\ &= A_{\underline{\lambda}}. \end{aligned}$$

When $\lambda = 1$, we let $\alpha < 1$, then $f(\lambda, f_1(\alpha, A)) = \phi = A_{\underline{1}}$.
Therefore $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, \mathrm{A}) = A_{\underline{\lambda}}$.

Property 3 *If the mapping f: [0, 1] $\times \mathcal{IF}(X) \to 3^X$ satisfies the following conditions:*

(1) $f(\lambda, \underset{t\in T}{\cap} A_t) = \underset{t\in T}{\cap} f(\lambda, A_t)$.
(2) When $\lambda > 0$ and A is a three-valued set, we have $f(\lambda, A) = A$.
(3)

$$f(\lambda, f_4(\alpha, A)) = \begin{cases} X & \alpha \ge \lambda \\ f(\lambda, A) & \alpha < \lambda \end{cases}.$$

Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, A) = A^{\lambda}$.

Proof When $\lambda > 0$,

$$\begin{aligned} f(\lambda, A) &= f(\lambda, \underset{\alpha}{\cap} f_4(\alpha, A^{\alpha})) \\ &= \underset{\alpha}{\cap} f(\lambda, f_4(\alpha, A^{\alpha})) \\ &= \underset{\alpha<\lambda}{\cap} f(\lambda, f_4(\alpha, A^{\alpha})) \\ &= \underset{\alpha<\lambda}{\cap} f(\lambda, A^{\alpha}) \\ &= \underset{\alpha<\lambda}{\cap} A^{\alpha} \\ &= A^{\lambda}. \end{aligned}$$

When $\lambda = 0$, we let $\alpha > 0$, then $f(\lambda, f_4(\alpha, A)) = X = A^0$.
Therefore $\forall A \in \mathcal{IF}(X)$, $\forall\lambda \in I$, we have $f(\lambda, A) = A^{\lambda}$.

Property 4 *If the mapping* f*: [0, 1]* $\times \mathcal{IF}(X) \to 3^X$ *satisfies the following conditions:*

(1) $f(\lambda, \underset{t\in T}{\cup} A_t) = \underset{t\in T}{\cup} f(\lambda, A_t)$.
(2) When $\lambda < 1$ *and A is a three-valued set, we have* $f(\lambda, A) = A$.
(3)

$$f(\lambda, f_3(\alpha, A)) = \begin{cases} \phi & \alpha \le \lambda \\ f(\lambda, A) & \alpha > \lambda \end{cases}.$$

Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, *we have* $f(\lambda, A) = A_{\underline{\lambda}}$.

Proof When $\lambda < 1$,

$$\begin{aligned} f(\lambda, A) &= f(\lambda, \underset{\alpha}{\cup} f_3(\alpha, A^{\underline{\alpha}})) \\ &= \underset{\alpha}{\cup} f(\lambda, f_3(\alpha, A^{\underline{\alpha}})) \\ &= \underset{\alpha>\lambda}{\cup} f(\lambda, f_3(\alpha, A^{\underline{\alpha}})) \\ &= \underset{\alpha>\lambda}{\cup} f(\lambda, A^{\underline{\alpha}}) \\ &= \underset{\alpha>\lambda}{\cup} A^{\underline{\alpha}} \\ &= A^{\underline{\lambda}}. \end{aligned}$$

When $\lambda = 1$, we let $\alpha = 1$, then $f(\lambda, f_3(\alpha, A)) = \phi = A_{\underline{1}}$.
Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, \text{A}) = A_{\underline{\lambda}}$.

Property 5 *If the mapping* f*: [0, 1]* $\times \mathcal{IF}(X) \to 3^X$ *satisfies the following conditions:*

(1) $f(\lambda, \underset{t\in T}{\cap} A_t) = \underset{t\in T}{\cap} f(\lambda, A_t)$.
(2) When $\lambda < 1$ *and A is a three-valued set, we have* $f(\lambda, A) = A$.
(3)

$$f(\lambda, f_6(\alpha, A)) = \begin{cases} X & \alpha \le \lambda \\ f(\lambda, A) & \alpha > \lambda \end{cases}.$$

Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, *we have* $f(\lambda, A) = A_{[\lambda]}$.

Proof When $\lambda < 1$,

$$\begin{aligned} f(\lambda, A) &= f(\lambda, \underset{\alpha}{\cap} f_6(\alpha, A_{[\alpha]})) \\ &= \underset{\alpha}{\cap} f(\lambda, f_6(\alpha, A_{[\alpha]})) \\ &= \underset{\alpha>\lambda}{\cap} f(\lambda, f_6(\alpha, A_{[\alpha]})) \\ &= \underset{\alpha>\lambda}{\cap} f(\lambda, A_{[\alpha]}) \\ &= \underset{\alpha>\lambda}{\cap} A_{[\alpha]} \\ &= A_{[\lambda]}. \end{aligned}$$

When $\lambda = 1$, we let $\alpha = 1$, then $f(\lambda, f_6(\alpha, A)) = X = A_{[1]}$.
Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, \text{A}) = A_{[\lambda]}$.

Property 6 *If the mapping $f: [0, 1] \times \mathcal{IF}(X) \to 3^X$ satisfies the following conditions:*

(1) $f(\lambda, \underset{t\in T}{\cup} A_t) = \underset{t\in T}{\cup} f(\lambda, A_t)$.
(2) When $\lambda > 0$ and A is a three-valued set, we have $f(\lambda, A) = A$.
(3)

$$f(\lambda, f_5(\alpha, A)) = \begin{cases} \phi & \alpha \geq \lambda \\ f(\lambda, A) & \alpha < \lambda \end{cases}.$$

Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, A) = A_{\underline{[\lambda]}}$.

Proof When $\lambda > 0$,

$$\begin{aligned} f(\lambda, A) &= f(\lambda, \underset{\alpha}{\cup} f_5(\alpha, A_{\underline{[\alpha]}})) \\ &= \underset{\alpha}{\cup} f(\lambda, f_5(\alpha, A_{\underline{[\alpha]}})) \\ &= \underset{\alpha<\lambda}{\cup} f(\lambda, f_5(\alpha, A_{\underline{[\alpha]}})) \\ &= \underset{\alpha<\lambda}{\cup} f(\lambda, A_{\underline{[\alpha]}}) \\ &= \underset{\alpha<\lambda}{\cup} A_{\underline{[\alpha]}} \\ &= A_{\underline{[\lambda]}}. \end{aligned}$$

when $\lambda = 0$, we let $\alpha = 0$, then $f(\lambda, f_5(\alpha, A)) = \phi = A_{[0]}$.
Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, \text{A}) = A_{\underline{[\lambda]}}$.

Property 7 *If the mapping $f: [0, 1] \times \mathcal{IF}(X) \to 3^X$ satisfies the following conditions:*

(1) $f(\lambda, \underset{t\in T}{\cap} A_t) = \underset{t\in T}{\cap} f(\lambda, A_t)$.
(2) When $\lambda < 1$ and A is a three-valued set, we have $f(\lambda, A) = A$.
(3)

$$f(\lambda, f_8(\alpha, A)) = \begin{cases} X & \alpha \leq \lambda \\ f(\lambda, A) & \alpha > \lambda \end{cases}.$$

Then $\forall A \in \mathcal{IF}(X)$, $\forall \lambda \in I$, we have $f(\lambda, A) = A^{[\lambda]}$.

Proof When $\lambda < 1$,

$$\begin{aligned} f(\lambda, A) &= f(\lambda, \underset{\alpha}{\cap} f_8(\alpha, A^{[\alpha]})) \\ &= \underset{\alpha}{\cap} f(\lambda, f_8(\alpha, A^{[\alpha]})) \\ &= \underset{\alpha>\lambda}{\cap} f(\lambda, f_8(\alpha, A^{[\alpha]})) \\ &= \underset{\alpha>\lambda}{\cap} f(\lambda, A^{[\alpha]}) \\ &= \underset{\alpha>\lambda}{\cap} A^{[\alpha]} \\ &= A^{[\lambda]}. \end{aligned}$$

When $\lambda = 1$, we let $\alpha = 1$, then $f(\lambda, f_8(\alpha, A)) = X = A^{[1]}$.
Then $\forall A \in \mathcal{IF}(X)$, $\forall\, \lambda \in I$, we have $f(\lambda, \mathrm{A}) = A^{[\lambda]}$.

Property 8 *If the mapping* f*: [0, 1]* $\times \mathcal{IF}(X) \rightarrow 3^X$ *satisfies the following conditions:*

(1) $f(\lambda, \bigcup_{t \in T} A_t) = \bigcup_{t \in T} f(\lambda, A_t)$.
(2) When $\lambda > 0$ *and* A *is a three-valued set, we have* $f(\lambda, A) = A$.
(3)

$$f(\lambda, f_7(\alpha, A)) = \begin{cases} \phi & \alpha \geq \lambda \\ f(\lambda, A) & \alpha < \lambda \end{cases}.$$

Then $\forall A \in \mathcal{IF}(X)$, $\forall\, \lambda \in I$, *we have* $f(\lambda, A) = A^{\underline{[\lambda]}}$.

Proof When $\lambda > 0$,

$$\begin{aligned} f(\lambda, A) &= f(\lambda, \bigcup_{\alpha} f_7(\alpha, A^{\underline{[\alpha]}})) \\ &= \bigcup_{\alpha} f(\lambda, f_7(\alpha, A^{\underline{[\alpha]}})) \\ &= \bigcup_{\alpha<\lambda} f(\lambda, f_7(\alpha, A^{\underline{[\alpha]}})) \\ &= \bigcup_{\alpha<\lambda} f(\lambda, A^{\underline{[\alpha]}}) \\ &= \bigcup_{\alpha<\lambda} A^{\underline{[\alpha]}} \\ &= A^{\underline{[\lambda]}}. \end{aligned}$$

When $\lambda = 0$, we let $\alpha = 0$, then $f(\lambda, f_7(\alpha, A)) = \phi = A^{\underline{[0]}}$.
Then $\forall A \in \mathcal{IF}(X)$, $\forall\, \lambda \in I$, we have $f(\lambda, \mathrm{A}) = A^{\underline{[\lambda]}}$.

4 Conclusions

This paper establishes a monomorphism which is from the three-valued fuzzy sets to intuitionistic fuzzy sets, and the three-valued fuzzy sets can be embedded in the intuitionistic fuzzy sets with this monomorphism. And the general definition of the cut sets on the intuitionistic fuzzy sets is proposed. Finally, the sufficient conditions are given which can make the f-cut sets be the existed sets.

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References

1. Zadeh, L.A.: Fuzzy Sets. Inform. Cont. **8**, 338–353 (1965)
2. Atanassov, K.: Intuitionistic Fuzzy Sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
3. Atanassov, K.: Intuitionistic Fuzzy Sets: Theory and Applications. Physica-Verlag, Heidelberg (1999)
4. Lai, Y.J., Hwang, C.L.: Fuzzy Mathematical Programming-Methods and Applications, pp. 1–156. Springer, Berlin (1992)
5. Xu, Z.S.: Uncertain Multiple Attribute Decision Making: Methods and Applications (in Chinese), pp. 3–236. Tsinghua University Press, Beijing (2004)
6. Mordeson, J.N., Malik, D.S.: Fuzzy Commutative Algebra, pp. 1–258. World Scientific Publishing, Singapore (1998)
7. Mordeson, J.N., Bhutani, K.R., Rosenfeld, A.: Fuzzy Group Theory, pp. 1–292. Springer, New York (2005)
8. Dubois, D., Hüllermeier, E.: On the Representation of Fuzzy Rules in Terms of Crisp Rules. Inform. Sci. **151**, 301–326 (2003)
9. Luo, C.Z., Wang, P.Z.: Representation of Compositional Relations in Fuzzy Reasoning. Fuzzy Sets Syst. **36**, 77–81 (1990)
10. Wang, G.J.: L-fuzzy Topology Space Theory (in Chinese), pp. 1–406. Shanxi Normal University Press, Xi'an (1988)
11. Liu, Y.M., Luo, M.K.: Fuzzy Topology, pp. 1–306. World Scientific Publishing, Singapore (1990)
12. Zhang, G.Q.: Fuzzy Measure Theory (in Chinese), pp. 1–265. Guizhou Science and Technology Press, Guiyang (1994)
13. Wu, C.X., Ma, M.: The Basis of Fuzzy Analysis (in Chinese), pp. 1–147. National Defence Industry Press, Beijing (1991)
14. Bertoluzza, C., Solci, M., Capodieci, M.L.: Measure of a Fuzzy Set: the α-cut Approach in the Finite Case. Fuzzy Sets Syst. **123**, 93–102 (2001)
15. Garcia, J.N., Kutalik, Z., Cho, K.H.: Level Sets and Minimum Volume Sets of Probability Density Functions. Int. J. Appr. Reason **34**, 25–47 (2003)
16. Pap, E., Surla, D.: Lebesgue Measure of α-cuts Approach for Finding the Height of the Membership Function. Fuzzy Sets Syst. **111**, 341–350 (2000)
17. Wang, G.J.: Non-classical Logic and Approximate Reasoning (in Chinese), pp. 24–185. Science Press, Beijing (2000)
18. Luo, C.Z.: Introduction to Fuzzy Sets (1) (in Chinese), pp. 1–486. Beijing Normal University Press, Beijing (1989)
19. Li, M.: Cut Sets of Intuitionistic Fuzzy Sets (in Chinese). J Liaoning Norm Univ. (Nat. Sci. Ed.) **30**, 152–154 (2007)
20. Yuan, X.H., Li, H.X., Sun, K.B.: The Cut Sets, Decomposition Theorems and Representation Theorems on Intuitionistic Fuzzy Sets and Interval Valued Fuzzy Sets. Sci. China Inf. Sci. **54**, 91–11 (2011)
21. Yuan, X.H., Li, H.X., Lee, E.S.: Three New Cut Sets of Fuzzy Sets and New Theories of Fuzzy Sets. Comput. Math. Appl. **57**, 691–701 (2009)

On the Fuzzy Fractional Posynomial Geometric Programming Problems

F. Zahmatkesh and Bing-yuan Cao

Abstract In this paper we consider the solution method for fuzzy fractional posynomial geometric programming (FFPGP) problems. The problem of concern involves positive trapezoidal fuzzy numbers in the objective function. The proposed approach relies on posing the FFPGP problem as a multi-objective posynomial geometric programming (MOPGP) problem by using simple transformation and condense technique. An illustrative example is included to demonstrate the correctness of the proposed solution algorithm.

Keywords Trapezoidal fuzzy number · Fractional programming · Posynomial function · Multi-objective posynomial geometric programming

1 Introduction

In various applications of nonlinear programming a ratio of two functions is to be maximized or minimized. Ratio optimization problems are commonly called fractional programming (FP) problems. Fractional objectives appear in many real world situations. For instance, we often need to optimize the efficiency of some activities like cost/time, cost/profit, and output/employee. One of the earliest fractional programs (though not called so) is an equilibrium model for an expanding economy introduced by Neumann [16] in 1937. There are different solution algorithms for determining the optimal solution of particular kinds of fractional programming prob-

F. Zahmatkesh · B. Cao (✉)
School of Mathematics and Information Science, Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong, Higher Education Institutes, Guangzhou University, Guangdong 510006, China
e-mail: caobingy@163.com

F. Zahmatkesh
e-mail: faeze_zahmatkesh@yahoo.com

B. Cao
Guangzhou University, Guangdong 510550, China

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lems. In 1962 Charnes and Cooper published their classical paper [8] in which they show that a linear fractional program can be reduced to a linear program with help of a nonlinear variable transformation. Jagannathan [12] supplied theoretical insight into the relationship between nonlinear fractional programming and nonlinear parametric programming. Dinkelbach [9] developed a method based on Jagannathan's theorem for solving nonlinear fractional problems. Chang [6] proposed an approximate approach to solving posynomial fractional programming problems by deriving the linear programming relaxation of the problem based on piecewise linearization techniques.

Sakawa et al. [20] used fuzzy mathematical programming to study linear fractional programming (LFP) problems with fuzzy goals or coefficients. Hladik [11] considered a linear FP problem with interval data and presented a method for computing the range of optimal values. A number of studies have been done by authors in the field of linear, nonlinear, integer fractional programming and multi-objective fractional programming problems under fuzziness [7, 15, 18, 19].

According to our experience, it is believed that the solution method in fuzzy fractional posynomial geometric programming has not been given comprehensive attention in the literature before. In the present paper, an attempt is made to study fractional posynomial geometric programming problem with positive trapezoidal fuzzy coefficients in objective function. The rest of this paper is organized as follows: fuzzy notations and definitions used in the remaining parts of the paper are presented in Sect. 2. Section 3 describes fuzzy fractional posynomial geometric programming problem and presents its solving procedure. Then one numerical example is presented to illustrate the tractability of the proposed approach in computational efficiency. Finally, brief conclusions are given in Sect. 5.

2 Preliminaries and Notations

In this section, we give some notions and definitions on which our research in this paper is based.

Fuzzy sets first introduced by Zadeh [21] in 1965 as a mathematical way of representing vagueness in everyday life. According to [14], the characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\widetilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\widetilde{A}} : X \to [0, 1]$. The assigned value indicates the *membership grade* of the element in the set A. The function $\mu_{\widetilde{A}}$ is called the *membership function* and the set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)); x \in X\}$ defined by $\mu_{\widetilde{A}}(x)$ for each $x \in X$ is called a *fuzzy set*. A fuzzy set $\widetilde{A}$, defined on the universal set of real numbers $\mathbb{R}$, is said to be a *fuzzy number* if its membership function has the following characteristics:

1. $\mu_{\widetilde{A}} : \mathbb{R} \to [0, 1]$ is continuous.
2. $\mu_{\widetilde{A}}(x) = 0$ for all $x \in (-\infty, a] \cup [d, \infty)$.

3. $\mu_{\widetilde{A}}(x)$ is strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$.
4. $\mu_{\widetilde{A}}(x) = 1$ for all $x \in [b, c]$, where $a < b < c < d$.

Definition 2.1 ([14]) A fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be a trapezoidal fuzzy number if its membership function is given by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a < x < b, \\ 1, & b \leq x \leq c, \\ \frac{(x-d)}{(c-d)}, & c < x < d. \end{cases} \tag{1}$$

Definition 2.2 ([10]) A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be positive (negative) trapezoidal fuzzy number, denoted by $\widetilde{A} > 0$ ($\widetilde{A} < 0$), if and only if $a > 0$ ($c < 0$).

Definition 2.3 ([13]) Let $\widetilde{M} = (a_1, b_1, c_1, d_1)$, $\widetilde{N} = (a_2, b_2, c_2, d_2)$ be two positive trapezoidal fuzzy numbers and $\lambda \in \mathbb{R}^+$. Then,

$$\lambda \cdot \widetilde{M} = (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1), \tag{2}$$

$$\widetilde{M} + \widetilde{N} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2). \tag{3}$$

Definition 2.4 ([3]) Let $x_1, \ldots, x_m$ denote m real positive variables, and $x = (x_1, \ldots, x_m)$ a vector with components x_l. A real valued function u of x, with the form

$$u(x) = c x_1^{\gamma_1} x_2^{\gamma_2} \ldots x_m^{\gamma_m}, \tag{4}$$

where $c > 0$ and $\gamma_l \in \mathbb{R}$ is called a monomial function, or more informally, a monomial (of the variables $x_1, \ldots, x_m$). A sum of one or more monomials, i.e., a function of the form

$$g(x) = \sum_{k=1}^{J} c_k x_1^{\gamma_{k1}} x_2^{\gamma_{k2}} \ldots x_m^{\gamma_{km}}, \tag{5}$$

where $c_k > 0$, is called a posynomial function or, more simply, a posynomial (with J terms, in the variables $x_1, \ldots, x_m$). Posynomials are closed under addition, multiplication, and positive scaling. Posynomials can be divided by monomials (with the result also a posynomial): if g is a posynomial and u is a monomial, then $\frac{g}{u}$ is a posynomial.

Definition 2.5 ([5]) A multi-objective posynomial geometric programming (MOPGP) problem can be stated as:

Find $x = (x_1, x_2, \ldots, x_m)^T$ so as to

$$\min \quad g_0^{(t)}(x) = \sum_{k=1}^{J_0^{(t)}} c_{0k}^{(t)} \prod_{l=1}^{m} x_l^{\gamma_{0kl}^{(t)}} \quad t \in \{1, 2, \ldots, n\},$$

subject to satisfying:

$$\begin{aligned} &g_i(x) = \sum\nolimits_{k=1}^{J_i} c_{ik} \prod\nolimits_{l=1}^{m} x_l^{\gamma_{ikl}} \le 1, \quad i = 1, 2, \ldots, p, \\ &x_l > 0, \quad l = 1, 2, \ldots, m, \end{aligned} \tag{6}$$

where $c_{0k}^{(t)}$ and c_{ik} are positive real constant coefficients for all t, i, k;
$\gamma_{0kl}^{(t)}$ and γ_{ikl} are arbitrary real constant exponents for all t, i, k, l;
$J_0^{(t)}$ is the number of terms present in the tth objective function $g_0^{(t)}(x)$, $t = 1, 2, \ldots, n$;
J_i is the number of terms present in the ith constraint, $i = 1, 2, \ldots, p$.

In the above multi-objective posynomial geometric programming problem there are n number of minimization type objective functions, p number of inequality type constraints and m number of strictly positive decision variables.

3 Problem Formulation and Solution Concept

In this section, a fuzzy fractional posynomial geometric programming problem and its solution strategy are described.

3.1 Problem Formulation

The problem to be considered in this paper is the following fuzzy fractional posynomial geometric programming (FFPGP) problem:

Find $x = (x_1, x_2, \ldots, x_m)^T$ so as to

$$\max \quad \frac{\widetilde{g_1}(x) = \sum_{k=1}^{J_1} \tilde{c}_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{g_2(x) = \sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

subject to satisfying:

$$\begin{aligned} &g_i(x) = \sum\nolimits_{k=1}^{J_i} c_{ik} \prod\nolimits_{l=1}^{m} x_l^{\gamma_{ikl}} \le 1, \quad i = 3, 4, \ldots, p+2, \\ &x_l > 0, \quad l = 1, 2, \ldots, m, \end{aligned} \tag{7}$$

where $x = (x_1, x_2, \ldots, x_m)^T$ is a variable vector, and T stands for transpose;
c_{2k} and c_{ik} are positive real constant coefficients for all k and $i = 3, 4, \ldots, p+2$;
$\tilde{c}_{1k} = (a_{1k}, b_{1k}, c_{1k}, d_{1k})$ are positive trapezoidal fuzzy numbers for all k;
$\gamma_{1kl}, \gamma_{2kl}$ and γ_{ikl} are arbitrary real constant exponents for all k, l and $i = 3, 4, \ldots, p+2$;

J_1 and J_2 represent the number of product terms of numerator and of denominator in the objective function, respectively;
J_i is the number of terms present in the ith constraint, $i = 3, 4, \ldots, p+2$;
$\widetilde{g_1}(x)$ is fuzzy posynomial function and $g_i(x)$ are posynomial functions for $i = 3, 4, \ldots, p+2$;
$g_2(x)$ is posynomial function and positive for all x in the feasible region.

3.2 Solution Concept

In this subsection, a solution method for the FFPGP problem involving positive trapezoidal fuzzy coefficient in objective function is presented.

At first based on Eq. (3), optimization problem (7) can be written as the following equivalent formulation:

$$\min \quad \frac{(\sum_{k=1}^{J_1} a_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}},\ \sum_{k=1}^{J_1} b_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}},\ \sum_{k=1}^{J_1} c_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}},\ \sum_{k=1}^{J_1} d_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}})}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

subject to satisfying:

$$\begin{aligned} & g_i(x) = \sum_{k=1}^{J_i} c_{ik} \prod_{l=1}^{m} x_l^{\gamma_{ikl}} \leq 1, \quad i = 3, 4, \ldots, p+2, \\ & x_l > 0, \quad l = 1, 2, \ldots, m. \end{aligned} \tag{8}$$

Next, by using Eq. (2) rearrange the problem (8) to obtain the following equivalent optimization problem:

$$\min \quad \left(\frac{\sum_{k=1}^{J_1} a_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}, \frac{\sum_{k=1}^{J_1} b_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}, \frac{\sum_{k=1}^{J_1} c_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}, \frac{\sum_{k=1}^{J_1} d_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}})}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}\right) \tag{9}$$

subject to satisfying:

$$\begin{aligned} & g_i(x) = \sum_{k=1}^{J_i} c_{ik} \prod_{l=1}^{m} x_l^{\gamma_{ikl}} \leq 1, \quad i = 3, 4, \ldots, p+2, \\ & x_l > 0, \quad l = 1, 2, \ldots, m. \end{aligned}$$

Now, let us define $\frac{g_1^1(x)}{g_2(x)} = \frac{\sum_{k=1}^{J_1} a_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$, $\frac{g_1^2(x)}{g_2(x)} = \frac{\sum_{k=1}^{J_1} b_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$, $\frac{g_1^3(x)}{g_2(x)} = \frac{\sum_{k=1}^{J_1} c_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$, $\frac{g_1^4(x)}{g_2(x)} = \frac{\sum_{k=1}^{J_1} d_{1k} \prod_{l=1}^{m} x_l^{\gamma_{1kl}})}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$, then the the problem (9) can be understood as the following multi-objective optimization problem:

$$\min \quad (\frac{g_1^1(x)}{g_2(x)}, \frac{g_1^2(x)}{g_2(x)}, \frac{g_1^3(x)}{g_2(x)}, \frac{g_1^4(x)}{g_2(x)})$$

subject to satisfying:

$$\begin{aligned} & g_i(x) = \sum\nolimits_{k=1}^{J_i} c_{ik} \prod\nolimits_{l=1}^{m} x_l^{\gamma_{ikl}} \leq 1, \quad i = 3, 4, \ldots, p+2, \\ & x_l > 0, \quad l = 1, 2, \ldots, m. \end{aligned} \tag{10}$$

In this problem the constraints $g_i(x)$ for all $i = 3, 4, \ldots, p+2$, are posynomial inequalities and constraints $x_l > 0$ for all $l = 1, 2, \ldots, m$, are monomial inequalities. They are all allowable expressions required in MOPGP problem. However, the objective functions are not allowable in MOPGP problem. To deal with this difficulty, we propose to approximate the denominator posynomials of objective functions with monomial functions, but leave their numerator as posynomial functions. The required monomial approximation can be computed using the following arithmetic-geometric mean approximation.

Consider a posynomial function $g(x) = \sum_{k=1}^{J} u_k(x)$ with $u_k(x)$ being the monomial terms. Then we have the following expression by the arithmetic-geometric mean inequality:

$$g(x) \geq \widehat{g}(x) = \prod\nolimits_{k=1}^{J} (\frac{u_k(x)}{\alpha_k(y)})^{\alpha_k(y)}, \tag{11}$$

where the parameters $\alpha_k(y)$ can be obtained by computing:

$$\alpha_k(y) = \frac{u_k(y)}{g(y)}, \quad \forall k, \tag{12}$$

where y is a fixed point with $y > 0$. It can be easily verified that $\widehat{g}(x)$ is the best local monomial approximation of $g(x)$ near y [3]. Then an objective function on a ratio of two posynomials $\min \frac{f(x)}{g(x)}$ can be approximated with $\min \frac{f(x)}{\widehat{g}(x)}$ and $\min \frac{f(x)}{g(x)} \leq \min \frac{f(x)}{\widehat{g}(x)}$ holds. Applying the presented monomial approximation method to the posynomial function $g_2(x)$, we get the following optimization problem:

$$\min \quad (\frac{g_1^1(x)}{\widehat{g}_2(x)}, \frac{g_1^2(x)}{\widehat{g}_2(x)}, \frac{g_1^3(x)}{\widehat{g}_2(x)}, \frac{g_1^4(x)}{\widehat{g}_2(x)}) \tag{13}$$

subject to satisfying:

$$\begin{aligned} & g_i(x) = \sum\nolimits_{k=1}^{J_i} c_{ik} \prod\nolimits_{l=1}^{m} x_l^{\gamma_{ikl}} \leq 1, \quad i = 3, 4, \ldots, p+2, \\ & x_l > 0, \quad l = 1, 2, \ldots, m, \end{aligned}$$

where $\widehat{g}_2(x)$ is the corresponding monomial function approximated by using Eqs. (11) and (12). Since posynomial function can be divided by monomial function and the result also is a posynomial function (based on Definition 2.4), so optimization problem (13) is a MOPGP problem that can be solved by using the following fuzzy programming method. It is assumed that the present problem (13) is feasible and has optimal solution.

3.3 Fuzzy Programming Method

Fuzzy programming problem due to Zimmermann [22] based on the concept given by Bellman and Zadeh [1] has been successfully applied to solve various types of multi-objective decision making problems. The following steps [5, 17] are used in solving a multi-objective optimization problem with a linear membership function by geometric programming technique to find an optimal compromise solution.

Step 1: Solve the MOPGP as a single objective posynomial geometric programming problem using only one objective at a time and ignoring the others by using geometric programming algorithm [4]. Repeat the process n times for n different objective functions.

Let $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ be the respective optimum solutions for n different posynomial geometric programming problems.

Step 2: Evaluate all these n objective functions over all n optimal solutions $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$.

Step 3: For each objective function, we find:

$$l_t = \min(g_0^{(t)}(x^{(1)}), g_0^{(t)}(x^{(2)}), \ldots, g_0^{(t)}(x^{(n)})), \tag{14}$$

$$u_t = \max(g_0^{(t)}(x^{(1)}), g_0^{(t)}(x^{(2)}), \ldots, g_0^{(t)}(x^{(n)})), \tag{15}$$

such that:

$$l_t \le g_0^{(t)}(x) \le u_t, \quad t = 1, 2, \ldots, n. \tag{16}$$

Step 4: Define a fuzzy membership function $\mu_t(x)$ for tth objective function $g_0^{(t)}(x), t = 1, 2, \ldots, n$, as:

$$\mu_t(g_0^{(t)}(x)) = \begin{cases} 1, g_0^{(t)}(x) \le l_t, \\ \frac{u_t - g_0^{(t)}(x)}{u_t - l_t}, l_t \le g_0^{(t)}(x) \le u_t, \\ 0, g_0^{(t)}(x) \ge u_t. \end{cases} \tag{17}$$

Step 5: Use the max-min operator [2] and formulate a crisp model as:

$$\max \quad x_{m+1} \; or \; \min \quad x_{m+1}^{-1} \tag{18}$$

subject to satisfying:

$$x_{m+1} \leq \frac{u_t - g_0^{(t)}(x)}{u_t - l_t}, \quad t = 1, 2, \ldots, n,$$
$$g_i(x) \leq 1, \quad (i = 1, 2, \ldots, p),$$
$$x_1, x_2, \ldots, x_{m+1} > 0.$$

Step 6: Solve the crisp posynomial geometric programming problem defined in Step 5 by using geometric programming algorithm to find optimal solution x^* and evaluate all the n number of objective functions at this optimal solution x^*.

3.4 Solution Algorithm

We now summarize the proposed approach for solving the FFPGP with positive trapezoidal fuzzy coefficients in this work and construct a solution algorithm.

The basic steps of the algorithm are given below:

Step 1: Convert the FFPGP problem into problem (9) with the help of arithmetic operations on fuzzy numbers (Eqs. (2) and (3)).

Step 2: Evaluate the monomial terms in the denominator posynomials of objective functions with the given y. Compute their corresponding parameter $\alpha_k(y)$ by using Eq. (12).

Step 3: Perform the condensation on the denominator posynomials of objective functions using Eq. (11) with parameter $\alpha_k(y)$.

Step 4: Change the problem to MOPGP problem (based on Definition 2.4).

Step 5: Solve the MOPGP problem by using fuzzy programming method.

4 Numerical Example

In this section, a numerical example is given to illustrate the approximation technique for solving the FFPGP with positive trapezoidal fuzzy coefficients in objective function.

Example 4.1 Consider the following FFPGP:

$$\min \quad \frac{(0.4, 0.5, 0.7, 0.8)x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + (2, 2.8, 6, 7)x_1^{-1}x_4x_6^{-1} + (1, 5, 6, 9)}{2x_3^{-0.71}x_6^{-1} + 3x_4x_5^{-2}x_7}$$

subject to satisfying:

$$\begin{aligned}
&0.01x_2x_3^{-1} + x_6^{-1}x_7 + 0.0005x_1x_2^{-1}x_5^{-1} \le 1,\\
&4x_3x_5^{-1} + 2x_3^{-0.71}x_5^{-1} + 0.0588x_3^{-1.3}x_7 \le 1,\\
&0.1 \le x_i \le 10, \quad i = 1, 2, \dots, 7.
\end{aligned}$$

According to Eqs. (2) and (3) in Sect. 2, Example can be transformed into the following form:

$$\begin{aligned}
\min \quad &(\frac{0.4x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 2x_1^{-1}x_4x_6^{-1} + 1}{2x_3^{-0.71}x_6^{-1} + 3x_4x_5^{-2}x_7}, \frac{0.5x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 2.8x_1^{-1}x_4x_6^{-1} + 5}{2x_3^{-0.71}x_6^{-1} + 3x_4x_5^{-2}x_7},\\
&\frac{0.7x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 6x_1^{-1}x_4x_6^{-1} + 6}{2x_3^{-0.71}x_6^{-1} + 3x_4x_5^{-2}x_7}, \frac{0.8x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 7x_1^{-1}x_4x_6^{-1} + 9}{2x_3^{-0.71}x_6^{-1} + 3x_4x_5^{-2}x_7})
\end{aligned}$$

subject to satisfying:

$$\begin{aligned}
&0.01x_2x_3^{-1} + x_6^{-1}x_7 + 0.0005x_1x_2^{-1}x_5^{-1} \le 1,\\
&4x_3x_5^{-1} + 2x_3^{-0.71}x_5^{-1} + 0.0588x_3^{-1.3}x_7 \le 1,\\
&0.1 \le x_i \le 10, \quad i = 1, 2, \dots, 7.
\end{aligned}$$

Now, we have the following approximation of this problem by using Eqs. (11) and (12) in previous section:

$$\begin{aligned}
\min \quad &(\frac{0.4x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 2x_1^{-1}x_4x_6^{-1} + 1}{\widehat{g}(x)}, \frac{0.5x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 2.8x_1^{-1}x_4x_6^{-1} + 5}{\widehat{g}(x)},\\
&\frac{0.7x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 6x_1^{-1}x_4x_6^{-1} + 6}{\widehat{g}(x)}, \frac{0.8x_1^{0.67}x_2x_3^{-1.3}x_7^{-0.67} + 7x_1^{-1}x_4x_6^{-1} + 9}{\widehat{g}(x)})
\end{aligned}$$

subject to satisfying:

$$\begin{aligned}
&0.01x_2x_3^{-1} + x_6^{-1}x_7 + 0.0005x_1x_2^{-1}x_5^{-1} \le 1,\\
&4x_3x_5^{-1} + 2x_3^{-0.71}x_5^{-1} + 0.0588x_3^{-1.3}x_7 \le 1,\\
&0.1 \le x_i \le 10, \quad i = 1, 2, \dots, 7,
\end{aligned}$$

where $\widehat{g}(x)$ is the corresponding monomial approximation of posynomial function $2x_3^{-0.71}x_6^{-1} + 3x_4x_5^{-2}x_7$. It has the following formulation by using Eq. (11):

$$\widehat{g}(x) = (\frac{2x_3^{-0.71}x_6^{-1}}{\alpha_1})^{\alpha_1}(\frac{3x_4x_5^{-2}x_7}{\alpha_2})^{\alpha_2},$$

where α_1 and α_2 can be computed by using Eq. (12) as follows:

$$\alpha_1 = \frac{2y_3^{-0.71}y_6^{-1}}{2y_3^{-0.71}y_6^{-1} + 3y_4y_5^{-2}y_7},$$
$$\alpha_2 = \frac{3y_4y_5^{-2}y_7}{2y_3^{-0.71}y_6^{-1} + 3y_4y_5^{-2}y_7}.$$

The following algorithm parameters were set in the implementation of the proposed method: $y_1 = 1, y_2 = 1, y_3 = 1, y_4 = 1, y_5 = 10, y_6 = 10, y_7 = 1$ and $\varepsilon = 10^{(-7)}$. By using the steps of fuzzy programming method, a crisp model is formulated as:

max θ

subject to satisfying:

$$\begin{aligned}
&0.5471592x_1^{0.67}x_2x_3^{-0.6826087}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.8004348}\\
&\quad + 2.7357966x_1^{-1}x_3^{0.6173913}x_4^{0.8695652}x_5^{0.2608696}x_6^{-0.1304348}x_7^{-0.1304348}\\
&\quad + 1.3678979x_3^{0.6173913}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.1304348}\\
&\quad + 0.1221097\theta \le 1,\\
&0.3148304x_1^{0.67}x_2x_3^{-0.6826087}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.8004348}\\
&\quad + 1.7630505x_1^{-1}x_3^{0.6173913}x_4^{0.8695652}x_5^{0.2608696}x_6^{-0.1304348}x_7^{-0.1304348}\\
&\quad + 3.1483044x_3^{0.6173913}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.1304348}\\
&\quad + 0.1221097\theta \le 1,\\
&0.3175095x_1^{0.67}x_2x_3^{-0.6826087}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.8004348}\\
&\quad + 2.7215098x_1^{-1}x_3^{0.6173913}x_4^{0.8695652}x_5^{0.2608696}x_6^{-0.1304348}x_7^{-0.1304348}\\
&\quad + 2.7215098x_3^{0.6173913}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.1304348}\\
&\quad + 0.1221097\theta \le 1,\\
&0.2697063x_1^{0.67}x_2x_3^{-0.6826087}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.8004348}\\
&\quad + 2.3599308x_1^{-1}x_3^{0.6173913}x_4^{0.8695652}x_5^{0.2608696}x_6^{-0.1304348}x_7^{-0.1304348}\\
&\quad + 3.0341968x_3^{0.6173913}x_4^{-0.1304348}x_5^{0.2608696}x_6^{0.8695652}x_7^{-0.1304348}\\
&\quad + 0.1221097\theta \le 1,\\
&0.01x_2x_3^{-1} + x_6^{-1}x_7 + 0.0005x_1x_2^{-1}x_5^{-1} \le 1,\\
&4x_3x_5^{-1} + 2x_3^{-0.71}x_5^{-1} + 0.0588x_3^{-1.3}x_7 \le 1,\\
&0.1 \le x_i \le 10, \quad i = 1, 2, \ldots, 7,\\
&0 < \theta.
\end{aligned}$$

The optimal compromise solution is obtained as: $\theta = 0.7569894$, $x_1 = 1.795215$, $x_2 = 0.1000000$, $x_3 = 0.2152608$, $x_4 = 0.1000000$, $x_5 = 7.120881$, $x_6 = 0.1005941$, $x_7 = 0.1000000$. So we can obtain the objective value (0.2136294, 0.4687867, 0.6641215, 0.8734667).

5 Conclusion

This paper has dealt with a fuzzified version of a fractional posynomial geometric programming problem in which fuzzy parameters are involved in the objective function. The algorithm presented here proposed a technique for changing the FFPGP problem to MOPGP problem and then solving the MOPGP problem. Based on the obtained results in the last section, we conclude that using the proposed solution algorithm is useful to solve a FFPGP problem.

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References

1. Bellman, R.E., Zadeh, L.A.: Decision making in fuzzy environment. Manag. Sci. **17B**, 141–164 (1970)
2. Boyd, S., Patil, D., Horowitz, M.: Digital circuit sizing via geometric programming. Oper. Res. **53**(6), 899–932 (2005)
3. Boyd, S., Kim, S.-J., Vandenberghe, L., Hassibi, A.: A tutorial on geometric programming. Optim. Eng. **8**(1), 67–127 (2007)
4. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge (2004)
5. Cao, B.-Y.: Fuzzy Geometric Programming. Kluwer Academic Publishers, Dordrecht (2001)
6. Chang, C.T.: On the posynomial fractional programming problems. Eur. J. Oper. Res. **143**(1), 42–52 (2002)
7. Chang, C.T.: A goal programming approach for fuzzy multiobjective fractional programming problems. Int. J. Syst. Sci. **40**(8), 867–874 (2009)
8. Charnes, A., Cooper, W.W.: Programming with linear fractional functionals. Naval Res. Logistics Quart. **9**, 181–186 (1962)
9. Dinkelbach, W.: On nonlinear fractional programming. Manag. Sci. **13**(7), 492–497 (1967)
10. Dubois, D., Prade, H.: Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York (1980)
11. Hladik, M.: Generalized linear fractional programming under interval uncertainty. Eur. J. Oper. Res. **205**, 42–46 (2010)
12. Jagannathan, R.: On some properties of programming problems in parametric form pertaining to fractional programming. Manage. Sci. **12**, 609–615 (1966)
13. Kauffmann, A., Gupta, M.M.: Introduction to Fuzzy Arithmetic: Theory and Applications. Van Nostrand Reinhold, New York (1991)
14. Kaufmann, A., Gupta, M.M.: Fuzzy Mathemaical Models in Engineering and Managment Science. Elseiver Science Publishers, Amsterdam (1988)
15. Mehra, A., Chandra, S., Bector, C.R.: Acceptable optimality in linear fractional programming with fuzzy coefficients. Fuzzy Opt. Dec. Making **6**, 5–16 (2007)
16. Neumann, J.V.: Uber Ein Es Gleichungs System and Eine Verallgemeinerung Des Brouwerschen Fixpuntsatzes. Ergebnisse Eines Mathematicschen **8**, 245–267 (1937)
17. Ojha, A.K., Biswal, K.K.: Multiobjective geometric programming problem with ϵ-constraint method. Appl. Math. Model. **29** (2013). doi:10.1016/j.apm.2013.07.003
18. Saad, O.M.: On solving single-objective fuzzy integer linear fractional programs. Appl. Math. Inf. Sci. **4**(3), 447–457 (2010)

19. Saad, O.M., Biltagy, M.Sh., Farag, T.B.: An algorithm for multiobjective integer nonlinear fractional programming under fuzziness. Ann. Fuzzy Math. Inf. **1**, 207–220 (2011)
20. Sakawa, M., Nishizaki, I., Uemura, Y.: Interactive fuzzy programming for two level linear and linear fractional production and assignment problems: a case study. Eur. J. Oper. Res. **135**, 142–157 (2001)
21. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
22. Zimmermann, H.J.: Fuzzy Set Theory and its Applications, 3rd edn. Kluwer Academic, Norwell (1996)

Generalized Fuzzy Imaginary Ideals of Complemented Semirings

Zu-hua Liao, Chan Zhu, Xiao-tang Luo, Shuang Hu and Wei-long Liu

Abstract Our aim in this paper is to introduce and study the new type of fuzzy ideals of a complemented semiring called generalized fuzzy imaginary right (resp.left) ideals and the direct products of them. The equivalence relation of them is given, besides, the fundamental properties of their intersection, union and level sets are discussed. Finally, we also investigated the properties of their homomorphic preimage.

Keywords $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary ideal generalized fuzzy imaginary ideal homomorphic preimage

1 Introduction

In 1965, Zadeh [1] introduced the concept of a fuzzy subset and studied its properties on the lines parallel to set theory. In 1971, Rosefeld [2] defined the notion of a fuzzy subgroup and gave some of its properties. Rosenfelds definition of a fuzzy group is a turning point for pure mathematicians. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector spaces and so on. In 1982, Liu [3] defined and studied the theory of fuzzy ideals in rings. In 1990, Ma [4] gave the definition of imaginary ideals of rings and obtained some results. Then in 1992, he discussed the relation between the imaginary ideals, pseudo ideals and ideals of rings [5]. Besides, some relevant examples were also given by him.

In 1992, Induab scholar Bhakat and Das [6] introduced the concept of (α, β)—fuzzy subgroup by using belong to $(\in)$ relation and quasi-coincident with (q) relation between the fuzzy point and the fuzzy sets. In fact, its an important and useful general-

Z. Liao (✉) · C. Zhu · X. Luo · S. Hu · W. Liu
School of Science, Jiangnan University, Wuxi 214122, China
e-mail: liaozuhua57@163.com

Z. Liao · C. Zhu · X. Luo · S. Hu
Institute of Intelligent Systems and Network Computing, Engineering of Internet of Things, Wuxi 214122, China

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ized of Rosenfelds fuzzy subgroup. In 2006, Liao etc. generalized "quasi-coinciednt with" relation (q) between a fuzzy point and a fuzzy set of Liu to "generalized quasi-coincident with" relation ($q_{(\lambda,\mu)}$) between a fuzzy point and a fuzzy set, and extended Rosenfeld's ($\in, \in$)—fuzzy algebra, Bhakat and Das's ($\in, \in \vee q$)—fuzzy algebra and ($\overline{\in}, \overline{\in} \vee \overline{q}$)—fuzzy algebra to ($\in, \in \vee q_{(\lambda,\mu)}$)—fuzzy algebra [7] with more abundant hierarchy [8].

Vandiver [9] in 1939 put forward the concept of semiring. The applications of semirings to areas such as optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, formal language theory, coding theory and analysis of computer programs have been extensively studied in the literature [10, 11]. Liu [12] introduced fuzzy ideals in a ring. Following this definition, Mukherjee and Sen [13, 14] obtained many interesting results in the theory of rings.

On the study of fuzzy semiring, there has been a large number of research at home and abroad. Then Feng and Zhan [15] proposed complemented semirings. They put the Boolean algebra as its proper class, and studied the algebraic structure of it. This paper is the continuation of the above work. Section 2 of this paper list some necessary preliminaries that support our results. Section 3 is the kernel of the whole paper, which display main results obtained by the authors, including the relationships among genenralized fuzzy imaginary ideal, ($\in, \in \vee q_{(\lambda,\mu)}$)—fuzzy imaginary ideal and level subsets of a fuzzy set and relative properties about the intersection, union and homomorphic preimage of such genenralized fuzzy imaginary ideal. In Sect. 4, we make a conclusion of the article.

2 Preliminaries

In this section we recall some basic notions and results which will be needed in the sequel.

Definition 2.1 ([16]) A semiring S is a structure consisting of a nonempty set S together with two binary operations on S called addition and multiplition (denoted in the usual manner) such that

(1) S together with addition is a semigroup. o is additive identity element;
(2) S together with multiplition is a semigroup. 1 is multiplitive identity element;
(3) $a(b + c) = ab + ac$, $(a + b)c = ac + bc$, $\forall a, b, c \in S$;
(4) $o \cdot a = a \cdot o = o$;

Definition 2.2 ([15]) Assume a is an element of semiring S, if there exists a complement $\overline{a}$ which makes $a\overline{a} = o$, $a + \overline{a} = 1$, a is called complemented. S is said to be a complemented semiring if every element of S has a complement.

Definition 2.3 ([17]) Let S be a semiring. A nonempty subset A of S is said to be a complemented subsemiring of S if A is closed under three binary operations on S:

(1) If $a, b \in A$, then $a + b \in A$;
(2) If $a, b \in A$, then $ab \in A$;
(3) If $a \in A$, then $\overline{a} \in A$.

Definition 2.4 ([18]) Let $\alpha, \lambda, \mu \in [0, 1]$ and $\lambda < \mu$. If $A(x) \geq \alpha$, then a fuzzy point x_α is said to belongs to a fuzzy subset A written $x_\alpha \in A$; if $\lambda < \alpha$ and $A(x)+\alpha > 2\mu$, then a fuzzy point x_α is called to be generalized quasi-coincident with a fuzzy subset A, denoted by $x_\alpha q_{(\lambda,\mu)} A$. If $x_\alpha \in A$ or $x_\alpha q_{(\lambda,\mu)} A$, then $x_\alpha \in \vee q_{(\lambda,\mu)} A$.

Definition 2.5 ([17]) Let $\alpha, \lambda, \mu \in [0, 1]$ and $\lambda < \mu$. A fuzzy subset A of S is called an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy complemented subsemiring of S, if $\forall t, r \in (\lambda, 1]$, $a, b \in S$, satisfy:

(1) If $a_t, b_r \in A$, we have $(a + b)_{t \wedge r} \in \vee q_{(\lambda,\mu)} A$;
(2) If $a_t, b_r \in A$, there exist $(ab)_{t \wedge r} \in \vee q_{(\lambda,\mu)} A$;
(3) If $a_t \in A$, $\overline{a_t} \in \vee q_{(\lambda,\mu)} A$ holds.

Definition 2.6 ([17]) Let $\alpha, \lambda, \mu \in [0, 1]$ and $\lambda < \mu$, A is a fuzzy set of S. We call A a generalized fuzzy complemented subsemiring of S if $\forall a, b \in S$ satisfy:

(1) $A(a + b) \vee \lambda \geq A(a) \wedge A(b) \wedge \mu$;
(2) $A(ab) \vee \lambda \geq A(a) \wedge A(b) \wedge \mu$;
(3) $A(\overline{a}) \vee \lambda \geq A(a) \wedge \mu$.

Definition 2.7 Let $S_i (1 \leq i \leq n)$ be complemented semirings and direct product: $\prod_{1 \leq i \leq n} S_i = \{(a_1, a_2, \dots a_n) | a_i \in S_i\}$. Then $\prod_{1 \leq i \leq n} S_i$ is a complemented semiring under the operations as following:

$(a_1, a_2, \dots a_n) + (b_1, b_2, \dots b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$;
$(a_1, a_2, \dots a_n)(b_1, b_2, \dots b_n) = (a_1 b_1, a_2 b_2, \dots a_n b_n)$;
$\overline{(a_1, a_2, \dots a_n)} = (\overline{a_1}, \overline{a_2}, \dots \overline{a_n})$.

Definition 2.8 Let $A_i (1 \leq i \leq n)$ be fuzzy subsets of S_i. Then a fuzzy set $\prod_{1 \leq i \leq n} A_i$ defined as $(\prod_{1 \leq i \leq n} A_i)(x_1, x_2, \dots x_n) = \inf_{1 \leq i \leq n} A_i(x_i)$ is called hboxfuzzy direct product.

Theorem 2.1 ([17]) *Let A be a fuzzy subset of S. Then A is a generalized fuzzy complemented subsemiring of S if and only if A is an* $(\in, \in \vee q_{(\lambda,\mu)})$*—fuzzy complemented subsemiring of S.*

Theorem 2.2 ([12]) *Let A be a fuzzy subset of S. Then A is a generalized fuzzy complemented subsemiring of S if and only if* $\forall \alpha \in (\lambda, \mu]$, *nonempty* A_α *is a subsemiring of S.*

Theorem 2.3 ([12]) *Let A and B be generalized fuzzy complemented subsemirings of S. Then* $A \cap B$ *is a generalized fuzzy complemented subsemiring of S.*

Theorem 2.4 ([17]) *A subset A of S is a complemented subsemiring of S if and only if* χ_A *is a generalized fuzzy complemented subsemiring of S.*

Theorem 2.5 ([17]) *Let* $f : S \to H$ *be a homomorphism. If B is a generalized fuzzy complemented subsemiring of H, then* $f^{-1}(B)$ *is a generalized fuzzy complemented subsemiring of S.*

3 $(\in, \in \vee q_{(\lambda,\mu)})$—Fuzzy Imaginary Ideals of Complemented Semiring

Definition 3.1 Let A be a complemented subsemiring of S. Then A is said to be a imaginary right (resp.left) ideal of S if for all $x \in A$, $y \in S$ and $xy \in A$(resp. $yx \in A$) imply $y \in A$.

Definition 3.2 Let A be a generalized fuzzy complemented subsemiring of S. Then A is called a generalized fuzzy imaginary right (resp.left) ideal of S if for all $x, y \in S$, $A(y) \vee \lambda \geq A(x) \wedge A(xy) \wedge \mu(A(y) \vee \lambda \geq A(x) \wedge A(yx) \wedge \mu)$.

Definition 3.3 Let A be an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy complemented subsemiring of S, $\alpha \in (\lambda, 1]$. Then A is called an $(\in, \in \vee q_{(\lambda,\mu)})$— fuzzy imaginary right (resp.left) ideal of S if $\forall x_\alpha \in A$, $y \in S$ and $(xy)_\alpha \in A$(resp.$(yx)_\alpha \in A$) implies $y_\alpha \in \vee q_{(\lambda,\mu)} A$.
We say A an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary ideal of S if A is both an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary right deal of S and an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary left ideal of S.

Theorem 3.1 *Let A be a fuzzy subset of S. Then the following conditions are equiavalent:*

(1) *A is an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary right (resp. left) ideal of S;*
(2) *A is a generalized fuzzy imaginary right (resp. left) ideal of S;*
(3) *nonempty set A_α is a imaginary right (resp. left) ideal of S for all $\alpha \in (\lambda, \mu]$.*

Proof (1) $\Rightarrow$ (2)
We obtain that A is a generalized fuzzy complemented subsemiring of S based on Theorem 2.1. Assume that $\exists x_0, y_0 \in S$ such that $A(y_0) \vee \lambda < A(x_0) \wedge A(x_0 y_0) \wedge \mu$. Choose α such that $A(y_0) \vee \lambda < \alpha < A(x_0) \wedge A(x_0 y_0) \wedge \mu$, then $A(y_0) < \alpha$, $A(x_0) > \alpha$, $A(x_0 y_0) > \alpha$ and $\lambda < \alpha < \mu$, so $(x_0)_\alpha \in A$ and $(x_0 y_0)_\alpha \in A$, thus $(y_0)_\alpha \in \vee q_{(\lambda,\mu)} A$ based on Definition 3.3. But $A(y_0) < \alpha$, thus $A(y_0) + \alpha < \alpha + \alpha \leq 2\mu$, a contradiction. So A is a generalized fuzzy imaginary right ideal of S.
(2) $\Rightarrow$ (1)
From Theorem 2.1, we know that A is an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy subsemiring of S. For all $x, y \in S$ and $\alpha \in (\lambda, 1]$, if $x_\alpha \in A$ and $(xy)_\alpha \in A$, then $A(x) \geq \alpha$ and $A(xy) \geq \alpha$. Since A is a generalized fuzzy imaginary right ideal of S, then $A(y) \vee \lambda \geq A(x) \wedge A(xy) \wedge \mu \geq \alpha \wedge \mu$. If $\alpha > \mu$, then $A(y) \vee \lambda \geq \mu$, by $\lambda < \mu$, so $A(y) \geq \mu$, then $A(y) + \alpha \geq \mu + \alpha > 2\mu$, i.e., $y_\alpha q_{(\lambda,\mu)} A$. On the other hand, if $\alpha \leq \mu$, then we can obtain that $A(y) \geq \alpha$, i.e., $y_\alpha \in A$, so $y_\alpha \in \vee q_{(\lambda,\mu)} A$.
Thus A is an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary right ideal of S.
(2) $\Rightarrow$ (3)
We know that A_α is a subsemiring of S based on Theorem 2.2. For all $x \in A_\alpha$ and $y \in S$, if $xy \in A_\alpha$, then $A(x) \geq \alpha$ and $A(xy) \geq \alpha$. Since A is a generalized fuzzy imaginary right ideal of S, then for all $\alpha \in (\lambda, \mu]$, we have $A(y) \vee \lambda \geq$

$A(x) \wedge A(xy) \wedge \mu \geq \alpha \wedge \mu = \alpha$, by $\lambda < \alpha$, so $A(y) \geq \alpha$, i.e., $y \in A_\alpha$. Therefore A_α is a imaginary right ideal of S for all $\alpha \in (\lambda, \mu]$.

(3) $\Rightarrow$ (2)

We obtain that A is a generalized fuzzy complemented subsemiring of S based on Theorem 2.2. Assume that there exists $x_0, y_0 \in S$ such that $A(y_0) \vee \lambda < A(x_0) \wedge A(x_0 y_0) \wedge \mu$. Choose α such that $A(y_0) \vee \lambda < \alpha < A(x_0) \wedge A(x_0 y_0) \wedge \mu$, then $A(y_0) < \alpha$, $A(x_0) > \alpha$, $A(x_0 y_0) > \alpha$ and $\lambda < \alpha < \mu$, i.e., $x_0 \in A_\alpha$ and $x_0 y_0 \in A_\alpha$. Since A_α is an imaginary right ideal of S, then $y_0 \in A_\alpha$, i.e., $A(y_0) \geq \alpha$, a contradiction to $A(y_0) < \alpha$. Therefore A is a generalized fuzzy imaginary right ideal of S.

Similarly, we can prove the other results.

The above theorems show that generalized fuzzy imaginary right (resp. left) ideal and $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary right (resp.left) ideal are equivalent. Thus we can prove a normal fuzzy set be an $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary right (resp.left) ideal by proving it to be a generalized fuzzy imaginary right (resp.left) ideal, which is easer than the former. Meanwhile, Theorem 3.1 establishes a kind of link between generalized fuzzy imaginary right (resp.left) ideal and ordinary imaginary right (resp.left) ideal.

Next we prove the theorem for imaginary right ideal. Analogous arguments apply to imaginary left ideal.

Theorem 3.2 *Let A and B be generalized fuzzy imaginary right (resp. left) ideal of S. Then $A \cap B$ is a generalized fuzzy imaginary right (resp.left) ideal of S.*

Proof We obtain that $A \cap B$ is a generalized fuzzy complemented subsemiring of S based on Theorem 2.3, for all $x, y \in S$, we have

$(A \cap B)(y) \vee \lambda = (A(y) \wedge B(y)) \vee \lambda = (A(y) \vee \lambda) \vee (B(y) \vee \lambda) \geq (A(x) \wedge A(xy) \wedge \mu) \wedge (B(x) \wedge B(xy) \wedge \mu) = (A \cap B)(x) \wedge (A \cap B)(xy) \wedge \mu$.

Therefore $A \cap B$ is a generalized fuzzy imaginary right ideal of S.

Corollary 3.1 *Let $A_i (i \in I)$ be generalized fuzzy imaginary right (resp.left) ideals of S. Then $\cap_{i \in I} A_i$ is a generalized fuzzy imaginary right (resp.left) ideal of S.*

Theorem 3.3 *Let $A_i (i \in I)$ be generalized fuzzy imaginary right (resp. left) ideals of S, and $\forall i, j \in I$, $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Then $\cup_{i \in I} A_i$ is a generalized fuzzy imaginary right (resp.left) ideals of S.*

Proof Firstly, we prove $\vee_{i \in I}(A_i(x) \wedge A_i(y) \wedge \mu) = (\cup_{i \in I} A_i)(x) \wedge (\cup_{i \in I} A_i)(y) \wedge \mu$. Obviously, $\vee_{i \in I}(A_i(x) \wedge A_i(y) \wedge \mu) \leq (\cup_{i \in I} A_i)(x) \wedge (\cup_{i \in I} A_i)(y) \wedge \mu$. Assume that $\vee_{i \in I}(A_i(x) \wedge A_i(y) \wedge \mu) \neq (\cup_{i \in I} A_i)(x) \wedge (\cup_{i \in I} A_i)(y) \wedge \mu$, then there exists r such that $\vee_{i \in I}(A_i(x) \wedge A_i(y) \wedge \mu) < r < (\cup_{i \in I} A_i)(x) \wedge (\cup_{i \in I} A_i)(y) \wedge \mu$. Since $A_i \subseteq A_j$ or $A_j \subseteq A_i$ for all $i, j \in I$, then exists $k \in I$ such that $r < A_k(x) \wedge A_k(y) \wedge \mu$. On the other hand $A_i(x) \wedge A_i(y) \wedge \mu < r$ for all $i \in I$, a contradiction. Thus $\vee_{i \in I}(A_i(x) \wedge A_i(y) \wedge \mu) = (\cup_{i \in I} A_i)(x) \wedge (\cup_{i \in I} A_i)(y) \wedge \mu$.

Next, for all $x, y \in S$, we have (1) $(\cup_{i\in I} A_i)(x+y) \vee \lambda = \vee_{i\in I} A_i(x+y) \vee \lambda = \vee_{i\in I}(A_i(x+y) \vee \lambda) \geq \vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu) = (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$;
Similarly, we can prove that (2) $(\cup_{i\in I} A_i)(\overline{x}) \vee \lambda = (\vee_{i\in I} A_i(\overline{x})) \vee \lambda = \vee_{i\in I}(A_i(\overline{x}) \vee \lambda) \geq \vee_{i\in I}(A_i(x) \wedge \mu) = (\cup_{i\in I} A_i)(x) \wedge \mu$;
(3) $(\cup_{i\in I} A_i)(xy) \vee \lambda = (\vee_{i\in I} A_i(xy)) \vee \lambda = \vee_{i\in I}(A_i(xy) \vee \lambda) \geq \vee_{i\in I}(A_i(x) \wedge A_i(y) \wedge \mu) = (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(y) \wedge \mu$;
(4) $(\cup_{i\in I} A_i)(y) \vee \lambda = (\vee_{i\in I} A_i(y)) \vee \lambda = \vee_{i\in I}(A_i(y) \vee \lambda) \geq \vee_{i\in I}(A_i(x) \wedge A_i(xy) \wedge \mu) = (\cup_{i\in I} A_i)(x) \wedge (\cup_{i\in I} A_i)(xy) \wedge \mu$.
Therefore $\cup_{i\in I} A_i$ is a generalized fuzzy imaginary right (resp.left) ideal of S.

Theorem 3.4 *Let A_1 and A_2 be generalized fuzzy imaginary right (resp.left) ideals of S_1 and S_2 respectively, then $A_1 \times A_2$ is a generalized fuzzy imaginary right (resp.left) ideal of $S_1 \times S_2$.*

Proof Firstly, we prove that $A_1 \times A_2$ is a generalized fuzzy subsemiring of $S_1 \times S_2$. For all $x, y \in S_1 \times S_2$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$, since A_1 and A_2 are generalized fuzzy subsemirings of S_1 and S_2 respectively, then $(A_1 \times A_2)(x+y) \vee \lambda = (A_1 \times A_2)((x_1, x_2) + (y_1, y_2)) \vee \lambda = (A_1 \times A_2)(x_1 + y_1, x_2 + y_2) \vee \lambda = (A_1(x_1 + y_1) \wedge A_2(x_2 + y_2)) \vee \lambda = (A_1(x_1 + y_1) \vee \lambda) \wedge (A_2(x_2 + y_2) \vee \lambda) \geq (A_1(x_1) \wedge A_1(y_1) \wedge \mu) \wedge (A_2(x_2) \wedge A_2(y_2) \wedge \mu) = (A_1 \times A_2)((x_1, x_2)) \wedge (A_1 \times A_2)((y_1, y_2)) \wedge \mu = (A_1 \times A_2)(x) \wedge (A_1 \times A_2)(y) \wedge \mu$.
Semilarly, we can prove that $(A_1 \times A_2)(\overline{x}) \vee \lambda \geq (A_1 \times A_2)(x) \wedge \mu$ and $(A_1 \times A_2)(xy) \vee \lambda \geq (A_1 \times A_2)(x) \wedge (A_1 \times A_2)(y) \wedge \mu$.
So $A_1 \times A_2$ is a generalized fuzzy subsemiring of $S_1 \times S_2$.
Next, for all $x, y \in S_1 \times S_2$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $(A_1 \times A_2)(y) \vee \lambda = (A_1 \times A_2)((y_1, y_2)) \vee \lambda = (A_1(y_1) \wedge A_2(y_2)) \vee \lambda = (A_1(y_1) \vee \lambda) \wedge (A_2(y_2) \vee \lambda) \geq (A_1(x_1) \wedge A_1(x_1 y_1) \wedge \mu) \wedge (A_2(x_2) \wedge A_2(x_2 y_2) \wedge \mu) = (A_1 \times A_2)((x_1, x_2)) \wedge (A_1 \times A_2)((x_1 y_1, x_2 y_2)) \wedge \mu = (A_1 \times A_2)((x_1, x_2)) \wedge (A_1 \times A_2)((x_1, x_2)(y_1, y_2)) \wedge \mu = (A_1 \times A_2)(x) \wedge (A_1 \times A_2)(xy) \wedge \mu$.
Thus $A_1 \times A_2$ is a generalized fuzzy imaginary right (resp.left) ideal of $S_1 \times S_2$.

Theorem 3.5 *Let A_i be generalized fuzzy imaginary right (resp.left) ideals of S_i, then $\prod_{1\leq i\leq n} A_i$ is a generalized fuzzy imaginary right (resp.left) ideal of $\prod_{1\leq i\leq n} S_i$.*

Proof Firstly, we prove that $\prod_{1\leq i\leq n} A_i$ is a generalized fuzzy sbusemiring of $\prod_{1\leq i\leq n} S_i$. For all $x, y \in \prod_{1\leq i\leq n} S_i$, where $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$, then $(\prod_{1\leq i\leq n} A_i)(x+y) \vee \lambda = (\inf_{1\leq i\leq n} A_i(x_i + y_i)) \vee \lambda = \inf_{1\leq i\leq n}(A_i(x_i + y_i) \vee \lambda) \geq \inf_{1\leq i\leq n}(A_i(x_i) \wedge A_i(y_i) \wedge \mu) = \inf A_i(x_i) \wedge \inf A_i(y_i) \wedge \mu = (\prod_{1\leq i\leq n} A_i)(x) \wedge (\prod_{1\leq i\leq n} A_i)(y) \wedge \mu$.
Similarly, $(\prod_{1\leq i\leq n} A_i)(xy) \vee \lambda \geq (\prod_{1\leq i\leq n} A_i)(x) \wedge (\prod_{1\leq i\leq n} A_i)(y) \wedge \mu$; $(\prod_{1\leq i\leq n} A_i)(\overline{x}) \vee \lambda = \inf_{1\leq i\leq n} A_i(\overline{x_i}) \vee \lambda = \inf_{1\leq i\leq n}(A_i(\overline{x_i}) \vee \lambda) \geq \inf_{1\leq i\leq n}(A_i(x_i) \wedge \mu) = \inf_{1\leq i\leq n} A_i(x_i) \wedge \mu = (\prod_{1\leq i\leq n} A_i)(x) \wedge \mu$.
Therefore $\prod_{1\leq i\leq n} A_i$ is a generalized fuzzy subsimiring of $\prod_{1\leq i\leq n} S_i$.
Next, For all $x, y \in \prod_{1\leq i\leq n} S_i$, where $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$, then $(\prod_{1\leq i\leq n} A_i)(y) \vee \lambda = \inf A_i(y_i) \vee \lambda = \inf(A_i(y_i) \vee \lambda) \geq \inf(A_i(x_i) \wedge$

$A_i(x_i y_i) \wedge \mu) = \inf A_i(x_i) \wedge \inf A_i(x_i y_i) \wedge \mu = (\prod_{1\leq i\leq n} A_i)(x) \wedge (\prod_{1\leq i\leq n} A_i)$ $(xy) \wedge \mu$.
Thus $\prod_{1\leq i\leq n} A_i$ is a generalized fuzzy imaginary right ideal of $\prod_{1\leq i\leq n} S_i$.

Theorem 3.6 *Let A be a subset of S. Then χ_A is a generalized fuzzy imaginary right (resp.left) ideal of S if and only if A is an imaginary right (resp.left) ideal of S.*

Proof Let χ_A be a generalized fuzzy imaginary right ideal of S. We know that A is a subsemiring of S based on Theorem 2.4. For all $x \in A$, $y \in S$ and $xy \in A$, then $\chi_A(x) = 1$ and $\chi_A(xy) = 1$. Since χ_A is a generalized fuzzy imaginary right ideal of S, so $\chi_A(y) \vee \lambda \geq \chi_A(x) \wedge \chi_A(xy) \wedge \mu = \mu$, by $\lambda < \mu$, so $\chi_A(y) \geq \mu\chi_A(y) = 1$, then $y \in A$. Thus A is a imaginary right ideal of S.

Conversely, we can easily obtain that χ_A is a generalized subsemiring of S based on Theorem 2.4. Assume that there exist $x_0, y_0 \in S$ satisfy $\chi_A(y_0) \vee \lambda < \chi_A(x_0) \wedge \chi_A$ $(x_0 y_0) \wedge \mu$. Choose α holds $\chi_A(y_0) \vee \lambda < \alpha < \chi_A(x_0) \wedge \chi_A(x_0 y_0) \wedge \mu$, then χ_A $(y_0) < \alpha, \chi_A(x_0) > \alpha, \chi_A(x_0 y_0) > \alpha$ and $\lambda < \alpha < \mu$. Thus $x_0 y_0 \in A$ and $x_0 \in A$. Since A is a imaginary right ideal of S, then $y_0 \in A$, so $\chi_A(y_0) = 1 > \alpha$, a contradiction.
Therefore χ_A is a generalized fuzzy imaginary right ideal of S.

Theorem 3.7 *Let A be a generalized fuzzy imaginary right (resp.left) ideal of S. Then $A_{\underline{\lambda}} = \{x | A(x) > \lambda\}$ is a imaginary right (resp.left) ideal of S.*

Proof Firstly, we prove that $A_{\underline{\lambda}}$ is a complemented subsemiring of S. Since A is a generalized fuzzy imaginary right ideal of S, then for all $x, y \in A_{\underline{\lambda}}$, $A(x+y) \vee \lambda \geq$ $A(x) \wedge A(y) \wedge \mu > \lambda$, $A(\overline{x}) \vee \lambda \geq A(x) \wedge \mu > \lambda$, and $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu >$ λ, i.e., $A(x+y) > \lambda$, $A(\overline{x}) > \lambda$ and $A(xy) > \lambda$, so $x + y, xy, \overline{x} \in A_{\underline{\lambda}}$.

Therefore $A_{\underline{\lambda}}$ is a complemented submiring of S.
Next, for all $x \in A_{\underline{\lambda}}$, $y \in S$ and $xy \in A_{\underline{\lambda}}$, then $A(x) > \lambda$ and $A(xy) > \lambda$. Since A is a generalized fuzzy imaginary right ideal of S, then $A(y) \vee \lambda \geq A(x) \wedge A(xy) \wedge \mu > \lambda$, so $A(y) > \lambda$, i.e., $y \in A_{\underline{\lambda}}$.
Therefore $A_{\underline{\lambda}}$ is a imaginary right ideal of S.

Theorem 3.8 *Let $f : S \to H$ be a homomorphism, if B is a gerneralized fuzzy imaginary right (resp.left) ideal of H, then $f^{-1}(B)$ is a generalized fuzzy imaginary right (resp.left) ideal of S.*

Proof Based on Theorem 2.5, we can obtain that $f^{-1}(B)$ is a gerneralized fuzzy complemented subsemiring of S. For all $x, y \in S$, then $f(x), f(y) \in H$, since B is a gerneralized fuzzy imaginary right ideal of H, then $f^{-1}(B)(y) \vee \lambda = B(f(y)) \vee \lambda \geq$ $B(f(x)) \wedge B(f(x)f(y)) \wedge \mu = B(f(x)) \wedge B(f(xy)) \wedge \mu = f^{-1}(B)(x) \wedge f^{-1}(B)$ $(xy) \wedge \mu$.

Therefore $f^{-1}(B)$ is a gerneralized fuzzy imaginary right ideal of S.

4 Conclusion

In this present investigation, the concept of gerneralized fuzzy imaginary right (resp.left) ideal and $(\in, \in \vee q_{(\lambda,\mu)})$—fuzzy imaginary right (resp.left) ideal are proposed. Moreover, the relevent properties are studied.

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References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965)
2. Rosenfeld, A.: Fuzzy groups. J. Math. Anal. Appl. **35**(3), 512–517 (1971)
3. Liu, W.J.: The character of fuzzy ideals and field. Fuzzy Math. **1**(1), 101–101 (1981)
4. Ma, H.C.: About imaginary ideals of rings. J. Xuchang Colledge. **9**(1), 8–10 (1990)
5. Ma, H.C., Chen, F.Z., Wu, Y.Z.: About pseudo ideals and imaginary ideals of rings. J. Harbin Normal Univ. (Nat. Sci.) **8**(1), 23–26 (1992)
6. Bhakat, S.K., Das, P.: On the definition of a fuzzy subgroup. Fuzzy Sets Syst. **51**(2), 235–241 (1992)
7. Liao, Z.H., Chen, M.: $(\in, \in \vee q_{(\lambda,\mu)})-$fuzzy subsemigroups and $(\in, \in \vee q_{(\lambda,\mu)})-$ fuzzy completely regular subsemigroups. J. Jiangnan Univ. (Nat. Sci.) **8**(2), 242–244 (2009)
8. Liao, Z.H., Hu, M.H., Chen, M., et al.: Gerneralized fuzzy interior ideals in semigroups. In: Third International Jiont Conference on Computational Science and Optimization, pp. 197–200 (2010)
9. Vandiver, H.S.: On some simple types of semirings. Am. Math. Mon. **46**(1), 22–26 (1939)
10. Glazek, K.: A Guide to the Literature on Semirings and Their Applications in Mathematics and Information Sciences. Kluwer, Dirdrecht (2002)
11. Golan, J.S.: Semirings and Affine Equations over Them: Theory and Applications. Kluwer, Dordrecht (2003)
12. Liu, W.J.: Fuzzy invariant subgroups and fuzzy ideals. Fuzzy Sets Syst. **8**(2), 133–139 (1982)
13. Mukherjee, T.K., Sen, M.K.: On fuzzy ideals of a ring. Fuzzy Sets Syst. **21**(1), 99–104 (1987)
14. Mukherjee, T.K., Sen, M.K.: Prime fuzzy ideals in rings. Fuzzy Sets Syst. **32**(3), 337–341 (1989)
15. Feng, J.W., Zhan, C.S.: Complemented semirings congruence. Fuzzy Syst. Math. **24**(6), 61–65 (2010)
16. Ming, P.P., Ming, L.Y.: Fuzzy topology: neighbourhood structure of a fuzzy point and Moore-Smith convergence. J. Math. Anal. Appl. **76**(2), 571–599 (1980)
17. Zhu, C., Liao, Z.H., Luo, X.T., et al.: Fuzzy complemented sumirings. Fuzzy Syst. Math. **27**(5), 47–54 (2013)
18. Liao, Z.H., Gu, H.: Fuzzy normal subgroup. Fuzzy Syst. Math. **20**(5), 340–343 (2006)
19. Yao, B.X.: Fuzzy Theory on Groups and Rings, pp. 1–125. Science Press, Beijing (2008)

Statistical Approximation of the q-Bernstein-Durrmeyer Type Operators

Mei-Ying Ren

Abstract In 2012, a kind of q-Bernstein-Durrmeyer type operators is introduced, and some approximate properties of these operators are studied by Ren. In this paper the statistical approximation properties of these operators are investigated. The Korovkin type statistical convergence theorem of these operators is established. Then the rates of statistical convergence of these operators are also studied by means of modulus of continuity and the help of functions of the Lipschitz class.

Keywords q-Bernstein-Durrmeyer type operators · q-integers · Korovkin type theorem · Rate of statistical convergence · Modulus of continuity

1 Introduction

After Philips [1] introduced and studied q analogue of Bernstein polynomials, the applications of q-calculus in the approximation theory become one of the main areas of research (such as, see [2–6]). Recently the statistical approximation properties have also been investigated for q-analogue polynomials. For instance, in [7] Kantorovich type q-Bernstein operators; in [8] q-Baskakov-Kantorovich operators; in [9] Kantorovich type q-Szász-Mirakjan operators; in [10] q-Bleimann, Butzer and Hahn operators; in [11] q-analogue of MKZ operators and in [12] modified q-Bernstein-Schurer operators were introduced and their statistical approximation properties were studied.

In 2012, Ren [13] introduced q-Bernstein-Durrmeyer type operators as follows:

$$M_{n,q}^{Q}(f;x) = f(0)p_{n,0}(q;x) + f(1)p_{n,n}(q;x) + \sum_{k=1}^{n-1} p_{n,k}(q;x)\int_0^1 \mu_{n,k}^{Q}(t)f(t)dt, \quad (1)$$

M.-Y. Ren (✉)
Department of Mathematics and Computer Science, Wuyi University,
Wuyishan 354300, China
e-mail: npmeiyingr@163.com

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where $f \in C[0,1], x \in [0,1]$, $0 < q < 1, Q > 0$, $n \in \mathbf{N}$, $p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k(1-x)_q^{n-k}$. For $1 \le k \le n-1$, $\mu_{n,k}^Q(t) = \frac{t^{Q[k]_q-1}(1-t)^{Q([n]_q-[k]_q)-1}}{B(Q[k]_q, Q([n]_q-[k]_q))}$. $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ $(x, y > 0)$ is a Euler Beta function.

In [13], some approximate properties of the q-Bernstein-Durrmeyer type operators are studied. The goal of this paper is to study the statistical approximation properties of these operators. The Korovkin type statistical convergence theorem of these operators is established. Then the rates of statistical convergence of these operators are also studied by means of modulus of continuity and the help of functions of the Lipschitz class.

Before, proceeding further, let us give some basic definitions and notations from q-calculus. Details on q-integers can be found in [14, 15].

Let $q > 0$, for each nonnegative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k, & q = 1 \end{cases}$$

and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1 \\ 1, & k = 0 \end{cases},$$

respectively.

Then for $q > 0$ and integers $n, k, n \geq k \geq 0$, we have

$$[k+1]_q = 1 + q[k]_q \quad and \quad [k]_q + q^k[n-k]_q = [n]_q.$$

For the integers $n, k, n \geq k \geq 0$, the q-binomial coefficients is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Let $q > 0$, for nonnegative integer n, the q-analogue of $(x-a)^n$ is defined by

$$(x-a)_q^n := \begin{cases} 1, & n = 0 \\ (x-a)(x-qa)\cdots(x-q^{n-1}a), & n \geq 1 \end{cases}.$$

Remark 1 For $f \in C[a,b]$, denote $\| f \|_{C[a,b]} = max\{|f(x)|; x \in [a,b]\}$.

2 Tow Lemmas

Now, we give two lemmas, which are necessary to prove our results.

Lemma 1 (see [13]) *For* $M_{n,q}^Q(t^s;x)$, $s = 0, 1, 2$, *we have*

$$M_{n,q}^{Q}(1;x) = 1; \tag{2}$$

$$M_{n,q}^{Q}(t;x) = x; \tag{3}$$

$$M_{n,q}^{Q}(t^2;x) = \frac{(1+Q)x}{1+Q[n]_q} + \frac{qQ[n-1]_q x^2}{1+Q[n]_q}. \tag{4}$$

Lemma 2 (see [13]) *For* $\{M_{n,q}^{Q}(f;x)\}$, *we have*

$$M_{n,q}^{Q}((t-x)^2;x) = \frac{(1+Q)x(1-x)}{1+Q[n]_q}. \tag{5}$$

3 Statistical Approximation of Korovkin Type

Now, let us recall the concept of the statistical convergence which was introduced by Fast [16].

Let set $K \subseteq \mathbf{N}$ and $K_n = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) := \lim_{n\to\infty} \frac{1}{n}|K_n|$ if the limit exists (see [17], where $|K_n|$ denotes the cardinality of the set K_n).

A sequence $x = \{x_k\}$ is call statistically convergent to a number L, if for every $\varepsilon > 0, \delta\{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\} = 0$. This convergence is denoted as $st - \lim_k x_k = L$.

Note that any convergent sequence is statistically convergent, but not conversely. Details can be found in [18].

Let $B[a,b]$ denote the class of all real valued bounded functions f on interval $[a,b]$, and Let $C[a,b]$ denote the class of all real valued continuous functions f on interval $[a,b]$. In approximation theory, the concept of statistically convergence was used by Gadjiev and Orhan [19]. They proved the following Bohman-Korovkin type approximation theorem for statistically convergence.

Theorem 1 (see [19]) *If the sequence of linear positive operators* $A_n : C[a,b] \to B[a,b]$ *satisfies the conditions*

$$st - \lim_n \|A_n(e_\upsilon;\cdot) - e_\upsilon\|_{C[a,b]} = 0,$$

for $e_\upsilon(t) = t^\upsilon,\ \upsilon = 0, 1, 2$. *Then for any* $f \in C[a,b]$,

$$st - \lim_n \|A_n(f;\cdot) - f\|_{C[a,b]} = 0.$$

Next, we give the statistical approximation properties of the q-Bernstein-Durrmeyer type operators.

Theorem 2 *Let* $q = \{q_n\},\ 0 < q_n < 1$ *be a sequence satisfying the following condition*

$$st-\lim_n q_n=1,\ st-\lim_n q_n^n=c\,(c<1). \tag{6}$$

Then for any $f\in C[0,1]$ *, we have*

$$st-\lim_n \|M^Q_{n,q_n}(f;\cdot)-f\|_{C[0,1]}=0.$$

Proof By the definition of the q-Bernstein-Durrmeyer type operators, for any $f\in C[0,1]$, $M^Q_{n,q}(f;x)$ has positivity and linearity. By Theorem 1, for any $f\in C[0,1]$, enough to prove that

$st-\lim_n \|M^Q_{n,q_n}(e_\upsilon;\cdot)-e_\upsilon\|_{C[0,1]}=0$, for $e_\upsilon(t)=t^\upsilon,\ \upsilon=0,1,2.$

From (2) and (3), we can easily get

$$st-\lim_n \|M^Q_{n,q_n}(e_0;\cdot)-e_0\|_{C[0,1]}=0, \tag{7}$$

$$st-\lim_n \|M^Q_{n,q_n}(e_1;\cdot)-e_1\|_{C[0,1]}=0. \tag{8}$$

From (4), we have

$$M^Q_{n,q_n}(e_2;x)-e_2(x)=\frac{(1+Q)x}{1+Q[n]_{q_n}}+(\frac{q_nQ[n-1]_{q_n}}{1+Q[n]_{q_n}}-1)x^2.$$

Since $q_n[n-1]_{q_n}=[n]_{q_n}-1$, for $x\in[0,1]$, by a simple computation, we have

$$\|M^Q_{n,q_n}(e_2;\cdot)-e_2\|_{C[0,1]}\le\frac{2(1+Q)}{1+Q[n]_{q_n}}.$$

For every given $\varepsilon>0$, let us define the following sets:

$$V=\{k:\|M^Q_{k,q_k}(e_2;\cdot)-e_2\|_{C[0,1]}\ge\varepsilon\},$$
$$V_1=\{k:\frac{2(1+Q)}{1+Q[k]_{q_k}}\ge\varepsilon\}.$$

It is clear that $V\subseteq V_1$, so we get

$$\delta\{k\le n:\|M^Q_{k,q_k}(e_2;\cdot)-e_2\|_{C[0,1]}\ge\varepsilon\}\le\delta\{k\le n:\frac{2(1+Q)}{1+Q[k]_{q_k}}\ge\varepsilon\}. \tag{9}$$

By condition (6), we have

$$st-\lim_n\frac{2(1+Q)}{1+Q[n]_{q_n}}=0,$$

so, by inequality (9), we get

$$st - \lim_{n} \|M_{n,q_n}^{Q}(e_2; \cdot) - e_2\|_{C[0,1]} = 0. \tag{10}$$

In view of the equalities (7), (8) and (10), the proof is complete.

4 Rates of Statistical Convergence

In this section, we will give the rates of statistical convergence of the operators $M_{n,q}^{Q}(f; x)$.

Let $f \in C[0, 1]$. For any $\delta > 0$, the usual modulus of continuity for f is defined as $\omega(f; \delta) = \sup\limits_{0<h\leq\delta} \sup\limits_{x,x+h\in[0,1]} |f(x+h) - f(x)|$.

By the property of the usual modulus of continuity, for $f \in C[0, 1]$ we have

$$\lim_{\delta\to 0^+} \omega(f; \delta) = 0. \tag{11}$$

Let $f \in C[0, 1]$. For any $t, x \in [0, 1]$ and $\delta > 0$, we have

$$|f(t) - f(x)| \leq \omega(f, |t - x|) \leq \begin{cases} \omega(f, \delta), & |t - x| < \delta \\ \omega(f, \frac{(t-x)^2}{\delta}), & |t - x| \geq \delta \end{cases}.$$

In the light of $\omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta)$ for $\lambda > 0$, it is clear that we have

$$|f(t) - f(x)| \leq (1 + \delta^{-2}(t - x)^2)\omega(f, \delta), \tag{12}$$

for any $t \in [0, 1],\ x \in [0, 1]$ and any $\delta > 0$.

First, we give the rates of convergence of the operators $M_{n,q}^{Q}(f; x)$ by means of modulus of continuity.

Theorem 3 *Let* $q = \{q_n\},\ 0 < q_n < 1$ *be a sequence satisfying the condition* (6). *Then for any* $f \in C[0, 1]$ *and* $x \in [0, 1]$, *we have*

$$|M_{n,q_n}^{Q}(f; x) - f(x)| \leq 2\omega(f, \delta_n(x)),$$

where

$$\delta_n(x) = \left\{ \frac{(1 + Q)x(1 - x)}{1 + Q[n]_{q_n}} \right\}^{1/2}. \tag{13}$$

Proof Using the linearity and positivity of these operators $M_{n,q}^{Q}(f; x)$, by Lemma 2, (2) and (12), for $f \in C[0, 1],\ x \in [0, 1]$, we get

$$
\begin{aligned}
|M^Q_{n,q}(f;x)-f(x)| &\le M^Q_{n,q}(|f(t)-f(x)|;x)\\
&\le (1+\delta^{-2}M^Q_{n,q}((t-x)^2;x))\omega(f,\delta)\\
&=\left[1+\delta^{-2}\frac{(1+Q)x(1-x)}{1+Q[n]_q}\right]\omega(f,\delta).
\end{aligned}
$$

Taking $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the condition (6) and choosing $\delta=\delta_n(x)$ as in (13), we have $|M^Q_{n,q_n}(f;x)-f(x)|\le 2\omega(f,\delta_n(x))$.

Remark 2 Form the condition (6) one can see that $st-\lim\limits_n M^Q_{n,q_n}((t-x)^2;x))=0$ which implies $st-\lim\limits_n \omega(f,\delta_n(x))=0$ from (11). This gives the pointwise rate of statistical coneergence of the operator $M^Q_{n,q_n}(f;x)$ to the function $f(x)$.

Corollary 1 *Let $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the condition* (6)*. Then for any $f\in C[0,1]$ and $x\in[0,1]$, we have*

$$
\|M^Q_{n,q_n}(f;\cdot)-f\|_{C[0,1]}\le 2\omega(f,\eta_n),
$$

where

$$
\eta_n=\left\{\frac{1+Q}{4(1+Q[n]_{q_n})}\right\}^{1/2}. \tag{14}
$$

Proof For $x\in[0,1]$, by Lemma 2, we have $M^Q_{n,q}((t-x)^2;x)\le\frac{1+Q}{4(1+Q[n]_q)}$, so, take $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the condition (6), we get $\delta_n(x)\le\eta_n$. Thus, for $f\in C[0,1]$ and $x\in[0,1]$, by using Theorem 3, one can get

$$
|M^Q_{n,q_n}(f;x)-f(x)|\le 2\omega(f,\delta_n(x))\le 2\omega(f,\eta_n),
$$

which implies the proof is complete.

Now, we give the rate of statistical convergence of the operators $M^Q_{n,q}(f;x)$ with the help of functions of the Lipschitz class.

Theorem 4 *Let $0<\alpha\le 1$, $M>0$, $f\in Lip^{\alpha}_M$ on [0,1], also let $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the conditions* (6)*. Then for any $x\in[0,1]$ we have*

$$
\|M^Q_{n,q_n}(f;\cdot)-f\|_{C[0,1]}\le M\eta_n^{\sigma},
$$

where η_n is given by (14).

Proof Let $0<\alpha\le 1$, $M>0$, $f\in Lip^{\alpha}_M$ on [0,1]. We obtain $f\in C[0,1]$, and for any $t,x\in[0,1]$, we have $|f(t)-f(x)|\le M|t-x|^{\alpha}$. Thus, using the linearity and positivity of the operators $M^Q_{n,q}(f;x)$, we obtain

$$
|M^Q_{n,q}(f;x)-f(x)|\le M^Q_{n,q}(|f(t)-f(x)|;x)\le MM^Q_{n,q}(|t-x|^{\sigma};x).
$$

Using the Hölder inequality with $m = \frac{2}{\alpha}$, $n = \frac{2}{2-\alpha}$ and (2), for $x \in [0, 1]$ we get

$$\begin{aligned} |M_{n,q}^{Q}(f; x) - f(x)| &\leq M M_{n,q}^{Q}(|t-x|^{\sigma}; x) \\ &\leq M\big[M_{n,q}^{Q}((t-x)^2; x)\big]^{\sigma/2} \\ &\leq M\Big[\frac{1+Q}{4(1+Q[n]_{q_n})}\Big]^{\sigma/2}. \end{aligned}$$

Taking $q = \{q_n\}$, $0 < q_n < 1$ be a sequence satisfying the condition (6), and choosing η_n as in (14), by Lemma 2, we have

$$|M_{n,q_n}^{Q}(f; x) - f(x)| \leq M\big[M_{n,q_n}^{Q}((t-x)^2; x)\big]^{\sigma/2} \leq M\eta_n^{\sigma},$$

the desired result follows immediately.

5 Conclusion

In this paper, some statistical approximation properties of the q-Bernstein-Durrmeyer type operators are studied. If we use King's approach to consider King type modification of the operators which is given by (1), we will obtain better statistical approximation (cf. [20, 21]).

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References

1. Phillips, G.M.: Bernstein polynomials based on the q-integers. Ann. Numer. Math. **4**, 511–518 (1997)
2. Gupta, V., Finta, Z.: On certain q-Durrmeyer type operators. Appl. Math. Comput. **209**, 415–420 (2009)
3. Örkcü, M., Doğru, O.: q-Szász-Mirakyan-Kantorovich type operators preserving some test functions. Appl. Math. Lett. **24**, 1588–1593 (2011)
4. Muraru, C.V.: Note on q-Bernstein-Schurer operators. Stud. Univ. Babeş-Bolyai Math. **56**, 489–495 (2011)
5. Doğru, O., Gupta, V.: Korovkin-type approximation properties of bivariate q-Meyer-König and Zeller operators. Calcolo **43**(1), 51–63 (2012)
6. Cai, Q.B., Zeng, X.M.: On the convergence of a modified q-Gamma operators. J. Comput. Anal. Appl. **15**(5), 826–832 (2013)
7. Dalmanoğlu, Ö., Doğru, O.: On statistical approximation properties of Kantorovich type q-Bernstein operators. Math. Comput. Model. **52**(5–6), 760–771 (2010)

8. Gupta, V., Radu, C.: Statistical approximation properties of q-Baskokov-Kantorovich operators. Cent. Eur. J. Math. **7**(4), 809–818 (2009)
9. Örkcü, M., Doğru, O.: Weighted statistical approximation by Kantorovich type q-Szász-Mirakjan operators. Appl. Math. Comput. **217**(20), 7913–7919 (2011)
10. Ersan, S., Doğru, O.: Statistical approximation properties of q-Bleimann, Butzer and Hahn operators. Math. Comput. Model. **49**(7–8), 1595–1606 (2009)
11. Mahmudov, N, Sabancigil, P.: A q-analogue of the Meyer-König and Zeller operators. Bull. Malays. Math. Sci. Soc. (2), **35**(1), 39–51 (2012)
12. Ren, M.Y., Zeng, X.M.: On statistical approximation properties of modified q-Bernstein-Schurer operators. Bull. Korean Math. Soc. **50**(4), 1145–1156 (2013)
13. Ren, M Y: Approximation properties of the q-Bernstein-Durrmeyer type operators. Fuzzy Syst. Math. **26**(5), 107–112 (2012) (Chinese)
14. Kac, V.G., Cheung, P.: Quantum Calculus. Universitext. Springer, New York (2002)
15. Gasper, G, Rahman, M.: Basic Hypergeometric Series. Encyclopedia of Mathematics and its Applications, vol. 35. Cambridge University Press, Cambridge (1990)
16. Fast, H.: Sur la convergence statistique. Collog. Math. **2**, 241–244 (1951)
17. Niven, I., Zuckerman, H.S., Montgomery, H.: An Introduction to the Theory of Numbers. Wiley, New york (1991)
18. Doğru O.: On statistical approximation properties of Stancu type bivariate generalization of q-Balás-Szabados operators. In: Seminar on Numerical Analysis and Approximation Theory, Cluj-Napoca, Univ. Babeş-Bolya, pp. 179–194 (2006)
19. Gadjiev, A.D., Orhan, C.: Some approximation theorems via statistical convergence. Rocky Mt. J. Math. **32**, 129–138 (2002)
20. King, J.P.: Positive linear operators which preserve x^2. Acta Math. Hungar. **99**(3), 203–208 (2003)
21. Mahmudov, N.I.: q-Szasz-Mirakjan operators which preserve x^2. J. Comput. Appl. Math. **235**, 4621–4628 (2011)

A New Similarity Measure Between Vague Sets

Bing-Jiang Zhang

Abstract Similarity measure is one of important, effective and widely-used methods in data processing and analysis. Vague set, as a generalized fuzzy set, has more powerful ability to process fuzzy information than fuzzy set. In this paper, we propose a new similarity measure between vague sets. Compared to existing similarity measures, our approach is far more reasonable, practical yet useful in measuring the similarity between vague sets.

Keywords Fuzzy sets · Vague sets · Similarity measure

1 Introduction

In the classical set theory introduced by Cantor, a German mathematician, values of elements in a set are only one of 0 and 1. That is, for any element, there are only two possibilities: in or not in the set. Therefore, the theory cannot handle the data with ambiguity and uncertainty.

Zadeh proposed fuzzy theory in 1965 [1]. The most important feature of a fuzzy set is that fuzzy set A is a class of objects that satisfy a certain (or several) property. Each object $u \in U$ is assigned a single value between 0 and 1, called the grade of membership, where U is a universe of discourse. Formally, a membership function $\mu_A : U \to [0, 1]$ is defined for a fuzzy set A, where $\mu_A(u)$, for each $u \in U$, denotes the degree of membership of u in the fuzzy set A. This membership function has the following characteristics: The single degree contains the evidences for both supporting and opposing u. It cannot only represent one of the two evidences, but it also cannot represent both at the same time too.

In order to deal with this problem, Gau and Buehrer proposed the concept of vague set in 1993 [2], by replacing the value of an element in a set with a sub-interval of

B.-J. Zhang (✉)
School of Applied Science, Beijing Information Science and Technology University,
Beijing 100192, China
e-mail: zbj2013ch@163.com

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[0, 1]. Namely, a truth-membership function $t_A(u)$ and a false-membership function $f_A(u)$ are used to describe the boundaries of membership degree. These two boundaries form a sub-interval $[t_A(u), 1 - f_A(u)]$ of [0, 1], to generalize the $\mu_A(u)$ of fuzzy sets, where $t_A(u) \leq \mu_A(u) \leq 1 - f_A(u)$. The vague set theory improves description of the objective real world, becoming a promising tool to deal with inexact, uncertain or vague knowledge. Many researchers have applies this theory to many situations, such as fuzzy control, decision-making, knowledge discovery and fault diagnosis, and the tool has presented more challenging than that with fuzzy sets theory in applications.

In intelligent activities, it is often needed to compare between two fuzzy concepts. That is, we need to check whether two knowledge patterns are identical or approximately same, to find out functional dependence relations between concepts in a data mining system. In order to solve this problem, many measure methods have been proposed to measure the similarity between two vague sets (values). Each of them is given from different side, having its own counterexamples.

After analyzing most existing vague sets and vague values similarity measures, this paper develops a method for the similarity measure of vague sets.

2 Preliminaries

In this section, we review some basic definitions of vague values and vague sets.

Definition 1 Let U be a space of points (objects), with a generic element of U denoted by u. A vague set A in U is characterized by a truth-membership function $t_A(u)$ and a false-membership function $f_A(u)$. $t_A(u)$ is a lower bound on the grade of membership of u derived from the evidence for u, and $f_A(u)$ is a lower bound on the negation of u derived from the evidence against u, $t_A(u)$ and $f_A(u)$ both associate a real number in the interval [0, 1] with each point in U, where $t_A(u) + f_A(u) \leq 1$. That is

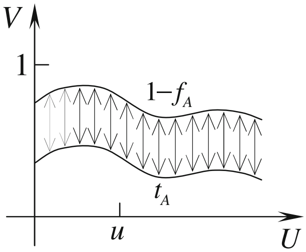
$$t_A : U \to [0, 1] \text{ and } f_A : U \to [0, 1]$$

This approach bounds the grade of membership of u to a subinterval $[t_A(u), 1 - f_A(u)]$ of [0, 1]. We depict these ideas in Fig. 1.

The precision of the knowledge about u is characterized by the difference $(1 - t_A(u) - f_A(u))$. If this is small, the knowledge about u is relatively precise; if it is large, we know correspondingly little. If $t_A(u)$ is equal to $(1 - f_A(u))$, the knowledge about u is exact, and the vague set theory reverts back to fuzzy set theory. If $t_A(u)$ and $(1 - f_A(u))$ are both equal to 1 or 0, depending on whether u does or does not belong to A, the knowledge about u is very exact and the theory reverts back to ordinary sets.

When U is continuous, a vague set A of the universe of discourse U can be represented by

Fig. 1 A vague set



$$A = \int_U [t_A(u), 1 - f_A(u)]/u,\ u \in U, \tag{1}$$

where $t_A(u) \leq \mu_A(u) \leq 1 - f_A(u)$.

When U is discrete, a vague set A of the universe of discourse U can be represented by

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i, \quad \forall u_i \in U, \tag{2}$$

where $t_A(u_i) \leq \mu_A(u_i) \leq 1 - f_A(u_i)$ and $1 \leq i \leq n$.

Definition 2 Let x be a vague value, where $x = [t_x, 1 - f_x]$. If $t_x = 1$ and $f_x = 0$ (i.e., $x = [1, 1]$). Then x is called a unit vague value.

Definition 3 Let x be a vague value, where $x = [t_x, 1 - f_x]$. If $t_x = 0$ and $f_x = 1$ (i.e., $x = [0, 0]$). Then x is called a zero vague value.

Definition 4 Let x and y be two vague values, where $x = [t_x, 1 - f_x]$ and $y = [t_y, 1 - f_y]$.

If $t_x = t_y$ and $f_x = f_y$, then the vague values x and y are called equal (i.e., $[t_x, 1 - f_x] = [t_y, 1 - f_y]$).

Definition 5 Let A be a vague set of the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$. If $\forall i, t_A(u_i) = 1$ and $t_A(u_i) = 0$, then A is called a unit vague set, where $1 \leq i \leq n$.

Definition 6 Let A be a vague set of the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$. If $\forall i, t_A(u_i) = 0$ and $f_A(u_i) = 1$, then A is called a zero vague set, where $1 \leq i \leq n$.

Definition 7 Let A be a vague set of the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$. If $\forall i,\ t_A(u_i) = 0$ and $f_A(u_i) = 0$, then A is called an empty vague set, where $1 \leq i \leq n$.

Definition 8 Let A and B be vague sets of the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$, $A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i$, $B = \sum_{i=1}^{n} [t_B(u_i), 1 - f_B(u_i)]/u_i$. If $\forall i, [t_A(u_i), 1 - f_A(u_i)] = [t_B(u_i), 1 - f_B(u_i)]$, then the vague sets A and B are called equal.

3 Research into Similarity Measure

Currently, there have been many similarity measurements for vague set (value).

Suppose that $x = [t_x, 1 - f_x]$ and $y = [t_y, 1 - f_y]$ are two vague values in the universe discourse over the discourse universe U. Let $S(x) = t_x - f_x$, $S(y) = t_y - f_y$. The M_C, M_H, M_L and M_O models are defined respectively in [3–6] as follows:

$$M_C(x, y) = 1 - \left|\frac{S(x) - S(y)}{2}\right| = 1 - \frac{\left|(t_x - t_y) - (f_x - f_y)\right|}{2} \tag{3}$$

$$M_H(x, y) = 1 - \frac{\left|t_x - t_y\right| + \left|f_x - f_y\right|}{2} \tag{4}$$

$$\begin{aligned} M_L(x, y) &= 1 - \frac{|S(x) - S(y)|}{4} - \frac{\left|t_x - t_y\right| + \left|f_x - f_y\right|}{4} \\ &= 1 - \frac{\left|(t_x - t_y) - (f_x - f_y)\right| + \left|t_x - t_y\right| + \left|f_x - f_y\right|}{4} \end{aligned} \tag{5}$$

$$M_O(x, y) = 1 - \sqrt{\frac{(t_x - t_y)^2 + (f_x - f_y)^2}{2}} \tag{6}$$

For the $M_C(x, y)$, $M_H(x, y)$ and $M_L(x, y)$, consider the vague values $x = [0, 1]$ and $y = [a, a]$, $0 \leq a \leq 1$. According to Formulas (3)–(5), we can be checked that $M_C(x, y) = M_H(x, y) = M_L(x, y) = 0.5$. This means that the similarity measure between the vague value with the most imprecise evidence (the precision of the evidence is equal to zero) and the vague value with the most precise evidence (the precision of the evidence is equal to one) is equal to 0.5. However, our intuition shows that the similarity measure in this case should be equal to 0. The $M_o(x, y)$ model does not consider whether the differences are positive or negative.

References [7, 8] adopt Hamming distance and Euclidean distance to measure the distances between intuitionistic fuzzy sets as follows:

1. Hamming distance is given by

$$D_H(x, y) = \frac{\left|t_x - t_y\right| + \left|f_x - f_y\right| + \left|(t_x - t_y) + (f_x - f_y)\right|}{2} \tag{7}$$

2. Euclidean distance is given by

$$D_E(x, y) = \sqrt{\frac{(t_x - t_y)^2 + (f_x - f_y)^2 + ((t_x - t_y) + (f_x - f_y))^2}{2}} \tag{8}$$

These methods also have some problems. We still consider the vague values $x = [0, 1]$ and $y = [a, a]$, $0 \leq a \leq 1$. For the Hamming distance, it can be calculated that $D_H(x, y) = 1$. This means that the Hamming distance between t_x and f_x are equal to that between t_y and f_y. For the Euclidean distance, consider the Euclidean distance between x and y, which is equal to $(\sqrt{a^2 - a + 1})$. This means that the distance between the vague value with the most imprecise evidence and the vague value with the most precise evidence is not equal to 1. (Actually, the Euclidean distance in this case is in the interval $[\sqrt{3}/2, 1)$.) However, our intuition shows that the distance in this case should always be equal to 1.

4 A New Similarity Measure

In order to solve all the problems mentioned above, we define a new similarity measure between the vague values.

Definition 9 Let x and y be two vague values, where $x = [t_x, 1 - f_x]$ and $x = [t_y, 1 - f_y]$. The similarity measure between the vague values x and y is defined by

$$M(x, y) = 1 - \frac{J + D}{2} \tag{9}$$

where

$$J = 1 - \frac{t_x t_y + (1 - f_x)(1 - f_y)}{\sqrt{t_x^2 + (1 - f_x^2)}\sqrt{t_y^2 + (1 - f_y^2)}}$$

and

$$D = \frac{\left|t_x - t_y\right| + \left|f_x - f_y\right|}{2}$$

The large the value of $M(x, y)$, the more the similarity between the vague values x and y.

From Definition 9, we can obtain the following theorem.

Theorem 1 *The following statements are true:*

1. *The similarity measure is bounded, i.e.,* $0 \leq M(x, y) \leq 1$.

Proof Since $0 \leq t_x \leq 1, 0 \leq t_y \leq 1, 0 \leq f_x \leq 1, 0 \leq f_y \leq 1$, we have $0 \leq D \leq 1$, $0 \leq J \leq 1$, that is $0 \leq M(x, y) \leq 1$.

2. $M(x, y) = 1$, if and only if, the vague values x and y are equal (i.e., $x = y$).

Proof If $x = y$, from the definition, it is clear that $M(x, y) = 1$. If $M(x, y) = 1$, then $D + J = 0$. Since $0 \leq D \leq 1$ and $0 \leq J \leq 1$, then $D = J = 0$, we have $t_x - t_y = 0$ and $f_x - f_y = 0$, that is $x = y$.

3. $M(x, y) = 0$, if and only if, the vague values x and y are $[0, 0]$ and $[1, 1]$ or $[1, 1]$ and $[0, 0]$.

Proof For $x = [0, 0]$ and $y = [1, 1]$ (or for $x = [1, 1]$ and $y = [0, 0]$), by the definition, we obviously have $M(x, y) = 0$. If $M(x, y) = 0$, we have $t_x - t_y = 1$ and $f_x - f_y = -1$; or $t_x - t_y = -1$ and $f_x - f_y = 1$. Hence, $x = [0, 0]$ and $y = [1, 1]$; or $x = [1, 1]$ and $y = [0, 0]$.

4. The similarity measure is commutative, i.e., $M(x, y) = M(y, x)$.

It is obtained directly from the definition of the $M(x, y)$ model.

Next we generalize the similarity measure to two given vague sets.

Definition 10 Let A and B be two vague sets of the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$, where $A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i$, and $B = \sum_{i=1}^{n} [t_B(u_i), 1 - f_B(u_i)]/u_i$; the similarity measure between the vague sets A and B can be evaluated as follows:

$$M(A, B) = 1 - \frac{M_J(A, B) + M_D(A, B)}{2}, \tag{10}$$

where

$$M_J(A, B) = \frac{1}{n} \sum_{i=1}^{n} (1 - \frac{t_A(u_i)t_B(u_i) + (1 - f_A(u_i))(1 - f_B(u_i))}{\sqrt{[t_A(u_i)]^2 + [(1 - f_A(u_i))]^2}\sqrt{[t_B(u_i)]^2 + [(1 - f_B(u_i))]^2}}),$$

and

$$M_D(A, B) = \frac{1}{n} \sum_{i=1}^{n} \frac{|t_A(u_i) - t_B(u_i)| + |f_A(u_i) - f_B(u_i)|}{2}$$

Similarly, we give the definition of distance between two vague sets as $D(A, B) = 1 - M(A, B)$.

From Definition 10, we obtain the following theorem for vague sets, which is similar to Theorem 1.

Theorem 2 *The following statements related to* $M(A, B)$ *are true:*

1. The similarity measure is bounded, i.e., $0 \leq M(A, B) \leq 1$.
2. $M(A, B) = 1$*, if and only if, the vague sets A and B are equal (i.e., $A = B$).*
3. $M(A, B) = 0$*, if and only if, all the vague sets* $A = \sum_{i=1}^{n} [0, 0]/u_i$ *and* $B = \sum_{i=1}^{n} [1, 1]/u_i$ *are* $[0, 0]$ *and* $[1, 1]$ *or* $A = \sum_{i=1}^{n} [1, 1]/u_i$ *and* $B = \sum_{i=1}^{n} [0, 0]/u_i$.
4. The similarity measure is commutative, i.e., sets $M(A, B) = M(B, A)$.

5 Numerical Experiments

We use the ideas of this paper for choosing the best supplier. There are five suppliers A_1, A_2, A_3, A_4, A_5 selected as alternatives, against three attributes C_1, C_2, C_3. We obtain decision matrix A as follows:

$$A = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{matrix} \begin{bmatrix} (C_1, [0.2, 0.8]) & (C_2, [0.3, 0.9]) & (C_3, [0.2, 1.0]) \\ (C_1, [0.3, 0.7]) & (C_2, [0.2, 0.8]) & (C_3, [0.3, 0.9]) \\ (C_1, [0.4, 0.6]) & (C_2, [0.5, 0.6]) & (C_3, [0.3, 0.8]) \\ (C_1, [0.5, 0.7]) & (C_2, [0.4, 0.6]) & (C_3, [0.5, 0.7]) \\ (C_1, [0.4, 0.6]) & (C_2, [0.6, 0.7]) & (C_3, [0.6, 0.6]) \end{bmatrix}$$

The paper takes the weight of each attribute as 1. The reference sequence A^* as follows is composed of the optimal interval value of indicator over all alternatives

$$A^* = [(C_1, [1, 1]) \ (C_2, [1, 1]) \ (C_3, [1, 1])]$$

Next, using the formulas (9) and (10), we can calculate the similarity measures between vague set A^* and other five vague sets $A_1 \ldots A_5$ to obtain an order vector as follows:

$$\begin{aligned} &(M(A_1, A^*), M(A_2, A^*), M(A_3, A^*), M(A_4, A^*), M(A_5, A^*)) \\ &\quad = (0.7140, 0.7134, 0.7478, 0.7756, 0.7879) \end{aligned}$$

Therefore, the ranking order of five suppliers will be as follows:

$$A_5 > A_4 > A_3 > A_1 > A_2$$

We can say that supplier A_5 is the best supplier among the five suppliers.

6 Conclusion

Many similarity measures have been proposed in literature for measuring the degree of similarity between fuzzy sets. Also several efforts have been made for the similarity measure between vague sets. After analyzing the limitations in current similarity measures for vague sets, we have proposed a new method for measuring the similarity between vague sets in this paper. The basic idea is to deeply understand the support, the difference of true-membership and the difference of false-membership, to significantly distinguish the directions of difference (positive and negative), and properly use varied-weights in the differences of true and false-membership, for two vague sets. The examples have illustrated that our approach is effective and practical.

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References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965)
2. Gau, W.L., Buehrer, D.J.: Vague sets. IEEE Trans. Syst. Man. Cybern. **23**(2), 610–614 (1993)
3. Chen, S.M.: Measures of similarity between vague sets. Fuzzy Sets Syst. **74**(2), 217–223 (1995)
4. Hong, D.H., Kim, C.: A note on similarity measures between vague sets and elements. Inf. Sci. **115**, 83–96 (1999)
5. Li, F., Xu, Z.Y.: Similarity measures between vague sets. Chinese J. Software **12**(6), 922–927 (2001)
6. Li, Y.H., Chi, Z.X., Yan, D.Q.: Vague similarity and vague entropy. Comput. Sci. **29**(12), 129–132 (2002). (Chinese journal)
7. Szmidt, E., Kacprzyk, J.: Distances between intuitionistic fuzzy sets. Fuzzy Sets Syst. **114**, 505–518 (2000)
8. Grzegorzewski, P.: Distances Between Intuitionistic Fuzzy Sets and/or Interval valued Fuzzy Sets Based on the Hausdorff Metric. Fuzzy Sets and Systems, 2003
9. Pappis, C.P., Karacapilidis, N.I.: A comparative assessment of measures of similarity of fuzzy values. Fuzzy Sets Syst. **56**, 171–174 (1993)
10. Zhang, W.N., Wang, K.: An efficient evaluation of a fuzzy equi-join using fuzzy equality indicators. IEEE Trans. Knowl. Data Eng. **12**(2), 225–237 (2000)
11. Ma, Z.M., Zhang, W.J., Ma, W.Y.: Semantic measure of fuzzy data in extended possibility-based fuzzy relational database. Int. J. Intell. Syst. **15**(8), 705–716 (2000)
12. Lu, A., Ng, W.: Managing merged data by vague functional dependencies. Lect. Notes Comput. Sci. **3288**, 259–272 (2004)
13. Li, Y.H., Olson, D.L., Qin, Z.: Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis. Pattern Recogn. Lett. **28**(2), 278–285 (2007)
14. Zhang, D.F., Zhang, J.L., Lai, K.K., Lu, Y.: An Novel approach to supplier selection based on vague sets group decision. Expert Syst. Appl. **36**(5), 9557–9563 (2009)
15. Zhang, Q.S., Jiang, S.Y.: A note on information entropy measures for vague sets and its applications. Inf. Sci. **178**, 4184–4191 (2008)
16. Ye, J.: Using an improved measure function of vague sets for multicriteria fuzzy decision-making. Expert Syst. Appl. **37**, 4706–4709 (2010)
17. Geng, X.L, Chu, X.N., Zhang, Z.F.: Test article sample title placed here. Expert Syst. Appl. **37**, 6629–6638 (2010)

Distributivity Equations Between Semi-t-operators Over Semi-uninorms

Feng Qin and Xiao-Quan Xu

Abstract The problem of distributivity was posed many years ago and investigated in families of certain operations, for example, t-norms, t-conorms, uninorms and nullnorms. In this paper, we continue to investigate the same topic as the above by focusing on semi-t-operators over semi-uninorms, which are generalizations of t-operators and uninorms by omitting commutativity, and associativity and commutativity, respectively. The obtained results are the full characterizations, and extend the previous ones about distributivity between nullnorms over uninorms, and also between semi-nullnorms over semi-uninorms.

Keywords Fuzzy connectives · Aggregation operators · Distributivity equation · Semi-uninorms · Semi-t-operators

1 Introduction

The problem of distributivity was posed many years ago [1]. In fact, This problem is related to the so-called pseudo-analysis, where the structure of **R** as a vector space is replaced by the structure of semi-ring on any interval $[a, b] \subseteq [-\infty, \infty]$, denoting the corresponding operations as pseudo-addition and pseudo-multiplication [2, 7]. In the semi-ring structure and some generalizations, the distributivity plays a fundamental and essential role. In this context, the known t-norms and t-conorms, more recently uninorms, have been used to model the mentioned pseudo-operations [8]. Hence, this leads to a new direction of investigations, namely, distributivity between t-norms and t-conorms [10], aggregation functions [3], fuzzy implications

F. Qin
College of Mathematics and Information Science, Jiangxi Normal University,
Nanchang 330022, People's Republic of China
e-mail: qinfeng923@163.com

X.-Q. Xu (✉)
Department of Mathematics, Nanchang Normal University, Nanchang
330032, People's Republic of China
e-mail: xiqxu2002@163.com

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[9, 14], uninorms and nullnorms [4, 13, 15], semi-uninorms and semi-nullnorms [5, 6].

In this paper, we will continue to investigate the above distributivity by considering semi-t-operators over semi-uninorms, which are generalizations of t-operators and uninorms by omitting commutativity, and associativity and commutativity, respectively. The obtained results are the full characterizations, and extend the previous ones about distributivity between nullnorms over uninorms, [13, 15], and also between semi-nullnorms over semi-uninorms [5, 6]. Since semi-uninorms and semi-t-operators can be used to model the pseudo-operations in pseudo-analysis, we hope that this work is helpful for developing pseudo-analysis, information aggregation and other practical applications.

The remainder of the paper is organized as follows. In Sect. 2, we present some results concerning basic fuzzy logic connectives and results of distributivity equation used later in the paper. In Sect. 3, we investigate the distributivity equation of semi-t-operators over semi-uninorms $\mathcal{N}_e^{\min}$. Section 4 is conclusion and further work.

2 Preliminaries

In this section, we recall basic notations and facts used later in the paper.

Definition 1 (*See* [8]) An operator $U : [0, 1]^2 \to [0, 1]$ is called a *semi-uninorm* if it is increasing in each variable and has a neutral element $e \in [0, 1]$, i.e., $\forall_{x\in[0,1]}, U(e, x) = U(x, e) = x$. We denote by $\mathcal{N}_e$ the family of all semi-uninorms with neutral element e. We call $U \in \mathcal{N}_1$ a *semi-t-norm* or a *semi-copula* and call $U \in \mathcal{N}_0$ a *semi-t-conorm*.

$U \in \mathcal{N}_e$ is a *uninorm* if it is a commutative and associative semi-uninorm (see [10]).

Theorem 1 (See [5]) *If an operator* $U : [0, 1]^2 \to [0, 1]$ *is a semi-uninorm with a neutral element* $e \in (0, 1)$*, then, for all* $x, y \in [0, 1]$*,*

$$U(x, y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}), & \textit{if } (x, y) \in [0, e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \textit{if } (x, y) \in [e, 1]^2, \\ C(x, y), & \textit{otherwise,} \end{cases} \tag{1}$$

where $T_U : [0, 1]^2 \to [0, 1]$ *is a semi-t-norm,* $S_U : [0, 1]^2 \to [0, 1]$ *is a semi-t-conorm, and* $C : [0, e) \times (e, 1] \cup (e, 1] \times [0, e) \to [0, 1]$ *is increasing and fulfils* $\min(x, y) \le C(x, y) \le \max(x, y)$ *for all* $(x, y) \in [0, e) \times (e, 1] \cup (e, 1] \times [0, e)$*. We call* T_U *and* S_U *the underlying semi-t-norm and semi-t-conorm of* U*, respectively.*

Definition 2 (*See* [17]) By $\mathcal{N}_e^{\max}$ ($\mathcal{N}_e^{\min}$) we denote the family of all semi-uninorms with the neutral element $e \in (0, 1)$ fulfilling the additional condition: $\forall_{x\in(e,1]} U(0, x) = U(x, 0) = x$ ($\forall_{x\in[0,e)} U(1, x) = U(x, 1) = x$).

From Theorem 1, we can obtain the structure of elements in $\mathcal{N}_e^{\max}$ and $\mathcal{N}_e^{\min}$.

Theorem 2 (See [16, 17]) *Let* $U \in \mathcal{N}_e$ *with neutral element* $e \in (0, 1)$.

(i) $U \in \mathcal{N}_e^{\min}$ *if and only if, for all* $x, y \in [0, 1]$,

$$U(x, y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}), & \text{if } (x, y) \in [0, e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y), & \text{otherwise.} \end{cases} \tag{2}$$

(ii) $U \in \mathcal{N}_e^{\max}$ *if and only if, for all* $x, y \in [0, 1]$,

$$U(x, y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}), & \text{if } (x, y) \in [0, e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \text{if } (x, y) \in [e, 1]^2, \\ \max(x, y), & \text{otherwise.} \end{cases} \tag{3}$$

It is clear that semi-uninorms have similar structure to uninorms and the members of $\mathcal{N}_e^{\min}$ and $\mathcal{N}_e^{\max}$ are conjunctive (i.e., $U(0, 1) = U(1, 0) = 0$) and disjunctive (i.e., $U(0, 1) = U(1, 0) = 1$), respectively.

Definition 3 (*See* [18]) An element $s \in [0, 1]$ is called an *idempotent* element of operation $G : [0, 1]^2 \to [0, 1]$ if $G(s, s) = s$. Operation G is called *idempotent* if all elements from $[0, 1]$ are idempotent.

Theorem 3 (See [18]) *Let* $e \in [0, 1]$. *Operations*

$$U(x, y) = \begin{cases} \max(x, y), & \text{if } (x, y) \in [e, 1]^2 \\ \min(x, y), & \text{otherwise} \end{cases} \tag{4}$$

and

$$U(x, y) = \begin{cases} \min(x, y), & \text{if } (x, y) \in [0, e]^2 \\ \max(x, y), & \text{otherwise} \end{cases} \tag{5}$$

are unique idempotent uninorms in $\mathcal{N}_e^{\min}$ *and* $\mathcal{N}_e^{\max}$*, respectively.*

Definition 4 (*See* [10]) An operation $V : [0, 1]^2 \to [0, 1]$ is called a *nullnorm* if it is commutative, associative, increasing, has a zero element $z \in [0, 1]$ such that

(i) $V(0, x) = V(x, 0) = x$ for all $x \leq z$,
(ii) $V(1, x) = V(x, 1) = x$ for all $x \geq z$.

By Definition 4, the case $z = 0$ leads back to t-norms, while the case $z = 1$ leads back to t-conorms.

Definition 5 (*See* [12]) Operation $F : [0, 1]^2 \to [0, 1]$ is called *t-operator* if it is commutative, associative, increasing and such that $F(0, 0) = 0$, $F(1, 1) = 1$ and the functions F_0 and F_1 are continuous, where $F_0(x) = F(0, x)$ and $F_1(x) = F(1, x)$.

The fact that Definitions 4 and 5 are equivalent was proved by Mas et al. [13], but their respective extensions semi-nullnorms and semi-t-operators are not. In this paper, only semi-t-operators will be recalled because we only consider the distributivity between semi-t-operators over semi-uninorms.

Definition 6 (*See* [6]) An operation $F : [0,1]^2 \to [0,1]$ is called a *semi-t-operator* if it is associative, increasing, fulfill $F(0,0) = 0$, $F(1,1) = 1$ and such that the functions F_0, F_1, F^0, F^1 are continuous, where $F_0(x) = F(0,x)$, $F_1(x) = F(1,x)$, $F^0(x) = F(x,0)$ and $F^1(x) = F(x,1)$.

Let $\mathcal{F}_{a,b}$ denote the family of all semi-t-operators such that $F(0,1) = a$ and $F(1,0) = b$.

Theorem 4 (See [11]) *Let* $F\colon [0,1]^2 \to [0,1]$, $F(0,1) = a$ *and* $F(1,0) = b$. *Operation* $F \in \mathcal{F}_{a,b}$ *if and only if there exists an associative semi-t-norm* T_F *and an associative semi-t-conorm* S_F *such that*

$$F(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & \text{if } (x,y) \in [0,a]^2 \\ b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}), & \text{if } (x,y) \in [b,1]^2 \\ a, & \text{if } (x,y) \in [0,a] \times [a,1] \\ b, & \text{if } (x,y) \in [b,1] \times [0,b] \\ x, & \text{otherwise} \end{cases} \tag{6}$$

for $a \le b$ *and*

$$F(x,y) = \begin{cases} bS_F(\frac{x}{b}, \frac{y}{b}), & \text{if } (x,y) \in [0,b]^2, \\ a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in [a,1]^2 \\ a, & \text{if } (x,y) \in [0,a] \times [a,1] \\ b, & \text{if } (x,y) \in [b,1] \times [0,b] \\ y, & \text{otherwise} \end{cases} \tag{7}$$

for $b \le a$.

Remark 1 (See [6]) Clearly, the class $\mathcal{F}_{z,z}$ is a class of associative semi-nullnorms with zero element z. But this paper does not involve it since the distributivity between semi-nullnorm and semi-uninorm is investigated by Drewniak et al. [5]. Hence, we always assume $a \ne b$ for any element of $\mathcal{F}_{a,b}$.

Next, for convenience sake, let us a new concept quasi-t-operator. Only difference with semi-t-operator is that quasi-t-operator has not associativity.

Definition 7 (*See* [6]) Let $F : [0,1]^2 \to [0,1]$, $F(0,1) = a$ and $F(1,0) = b$. We call F a *quasi-t-operator* if there exists a semi-t-norm T_F and a semi-t-conorm S_F such that F has the form (6) when $a \le b$ and the form (7) when $b \le a$.

Let $\mathcal{QF}_{a,b}$ denote the family of all quasi-t-operators such that $F(0,1) = a$ and $F(1,0) = b$. Now, we recall the distributivity equations (see [1], p. 318).

Definition 8 (*See* [1]) Let $F, G : [0,1]^2 \to [0,1]$. We say that

(i) F is *left distributive* over G, if for all $x, y, z \in [0,1]$,

$$F(x, G(y,z)) = G(F(x,y), F(x,z)). \tag{8}$$

(ii) F is *right distributive* over G, if for all $x, y, z \in [0,1]$,

$$F(G(y,z), x) = G(F(y,x), F(z,x)). \tag{9}$$

If Eqs. (8) and (9) are fulfilled simultaneously, for example, F is commutative, we say that F is distributive over G.

Lemma 1 (See [6]) *Let $F : X^2 \to X$ have right (left) neutral element e in a subset $\emptyset \neq Y \subset X$ (i.e., $\forall_{x \in Y}, F(x,e) = x$ $(F(e,x) = x)$). If operation F is distributive over operation $G : X^2 \to X$ fulfilling $G(e,e) = e$, then G is idempotent in Y.*

Corollary 1 (See [6]) *If operation $F : [0,1]^2 \to [0,1]$ with neutral element $e \in [0,1]$ is distributive over operation $G : [0,1]^2 \to [0,1]$ fulfilling $G(e,e) = e$, then G is idempotent.*

Lemma 2 (See [6]) *Every increasing operation $F : [0,1]^2 \to [0,1]$ is distributive over* max *and* min.

3 Distributivity of $F \in \mathcal{F}_{a,b}$ over $G \in \mathcal{N}_e^{\min} \cup \mathcal{N}_e^{\max}$

In this section, we will discuss the distributivity of semi-t-operators $F \in \mathcal{F}_{a,b}$ over $G \in \mathcal{N}_e^{\min} \cup \mathcal{N}_e^{\max}$. Depending on the inequality between a and b of operation F, the assumption $a \neq b$, and the membership between G and $\mathcal{N}_e^{\min} \cup \mathcal{N}_e^{\max}$, there are four different cases: (1) $a < b$ and $G \in \mathcal{N}_e^{\min}$, (2) $a < b$ and $G \in \mathcal{N}_e^{\max}$, (3) $a > b$ and $G \in \mathcal{N}_e^{\min}$, (4) $a > b$ and $G \in \mathcal{N}_e^{\min}$. In this paper, only Case (1) are discussed in detail, the results of Case (2) are listed because proof are similar. For Cases (3) and (4), their results are omitted because they can be obtained by exchanging a and b, (4) and (5), (8) and (9). At first, let us prove the following two lemmas which are useful in whole section.

Lemma 3 *Let $a, b, e \in [0,1]$. If $F \in \mathcal{F}_{a,b}$ is left or right distributive over $G \in \mathcal{N}_e^{\min}$, then G is the idempotent uninorm* (4).

Proof Let $a, b, e \in [0,1]$. Then $G(0,0) = 0$ and $G(1,1) = 1$. If $a < b$, then F has the form (6) and 0 is the right neutral element on the set $[0,b]$, 1 is the right neutral element on the set $[a,1]$. Applying twice Lemma 1, operation G is an idempotent uninorm and it is given by (4). Similar, if $b < a$ then F has the form (7) and 0 is the left neutral element on the set $[0, a]$, 1 is the left neutral element on the set $[b, 1]$. Applying again twice Lemma 1, operation G is yet an idempotent uninorm and it is given by (4).

Similar to Lemma 3, we have the following lemma.

Lemma 4 *Let $a, b, e \in [0, 1]$. If $F \in \mathcal{F}_{a,b}$ is left or right distributive over $G \in \mathcal{N}_e^{\max}$, then G is the idempotent uninorm (5).*

Next, let us consider Case (1), i.e., $a < b$ and $G \in \mathcal{N}_e^{\min}$.

3.1 $a < b$ and $G \in \mathcal{N}_e^{\min}$

Lemma 5 *Let $a, b, e \in [0, 1]$ and $a < b$. If $F \in \mathcal{F}_{a,b}$ is left or right distributive over $G \in \mathcal{N}_e^{\min}$, then $e \geq a$.*

Proof Over there, we only prove the left distributivity of $F \in \mathcal{F}_{a,b}$ over $G \in \mathcal{N}_e^{\min}$ because it is similar to the right distributivity. On the contrary, let us assume that $e < a$, then it follows that $e = F(e, 0) = F(e, G(1, 0)) = G(F(e, 1), F(e, 0)) = G(F(e, 1), e) = G(a, e) = a$. This is a contradiction. So $e \geq a$.

By Lemma 5 and assumption $a < b$, we only need to consider the following two subcases: $a \leq e \leq b$ and $a < b < e$. Now, let us study the subcase $a \leq e \leq b$.

Proposition 1 *Let $a, b, e \in [0, 1]$ and $a \leq e \leq b$. $F \in \mathcal{F}_{a,b}$ is left or right distributive over $G \in \mathcal{N}_e^{\min}$ if and only if G is the idempotent uninorm (4).*

Finally, let us discuss the case $a < b < e$.

Proposition 2 *Let $a, b, e \in [0, 1]$ and $a < b < e$. $F \in \mathcal{QF}_{a,b}$ is left distributive over $G \in \mathcal{N}_e^{\min}$ if and only if G is the idempotent uninorm (4) and F has the following form:*

$$F(x, y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ A_1, & \text{if } (x, y) \in [b, e]^2, \\ A_2, & \text{if } (x, y) \in [e, 1]^2, \\ A_3, & \text{if } (x, y) \in [e, 1] \times [b, e], \\ \min(x, y), & \text{if } (x, y) \in [b, e] \times [e, 1], \\ a, & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b, & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x, & \text{otherwise,} \end{cases} \tag{10}$$

such that $F(x, 1) = F(1, x) = x$ and $F(x, b) = F(b, x) = b$ hold for all $x \in [b, 1]$, where S_F is a semi-t-conorm, e is a right side neutral element of $A_1 : [b, e]^2 \to [b, e]$, $A_2 : [e, 1]^2 \to [e, 1]$, $A_3 : [e, 1] \times [b, e] \to [b, e]$ and S_F, A_1, A_2, A_3 are increasing operations having common boundary values.

Proposition 3 *Let $a, b, e \in [0, 1]$ and $a < b < e$. $F \in \mathcal{QF}_{a,b}$ is right distributive over $G \in \mathcal{N}_e^{\min}$ if and only if G is the idempotent uninorm (4) and F has the following form:*

$$F(x, y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ A_1, & \text{if } (x, y) \in [b, e]^2, \\ A_2, & \text{if } (x, y) \in [e, 1]^2, \\ A_3, & \text{if } (x, y) \in [b, e] \times [e, 1], \\ \min(x, y), & \text{if } (x, y) \in [e, 1] \times [b, e], \\ a, & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b, & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x, & \text{otherwise,} \end{cases} \tag{11}$$

such that $F(x, 1) = F(1, x) = x$ and $F(x, b) = F(b, x) = b$ hold for all $x \in [b, 1]$, where S_F is a semi-t-conorm, e is a right side neutral element of $A_1 : [b, e]^2 \to [b, e]$, $A_2 : [e, 1]^2 \to [e, 1]$, $A_3 : [b, e] \times [e, 1] \to [b, e]$ and S_F, A_1, A_2, A_3 are increasing operations having common boundary values.

By Propositions 1, 2, 3 and Theorem 3.42 in [10], we have the following theorem.

Theorem 5 *Let $a, b, e \in [0, 1]$ and $a < b$. $F \in \mathcal{F}_{a,b}$ is distributive over $G \in \mathcal{N}_e^{\min}$ if and only if one of two results holds.*

(i) If $a \le e \le b$, then G is the idempotent uninorm (4).
(ii) If $a < b < e$ then G is the idempotent uninorm (4) and F has the following form:

$$F(x, y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & \text{if } (x, y) \in [0, a]^2, \\ b + (e - b)T_1(\frac{x-b}{e-b}, \frac{y-b}{e-b}), & \text{if } (x, y) \in [b, e]^2, \\ e + (1 - e)T_2(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y), & \text{if } (x, y) \in [b, e] \times [e, 1] \cup [e, 1] \times [b, e], \\ a, & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b, & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x, & \text{otherwise,} \end{cases} \tag{12}$$

where S_F is a associative semi-t-conorm, T_1 and T_2 are associative semi-t-norms.

3.2 $a < b$ and $G \in \mathcal{N}_e^{\max}$

In this subsection, we only list the results of the case $a < b$ and $G \in \mathcal{N}_e^{\max}$ because their proofs are similar Sect. 3.1.

Lemma 6 *Let $a, b, e \in [0, 1]$ and $a < b$. If $F \in \mathcal{F}_{a,b}$ is left or right distributive over $G \in \mathcal{N}_e^{\max}$, then $b \geq e$.*

By Lemma 6 and assumption $a < b$, we only need to consider the following two subcases: $a \leq e \leq b$ and $e < a$.

Proposition 4 *Let $a, b, e \in [0, 1]$ and $a \leq e \leq b$. $F \in \mathcal{F}_{a,b}$ is left or right distributive over $G \in \mathcal{N}_e^{\max}$ if and only if G is the idempotent uninorm (5).*

Proposition 5 *Let $a, b, e \in [0, 1]$ and $e < a < b$. $F \in \mathcal{QF}_{a,b}$ is left distributive over $G \in \mathcal{N}_e^{\max}$ if and only if G is the idempotent uninorm (5) and F has the following form:*

$$F(x, y) = \begin{cases} b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \text{if } (x, y) \in [b, 1]^2, \\ A_1, & \text{if } (x, y) \in [0, e]^2, \\ A_2, & \text{if } (x, y) \in [e, a]^2, \\ A_3, & \text{if } (x, y) \in [0, e] \times [e, a], \\ \max(x, y), & \text{if } (x, y) \in [e, a] \times [0, e], \\ a, & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b, & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x, & \text{otherwise,} \end{cases} \tag{13}$$

such that $F(x, 0) = F(0, x) = x$ and $F(x, a) = F(a, x) = a$ hold for all $x \in [0, a]$, where T_F is a semi-t-norm, $A_1 : [0, e]^2 \to [0, e]$, e is a right side neutral element of $A_2 : [e, a]^2 \to [e, a]$, $A_3 : [0, e] \times [e, a] \to [e, a]$ and A_1, A_2, A_3, T_F are increasing operations having common boundary values.

Proposition 6 *Let $a, b, e \in [0, 1]$ and $e < a < b$. $F \in \mathcal{QF}_{a,b}$ is left distributive over $G \in \mathcal{N}_e^{\max}$ if and only if G is the idempotent uninorm (5) and F has the following form:*

$$F(x, y) = \begin{cases} b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}), & \text{if } (x, y) \in [b, 1]^2, \\ A_1, & \text{if } (x, y) \in [0, e]^2, \\ A_2, & \text{if } (x, y) \in [e, a]^2, \\ A_3, & \text{if } (x, y) \in [e, a] \times [0, e], \\ \max(x, y), & \text{if } (x, y) \in [0, e] \times [e, a], \\ a, & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b, & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x, & \text{otherwise,} \end{cases} \tag{14}$$

such that $F(x, 0) = F(0, x) = x$ and $F(x, a) = F(a, x) = a$ hold for all $x \in [0, a]$, where T_F is a semi-t-norm, $A_1 : [0, e]^2 \to [0, e]$, e is a right side neutral element of $A_2 : [e, a]^2 \to [e, a]$, $A_3 : [0, e] \times [e, a] \to [e, a]$ and A_1, A_2, A_3, T_F are increasing operations having common boundary values.

By Propositions 4, 5, 6 and Theorem 3.42 in [10], we have the following theorem.

Theorem 6 *Let $a, b, e \in [0, 1]$ and $a < b$. $F \in \mathcal{F}_{a,b}$ is distributive over $G \in \mathcal{N}_e^{\max}$ if and only if one of two results holds.*

(i) If $a \le e \le b$, then G is the idempotent uninorm (5).
(ii) If $e < a < b$ then G is the idempotent uninorm (5) and F has the following form:

$$F(x, y) = \begin{cases} eS_1(\frac{x}{e}, \frac{y}{e}), & \text{if } (x, y) \in [0, e]^2, \\ e + (a-e)S_2(\frac{x-e}{a-e}, \frac{y-e}{a-e}), & \text{if } (x, y) \in [e, a]^2, \\ \max(x, y), & \text{if } (x, y) \in [0, e] \times [e, a] \cup [e, a] \times [0, e], \\ b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}), & \text{if } (x, y) \in [b, 1]^2, \\ a, & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b, & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x, & \text{otherwise,} \end{cases} \tag{15}$$

where T_F is a associative semi-t-norm, S_1 and S_2 are associative semi-t-conorms.

4 Conclusion and Further Work

In this paper, we investigated the distributivity equations between semi-uninorms and semi-t-operator, which are generalizations of uninorms and t-operators by omitting their commutativity, respectively. The obtained results extend the previous ones about distributivity between uninorms and nullnorms or between semi-uninorms and semi- nullnorms. For examples, Propositions 4.2 and 4.3 in [13] are special cases of Theorems 6 and 5 of this paper. In the future work, we will concentrate on the distributivity equations among other cases.

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References

1. Aczél, J.: Lectures on Functional Equations and Their Applications. Acad. Press, New York (1966)
2. Baczyński, M., Jayaram, B.: Fuzzy Implications, Studies in Fuzziness and Soft Computing, vol. 231. Springer, Berlin (2008)

3. Bustince, H., Burillo, P., Soria, F.: Automorphisms, negations and implication operators. Fuzzy Sets Syst. **134**, 209–229 (2003)
4. Calvo, T., De Baets, B., Fodor, J.: The functional equations of Frank and Alsina for uninorms and nullnorms. Fuzzy Sets Syst. **120**, 385–394 (2001)
5. Drewniak, J., Drygaś, P., Rak, E.: Distributivity between uninorms and nullnorms. Fuzzy Sets Syst. **159**, 1646–1657 (2008)
6. Drygaś, P.: Distributivity between semi-t-operators and semi-nullnorms. Fuzzy Sets. Syst. **264**, 100–109 (2015)
7. Fodor, J., Roubens, M.: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer, Dordrecht (1994)
8. Liu, H.W.: Distributivity and conditional distributivity of semi-uninorms over continuous t-conorms and t-norms. Fuzzy Sets Syst. **268**, 27–43 (2015)
9. Jayaram, B., Rao, C.J.M.: On the distributivity of implication operators over T- and S-norms. IEEE Trans. Fuzzy Syst. **12**, 194–198 (2004)
10. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer, Dordrecht (2000)
11. Marichal, J.L.: On the associativity functional equation. Fuzzy Sets Syst. **114**, 381–389 (2000)
12. Mas, M., Mayor, G., Torrens, J.: t-Operators. Int. J. Uncertain. Fuzziness Knowl. Based Syst. **7**, 31–50 (1999)
13. Mas, M., Mayor, G., Torrens, J.: The distributivity condition for uninorms and t-operators. Fuzzy Sets. Syst. **128**, 209–225 (2002)
14. Qin, F., Baczyński, M.: Distributivity equations of implications based on continuous triangular norms (I). IEEE Trans. Fuzzy Syst. **20**, 153–167 (2012)
15. Qin, F., Zhao, B.: The distributivity equations for idempotent uninorms and nullnorms. Fuzzy Sets. Syst. **155**, 446–458 (2005)
16. Rak, E.: Distributivity equation for nullnorms. J. Electr. Eng. **56**(12/s), 53–55 (2005)
17. Rak, E., Drygaś, P.: Distributivity between uninorms. J. Electr. Eng. **57**(7/s), 35–38 (2006)
18. Ruiz, D., Torrens, J.: Distributive idempotent uninorms. Int. J. Uncertain. Fuzziness Knowl. Based Syst. **11**, 413–428 (2003)

Complex Fuzzy Set-Valued Complex Fuzzy Integral and Its Convergence Theorem

Sheng-quan Ma and Sheng-gang Li

Abstract This paper is devoted to propose the convergence problem of complex fuzzy set-valued complex fuzzy integral base on the complex fuzzy sets values complex fuzzy measure. We introduces the concepts of the complex fuzzy set-valued complex fuzzy measure in [1], the complex fuzzy set-valued measurable function in [2], and the complex fuzzy set-valued complex fuzzy integral in [3]. And then, we focuses on convergence problem of complex fuzzy set-valued complex fuzzy integral, obtained some convergence theorems.

Keywords Complex fuzzy set-valued measure · Complex fuzzy set-valued measurable function · Complex fuzzy set-valued complex fuzzy integral · Convergence theorem

1 Introductions

In 1998, fuzzy measure range is extended to the fuzzy real number field by Wu et al. [4] etc., which give the definition of Sugeno integral base on fuzzy number fuzzy measure, Guo et al. [5] etc. Also give the definition of (G) integral on fuzzy measure of fuzzy valued functions, which will be generalized the Sugeno integral to fuzzy sets [6]. In 1989 Buckley [7] proposed the concepts of fuzzy complex number, including people need to consider the measure and integration problems of fuzzy complex numbers, introduction fuzzy distance by Zhang [8–12], which discussed the fuzzy real valued measure problem of fuzzy sets, and give the fuzzy real valued

S. Ma (✉) · S. Li
College of Mathematics and Information Science, Shanxi Normal University,
Xi'an 710062, China
e-mail: mashengquan@163.com

S. Li
e-mail: shengganglinew@126.com

S. Ma
School of Information and Technology, Hainan Normal University, Haikou 571158, China

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fuzzy integral; in 1996, fuzzy measure and measurable function concept is extended to fuzzy complex sets by Qiu et al. [13], which given the concept of complex fuzzy measure, complex fuzzy measurable function and complex fuzzy integral, Wang and Li [14] etc. in 1999 based on the concepts of fuzzy number of Buckley, gives the concept of fuzzy complex valued measures and fuzzy complex valued integral, obtain some important results. The [15–17] study measurable function and its integral of complex fuzzy number set, especially Sugeno and Choquet type fuzzy complex numerical integral and its properties, and its application in classification technique. In this paper, base on the research work on the basis of [1–3], gives the convergence theorem of complex fuzzy set-valued complex fuzzy integral, lays a foundation for the complex fuzzy set-valued complex fuzzy integral theory.

2 Complex Fuzzy Set-Valued Complex Fuzzy Measure

Definition 1 ([18]) Suppose (X, F) is a classic measurable space, $E \to F$, mapping $f : X \to [-\infty, +\infty]$, f is called real valued measurable function of (X, F) on $\tilde{E}$, if and only if $\forall\alpha \in (-\infty, +\infty)$, $\tilde{E} \cap \chi_{F_\alpha} \in \mathcal{F}$, $\widetilde{E} \cap x^c_{F_\alpha} \in \mathcal{F}$, where $F_\alpha = \{x : f(x) \geq \alpha\}$, mapping $\tilde{f} : X \to F^*(R)$, f is called real valued measurable function of $(X, \mathcal{F})$ on $\tilde{E}$, if and only if $\forall\lambda \in (0, 1]$, $f_\lambda{}^-(x)$, $f_\lambda{}^+(x)$ is a real valued measurable function, where

$$\tilde{f}(x) = \bigcup_{\lambda\in[0,1]} \lambda[(f(x))^-_\lambda, (f(x))^+_\lambda] \overset{\Delta}{=} \bigcup_{\lambda\in[0,1]} \lambda[f^-_\lambda(x), f^+_\lambda(x)].$$

Definition 2 ([1]) Suppose Z is a non-empty complex numbers set, $F(Z)$ is set kinds on Z that consisting of by all complex fuzzy set $\tilde{\rho}$ is fuzzy complex valued distance that defined in $F(Z)$, set function

$$\tilde{\mu} : F(Z) \to F^*_+(K) = \{\tilde{A} + i\tilde{B} : \tilde{A}, \tilde{B} \in F^*_+(R), i = \sqrt{-1}\},$$

$$\tilde{A} + i\tilde{B} \mapsto \tilde{\mu}(\tilde{A} + i\tilde{B}) \in F^*_+(K)$$

called complex fuzzy set-valued complex fuzzy measure on $(Z, F(Z))$ if and only if

1. $\tilde{\mu}(\phi) = \tilde{0}, \tilde{0} = (\tilde{0}, \tilde{0}), \tilde{0} \in F^*(K)$,
2. $\forall\tilde{A}, \tilde{B} \in F(Z), \tilde{A} \subset \tilde{B} \Rightarrow \tilde{\mu}(\tilde{A}) \leq \tilde{\mu}(\tilde{B})$, where $\mathrm{Re}\tilde{\mu}(\tilde{A}) \leq \mathrm{Re}\tilde{\mu}(\tilde{B}), \mathrm{Im}\tilde{\mu}(\tilde{A}) \leq \mathrm{Im}\tilde{\mu}(\tilde{B})$,
3. $\{\tilde{A}_n\} \subset F(Z)\, \tilde{A}_n \subset \tilde{A}_{n+1}\, (n = 1, 2 \ldots) \Rightarrow \tilde{\rho} \lim_{n\to\infty} \tilde{\mu}\tilde{A}_n = \tilde{\mu}\left(\bigcup_{n=1}^{\infty} \tilde{A}_n\right)$,
4. $\left\{\tilde{A}_n\right\} \subset F(Z)\, \tilde{A}_n \supset \tilde{A}_{n+1}\, (n = 1, 2 \ldots)$,

note as $\tilde{\mu} = \mathrm{Re}\tilde{\mu} + i\mathrm{Im}\tilde{\mu} \overset{\Delta}{=} \tilde{\mu}_R + i\tilde{\mu}_I$, $(i = \sqrt{-1})$.

Definition 3 ([2]) Suppose $Z \subset K$ is a non-empty set of complex numbers, $(Z, \mathcal{F}(Z))$ is a classical complex measurable space, $\tilde{E} \in \mathcal{F}(Z)$, mapping $f : Z \to K$, called f is a complex valued measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z))$, if and only if $\forall a + ib \in K$, $\tilde{E} \cap \chi_{F_{a,b}} \in \mathcal{F}(Z)$, and $\tilde{E} \cap {\chi^c}_{F_{a,b}} \in \mathcal{F}(Z)$, where

$$F_{a,b} = \{z \in K \,|\mathrm{Re}\,[f(z)] \geq a, \mathrm{Im}\,[f(z)] \geq b\}\,.$$

Definition 4 ([2]) Suppose Z is a non-empty complex numbers set, $\tilde{E} \in \mathcal{F}(Z)$, mapping $\tilde{f} : Z \to F_0(K)$, $z \mapsto \tilde{f}(z) = \mathrm{Re}\tilde{f}(z) + i\mathrm{Im}\tilde{f}(z) \in F_0(K)$, $i = \sqrt{-1}$,

$$\tilde{f}(z) = \bigcup_{\lambda \in [0,1]} \lambda \left[\left(\mathrm{Re}\tilde{f}(z)\right)\right]_\lambda + i \bigcup_{\lambda \in [0,1]} \lambda \left[\mathrm{Im}\tilde{f}(z)\right]_\lambda$$

$$\triangleq \bigcup_{\lambda \in [0,1]} \lambda \mathrm{Re}\tilde{f}_\lambda(z) + i \bigcup_{\lambda \in [0,1]} \lambda \mathrm{Im}\tilde{f}_\lambda(z)$$

$$\triangleq \bigcup_{\lambda \in [0,1]} \lambda \left[\mathrm{Re}\tilde{f}_\lambda^-(z), \mathrm{Re}\tilde{f}_\lambda^+(z)\right] + i \bigcup_{\lambda \in [0,1]} \lambda \left[\mathrm{Im}\tilde{f}_\lambda^-(z), \mathrm{Im}\tilde{f}_\lambda^+(z)\right],$$

then $(Z, \mathcal{F}(Z), \tilde{\mu})$ is complex fuzzy valued fuzzy measure space, called $\tilde{f}$ is complex fuzzy valued complex fuzzy measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z), \tilde{\mu})$ if and only if $\forall\lambda \in [0, 1]$, $\mathrm{Re}\tilde{f}_\lambda(z)$, $\mathrm{Im}\tilde{f}_\lambda(z)$, which are complex valued measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z))$

record

$$F_{(\alpha,\beta),\lambda} \triangleq \left\{z = \alpha + i\beta : \mathrm{Re}f_\lambda^\pm(z) \geq \alpha, \mathrm{Im}f_\lambda^\pm(z) \geq \beta\right\},$$

where

$$\mathrm{Re}f_\lambda^\pm(z) \geq \alpha,$$

express

$$\mathrm{Re}f_\lambda^+(z) \geq \alpha \text{ and } \mathrm{Re}f_\lambda^-(z) \geq \alpha, \mathrm{Im}f_\lambda^\pm(z) \geq \beta,$$

express

$$\mathrm{Im}f_\lambda^+(z) \geq \beta \text{ and } \mathrm{Im}f_\lambda^-(z) \geq \beta, \forall\alpha, \beta \in [0, \infty),$$

thus, $\tilde{f}$ is complex fuzzy valued complex fuzzy measurable function on $\tilde{E}$ about $(Z, \mathcal{F}(Z), \tilde{\mu})$ if and only if $\forall\lambda \in [0, 1]$, $\tilde{E} \cap \chi_{F_{(\alpha,\beta))\lambda}} \in \mathcal{F}(Z)$, $\tilde{E} \cap {\chi^c}_{F_{(\alpha,\beta),\lambda}} \in \mathcal{F}(Z)$ $\tilde{M}\left(\tilde{E}\right)$ express all of the complex fuzzy set-valued measurable function on $\tilde{E}$.

3 Complex Fuzzy Set-valued Complex Fuzzy Integral and Its Properties

Definition 5 ([3]) Suppose $(Z, \mathcal{F}(Z), \tilde{\mu})$ is complex fuzzy set-valued fuzzy measure space, $\tilde{E} \in \mathcal{F}(Z)$, $\tilde{f} : Z \to F_0(K)$, define $\tilde{f}$ is complex fuzzy set-valued complex fuzzy integral on $\tilde{E}$ about $\tilde{\mu}$,

$$\int_{\tilde{E}} \tilde{f} d\tilde{\mu} \triangleq \left(\int_{\tilde{E}} \mathrm{Re} \tilde{f} d\tilde{\mu}_R, \int_{\tilde{E}} \mathrm{Im} \tilde{f} d\tilde{\mu} \right)_I,$$

where

$$\int_{\tilde{E}} \mathrm{Re} \tilde{f} d\tilde{\mu}_R = \bigcup_{\lambda \in [0,1]} \lambda \left[\int_{\tilde{E}} \mathrm{Re} f_\lambda{}^- d\tilde{\mu}_R, \int_{\tilde{E}} \mathrm{Re} f_\lambda{}^+ d\tilde{\mu}_R \right]$$

$$= \bigcup_{\lambda \in [0,1]} \lambda \left[\sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Re} \tilde{\mu}_\lambda^- \left(\tilde{A} \cap \chi_{F_{\lambda,\alpha,1}{}^-} \right), \sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Re} \tilde{\mu}_\lambda^+ \left(\tilde{A} \cap \chi_{F_{\lambda,\alpha,1}{}^+} \right) \right]$$

$$\int_{\tilde{E}} \mathrm{Im} \tilde{f} d\tilde{\mu}_I = \bigcup_{\lambda \in [0,1]} \lambda \left[\int_{\tilde{E}} \mathrm{Im} f_\lambda{}^- d\tilde{\mu}_I, \int_{\tilde{E}} \mathrm{Im} f_\lambda{}^+ d\tilde{\mu}_I \right]$$

$$= \bigcup_{\lambda \in [0,1]} \lambda \left[\sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Im} \tilde{\mu}_\lambda^- (\tilde{A} \cap \chi_{F_{\lambda,\alpha,2}{}^-}, \sup_{\alpha \in [0,\infty)} \alpha \wedge \mathrm{Im} \tilde{\mu}_\lambda^+ (\tilde{A} \cap \chi_{F_{\lambda,\alpha,2}{}^+} \right],$$

where

$$F_{\lambda,\alpha,1}{}^- = \left\{ z | \mathrm{Re} f_\lambda{}^- (z) \geq \alpha \right\},$$

$$F_{\lambda,\alpha,1}{}^+ = \left\{ z | \mathrm{Re} f_\lambda{}^+ (z) \geq \alpha \right\},$$

$$F_{\lambda,\alpha,2}{}^- = \left\{ z | \mathrm{Im} f_\lambda{}^- (z) \geq \alpha \right\},$$

$$F_{\lambda,\alpha,2}{}^+ = \left\{ z | \mathrm{Im} f_\lambda{}^+ (z) \geq \alpha \right\},$$

now called $\tilde{f}$ complex fuzzy set-value complex fuzzy integrable in $\tilde{E}$ about $\tilde{\mu}$.

Complex fuzzy set-valued complex fuzzy integral has the following important properties:

Theorem 1 *([3]) Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space:*

1. *$\forall \tilde{E} \in \mathcal{F}(Z)$, $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, then*

$$\int_{\tilde{E}} \tilde{f} d\tilde{\mu} \in F^{*}(K),$$

2. *Suppose $\tilde{E} \in \mathcal{F}(Z)$, if $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, $\chi_{\tilde{E}}$ is the characteristic function of $\tilde{E}$, then $\int_{\tilde{E}} \tilde{f} d\tilde{\mu} = \int \tilde{f} \chi_{\tilde{E}} d\tilde{\mu}$,*
3. *Let $\tilde{E} \in \mathcal{F}(Z)$. If $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, if $\tilde{\mu}\left(\tilde{E}\right) = \tilde{0}$, then*

$$\int_{\tilde{E}} \tilde{f} d\tilde{\mu} = \tilde{0},$$

4. *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$. If $\tilde{f} \in \tilde{M}\left(\tilde{E}\right)$, if $\tilde{A} \subset \tilde{B}$, then*

$$\int_{\tilde{A}} \tilde{f} d\tilde{\mu} \subseteq \int_{\tilde{B}} \tilde{f} d\tilde{\mu},$$

5. *Let $\tilde{A} \in \mathcal{F}(Z)$. If $\tilde{f}_1, \tilde{f}_2 \in \tilde{M}\left(\tilde{E}\right)$, if $\tilde{f}_1 \subseteq \tilde{f}_2$ in $\tilde{A}$, then*

$$\int_{\tilde{A}} \tilde{f}_1 d\tilde{\mu} \subseteq \int_{\tilde{A}} \tilde{f}_2 d\tilde{\mu},$$

these properties are demonstrated in [18]. Here ignore.

4 Complex Fuzzy Set-Value Complex Fuzzy Integral and Its Convergence Theorem

Theorem 2 *Suppose $(Z, \mathcal{F}(Z), \tilde{\mu})$ is complex fuzzy set-valued fuzzy measure space, $\left\{\tilde{f}_n\right\}$ is non-negative complex fuzzy set-valued complex fuzzy integrable function sequence in $(Z, \mathcal{F}(Z), \tilde{\mu})$, $A \in \mathcal{F}(Z)$, if in A, $\left\{\tilde{f}_n\right\}$ monotone convergence in $\tilde{f}$ incrementing, then $\lim\limits_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$.*

Proof Suppose $A = X$, because $\tilde{f}_n \leq \tilde{f}$, $(n = 1, 2, \ldots)$, so by the generalized complex fuzzy set-valued complex fuzzy integral properties have the following result

$$\lim_{n\to\infty} \int_X \tilde{f}_n d\tilde{\mu} \leq \int_X \tilde{f} d\tilde{\mu},$$

now to proof the opposite inequality.

Let $I = \int_X \tilde{f} d\tilde{\mu}$. Then

1. if $I = 0$, conclusion obvious,
2. if $0 < I < \infty + i\infty$ then

$$I = \sup_{\mathrm{Re}\alpha \in [0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}\left(\tilde{f}\right)_\alpha\right) + i \sup_{\mathrm{Im}\alpha \in [0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_\alpha\right)\right)$$

to know the exist $\alpha_k > 0$ makes

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > ReI - \frac{1}{2k}, S\left(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > ImI - \frac{1}{2k},$$
$$(k = 1, 2, \ldots), \quad another \quad \tilde{f}_n \quad \uparrow \tilde{\ } f \quad have$$

$$\left(\tilde{f}_n\right)_{\alpha_k} \uparrow \left(\tilde{f}\right)_{\alpha_k},$$

then using the properties of generalized triangle norm, know exist n_k, such that, when $n \geq n_k$,

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right)$$

$$> ReI - \frac{1}{2k}, S\left(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right), > ImI - \frac{1}{2k}, (k = 1, 2, \ldots),$$

when $n \geq n_k$,

$$\int_X \tilde{f}_n d\tilde{\mu} > I - \frac{1}{k}, (k = 1, 2, \ldots),$$

by the arbitrariness of k have

$$\int_X \tilde{f} d\tilde{\mu} \leq \lim_{n\to\infty} \int_X \tilde{f}_n d\tilde{\mu}.$$

3. if $I = \infty + i\infty$, then $\alpha_k > 0$ makes

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\left(\tilde{f}_{\alpha_k}\right)\right)\right) > \mathrm{k},$$

$$S(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}(\tilde{f}_{\alpha_k})) > \mathrm{k}, (k = 1, 2, \ldots),$$

exit n_k, such that when $n \geq n_k$,

$$S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > \mathrm{k},$$

$$S\left(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > \mathrm{k},$$

then

$$\lim_{n\to\infty}\int_X \tilde{f}_n d\tilde{\mu} \geq \int_X \tilde{f} d\tilde{\mu} \geq S\left(\mathrm{Re}\alpha_k, \mathrm{Re}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right)$$

$$+iS\left(\mathrm{Im}\alpha_k, \mathrm{Im}\tilde{\mu}\left(\tilde{f}_{\alpha_k}\right)\right) > k + ik\,(k = 1, 2, \ldots),$$

that is $\lim_{n\to\infty}\int_X \tilde{f}_n d\tilde{\mu} \geq \int_X \tilde{f} d\tilde{\mu}$.

Theorem 3 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space,* $\left\{\tilde{f}_n\right\}$ *is complex fuzzy set value complex fuzzy integrable function sequence in* $(Z, \mathcal{F}(Z), \tilde{\mu})$*,* $A \in \mathcal{F}(Z)$*, if in A,* $\left\{\tilde{f}_n\right\}$ *decrease monotonically converges to* $\tilde{f}$*, and for arbitrarily* $\varepsilon_i > 0, (i = 1, 2)$*, where* $\varepsilon = \varepsilon_1 + i\varepsilon_2$ *exit* n_0 *makes*

$$\tilde{\mu}\left(\left\{x|\tilde{f}_{n_0} > \int_A \tilde{f} d\tilde{\mu} + \varepsilon\right\} \cap A\right) < \infty + i\infty,$$

then

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}.$$

Proof $\tilde{f}_1 \geq \tilde{f}_2 \geq \cdots$, so $\int_A \tilde{f}_1 d\tilde{\mu} \geq \int_A \tilde{f}_2 d\tilde{\mu} \geq \cdots$, then

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu},$$

there

$$\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} = \mathop{\wedge}_{n=1}^{\infty}\mathrm{Re}\int_A \tilde{f}_n d\tilde{\mu} + i\mathop{\wedge}_{n=1}^{\infty}\mathrm{Im}\int_A \tilde{f}_n d\tilde{\mu},$$

due to $\forall n$, $\tilde{f}_n \geq \tilde{f}$, so

$$\int_A \tilde{f}_n d\tilde{\mu} \geq \int_A \tilde{f} d\tilde{\mu}$$

have

$$\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} \geq \int_A \tilde{f} d\tilde{\mu}$$

if $\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} > \int_A \tilde{f} d\tilde{\mu}$, then $\int_A \tilde{f} d\tilde{\mu} = \lambda < \infty + i\infty$ where $\lambda = \lambda_1 + i\lambda_2$, and there is $\gamma_i \in (0, \infty)$ where $\gamma = \gamma_1 + i\gamma_2$, $(i = 1, 2)$ makes

$$\mathop{\wedge}_{n=1}^{\infty}\int_A \tilde{f}_n d\tilde{\mu} > \gamma > \lambda,$$

$$\Rightarrow \forall n, \sup_{\mathrm{Re}\alpha \in [0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}[\left(\tilde{f}_n\right)_\alpha \cap A]\right)$$

$$+i \sup_{\mathrm{Im}\alpha \in [0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left(\left(\tilde{f}_n\right)_\alpha \cap A]\right)\right) > \gamma,$$

$$\Rightarrow \forall n, S\left(\gamma, \tilde{\mu}\left(\left(\tilde{f}_n\right)_\gamma \cap A\right)\right) = S\left(\mathrm{Re}\gamma, \mathrm{Re}\tilde{\mu}\left(\left(\tilde{f}_n\right)_\gamma \cap A\right)\right)$$

$$+iS\left(\mathrm{Im}\gamma, \mathrm{Im}\tilde{\mu}\left(\left(\tilde{f}_n\right)_\gamma \cap A\right)\right) > \gamma,$$

take $\varepsilon_l = \frac{\gamma_l+\lambda_l}{2}$ $(l = 1, 2)$ exit n_0,

$$\tilde{\mu}\left(\left\{x|\tilde{f}_{n_0} > \int_A \tilde{f}d\tilde{\mu} + \varepsilon\right\} \cap A\right) < \infty + i\infty, \gamma_l = \lambda_l + 2\varepsilon_l > \lambda_l + \varepsilon_l$$

$$\Rightarrow \left\{x|\tilde{f}_{n_0} \geq \gamma\right\} \subseteq \left\{x|\tilde{f}_{n_0} > \lambda + \varepsilon\right\},$$

$$\Rightarrow \tilde{\mu}\left(A \cap \left(\tilde{f}_{n_0}\right)_\gamma\right) \leq \tilde{\mu}\left(\left\{x|\tilde{f}_{n_0} > \lambda + \varepsilon\right\} \cap A\right) < \infty + i\infty.$$

By the continuity of $\tilde{\mu}$, $A \cap \left(\tilde{f}_{n_1}\right)_\gamma \supseteq A \cap \left(\tilde{f}_{n_2}\right)_\gamma \supseteq \ldots$,

$$\tilde{\mu}\left(A \cap \left(\tilde{f}\right)_\gamma\right)$$

$$= \tilde{\mu}\left(\bigcap_{n=1}^{\infty}\left[A \cap \left(\tilde{f}_n\right)_\gamma\right]\right) = \lim_{n\to\infty} \tilde{\mu}\left(A \cap \left(\tilde{f}_n\right)_\gamma\right) \geq \gamma \geq \lambda$$

$$\int_A \tilde{f}d\tilde{\mu} \stackrel{\Delta}{=} \sup_{\mathrm{Re}\alpha \in [0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right)$$

$$+i \sup_{\mathrm{Im}\alpha \in [0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right)$$

$$\geq S\left(\mathrm{Re}\gamma, \mathrm{Re}\tilde{\mu}\left[\tilde{f}_\gamma \cap A\right]\right) + iS\left(\mathrm{Im}\gamma, \mathrm{Im}\tilde{\mu}\left[\tilde{f}_\gamma \cap A\right]\right) > \lambda = \lambda_1 + i\lambda_2,$$

conflicting with $\int_A \tilde{f}d\tilde{\mu} = \lambda$, so $\bigwedge_{n=1}^{\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f}d\tilde{\mu}$, that is

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}.$$

Theorem 4 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space,* $\left\{\tilde{f}_n\right\}$ *is complex fuzzy set value complex fuzzy integrable function sequence in* $(Z, \mathcal{F}(Z), \tilde{\mu})$ $A \in \mathcal{F}(Z)$*, if in* A*,* $\left\{\tilde{f}_n\right\}$ *convergence in* $\tilde{f}$*, and for arbitrary* $\varepsilon_k > 0$, $(k = 1, 2)$*, where,* $\varepsilon = \varepsilon_1 + i\varepsilon_2$ *exit* n_0 *makes*

$$\tilde{\mu}\left(\left\{x \mid \sup_{n\geq n_0} \tilde{f}_n > \int_A \tilde{f} d\tilde{\mu} + \varepsilon\right\} \cap A\right) < \infty + i\infty.$$

then $\lim\limits_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$.

Proof Let $\tilde{h}_n = \overset{n}{\underset{k=1}{\vee}} \tilde{f}_k$, $\tilde{g}_k = \overset{n}{\underset{k=1}{\wedge}} \tilde{f}_k$. Then $\left\{\tilde{h}_n\right\} \downarrow \tilde{f}$ and $\{\tilde{g}_n\} \uparrow \tilde{f}$, $\tilde{g}_n \leq \tilde{f}_n \leq \tilde{h}_n$, so

$$\int_A \tilde{g}_n d\tilde{\mu} \leq \int_A \tilde{f}_n d\tilde{\mu} \leq \int_A \tilde{h}_n d\tilde{\mu}$$

by theory 3,

$$\lim_{n\to\infty}\int_A \tilde{g}_n d\tilde{\mu} = \lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu}$$

$$= \lim_{n\to\infty}\int_A \tilde{h}_n d\tilde{\mu} = \lim_{n\to\infty}\int_A \tilde{f} d\tilde{\mu}.$$

Definition 6 ([3]) Given a fuzzy complex value fuzzy measure space $\left(C, \tilde{F}, \tilde{\mu}\right)$, let $\tilde{f}_n$ $(n = 1, 2, \ldots)$ and $\tilde{f} : C \to F^*(C)$ are fuzzy complex value fuzzy measurable function, $\tilde{A} \in \tilde{F}$ then

(1) $\left\{\tilde{f}_n\right\}$ almost everywhere converges to $\tilde{f}$ on $\tilde{A}$, if $\tilde{\mu}\left(\tilde{E}\right) = \tilde{0}$ for $\tilde{E} \in \tilde{F}$ and makes $\left\{\tilde{f}_n\right\}$ converges to $\tilde{f}$ point by point on $\tilde{A} - \tilde{E}$, note that $\tilde{f}_n \overset{a.e}{\to} \tilde{f}$;

(2) $\left\{\tilde{f}_n\right\}$ almost uniform converges to $\tilde{f}$ on $\tilde{A}$, if $\exists\tilde{E} \in \tilde{F}$ for $\varepsilon > 0$, $\left|\tilde{\mu}_\alpha\left(\tilde{E}\right)\right| < \varepsilon$ and makes $\left\{\tilde{f}_n\right\}$ uniform converges to $\tilde{f}$ point by point, note that $\tilde{f}_n \overset{a.u}{\to} \tilde{f}$;

(3) $\left\{\tilde{f}_n\right\}$ pseudo almost everywhere converges to $\tilde{f}$, if $\hat{\mu}\left(\tilde{A} - \tilde{E}\right) = \tilde{\mu}\left(\tilde{E}\right)$ for $\tilde{E} \in \tilde{F}$ and makes $\left\{\tilde{f}_n\right\}$ converges to $\tilde{f}$ point by point on $\tilde{A} - \tilde{E}$, note that $\tilde{f}_n \overset{p.a.e}{\to} \tilde{f}$;

(4) $\left\{\tilde{f}_n\right\}$ pseudo almost uniform converges to $\tilde{f}$, if $\lim\limits_{n\to\infty}\tilde{\mu}\left(\tilde{A} - \tilde{E}_k\right) = \tilde{\mu}\left(\tilde{A}\right)$ for $\left\{\tilde{E}_k\right\} \subset \tilde{F}$ and makes $\left\{\tilde{f}_n\right\}$ uniform converges to $\tilde{f}$ point by point on $\tilde{A} - \tilde{E}$ for any fixed point, $k = 1, 2, 3\ldots$ note that $\tilde{f}_n \overset{p.a.u}{\to} \tilde{f}$;

(5) $\left\{\tilde{f}_n\right\}$ converges in measure to $\tilde{f}$, if

$$\lim_{n\to\infty}\tilde{\mu}\left(\left\{x|\left\{\tilde{f}_n(x)-\tilde{f}(x)\right\}>\varepsilon\right\}\cap\tilde{A}\right)=0$$

for any $\varepsilon > 0$, note that $\tilde{f}_n \overset{\tilde{\mu}}{\to} \tilde{f}$;

(6) $\left\{\tilde{f}_n\right\}$ pseudo converges in measure to $\tilde{f}$, if

$$\lim_{n\to\infty}\tilde{\mu}\left(\left\{x|\left\{\tilde{f}_n(x)-\tilde{f}(x)\right\}>\varepsilon\right\}\cap\tilde{A}\right)=\tilde{\mu}\left(\tilde{A}\right),$$

note that $\tilde{f}_n \overset{p.\tilde{\mu}}{\to} \tilde{f}$.

Theorem 5 *Suppose* $(Z,\mathcal{F}(Z),\tilde{\mu})$ *is Complex fuzzy set-valued fuzzy measure space* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *is complex fuzzy set-valued complex fuzzy measurable function in* $(Z,\mathcal{F}(Z),\tilde{\mu})$, $A\in\mathcal{F}(Z)$, *if* $\tilde{f}_n \overset{a.e}{\to} \tilde{f}$, $\tilde{\mu}$ *is zero-additive, and for arbitrarily* $\varepsilon_k > 0$, $(k=1,2)$, *where* $\varepsilon=\varepsilon_1+i\varepsilon_2$ *exit* n_0 *makes*

$$\tilde{\mu}\left(\left\{x|\sup_{n\geq n_0}\tilde{f}_n\rangle\right\}\int_A \tilde{f}d\tilde{\mu}+\varepsilon\cap A\right)<\infty+i\infty,$$

then $\lim\limits_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu}=\int_A \tilde{f}d\tilde{\mu}$.

Proof Because $\tilde{f}_n \overset{a.e}{\to} \tilde{f}$ in A, then exit $B\in\mathcal{F}(Z)$, $\tilde{\mu}(B)=0$, $\tilde{\mu}$ is zero-additive, $\int_{A\backslash B}\tilde{f}d\tilde{\mu}\overset{\Delta}{=}$

$$\sup_{\mathrm{Re}\alpha\in[0,\infty)} S\left(\mathrm{Re}\alpha,\mathrm{Re}\tilde{\mu}\left[\tilde{f}_\alpha\cap(A\backslash B)\right]\right)+i\sup_{\mathrm{Im}\alpha\in[0,\infty)} S\left(\mathrm{Im}\alpha,\mathrm{Im}\tilde{\mu}\left[\tilde{f}_\alpha\cap(A\backslash B)\right]\right)$$
$$=\sup_{\mathrm{Re}\alpha\in[0,\infty)} S\left(\mathrm{Re}\alpha,\mathrm{Re}\tilde{\mu}\left[\left(A\cap\tilde{f}_\alpha\right)\backslash B\right]\right)+i\sup_{\mathrm{Im}\alpha\in[0,\infty)} S\left(\mathrm{Im}\alpha,\mathrm{Im}\tilde{\mu}\left[\left(A\cap\tilde{f}_\alpha\right)\backslash B\right]\right)$$
$$=\sup_{\mathrm{Re}\alpha\in[0,\infty)} S\left(\mathrm{Re}\alpha,\mathrm{Re}\tilde{\mu}\left(A\cap\tilde{f}_\alpha\right)\right)+i\sup_{\mathrm{Re}\alpha\in[0,\infty)} S\left(\mathrm{Im}\alpha,\mathrm{Im}\tilde{\mu}\left(A\cap\tilde{f}_\alpha\right)\right)$$
$$=\int_A \tilde{f}d\tilde{\mu}.$$

Similarly

$$\int_A \tilde{f}_n d\tilde{\mu}=\int_{A\backslash B}\tilde{f}_n d\tilde{\mu},$$

and because

$$\tilde{\mu}\left(\left\{x|\sup_{n\geq n_0}\tilde{f}_n>\int_A \tilde{f}d\tilde{\mu}+\varepsilon\right\}\cap A\right)<\infty+i\infty,$$

and from the Theorem 2, obtained $\lim\limits_{n\to\infty}\int_{A\backslash B}\tilde{f}_n d\tilde{\mu}=\int_{A\backslash B}\tilde{f}d\tilde{\mu}$, so $\lim\limits_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu}=\int_A \tilde{f}d\tilde{\mu}$.

Theorem 6 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *is complex fuzzy set-valued complex fuzzy measurable function in* $(Z, \mathcal{F}(Z), \tilde{\mu})$, $A \in \mathcal{F}(Z)$, *if* $\left\{\tilde{f}_n\right\}$ *uniform convergence in* $\tilde{f}$ *in A, then*

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}.$$

Proof

(1) If $\int_A \tilde{f} d\tilde{\mu} = \infty + i\infty$, let $\tilde{g}_n = \overset{n}{\underset{k=1}{\wedge}} \tilde{f}_k$ then

$$\lim_{n\to\infty}\int_A \tilde{f}_n d\tilde{\mu} \geq \int_A \tilde{g}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu} = \infty + i\infty.$$

(2) If

$$\int_A \tilde{f} d\tilde{\mu} = \sup_{\mathrm{Re}\alpha\in[0,\infty)} S\left(\mathrm{Re}\alpha, \mathrm{Re}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right) + i \sup_{\mathrm{Im}\alpha\in[0,\infty)} S\left(\mathrm{Im}\alpha, \mathrm{Im}\tilde{\mu}\left[\tilde{f}_\alpha \cap A\right]\right)$$

$= \lambda < \infty + i\infty$, then,

$$\forall\alpha \leq \lambda, \tilde{\mu}\left[\tilde{f}_\alpha \cap A\right] \geq \lambda;$$

$$\forall\alpha > \lambda, \tilde{\mu}\left[\tilde{f}_\alpha \cap A\right] \leq \lambda;$$

α_n monotone decreasing trend to λ, then $\tilde{f}_{\alpha_1} \subseteq \tilde{f}_{\alpha_2} \subseteq \cdots$ and $\overset{\infty}{\underset{n=1}{\cap}} \tilde{f}_{\alpha_n} = \tilde{f}_{\dot{\lambda}}$ by under continuous of $\tilde{\mu}$, $\tilde{\mu}\left(\tilde{f}_{\dot{\lambda}} \cap A\right) = \lim_{n\to\infty} \tilde{\mu}\left(\tilde{f}_{\alpha_n} \cap A\right) \leq \lambda$, uniform convergence in $\tilde{f}$ in A, arbitrarily $\varepsilon'_k > 0$, $(k = 1, 2)$ where $\varepsilon' = \varepsilon'_1 + i\varepsilon'_2$ exit n_0, $\forall x \in A$,

$$\tilde{f}_n(x) \leq \tilde{f}(x) + \varepsilon \Rightarrow \sup_{n\geq n_0} \tilde{f}_n(x) \leq \tilde{f}(x) + \varepsilon$$

$$\Rightarrow \left\{x \middle| \sup_{n\geq n_0} \tilde{f}_n(x) \geq \lambda + \varepsilon\right\} \cap A \subseteq \left\{x \middle| \tilde{f}(x) \geq \lambda\right\} \cap A = \tilde{f}_{\dot{\lambda}} \cap A$$

$$\Rightarrow \tilde{\mu}\left(\left\{x \middle| \sup_{n\geq n_0} \tilde{f}_n(x) \geq \lambda + \varepsilon\right\} \cap A\right)$$

$$\leq \lambda < \infty + i\infty$$

by Theorem 3 has $\lim_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$.

Theorem 7 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space,* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *are complex fuzzy set-valued complex fuzzy measurable function on* $(Z, \mathcal{F}(Z), \tilde{\mu})$, $A \in \mathcal{F}(Z)$, *if* $\tilde{f}_n \overset{a.e.u}{\to} \tilde{f}$ *on A, and* $\tilde{\mu}$ *is zero-additive, then* $\lim\limits_{n\to\infty} \int_A \tilde{f}_n d\tilde{\mu} = \int_A \tilde{f} d\tilde{\mu}$.

Proof Similar to the proof method of Theorem 6.

Theorem 8 *Suppose* $(Z, \mathcal{F}(Z), \tilde{\mu})$ *is complex fuzzy set-valued fuzzy measure space* $\left\{\tilde{f}_n\right\}$, $\tilde{f}$ *are complex fuzzy set-valued complex fuzzy measurable function on* $(Z, \mathcal{F}(Z), \tilde{\mu})$, $A \in \mathcal{F}(Z)$, *if* $\tilde{f}_n \overset{a.u}{\to} \tilde{f}$ *in A, and* $\tilde{\mu}$ *is zero-additive, and exit*

$$\{B_k\} \subseteq \mathcal{F}(Z),\, B_1 \supseteq B_2 \supseteq \cdots,\, \tilde{\mu}(B_k) \to 0$$

makes

$$\lim_{n\to\infty} \int_{A\backslash B_k} \tilde{f}_n d\tilde{\mu} = \int_{A\backslash B_k} \tilde{f} d\tilde{\mu}.$$

Proof If $\tilde{f}_n \overset{a.u}{\to} \tilde{f}$ in A, then exit $\{E_k\} \subseteq \mathcal{F}(Z)$, $\tilde{\mu}(E_k) \to 0$, $\tilde{f}_n \overset{u}{\to} \tilde{f}$ in $A\backslash E_k$, let

$$B_k = \bigcap_{i=1}^{k} E_i \subseteq E_k.$$

Then

$$A\backslash B_k = \bigcap_{i=1}^{k} (A\backslash E_i),$$

$\forall k$, $\tilde{f}_n \overset{u}{\to} \tilde{f}$ in $A\backslash E_k$, $\tilde{f}_n \overset{u}{\to} \tilde{f}$ in $A\backslash B_k$, by Theorem 6 know the conclusion is right.

5 Conclusion

In this paper, the fuzzy measure concepts was extended from general classical set to ordinary complex fuzzy set, fuzzy, research complex fuzzy set-valued complex fuzzy measure and its properties, and measurable function in complex fuzzy set value complex fuzzy measure space and its properties was studied, of the of extension of the scope of classical measure theory, generalization of the corresponding conclusion of classical measure theory; research Integral theory problem of complex fuzzy set-valued function base on complex fuzzy set-valued measure, establish complex fuzzy set-valued complex fuzzy Integral theory, which is important work in fuzzy complex analysis. This work extends the fuzzy measure and fuzzy integral theory, to lay a solid foundation for our future research on complex fuzzy set-valued complex fuzzy integral application problem.

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References

1. Ma, S., Li, S.: Complex fuzzy set-valued complex fuzzy measures and their properties. Sci. World J. **2014**, Article ID 493703, 7 paper
2. Ma, S., Zhao, H., Li, S.: Measurable function on complex fuzzy set-valued complex fuzzy measure space and its properties. Fuzzy Syst. Math. (2014) (has been finalized)
3. Ma, S., Li, S., Zhao, H., Zhou, M.: Complex fuzzy set-value complex fuzzy integrals and their properties. J. Lanzhou Univ. (Nat. Sci. Ed.) (has been finalized)
4. Wu, C.X., Zhang, D.L., Guo, C.M.: Fuzzy number fuzzy measure and fuzzy integrals (I): fuzzy integrals of functions with respect to fuzzy number fuzzy measure. Fuzzy Sets Syst. **98**(3), 355–360 (1998)
5. Guo, C.M., Zhang, D.L., Wu, C.X.: Fuzzy-valued fuzzy measures and generalized fuzzy integrals. Fuzzy Sets Syst. **97**(2), 255–260 (1998)
6. Wu, C.X., Zhang, D.L., Zhang, B., et al.: Fuzzy number fuzzy measure and fuzzy integrals (I): fuzzy-valued functions with respect to fuzzy number fuzzy measure on fuzzy sets. Fuzzy Sets Syst. **107**(2), 219–226 (1999)
7. Buckley, J.J.: Fuzzy complex numbers. Fuzzy Sets Syst. **33**(3), 333–345 (1989)
8. Zhang, G.: Fuzzy number-valued fuzzy measure and fuzzy number-valued fuzzy Integral on the fuzzy set. Fuzzy Sets Syst. **49**(3), 357–376 (1992)
9. Zhang, G.: The structural characteristics of the fuzzy number-valued fuzzy measure on the fuzzy algebra and their applications. Fuzzy Sets Syst. **52**(1), 69–81 (1992)
10. Zhang, G.: The convergence for a sequence of fuzzy integrals of fuzzy number-valued function on the fuzzy set. Fuzzy Sets Syst. **59**(1), 43–57 (1993)
11. Zhang, G.: On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals on the fuzzy set. Fuzzy Sets Syst. **45**(2), 227–237 (1992)
12. Zhang, G.: On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals on the fuzzy set. Fuzzy Sets Syst. **48**(2), 257–265 (1992)
13. Qiu, J., Li, F., Su, L.: Complex fuzzy measure and complex fuzzy integral. J. Lanzhou Univ. (Nat. Sci. Ed.) **32**, 155–158 (1996)
14. Wang, G., Li, X.: Lebesugue integral with respect to the fuzzy complex-valued measure. J. Harbin Normal Univ. (Nat. Sci. Ed.) **15**(2), 20–26 (1999)
15. Ma, S., Chen, F., Wang, Q., Zhao, Z.: The design of fuzzy classifier base on Sugeno type fuzzy complex-value integral. In: Proceedings of 2011 Seventh International Conferen on Computational Intelligence and Security (published by CPS, IEEE computer society), pp. 338–342 (2011)
16. Ma, S., Chen, F., Zhao, Z.: Choquet type fuzzy complex-valued integral and its application in classification. Fuzzy Engineering and Operations Research, AISC, vol. 147, pp. 229–237. Springer (2012)
17. Ma, S., Chen, F., Wang, Q., Zhao, Z.: Sugeno type fuzzy complex-value integral and its application in classification. Procedia Eng. ELSEVIER **29**, 4140–4151 (2012)
18. Yao, O., Jun, L.: Note on the definition of fuzzy number-valued fuzzy measurable functions. J. Southeast Univ. (Nat. Sci. Ed.) **33**(6), 801–803 (2003)

Complex Fuzzy Matrix and Its Convergence Problem Research

Zhi-qing Zhao and Sheng-quan Ma

Abstract In this paper, gives the definition of complex fuzzy matrix, and study its convergence problems based on the fuzzy matrix theory, which included the Convergence in norm and the Convergence in Power, some important conclusions are obtained, to build and to improve the solid foundation for complex fuzzy matrix theory.

Keywords Complex fuzzy matrix · Convergence in norm · Convergence in power

1 Introduction

The fuzzy matrix is an important part of fuzzy mathematics, which plays an important role in the fuzziness expression of two dimensional relationship, which has important applications in the circuit design the exchange of information, cluster analysis, etc., but, because the diversity of the research fields and the target, to solve complicated system problems will require higher dimensions, in this paper, based on the fuzzy matrix theory, gives the definition of complex fuzzy matrix, and study its convergence problems of complex fuzzy matrix, to establish complex fuzzy matrix theory and to laid a solid foundation for the further research.

2 Complex Fuzzy Sets and Complex Fuzzy Number

Definition 1 ([1]) Let R be the real field, C is complex field. For $\forall X, Y \in F(R)$, $Z = X + iY$, called the complex fuzzy sets, referred to as the complex fuzzy sets.

Definition 2 ([1]) Suppose $Z_1 = X_1 + iY_1, Z_2 = X_2 + iY_2 \in C^F(C)$, the Operation of intersection and union be defined as: $\forall z = x + iy \in C$

Z. Zhao · S. Ma (✉)
School of Information and Technology, Hainan Normal University, Haikou 571158, China
e-mail: 964976605@qq.com

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$$(Z_1 \cap Z_2)(z) \overset{\Delta}{=} (X_1 \cap X_2)(x) \wedge (Y_1 \cap Y_2)(y) = (X_1(x) \wedge X_2(x)) \wedge (Y_1(y) \wedge Y_2(y))$$

$$(Z_1 \cup Z_2)(z) \overset{\Delta}{=} (X_1 \cup X_2)(x) \wedge (Y_1 \cup Y_2)(y) = (X_1(x) \vee X_2(x)) \wedge (Y_1(y) \vee Y_2(y)).$$

Definition 3 Suppose $\{Z_\gamma = X_\gamma + iY_\gamma, \gamma \in \Gamma\} \subseteq C^F(C)$, the operations of infinite intersection and infinite unit of the complex fuzzy sets are defined as: $\forall z = x + iy \in C$,

$$(\bigcap_{\gamma\in\Gamma} Z_\gamma)(z) \overset{\Delta}{=} (\bigcap_{\gamma\in\Gamma} X_\gamma)(x) \wedge (\bigcap_{\gamma\in\Gamma} Y_\gamma)(y) = (\bigwedge_{\gamma\in\Gamma} X_\gamma(x)) \wedge (\bigwedge_{\gamma\in\Gamma} Y_\gamma(y))$$

$$(\bigcap_{\gamma\in\Gamma} Z_\gamma)(z) \overset{\Delta}{=} (\bigcup_{\gamma\in\Gamma} X_\gamma)(x) \wedge (\bigcup_{\gamma\in\Gamma} Y_\gamma)(y) = (\bigvee_{\gamma\in\Gamma} X_\gamma(x)) \wedge (\bigvee_{\gamma\in\Gamma} Y_\gamma(y))$$

$$Z_1 \cap Z_2, Z_1 \cup Z_2, \bigcap_{\gamma\in\Gamma} Z_\gamma, \bigcup_{\gamma\in\Gamma} Z_\gamma \in C^F(C).$$

Definition 4 The regular convex complex fuzzy sets in the complex field, $Z = X + iY$ is called a complex fuzzy numbers.

3 Fuzzy Matrix

Definition 5 ([2]) All the elements of a matrix are in the closed interval [0, 1], which called the fuzzy matrix.

Union, intersection, complement operation, the corresponding element of the two fuzzy matrix: take big, take small, take up, which obtain a new element matrix called their union, intersection, complement operation.

Concrete concepts and properties of fuzzy matrix, please refer to the Ref. [2].

4 Complex Fuzzy Matrix

Definition 6 Suppose X is a non empty sets of real numbers, called

$$C = \{A(x) + iB(x), x \in X\}$$

is complex fuzzy sets on X, $A(x)$, $B(x)$ are real fuzzy number, $A(x) + iB(x)$ is complex fuzzy number, where $A(x) : X \to [0, 1]$, $B(x) : X \to [0, 1]$, which expressed the real parts and the imaginary parts of C separately, $0 \le A(x) + B(x) \le 1$.

Definition 7 Suppose $C = \left(A_{ij}(x) + iB_{ij}(x)\right)_{n\times m}$ is matrix, all of the $A_{ij}(x) + iB_{ij}(x)$ is complex fuzzy number for i, j $(1 \le i \le n, 1 \le j \le n)$, then called C is complex fuzzy matrix, note as $CFM(n, m)$.

Definition 8 Suppose

$$C_1 = \left(A_{ij}(x) + iB_{ij}(x)\right)_{n\times m},$$

$$C_2 = \left(D_{ij}(x) + iE_{ij}(x)\right)_{n\times m}$$

are complex fuzzy matrix, the operation of the A and B is defined as:

$$C_1 \cup C_2 = \left(A_{ij}(x) \vee D_{ij}(x), B_{ij}(x) \wedge E_{ij}(x)\right)_{n\times m}.$$

Definition 9 Suppose

$$C_1 = \left(A_{ij}(x) + iB_{ij}(x)\right)_{n\times m},$$

$$C_2 = \left(D_{ij}(x) + iE_{ij}(x)\right)_{n\times m}$$

are complex fuzzy matrix, the Product operation of the A and B is defined as:

$$C_1C_2 = \left(C_1C_2R_{ij}(x) + iC_1C_2I_{ij}(x)\right)_{n\times m},$$

$$C_1C_2R_{ij}(x) = \underset{1\le k\le m}{\vee} \left(A_{ik}(x) \wedge D_{kj}(x)\right),$$

$$C_1C_2I_{ij}(x) = \underset{1\le k\le m}{\wedge} \left(B_{ik}(x) \vee E_{kj}(x)\right).$$

5 Complex Fuzzy Matrix Norm and Convergence in Norm

Definition 10 ([3]) Let $F = R$ or C, V is a linear space over F. If the real vector function $\|*\|$ on V satisfies the following properties:

1. For arbitrary $x \in V$, $\|x\| \ge 0$, and $\|x\| = 0 \Leftrightarrow x = 0$,
2. For arbitrary $k \in F$, $x \in V$ get $\|kx\| = \|k\| \|x\|$,
3. For arbitrary $x, y \in V$, get $\|x + y\| \le \|x\| + \|y\|$,

then $\|x\|$ is called vector norm of X in V.

Definition 11 Suppose $\|*\|$ is non negative real function on $F^{n\times n}$, if

$$\left\|C_1C_2R_{ij}(x)\right\| \le \left\|C_1R_{ij}(x)\right\| \left\|C_2R_{ij}(x)\right\|,$$

$$\left\|C_1C_2I_{ij}(x)\right\| \le \left\|C_1I_{ij}(x)\right\| \left\|C_2I_{ij}(x)\right\|,$$

then called $\|*\|$ is $CFM(n,m)$.

Definition 12 ([4]) Suppose $(V, \|*\|)$ is a n-dimensional normed linear space, $x_1, x_2, \ldots, x_k, \ldots$ is a vector sequence of V, α is a fixed vector V, if

$$\lim_{k\to\infty} \|x_k - \alpha\| = 0,$$

then called vector sequence $x_1, x_2, \ldots, x_k, \ldots$ convergence in norm, A is the limit of a sequence, note as:

$$\lim_{k\to\infty} x_k = \alpha \text{ or } x_k \to \alpha,$$

vector sequence does not converge called divergence.

Definition 13 Suppose $(V, \|*\|)$ is a n-dimensional normed linear space, $c_1, c_2, \ldots, c_k, \ldots$ is a complex fuzzy matrix sequence of V, $c(k)$ constitutes a complex fuzzy matrix function, $\alpha = \alpha R + i\alpha I$ is a fixed complex fuzzy matrix of V, if

$$\lim_{k\to\infty} \|cR(k) - \alpha R\| = 0, \lim_{k\to\infty} \|cI(k) - \alpha I\| = 0,$$

then called complex fuzzy matrix sequence $c_1, c_2, \ldots, c_k, \ldots$ convergence in norm, $\alpha = \alpha R + i\alpha I$ is the limit of a sequence, note to: $\lim\limits_{k\to\infty} x(k) = \alpha$ or $x_k \to \alpha$.

Definition 14 Suppose $(V, \|*\|)$ is a n-dimensional normed linear space $C(k)$, $(n = 1, 2, \ldots)$, $c : C\text{FM}(n,m) \to C\text{FM}(n,m)$, then

1. $\{c(k)\}$ almost everywhere convergence c, in V, if there is $E \in V$, $c(E) = 0$, makes $\{c(k)\}$ pointiest convergence on c in $V - E$, note as $c(k) \overset{a.e}{\to} c$ in V,
2. $\{c(k)\}$ almost uniform convergence c in V, if there is $E \in V$, for any $\varepsilon > 0$, $\|c(E)\| < \varepsilon$, makes $\{c(k)\}$ pointiest uniform convergence on c in $V - E$, note as $c(k) \overset{a.u}{\to} c$ in V,
3. $\{c(k)\}$ pseudo almost everywhere convergence c in V, if there is $E \in V$, $c(V - E) = c(E)$, makes $\{c(k)\}$ pointiest convergence on c in $V - E$, note as $c(k) \overset{p.a.e}{\to} c$ in V,
4. $\{c(k)\}$ pseudo almost uniform convergence c in V, if there is $\{E_k\} \subset V$, $\lim\limits_{n\to\infty} c(V - E_k) = c(A)$ makes $\{c(k)\}$ pointiest uniform convergence on c in $V - E$, note as $c(k) \overset{p.a.u}{\to} c$ in V.

Theorem 1 *A necessary and sufficient condition of null additives of c is for any* $A \in V$, $c \in C\text{FM}(n,m)$, $c(k) \in C\text{FM}(n,m)$ *and* $\|CMF(n,m)\| < \infty$ *has*

$$c(k) \underset{A}{\overset{a.e}{\to}} c \Rightarrow c(k) \underset{A}{\overset{p.a.u}{\to}} c.$$

***Proof*: *Necessity*:**

$$c(k) \underset{V}{\overset{a.e}{\to}} c \Rightarrow c(k) \underset{V}{\overset{a.u}{\to}} c,$$

by consistency theorem has

$$c(k) \underset{V}{\overset{a.e}{\to}} c \Rightarrow c(k) \underset{V}{\overset{p.a.u}{\to}} c.$$

Sufficiency: *Suppose for any $B \in V$ has $c(B) = 0$, let $x \in V - B$, $c(x) = 0$, so obviously there is $c(k) \underset{V}{\overset{a.e}{\to}} 0$, exist P_k, when $k \to \infty$ has $c(V - P_k) \to C(V)$, and $\{c(k)\}$ uniform convergence 0 on $V - p_k$, $n = 1, 2, \ldots$, at this time there will be $V - B \supseteq V - P_k$, $k = 1, 2, \ldots$, $c(V - B) \geq c(V - P_k) \to c(V)$, therefore it is $c(V - B) = c(V)$.*

6 The Power of Complex Fuzzy Matrix and Its Power Convergence

Definition 15 Suppose $A \in CMF(n, n)$ power K of A is defined as A^k, among them $A^1 = A$, $A^k = A^{k-1}A$.

Theorem 2 *Suppose $A \in CFM(n, n)$, there are must be a positive integer P and K, making $\forall k \geq K$ has $A^{k+p} = A^k$.*

Proof: *Let $\forall k \geq 1$, $A = \left(A_{ij}(x) + iB_{ij}(x)\right)$,*

$$A^k = \left(A_{ij}(x) + iB_{ij}(x)\right)_{n\times n}^k = \left(A_{ij}^k(x) + iB_{ij}^k(x)\right)_{n\times n},$$

$$A_{ij}^k(x) = \bigvee_{1\leq t_1,\ldots,t_{k-1}\leq n} \left(A_{it_1} \wedge A_{it_2} \wedge \cdots \wedge A_{it_{k-1}}\right),$$

$$B_{ij}^k(x) = \bigwedge_{1\leq t_1,\ldots,t_{k-1}\leq n} \left(A_{it_1} \vee A_{it_2} \vee \cdots \vee A_{it_{k-1}}\right),$$

because the $\wedge, \vee$ is closed, so the elements number of $\left\{A^k, k \geq 1\right\}$ in not more than $\left(n^{4n}\right)^n$ there must be a positive integer P and K, make $A^{k+p} = A^k$, so for any $k > K$ has

$$A^{k+p} = A^{(k-k)+k+p} = A^{k-k}A^{k+p} = A^{k-k}A^k = A^k.$$

Theorem 3 *Suppose $A \in CFM(n, n)$, which is power sequence monotone increasing of A, hence*

$$A^n = A^{n+1} = A^{n+2} = \cdots = \lim_{k\to\infty} A^k,$$

that is, power sequences of A is Convergence.

Proof: *Because power sequence of A monotone increasing, then* $A^n \le A^{n+1}$, *and from the* $A^{n+1} \le A \vee A^2 \vee \cdots \vee A^n = A^n$, *so* $A^n = A^{n+1}$.

7 Conclusion

In this paper, by studied complex fuzzy matrix convergence problems based on the fuzzy matrix theory, which included the Convergence in norm and the Convergence in Power, which is important work in fuzzy complex analysis, to build and to lay a solid foundation for our future research on complex fuzzy matrix theory.

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References

1. Ma, S.: The Theory of Fuzzy Complex Analysis Foundation. Science Press, Beijing (2010)
2. Fan, Z.: Theory and Application of Fuzzy Matrix. Science Press, Beijing (2006)
3. Yan, L., Gao, Y.: The Principle of Fuzzy Mathematics and Applications, 3rd edn., pp. 90–94. South China University of Technology Press, Guangzhou (2001)
4. Zhang, Y.: Matrix Theory and Application. Science Press, Beijing (2011)

Rough Fuzzy Concept Lattice and Its Properties

Chang Shu and Zhi-wen Mo

Abstract Concept lattice and rough set theory, two different methods for knowledge representation and knowledge discovery, are successfully applied to many fields. Methods of fuzzy rule extraction based on rough set theory are rarely reported in incomplete interval-valued fuzzy information systems. Thus, this paper deals with the relationship of such systems and fuzzy concept lattice. The purpose of this paper is to study a new model called rough fuzzy concept lattice (RFCL) and its properties.

Keywords Rough fuzzy concept lattice · Fuzzy concept lattice · Rough set theory · Fuzzy formal context · Fuzzy formal concept · Fuzzy equivalence class

1 Introduction

With the development of computer science, more and more attention is paid to the research of its mathematical foundations which have been the common field of mathematicians and computer scientists. Domain theory (DT), formal concept analysis (FCA) and rough set theory (RST) are three important crossing fields based on relations (orders) and simultaneously related to topology, algebra, logic, etc., and provide mathematical foundations for computer science and information science.

In this paper, a covariant Galois connection is put forward by approximation operators of rough sets, thus rough fuzzy concept lattice is established. We also discuss its properties and attribute reduction of fuzzy format concept based on rough fuzzy concept lattice. So profoundly reveals the connection of two knowledge discovery tool, to prepare for the better application prospect.

C. Shu (✉)
Department of Mathematics Teaching, Chengdu University of Technology,
Chengdu 610059, Sichuan, People's Republic of China
e-mail: 20812229@qq.com

Z. Mo
College of Mathematics and Software Science, Sichuan Normal University,
Chengdu 610066, Sichuan, People's Republic of China
e-mail: mozhiwen@263.net

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2 Rough Fuzzy Concept Lattices

Definition 1 $\left(U, A, \tilde{I}\right)$ is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties and $\tilde{I}(U \times A \to [0, 1])$ is a fuzzy binary relations defined in object set to the set of properties. For $x \in U, a \in A$, $I_\delta(x, a) = \begin{cases} 1, \tilde{I}(x, a) \geq \delta \\ 0, \tilde{I}(x, a) < \delta \end{cases}$ $I_{\overline{\delta}}(x, a) = \begin{cases} 1, \tilde{I}(x, a) > \delta \\ 0, \tilde{I}(x, a) \leq \delta \end{cases}$ $0 \leq \delta \leq 1$ is respectively called δ -cross-sectional relationship and δ -strong cross sectional relationship.

Definition 2 $\left(U, A, \tilde{I}\right)$ is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, $I_\delta(x, a)$ is a δ-cross-sectional relationship. A pair of dual operator between attributes and object set is defined as:

$$X_\delta^{\diamond} = \{a | a \in A, \forall x \in X, (x, a) \in I_\delta\}$$

$$B_\delta^{\circ} = \{x | x \in U, \forall a \in B, (x, a) \in I_\delta\}$$

$X \subseteq U, B \subseteq A.$

Theorem 1 $\left(U, A, \tilde{I}\right)$ *is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties,* $\forall X_1, X_2, X \subseteq U, \forall B_1, B_2, B \subseteq A$, *the following properties are below:*

(1) $X \subseteq B_\delta^{\circ} \Leftrightarrow X_\delta^{\diamond} \subseteq B$;
(2) $X_1 \subseteq X_2 \Rightarrow X_{1\delta}^{\diamond} \supseteq X_{2\delta}^{\diamond}, B_1 \subseteq B_2 \Rightarrow B_{1\delta}^{\circ} \supseteq B_{2\delta}^{\circ}$;
(3) $X \subseteq X_{\delta_\delta}^{\diamond^\circ}, B_{\delta_\delta}^{\circ^\diamond} \subseteq B$;
(4) $X_\delta^{\diamond} = X_{\delta_{\delta\delta}}^{\diamond^{\circ\diamond}}, B_\delta^{\circ} = B_{\delta_{\delta\delta}}^{\circ^{\diamond\circ}}$;
(5) $(X_1 \cup X_2)_\delta^{\diamond} = X_{1\delta}^{\diamond} \cap X_{2\delta}^{\diamond}, (B_1 \cup B_2)_\delta^{\circ} = B_{1\delta}^{\circ} \cap B_{2\delta}^{\circ}$.

Proof (1) $X \subseteq B_\delta^{\circ} \Leftrightarrow \forall x \in X, x \in B_\delta^{\circ} \Leftrightarrow \forall x \in B_\delta^{\circ}, so\ (x, a) \in I_\delta a \in B \Leftrightarrow$ $\forall x \in X, when\ (x, a) \in I_\delta, then\ a \in B \Leftrightarrow X_\delta^{\diamond} \subseteq B$.

(2) $X_1 \subseteq X_2 \Rightarrow \forall x \in X_1 \subseteq X_2, when\ (x, a) \in I_\delta, so\ a \in X_{2\delta}^{\diamond} \subseteq X_{1\delta}^{\diamond}$. $B_1 \subseteq B_2 \Rightarrow \forall a \in B_1 \subseteq B_2 \Rightarrow when\ (x, a) \in I_\delta, then\ x \in B_{2\delta}^{\circ} \subseteq B_{1\delta}^{\circ}$.

(3) $\forall x \in X \Rightarrow when\ (x, a) \in I_\delta, then\ a \in X_\delta^{\diamond} \Rightarrow \forall a \in X_\delta^{\diamond}, when\ (x, a) \in I_\delta$, then $x \in X_\delta^{\diamond} \Rightarrow X \subseteq X_\delta^{\diamond}$. $\forall a \in B_{\delta_\delta}^{\circ^\diamond} \Rightarrow when\ (x, a) \in I_\delta, then\ x \in B_\delta^{\circ} \Rightarrow$ $\forall x \in B_\delta^{\circ}, when\ (x, a) \in I_\delta$, then $a \in B \Rightarrow B_{\delta_\delta}^{\circ^\diamond} \subseteq B$.

(4) For (3) there are clearly $X_{\delta_{\delta\delta}}^{\diamond^{\circ\diamond}} \subseteq X_\delta^{\diamond}$. On the other hand, $\forall x \in X, when\ (x, a) \in I_\delta, then\ a \in X_\delta^{\diamond}$ $\Rightarrow \forall a \in X_\delta^{\diamond}, when\ (x, a) \in I_\delta, so\ x \in X_{\delta_\delta}^{\diamond^\circ} \Rightarrow \forall x \in X_{\delta_\delta}^{\diamond^\circ}, (x, a) \in I_\delta so\ a \in X_{\delta_{\delta\delta}}^{\diamond^{\circ\diamond}} \Rightarrow X_{\delta_{\delta\delta}}^{\diamond^{\circ\diamond}} \supseteq X_\delta^{\diamond}, then\ X_{\delta_{\delta\delta}}^{\diamond^{\circ\diamond}} = X_\delta^{\diamond}$. $B_\delta^{\circ} = B_{\delta_{\delta\delta}}^{\circ^{\diamond\circ}}$ that such.

(5) $\forall a \in (X_1 \cup X_2)^{\diamond}$, *when* $(x, a) \in I_\delta, x \in X_1 \cup X_2 \Rightarrow x \in X_1$, then $(x, a) \in I_\delta \Rightarrow a \in X_{1\delta}^{\diamond}, x \in X_2$, *when* $(x, a) \in I_\delta$, *then* $a \in X_{2\delta}^{\diamond} \Rightarrow a \in X_{1\delta}^{\diamond} \cap X_{2\delta}^{\diamond} \Rightarrow (X_1 \cup X_2)_\delta^{\diamond} \subseteq X_{1\delta}^{\diamond} \cap X_{2\delta}^{\diamond}$
$\forall a \in X_{1\delta}^{\diamond} \cap X_{2\delta}^{\diamond} \Rightarrow a \in X_{1\delta}^{\diamond}((x, a) \in I_\delta \Rightarrow x \in X_1)$, *and* $a \in X_{2\delta}^{\diamond}$ $((x, a) \in I_\delta \Rightarrow x \in X_2) \Rightarrow \forall x \in X_1 \cup X_2, (x, a) \in I_\delta \Rightarrow a \in (X_1 \cup X_2)_\delta^{\diamond}$. $(B_1 \cup B_2)_\delta^{\circ} = B_{1\delta}^{\circ} \cap B_{2\delta}^{\circ}$
is similar to be proved, so omit.

Theorem 2 $\left(U, A, \tilde{I}\right)$ *is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, for* $X \subseteq U, B \subseteq A$, *when* $0 < \delta_1 < \delta_2 \leq 1$, *then* $X_{\delta_2}^{\diamond} \subseteq X_{\delta_1}^{\diamond}, B_{\delta_2}^{\circ} \subseteq B_{\delta_1}^{\circ}$.

Proof $\forall a \in X_{\delta_2}^{\diamond}, \exists x \in X \Rightarrow (x, a) \in I_{\delta_2}$ *thus* $\tilde{I}(x, a) \geq \delta_2 > \delta_1 \Rightarrow (x, a) \in I_{\delta_1} \Rightarrow a \in X_{\delta_1}^{\diamond} \Rightarrow X_{\delta_2}^{\diamond} \subseteq X_{\delta_1}^{\diamond}$. 证 $B_{\delta_2}^{\circ} \subseteq B_{\delta_1}^{\circ}$ is similar to be proved, so omit.

Definition 6 $\left(U, A, \tilde{I}\right)$ is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, for the object set $X \subseteq U$ and the set of properties set $B \subseteq A$, the up and down approximate are respectively defined as:

$$X_\delta^{\nabla} = \left\{a \in A \middle| a_\delta^{\circ} \subseteq X\right\}, X_\delta^{\Delta} = \left\{a \in A \middle| a_\delta^{\circ} \bigcap X \neq \phi\right\}$$

$$B_\delta^{\nabla} = \left\{x \in U \middle| x_\delta^{\diamond} \subseteq B\right\}$$

$$B_\delta^{\Delta} = \left\{x \in U \middle| x_\delta^{\diamond} \bigcap B \neq \phi\right\}$$

where $a_\delta^{\circ} = \{x | x \in U, (x, a) \in I_\delta\}, x_\delta^{\diamond} = \{a | a \in A, (x, a) \in I_\delta\}$.

Theorem 3 $\left(U, A, \tilde{I}\right)$ *is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, for the object set* $X \subseteq U$ *and the set of properties set* $B \subseteq A$, *then* $(\nabla_\delta, \Delta_\delta), (\Delta_\delta, \nabla_\delta)$ *are all Galois connections.*

Proof $\forall X \subseteq U, B \subseteq A \;\; X \subseteq B_\delta^{\Delta} \Leftrightarrow x \in X, x_\delta^{\diamond} \cap B \neq \phi \Leftrightarrow$ *when* $(x, a) \in I_\delta$, *so* $a \in B \Leftrightarrow X_\delta^{\nabla} \subseteq B$, then $(\nabla_\delta, \Delta_\delta)$ is Galois connection; $(\Delta_\delta, \nabla_\delta)$ is similar to be proved, so omit.

Theorem 4 $\left(U, A, \tilde{I}\right)$ *is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, for the object set* $X \subseteq U$ *and the set of properties set* $B \subseteq A$, *when* $0 < \delta_1 < \delta_2 \leq 1$, *then*

$$X_{\delta_1}^{\nabla} \subseteq X_{\delta_2}^{\nabla}, B_{\delta_1}^{\Delta} \subseteq B_{\delta_2}^{\Delta},$$
$$X_{\delta_1}^{\Delta} \subseteq X_{\delta_2}^{\Delta}, B_{\delta_1}^{\nabla} \subseteq B_{\delta_2}^{\nabla}.$$

Proof By the definition, conclusion is easy to proved.

Theorem 5 $\left(U, A, \tilde{I}\right)$ *is a fuzzy formal context, where* U *is a limited non-null object set,* A *is a limited non-empty set of properties, for* $\forall X_1, X_2, X \subseteq U, \forall B_1, B_2, B \subseteq A$, *the up and down approximation have the following properties:*

(1) $X_1 \subseteq X_2 \Rightarrow X_{1\delta}^{\nabla} \subseteq X_{2\delta}^{\nabla}, X_1 \subseteq X_2 \Rightarrow X_{1\delta}^{\Delta} \subseteq X_{2\delta}^{\Delta}$,
(2) $B_1 \subseteq B_2 \Rightarrow B_{1\delta}^{\nabla} \subseteq B_{2\delta}^{\nabla}, B_1 \subseteq B_2 \Rightarrow B_{1\delta}^{\Delta} \subseteq B_{2\delta}^{\Delta}$,
(3) $X_{\delta}^{\Delta\nabla} \subseteq X \subseteq X_{\delta}^{\Delta\nabla}, B_{\delta}^{\Delta\nabla} \subseteq B \subseteq B_{\delta}^{\Delta\nabla}$,
(4) $X_{\delta}^{\Delta\nabla\Delta} = X_{\delta}^{\Delta}, X_{\delta}^{\nabla\Delta\nabla} = X_{\delta}^{\nabla}, B_{\delta}^{\Delta\nabla\Delta} = B_{\delta}^{\Delta}, B_{\delta}^{\nabla\Delta\nabla} = B_{\delta}^{\Delta}$,
(5) $(X_1 \cap X_2)_{\delta}^{\nabla} = X_{1\delta}^{\nabla} \cap X_{2\delta}^{\nabla}, (X_1 \cup X_2)_{\delta}^{\Delta} = X_{1\delta}^{\Delta} \cup X_{2\delta}^{\Delta}$,
$(B_1 \cap B_2)_{\delta}^{\nabla} = B_{1\delta}^{\nabla} \cap B_{2\delta}^{\nabla}, (B_1 \cup B_2)_{\delta}^{\Delta} = B_{1\delta}^{\Delta} \cup B_{2\delta}^{\Delta}$

Proof (1) $\forall a \in X_{1\delta}^{\nabla} \Rightarrow a_{\delta}^{\circ} \subseteq X_1 \subseteq X_2 \Rightarrow a \in X_{2\delta}^{\nabla} \Rightarrow X_{1\delta}^{\nabla} \subseteq X_{2\delta}^{\nabla}$.

$$\forall a \in X_{1\delta}^{\Delta} \Rightarrow a_{\delta}^{\circ} \cap X_1 \neq \phi \Rightarrow a_{\delta}^{\circ} \cap X_2 \neq \phi \Rightarrow a \in X_{2\delta}^{\Delta} \Rightarrow X_{1\delta}^{\Delta} \subseteq X_{2\delta}^{\Delta}.$$

This proof is similar to (1), so omit.

(3)$\forall x \in X_{\delta_\delta}^{\nabla\Delta} \Rightarrow x \in \left\{x \in U \middle| x_{\delta}^{\diamond} \cap X_{\delta}^{\nabla} \neq \phi\right\}$, then $a \in A$, 对 $\forall x \in X, (x, a) \in I_\delta \Rightarrow X_{\delta_\delta}^{\nabla\Delta} \subseteq X$.

$$\forall x \in X, so\,(x, a) \in I_\delta, a \in X_{\delta}^{\diamond} \Rightarrow X_{\delta}^{\diamond} \subseteq X_{\delta}^{\Delta} \Rightarrow x \in X_{\delta_\delta}^{\Delta\nabla}.$$

$B_{\delta}^{\nabla\Delta} \subseteq B \subseteq B_{\delta}^{\Delta\nabla}$ is omitted.

(4) $a \in X_{\delta}^{\Delta\nabla\Delta} \Leftrightarrow a_{\delta}^{\circ} \cap X_{\delta}^{\Delta\nabla} \neq \phi \Leftrightarrow \{x \in U \,|(x, a) \in I_\delta\} \cap \left\{x \in U \middle| x_{\delta}^{\diamond} \subseteq X_{\delta}^{\Delta}\right\} \neq \phi \Leftrightarrow a_{\delta}^{\circ} \cap X \neq \phi, a \in X_{\delta}^{\Delta}$, so $X_{\delta}^{\Delta\nabla\Delta} = X_{\delta}^{\Delta}$. Other proof is similar, so omitted.

(5) $\Leftrightarrow a \in X_{1\delta}^{\nabla} \cap X_{2\delta}^{\nabla}$, then $(X_1 \cap X_2)_{\delta}^{\nabla} = X_{1_\delta}^{\nabla} \cap X_{2_\delta}^{\nabla}$. Other proof is $a \in (X_1 \cap X_2)_{\delta}^{\nabla} \Leftrightarrow a_{\delta}^{\circ} \subseteq X_1 \cap X_2 \Leftrightarrow a_{\delta}^{\circ} \subseteq X_1, a_{\delta}^{\circ} \subseteq X_2 \Leftrightarrow a \in X_{1\delta}^{\nabla}, a \in X_{2\delta}^{\nabla}$ similar, so omitted.

Definition 7 $\left(U, A, \tilde{I}\right)$ is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, for the object set $X \subseteq U$ and the set of properties set $B \subseteq A$. If $X = B_{\delta}^{\Delta}, B = X_{\delta}^{\nabla}$, then (X, B) is referred to as the fuzzy concept of object; if $X = B_{\delta}^{\nabla}, B = X_{\delta}^{\Delta}$, then (X, B) is referred to as the fuzzy concept of attribution, where X is referred as the extension of fuzzy concept, B is referred as the intension of fuzzy concept.

Definition 8 $\left(U, A, \tilde{I}\right)$ is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, all the concept of fuzzy object and concept of fuzzy attribute sets of $\left(U, A, \tilde{I}\right)$ are recorded as:

$$L_O\,(U, A, I_\delta) = \left\{(X, B) \middle| X = B_{\delta}^{\Delta}, B = X_{\delta}^{\nabla}\right\},$$

$$L_P(U, A, I_\delta) = \left\{(X, B) \middle| X = B_\delta^\nabla, B = X_\delta^\Delta\right\}.$$

Theorem 6 $\left(U, A, \tilde{I}\right)$ *is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, on the concept of fuzzy object sets* $L_O(U, A, I_\delta) = \left\{(X, B) \middle| X = B_\delta^\Delta, B = X_\delta^\nabla\right\}$ *and the concept of fuzzy attribute sets* $L_P(U, A, I_\delta) = \left\{(X, B) \middle| X = B_\delta^\nabla, B = X_\delta^\Delta\right\}$*, binary relation are recorded as:*

$$(X_1, B_1) \le (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (B_2 \subseteq B_1).$$

Then $L_O(U, A, I_\delta)$*,* $L_P(U, A, I_\delta)$ *are all partial order sets.*

Proof (1) reflexivity: $(X_1, B_1) \le (X_1, B_1)$,

(2) symmetry: $(X_1, B_1) \le (X_2, B_2), (X_2, B_2) \le (X_1, B_1) \Rightarrow (X_1, B_1) = (X_2, B_2)$,

(3) transitivity: $(X_1, B_1) \le (X_2, B_2), (X_2, B_2) \le (X_3, B_3) \Rightarrow (X_1, B_1) = (X_3, B_3)$.

Then binary relation $\le$ is partial order relation, so $L_O(U, A, I_\delta)$, $L_P(U, A, I_\delta)$ are all partial order sets.

Theorem 7 *Let* $L_O(U, A, I_\delta)$*,* $L_P(U, A, I_\delta)$ *be partial order sets. For*

$$\forall (X_i, B_i) \in L_0(U, A, I_\delta)(i \in I); \forall \left(X_j, B_j\right) \in L_p(U, A, I_\delta)(j \in J),$$

the join and intersect operation are defined as:

$$\mathop{\wedge}_{i \in I}(X_i, B_i) = \left(\left(\mathop{\cap}_{i \in I} X_i\right)_\delta^{\nabla\Delta}, \mathop{\cap}_{i \in I} B_i\right),$$

$$\mathop{\vee}_{i \in I}(X_i, B_i) = \left(\mathop{\cup}_{i \in I} X_i, \left(\mathop{\cup}_{i \in I} B_i\right)_\delta^{\Delta\nabla}\right),$$

$$\mathop{\wedge}_{j \in J}\left(X_j, B_j\right) = \left(\mathop{\cap}_{j \in J} X_j, \left(\mathop{\cap}_{j \in J} B_j\right)_\delta^{\nabla\Delta}\right),$$

$$\mathop{\vee}_{j \in J}\left(X_j, B_j\right) = \left((\mathop{\cup}_{j \in J} X_j)_\delta^{\Delta\nabla}, \mathop{\cup}_{j \in J} B_j\right).$$

Then $L_O(U, A, I_\delta)$ $L_P(U, A, I_\delta)$ *are all complete lattices.*

Proof For $\forall (X_i, B_i), (X_k, B_k) \in L_O(U, A, I_\delta), i, k \in I$, then

$$\mathop{\cap}_{i \in I} B_i \subseteq B_k \Rightarrow \left(\mathop{\cap}_{i \in I} B_i\right)_\delta^{\Delta\nabla} \subseteq B_{k_\delta}^{\Delta\nabla} = B_k \Rightarrow \left(\mathop{\cap}_{i \in I} B_i\right)_\delta^{\Delta\nabla} \subseteq \mathop{\cap}_{k \in I} B_k = \mathop{\cap}_{i \in I} B_i,$$

$\bigcap_{i=I} B_i \subseteq \left(\bigcap_{i=I} B_i\right)_\delta^{\Delta\nabla}$ then $\bigcap_{i=I} B_i = \left(\bigcap_{i=I} B_i\right)_\delta^{\Delta\nabla} . \left(\bigcap_{i\in I} B_i\right)_\delta^{\Delta} = \left(\bigcap_{i\in I} X_{i_\delta}^{\nabla}\right)_\delta^{\Delta} =$ $\left(\bigcap_{i\in I} X_i\right)_\delta^{\nabla\Delta}$

$\left(\left(\bigcap_{i\in I} X_i\right)_\delta^{\nabla\Delta}, \bigcap_{i\in I} B_i\right) = (\left(\bigcap_{i\in I} B_i\right)_\delta^{\Delta}, \bigcap_{i\in I} B_i) \in L_O\,(U, A, I_\delta)$, then $\left(\left(\bigcap_{i\in I} X_i\right)_\delta^{\nabla\Delta}, \bigcap_{i\in I} B_i\right)$ is lower bound of $L_O\,(U, A, I_\delta)$.

On the other hand, if (X, B) is a any lower bound of $L_O\,(U, A, I_\delta)$, then

$$X \subseteq X_i \Rightarrow X \subseteq \left(\bigcap_{i\in I} X_i\right)_\delta^{\nabla\Delta} \Rightarrow (X, B) \le \left(\left(\bigcap_{i\in I} X_i\right)_\delta^{\nabla\Delta}, \bigcap_{i\in I} B_i\right),$$

so $\bigwedge_{i\in I} (X_i, B_i) = \left(\left(\bigcap_{i\in I} X_i\right)_\delta^{\nabla\Delta}, \bigcap_{i\in I} B_i\right)$
is infimum of $L_O\,(U, A, I_\delta)$; for

$$\bigvee_{i\in I} (X_i, B_i) = \left(\bigcup_{i\in I} X_i, \left(\bigcup_{i\in I} B_i\right)_\delta^{\Delta\nabla}\right)$$

is supremum of $L_O\,(U, A, I_\delta)$, this proof is omitted.

From what has been discussed above, $L_O\,(U, A, I_\delta)$ is complete lattices.

The proof about $L_P\,(U, A, I_\delta)$ is omitted.

Definition 9 $\left(U, A, \tilde{I}\right)$ is a fuzzy formal context, where U is a limited non-null object set, A is a limited non-empty set of properties, complete lattices

$$L_O\,(U, A, I_\delta)\,, L_P\,(U, A, I_\delta)$$

are respectively referred to as fuzzy object concept lattice and fuzzy attribute concept lattice.

3 Conclusion

Concept lattice and rough set theory, two different methods for knowledge representation and knowledge discovery, are successfully applied to many fields. Methods of fuzzy rule extraction based on rough set theory are rarely reported in incomplete interval-valued fuzzy information systems. Thus, this paper deals with the relationship of such systems and fuzzy concept lattice. The purpose of this paper is to study a new model called fuzzy rough concept lattice (FRCL) and its properties. We study four models of FRCL and the relationship of those. Meanwhile transformation algorithms and examples are be given.

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References

1. Belohlavek, R.: Fuzzy galois connection. Math. Log. Quart. **45**(4), 497–504 (1991)
2. Belohlavek, R.: Fuzzy Relational System: Foundation and Principles. Kluwer Academic/Plenum, New York (2002)
3. Bělohlávek, R., Sklenář, V., Zacpal, J.: Crisply generated fuzzy concepts. In: Ganter, B., Godin, R. (eds.) ICFCA 2005. LNCS (LNAI), vol. 3403, pp. 269–284. Springer, Heidelberg (2005)
4. Belohlavek, R., Vychodil, V.: Reducing the size of fuzzy concept lattices by hedge. In: FUZZ-IEEE 2005. The IEEE International Conference on Fuzzy Systems, pp. 663–668. Reno (Nevada, USA), 22–25 May 2005
5. Burusco, A., Fuentes-Gonzales, R.: The study of the L-fuzzy concept lattice. Mathware Soft Comput **1**(3), 209–218 (1994)
6. Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundations. Springer, Berlin (1999)
7. Zadeh, L.A.: Fuzzy sets. Inform. Control **8**, 338–353 (1965)
8. Yao, Y.Y.: A comparative study of formal concept analysis and rough set theory in data analysis. LNAI **3066**, 59–68 (2004)
9. Yao, Y.Y.: Concept lattices in rough set theory. In: Proceedings of the 23rd International Meeting of the North American Fuzzy Information Processing Society (2004)
10. Sigayret, A., Berry, A.: Representating a concept lattice by a graph. Discrete Appl. Math. **144**, 27–42 (2004)
11. Yahia, S., Jaoua, A.: Discovering knowledge from fuzzy concept lattice. In: Kandel, A., Last, M., Bunke, H. (eds.) Data Mining and Computational Intelligence, pp. 167–190. Physica-Verlag, Wurzburg (2002)
12. Wille, R.: Restructuring lattice theory: an approach based on hierarchies of concept. In: Rival, I. (ed.) Ordered Sets, pp. 445–470. BOSTON, Reidel, Dordrecht (1982)
13. Gediga, G., Wille, R.: Formal Concept Analysis. Mathematic Foundations, Springer, Berlin (1999)
14. Faid, M., Missaoi, R., Godin, R.: Mining complex structures using context concatenation in formal concept analysis. In: International KRUSE Symposium, Vancouver, BC, 11–13 Aug 1997
15. Godin, R., Missaoi, R.: An incremental concept formation approach for learning from databases. In: Formal Methods in Databases and Software Engineering, Theoretical Computer Science 133, pp. 387–419 (special issue) (1994)
16. Wille, R.: Knowledge acquisition by methods of formal concept analysis. In: Diday, E. (ed.) Data Analysis, Learning Symbolic and Numeric Knowledge, pp. 365–380. Nova Science, New York (1989)
17. Belohlavek, R.: Fuzzy closure operators. J. Math. Anal. Appl. **262**, 473–489 (2001)
18. Belohlavek, R.: Lattices of fixed points of fuzzy Galois connection. Math. Logic Quart. **47**, 111–116 (2001)
19. Belohlavek, R.: Concept lattices and order in fuzzy logic. Ann. Pure Appl. Logic **128**, 277–298 (2004)
20. Pawlak, Z.: Rough sets. Int. J. Comput. Inform. Sci. **11**, 341–356 (1982)

21. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning About Data. Kluwer Academic Publishers, Boston (1991)
22. Burusco, A., Fuentes-Gonzalez, R.: Construction of the L-fuzzy concept lattices. Fuzzy Sets Syst. **97**, 109–114 (1998)

Dual Hesitant Fuzzy Soft Set and Its Properties

Hai-dong Zhang and Lan Shu

Abstract The soft set theory, originally proposed by Molodtsov, can be used as a general mathematical tool for dealing with uncertainty. By combining the dual hesitant fuzzy set and soft set models, we introduce the concept of dual hesitant fuzzy soft sets. Further some operations on the dual hesitant fuzzy soft sets are investigated, such as complement operation, "AND" and "OR" operations, sum and product operations.

Keywords Soft set · Dual hesitant fuzzy set · Dual hesitant fuzzy soft set · Operations

1 Introduction

The soft set theory, originally proposed by Molodtsov [8], can be regarded as a general mathematical tool to deal with uncertainty. It has been found that existing uncertainty theories such as fuzzy sets [23], rough sets [14], intuitionistic fuzzy sets [1], vague sets theory [6] and interval-valued fuzzy sets [4] have their inherent difficulties, but soft set theory as a new mathematical tool for dealing with uncertainties is free from the difficulties affecting existing methods as pointed out in [8]. In recent years, research works on soft sets are very active and progressing rapidly. Especially, the study of hybrid models combining soft sets with other mathematical structures is also emerging as an active research topic of soft set theory. A lot of extensions of soft set model have also been developed recently [2, 7, 9–12, 18, 20, 21].

H. Zhang (✉) · L. Shu
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China
e-mail: lingdianstar@163.com

H. Zhang
School of Mathematics and Computer Science, Northwest University for Nationalities, Lanzhou 730030, China

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Zhu et al. [22] introduced the dual hesitant fuzzy set which is a comprehensive set encompassing fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets [15, 16], and fuzzy multisets [13] as special cases, and whose membership degrees and nonmembership degrees are represented by two sets of possible values. By several possible values for the membership and nonmembership degrees, respectively, dual hesitant fuzzy sets can take much more information given by decision makers into account in multiple attribute decision making. More recently, many authors have developed multiple attribute decision-making theories and methods under dual hesitant fuzzy environment [3, 5, 17, 19].

Up to now, many of researches about decision making based on dual hesitant fuzzy set are mainly focusing on the dual hesitant fuzzy set itself. Like the above mentioned uncertainty theories, dual hesitant fuzzy set has also its inherent difficulties of inadequacy to the parameterization tools, while soft set theory is free from the difficulties. Therefore, the research about fusions of dual hesitant fuzzy set and soft set is important and necessary to us. So we present the new hybrid model called dual hesitant fuzzy soft sets by combining dual hesitant fuzzy set and soft set, and investigate its some interesting properties.

2 Preliminaries

2.1 Soft Sets

The pair (U, E) will be called a soft universe. Throughout this paper, unless otherwise stated, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subseteq E$. According to [8], the concept of soft sets is defined as follows.

Definition 1 ([8]) A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

2.2 Dual Hesitant Fuzzy Sets

Zhu et al. [22] defined a dual hesitant fuzzy set, which is an extension of a hesitant fuzzy set, in terms of two functions that return two sets of membership values and nonmembership values, respectively, for each element in the domain as follows.

Definition 2 ([22]) Let U be a fixed set, a dual hesitant fuzzy (DHF, for short) set $\mathbb{D}$ on U is described as:

$$\mathbb{D} = \{\langle x, h_{\mathbb{D}}(x), g_{\mathbb{D}}(x)\rangle | x \in U\},$$

in which $h_{\mathbb{D}}(x)$ and $g_{\mathbb{D}}(x)$ are two sets of some values in [0,1], denoting the possible membership degrees and non-membership degrees of the element $x \in U$ to the set $\mathbb{D}$ respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where for all $x \in U$, $\gamma \in h_{\mathbb{D}}(x)$, $\eta \in g_{\mathbb{D}}(x)$, $\gamma^+ \in h^+_{\mathbb{D}}(x) = \cup_{\gamma \in h_{\mathbb{D}}(x)} max\{\gamma\}$, $\eta^+ \in g^+_{\mathbb{D}}(x) = \cup_{\eta \in g_{\mathbb{D}}(x)} max\{\eta\}$.

For convenience, the pair $d(x) = (h_{\mathbb{D}}(x), g_{\mathbb{D}}(x))$ is called a DHF element denoted by $d = (h, g)$. The set of all DHF sets on U is denoted by $DHF(U)$.

Example 1 Let $U = \{x_1, x_2\}$ be a reference set, $d(x_1) = (h_{\mathbb{D}}(x_1), g_{\mathbb{D}}(x_1)) = (\{0.2, 0.4\}, \{0.6\})$ and $d(x_2) = (h_{\mathbb{D}}(x_2), g_{\mathbb{D}}(x_2)) = (\{0.3, 0.4\}, \{0.1, 0.5\})$ be the DHF elements of $x_i (i = 1, 2)$ to a set $\mathbb{D}$, respectively. Then $\mathbb{D}$ can be considered as a DHF set, that is,

$$\mathbb{D} = \{\langle x_1, \{0.2, 0.4\}, \{0.6\}\rangle, \langle x_2, \{0.3, 0.4\}, \{0.1, 0.5\}\rangle\}.$$

Let U be a discrete universe of discourse, $\mathbb{A}$ and $\mathbb{B}$ be two DHF sets on U denoted as $\mathbb{A} = \{\langle x, h_{\mathbb{A}}(x), g_{\mathbb{A}}(x)\rangle | x \in U\}$ and $\mathbb{B} = \{\langle x, h_{\mathbb{B}}(x), g_{\mathbb{B}}(x)\rangle | x \in U\}$ respectively.

It is noted that the number of values in different DHF elements may be different and the values of DHF elements are usually given in a disorder. Suppose that $l(h_{\mathbb{A}}(x))$ and $l(g_{\mathbb{A}}(x))$ stand for the number of values in $h_{\mathbb{A}}(x)$ and $g_{\mathbb{A}}(x)$, respectively. To operate correctly, Ye [19] and Chen [3] gave the following assumptions:

(A1) For a DHF set $\mathbb{A} = \{\langle x, h_{\mathbb{A}}(x), g_{\mathbb{A}}(x)\rangle | x \in U\}$, let $\sigma : (1, 2, \ldots, n) \longrightarrow (1, 2, \ldots, n)$ be a permutation satisfying $h^{\sigma(s)}_{\mathbb{A}}(x) \geq h^{\sigma(s+1)}_{\mathbb{A}}(x)$ for $s = 1, 2, \ldots, n-1$, and $h^{\sigma(s)}_{\mathbb{A}}(x)$ be the sth largest value in $h_{\mathbb{A}}(x)$; let $\sigma : (1, 2, \ldots, m) \longrightarrow (1, 2, \ldots, m)$ be a permutation satisfying $g^{\sigma(t)}_{\mathbb{A}}(x) \geq g^{\sigma(t+1)}_{\mathbb{A}}(x)$ for $t = 1, 2, \ldots, m-1$, and $g^{\sigma(t)}_{\mathbb{A}}(x)$ be the tth largest value in $g_{\mathbb{A}}(x)$.

(A2) For two DHF sets $\mathbb{A}$ and $\mathbb{B}$, when $l(h_{\mathbb{A}}(x)) \neq l(h_{\mathbb{B}}(x))$, $l(g_{\mathbb{A}}(x)) \neq l(g_{\mathbb{B}}(x))$, one can make them have the same number of elements through adding some elements to the DHF element which has less number of elements. In terms of the pessimistic principle, the smallest element will be added while in the opposite case, the optimistic principle may be adopted. In the present work, we use the former case. Therefore, if $l(h_{\mathbb{A}}(x)) < l(h_{\mathbb{B}}(x))$, then $h_{\mathbb{A}}(x)$ should be extended by adding the minimum value in it until it has the same length as $h_{\mathbb{B}}(x)$; if $l(g_{\mathbb{A}}(x)) < l(g_{\mathbb{B}}(x))$, then $g_{\mathbb{A}}(x)$ should be extended by adding the minimum value in it until it has the same length as $g_{\mathbb{B}}(x)$.

In the following, we develop some new methods to decrease the dimension of the derived DHF element when operating the DHF elements on the premise of assumptions given by Ye [19] and Chen [3], which are slightly different from some operations on DHF sets introduced by Farhadinia in [5]. The adjusted operational laws are defined as follows.

Definition 3 Let U be a nonempty and finite universe of discourse, and $\mathbb{A}$ and $\mathbb{B} \in DHF(U)$. Denote $k = max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$ and $l = max\{l(g_{\mathbb{A}}(x)), l(g_{\mathbb{B}}(x))\}$, then for all $1 \leq s \leq k$, $1 \leq t \leq l$,

(1) the complement of $\mathbb{A}$, denoted by $\mathbb{A}^c$, is given by

$$\mathbb{A}^c = \{\langle x, g_{\mathbb{A}}(x), h_{\mathbb{A}}(x)\rangle | x \in U\} = \{\langle x, \{g^{\sigma(t)}_{\mathbb{A}}(x)\}, \{h^{\sigma(s)}_{\mathbb{A}}(x)\}\rangle | x \in U\},$$

(2) the union of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cup \mathbb{B}$, is given by

$$\begin{aligned}\mathbb{A} \Cup \mathbb{B} &= \{\langle x, h_{\mathbb{A}}(x) \cup h_{\mathbb{B}}(x), g_{\mathbb{A}}(x) \cap g_{\mathbb{B}}(x)\rangle | x \in U\} \\ &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) \vee h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) \wedge g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\};\end{aligned}$$

(3) the intersection of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cap \mathbb{B}$, is given by

$$\begin{aligned}\mathbb{A} \Cap \mathbb{B} &= \{\langle x, h_{\mathbb{A}}(x) \cap h_{\mathbb{B}}(x), g_{\mathbb{A}}(x) \cup g_{\mathbb{B}}(x)\rangle | x \in U\} \\ &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) \wedge h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) \vee g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\};\end{aligned}$$

(4) the sum of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxplus \mathbb{B}$, is given by

$$\begin{aligned}\mathbb{A} \boxplus \mathbb{B} &= \{\langle x, h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x), g_{\mathbb{A}}(x) \otimes g_{\mathbb{B}}(x)\rangle | x \in U\} \\ &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) + h_{\mathbb{B}}^{\sigma(s)}(x) - h_{\mathbb{A}}^{\sigma(s)}(x) h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\};\end{aligned}$$

(5) the product of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxtimes \mathbb{B}$, is given by

$$\begin{aligned}\mathbb{A} \boxtimes \mathbb{B} &= \{\langle x, h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x), g_{\mathbb{A}}(x) \oplus g_{\mathbb{B}}(x)\rangle | x \in U\} \\ &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) + g_{\mathbb{B}}^{\sigma(t)}(x) - g_{\mathbb{A}}^{\sigma(t)}(x) g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\},\end{aligned}$$

where $g_{\mathbb{A}}^{\sigma(t)}(x)$ denotes the tth largest value in $g_{\mathbb{A}}(x)$ and $h_{\mathbb{A}}^{\sigma(s)}(x)$ denotes the sth largest value in $h_{\mathbb{A}}(x)$.

Example 2 Let $U = \{x_1, x_2\}$ be a reference set. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two DHF sets on U defined as follows:

$$\begin{aligned}&\mathbb{A} = \{\langle x_1, \{0.6, 0.4, 0.2\}, \{0.4, 0.3\}\rangle, \langle x_2, \{0.5, 0.4\}, \{0.3, 0.1\}\rangle, \\ &\mathbb{B} = \{\langle x_1, \{0.7, 0.3\}, \{0.2\}\rangle, \langle x_2, \{0.6, 0.3, 0.1\}, \{0.4, 0.0\}\rangle.\end{aligned}$$

According to Definition 3 and assumptions given by Ye [19] and Chen [3], we have

$$\begin{aligned}\mathbb{A} \Cup \mathbb{B} &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) \vee h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) \wedge g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\} \\ &= \{\langle x_1, \{0.7, 0.4, 0.3\}, \{0.2, 0.2\}\rangle, \langle x_2, \{0.6, 0.4, 0.4\}, \{0.3, 0.0\}\rangle\};\end{aligned}$$

$$\begin{aligned}\mathbb{A} \Cap \mathbb{B} &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) \wedge h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) \vee g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\} \\ &= \{\langle x_1, \{0.6, 0.3, 0.2\}, \{0.4, 0.3\}\rangle, \langle x_2, \{0.5, 0.3, 0.1\}, \{0.4, 0.1\}\rangle\};\end{aligned}$$

$$\begin{aligned}\mathbb{A} \boxplus \mathbb{B} &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) + h_{\mathbb{B}}^{\sigma(s)}(x) - h_{\mathbb{A}}^{\sigma(s)}(x) h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\} \\ &= \{\langle x_1, \{0.88, 0.58, 0.44\}, \{0.08, 0.06\}\rangle, \langle x_2, \{0.8, 0.58, 0.46\}, \{0.12, 0.0\}\rangle\};\end{aligned}$$

$$\begin{aligned}\mathbb{A} \boxtimes \mathbb{B} &= \{\langle x, \{h_{\mathbb{A}}^{\sigma(s)}(x) h_{\mathbb{B}}^{\sigma(s)}(x)\}, \{g_{\mathbb{A}}^{\sigma(t)}(x) + g_{\mathbb{B}}^{\sigma(t)}(x) - g_{\mathbb{A}}^{\sigma(t)}(x) g_{\mathbb{B}}^{\sigma(t)}(x)\}\rangle | x \in U\} \\ &= \{\langle x_1, \{0.42, 0.12, 0.06\}, \{0.52, 0.44\}\rangle, \langle x_2, \{0.3, 0.12, 0.04\}, \{0.58, 0.1\}\rangle\}.\end{aligned}$$

In what follows, we first introduce the concept of the DHF subset.

Definition 4 Let U be a nonempty and finite universe of discourse. Denote $k = max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$, and $l = max\{l(g_{\mathbb{A}}(x)), l(g_{\mathbb{B}}(x))\}$. For all $\mathbb{A}, \mathbb{B} \in DHF(U)$, $\mathbb{A}$ is said to be a DHF subset of $\mathbb{B}$, if $h_{\mathbb{A}}(x) \preceq h_{\mathbb{B}}(x)$ and $g_{\mathbb{A}}(x) \succeq g_{\mathbb{B}}(x)$ holds for any $x \in U$ such that

$h_{\mathbb{A}}(x) \preceq h_{\mathbb{B}}(x),\ g_{\mathbb{A}}(x) \succeq g_{\mathbb{B}}(x) \Leftrightarrow h_{\mathbb{A}}^{\sigma(s)}(x) \leq h_{\mathbb{B}}^{\sigma(s)}(x), g_{\mathbb{A}}^{\sigma(t)}(x) \geq g_{\mathbb{B}}^{\sigma(t)}(x)$,
for all $1 \leq s \leq k$ and $1 \leq t \leq l$. We denote it by $\mathbb{A} \Subset \mathbb{B}$.

Obviously, the notation $\Subset$ is reflexive, transitive and antisymmetric on DHF(U).

3 Dual Hesitant Fuzzy Soft Sets

In this subsection, by combining dual hesitant fuzzy sets and soft sets, we first introduce a new hybrid model called dual hesitant fuzzy soft sets.

Definition 5 Let (U, E) be a soft universe and $A \subseteq E$. A pair $\mathfrak{S} = (\tilde{F}, A)$ is called a dual hesitant fuzzy soft set over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : A \to DHF(U)$.

Generally speaking, for any parameter $e \in A$, $\tilde{F}(e)$ is a dual hesitant fuzzy subsets in U. Following the standard notations, $\tilde{F}(e)$ can be written as

$$\tilde{F}(e) = \{\langle x, h_{\tilde{F}(e)}(x), g_{\tilde{F}(e)}(x)\rangle | x \in U\},$$

where $h_{\tilde{F}(e)}(x)$ and $g_{\tilde{F}(e)}(x)$ are two sets of some values in [0,1], denoting the possible membership degrees and non-membership degrees that object x holds on parameter e, respectively.

Sometimes for convenience, we write $\tilde{F}$ as $(\tilde{F}, E)$. If $A \subseteq E$, we can also have a dual hesitant fuzzy soft set $(\tilde{F}, A)$.

Example 3 Let U be a set of four participants performing dance programme, which is denoted by $U = \{x_1, x_2, x_3, x_4\}$. Let E be a parameter set, where $E = \{e_1, e_2, e_3\}$ = {confident; creative; graceful}. Suppose that there are three judges who are invited to evaluate the membership degrees and non-membership degrees of a candidate x_j to a parameter e_i with several possible values in [0,1]. Then dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$ defined as below gives the evaluation of the performance of candidates by three judges.

$$\tilde{F}(e_1) = \{\langle x_1, \{0.6, 0.7, 0.8\}, \{0.3, 0.2, 0.1\}\rangle, \langle x_2, \{0.4, 0.5, 0.6\}, \{0.3, 0.2, 0.1\}\rangle, \langle x_3, \{0.8, 0.7, 0.7\}, \{0.2, 0.1, 0.1\}\rangle, \langle x_4, \{0.3, 0.4, 0.4\}, \{0.6, 0.5, 0.4\}\rangle\},$$

$$\tilde{F}(e_2) = \{\langle x_1, \{0.5, 0.6, 0.4\}, \{0.4, 0.3, 0.2\}\rangle, \langle x_2, \{0.5, 0.4, 0.3\}, \{0.5, 0.3, 0.3\}\rangle, \langle x_3, \{0.7, 0.8, 0.8\}, \{0.2, 0.2, 0.1\}\rangle, \langle x_4, \{0.5, 0.6, 0.6\}, \{0.4, 0.3, 0.2\}\rangle\},$$

$$\tilde{F}(e_3) = \{\langle x_1, \{0.4, 0.4, 0.3\}, \{0.7, 0.6, 0.6\}\rangle, \langle x_2, \{0.5, 0.7, 0.7\}, \{0.3, 0.2, 0.2\}\rangle, \langle x_3, \{0.5, 0.6, 0.7\}, \{0.3, 0.2, 0.1\}\rangle, \langle x_4, \{0.7, 0.6, 0.8\}, \{0.2, 0.1, 0.1\}\rangle\}.$$

4 Operations on Dual Hesitant Fuzzy Soft Sets

In what follows, we define some operations of dual hesitant fuzzy soft sets and then some properties can be further established for such operations on dual hesitant fuzzy soft sets. Firstly, we introduce the concept of dual hesitant fuzzy soft subsets.

Definition 6 Let U be an initial universe and E be a set of parameters. Supposing that $A, B \subseteq E$, $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two dual hesitant fuzzy soft sets, we say that $(\tilde{F}, A)$ is a dual hesitant fuzzy soft subset of $(\tilde{G}, B)$ if and only if:

(1) $A \subseteq B$, and
(2) $\forall e \in A$, $\tilde{F}(e)$ is a DHF subset of $\tilde{G}(e)$, i.e. for all $x \in U$ and $e \in A$, $h^{\sigma(s)}_{\tilde{F}(e)}(x) \leq h^{\sigma(s)}_{\tilde{G}(e)}(x)$, $g^{\sigma(t)}_{\tilde{F}(e)}(x) \geq g^{\sigma(t)}_{\tilde{G}(e)}(x)$.

This relationship is denoted by $(\tilde{F}, A) \sqsubseteq (\tilde{G}, B)$. Similarly, $(\tilde{F}, A)$ is said to be a dual hesitant fuzzy soft super set of $(\tilde{G}, B)$ if $(\tilde{G}, B)$ is a dual hesitant fuzzy soft subset of $(\tilde{F}, A)$. We denote it by $(\tilde{F}, A) \sqsupseteq (\tilde{G}, B)$.

Example 4 Suppose that $U = \{x_1, x_2, x_3, x_4\}$ is an initial universe and $E = \{e_1, e_2, e_3\}$ is a set of parameters. Let $A = \{e_1, e_2\}$, $B = E = \{e_1, e_2, e_3\}$. Given two dual hesitant fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ as follows:

$$\tilde{F}(e_1) = \{\langle x_1, \{0.6, 0.5, 0.3\}, \{0.4, 0.3\}\rangle, \langle x_2, \{0.7, 0.5, 0.5\}, \{0.3, 0.2\}\rangle, \langle x_3, \{0.6, 0.5, 0.4\}, \{0.3, 0.1\}\rangle, \langle x_4, \{0.5, 0.4, 0.2\}, \{0.5, 0.4\}\rangle\},$$

$$\tilde{F}(e_2) = \{\langle x_1, \{0.4, 0.3, 0.2\}, \{0.6, 0.2\}\rangle, \langle x_2, \{0.5, 0.4, 0.4\}, \{0.5, 0.2\}\rangle, \langle x_3, \{0.4, 0.2\}, \{0.6, 0.5, 0.3\}\rangle, \langle x_4, \{0.7, 0.6, 0.4\}, \{0.3, 0.2\}\}\rangle\}.$$

$$\tilde{G}(e_1) = \{\langle x_1, \{0.7, 0.6, 0.4\}, \{0.3, 0.2\}\rangle, \langle x_2, \{0.9, 0.7, 0.6\}, \{0.1\}\rangle, \langle x_3, \{0.8, 0.7, 0.6\}, \{0.2, 0.1\}\rangle, \langle x_4, \{0.6, 0.5, 0.5\}, \{0.4, 0.2\}\rangle\},$$

$$\tilde{G}(e_2) = \{\langle x_1, \{0.6, 0.4, 0.3\}, \{0.3, 0.1\}\rangle, \langle x_2, \{0.7, 0.5, 0.4\}, \{0.3, 0.2\}\rangle, \langle x_3, \{0.5, 0.4\}, \{0.2, 0.1\}\rangle, \langle x_4, \{0.8, 0.7, 0.5\}, \{0.2, 0.1\}\rangle\},$$

$$\tilde{G}(e_3) = \{\langle x_1, \{0.6, 0.4, 0.3\}, \{0.4, 0.1\}\rangle, \langle x_2, \{0.4, 0.3, 0.3\}, \{0.6, 0.5\}\rangle, \langle x_3, \{0.9, 0.8, 0.6\}, \{0.1\}\rangle, \langle x_4, \{0.6, 0.4, 0.3\}, \{0.3, 0.1\}\rangle.$$

According to Definition 6, we have $(\tilde{F}, A) \sqsubseteq (\tilde{G}, B)$.

Definition 7 The complement of $(\tilde{F}, A)$, denoted by $(\tilde{F}, A)^c$, is defined by $(\tilde{F}, A)^c = (\tilde{F}^c, A)$, where $\tilde{F}^c : A \to DHF(U)$ is a mapping given by $\tilde{F}^c(e)$, for all $e \in A$ such that $\tilde{F}^c(e)$ is the complement of dual hesitant fuzzy set $\tilde{F}(e)$ on U.

Clearly, we have $((\tilde{F}, A)^c)^c = (\tilde{F}, A)$.

Example 5 Consider the dual hesitant fuzzy soft set $(\tilde{G}, B)$ over U defined in Example 4. Thus, by Definition 7, we have

$$\begin{aligned}\tilde{G}^c(e_1) =&\{\langle x_1, \{0.3, 0.2\}, \{0.7, 0.6, 0.4\}\rangle, \langle x_2, \{0.1\}, \{0.9, 0.7, 0.6\}\rangle,\\ &\langle x_3, \{0.2, 0.1\}, \{0.8, 0.7, 0.6\}\rangle, \langle x_4, \{0.4, 0.2\}, \{0.6, 0.5, 0.5\}\rangle\},\end{aligned}$$

$$\begin{aligned}\tilde{G}^c(e_2) =&\{\langle x_1, \{0.3, 0.1\}, \{0.6, 0.4, 0.3\}, \rangle, \langle x_2, \{0.3, 0.2\}, \{0.7, 0.5, 0.4\}\rangle,\\ &\langle x_3, \{0.2, 0.1\}, \{0.5, 0.4\}\rangle, \langle x_4, \{0.2, 0.1\}, \{0.8, 0.7, 0.5\}\rangle\},\end{aligned}$$

$$\begin{aligned}\tilde{G}^c(e_3) =&\{\langle x_1, \{0.4, 0.1\}, \{0.6, 0.4, 0.3\}\rangle, \langle x_2, \{0.6, 0.5\}, \{0.4, 0.3, 0.3\}\rangle,\\ &\langle x_3, \{0.1\}, \{0.9, 0.8, 0.6\}\rangle, \langle x_4, \{0.3, 0.1\}, \{0.6, 0.4, 0.3\}\rangle.\end{aligned}$$

By the suggestions given by Molodtsov in [8], we present the notion of AND and OR operations on two dual hesitant fuzzy soft sets as follows.

Definition 8 Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two dual hesitant fuzzy soft sets over U. Suppose that $k = max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$ and $l = max\{l(g_{\mathbb{A}}(x)), l(g_{\mathbb{B}}(x))\}$, then the "$(\tilde{F}, A)$ AND $(\tilde{G}, B)$", denoted by $(\tilde{F}, A) \wedge (\tilde{G}, B)$, is defined by

$$(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B),$$

where for all $1 \le s \le k$, $1 \le t \le l$ and $(\alpha, \beta) \in A \times B$,

$$\tilde{H}(\alpha, \beta) = \{\langle x, h_{\tilde{F}(\alpha)}(x) \cap h_{\tilde{G}(\beta)}(x), g_{\tilde{F}(\alpha)}(x) \cup g_{\tilde{G}(\beta)}(x)\rangle : x \in U\}.$$

Definition 9 Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two dual hesitant fuzzy soft sets over U. Suppose that $k = max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$ and $l = max\{l(g_{\mathbb{A}}(x)), l(g_{\mathbb{B}}(x))\}$, then the "$(\tilde{F}, A)$ OR $(\tilde{G}, B)$", denoted by $(\tilde{F}, A) \vee (\tilde{G}, B)$, is defined by

$$(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{I}, A \times B),$$

where for all $1 \le s \le k$, $1 \le t \le l$ and $(\alpha, \beta) \in A \times B$,

$$\tilde{I}(\alpha, \beta) = \{\langle x, h_{\tilde{F}(\alpha)}(x) \cup h_{\tilde{G}(\beta)}(x), g_{\tilde{F}(\alpha)}(x) \cap g_{\tilde{G}(\beta)}(x)\rangle : x \in U\}.$$

Example 6 Reconsider Example 4. Then we have $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$ and $(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{I}, A \times B)$ as follows:

$$\begin{aligned}\tilde{H}(e_1, e_1) =&\{\langle x_1, \{0.6, 0.5, 0.3\}, \{0.4, 0.3\}\rangle, \langle x_2, \{0.7, 0.5, 0.5\}, \{0.3, 0.2\}\rangle,\\ &\langle x_3, \{0.6, 0.5, 0.4\}, \{0.3, 0.1\}\rangle, \langle x_4, \{0.5, 0.4, 0.2\}, \{0.5, 0.4\}\rangle\},\end{aligned}$$

$$\tilde{H}(e_1, e_2) = \{\langle x_1, \{0.6, 0.4, 0.3\}, \{0.4, 0.3\}\rangle, \langle x_2, \{0.7, 0.5, 0.4\}, \{0.3, 0.2\}\rangle,$$
$$\langle x_3, \{0.5, 0.4, 0.4\}, \{0.3, 0.1\}\rangle, \langle x_4, \{0.5, 0.4, 0.2\}, \{0.5, 0.4\}\rangle\},$$

$$\tilde{H}(e_1, e_3) = \{\langle x_1, \{0.6, 0.4, 0.3\}, \{0.4, 0.3\}\rangle, \langle x_2, \{0.4, 0.3, 0.3\}, \{0.6, 0.5\}\rangle,$$
$$\langle x_3, \{0.6, 0.5, 0.4\}, \{0.3, 0.1\}\rangle, \langle x_4, \{0.5, 0.4, 0.2\}, \{0.5, 0.4\}\rangle\},$$

$$\tilde{H}(e_2, e_1) = \{\langle x_1, \{0.4, 0.3, 0.2\}, \{0.6, 0.2\}\rangle, \langle x_2, \{0.5, 0.4, 0.4\}, \{0.5, 0.2\}\rangle,$$
$$\langle x_3, \{0.4, 0.2, 0.2\}, \{0.6, 0.5, 0.3\}\rangle, \langle x_4, \{0.6, 0.5, 0.4\}, \{0.4, 0.2\}\rangle\},$$

$$\tilde{H}(e_2, e_2) = \{\langle x_1, \{0.4, 0.3, 0.2\}, \{0.6, 0.2\}\rangle, \langle x_2, \{0.5, 0.4, 0.4\}, \{0.5, 0.2\}\rangle,$$
$$\langle x_3, \{0.4, 0.2\}, \{0.6, 0.5, 0.3\}\rangle, \langle x_4, \{0.7, 0.6, 0.4\}, \{0.3, 0.2\}\rangle\},$$

$$\tilde{H}(e_2, e_3) = \{\langle x_1, \{0.4, 0.3, 0.2\}, \{0.6, 0.2\}\rangle, \langle x_2, \{0.4, 0.3, 0.3\}, \{0.6, 0.5\}\rangle,$$
$$\langle x_3, \{0.4, 0.2, 0.2\}, \{0.6, 0.5, 0.3\}\rangle, \langle x_4, \{0.6, 0.4, 0.3\}, \{0.3, 0.2\}\rangle\},$$

and

$$\tilde{I}(e_1, e_1) = \{\langle x_1, \{0.7, 0.6, 0.4\}, \{0.3, 0.2\}\rangle, \langle x_2, \{0.9, 0.7, 0.6\}, \{0.1\}\rangle,$$
$$\langle x_3, \{0.8, 0.7, 0.6\}, \{0.2, 0.1\}\rangle, \langle x_4, \{0.6, 0.5, 0.5\}, \{0.4, 0.2\}\rangle\},$$

$$\tilde{I}(e_1, e_2) = \{\langle x_1, \{0.6, 0.5, 0.3\}, \{0.3, 0.1\}\rangle, \langle x_2, \{0.7, 0.5, 0.5\}, \{0.3, 0.2\}\rangle,$$
$$\langle x_3, \{0.6, 0.5, 0.4\}, \{0.2, 0.1\}\rangle, \langle x_4, \{0.8, 0.7, 0.5\}, \{0.2, 0.1\}\rangle\},$$

$$\tilde{I}(e_1, e_3) = \{\langle x_1, \{0.6, 0.5, 0.3\}, \{0.4, 0.1\}\rangle, \langle x_2, \{0.7, 0.5, 0.5\}, \{0.3, 0.2\}\rangle,$$
$$\langle x_3, \{0.9, 0.8, 0.6\}, \{0.1\}\rangle, \langle x_4, \{0.6, 0.4, 0.3\}, \{0.3, 0.1\}\rangle\},$$

$$\tilde{I}(e_2, e_1) = \{\langle x_1, \{0.7, 0.6, 0.4\}, \{0.3, 0.2\}\rangle, \langle x_2, \{0.9, 0.7, 0.6\}, \{0.1\}\rangle,$$
$$\langle x_3, \{0.8, 0.7, 0.6\}, \{0.2, 0.1, 0.1\}\rangle, \langle x_4, \{0.7, 0.6, 0.5\}, \{0.3, 0.2\}\rangle\},$$

$$\tilde{I}(e_2, e_2) = \{\langle x_1, \{0.6, 0.4, 0.3\}, \{0.3, 0.1\}\rangle, \langle x_2, \{0.7, 0.5, 0.4\}, \{0.3, 0.2\}\rangle,$$
$$\langle x_3, \{0.5, 0.4\}, \{0.2, 0.1, 0.1\}\rangle, \langle x_4, \{0.8, 0.7, 0.5\}, \{0.2, 0.1\}\rangle\},$$

$$\tilde{I}(e_2, e_3) = \{\langle x_1, \{0.6, 0.4, 0.3\}, \{0.4, 0.1\}\rangle, \langle x_2, \{0.5, 0.4, 0.4\}, \{0.5, 0.2\}\rangle,$$
$$\langle x_3, \{0.9, 0.8, 0.6\}, \{0.1\}\rangle, \langle x_4, \{0.7, 0.6, 0.4\}, \{0.3, 0.1\}\rangle\}.$$

Theorem 1 *Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two dual hesitant fuzzy soft sets over U. Then*
(1) $((\tilde{F}, A) \wedge (\tilde{G}, B))^c = (\tilde{F}, A)^c \vee (\tilde{G}, B)^c$,
(2) $((\tilde{F}, A) \vee (\tilde{G}, B))^c = (\tilde{F}, A)^c \wedge (\tilde{G}, B)^c$.

Proof (1) Suppose that $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$. Therefore, by Definitions 7 and 8, we have $((\tilde{F}, A) \wedge (\tilde{G}, B))^c = (\tilde{H}, A \times B)^c = (\tilde{H}^c, A \times B)$, where for all $(\alpha, \beta) \in A \times B$, $\tilde{H}^c(\alpha, \beta) = \{\langle x, g_{\tilde{H}(\alpha,\beta)}(x), h_{\tilde{H}(\alpha,\beta)}(x)\rangle : x \in U\} = \{\langle x, g_{\tilde{F}(\alpha)}(x) \cup g_{\tilde{G}(\beta)}(x), h_{\tilde{F}(\alpha)}(x) \cap h_{\tilde{G}(\beta)}(x)\rangle : x \in U\}$.

On the other hand, by Definitions 7 and 9, we have $(\tilde{F}, A)^c \vee (\tilde{G}, B)^c = (\tilde{F}^c, A) \vee (\tilde{G}^c, B) = (\tilde{I}, A \times B)$, where for all $(\alpha, \beta) \in A \times B$, $\tilde{I}(\alpha, \beta) = \{\langle x, g_{\tilde{F}(\alpha)}(x) \cup g_{\tilde{G}(\beta)}(x), h_{\tilde{F}(\alpha)}(x) \cap h_{\tilde{G}(\beta)}(x)\rangle : x \in U\}$. Hence, $(\tilde{H}^c, A \times B) = (\tilde{I}, A \times B)$.

(2) The result can be proved in the similar way.

Theorem 2 *Let $(\tilde{F}, A)$, $(\tilde{G}, B)$ and $(\tilde{H}, C)$ be three dual hesitant fuzzy soft sets over U. Then we have*

(1) $(\tilde{F}, A) \wedge ((\tilde{G}, B) \wedge (\tilde{H}, C)) = ((\tilde{F}, A) \wedge (\tilde{G}, B)) \wedge (\tilde{H}, C)$,
(2) $(\tilde{F}, A) \vee ((\tilde{G}, B) \vee (\tilde{H}, C)) = ((\tilde{F}, A) \vee (\tilde{G}, B)) \vee (\tilde{H}, C)$,
(3) $(\tilde{F}, A) \wedge ((\tilde{G}, B) \vee (\tilde{H}, C)) = ((\tilde{F}, A) \wedge (\tilde{G}, B)) \vee ((\tilde{F}, A) \wedge (\tilde{H}, C))$,
(4) $(\tilde{F}, A) \vee ((\tilde{G}, B) \wedge (\tilde{H}, C)) = ((\tilde{F}, A) \vee (\tilde{G}, B)) \wedge ((\tilde{F}, A) \vee (\tilde{H}, C))$.

Proof It can be directly followed from Definitions 8 and 9.

Remark 1 Suppose that $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two dual hesitant fuzzy soft sets over U. It is noted that for all $(\alpha, \beta) \in A \times B$, if $\alpha \neq \beta$, then $(\tilde{G}, B) \wedge (\tilde{F}, A) \neq (\tilde{F}, A) \wedge (\tilde{G}, B)$, and $(\tilde{G}, B) \vee (\tilde{F}, A) \neq (\tilde{F}, A) \vee (\tilde{G}, B)$.

Next, based on the operations in Definition 3, we first present sum and product operations of dual hesitant fuzzy soft sets.

Definition 10 The sum operation on the two dual hesitant fuzzy soft sets $\tilde{F}$ and $\tilde{G}$ over U, denoted by $\tilde{F} \boxplus \tilde{G} = \tilde{H}$, is a mapping given by $\tilde{H} : E \to DHF(U)$, such that for all $e \in E$,

$$\begin{aligned} \tilde{H}(e) &= \{\langle x, h_{\tilde{H}(e)}(x), g_{\tilde{H}(e)}(x)\rangle : x \in U\} \\ &= \{\langle x, h_{\tilde{F}(e)}(x) \oplus h_{\tilde{G}(e)}(x), g_{\tilde{F}(e)}(x) \otimes g_{\tilde{G}(e)}(x)\rangle : x \in U\}. \end{aligned}$$

Definition 11 The product operation on the two dual hesitant fuzzy soft sets $\tilde{F}$ and $\tilde{G}$ over U, denoted by $\tilde{F} \boxtimes \tilde{G} = \tilde{I}$, is a mapping given by $\tilde{I} : E \to DHF(U)$, such that for all $e \in E$,

$$\begin{aligned} \tilde{I}(e) &= \{\langle x, h_{\tilde{I}(e)}(x), g_{\tilde{I}(e)}(x)\rangle : x \in U\} \\ &= \{\langle x, h_{\tilde{F}(e)}(x) \otimes h_{\tilde{G}(e)}(x), g_{\tilde{F}(e)}(x) \oplus g_{\tilde{G}(e)}(x)\rangle : x \in U\}. \end{aligned}$$

Example 7 Let us consider the dual hesitant fuzzy soft set $\tilde{G}$ in Example 4. Let $\tilde{F}$ be another dual hesitant fuzzy soft set over U defined as follows:

$$\begin{aligned} \tilde{F}(e_1) =& \{\langle x_1, \{0.4, 0.3, 0.2\}, \{0.6, 0.5\}\rangle, \langle x_2, \{0.6, 0.4, 0.3\}, \{0.4, 0.2\}\rangle, \\ & \langle x_3, \{0.5, 0.3, 0.1\}, \{0.4, 0.3\}\rangle, \langle x_4, \{0.7, 0.6, 0.5\}, \{0.3, 0.1\}\rangle\}, \end{aligned}$$

$$\tilde{F}(e_2) = \{\langle x_1, \{0.8, 0.6, 0.4\}, \{0.2, 0.1\}\rangle, \langle x_2, \{0.7, 0.6, 0.5\}, \{0.3, 0.2\}\rangle,$$
$$\langle x_3, \{0.7, 0.4, 0.3\}, \{0.3, 0.1\}\rangle, \langle x_4, \{0.4, 0.2, 0.1\}, \{0.5, 0.4\}\rangle\},$$

$$\tilde{F}(e_3) = \{\langle x_1, \{0.4, 0.3, 0.3\}, \{0.6, 0.4\}\rangle, \langle x_2, \{0.8, 0.6, 0.5\}, \{0.2, 0.1\}\rangle,$$
$$\langle x_3, \{0.5, 0.4, 0.4\}, \{0.5, 0.4\}\rangle, \langle x_4, \{0.7, 0.5, 0.2\}, \{0.2, 0.0\}\rangle.$$

Then by Definitions 10 and 11, we have

$$(\tilde{F} \boxplus \tilde{G})(e_1) = \{\langle x_1, \{0.82, 0.72, 0.52\}, \{0.18, 0.10\}\rangle, \langle x_2, \{0.96, 0.82, 0.72\}, \{0.04, 0.02\}\rangle,$$
$$\langle x_3, \{0.90, 0.79, 0.64\}, \{0.08, 0.03\}\rangle, \langle x_4, \{0.88, 0.80, 0.75\}, \{0.12, 0.02\}\rangle\},$$

$$(\tilde{F} \boxplus \tilde{G})(e_2) = \{\langle x_1, \{0.92, 0.58, 0.58\}, \{0.06, 0.01\}\rangle, \langle x_2, \{0.91, 0.80, 0.70\}, \{0.09, 0.04\}\rangle,$$
$$\langle x_3, \{0.85, 0.64, 0.58\}, \{0.06, 0.01\}\rangle, \langle x_4, \{0.88, 0.76, 0.55\}, \{0.10, 0.04\}\rangle\},$$

$$(\tilde{F} \boxplus \tilde{G})(e_3) = \{\langle x_1, \{0.76, 0.58, 0.51\}, \{0.24, 0.04\}\rangle, \langle x_2, \{0.88, 0.72, 0.65\}, \{0.12, 0.05\}\rangle,$$
$$\langle x_3, \{0.95, 0.88, 0.76\}, \{0.05, 0.04\}\rangle, \langle x_4, \{0.88, 0.70, 0.44\}, \{0.06, 0.00\}\rangle\}.$$

$$(\tilde{F} \boxtimes \tilde{G})(e_1) = \{\langle x_1, \{0.28, 0.18, 0.08\}, \{0.72, 0.06\}\rangle, \langle x_2, \{0.54, 0.28, 0.18\}, \{0.46, 0.28\}\rangle,$$
$$\langle x_3, \{0.40, 0.21, 0.06\}, \{0.52, 0.37\}\rangle, \langle x_4, \{0.42, 0.30, 0.25\}, \{0.58, 0.28\}\rangle\},$$

$$(\tilde{F} \boxtimes \tilde{G})(e_2) = \{\langle x_1, \{0.48, 0.24, 0.12\}, \{0.44, 0.19\}\rangle, \langle x_2, \{0.49, 0.30, 0.20\}, \{0.51, 0.36\}\rangle,$$
$$\langle x_3, \{0.45, 0.16, 0.12\}, \{0.44, 0.19\}\rangle, \langle x_4, \{0.32, 0.14, 0.05\}, \{0.60, 0.46\}\rangle\},$$

$$(\tilde{F} \boxtimes \tilde{G})(e_3) = \{\langle x_1, \{0.42, 0.12, 0.09\}, \{0.76, 0.46\}\rangle, \langle x_2, \{0.32, 0.18, 0.15\}, \{0.68, 0.55\}\rangle,$$
$$\langle x_3, \{0.45, 0.32, 0.24\}, \{0.55, 0.46\}\rangle, \langle x_4, \{0.42, 0.20, 0.06\}, \{0.44, 0.10\}\rangle\}.$$

Theorem 3 *Let $\tilde{F}$ and $\tilde{G}$ be two dual hesitant fuzzy soft sets over U. Then the following laws are valid:*

(1) $\tilde{F} \boxplus \tilde{G} = \tilde{G} \boxplus \tilde{F}$,
(2) $\tilde{F} \boxtimes \tilde{G} = \tilde{G} \boxtimes \tilde{F}$,
(3) $(\tilde{F} \boxplus \tilde{G})^c = \tilde{F}^c \boxtimes \tilde{G}^c$,
(4) $(\tilde{F} \boxtimes \tilde{G})^c = \tilde{F}^c \boxplus \tilde{G}^c$.

Proof (1) and (2) are straightforward.
(3) According to Definitions 10 and 3, for all $e \in E$ we have

$$\begin{aligned}(\tilde{F} \boxplus \tilde{G})^c &= \{\langle x, g_{\tilde{F}(e)}(x) \otimes g_{\tilde{G}(e)}(x), h_{\tilde{F}(e)}(x) \oplus h_{\tilde{G}(e)}(x)\rangle : x \in U\} \\ &= \{\langle x, g_{\tilde{F}(e)}(x), h_{\tilde{F}(e)}(x)\rangle : x \in U\} \boxtimes \{\langle x, g_{\tilde{G}(e)}(x), \oplus h_{\tilde{G}(e)}(x)\rangle : x \in U\} \\ &= \tilde{F}^c \boxtimes \tilde{G}^c.\end{aligned}$$

(4) The result can be proved in the similar way.

Theorem 4 *Let $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ be any three dual hesitant fuzzy soft sets over U. Then the following holds:*

(1) $(\tilde{F} \boxplus \tilde{G}) \boxplus \tilde{H} = \tilde{F} \boxplus (\tilde{G} \boxplus \tilde{H})$,
(2) $(\tilde{F} \boxtimes \tilde{G}) \boxtimes \tilde{H} = \tilde{F} \boxtimes (\tilde{G} \boxtimes \tilde{H})$.

Proof The properties follow from Definitions 10 and 11.

5 Conclusion

In this paper, we propose the concept of dual hesitant fuzzy soft sets, and establish some operations of dual hesitant fuzzy soft sets and then some properties can be further investigated for such operations on dual hesitant fuzzy soft sets. In the future, we further investigate relationships between dual hesitant fuzzy soft sets and other mathematical structures, such as lattice structures and topological structures.

References

1. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**(1), 87–96 (1986)
2. Babitha, K.V., John, S.J.: Hesitant fuzzy soft sets. J. New Results Sci. **3**, 98–107 (2013)
3. Chen, Y.F., Peng, X.D., Guan, G.H., Jiang, H.D.: Approaches to multiple attribute decision making based on the correlation coefficient with dual hesitant fuzzy information. J. Intell. Fuzzy Syst. **26**, 2547–2556 (2014)
4. Deschrijver, G., Kerre, E.E.: Implicators based on binary aggregation operators in interval-valued fuzzy set theory. Fuzzy Sets Syst. **153**, 229–248 (2005)
5. Farhadinia, B.: Correlation for dual hesitant fuzzy sets and dual interval-valued hesitant fuzzy sets. Int. J. Intell. Syst. **29**, 184–205 (2014)
6. Gau, W.L., Buehrer, D.J.: Vague sets. IEEE Trans. Syst. Man Cybern. **23**(2), 610–614 (1993)
7. Jiang, Y., Tang, Y., Chen, Q., Liu, H., Tang, J.: Interval-valued intuitionistic fuzzy soft sets and their properties. Comput. Math. Appl. **60**, 906–918 (2010)
8. Molodtsov, D.: Soft set theory-First results. Comput. Math. Appl. **37**, 19–31 (1999)
9. Maji, P.K., Biswas, R., Roy, A.R.: Fuzzy soft set. J. Fuzzy Math. **9**(3), 589–602 (2001)
10. Maji, P.K.: More on intuitionistic fuzzy soft sets. Lect. Notes Comput. Sci **5908**, 231–240 (2009)
11. Maji, P.K., Biswas, R., Roy. A.R.: Intuitionistic fuzzy soft sets. J. Fuzzy Math. **9**, 677–692 (2001)
12. Maji, P.K., Biswas, R., Roy, A.R.: On intuitionistic fuzzy soft sets. J. Fuzzy Math. **12**, 669–683 (2004)
13. Miyamoto, S.: Multisets and fuzzy multisets. In: Liu, Z.Q., Miyamoto, S. (eds.) Soft Computing and Human-Centered Machines, pp. 9–33. Springer, Berlin, Germany (2000)
14. Pawlak, Z.: Rough sets. Int. J. Comput. Inf. Sci. **11**, 145–172 (1982)
15. Torra, V., Narukawa, Y.: On hesitant fuzzy sets and decision. The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, pp. 1378–1382 (2009)
16. Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**, 529–539 (2010)
17. Wang, H.J., Zhao, X.F., Wei, G.W.: Dual hesitant fuzzy aggregation operators in multiple attribute decision making. J. Intell. Fuzzy Syst. **26**, 2281–2290 (2014)

18. Xu, W., Ma, J., Wang, S.Y., Hao, G.: Vague soft sets and their properties. Comput. Math. Appl. **59**, 787–794 (2010)
19. Ye, J.: Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making. Appl. Math. Modell. **38**, 659–666 (2014)
20. Yang, X.B., Lin, T.Y., Yang, J.Y., Li, Y., Yu, D.: Combination of interval-valued fuzzy set and soft set. Comput. Math. Appl. **58**(3), 521–527 (2009)
21. Yang, Y., Tan, X., Meng, C.C.: The multi-fuzzy soft set and its application in decision making. Appl. Math. Modell. **37**, 4915–4923 (2013)
22. Zhu, B., Xu, Z.S., Xia, M.M.: Dual hesitant fuzzy sets. J. Appl. Math. **13**, Article ID 879629 (2012)
23. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 378–352 (1965)

Three-Valued Random Fuzzy Sets

Bin Yu and Xue-Hai Yuan

Abstract The theory of fuzzy sets was founded by Zadeh as an approach to cope with the pressing need to deal with phenomena which cannot be modelled properly by conventional mathematics because they contain factors which are fuzzy in nature. However, in this frame of the theory of fuzzy sets, we cannot discern two conflicting fuzzy conceptions properly, and therefore, the excluded middle law is violated. In this theory, one does not differ fuzzy sets from their membership functions. That is, a fuzzy set in the sense of Zadeh is equivalent to its membership function. This may be the main reason why the excluded middle law was violated. In this paper, we use the concept of three-valued fuzzy sets to provide a mathematical frame of fuzzy sets theory, such that it can eliminate the suspicion for the objective reality of membership functions of fuzzy sets. In our frame, fuzzy sets and their membership functions are not equivalent conceptions.

Keywords Three-valued random fuzzy set · Random set · Fuzzy set · Random variable

1 Introduction

Zadeh [1] introduced the theory of fuzzy sets was first in 1965 as an approach to deal with the phenomena which cannot be modelled properly by conventional mathematics because they are fuzzy in nature. Now the theory of fuzzy sets has become a powerful application tool in many area such as economy, psychology, sociology, linguistics, management science, systems science, control theory, knowledge engineering, picture processing, pattern recognition, etc. However the controversies over

B. Yu (✉) · X.-H. Yuan
Faculty of Electronic Information and Electrical Engineering,
Dalian University of Technology, Dalian 116024, China
e-mail: yub@dlou.edu.cn

B. Yu
School of Science, Dalian Ocean University, Dalian 116023, China

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the mathematical theory of fuzzy sets is far from an end, especially for the objective reality of membership functions of fuzzy sets. In the application of the fuzzy sets, to give a proper, objective membership function of a fuzzy set is very important; it will directly affect the correction of the results we get from the mathematical model based on fuzzy sets theory. But by far, we do not have a commonly accepted method of deciding this kind of function. Can we reframe the theory of fuzzy sets theory such that it can not only eliminate the suspicion for the objective reality of membership functions of fuzzy sets? In the past twenty years, one has got many valuable results in this aspect. In this paper, we will continue study this problem based on the work of others [2–8]. Especially, Yuan's work [9] provide a way to continue study this problem. That is our primary motive to introduce the concept of three-valued random fuzzy set.

2 Three-Valued Random Fuzzy Sets

In this paper, $(\Omega, \mathcal{A}, P)$ denotes a probability space, and ξ a random variable on (r.v.) Ω. Set $\Xi(\{0, 1/2, 1\}) = \{\xi|\xi$ is r.v. with distribution: $\frac{\xi \mid 0 \mid 1/2 \mid 1}{p \mid p_1 \mid p_2 \mid p_3}\}$. Then the following properties are evident.

Proposition 1

(1) 0,1/2,1 $\in \Xi(\{0, 1/2, 1\})$.

(2) For arbitrary $\xi, \eta \in \Xi(\{0, 1/2, 1\})$, *we have*

$$\xi \vee \eta \quad \textit{and} \quad \xi \wedge \eta \in \Xi(\{0, 1/2, 1\}).$$

(3) For arbitrary $\xi_i \in \Xi(\{0, 1/2, 1\}), i = 1, 2, \ldots,$

$$\bigvee_{i=1}^{\infty} \xi_i \quad \textit{and} \quad \bigwedge_{i=1}^{\infty} \xi_i \in \Xi(\{0, 1/2, 1\}),$$

$$\underline{lim}_{n\to\infty}\xi_n \quad \textit{and} \quad \overline{lim}_{n\to\infty}\xi_n \in \Xi(\{0, 1/2, 1\}).$$

Let

$$\mathcal{F}'_{\Omega}(X) = \{A|A : X \to \Xi(\{0, 1/2, 1\})\}.$$

For $A, B \in \mathcal{F}'_{\Omega}(X)$, *we define*

$$(A \bigcup B)(x) = A(x) \bigvee B(x),$$

$$(A \bigcap B)(x) = A(x) \bigwedge B(x),$$

$$A^c(x) = 1 - A(x).$$

It is easy to see that $A \bigcap B$, $A \bigcap B$, $A^c \in \mathcal{F}'_\Omega(X)$. *They are called the union and the intersection of A and B, and the complement of A, respectively.*

Now we introduce a relation '$\sim$' in $\mathcal{F}'_\Omega(X)$*: for any element* A, B *in* $\mathcal{F}'_\Omega(X)$*, we put '*$A \sim B$*' whenever* $P\{A(x) = B(x)\} = 1$ *for every x in X. For every A in* $\mathcal{F}'_\Omega(X)$*, we denote by* $\widetilde{A}$ *the equivalence class containing A:*

$$\widetilde{A} = \{A' | A' \in \mathcal{F}'_\Omega(X), A' \sim A\}.$$

Put

$$\mathcal{F}_\Omega(X) = \{A' | A \in \mathcal{F}'_\Omega(X)\}.$$

Proposition 2 *Let* $\widetilde{A}, \widetilde{B} \in \mathcal{F}_\Omega(X)$.

(1) If $\widetilde{A} = \widetilde{B}$*, then*

$$E(A(x)) = E(B(x)), (\forall x \in X),$$

$$i.e.\ P\{A(x) = 1\} + 1/2P\{A(x) = 1/2\} = P\{B(x) = 1\} + 1/2P\{B(x) = 1/2\}.$$

(2) If for every $x \in X$, $P\{A(x) \le B(x)\} = 1$*, then for any* $A' \in \widetilde{A}$, $B' \in \widetilde{B}$ *we have*

$$P\{A'(x) \le B'(x)\} = 1, (\forall x \in X).$$

(3) For any $A' \in \widetilde{A}$, $B' \in \widetilde{B}$,

$$A' \bigcup B' \sim A \bigcup B,$$

$$A' \bigcap B' \sim A \bigcap B.$$

Generally, we have

$$\bigcup_{n=1}^{\infty} A'_n \sim \bigcup_{n=1}^{\infty} A_n,$$

$$\bigcap_{n=1}^{\infty} A'_n \sim \bigcap_{n=1}^{\infty} A_n,$$

$$\underline{lim}_{n\to\infty} A'_n \sim \underline{lim}_{n\to\infty} A_n,$$

$$\overline{lim}_{n\to\infty} A'_n \sim \overline{lim}_{n\to\infty} A_n,$$

whenever $A'_n \sim A_n, n = 1, 2, \ldots.$

(4) For any $A' \in \widetilde{A}$, $1 - A' \sim 1 - A$.

Definition 1 For every $A \in \mathcal{F}'_{\Omega}(X)$, $\widetilde{A}$ is called a three-valued random fuzzy set of X with probability space $(\Omega, \mathcal{A}, P)$. Let

$$\mu_{\widetilde{A}}(x) = E(A(x)) = P\{A(x) = 1\} + 1/2P\{A(x) = 1/2\}.$$

$\mu_{\widetilde{A}}(x)$ is called the membership function of three-valued random fuzzy set $\widetilde{A}$.

Definition 2 Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}_{\Omega}(X)$,

(1) $\widetilde{A}$ is said to be included in $\widetilde{B}$, denoted by $\widetilde{A} \subset \widetilde{B}$ if $\forall x \in X$, we have

$$P\{A(x) \leq B(x)\} = 1.$$

(2) $\widetilde{A}$ is said to be equal to $\widetilde{B}$, denoted by $\widetilde{A} = \widetilde{B}$, if both $\widetilde{A} \subset \widetilde{B}$ and $\widetilde{B} \subset \widetilde{A}$ are satisfied.

Definition 3 Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}_{\Omega}(X)$, we define that

$$\widetilde{A} \bigcup \widetilde{B} = \widetilde{A \bigcup B},$$

$$\widetilde{A} \bigcap \widetilde{B} = \widetilde{A \bigcap B},$$

$$\widetilde{A}^c = \widetilde{1 - A}.$$

They are called the union and the intersection of $\widetilde{A}$ and $\widetilde{B}$, and the complement of $\widetilde{A}$, respectively.

With Proposition 2, we know that all these definitions are well defined.

A three-valued random fuzzy set is also a generalization of a classical set. Let $\widetilde{A} \in \mathcal{F}_{\Omega}(X)$, such that $\forall x \in X$, $P\{A(x) = 1\} = 1$ or $P\{A(x) = 0\} = 1$, then the relation between x and the three-valued random fuzzy set $\widetilde{A}$ is determined. Thus $\widetilde{A}$ represents a conventional set. On the other hand, the characteristic function of a classical set can also be regarded as a mapping from X to $\Xi(\{0, 1/2, 1\})$, and thus corresponds to an element in $\mathcal{F}_{\Omega}(X)$. Set

$$\mathcal{P}_{\Omega}(X) = \{\widetilde{A} \in \mathcal{F}_{\Omega}(X) | \forall x \in X,$$

either

$$P\{A(x) = 1\} = 1 \quad \text{or} \quad P\{A(x) = 0\} = 1\}.$$

It is obvious that $(\mathcal{P}_{\Omega}(X), \cup, \cap, c)$ is a subalgebra of $(\mathcal{F}_{\Omega}(X), \cup, \cap, c)$. Let $\mathcal{P}(X)$ denote the collection of all conventional set of X, then we have the following proposition.

Proposition 3 *The mapping*

$$T : \mathcal{P}_\Omega(X) \to \mathcal{P}(X),$$

$$\widetilde{A} \mapsto \{x|P\{A(x) = 1\} = 1\},$$

is an isomorphic map from $(\mathcal{P}_\Omega(X), \cup, \cap, c)$ *to* $(\mathcal{P}(X), \cup, \cap, c)$.

Proof We need only to show that

$$\widetilde{A} = \widetilde{B} \quad \text{iff} \quad \{x|P\{A(x) = 1\} = 1\} = \{x|P\{B(x) = 1\} = 1\}.$$

With Proposition 2, we know that

$$\{x|P\{A(x) = 1\} = 1\} = \{x|P\{B(x) = 1\} = 1\},$$

when $\widetilde{A} = \widetilde{B}$. On the other hand, if

$$\{x|P\{A(x) = 1\} = 1\} = \{x|P\{B(x) = 1\} = 1\},$$

since $\widetilde{A}, \widetilde{B} \in \mathcal{P}_\Omega(X)$, for any $x \in X$, we have

$$P\{A(x) = 0\} = 1, \quad \text{or} \quad P\{A(x) = 1/2\} = 1, \quad \text{or} \quad P\{A(x) = 1\} = 1,$$

and

$$P\{B(x) = 0\} = 1, \quad \text{or} \quad P\{B(x) = 1/2\} = 1, \quad \text{or} \quad P\{B(x) = 1\} = 1.$$

Thus if $x \in \{x|P\{A(x) = 1\} = 1\} = \{x|P\{B(x) = 1\} = 1\}$, then

$$\begin{aligned}&P\{A(x) = 1, B(x) = 0\} + P\{A(x) = 1, B(x) = 1/2\} + P\{A(x) = 0, B(x) = 1\}\\&+ P\{A(x) = 0, B(x) = 1/2\} + P\{A(x) = 1/2, B(x) = 0\} + P\{A(x) = 1/2, B(x) = 1\} = 0,\end{aligned}$$

and hence,

$$\begin{aligned}&P\{A(x) = B(x)\}\\&\quad = P\{A(x) = 1, B(x) = 1\} + P\{A(x) = 1/2, B(x) = 1/2\} + P\{A(x) = 0, B(x) = 0\}\\&\quad = P\{A(x) = 1, B(x) = 1\} + P\{A(x) = 1/2, B(x) = 1/2\} + P\{A(x) = 0, B(x) = 0\}\\&\qquad + P\{A(x) = 1, B(x) = 0\} + P\{A(x) = 1, B(x) = 1/2\} + P\{A(x) = 0, B(x) = 1\}\\&\qquad + P\{A(x) = 0, B(x) = 1/2\} + P\{A(x) = 1/2, B(x) = 0\}\\&\qquad + P\{A(x) = 1/2, B(x) = 1\} = 1\end{aligned}$$

If $x \overline{\in} \{x|P\{A(x) = 1\} = 1\} = \{x|P\{B(x) = 1\} = 1\}$, then $P\{A(x) = 1\} = 0$, $P\{B(x) = 1\} = 0$, and thus

$$\begin{aligned}1 = &P\{A(x) = 1, B(x) = 1\} + P\{A(x) = 1/2, B(x) = 1/2\} + P\{A(x) = 0, B(x) = 0\}\\ &+ P\{A(x) = 1, B(x) = 0\} + P\{A(x) = 1, B(x) = 1/2\} + P\{A(x) = 0, B(x) = 1\}\\ &+ P\{A(x) = 0, B(x) = 1/2\} + P\{A(x) = 1/2, B(x) = 0\} + P\{A(x) = 1/2,\\ &B(x) = 1\} = P\{A(x) = B(x)\}.\end{aligned}$$

Because of this conclusion, we will not differ $\widetilde{A}$ and the set $\{x|P\{A(x) = 1\} = 1\}$ when $\widetilde{A} \in \mathcal{P}_\Omega(X)$. Especially, if for all $x \in X$, $P\{A(x) = 1\} = 1$, we have $\widetilde{A} = X$, if for all $x \in X$, $P\{A(x) = 0\} = 1$, we have $\widetilde{A} = \emptyset$.

Proposition 4 *Let $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{F}_\Omega(X)$. Then*
(1) Idempotent law

$$\widetilde{A} \cup \widetilde{A} = \widetilde{A},$$

$$\widetilde{A} \cap \widetilde{A} = \widetilde{A}.$$

(2) Commutative law

$$\widetilde{A} \cup \widetilde{B} = \widetilde{B} \cup \widetilde{A},$$

$$\widetilde{A} \cap \widetilde{B} = \widetilde{B} \cap \widetilde{A}.$$

(3) Associative law

$$(\widetilde{A} \cup \widetilde{B}) \cup \widetilde{C} = \widetilde{A} \cup (\widetilde{B} \cup \widetilde{C}),$$

$$(\widetilde{A} \cap \widetilde{B}) \cap \widetilde{C} = \widetilde{A} \cap (\widetilde{B} \cap \widetilde{C}).$$

(4) Absorption law

$$(\widetilde{A} \cup \widetilde{B}) \cap \widetilde{A} = \widetilde{A},$$

$$(\widetilde{A} \cap \widetilde{B}) \cup \widetilde{A} = \widetilde{A}.$$

(5) Distributive law

$$(\widetilde{A} \cup \widetilde{B}) \cap \widetilde{C} = (\widetilde{A} \cap \widetilde{C}) \cup (\widetilde{B} \cap \widetilde{C}),$$

$$(\widetilde{A} \cap \widetilde{B}) \cup \widetilde{C} = (\widetilde{A} \cup \widetilde{C}) \cap (\widetilde{B} \cup \widetilde{C}).$$

(6) Identity law

$$\widetilde{A} \cup X = X, \widetilde{A} \cap X = \widetilde{A},$$

$$\widetilde{A} \cup \emptyset = \widetilde{A}, \widetilde{A} \cap \emptyset = \emptyset.$$

(7) Involution law

$$\widetilde{A}^{cc} = \widetilde{A}.$$

(8) Dualization law

$$(\widetilde{A} \cup \widetilde{B})^c = \widetilde{A}^c \cap \widetilde{B}^c,$$

$$(\widetilde{A} \cap \widetilde{B})^c = \widetilde{A}^c \cup \widetilde{B}^c.$$

3 The Properties of $\bigcup, \bigcap$, c

Let x be a fixed element of X. Set

$$f_x(\widetilde{A}) = \mu_{\widetilde{A}}(x).$$

Proposition 5 *Let* $\widetilde{A}, \widetilde{B} \in \mathcal{F}_\Omega(X)$. *Then*

$$f_x(\widetilde{A} \cup \widetilde{B}) = f_x(\widetilde{A}) + f_x(\widetilde{B}) - f_x(\widetilde{A} \cap \widetilde{B}).$$

Proof $f_x(\widetilde{A} \cup \widetilde{B}) = \mu_{\widetilde{A} \cup \widetilde{B}}(x) = E(\widetilde{A} \cup \widetilde{B}(x))$
$= P\{(A \cup B)(x) = 1\} + 1/2P\{(A \cup B)(x) = 1/2\}$
$= P\{A(x) = 1 \text{ or } B(x) = 1\} + 1/2P\{A(x) = 1/2 \text{ or } B(x) = 1/2\}$
$= P\{A(x) = 1\} + P\{B(x) = 1\} - P\{A(x) = 1, B(x) = 1\}$
$+ 1/2P\{A(x) = 1/2\} + 1/2P\{B(x) = 1/2\} - 1/2P\{(A \cap B)(x) = 1/2\}$
$= [P\{A(x) = 1\} + 1/2P\{A(x) = 1/2\}] + [P\{B(x) = 1\} + 1/2P\{B(x)$
$= 1/2\}] - [P\{(A \cap B)(x) = 1\} + 1/2P\{(A \cap B)(x) = 1/2\}]$
$= \mu_{\widetilde{A}}(x) + \mu_{\widetilde{B}}(x) - \mu_{\widetilde{A} \cap \widetilde{B}}(x)$
$= f_x(\widetilde{A}) + f_x(\widetilde{B}) - f_x(\widetilde{A} \cap \widetilde{B}).$

Proposition 6
(1) Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}_\Omega(X)$, *and* $\widetilde{A} \subset \widetilde{B}$. *Then*

$$f_x(\widetilde{A}) \le f_x(\widetilde{B}).$$

(2) For arbitrary $\widetilde{A}, \widetilde{B} \in \mathcal{F}_\Omega(X)$,

$$f_x(\widetilde{A} \cup \widetilde{B}) \ge f_x(\widetilde{A}) \vee f_x(\widetilde{B}).$$

$$f_x(\widetilde{A} \cap \widetilde{B}) \geq f_x(\widetilde{A}) \wedge f_x(\widetilde{B}).$$

(3) For any $\widetilde{A} \in \mathcal{F}_\Omega(X)$,

$$f_x(\widetilde{A}^c\} = 1 - f_x(\widetilde{A}).$$

(4) For $\widetilde{A_n} \in \mathcal{F}_\Omega(X), n = 1, 2, \ldots,$ *and* $\{\widetilde{A_n}\}$ *is monotone, then*

$$\lim_{n\to\infty} f_x(\widetilde{A_n}) = f_x\{\lim_{n\to\infty} \widetilde{A_n}\}.$$

Proof

(1) Since $\widetilde{A} \subset \widetilde{B}$ is equivalent to $P\{A(x) \leq B(x)\} = 1$, thus with Proposition 1, we have

$$f_x(\widetilde{A}) \leq f_x(\widetilde{B}).$$

(2) Follows from (1) directly.

(3) Obvious.

(4) Given that $\{\widetilde{A_n}\}$ is monotone increasing. Thus we have

$$P\{A_n(x) \leq A_{n+1}(x)\} = 1, n = 1, 2, \ldots,$$

Set $E_n = \{A_n(x) > A_{n+1}(x)\}$, then $P(E_n) = 0, n = 1, 2, \ldots$. Denoted by Ω' the set $\Omega \setminus \cup_{n=1}^{\infty} E_n$, then

$$\{A_1(x) = 1\} \cap \Omega' \subset \{A_2(x) = 1\} \cap \Omega' \subset \ldots$$

and

$$\{A_1(x) = 1/2\} \cap \Omega' \subset \{A_2(x) = 1/2\} \cap \Omega' \subset \ldots.$$

Hence, with the property of probability measure P,

$$P\{\lim_{n\to\infty} \{A_n(x) = 1\} \cap \Omega'\} = \lim_{n\to\infty} P\{\{A_n(x) = 1\} \cap \Omega'\},$$

and

$$P\{\lim_{n\to\infty} \{A_n(x) = 1/2\} \cap \Omega'\} = \lim_{n\to\infty} P\{\{A_n(x) = 1/2\} \cap \Omega'\}.$$

Noting that $P(\cup_{n=1}^{\infty} E_n) = 0$, we obtain that

$$P\{\lim_{n\to\infty} A_n(x) = 1\} = \lim_{n\to\infty} P\{A_n(x) = 1\},$$

and

$$P\{\lim_{n\to\infty} A_n(x) = 1/2\} = \lim_{n\to\infty} P\{A_n(x) = 1/2\},$$

so

$$P\{\lim_{n\to\infty} A_n(x) = 1\} + 1/2P\{\lim_{n\to\infty} A_n(x) = 1/2\}$$

$$= \lim_{n\to\infty} P\{A_n(x) = 1\} + 1/2 \lim_{n\to\infty} P\{A_n(x) = 1/2\}.$$

and thus,

$$\lim_{n\to\infty} f_x(\widetilde{A_n}) = f_x \lim_{n\to\infty} (\widetilde{A_n}).$$

For the case when $\widetilde{A_n}$ is monotone decreasing, we can show in the same way. For a fixed x in X, let

$$F_x(\widetilde{A}, \widetilde{B}) = \mu_{\widetilde{A}\cup\widetilde{B}}(x),$$

$$G_x(\widetilde{A}, \widetilde{B}) = \mu_{\widetilde{A}\cap\widetilde{B}}(x),$$

$$C_x(\widetilde{A}) = \mu_{\widetilde{A}^c}(x).$$

Proposition 7

(1) *As a mapping from* $\mathcal{F}_\Omega(X) \times \mathcal{F}_\Omega(X)$ *to* $[0, 1]$*,* F_x, G_x *have the following properties:*

(1) both of F_x *and* G_x *are nondecreasing,*

(2) $F_x(\widetilde{A}, \widetilde{B}) = F_x(\widetilde{B}, \widetilde{A}), G_x(\widetilde{A}, \widetilde{B}) = G_x(\widetilde{B}, \widetilde{A})$*,*

(3) $F_x(\widetilde{A}, \widetilde{A}) = G_x(\widetilde{B}, \widetilde{B}) = f_x(\widetilde{A})$*,*

(4) $F_x(\widetilde{A}, \widetilde{B}) \geq f_x(\widetilde{A}) \vee f_x(\widetilde{B}), G_x(\widetilde{A}, \widetilde{B}) \leq f_x(\widetilde{A}) \wedge f_x(\widetilde{B})$*,*

(5) $F_x(X, X) = 1, G_x(\emptyset, \emptyset) = 0$*,*

(6) for $\widetilde{A_n} \in \mathcal{F}_\Omega(X), n = 1, 2, \ldots,$ *and* $\{\widetilde{A_n}\}$ *is monotone, then*

$$\lim_{n\to\infty} F_x(\widetilde{A}_n, \widetilde{B}) = F_x(\lim_{n\to\infty} \widetilde{A}_n, \widetilde{B}),$$

$$\lim_{n\to\infty} F_x(\widetilde{B}, \widetilde{A}_n) = F_x(\widetilde{B}, \lim_{n\to\infty} \widetilde{A}_n),$$

$$\lim_{n\to\infty} G_x(\widetilde{A}_n, \widetilde{B}) = G_x(\lim_{n\to\infty} \widetilde{A}_n, \widetilde{B}),$$

$$\lim_{n\to\infty} G_x(\widetilde{B}, \widetilde{A}_n) = G_x(\widetilde{B}, \lim_{n\to\infty} \widetilde{A}_n).$$

(2) *The mapping* $C_x : \mathcal{F}_\Omega(X) \to [0, 1]$ *has the following properties:*
(1) $C_x(X) = 0, C_x(\emptyset) = 1$.
(2) C_x *is nonincreasing.*
(3) For $\widetilde{A_n} \in \mathcal{F}_\Omega(X), n = 1, 2 \ldots,$ *and* $\{\widetilde{A_n}\}$ *is monotone, then*

$$\lim_{n\to\infty} C_x(\widetilde{A}_n) = C_x(\lim_{n\to\infty} \widetilde{A}_n).$$

4 $(\mathcal{F}_{[0,1]}(X), \bigcup, \bigcap, c)$ and $(\mathcal{F}(X), \bigcup, \bigcap, c)$

In this section, we will let $\mathcal{F}(X)$ denote the collection of all fuzzy sets in X defined by Zadeh, and $\mathcal{F}_{[0,1]}(X)$ the collection of all three-valued random Fuzzy Sets in X with probability space $([0, 1], \mathcal{B}_0, P)$, where $\mathcal{B}_0$ is composed of all Borel set in [0, 1] and measure P is the Lebesgue measure on [0, 1].

In this part of the paper, we will examine the relations between three-valued random fuzzy sets in X with probability space $([0, 1], \mathcal{B}_0, P)$. As we can see, the fuzzy sets defined by Zadeh can also be regarded as the equivalent class of $\mathcal{F}_{[0,1]}(X)$, but it is a coarse classification. The operations for three-valued random fuzzy sets and those in the sense given by Zadeh are also different, they coincide only in some special cases.
Let

$$\Xi_0 = \{\xi : [0, 1] \Rightarrow \{0, 1/2, 1\} | \xi \quad \text{is measurable}\}.$$

Definition 5 $\xi \in \Xi_0$ is called nonincreasing if $\forall \omega_1, \omega_2 \in [0, 1], \omega_1 \le \omega_2$, then

$$\xi(\omega_1) \ge \xi(\omega_2).$$

$\xi \in \Xi_0$ is called almost certain nonincreasing if there exists a set $E \subset [0, 1]$ such that $P(E) = 0$, and ξ is nonincreasing on $[0, 1]\backslash E$.

It is easy to see, if ξ is nonincreasing, and $P\{\xi = \eta\} = 1$, then η is almost certain nonincreasing. The collection of all almost certain nonincreasing r.v. in Ξ_0 is denoted by Ξ_0^*.

Lemma 1 *Given that* $\xi \in \Xi_0$ *is nonincreasing, let* $\alpha = sup\{\xi = 1\}, \beta = inf\{\xi = 0\}$, *then*

$$(1)\{\xi = 0\} = \begin{cases} [\beta, 1], \ \xi(\beta) = 0, \\ (\beta, 1], \ \xi(\beta) = 1; \end{cases}$$

$$(2)\{\xi = 1\} = \begin{cases} [0, \alpha), \ \xi(\alpha) = 0, \\ [0, \alpha], \ \xi(\alpha) = 1; \end{cases}$$

$$(3)\{\xi = 1/2\} = \langle \alpha, \beta \rangle.$$

Proof (2) If $\beta < \alpha$, then $\inf\{\xi = 0\} < \sup\{\xi = 1\}$, it is a contradiction to the condition that $\xi \in \Xi_0$ is nonincreasing. So $\beta \geq \alpha$.

$\beta = inf\{\xi = 0\}$, so $\{\xi = 0\} \subset [\beta, 1]$; for $\beta \in [0, 1]$, if $\gamma > \beta = inf\{\xi = 0\}$, then there exists a $\gamma' \in [0, 1]$, $\gamma' < \gamma$, and $\xi(\gamma') = 0$.

Since ξ is nonincreasing, $\xi(\gamma) = 0$. Thus $[\beta, 1) \subset \{\xi = 0\}$. Therefore (2) is true. In the sequel of the paper, we will use $\langle a, b\rangle$ to denote an interval with ends a, b, when we are not sure whether a and b are included in the interval.

Definition 6 $\widetilde{A} \in \mathcal{F}_{[0,1]}(X)$ is said to be a three-valued random* fuzzy set if $\forall x \in X$, $A(x) \in \Xi_0^*$.

The collection of all three-valued random* fuzzy sets are denoted by $\mathcal{F}^*(X)$. In the first section of this paper, we introduce an equivalence relation '$\sim$' in $\mathcal{F}_\Omega(X)$, where we see that $\widetilde{A} \sim \widetilde{B}$ is not equivalent to $\mu_{\widetilde{A}} = \mu_{\widetilde{B}}$. However, for three-valued random* fuzzy sets we have the following proposition.

Proposition 8 *Let $\widetilde{A}$, $\widetilde{B}$ be three-valued random*-fuzzy sets. Then*

$$\widetilde{A} = \widetilde{B}, \quad \textit{iff} \;\; \mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x), (\forall x \in X).$$

Proof With Proposition 1 we need only to show that $\widetilde{A}(x) = \widetilde{B}(x)$ is sufficient for $\widetilde{A} = \widetilde{B}$.

To show $\widetilde{A} = \widetilde{B}$, it suffices to show $\forall x \in X$, $P\{A(x) = B(x)\} = 1$. Without loss of generality we assume that all the values taken by A and B are nonincreasing.

For $A(x)$, let $\alpha = \sup\{\xi = 1\}$, $\beta = \inf\{\xi = 0\}$.

For $B(x)$, let $\alpha' = \sup\{\xi' = 1\}$, $\beta' = \inf\{\xi' = 0\}$.

By Lemma 1,

$$\{A(x) = 0\} = \begin{cases} [\beta, 1], & A(x)(\beta) = 0, \\ (\beta, 1], & A(x)(\beta) = 1; \end{cases}$$

$$\{B(x) = 0\} = \begin{cases} (\beta', 1], & A(x)(\beta') = 0, \\ [\beta', 1], & A(x)(\beta') = 1; \end{cases}$$

$$\{A(x) = 1\} = \begin{cases} [0, \alpha), & A(x)(\alpha) = 0, \\ [0, \alpha], & A(x)(\alpha) = 1; \end{cases}$$

$$\{B(x) = 1\} = \begin{cases} [0, \alpha'), & A(x)(\alpha') = 0, \\ [0, \alpha'], & A(x)(\alpha') = 1; \end{cases}$$

$$\{A(x) = 1/2\} = \langle \alpha, \beta \rangle,$$

$$\{B(x) = 1/2\} = \langle \alpha', \beta' \rangle.$$

Therefore,

$$P\{A(x) = B(x)\} = P\{A(x) = 1, B(x) = 1\} + P\{A(x) = 0, B(x) = 0\} + P\{A(x) = 1/2, B(x) = 1/2\}$$
$$= P\{[0, \alpha \wedge \beta\rangle\} + P\{\langle\alpha' \vee \beta'\rangle, 1\} + P\{\langle\alpha \wedge \alpha', \beta \wedge \beta'\rangle\}$$
$$= (\alpha \wedge \beta) + (1 - \alpha' \vee \beta') + [(\beta \wedge \beta') - (\alpha \wedge \alpha')].$$

When $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x)$, let $\alpha < \beta$, we have

$$\alpha = \alpha', \beta = \beta'.$$

So,

$$P\{A(x) = B(x)\} = \alpha \wedge \beta + 1 - (\alpha \vee \beta) + \beta - \alpha = 1.$$

Proposition 9 *Let $\widetilde{A}$, $\widetilde{B}$ be three-valued random*-fuzzy sets. Then*

$$\widetilde{A} \subset \widetilde{B}, \textit{iff} \ \ \mu_{\widetilde{A}}(x) \le \mu_{\widetilde{B}}(x), (\forall x \in X).$$

Proof Let $\alpha = sup\{\xi = 1\}, \beta = inf\{\xi = 0\}$.

By Proposition 1, we need only to show that $P\{A(x) \le B(x)\} = 1$ for all $x \in X$ when $\mu_{\widetilde{A}} \le \mu_{\widetilde{B}}$.

Without loss of generality, we assume that all the values taken by A and B are nonincreasing. Then

$$\{A(x) \le B(x)\} = [0, 1]\backslash\{A(x) > B(x)\}$$
$$= [0, 1]\backslash[\{A(x) = 1, B(x) = 0\}, \{A(x) = 1, B(x) = 1/2\}, \{A(x) = 1/2, B(x) = 0\}].$$

But $\mu_{\widetilde{A}}(x) \le \mu_{\widetilde{B}}(x)$, we have

$$\alpha \le \alpha', \beta \le \beta',$$

so

$$\alpha \le \beta, \alpha' \le \beta',$$

and thus

$$P\{A(x) = 1, B(x) = 0\} = [0, \alpha\rangle \cap [\beta', 1] = 0;$$

$$P\{A(x) = 1, B(x) = 1/2\} = [0, \alpha\rangle \cap [\alpha', \beta' >= 0;$$

$$P\{A(x) = 1/2, B(x) = 0\} = [\alpha, \beta\rangle \cap [\beta', 1] = 0.$$

This shows

$$P\{A(x) \leq B(x)\} = 1.$$

Proposition 10 *Let $\widetilde{A}$, $\widetilde{B}$ be three-valued random*-fuzzy sets. Then*

$$\mu_{\widetilde{A}\cup\widetilde{B}}(x) = \mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x),$$

$$\mu_{\widetilde{A}\cap\widetilde{B}}(x) = \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x).$$

Proof Let $\alpha = sup\{\xi = 1\}$, $\beta = inf\{\xi = 0\}$.

$$\begin{aligned}
\mu_{\widetilde{A}\cup\widetilde{B}}(x) &= E(A \cup B)(x)) \\
&= P\{(A \cup B)(x) = 1\} + 1/2P\{(A \cup B)(x) = 1/2\} \\
&= P\{[0, \alpha\rangle \cup [0, \alpha\rangle\} + 1/2P\{[\alpha, \beta\rangle \cup [\alpha', \beta'\rangle\} \\
&= (\alpha \vee \alpha') + [1/2(\beta - \alpha) \vee 1/2(\beta' - \alpha')] \\
&= [\alpha + 1/2(\beta - \alpha)] \vee [\alpha' + 1/2(\beta' - \alpha')] \\
&= E(A(x)) \vee E(B(x)) = \mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x).
\end{aligned}$$

In the same way, we can show $\mu_{\widetilde{A}\cap\widetilde{B}}(x) = \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x)$.

Theorem 1 *The mapping*

$$\Lambda : \mathcal{F}^*(X) \to \mathcal{F}(X),$$

$$\widetilde{A} \mapsto \Lambda(\widetilde{A}),$$

$$\mu_{\Lambda(\widetilde{A}}(x) = \mu_{\widetilde{A}}(x)$$

is an isomorphic map from $(\mathcal{F}^*(X), \cup, \cap)$ *to* $(\mathcal{F}(X), \cup, \cap)$.

Proof

(1) Λ is injective. Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}^*(X)$, and $\widetilde{A} \neq \widetilde{B}$. By Proposition 8, $\Lambda(\widetilde{A}) \neq \Lambda(\widetilde{B})$.

(2) Λ is surjective. Let $\alpha = sup\{\xi = 1\}$, $\beta = inf\{\xi = 0\}$. For arbitrary $F \in \mathcal{F}(X)$, let $f_x(\alpha) = \chi_{F_\alpha}(x)$, where F_α denotes the $\alpha - cut$ of fuzzy set F. Then $f_x(\alpha)$ as a function from [0, 1] to {0, 1/2, 1} is measurable because $\{f_x = 1\} = [0, \alpha]$, $\{f_x = 1/2\} = \langle\alpha, \beta\rangle$, $\{f_x = 0\} = [\beta, 1]$. Furthermore, $\forall\alpha', \alpha'' \in [0, 1]$, $\alpha' < \alpha''$,

$$f_x(\alpha') = \chi_{F_{\alpha'}}(x) \geq \chi_{F_{\alpha''}}(x) = f_x(\alpha'').$$

Thus F is nonincreasing. Let $\widetilde{A}$ be a three-valued random*-fuzzy set, such that $A(x)(\alpha) = \chi_{F_\alpha}(x)$; then

$$\Lambda(\widetilde{A})(x) = \mu_{\widetilde{A}}(x) = E(A(x))$$

$$= P\{A(x) = 1\} + 1/2P\{A(x) = 1/2\}$$

$$= P\{[0, \alpha]\} + 1/2P\{\langle\alpha, \beta\rangle\} = \mu_F(x).$$

That is $\Lambda(\widetilde{A}) = F$.

(3) For $\widetilde{A}, \widetilde{B} \in \mathcal{F}^*(X)$, with Proposition 8, we have

$$\Lambda(\widetilde{A} \cup \widetilde{B}) = \Lambda(\widetilde{A}) \cup \Lambda(\widetilde{B}),$$

$$\Lambda(\widetilde{A} \cap \widetilde{B}) = \Lambda(\widetilde{A}) \cap \Lambda(\widetilde{B}).$$

To sum up, we know Λ is an isomorphic mapping.

Note that for $\xi \in \Xi_0^*$, $1 - \xi$ will not belong to Ξ_0^*, Ξ_0^* is not closed for the complement of three-valued random-fuzzy set. In order to see the difference between $(\mathcal{F}^*(X), \cup, \cap, c^*)$ and $(\mathcal{F}(X), \cup, \cap, c)$, we give following definition.

Definition 7 Let $\xi \in \Xi_0^*$ and set $\xi^{c^*}(\alpha) = 1 - \xi(1 - \alpha)$. Then it is evident that ξ^{c^*} is almost certain nonincreasing when ξ^{c^*} is. We call ξ^{c^*} the c^*—*complement* of ξ.

Definition 8 Let $\widetilde{A} \in \mathcal{F}^*(X)$ and set $\widetilde{A}^{c^*} = \widetilde{A^{c^*}}$, where A^{c^*} is defined by $(A^{c^*})(x) = A(x)^{c^*}$.

We call $A^{c^*}(x)$ the c^*—*complement* of $\widetilde{A}$.

Proposition 11 *For every* $\widetilde{A} \in \mathcal{F}_{[0,1]}(X)$,

$$\mu_{\widetilde{A}^c}(x) = \mu_{\widetilde{A}^{c*}}(x) = 1 - \mu_{\widetilde{A}}(x).$$

Proof Without loss of generality we assume that $\forall x \in X$, $A(x)$ is nonincreasing.

Let $\alpha = sup\{\xi = 1\}$, $\beta = inf\{\xi = 0\}$.

$\mu_{\widetilde{A}^{c*}}(x) = E(A^{c^*}(x))$
$= P\{A^{c^*}(x) = 1\} + 1/2P\{A^{c^*}(x) = 1/2\}$
$= P\{1 - \alpha | A(x)(\alpha) = 0\} + 1/2P\{\alpha | 1 - A(x)(1 - \alpha) = 1/2\}$
$= P\{1 - \alpha | A(x)(\alpha) = 0\} + 1/2P\{1 - \alpha | A(x)(1 - \alpha) = 1/2\}$
$= P\{\langle\beta, 1]\} + 1/2P\{\langle\alpha, \beta\rangle\}$
$= 1 - \beta + 1/2(\beta - \alpha) = 1 - 1/2(\alpha + \beta)$
$= 1 - \{[0, \alpha] + 1/2\langle\alpha, \beta\langle\}$
$= 1 - [P\{\widetilde{A}(x) = 1\} + 1/2P\{\widetilde{A}(x) = 1/2\}]$
$= 1 - \mu_{\widetilde{A}}(x).$

Theorem 1 and Proposition 11 give the following Theorem.

Theorem 2 *The mapping*

$$\Lambda : \mathcal{F}^*(X) \to \mathcal{F}(X),$$

$$\widetilde{A} \mapsto \Lambda(\widetilde{A}),$$

$$\mu_{\Lambda(\widetilde{A})}(x) = \mu_{\widetilde{A}}(x)$$

is an isomorphic map from $\mathcal{F}^*(X), \cup, \cap, c^*)$ *to* $(\mathcal{F}(X), \cup, \cap, c)$.

The above results indicate that the fuzzy sets in the sense of Zadeh correspond to the coarse classification of the element of $\mathcal{F}'_{[0,1]}(X)$, where we do not differ A from B whenever $E(A(x)) = E(B(x))$. To see this more clearly, we introduce an equivalence relation $\sim'$ in $\mathcal{F}_{[0,1]}(X)$.

For $A, B \in \mathcal{F}_{[0,1]}(X)$, we put $A \sim' B$ if $E(A(x)) = E(B(x))$. Let

$$\hat{A} = \{B | B \sim' A\},$$

$$\mathcal{F}^*{}_0(X) = \{\hat{A} | A \in \mathcal{F}'_{[0,1]}(X)\},$$

and let ξ_α be the element of Ξ_0^*, such that $\forall \beta \in [0, 1]$.

Let $\alpha = sup\{\xi = 1\}, \beta = inf\{\xi = 0\}$.

$$\xi_{\alpha_\beta}(\gamma) = \begin{cases} 1, & \gamma \le \alpha, \\ 1/2, & \alpha < \gamma \le \beta, \\ 0, & \gamma > \beta. \end{cases}$$

For each $\hat{A} \in \mathcal{F}^*{}_0(X)$, let

$$\triangle_A(x) = \xi_{E(A(x))}, (x \in X).$$

Definition 9 For $\hat{A}, \hat{B} \in \mathcal{F}^*{}_0(X)$, we define

$$\hat{A} \cup \hat{B} = \Delta_A \,\widehat{\cup}\, \Delta_B,$$

$$\hat{A} \cap \hat{B} = \Delta_A \,\widehat{\cap}\, \Delta_B,$$

$$\hat{A}^c = \widehat{\Delta_{A^c}}.$$

Theorem 3 $(\mathcal{F}_0^*(X), \cup, \cap, c)$ *is isomorphic to* $(\mathcal{F}(X), \cup, \cap, c)$.

Proof Let

$$G : \mathcal{F}_0^*(X) \to \mathcal{F}(X),$$

$$\hat{A} \mapsto G(\hat{A}),$$

$$\mu_{G(\hat{A})}(x) = E(A(x)).$$

It is evident that G is an isomorphic mapping from $(\mathcal{F}_0^*(X), \cup, \cap, c)$ to $(\mathcal{F}(X), \cup, \cap, c)$.

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References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
2. Landles, R.H., Wolfe, D.A.: Introduction to the Theory of Nonparamelvic Statistics. Wiley, New York (1979)
3. Hirota, K.: Concepts of probabilistic sets. Fuzzy Set Syst. **5**, 31–46 (1981)
4. Luo, C.Z.: Fuzzy sets and set-embeddings. Fuzzy Math. **4**, (Chinese) (1983)
5. Wang, P.Z., Liu, X.H.: Set valued statistics. J. Eng. Math. (first issue), (Chinese)
6. Wang, P.Z.: Fuzzy Sets and Random Falling Shadows. Beijing Normal University, Beijing (Chinese) (1985)
7. Suzuki, H.: Fuzzy sets and membership functions. Fuzzy Sets Syst. **58**, 123–132 (1993)
8. Li, Q., Wang, P., Lee, E.S.: r-Fuzzy Set. Comput. Math. Appl. **31**(2), 49–61 (1996)
9. Yuan, X.H., Li, H.X., Sun, K.B.: The cut sets, decomposition theorems and representation theorems on Intuitionistic fuzzy Sets and interval-valued fuzzy sets. Sci. China Inf. Sci. **53**, 1–18 (2010). doi:10.1007/s11432-010-4078-6

Part III
Fuzzy Information and Computer

Improved Interval-Valued Intuitionistic Fuzzy Entropy and Its Applications in Multi-attribute Decision Making Problems

Xiang-jun Xie and Xiao-xia Lv

Abstract According to the defect of not fully reflecting the uncertainty, the improved definition and formula, which consider both aspects of uncertainty (fuzziness and lack of knowledge), of entropy for interval-valued intuitionistic fuzzy sets are proposed. The proposed entropy is used to solve multi-attribute decision making problems. Two numerical example verifies the appropriateness and effectiveness of the improved entropy for solving multi-attribute decision making problems.

Keywords Interval-valued intuitionistic fuzzy sets · Interval-valued intuitionistic fuzzy entropy · Uncertainty · Hamming distance · Multi-attribute decision making

1 Introduction

In order to describe the uncertainty and fuzziness better, professor Zadeh [1] proposed the theory of fuzzy sets (FSs) in 1965. In 1986, Antanassov [2] proposed the concept of intuitionistic fuzzy sets (IFSs). An IFS, which is a generalization of fuzzy sets, takes into account the neutral state 'neither this nor that' and provides a more precise description of problems' vagueness through two indexes-the degree of non-membership and membership. In practical decision making problems, decision makers often quantify the amount of decision information by interval numbers instead of crisp numbers for just being able to provide the approximate range of the degree of membership and non-membership. In other words, the degree of membership and non-membership are usually expressed by interval numbers. Therefore, based on the IFSs, Atanassov and Gargov [3] further proposed the concept of interval-valued intuitionistic fuzzy sets (IVIFSs). Recent years, many scholars have being studying the IFSs and IVIFSs, the entropy is a hotspot of the research field.

X. Xie · X. Lv (✉)
School of Sciences, Southwest Petroleum University, Chengdu 610500, People's Republic of China
e-mail: Lvye_1988@126.com

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The concept of entropy, which is originated in the Thermodynamics, was introduced into information theory by Shannon to measure the uncertainty of information. In 1965, the entropy was first used to measure the fuzziness of a fuzzy set by Zadeh [1]. Later, Burillo and Bustince [4] defined the entropy for intuitionistic fuzzy sets to measure the degree of hesitation in 1966. In 2001, Szmidt and Kacprzyk gave the definition of a new non-probabilistic intuitionistic fuzzy entropy based on the geometric interpretation for intuitionistic fuzzy sets. About entropy on IVIFSs, Guo [5] and Liu [6] presented the axiomatic definition of interval-valued intuitionistic fuzzy entropy. Wang and Wei [7] extended the formula of entropy for IFSs and proposed a new formula for inteval-valued intuitionistic fuzzy entropy based on the Guo's axiomatic definition. Gao and Wei [8] defined a new formula based on the improved Hamming distance for IVIFSs. However, the definition in paper [5] has some defects. The constraint for the maximum values of entropy consider only one aspect of uncertainty-fuzziness and neglect the other aspect of uncertainty-lack of knowledge.

By analyzing the papers about intuitionistic fuzzy entropy, we point out that the two aspects of uncertainty should both be taken into account to measure the knowledge of an IFS adequately. As an extension of IFSs, IVIFSs also include both fuzziness and lack of knowledge. So this paper improved the axiomatic definition of entropy for IVIFSs and proposed a new formula, which can reflect the amount of information better. Two applications are illustrated to verify the rationality of the proposed views in the end.

2 Preliminaries

Definition 1 [2] Let X be a finite and non-empty universe of discourse. An intuitionistic fuzzy set A is given by:

$$A = \{\langle x, u_A(x), v_A(x)\rangle \mid x \in X\},$$

where $u_A(x) \in [0, 1]$ denotes the degree of membership of x to A, $v_A(x) \in [0, 1]$ denotes the degree of non-membership of x to A. For every x to A, it satisfies the following condition:

$$0 \le u_A(x) + v_A(x) \le 1.$$

For a given $x \in X$, $\pi_A(x) = 1 - u_A(x) - v_A(x)$ is called the intuitionistic fuzzy index or the hesitation margin.

Definition 2 [3] Let X be a finite and non-empty universe of discourse. An interval-valued intuitionistic fuzzy set A is given by:

$$\begin{aligned} \widetilde{A} &= \left\{ \langle x, u_A(x), \widetilde{v}_A(x) \rangle | x \in X \right\} \\ &= \left\{ \langle x, \left[u_A^-(x), u_A^+(x) \right], \left[v_A^-(x), v_A^+(x) \right] \rangle | x \in X \right\} \end{aligned},$$

where $u_A^-(x) \in [0,1], u_A^+(x) \in [0,1], v_A^-(x) \in [0,1], v_A^+(x) \in [0,1]$. $\left[u_A^-(x), u_A^+(x) \right]$ and $\left[v_A^-(x), v_A^+(x) \right]$ denote the degree of membership and non-membership of x to A, with the condition:

$$u_A^+(x) + v_A^+(x) \leq 1.$$

For a given $x \in X, \widetilde{\pi}_A(x) = \left[1 - u_A^+(x) - v_A^+(x), 1 - u_A^-(x) - v_A^-(x) \right]$ is called the interval-valued intuitionistic fuzzy index or the hesitation margin.

Definition 3 [3] Let $\widetilde{A}, \widetilde{B} \in IVIFS(X)$, $\widetilde{A} = \left\{ \langle x, \left[u_A^-(x), u_A^+(x) \right], \left[v_A^-(x), v_A^+(x) \right] \rangle | x \in X \right\}$, $\widetilde{B} = \left\{ \langle x, \left[u_B^-(x), u_B^+(x) \right], \left[v_B^-(x), v_B^+(x) \right] \rangle | x \in X \right\}$. The following basic operations can be defined:

(1) $\widetilde{A} \subseteq \widetilde{B}$ if and only if $\begin{cases} u_A^-(x) \leq u_B^-(x), u_A^+(x) \leq u_B^+(x) \\ v_A^-(x) \leq v_B^-(x), v_A^+(x) \leq v_B^+(x) \end{cases}$;

(2) $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A} \subseteq \widetilde{B}, \widetilde{A} \supseteq \widetilde{B}$;

(3) $\widetilde{A}^C = \left\{ \langle x, \left[v_A^-(x), v_A^+(x) \right], \left[u_A^-(x), u_A^+(x) \right] \rangle | x \in X \right\}$.

Definition 4 [8] For two IVIFSs $\widetilde{A} = \left\{ \langle x_i, \left[u_A^-(x_i), u_A^+(x_i) \right], \left[v_A^-(x_i), v_A^+(x_i) \right] \rangle | x_i \in X \right\}$ and $\widetilde{B} = \left\{ \langle x_i, \left[u_B^-(x_i), u_B^+(x_i) \right], \left[v_B^-(x_i), v_B^+(x_i) \right] \rangle | x_i \in X \right\}$, the Hamming distance measure and the weighted Hamming distance measure between $\widetilde{A}$ and $\widetilde{B}$ are defined as follows:

$$\begin{aligned} d\left(\widetilde{A}, \widetilde{B}\right) = \tfrac{1}{4n} \textstyle\sum_{i=1}^{n} \Big[&\left| u_A^-(x_i) - u_B^-(x_i) \right| + \left| u_A^+(x_i) - u_B^+(x_i) \right| \\ &+ \left| v_A^-(x_i) - v_B^-(x_i) \right| \left| v_A^+(x_i) - v_B^+(x_i) \right| \\ &+ \left| \pi_A^-(x_i) - \pi_B^-(x_i) \right| + \left| \pi_A^+(x_i) - \pi_B^+(x_i) \right| \Big], \end{aligned} \tag{1}$$

$$\begin{aligned} d\left(\widetilde{A}, \widetilde{B}\right)_w = \tfrac{1}{4n} \textstyle\sum_{i=1}^{n} w_i \Big[&\left| u_A^-(x_i) - u_B^-(x_i) \right| + \left| u_A^+(x_i) - u_B^+(x_i) \right| \\ &+ \left| v_A^-(x_i) - v_B^-(x_i) \right| \left| v_A^+(x_i) - v_B^+(x_i) \right| \\ &+ \left| \pi_A^-(x_i) - \pi_B^-(x_i) \right| + \left| \pi_A^+(x_i) - \pi_B^+(x_i) \right| \Big]. \end{aligned} \tag{2}$$

3 Improved Axiomatic Definition and Formula of the Entropy for Interval-Valued Intuitionistic Fuzzy Sets

3.1 The Necessity of Improving the Axiomatic Definition and Formula of Interval-Valued Intuitionistic Fuzzy Entropy

Definition 5 [5, 6] $\forall \widetilde{A} \in IVIFS(X)$, the mapping $E : IVIFS(X) \to [0, 1]$ is called as entropy if E satisfies the following conditions:

Condition 1: $E\left(\widetilde{A}\right) = 0$ if and only if $\widetilde{A}$ is a crisp set;

Condition 2: $E\left(\widetilde{A}\right) = 1$ if and only if $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right]$ for every $x_i \in X$;

Condition 3: $E\left(\widetilde{A}\right) = E\left(\widetilde{A}^C\right)$ for every $\widetilde{A} \in IVIFS(X)$;

Condition 4: For any $\widetilde{B} \in IVIFS(X)$, if $\widetilde{A} \subseteq \widetilde{B}$ when $u_B^-(x_i) \leq v_B^-(x_i), u_B^+(x_i) \leq v_B^+(x_i)$ for every $x_i \in X$, or $\widetilde{A} \supseteq \widetilde{B}$ when $u_B^-(x_i) \geq v_B^-(x_i), u_B^+(x_i) \geq v_B^+(x_i)$ for every $x_i \in X$, then $E\left(\widetilde{A}\right) \leq E\left(\widetilde{B}\right)$.

Based on the Definition 5, paper [7] and paper [8] gave the concrete formulas of entropy respectively:

$$E_1\left(\widetilde{A}\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{\min\left\{u_A^-(x_i), v_A^-(x_i)\right\} + \min\left\{u_A^+(x_i), v_A^+(x_i)\right\} + \pi_A^-(x_i) + \pi_A^+(x_i)}{\max\left\{u_A^-(x_i), v_A^-(x_i)\right\} + \max\left\{u_A^+(x_i), v_A^+(x_i)\right\} + \pi_A^-(x_i) + \pi_A^+(x_i)}$$
$$= \frac{1}{n}\sum_{i=1}^{n} \frac{2 - \left|u_A^-(x_i) - v_A^-(x_i)\right| - \left|u_A^+(x_i) - v_A^+(x_i)\right| + \pi_A^-(x_i) + \pi_A^+(x_i)}{2 + \left|u_A^-(x_i) - v_A^-(x_i)\right| + \left|u_A^+(x_i) - v_A^+(x_i)\right| + \pi_A^-(x_i) + \pi_A^+(x_i)},$$

$$E_2\left(\widetilde{A}\right) = \frac{\min\left\{d\left(\widetilde{A}, \widetilde{P}\right), d\left(\widetilde{A}, \widetilde{Q}\right)\right\}}{\max\left\{d\left(\widetilde{A}, \widetilde{P}\right), d\left(\widetilde{A}, \widetilde{Q}\right)\right\}},$$

where $\widetilde{P} = \{\langle x_i, [1, 1], [0, 0]\rangle \,|x_i \in X\}$ and $\widetilde{Q} = \{\langle x_i, [0, 0], [1, 1]\rangle \,|x_i \in X\}$, $d\left(\widetilde{A}, \widetilde{P}\right)$ and $d\left(\widetilde{A}, \widetilde{Q}\right)$ are calculated by formula (1).

Moreover, different definitions, which are equal to Definition 5 in essence, are proposed by Zhang [9], Szmidt [10] and Ye [11]. They also define the formulas of entropy for IVIFSs, the entropy formula in [11] is an extension of the intuitionistic fuzzy entropy formula based on the trigonometric function proposed in [12].

Fuzzy entropy measures the fuzziness of a fuzzy set, but researchers have differences in the definitions of the intuitionistic fuzzy entropy. Burrillo and Bustince [4] defined the intuitionistic fuzzy entropy as a measure of the degree of the hesitation, namely the entropy measures how far the intuitionistic fuzzy sets from the fuzzy sets. We called it B-B axiom for short. The constraint of the maximum value is: $u_A(x_i) = v_A(x_i) = 0$ (i.e. $\pi_A(x_i) = 0$); Szmidt and Kacprzykv [9] defined the entropy on intuitionistic fuzzy sets to measure how far the intuitionistic fuzzy sets from the crisp sets and it (S-K axiom for short) was a measure of the fuzziness. The intuitionistic fuzzy entropy is maximum if and only if $u_A(x_i) = v_A(x_i)$. Pal and Bustince [13] point out that the uncertainty of a intuitionistic fuzzy set includes fuzziness and lack of knowledge. In order to quantify the uncertainty of an IFS better, they propose the concept of two-tuple entropy, which is a pair (E_I, E_F) consisted of E_I based on B-B axiom and E_F based on S-K axiom; Szmidt and Kacprzyk [14] point out that: the two situations, one with the maximal entropy when $u_A(x_i) = v_A(x_i)$ and another when $u_A(x_i) = v_A(x_i) = 0$ are equivalent from the point of view of the entropy measure. But the two situations are completely different from the perspective of decision making. The degree of lack of information should also be considered when we deal with decision making problems. Then, they propose a new index (i.e., the measure of lack of information: $K(x) = 1 - 0.5(E(x) + \pi(x))$) with not changing the axiom in paper [9], where the $E(x)$ is the ratio-based entropy in [9]. The greater of the $K(x)$, the more amount of information the IFS presents. It's difficult to cope with the practical decision making problems using the two-tuple entropy. Furthermore, the constraint of the minimal entropy in B-B axiom neglect the inherent fuzziness of an IFS. Relatively speaking, the measure $K(x)$ is a more better choice.

Example 1 Let $A_1 = \langle 0.3, 0.3\rangle$ and $A_2 = \langle 0.1, 0.1\rangle$. We can calculate that: $K_1(x) = 0.3$, $K_2(x) = 0.1$, namely the amount of information of A_1 is more than A_2. It's obvious that we can not distinguish A_1 and A_2 if considering only fuzziness.

In fact, there is a third view about the intuitionistic fuzzy entropy which considers both fuzziness and lack of information. Lv [15] defined a measure of fuzziness (i.e., $f_A(x) = 1 - |u_A(x) - v_A(x)|$) and defined the new axiom formula of intuitionistic fuzzy entropy, which satisfies the following conditions. Firstly, the entropy gets the minimal value if and only if the IFS A is a crisp set; Secondly, the constraint of the maximal entropy is: $u_A(x_i) = v_A(x_i) = 0$; Thirdly, the entropy increases with the fuzziness and degree of hesitation (i.e., degree of lack of information) increasing (which is called the monotonicity of intuitionistic fuzzy entropy).

The definitions of intuitionistic fuzzy entropy in papers [16–19] also take two aspects-fuzziness and degree of hesitation-into account. Although the constraints in the four papers have some small differences, the definitions are essentially the same. The four formulas are respectively as follows:

$$E_1(A) = \frac{1}{n}\sum\nolimits_{i=1}^{n}\left[1 - \sqrt{(1 - \pi_A(x_i))^2 - u_A(x_i)\, v_A(x_i)}\right], \qquad (3)$$

$$E_2(A) = \frac{1}{n}\sum\nolimits_{i=1}^{n}\sqrt{\frac{(1-|u_A(x_i)-v_A(x_i)|)^2+\pi_A^2(x)}{2}}, \quad (4)$$

$$E_3(A) = \frac{1}{n}\sum\nolimits_{i=1}^{n}\frac{1-|u_A(x_i)-v_A(x_i)|+\pi_A(x_i)}{2}, \quad (5)$$

$$E_4(A) = \frac{1}{n}\sum\nolimits_{i=1}^{n}\frac{1-|u_A(x_i)-v_A(x_i)|^2+\pi_A^2(x_i)}{2}. \quad (6)$$

Example 2 Take the above $A_1 = \langle 0.3, 0.3\rangle$ and $A_2 = \langle 0.1, 0.1\rangle$ for example, we calculate that: $E_1(A_1) = 0.4804$, $E_1(A_2) = 0.8268$; $E_2(A_1) = 0.7616$, $E_2(A_2) = 0.9055$; $E_3(A_1) = 0.7$, $E_3(A_2) = 0.9$, $E_4(A_1) = 0.58$, $E_4(A_2) = 0.82$. The greater the entropy, the less amount of information the intuitionistic expresses, so the results calculated through formulas (3)–(6) are the same with the result calculated by$K$, namely the intuitionistic fuzzy entropy defined in papers [15–19] can quantify the amount of information better.

IVIFSs are extensions of IFSs, so it's same that the entropy for IVIFSs differentiate when $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right]$ and $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right] = [0, 0]$ from the point of view of decision making. It just means that the fuzziness of an IVIFS is maximal when $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right]$, while fuzziness and degree of lack of knowledge both come to maximum when $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right] = [0, 0]$. So it will be more reasonable to use $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right] = [0, 0]$ as the constraint of maximal value when we define the entropy for IVIFSs.

3.2 The Improved Axiomatic Definition and Formula for Interval-Valued Intuitionistic Fuzzy Sets

Definition 6 $\forall \widetilde{A} \in IVIFS(X)$, the mapping $E : IVIFS(X) \to [0, 1]$ is called as entropy if E satisfies the following conditions:

Condition 1: $E\left(\widetilde{A}\right) = 0$ if and only if $\widetilde{A}$ is a crisp set;

Condition 2: $E\left(\widetilde{A}\right) = 1$ if and only if $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right] = [0, 0]$ for every $x_i \in X$;

Condition 3: $E\left(\widetilde{A}\right) = E\left(\widetilde{A}^C\right)$ for every $\widetilde{A} \in IVIFS(X)$;

Condition 4: For any $\widetilde{B} \in IVIFS(X)$, if $\widetilde{A} \subseteq \widetilde{B}$ when $u_B^-(x_i) \le v_B^-(x_i)$, $u_B^+(x_i) \le v_B^+(x_i)$ for every $x_i \in X$, or $\widetilde{A} \supseteq \widetilde{B}$ when $u_B^-(x_i) \ge v_B^-(x_i)$, $u_B^+(x_i) \ge v_B^+(x_i)$ for every $x_i \in X$, then $E\left(\widetilde{A}\right) \le E\left(\widetilde{B}\right)$.

Theorem 1 *Let* $X = \{x_1, x_2, \ldots, x_n\}$ *be a universe.* $\widetilde{A} = \left\{\langle x_i, \left[u_A^-(x_i), u_A^+(x_i)\right], \left[v_A^-(x_i), v_A^+(x_i)\right]\rangle \,|x_i \in X\right\}$*, the formula of the entropy is as follows:*

$$E\left(\widetilde{A}\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{2 - \left|u_A^+(x_i) - v_A^+(x_i)\right|^2 - \left|u_A^-(x_i) - v_A^-(x_i)\right|^2 + \left(\pi_A^-(x_i)\right)^2 + \left(\pi_A^+(x_i)\right)^2}{4} \tag{7}$$

Proof **Condition 1**:

$\widetilde{A}$ is a crisp set, namely

$$\left[u_A^-(x_i), u_A^+(x_i)\right] = [0, 0], \quad \left[v_A^-(x_i), v_A^+(x_i)\right] = [1, 1]$$

or

$$\left[u_A^-(x_i), u_A^+(x_i)\right] = [1, 1], \left[v_A^-(x_i), v_A^+(x_i)\right] = [0, 0].$$

Then $E\left(\widetilde{A}\right) = 0$.

If $E\left(\widetilde{A}\right) = 0$, since

$$2 + \left(\pi_A^-(x_i)\right)^2 + \left(\pi_A^+(x_i)\right)^2 \geq 2, \left|u_A^+(x_i) - v_A^+(x_i)\right|^2 + \left|u_A^-(x_i) - v_A^-(x_i)\right|^2 \leq 2,$$

So

$$\left[\pi_A^-(x_i), \pi_A^+(x_i)\right] = [0, 0] \text{ and } \left|u_A^+(x_i) - v_A^+(x_i)\right| = \left|u_A^-(x_i) - v_A^-(x_i)\right| = 1,$$

namely $\widetilde{A}$ is a crisp set.

Condition 2: If $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right] = [0, 0]$, it's obvious that $E\left(\widetilde{A}\right) = 1$. If $E\left(\widetilde{A}\right) = 1$, since

$$2 + \left(\pi_A^-(x_i)\right)^2 + \left(\pi_A^+(x_i)\right)^2 \leq 4, \left|u_A^+(x_i) - v_A^+(x_i)\right|^2 + \left|u_A^-(x_i) - v_A^-(x_i)\right|^2 \geq 0,$$

So $\left[\pi_A^-(x_i), \pi_A^+(x_i)\right] = [1, 1]$ and $\left[v_A^-(x_i), v_A^+(x_i)\right] = \left[u_A^-(x_i), u_A^+(x_i)\right] = [0, 0]$.

Condition 3: For the two IVIFSs $\widetilde{A}$ and $\widetilde{A}^C$, $\left[\pi_A^-(x_i), \pi_A^+(x_i)\right] = \left[\pi_{A^C}^-(x_i), \pi_{A^C}^+(x_i)\right]$, so it's obvious that the condition 3 is right.

Condition 4:

$$\begin{aligned} E\left(\widetilde{A}\right) &= \frac{1}{n}\sum_{i=1}^{n} \frac{2 - \left|u_A^+(x_i) - v_A^+(x_i)\right|^2 - \left|u_A^-(x_i) - v_A^-(x_i)\right|^2 + \left(\pi_A^-(x_i)\right)^2 + \left(\pi_A^+(x_i)\right)^2}{4} \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{2 + u_A^+(x_i)\left(v_A^+(x_i) - 1\right) + v_A^+(x_i)\left(u_A^+(x_i) - 1\right) + u_A^-(x_i)\left(v_A^-(x_i) - 1\right) + v_A^-(x_i)\left(u_A^-(x_i) - 1\right)}{2} \end{aligned} \tag{8}$$

and

if $u_B^-(x_i) \le v_B^-(x_i)$, $u_B^+(x_i) \le v_B^+(x_i)$ and $\widetilde{A} \subseteq \widetilde{B}$ for every $x_i \in A$,

then $v_A^-(x_i) \ge v_B^-(x_i) \ge u_B^-(x_i) \ge u_A^-(x_i)$ and $v_A^+(x_i) \ge v_B^+(x_i) \ge u_B^+(x_i) \ge u_A^+(x_i)$,

so

$$u_A^+(x_i)\left(v_A^+(x_i)-1\right) \le u_B^+(x_i)\left(v_B^+(x_i)-1\right), v_A^+(x_i)\left(u_A^+(x_i)-1\right) \le v_B^+(x_i)\left(u_B^+(x_i)-1\right),$$

$$u_A^-(x_i)\left(v_A^-(x_i)-1\right) \le u_B^-(x_i)\left(v_B^-(x_i)-1\right), v_A^-(x_i)\left(u_A^-(x_i)-1\right) \le v_B^-(x_i)\left(u_B^-(x_i)-1\right),$$

then

$$E\left(\widetilde{A}\right) \le E\left(\widetilde{B}\right).$$

As the above method, when $u_B^-(x_i) \ge v_B^-(x_i)$, $u_B^+(x_i) \ge v_B^+(x_i)$ and $\widetilde{A} \supseteq \widetilde{B}$ for every $x_i \in A$, we can conclude that:

$$E\left(\widetilde{A}\right) \le E\left(\widetilde{B}\right).$$

So, the condition 4 is correct.

4 Applications in Multi-attribute Decision Making Problems

Consider a multi-attribute decision making problem with the attribute set $C = \{c_1, c_2, \ldots, c_n\}$ and the alternative set $A = \{a_1, a_2, \ldots, a_m\}$. Let $W = \{w_1, w_2, \ldots, w_n\}$ be the weight set, which is unknown, where $\sum_{j=1}^{n} w_j = 1$, $w_j \in [0, 1]$. $\underset{ij}{\widetilde{d}} = \left\{\langle x_{ij}, \left[u_A^-\left(x_{ij}\right), u_A^+\left(x_{ij}\right)\right], \left[v_A^-\left(x_{ij}\right), v_A^+\left(x_{ij}\right)\right]\rangle\right\}$ means the evaluation of the ith alternative a_i satisfies the jth attribute. Then the steps in dealing with the problem is as follows:

Step 1: construct the decision matrix;
Step 2: calculate the entropy of every attribute $E_j = \sum_{i=1}^{m} e_{ij}$ by using (7), where $j = 1, 2, \ldots, n$;
Step 3: calculate the weight value of each attribute by using the model as follows [16]:

$$w_j = \frac{E_j^{-1}}{\sum_{j=1}^{n} E_j^{-1}}. \tag{9}$$

Step 4: let $A^* = \left\{\langle c_j, [1, 1], [0, 0]\rangle \,\middle|\, c_j \in C\right\}$ be the positive ideal point. Calculate the weighted Hamming distance d_i between every alternative and the positive ideal point;

Table 1 Decision matrix 1

Alternative\attribute	c_1	c_2	c_3
a_1	([0.4, 0.5] , [0.3, 0.4])	([0.4, 0.6] , [0.2, 0.4])	([0.1, 0.3] , [0.5, 0.6])
a_2	([0.6, 0.7] , [0.2, 0.3])	([0.6, 0.7] , [0.2, 0.3])	([0.4, 0.7] , [0.1, 0.2])
a_3	([0.3, 0.4] , [0.3, 0.6])	([0.5, 0.6] , [0.3, 0.4])	([0.5, 0.6] , [0.1, 0.3])
a_4	([0.7, 0.8] , [0.1, 0.2])	([0.6, 0.7] , [0.1, 0.3])	([0.3, 0.4] , [0.1, 0.2])

Step 5: sort the $d_i, i = 1, 2, \ldots, m$, get the best alternative. The smaller the d_i, which means the closer the alternative a_i is to the positive ideal point, the better the alternative.

Example 3 Consider a company is to invest in one of the following four projects: (1) a_1 be a automaker; (2) a_2 be a food company; (3) a_3 be a computer company; (4) a_4 be a firearms manufacturer. Three factors should be considered: (i) c_1 risk analysis; (ii) c_2 the development; (iii) c_3 the production environment pressure analysis [8].

Step 1: the decision matrix is as follows [8] (Table 1):
Step 2: calculate entropy of each attribute by using (7): $E_1 = 0.4525$, $E_2 = 0.4650$, $E_3 = 0.5025$;
Step 3: calculate the weight value of each attribute by using (9): $w_1 = 0.3480$, $w_2 = 0.3386$, $w_3 = 0.3134$;
Step 4: from (2), we get the weighted Hamming distances: $d_1 = 0.6114$. $d_2 = 0.3813$. $d_3 = 0.5196$. $d_4 = 0.4092$;
Step 5: the order of the distances is: $d_2 < d_4 < d_3 < d_1$, so $a_2 > a_4 > a_3 > a_1$, the second alternative is the best alternative.

The above result is equal to the one in [8]. Taking the degree of lack of knowledge into account while calculating the weight of the attribute leads to this situation, which further verifies the impact of the degree of lack of knowledge should not be neglected in decision making problems.

Example 4 Consider a manufacturer selection problem. The supplier is to choose one from three manufacturers (i.e., Alternatives a_j ($j = 1, 2, 3$)). Five evaluating indexes (i.e., attributes) should be considered: quality of product (c_1), cost of product (c_2), time of delivery (c_3), transportation cost (c_4), service attitude (c_5) [20].

Step 1: decision matrix is as follows [20] (Table 2):
Step 2: from (7), the entropy of each attribute: $E_1 = 0.4833$, $E_2 = 0.5133$, $E_3 = 0.5400$, $E_4 = 0.5333$, $E_5 = 0.4800$;
Step 3: from (9), we can get each weight of attributes: $w_1 = 0.2105$, $w_2 = 0.1982$, $w_3 = 0.1884$, $w_4 = 0.1908$, $w_5 = 0.2120$;
Step 4: from (2), the weighted Hamming distance between each attribute and the positive ideal point: $d_1 = 0.6077$, $d_2 = 0.6600$,$d_3 = 0.5278$;
Step 5: the order of the distances is: $d_2 > d_1 > d_3$, so the order of the alternatives is: $a_3 > a_1 > a_2$. The second alternative is the best, which is the same with the result in [20] when $q = 1$, $q = 2$, $q \to 0$ and $q \to \infty$

Table 2 Decision matrix 2

Attribute\alternative	a_1	a_2	a_3
c_1	$([0.4, 0.5], [0.2, 0.3])$	$([0.3, 0.4], [0.4, 0.5])$	$([0.6, 0.7], [0.1, 0.2])$
c_2	$([0.2, 0.3], [0.5, 0.6])$	$([0.4, 0.5], [0.3, 0.4])$	$([0.3, 0.4], [0.3, 0.5])$
c_3	$([0.4, 0.5], [0.1, 0.3])$	$([0.2, 0.4], [0.3, 0.5])$	$([0.4, 0.6], [0.1, 0.3])$
c_4	$([0.2, 0.4], [0.4, 0.5])$	$([0.4, 0.5], [0.2, 0.3])$	$([0.3, 0.5], [0.3, 0.4])$
c_5	$([0.4, 0.6], [0.1, 0.2])$	$([0.1, 0.2], [0.6, 0.7])$	$([0.4, 0.5], [0.2, 0.3])$

5 Conclusion

By analyzing the research about the intuitionistic fuzzy entropy, we explain that the uncertainty of an IFS should include both fuzziness and the degree of lack of information. The interval-valued intuitionistic fuzzy sets are extension of the intuitionistic fuzzy sets, so we should neglect neither fuzziness nor lack of knowledge when we define the measure of entropy. This paper improved the existing axiomatic definitions and formulas for IVIFSs and the improved entropy is used to deal with two decision making problems. The examples further demonstrate the correctness of the new entropy and its effectiveness in tackling practical problems.

References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–356 (1965)
2. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**(1), 87–96 (1986)
3. Atanassov, K., Gargov, G.: Interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst. **31**(3), 343–349 (1989)
4. Burillo, P., Bustince, H.: Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. Fuzzy Sets Syst. **78**(3), 305–316 (1996)
5. Guo, X.: The discussion an expansion on measurement of fuzzy uncertainty. Xi an: Northwest University, pp. 43–45 (2004)
6. Liu, X., Zheng, S., Xiong, F.: Entropy and Subsethood for General Interval-valued Intuitionistic Fuzzy sets//LNAI 3613:FSKD. Springer, Berlin (2005)
7. Wang, P., Wei, C.: Constructing method of interval-valued intuitionistic fuzzy entropy. Comput. Eng. Appl. **47**(2), 43–45 (2011)
8. Gao, Z., Wei, C.: Formula of interval-valued intuitionistic fuzzy entropy and its application. Comput. Eng. Appl. **48**(2), 53–55 (2012)
9. Szmidt, E., Kacpryzk, J.: Entropy for intuitionistic fuzzy sets. Fuzzy Sets Syst. **118**(3), 467–477 (2001)
10. Zhang, Q., Jiang, S., Jia, B., et al.: Some information measures for interval-valued intuitionistic fuzzy sets. Inf. Sci. **180**(24), 5130–5145 (2010)
11. Ye, J.: Multiattribute fuzzy decision-making method using entropy weights-based correlation coefficients for interval-valued intuitionistic fuzzy sets. Appl. Math. Model. **34**(12), 3864–3870 (2010)
12. Ye, J.: Two effective measures of intuitionistic fuzzy entropy. Computing **87**(1/2), 55–62 (2010)
13. Pal, N.R., Bustince, H., Pagola, M., et al.: Uncertainties with Atanassov's intuitionistic fuzzy sets: fuzziness and lack of knowledge. Inf. Sci. **228**, 61–74 (2013)

14. Szmidt, E., Kacpryzk, J., Bujnowski, P.: How to measure the amount of knowledge conveyed by Atanassov's intuitionistic fuzzy sets. Inf. Sci. **257**, 276–285 (2014)
15. Lv, Y., Guo, S.: Entropy of intuitionistic fuzzy sets and its general forms. Comput. Eng. Appl. **47**(28), 52–55 (2011)
16. Wang, C., Yao, D., Mao, J., et al.: Intuitionistic fuzzy multiple attributes decision making method based on entropy and correlation coefficient. J. Comput. Appl. **32**(11), 3002–3004, 3017 (2012)
17. Shen, X., Guo, S.: New entropy formula for intuitionistic fuzzy sets and its applications. Comput. Eng. Appl. **49**(24), 28–31 (2013)
18. Wu, T., Bai, L., Liu, R., et al.: New entropy formula of intuitionistic fuzzy sets and its application. Comput. Eng. Appl. **49**(23), 48–51 (2013)
19. Gao, M., Sun, T., Zhu, J.: Reised axiomatic definition and structured formula of intuitionistic fuzzy entropy. Control Decis. **29**(3), 470–474 (2014)
20. Li, D.: Intuitionistic Fuzzy Set Decision and Game Analysis Methodologies, pp. 116–123. National Defence Industry Press, Beijing (2012)

$(\in, \in \vee q)$-Fuzzy Filter Theory in FI-algebras

Chun-hui Liu

Abstract The notions of $(\in, \in \vee q)$-fuzzy filters and $(\in, \in \vee q)$-fuzzy implicative (positive implicative, commutative, involution) filters in FI-algebras are introduced and their properties are investigated. Equivalent characterizations of various $(\in, \in \vee q)$-fuzzy filters are obtained. Relations among various $(\in, \in \vee q)$-fuzzy filters are discussed and it is proved that a fuzzy set is $(\in, \in \vee q)$-fuzzy positive implicative filter iff it is both $(\in, \in \vee q)$-fuzzy implicative and $(\in, \in \vee q)$-fuzzy commutative (or involution) filter.

Keywords Fuzzy logic · FI-algebra · $(\in, \in \vee q)$-fuzzy filter · $(\in, \in \vee q)$-fuzzy implicative (positive implicative, commutative, involution) filter

1 Introduction

Non-classical logic, extension and development of classical logic, has become a formal tool for computer science and artificial intelligence to deal with uncertainty information. To satisfy needs of fuzzy reasoning, many kinds of fuzzy logic algebras have been established. Among them, fuzzy implication algebras (in short, FI-algebras) posed in Ref. [1] are more extensive. Precisely, MV-algebras, bounded BCK-algebras, BL-algebras, lattice implication algebras and R_0-algebras are all FI-algebras and implications in FI-algebras generalize implication operators in various logic algebras. Thus, it is meaningful to deeply study properties of FI-algebras. Fuzzy sets introduced firstly by Zadeh in Ref. [2]. At present, some types of generalized fuzzy filters were introduced in various logical algebras by using the ideas of fuzzy sets [3–5].

In this paper, we apply the ideas and methods of fuzzy sets to FI-algebras. We introduce various concepts of $(\in, \in \vee q)$-fuzzy filters and discuss their properties. Some significative results are obtained.

C. Liu (✉)
College of Mathematics and Statistics, Chifeng University, 024000 Chifeng, China
e-mail: chunhuiliu1982@163.com

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2 Preliminaries

By an fuzzy implication algebra (in short, FI-algebra), we mean an algebra $(X, \to, 0)$ such that the following conditions hold: for all $x, y, z \in X$,

(I_1) $x \to (y \to z) = y \to (x \to z)$;
(I_2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;
(I_3) $x \to x = 1$;
(I_4) $x \to y = y \to x = 1 \Rightarrow x = y$;
(I_5) $0 \to x = 1$.

In any FI-algebra X, The following statements are true, for all $x, y, z \in X$,

(I_6) $1 \to x = x, x \to 1 = 1$;
(I_7) $((x \to y) \to y) \to y = x \to y$;
(I_8) $(y \to z) \to ((x \to y) \to (x \to z)) = 1$;
(I_9) $x \leq y \Leftrightarrow x \to y = 1$;
(I_{10}) $x \leq y \Rightarrow (z \to x \leq z \to y$ and $y \to z \leq x \to z)$;
(I_{11}) $0' = 1, 1' = 0, x \leq x'', x''' = x'$;
(I_{12}) $x \leq y \Rightarrow y' \leq x'$,

where $1 = 0 \to 0$ and $x' = x \to 0$.

A fuzzy set on X is a mapping $f : X \to [0, 1]$. A fuzzy set on X with the form

$$f(y) = \begin{cases} t \in (0, 1], & y = x \\ 0, & y \neq x \end{cases} \tag{1}$$

is said to be a fuzzy point with support x and value t witch is denoted by x_t. For a fuzzy point x_t and a fuzzy set f we have

(1) x_t is said to belong to f, written as $x_t \in f$, if and only if $f(x) \geqslant t$;
(2) x_t is said to be quasi-coincident with f, written as $x_t q f$, iff $f(x) + t > 1$;
(3) If $x_t \in f$ or $x_t q f$, then we write $x_t \in\vee\, q f$;
(4) The symbol $\overline{\in\vee\, q}$ means that $\in\vee\, q$ does not hold.

3 $(\in, \in\vee\, q)$-Fuzzy Filters in FI-algebras

Definition 3.1 Let X be an FI-algebra. A fuzzy set f on X is called a $(\in, \in\vee\, q)$-fuzzy filter in X, if it satisfies the following conditions:

(F1) $(\forall t \in (0, 1])(\forall x \in X)(x_t \in f \Rightarrow 1_t \in\vee\, q f)$;
(F2) $(\forall t, r \in (0, 1])(\forall x, y \in X)((x_t \in f$ and $(x \to y)_r \in f) \Rightarrow y_{\min\{t,r\}} \in\vee\, q f)$.

The set of all $(\in, \in\vee\, q)$-fuzzy filters in X is denoted by $\mathbf{FF}(X)$.

Theorem 3.1 *The conditions (F1) and (F2) in Definition 3.1 are equivalent to the following conditions (F3) and (F4), respectively:*

(F3) $(\forall x \in X)(f(1) \geqslant \min\{f(x), 0.5\})$;
(F4) $(\forall x, y \in X)(f(y) \geqslant \min\{f(x), f(x \to y), 0.5\})$.

Proof It is similar to the proof of Theorem 1 and Theorem 2 in Ref. [5].

Remark 3.1 Let X be an FI-algebra and f a fuzzy set on X.

(1) Theorem 3.1 shows that $f \in \mathbf{FF}(X)$ iff f satisfies conditions (F3) and (F4).
(2) Let $f \in \mathbf{FF}(X)$, then it is easy to see by (1) of this remark that f satisfies the following condition:

(F5) $(\forall x, y \in X)(x \leq y \Rightarrow f(y) \geqslant \min\{f(x), 0.5\})$.

Theorem 3.2 *Let X be an FI-algebra and f a fuzzy set on X. Then $f \in \boldsymbol{FF}(X)$ iff f satisfies the following condition:*

(F6) $(\forall x, y, z \in X)(x \to (y \to z) = 1 \Rightarrow f(z) \geqslant \min\{f(x), f(y), 0.5\})$.

Proof Suppose $f \in \mathbf{FF}(X)$. Let $x, y, z \in X$ such that $x \to (y \to z) = 1$. Then $x \leq y \to z$, and so we have $f(y \to z) \geqslant \min\{f(x), 0.5\}$ by (F5). It follows from (F4) that $f(z) \geqslant \min\{f(y), f(y \to z), 0.5\} \geqslant \min\{f(y), f(x), 0.5\}$. Conversely, since for all $x \in X$, $x \to (x \to 1) = 1$, we have $f(1) \geqslant \min\{f(x), f(x), 0.5\} = \min\{f(x), 0.5\}$, i.e., (F3) holds. By ($I_3$) we have $(x \to y) \to (x \to y) = 1$, thus $f(y) \geqslant \min\{f(x), f(x \to y), 0.5\}$, i.e., (F4) holds. So, $f \in \mathbf{FF}(X)$ by Theorem 3.1.

4 Various $(\in, \in\vee q)$-Fuzzy Filters

4.1 $(\in, \in\vee q)$-Fuzzy Implicative Filters

Definition 4.1 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy implicative filter, if it satisfies the following condition:

(F7) $(\forall x, y, z \in X)(f(x \to z) \geqslant \min\{f(x \to (y \to z)), f(x \to y), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy implicative filters in X is denoted by $\mathbf{FIF}(X)$.

Example 4.1 Let $X = \{0, a, b, 1\}$. The operator $\to_1$ on X is defined in Table 1.
Then $(X, \to_1, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.3$, $f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FIF}(X)$.

Table 1 Definition of $\to_1$ on X

$\to_1$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Theorem 4.1 *Let X be an FI-algebra, $f \in \boldsymbol{FF}(X)$. Consider the conditions:*

(F8) $(\forall x, y, z \in X)(f(y \to z) \geqslant \min\{f(x), f(x \to (y \to (y \to z))), 0.5\})$;
(F9) $(\forall x, y \in X)(f(x \to y) \geqslant \min\{f(x \to (x \to y)), 0.5\})$;
(F10) $(\forall x, y, z \in X)(f((x \to y) \to (x \to z)) \geqslant \min\{f(x \to (y \to z)), 0.5\})$.

Then $f \in \boldsymbol{FIF}(X) \Leftrightarrow$ *(F8)*$\Leftrightarrow$ *(F9)*$\Leftrightarrow$ *(F10).*

Proof $f \in \mathbf{FIF}(X) \Rightarrow$(F8): Let $f \in \mathbf{FIF}(X)$. Take $x = y$ in (F7) first, then by (F3) and (F4) we have that $f(y \to z) \geqslant \min\{f(y \to (y \to z)), f(y \to y), 0.5\} = \min\{f(y \to (y \to z)), f(1), 0.5\} = \min\{f(y \to (y \to z)), 0.5\} \geqslant \min\{f(x), f(x \to (y \to (y \to z))), 0.5\}$. That is, (F8) holds.

(F8) $\Rightarrow$ (F9): Assume $f \in \mathbf{FF}(X)$ and f satisfies (F8). Then for all $x, y \in X$ we can obtain by (F3) that $f(x \to y) \geqslant \min\{f(1), f(1 \to (x \to (x \to y))), 0.5\} = \min\{f(x \to (x \to y)), 0.5\}$. So, f satisfies (F9).

(F9) $\Rightarrow$ (F10): Assume (F9) holds. Since for all $x, y, z \in X$ by (I_1) and (I_2) we have $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)) = x \to ((x \to y) \to (x \to z))$. By ($I_1$), (F9) and (F5) we further have that $f((x \to y) \to (x \to z)) = f(x \to ((x \to y) \to z)) \geqslant \min\{f(x \to (x \to ((x \to y) \to z))), 0.5\} = \min\{f(x \to ((x \to y) \to (x \to z))), 0.5\} \geqslant \min\{\min\{f(x \to (y \to z)), 0.5\}, 0.5\} = \min\{f(x \to (y \to z)), 0.5\}$. That is (F10) holds.

(F10)$\Rightarrow$ $f \in \mathbf{FIF}(X)$: Assume $f((x \to y) \to (x \to z)) \geqslant \min\{f(x \to (y \to z)), 0.5\}$ for all $x, y \in X$. By $((x \to y) \to (x \to z)) \to ((x \to y) \to (x \to z)) = 1$ and Theorem 3.2 we have that $f(x \to z) \geqslant \min\{f((x \to y) \to (x \to z)), f(x \to y), 0.5\}$. So, by (F10) we have that $f(x \to z) \geqslant \min\{f(x \to (y \to z)), f(x \to y), 0.5\}$. That is (F7) holds, thus, $f \in \mathbf{FIF}(X)$.

4.2 (∈, ∈∨ *q*)-Fuzzy Positive Implicative Filters

Definition 4.2 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy positive implicative filter, if it satisfies the following condition:

(F11) $(\forall x, y, z \in X)(f(y) \geqslant \min\{f(x), f(x \to ((y \to z) \to y)), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy positive implicative filters in X is denoted by $\mathbf{FPIF}(X)$.

Example 4.2 Let $X = \{0, a, b, 1\}$. The operator $\to_2$ on X is defined in Table 2.

Then $(X, \to_2, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = 0.3$, $f(a) = f(b) = f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FPIF}(X)$.

Table 2 Definition of $\to_2$ on X

$\to_2$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Theorem 4.2 *Let X be an FI-algebra, $f \in$ **FF**(X). Consider the conditions:*

(F12) $(\forall x, y \in X)(f(x) \geqslant \min\{f((x \to y) \to x), 0.5\})$;
(F13) $(\forall x \in X)(f(x) \geqslant \min\{f(x' \to x), 0.5\})$;
(F14) $(\forall x, y, z \in X)(f((x' \to x) \to x) \geqslant \min\{f(1), 0.5\})$.

*Then $f \in$ **FPIF**(X) ⇔(F12)⇔(F13)⇔ (F14).*

Proof $f \in$ **FPIF**(X) ⇒(F12): Assume that $f \in$ **FPIF**(X). Then by using (F11) and (F3) we have that $f(x) \geqslant \min\{f(1), f(1 \to ((x \to y) \to x)), 0.5\} = \min\{f(1), f((x \to y) \to x), 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. So, (F12) holds.

(F12)⇒ $f \in$ **FPIF**(X): Assume $f \in$ **FF**(X) and (F12) holds. Then for all $x, y, z \in X$, by (F4) we have that $f(y) \geqslant \min\{f((y \to z) \to y), 0.5\} \geqslant \min\{f(x), f(x \to ((y \to z) \to y)), 0.5\}$. That is (F11) holds and so $f \in$ **FPIF**(X).

(F12)⇒(F13): Putting $y = 0$ in (F12), one gets (F13).

(F13)⇒(F12): By (I_{10}) we have $x' = x \to 0 \leq x \to y$. By (I_{10}) again we get $(x \to y) \to x \leq x' \to x$. By (F13) and (F5) we further have $f(x) \geqslant \min\{f(\neg x \to x), 0.5\} \geqslant \min\{f((x \to y) \to x), 0.5\}$, i.e., (F12) holds.

(F14)⇒(F12): Assume $f \in$ **FF**(X) and (F14) holds. Since by (I_{10}) we have $x \to 0 \leq x \to y$ and $(x \to y) \to x \leq (x \to 0) \to x = x' \to x$. By (F5) we get $f(x' \to x) \geqslant \min\{f((x \to y) \to x), 0.5\}$. Thus by (F4) and (F14), we have $f(x) \geqslant \min\{f(\neg x \to x), f((x' \to x) \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, \min\{f(1), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), f(1), 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. So, (F12) holds.

(F13)⇒(F14): Assume (F13) holds. Then by (I_1), (I_2) and (F5) we obtain that $f((x' \to x) \to x) = f(\alpha) \geqslant \min\{f((\alpha \to 0) \to \alpha), 0.5\} = \min\{f((((x' \to x) \to x) \to 0) \to ((x' \to x) \to x)), 0.5\} = \min\{f((x' \to x) \to ((((x' \to x) \to x) \to 0) \to x)), 0.5\} \geqslant \min\{f((((x' \to x) \to x) \to 0) \to x'), 0.5\} = \min\{f((((x' \to x) \to x) \to 0) \to (x \to 0)), 0.5\} \geqslant \min\{f(x \to ((x' \to x) \to x)), 0.5\} = \min\{f((x' \to x) \to (x \to x)), 0.5\} = \min\{f(1), 0.5\}$. So (F14) holds and the proof is thus finished.

Theorem 4.3 *Let X be an FI-algebra. Then **FPIF**$(X) \subseteq$ **FIF**(X).*

Proof Assume $f \in$ **FPIF**(X). Since $z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to (z \to (z \to x))$ by (I_8), we have by (F5) that $f((z \to y) \to (z \to (z \to x))) \geqslant \min\{f(z \to (y \to x)), 0.5\}$. By (F11), (I_1) and (I_7) we have further that $f(z \to x) \geqslant \min\{f(1), f(1 \to (((z \to x) \to x) \to (z \to x))), 0.5\} = \min\{f(((z \to x) \to x) \to (z \to x)), 0.5\} = \min\{f(z \to (((z \to x) \to x) \to x)), 0.5\} = \min\{f(z \to (z \to x)), 0.5\} \geqslant \min\{f(z \to y), f((z \to y) \to (z \to (z \to x))), 0.5\} \geqslant \min\{f(z \to y), \min\{f(z \to (y \to x)), 0.5\}, 0.5\} = \min\{f(z \to y), f(z \to (y \to x)), 0.5\}$, i.e., (F7) holds. Thus $f \in$ **FIF**(X) and thus **FPIF**$(X) \subseteq$ **FIF**(X).

Remark 4.1 The converse of Theorem 4.3 may not be true. Consider the FI-algebra $(X, \to_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.7$, it is easy to verify that $f \in$ **FIF**(X). But $f \notin$ **FPIF**(X), since $f((a' \to a) \to a) = f((0 \to a) \to a) = f(a) = 0.4 \ngeqslant 0.5 = \min\{f(1), 0.5\}$.

Theorem 4.4 *Let X be an FI-algebra. Then* $\textbf{\textit{FIF}}(X) \subseteq \textbf{\textit{FPIF}}(X)$ *if and only if the following condition holds for all* $f \in \textbf{\textit{FIF}}(X)$:

(F15) $(\forall x, y \in X)(f((x \to y) \to y) \geqslant \min\{f((y \to x) \to x), 0.5\})$.

Proof Assume $\mathbf{FIF}(X) \subseteq \mathbf{FPIF}(X)$. Then for all $f \in \mathbf{FIF}(X)$, we have that $f \in \mathbf{FPIF}(X)$. Since $y \leq (x \to y) \to y$, we have $((x \to y) \to y) \to x \leq y \to x$. By ($I_1$) and ($I_2$), we have further that $(((x \to y) \to y) \to x) \to ((x \to y) \to y) \geq (y \to x) \to ((x \to y) \to y) = (x \to y) \to ((y \to x) \to y) \geq (y \to x) \to x$. Hence by (F11) and (F5) we have that $f((x \to y) \to y) \geqslant \min\{f(1), f(1 \to ((((x \to y) \to y) \to x) \to ((x \to y) \to y))), 0.5\} = \min\{f((((x \to y) \to y) \to x) \to ((x \to y) \to y)), 0.5\} \geqslant \min\{\min\{f((y \to x) \to x), 0.5\}, 0.5\} = \min\{f((y \to x) \to x), 0.5\}$, that is f satisfies (F15).

Conversely, let $f \in \mathbf{FIF}(X)$ and for any $x, y \in X$, it satisfies that $f((x \to y) \to y) \geqslant \min\{f((y \to x) \to x), 0.5\}$. It follows from ($I_2$) that $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$. Thus by (F5) and (F9), we have that $f((y \to x) \to x) \geqslant \min\{f((x \to y) \to y), 0.5\} \geqslant \min\{\min\{f((x \to y) \to ((x \to y) \to y)), 0.5\}, 0.5\} = \min\{f((x \to y) \to ((x \to y) \to y)), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. Since $y \leq x \to y$, we have $(x \to y) \to x \leq y \to x$ and by (F5) $f(y \to x) \geqslant \min\{f((x \to y) \to x), 0.5\}$. So, by (F4) we can obtain that $f(x) \geqslant \min\{f(y \to x), f((y \to x) \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, \min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. Hence $f \in \mathbf{FPIF}(X)$ by (F12), and so $\mathbf{FIF}(X) \subseteq \mathbf{FPIF}(X)$.

4.3 $(\in, \in\vee q)$-Fuzzy Commutative Filters

Definition 4.3 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy commutative filter, if it satisfies the following condition:

(F16) $(\forall x, y, z \in X)(f(((x \to y) \to y) \to x) \geqslant \min\{f(z), f(z \to (y \to x)), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy commutative filters in X is denoted by $\mathbf{FCF}(X)$.

Example 4.3 Let $X = \{0, a, b, 1\}$. The operator $\to_3$ on X is defined in Table 3.

Then $(X, \to_3, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.3$, $f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FCF}(X)$.

Table 3 Definition of $\to_3$ on X

$\to_3$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Theorem 4.5 *Let X be an FI-algebra, $f \in \boldsymbol{FF}(X)$. Then $f \in \boldsymbol{FCF}(X)$ if and only if f satisfies the following condition:*

(F17) $(\forall x, y \in X)(f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\})$.

Proof Assume $f \in \mathbf{FCF}(X)$. Then taking $z = 1$ in (F16) we can obtain that $f(((x \to y) \to y) \to x) \geqslant \min\{f(1), f(1 \to (y \to x)), 0.5\} = \min\{f(1), f(y \to x), 0.5\} = \min\{f(y \to x), 0.5\}$. That is (F17) holds.

Conversely, assume (F17) holds, i.e., for all $x, y \in X$, $f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\}$. Then for all $x, y, z \in X$, since $f \in \mathbf{FF}(X)$, by (F4), we have $f(y \to x) \geqslant \min\{f(z), f(z \to (y \to x)), 0.5\}$. So, $f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\} \geqslant \min\{\min\{f(z), f(z \to (y \to x)), 0.5\}, 0.5\} = \min\{f(z), f(z \to (y \to x)), 0.5\}$, i.e., (F16) holds. Thus $f \in \mathbf{FCF}(X)$.

Theorem 4.6 *Let X be an FI-algebra. Then $\boldsymbol{FPIF}(X) \subseteq \boldsymbol{FCF}(X)$.*

Proof Assume $f \in \mathbf{FPIF}(X)$. Since $x \leq ((x \to y) \to y) \to x$, by ($I_{10}$) we have that $(((x \to y) \to y) \to x) \to y \leq x \to y$. Letting $\alpha = ((x \to y) \to y) \to x$, by ($I_1$) and ($I_{10}$), we have that $(\alpha \to y) \to \alpha = ((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$. So, $f((\alpha \to y) \to \alpha) \geqslant \min\{f(y \to x), 0.5\}$ by (F5). Then by (F12) we have that $f(((x \to y) \to y) \to x) = f(\alpha) \geqslant \min\{f((\alpha \to y) \to \alpha), 0.5\} \geqslant \min\{\min\{f(y \to x), 0.5\}, 0.5\} = \min\{f(y \to x), 0.5\}$, that is, f satisfies (F17). Thus $f \in \mathbf{FCF}(X)$ by Theorem 4.5.

Remark 4.2 The converse of Theorem 4.6 may not be true. Consider the FI-algebra $(X, \to_3, 0)$ in Example 4.3, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.7$, it is easy to verify that $f \in \mathbf{FCF}(X)$. But $f \notin \mathbf{FPIF}(X)$, since $f(b) = 0.4 \ngeqslant 0.5 = \min\{f((b \to a) \to b), 0.5\}$.

Theorem 4.7 *Let X be an FI-algebra. Then $\boldsymbol{FPIF}(X) = \boldsymbol{FIF}(X) \cap \boldsymbol{FCF}(X)$.*

Proof By Theorems 4.3 and 4.6 we have that $\mathbf{FPIF}(X) \subseteq \mathbf{FIF}(X) \cap \mathbf{FCF}(X)$.

To show that $\mathbf{FIF}(X) \cap \mathbf{FCF}(X) \subseteq \mathbf{FPIF}(X)$, assume $f \in \mathbf{FIF}(X) \cap \mathbf{FCF}(X)$. Then for all $x, y \in X$, since $x \leq (x \to y) \to y$, we have $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$ by (I_{10}). Thus by (F9) and (F5) we obtain that $f((x \to y) \to y) \geqslant \min\{f((x \to y) \to ((x \to y) \to y)), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. Since $y \leq x \to y$, we have $(x \to y) \to x \leq y \to x$ by (I_8). Thus by (F17) and (F5) we obtain further that $f(((x \to y) \to y) \to x) \geqslant \min\{f(y \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. So, by (F4) we have that $f(x) \geqslant \min\{f((x \to y) \to y), f(((x \to y) \to y) \to x), 0.5\} \geqslant \min\{\min\{f((x \to y) \to x), 0.5\}, \min\{f((x \to y) \to x), 0.5\}, 0.5\} = \min\{f((x \to y) \to x), 0.5\}$. It follows from Theorem 4.2 that $f \in \mathbf{FPIF}(X)$. Hence $\mathbf{FIF}(X) \cap \mathbf{FCF}(X) \subseteq \mathbf{FPIF}(X)$.

Remark 4.3 Let X be an FI-algebra. We claim that $\mathbf{FIF}(X) \nsubseteq \mathbf{FCF}(X)$ and $\mathbf{FCF}(X) \nsubseteq \mathbf{FIF}(X)$. In fact, on the one hand, consider the FI-algebra $(X, \rightarrow_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.6$, then $f \in \mathbf{FIF}(X)$, but $f \notin \mathbf{FCF}(X)$, since $f(((b \rightarrow a) \rightarrow a) \rightarrow b) = f(b) = 0.4 < 0.5 = \min\{f(a \rightarrow b), 0.5\}$. On the other hand, consider the FI-algebra $(X, \rightarrow_3, 0)$ in Example 4.3, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.4$, $f(1) = 0.6$, then $f \in \mathbf{FCF}(X)$, but $f \notin \mathbf{FIF}(X)$, since $f(b \rightarrow a) = f(b) = 0.4 < 0.5 = \min\{f(b \rightarrow (b \rightarrow a)), f(b \rightarrow b), 0.5\}$.

4.4 $(\in, \in\vee q)$-Fuzzy Involution Filters

Definition 4.4 Let X be an FI-algebra. A $(\in, \in\vee q)$-fuzzy filter f in X is called a $(\in, \in\vee q)$-fuzzy involution filter, if it satisfies the following condition:

(F18) $(\forall x, y \in X)(f(y) \geqslant \min\{f(x), f((x \rightarrow y)''), 0.5\})$.

The set of all $(\in, \in\vee q)$-fuzzy involution filters in X is denoted by $\mathbf{FINF}(X)$.

Example 4.4 Let $X = \{0, a, b, 1\}$. The operator $\rightarrow_4$ on X is defined in Table 4.

Then $(X, \rightarrow_4, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = 0.3$, $f(b) = f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FINF}(X)$.

Remark 4.4 Let X be an FI-algebra. Then $\mathbf{FF}(X) \subseteq \mathbf{FINF}(X)$ general does not hold. In fact, consider the FI-algebra $(X, \rightarrow_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = 0.4$, $f(b) = f(1) = 0.6$, then $f \in \mathbf{FF}(X)$, but $f \notin \mathbf{FINF}(X)$, since $f(a) = 0.4 < 0.5 = \min\{f(b), f((b \rightarrow a)''), 0.5\}$.

Theorem 4.8 *Let X be an FI-algebra, $f \in \mathbf{FF}(X)$. Then $f \in \mathbf{FINF}(X)$ if and only if f satisfies the following condition:*

(F19) $(\forall x \in X)(f(x) \geqslant \min\{f(x''), 0.5\})$.

Proof Assume $f \in \mathbf{FINF}(X)$. Then for all $x \in L$, by (F18) and (F3) we can obtain that $f(x) \geqslant \min\{f(1), f((1 \rightarrow x)''), 0.5\} = \min\{f(1), f(x''), 0.5\} = \min\{f(x''), 0.5\}$. That is (F19) holds.

Conversely, assume (F19) holds, i.e., for all $x \in X$, $f(x) \geqslant \min\{f(x''), 0.5\}$. Then for all $x, y \in X$, since $f \in \mathbf{FF}(X)$, by (F4), we have $f(y) \geqslant \min\{f(x), f(x \rightarrow y), 0.5\} \geqslant \min\{f(x), \min\{f((x \rightarrow y)''), 0.5\}, 0.5\} = \min\{f(x), f((x \rightarrow y)''), 0.5\}$. i.e., (F18) holds. Thus $f \in \mathbf{FINF}(X)$.

Table 4 Definition of $\rightarrow_4$ on X

$\rightarrow_4$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Remark 4.5 Let X be an FI-algebra. We claim that $\mathbf{FINF}(X) \nsubseteq \mathbf{FIF}(X)$ and $\mathbf{FIF}(X) \nsubseteq \mathbf{FINF}(X)$. In fact, on the one hand, consider the FI-algebra $(X, \rightarrow_4, 0)$ in Example 4.4, define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.2$, $f(1) = 0.9$, then $f \in \mathbf{FINF}(X)$, but $f \notin \mathbf{FIF}(X)$, since $f(a \rightarrow 0) = f(b) = 0.2 < 0.5 = \min\{f(a \rightarrow a), f(a \rightarrow (a \rightarrow 0)), 0.5\}$. On the other hand, consider the FI-algebra $(X, \rightarrow_1, 0)$ in Example 4.1, define a fuzzy set on X by $f(0) = f(a) = 0.2$, $f(b) = f(1) = 0.9$, then $f \in \mathbf{FIF}(X)$, but $f \notin \mathbf{FINF}(X)$, since $f(a) = 0.2 < 0.5 = \min\{f(b), f((b \rightarrow a)''), 0.5\}$.

Theorem 4.9 *Let X be an FI-algebra. Then $\boldsymbol{FCF}(X) \subseteq \boldsymbol{FINF}(X)$.*

Proof Assume $f \in \mathbf{FCF}(X)$. Since $0 \rightarrow (x \rightarrow y) = 1$, by (F17), we have that $f((x \rightarrow y)'' \rightarrow (x \rightarrow y)) = f((((x \rightarrow y) \rightarrow 0) \rightarrow 0) \rightarrow (x \rightarrow y)) \geqslant \min\{f(0 \rightarrow (x \rightarrow y)), 0.5\} = \min\{f(1), 0.5\}$. Thus by (F4) we can obtain that $f(y) \geqslant \min\{f(x), f(x \rightarrow y), 0.5\} \geqslant \min\{f(x), \min\{f((x \rightarrow y)''), f((x \rightarrow y)'' \rightarrow (x \rightarrow y)), 0.5\}, 0.5\} \geqslant \min\{f(x), \min\{f((x \rightarrow y)''), \min\{f(1), 0.5\}, 0.5\}, 0.5\} = \min\{f(x), f((x \rightarrow y)''), f(1), 0.5\} = \min\{f(x), f((x \rightarrow y)''), 0.5\}$, that is, f satisfies (F18). Thus $f \in \mathbf{FINF}(X)$ by Definition 4.4.

Corollary 4.1 *Let X be an FI-algebra. Then $\boldsymbol{FPIF}(X) \subseteq \boldsymbol{FINF}(X)$.*

Remark 4.6 The converse of Theorem 4.9 and Corollary 4.1 may not be true. Let $X = \{0, a, b, c, 1\}$. The operator $\rightarrow_5$ on X is defined in Table 5.

Then $(X, \rightarrow_5, 0)$ is an FI-algebra. Define a fuzzy set on X by $f(0) = f(a) = f(b) = 0.3$, $f(c) = f(1) = 0.8$, it is easy to verify that $f \in \mathbf{FINF}(X)$. But $f \notin \mathbf{FCF}(X)$, and thus $f \notin \mathbf{FPIF}(X)$, since $f(((b \rightarrow a) \rightarrow a) \rightarrow b) = f((a \rightarrow a) \rightarrow b) = f(1 \rightarrow b) = f(b) = 0.3 < 0.5 = \min\{f(a \rightarrow b), 0.5\}$.

Theorem 4.10 *Let X be an FI-algebra. Then $\boldsymbol{FPIF}(X) = \boldsymbol{FIF}(X) \cap \boldsymbol{FINF}(X)$.*

Proof By Theorem 4.3 and Corollary 4.1 we have that $\mathbf{FPIF}(X) \subseteq \mathbf{FIF}(X) \cap \mathbf{FINF}(X)$. To show that $\mathbf{FIF}(X) \cap \mathbf{FINF}(X) \subseteq \mathbf{FPIF}(X)$, assume $f \in \mathbf{FIF}(X) \cap \mathbf{FINF}(X)$, Then for all $x \in X$, since $x \leqslant x''$, we have $x' \rightarrow x \leqslant x' \rightarrow x''$ by (I_{10}). Thus by (F5) we have that $f(x' \rightarrow x'') \geqslant \min\{f(x' \rightarrow x), 0.5\}$, and thus by $f \in \mathbf{FIF}(X)$ and (F9) we have $f(x'') = f(x' \rightarrow 0) \geqslant \min\{f(x' \rightarrow (x' \rightarrow 0)), 0.5\} = \min\{f(x' \rightarrow x''), 0.5\}$. This together with $f \in \mathbf{FINF}(X)$ and (F19) we can obtain that $f(x) \geqslant \min\{f(x''), 0.5\} \geqslant \min\{f(x' \rightarrow x''), 0.5\} \geqslant \min\{f(x' \rightarrow x), 0.5\}$, i.e., f satisfies (F13). It follows from Theorem 4.2 that $f \in \mathbf{FPIF}(X)$. Hence $\mathbf{FIF}(X) \cap \mathbf{FINF}(X) \subseteq \mathbf{FPIF}(X)$.

Table 5 Definition of $\rightarrow_5$ on X

$\rightarrow_5$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

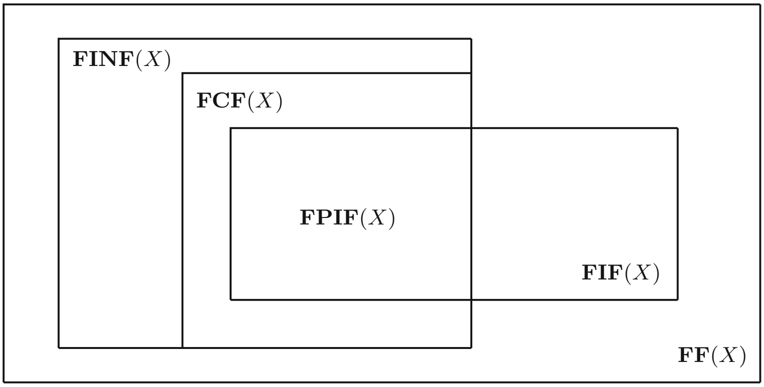

Fig. 1 Relations among sets of various $(\in, \in\vee q)$-fuzzy filters in FI-algebras

5 Conclusion

In this paper, we introduced the notions of $(\in, \in\vee q)$-fuzzy filters and $(\in, \in\vee q)$-fuzzy implicative (resp., positive implicative, commutative, involution) filters in FI-algebras and investigated their properties and relationships (as Fig. 1). Some significant characterizations of the mentioned kinds of filters are obtained. Since FI-algebras are more extensive, the results obtained in this paper can be applied to some other logical algebras such as MV-algebras, BL-algebras, R_0-algebras, MTL-algebras and lattice implication algebras and so on. So, it is hoped that more research topics of Non-classical Mathematical Logic will arise with this work.

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References

1. Wu, W.M.: Fuzzy implication algebras. Fuzzy Syst. Math. **4**(1), 56–64 (1990)
2. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
3. Jun, Y.B., Xu, Y., Qin, K.Y.: Fuzzy filters redefined in lattice implication algebras. J. Fuzzy Math. **14**(2), 345–353 (2006)
4. Ma, X.L., Zhan, J.M.: On $(\in, \in\vee q)$-fuzzy filters of BL-algebras. J. Syst. Sci. Complex. **21**, 144–158 (2008)
5. Liu, C.H.: $(\in, \in\vee q)$-fuzzy filters of UB algebras in universal logic. Comput. Eng. Appl. **45**(34), 29–31 (2009)

Approach to Group Decision Making Based on Intuitionistic Uncertain Linguistic Aggregation Operators

Xin-Fan Wang and Jian-Qiang Wang

Abstract In this paper, we define some new additive operational laws of intuitionistic uncertain linguistic numbers (IULNs). Based on these operational laws, we propose some new arithmetic aggregation operators, such as the intuitionistic uncertain linguistic number weighted averaging (IULNWA) operator, intuitionistic uncertain linguistic number ordered weighted averaging (IULNOWA) operator and intuitionistic uncertain linguistic number hybrid aggregation (IULNHA) operator. Furthermore, based on the IULNWA and IULNHA operators, we develop a group decision making approach in which the criterion values take the form of IULNs.

Keywords Group decision making · IULNWA operator · IULNOWA operator · IULNHA operator

1 Introduction

In many multi-criteria decision making (MCDM) situations, a realistic approach may be to use linguistic assessments instead of precise numerical values by means of linguistic variables, that is, variables whose values are not exact numbers but linguistic terms. Over the last decades, the linguistic MCDM has been attracting more and more attention [1–12].

When using linguistic variables, there exists an assumption that the membership degree of an element to a linguistic term is 1, which cannot describe the confidence level of the judgment of a decision-maker. Based on the concept of intuitionistic fuzzy set [13] and its applications in MCDM field [14–21], Wang and Li [22] defined the concept of intuitionistic linguistic number (ILN). They described the membership degree and non-membership degree of an element to a linguistic label, which can

X.-F. Wang (✉)
School of Science, Hunan University of Technology, Zhuzhou 412007, Hunan, China
e-mail: zzwxfydm@126.com

J.-Q. Wang
School of Business, Central South University, Changsha 410083, Hunan, China
e-mail: jqwang@csu.edu.cn

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reflect decision-maker's confidence level for his/her evaluation. Many scholars have shown their great interests in intuitionistic linguistic MCDM problems [22–28].

From the works of Refs. [22–27], it can be known that an ILN is characterized by a linguistic term, a membership degree and a non-membership degree. Sometimes, however, the given linguistic evaluation may not match any of the original linguistic terms, and they may be located between two of them. In such case, it is more suitable to deal with vagueness and uncertainty taking the form of uncertain linguistic variables [29, 30]. Therefore, based on the ILN [22] and uncertain linguistic variable [29, 30], Liu and Zhang [31] defined the notion of intuitionistic uncertain linguistic number (IULN), which is characterized by a uncertain linguistic variable, a membership degree and a non-membership degree. Some scholars have begun to address the IULN set theory, especially the intuitionistic uncertain linguistic information aggregation problems [31, 32]. Liu and Zhang [31] proposed some intuitionistic uncertain linguistic aggregation operators, such as the intuitionistic uncertain linguistic weighted arithmetic averaging (IULWAA) operator, intuitionistic uncertain linguistic ordered weighted averaging (IULOWA) operator and intuitionistic uncertain linguistic hybrid averaging (IULHA) operator, and applied them to group decision making. Liu and Jin [32] developed some intuitionistic uncertain linguistic geometric operators, such as the intuitionistic uncertain linguistic weighted geometric average (IULWGA) operator, intuitionistic uncertain linguistic ordered weighted geometric (IULOWG) operator and intuitionistic uncertain linguistic hybrid geometric (IULHG) operator, and proposed a group decision making method based on these operators. However, the additive operational laws of IULNs in [31] maybe have some shortcomings (See Example 1). To overcome the drawback, in this paper, we shall define some new additive operational laws of IULNs, and develop some new intuitionistic uncertain linguistic aggregation operators and apply them to group decision making.

In order to do that, we organize this paper as follows. In Sect. 2, we define some new additive operational laws of IULNs, and present a simple formula of the degree of possibility for the comparison between two IULNs. In Sect. 3, we propose some new arithmetic aggregation operators, such as the intuitionistic uncertain linguistic number weighted averaging (IULNWA) operator, intuitionistic uncertain linguistic number ordered weighted averaging (IULNOWA) operator and intuitionistic uncertain linguistic number hybrid aggregation (IULNHA) operator. In Sect. 4, based on the IULNWA and IULNHA operators, we develop an approach to group decision making, in which the criterion values are expressed as IULNs and the criterion weights are known completely. Finally, conclusions of this paper are presented in Sect. 5.

2 Preliminaries

Suppose that $S = \{s_\theta | \theta = 0, 1, \ldots, 2l\}$ is a discrete linguistic term set [33], where l is a positive integer, and s_θ has the following characteristics: ① if $a > b$, then $s_a > s_b$; ② the negation operator is defined: $\mathrm{neg}(s_a) = s_b$ such that $a + b = 2l$. For example, when $l = 4$, S can be defined as:

$S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ ={extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good}.

To preserve all the given information, the discrete linguistic term set S should be extended to a continuous linguistic term set $\bar{S} = \{s_\theta | \theta \in [0, q]\}$, in which $s_a > s_b$ if $a > b$, and $q(q > 2l)$ is a sufficiently large positive integer.

Definition 1 ([34]) Let $\tilde{s} = [s_\theta, s_\phi]$, where $s_\theta, s_\phi \in \bar{S}$, s_θ and s_ϕ are the lower and the upper limits, respectively. Then we call $\tilde{s}$ the uncertain linguistic variable. Let $\tilde{S}$ be the set of all the uncertain linguistic variables.

Definition 2 ([22]) Let X be a universe of discourse, $s_{\theta(x)} \in \bar{S}$. Then an ILN set A in X is an object having the following form:

$$A = \left\{ \left(x, \left\langle s_{\theta(x)}, \mu_A(x), \nu_A(x) \right\rangle\right) | x \in X \right\},$$

where $s_{\theta(x)}$ is a linguistic term, the functions $\mu_A(x)$ and $\nu_A(x)$ determine the degree of membership and the degree of non-membership of the element x to the linguistic evaluation $s_{\theta(x)}$, respectively, and for each $x \in X$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1$$

Based on Definitions 1 and 2, Liu and Zhang [31] extended the ILN set, and defined the concept of IULN set as follows.

Definition 3 ([31]) Let X be fixed, $\tilde{s}(x) \in \tilde{S}$. Then an IULN set $\tilde{A}$ in X is defined as:

$$\tilde{A} = \left\{ \left(x, \left\langle \tilde{s}(x), \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \right\rangle\right) | x \in X \right\},$$

which is characterized by a uncertain linguistic variable $\tilde{s}(x)$, a membership function $\mu_{\tilde{A}}(x)$ and a non-membership function $\nu_{\tilde{A}}(x)$ of the element x to the linguistic evaluation $\tilde{s}(x)$, and for each $x \in X$: $0 \le \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \le 1$.

For convenience, Liu and Zhang [31] called $\langle \tilde{s}(x), \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle$ the IULN, and denoted it as $\tilde{\beta} = \langle [s_{\theta(\tilde{\beta})}, s_{\phi(\tilde{\beta})}], \mu_{\tilde{\beta}}, \nu_{\tilde{\beta}} \rangle$, where $[s_{\theta(\tilde{\beta})}, s_{\phi(\tilde{\beta})}] \in \tilde{S}$, $\mu_{\tilde{\beta}} \in [0, 1]$, $\nu_{\tilde{\beta}} \in [0, 1]$ and $\mu_{\tilde{\beta}} + \nu_{\tilde{\beta}} \le 1$. Let Ω be the set of all IULNs.

Definition 4 Let $\tilde{\beta}_1 = \langle [s_{\theta(\tilde{\beta}_1)}, s_{\phi(\tilde{\beta}_1)}], \mu_{\tilde{\beta}_1}, \nu_{\tilde{\beta}_1} \rangle$ and $\tilde{\beta}_2 = \langle [s_{\theta(\tilde{\beta}_2)}, s_{\phi(\tilde{\beta}_2)}], \mu_{\tilde{\beta}_2}, \nu_{\tilde{\beta}_2} \rangle$ be two IULNs. Then

$$(1)\quad \tilde{\beta}_1 \oplus \tilde{\beta}_2 = \left\langle \left[s_{\theta(\tilde{\beta}_1)+\theta(\tilde{\beta}_2)}, s_{\phi(\tilde{\beta}_1)+\phi(\tilde{\beta}_2)} \right], \frac{[\theta(\tilde{\beta}_1)+\phi(\tilde{\beta}_1)]\mu_{\tilde{\beta}_1} + [\theta(\tilde{\beta}_2)+\phi(\tilde{\beta}_2)]\mu_{\tilde{\beta}_2}}{\theta(\tilde{\beta}_1)+\phi(\tilde{\beta}_1)+\theta(\tilde{\beta}_2)+\phi(\tilde{\beta}_2)}, \frac{[\theta(\tilde{\beta}_1)+\phi(\tilde{\beta}_1)]\nu_{\tilde{\beta}_1} + [\theta(\tilde{\beta}_2)+\phi(\tilde{\beta}_2)]\nu_{\tilde{\beta}_2}}{\theta(\tilde{\beta}_1)+\phi(\tilde{\beta}_1)+\theta(\tilde{\beta}_2)+\phi(\tilde{\beta}_2)} \right\rangle;$$

(2) $\lambda\tilde{\beta}_1 = \left\langle \left[s_{\lambda\theta(\tilde{\beta}_1)}, s_{\lambda\phi(\tilde{\beta}_1)} \right], \mu_{\tilde{\beta}_1}, \nu_{\tilde{\beta}_1} \right\rangle, \lambda \in [0, 1]$.

It can be easily proved that all the above results are also IULNs. Based on Definition 4, the following relations can be further obtained.

(1) $\tilde{\beta}_1 \oplus \tilde{\beta}_2 = \tilde{\beta}_2 \oplus \tilde{\beta}_1$;
(2) $(\tilde{\beta}_1 \oplus \tilde{\beta}_2) \oplus \tilde{\beta}_3 = \tilde{\beta}_1 \oplus (\tilde{\beta}_2 \oplus \tilde{\beta}_3)$;
(3) $\lambda(\tilde{\beta}_1 \oplus \tilde{\beta}_2) = \lambda\tilde{\beta}_1 \oplus \lambda\tilde{\beta}_2, \lambda \in [0, 1]$;
(4) $\lambda_1\tilde{\beta}_1 \oplus \lambda_2\tilde{\beta}_1 = (\lambda_1 + \lambda_2)\tilde{\beta}_1, \lambda_1, \lambda_2 \in [0, 1]$.

Example 1 Let $\tilde{\beta}_1 = \langle [s_4, s_5], a, b\rangle (a \in [0, 1], b \in [0, 1], a + b \leq 1)$, $\tilde{\beta}_2 = \langle [s_2, s_4], 1, 0\rangle$ and $\tilde{\beta}_3 = \langle [s_4, s_5], 0, 1\rangle$ be three IULNs, by Definition 4. We have

$$\tilde{\beta}_1 \oplus \tilde{\beta}_2 = \langle [s_6, s_9], 0.4 + 0.6a, 0.6b\rangle, \tag{1}$$

$$\tilde{\beta}_1 \oplus \tilde{\beta}_3 = \langle [s_6, s_9], 0.6a, 0.4 + 0.6b\rangle. \tag{2}$$

According to the operational laws of IULNs in [31], we have

$$\tilde{\beta}_1 \oplus \tilde{\beta}_2 = \langle [s_6, s_9], 1, 0\rangle, \tag{3}$$

$$\tilde{\beta}_1 \oplus \tilde{\beta}_3 = \langle [s_6, s_9], a, b\rangle. \tag{4}$$

Comparing the aforementioned results, it can be known that Eqs. (1) and (2) may be more easily accepted, and Eqs. (3) and (4) cannot be accepted.

In addition, from the above, we know that $[s_4, s_5]$ is closer to the sum $[s_6, s_9]$ than $[s_2, s_4]$, so its confidence degree should be higher, that is, the weight of its membership degree and non-membership degree should be bigger, thus, it is more suitable that we weight the membership degree and non-membership degree of $\tilde{\beta}_1$ by $\frac{4+5}{4+5+2+4} = \frac{9}{15}$ and $\tilde{\beta}_2$ by $\frac{2+4}{4+5+2+4} = \frac{6}{15}$. Thus, the additional operation laws in Definition 4 is more reasonable than that in [31].

Based on the degree of possibility of interval numbers [35], in the following, we define a simple formula for the comparison between any two IULNs.

Definition 5 Let $\tilde{\beta}_1 = \langle [s_{\theta(\tilde{\beta}_1)}, s_{\phi(\tilde{\beta}_1)}], \mu_{\tilde{\beta}_1}, \nu_{\tilde{\beta}_1}\rangle$ and $\tilde{\beta}_2 = \langle [s_{\theta(\tilde{\beta}_2)}, s_{\phi(\tilde{\beta}_2)}], \mu_{\tilde{\beta}_2}, \nu_{\tilde{\beta}_2}\rangle$ be two IULNs, and let $l_{\tilde{\beta}_1} = [\phi(\tilde{\beta}_1) - \theta(\tilde{\beta}_1)](\mu_{\tilde{\beta}_1} - \nu_{\tilde{\beta}_1})$ and $l_{\tilde{\beta}_2} = [\phi(\tilde{\beta}_2) - \theta(\tilde{\beta}_2)](\mu_{\tilde{\beta}_2} - \nu_{\tilde{\beta}_2})$. Then the degree of possibility of $\tilde{\beta}_1 \geq \tilde{\beta}_2$ is defined as:

$$p(\tilde{\beta}_1 \geq \tilde{\beta}_2) = \min\left\{\max\left(\frac{\phi(\tilde{\beta}_1)(\mu_{\tilde{\beta}_1} - \nu_{\tilde{\beta}_1}) - \theta(\tilde{\beta}_2)(\mu_{\tilde{\beta}_2} - \nu_{\tilde{\beta}_2})}{l_{\tilde{\beta}_1} + l_{\tilde{\beta}_2}}, 0\right), 1\right\}. \tag{5}$$

From Definition 5, the following useful results can easily be obtained.

(1) $0 \leq p(\tilde{\beta}_1 \geq \tilde{\beta}_2) \leq 1, 0 \leq p(\tilde{\beta}_2 \geq \tilde{\beta}_1) \leq 1$;
(2) $p(\tilde{\beta}_1 \geq \tilde{\beta}_2) + p(\tilde{\beta}_2 \geq \tilde{\beta}_1) = 1$. Especially, $p(\tilde{\beta}_i \geq \tilde{\beta}_i) = 0.5, i = 1, 2$.

3 Intuitionistic Uncertain Linguistic Number Aggregation Operators

To aggregate the intuitionistic uncertain linguistic information, in the following, we develop some new aggregation operators, such as the IULNWA operator, the IULNOWA operator and the IULNHA operator.

Definition 6 Let $\tilde{\beta}_j = \langle [s_{\theta(\tilde{\beta}_j)}, s_{\phi(\tilde{\beta}_j)}], \mu_{\tilde{\beta}_j}, \nu_{\tilde{\beta}_j} \rangle (j = 1, 2, \ldots, n)$ be a collection of IULNs and let IULNWA: $\Omega^n \to \Omega$. If

$$\text{IULNWA}_{\boldsymbol{w}}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) = w_1\tilde{\beta}_1 \oplus w_2\tilde{\beta}_2 \oplus \cdots \oplus w_n\tilde{\beta}_n, \tag{6}$$

then IULNWA is called an IULNWA operator of dimension n, where $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{\beta}_j \quad (j = 1, 2, \ldots, n)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Theorem 1 *Let* $\tilde{\beta}_j = \langle [s_{\theta(\tilde{\beta}_j)}, s_{\phi(\tilde{\beta}_j)}], \mu_{\tilde{\beta}_j}, \nu_{\tilde{\beta}_j} \rangle (j = 1, 2, \ldots, n)$ *be a collection of IULNs, and* $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* $\tilde{\beta}_j \quad (j = 1, 2, \ldots, n)$, *with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^n w_j = 1$. *Then their aggregated value by using the IULNWA operator is also an IULN, and*

$$\begin{aligned} &\text{IULNWA}_{\boldsymbol{w}}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) \\ &= \left\langle \left[s_{\sum_{j=1}^n w_j \theta(\tilde{\beta}_j)}, s_{\sum_{j=1}^n w_j \phi(\tilde{\beta}_j)} \right], \right. \\ &\left. \frac{\sum_{j=1}^n w_j [\theta(\tilde{\beta}_j) + \phi(\tilde{\beta}_j)] \mu_{\tilde{\beta}_j}}{\sum_{j=1}^n w_j [\theta(\tilde{\beta}_j) + \phi(\tilde{\beta}_j)]}, \frac{\sum_{j=1}^n w_j [\theta(\tilde{\beta}_j) + \phi(\tilde{\beta}_j)] \nu_{\tilde{\beta}_j}}{\sum_{j=1}^n w_j [\theta(\tilde{\beta}_j) + \phi(\tilde{\beta}_j)]} \right\rangle. \end{aligned} \tag{7}$$

Definition 7 Let $\tilde{\beta}_j = \langle [s_{\theta(\tilde{\beta}_j)}, s_{\phi(\tilde{\beta}_j)}], \mu_{\tilde{\beta}_j}, \nu_{\tilde{\beta}_j} \rangle (j = 1, 2, \ldots, n)$ be a collection of IULNs. An IULNOWA operator of dimension n is a mapping IULNOWA: $\Omega^n \to \Omega$, that has an associated weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Furthermore,

$$\text{IULNOWA}_{\boldsymbol{\omega}}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) = \omega_1\tilde{\beta}_{\tau(1)} \oplus \omega_2\tilde{\beta}_{\tau(2)} \oplus \cdots \oplus \omega_n\tilde{\beta}_{\tau(n)} \tag{8}$$

where $(\tau(1), \tau(2), \ldots, \tau(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{\beta}_{\tau(j-1)} \geq \tilde{\beta}_{\tau(j)}$ for all j.

Theorem 2 *Let* $\tilde{\beta}_j = \langle [s_{\theta(\tilde{\beta}_j)}, s_{\phi(\tilde{\beta}_j)}], \mu_{\tilde{\beta}_j}, \nu_{\tilde{\beta}_j} \rangle \quad (j = 1, 2, \ldots, n)$ *be a collection of IULNs. Then their aggregated value by using the IULNOWA operator is also an IULN, and*

$$\text{IULNOWA}_{\boldsymbol{\omega}}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n)$$
$$= \left\langle \left[s_{\sum_{j=1}^{n} \omega_j \theta(\tilde{\beta}_{\tau(j)})}, s_{\sum_{j=1}^{n} \omega_j \phi(\tilde{\beta}_{\tau(j)})} \right], \right.$$
$$\left. \frac{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}_{\tau(j)}) + \phi(\tilde{\beta}_{\tau(j)})] \mu_{\tilde{\beta}_{\tau(j)}}}{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}_{\tau(j)}) + \phi(\tilde{\beta}_{\tau(j)})]}, \frac{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}_{\tau(j)}) + \phi(\tilde{\beta}_{\tau(j)})] \nu_{\tilde{\beta}_{\tau(j)}}}{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}_{\tau(j)}) + \phi(\tilde{\beta}_{\tau(j)})]} \right\rangle, \tag{9}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weight vector related to the IULNOWA operator, with* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1$*, which can be determined similar to the OWA weights.*

From Definitions 6 and 7, it can be known that the IULNWA operator weights only the IULNs, whereas the IULNOWA operator weights only the ordered positions of them. To overcome this limitation, in what follows, we develop an IULNHA operator, which weights both the given IULNs and their ordered positions.

Definition 8 Let $\tilde{\beta}_j = \langle [s_{\theta(\tilde{\beta}_j)}, s_{\phi(\tilde{\beta}_j)}], \mu_{\tilde{\beta}_j}, \nu_{\tilde{\beta}_j} \rangle \quad (j = 1, 2, \ldots, n)$ be a collection of IULNs. An IULNHA operator of dimension n is a mapping IULNHA: $\Omega^n \rightarrow \Omega$, which has an associated vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, such that

$$\text{IULNHA}_{\boldsymbol{w},\boldsymbol{\omega}}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) = \omega_1 \tilde{\beta}'_{\tau(1)} \oplus \omega_2 \tilde{\beta}'_{\tau(2)} \oplus \cdots \oplus \omega_n \tilde{\beta}'_{\tau(n)}, \tag{10}$$

where $\tilde{\beta}'_{\tau(j)}$ is the jth largest of weighted IULNs $(nw_1\tilde{\beta}_1, nw_2\tilde{\beta}_2, \ldots, nw_n\tilde{\beta}_n)$, $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{\beta}_j \quad (j = 1, 2, \ldots, n)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient.

Furthermore, we have the following:

$$\text{IULNHA}_{\boldsymbol{w},\boldsymbol{\omega}}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) =$$
$$\left\langle \left[s_{\sum_{j=1}^{n} \omega_j \theta(\tilde{\beta}'_{\tau(j)})}, s_{\sum_{j=1}^{n} \omega_j \phi(\tilde{\beta}'_{\tau(j)})} \right], \right.$$
$$\left. \frac{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}'_{\tau(j)}) + \phi(\tilde{\beta}'_{\tau(j)})] \mu_{\tilde{\beta}'_{\tau(j)}}}{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}'_{\tau(j)}) + \phi(\tilde{\beta}'_{\tau(j)})]}, \frac{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}'_{\tau(j)}) + \phi(\tilde{\beta}'_{\tau(j)})] \nu_{\tilde{\beta}'_{\tau(j)}}}{\sum_{j=1}^{n} \omega_j [\theta(\tilde{\beta}'_{\tau(j)}) + \phi(\tilde{\beta}'_{\tau(j)})]} \right\rangle \tag{11}$$

and the aggregated value derived by using the IULNHA operator is also an IULN.

Theorem 3 *If* $\boldsymbol{\omega} = (1/n, 1/n, \ldots, 1/n)^T$*, then the IULNHA operator is reduced to the IULNWA operator; if* $\boldsymbol{w} = (1/n, 1/n, \ldots, 1/n)^T$*, then the IULNHA operator is reduced to the IULNOWA operator.*

From Theorem 3 it can be known that the IULNHA operator generalizes both the IULNWA operator and the IULNOWA operator at the same time, and reflects the importance degrees of both the given IULNs and their ordered positions.

4 Group Decision Making Method Based on the IULNWA and IULNHA Operators

In the following, we apply the IULNWA and IULNHA operators to develop a group decision making approach, in which the criterion values take the form of IULNs.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of alternatives, $I = \{I_1, I_2, \ldots, I_n\}$ be the set of criteria, and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of I_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Let $D = \{d_1, d_2, \ldots, d_t\}$ be the set of decision-makers, and $\boldsymbol{e} = (e_1, e_2, \ldots, e_t)^T$ be the weight vector of d_k $(k = 1, 2, \ldots, t)$, with $e_k \geq 0$ and $\sum_{k=1}^{t} e_k = 1$. Suppose that $\tilde{\boldsymbol{R}}^{(k)} = (\tilde{\beta}_{ij}^{(k)})_{m\times n}$ $(k = 1, 2, \ldots, t)$ are the decision matrixes, where $\tilde{\beta}_{ij}^{(k)} = \langle [s_{\theta(\tilde{\beta}_{ij}^{(k)})}, s_{\phi(\tilde{\beta}_{ij}^{(k)})}], \quad \mu_{\tilde{\beta}_{ij}^{(k)}}, \nu_{\tilde{\beta}_{ij}^{(k)}} \rangle$ take the form of IULNs, given by the decision-makers d_k $(k = 1, 2, \ldots, t)$, for the alternatives A_i $(i = 1, 2, \ldots, m)$ with respect to the criteria I_j $(j = 1, 2, \ldots, n)$. Then the ranking of the alternatives is required.

Step 1 Normalize the decision matrices $\tilde{\boldsymbol{R}}^{(k)}$ $(k = 1, 2, \ldots, t)$. For the benefit-type criteria, we do nothing; for the cost-type criteria, we utilize the linguistic negation operator $\bar{s}_{\theta(\tilde{\beta}_{ij}^{(k)})} = \text{neg}(s_{\theta(\tilde{\beta}_{ij}^{(k)})}) = s_{2l-\theta(\tilde{\beta}_{ij}^{(k)})}$ and $\bar{s}_{\phi(\tilde{\beta}_{ij}^{(k)})} = \text{neg}(s_{\phi(\tilde{\beta}_{ij}^{(k)})}) = s_{2l-\phi(\tilde{\beta}_{ij}^{(k)})}$ to make the uncertain linguistic evaluation values be normalized.

For convenience, the normalized criterion values of the alternatives A_i with respect to the criteria I_j are also denoted by $\tilde{\beta}_{ij}^{(k)}(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$.

Step 2 Utilize the IULNWA operator

$$\tilde{\beta}_i^{(k)} = \text{IULNWA}_{\boldsymbol{w}}(\tilde{\beta}_{i1}^{(k)}, \tilde{\beta}_{i2}^{(k)}, \ldots, \tilde{\beta}_{in}^{(k)}) \tag{12}$$

to aggregate the criterion values of the ith line of the decision matrices $\tilde{\boldsymbol{R}}^{(k)}$ $(k = 1, 2, \ldots, t)$ and derive the individual overall evaluation values $\tilde{\beta}_i^{(k)}$ of the alternatives A_i $(i = 1, 2, \ldots, m)$, given by the decision-makers d_k $(k = 1, 2, \ldots, t)$.

Step 3 Utilize the IULNHA operator

$$\begin{aligned}\tilde{\beta}_i &= \text{IULNHA}_{\boldsymbol{e},\boldsymbol{v}}(\tilde{\beta}_i^{(1)}, \tilde{\beta}_i^{(2)}, \ldots, \tilde{\beta}_i^{(t)}) \\ &= v_1 \tilde{\beta'}_i^{(\tau(1))} \oplus v_2 \tilde{\beta'}_i^{(\tau(2))} \oplus \cdots \oplus v_t \tilde{\beta'}_i^{(\tau(t))}\end{aligned} \tag{13}$$

to derive the collective overall evaluation values $\tilde{\beta}_i$ of the alternatives A_i $(i = 1, 2, \ldots, m)$, where $\boldsymbol{v} = (v_1, v_2, \ldots, v_t)^T$ is the weighting vector of the IULNHA operator with $v_k \geq 0$ and $\sum_{k=1}^{t} v_k = 1$, and $\tilde{\beta}'^{(\tau(k))}_i$ is the kth largest of the weighted IULNs $(te_1\tilde{\beta}_i^{(1)}, te_2\tilde{\beta}_i^{(2)}, \ldots, te_t\tilde{\beta}_i^{(t)})$, $(\tau(1), \tau(2), \ldots, \tau(t))$ is a permutation of $(1, 2, \ldots, t)$, and t is the balancing coefficient.

Step 4 Utilize Eq. (5) to compare each $\tilde{\beta}_i$ $(i = 1, 2, \ldots, m)$ with all $\tilde{\beta}_r$ $(r = 1, 2, \ldots, m)$, and construct a complementary matrix $\boldsymbol{P} = (p_{ir})_{m\times m}$, where $p_{ir} = p(\tilde{\beta}_i \geq \tilde{\beta}_r)$. By summing all elements in each line of matrix $\boldsymbol{P}$, we derive $p_i = \sum_{r=1}^{n} p_{ir}(i = 1, 2, \ldots, m)$. Then we rank these $\tilde{\beta}_i$ $(i = 1, 2, \ldots, m)$ in descending order in accordance with the values of p_i $(i = 1, 2, \ldots, m)$.

Step 5 Rank all the alternatives A_i $(i = 1, 2, \ldots, m)$, and then select the best one in accordance with $\tilde{\beta}_i$ $(i = 1, 2, \ldots, m)$.

5 Conclusion

In this paper, we have defined some new additive operational laws of IULNs, which can overcome the drawback of additive operational laws proposed in [31]. Based on these operational laws, we have proposed some new arithmetic aggregation operators, such as the IULNWA operator, the IULNOWA operator and the IULNHA operator. Furthermore, based on the IULNWA and IULNHA operators, we have developed a group decision making method with intuitionistic uncertain linguistic information, which develops the theories of aggregation operators and linguistic MCDM.

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References

1. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning. Part 1. Inf. Sci. **8**(3), 199–249 (1975)
2. Herrera, F., Herrera-Viedma, E.: Aggregation operators for linguistic weighted information. IEEE Trans. Syst. Man Cybern. part A: Syst. Hum. **27**(5), 646–656 (1997)
3. Xu, Z.S.: An approach to pure linguistic multiple attribute decision making under uncertainty. Int. J. Inf. Technol. Decis. Making **4**(2), 197–206 (2005)
4. Merigó, J.M., Casanovas, M., Martínez, L.: Linguistic aggregation operators for linguistic decision making based on the Dempster-Shafer theory of evidence. Int. J. Uncertainty Fuzziness Knowl. Based Syst. **18**(3), 287–304 (2010)
5. Wei, G.W.: Some generalized aggregating operators with linguistic information and their application to multiple attribute group decision making. Comput. Ind. Eng. **61**(1), 32–38 (2011)
6. Xu, Z.S.: Linguistic Decision Making: Theories and Methods. Science Press, Beijing (2012)
7. Xu, Y.J., Merigó, J.M., Wang, H.M.: Linguistic power aggregation operators and their application to multiple attribute group decision making. Appl. Math. Model. **36**(11), 5427–5444 (2012)

8. Yang, W.-E., Wang, J.-Q.: Multi-criteria semantic dominance: a linguistic decision aiding technique based on incomplete preference information. Eur. J. Oper. Res. **231**(1), 171–181 (2013)
9. Yang, W.-E., Wang, J.-Q.: Vague linguistic matrix game approach for multi-criteria decision making with uncertain weights. J. Intell. Fuzzy Syst. **25**(2), 315–324 (2013)
10. Yang, W.-E., Wang, J.-Q., Wang, X.-F.: An outranking method for multi-criteria decision making with duplex linguistic information. Fuzzy Sets Syst. **198**, 20–33 (2012)
11. Wang, J.Q., Peng, L., Zhang, H.Y., Chen, X.H.: Method of multi-criteria group decision-making based on cloud aggregation operators with linguistic information. Inf. Sci. **274**, 177–191 (2014)
12. Wang, J.Q., Wang, D.D., Zhang, H.Y., Chen, X.H.: Multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators. Soft Comput. **19**(2), 389–405 (2015)
13. Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**(1), 87–96 (1986)
14. Xu, Z.: Intuitionistic fuzzy aggregation operators. IEEE Trans. Fuzzy Syst. **15**(6), 1179–1187 (2007)
15. Wan, S.P., Li, D.F.: Atanassov's intuitionistic fuzzy programming method for heterogeneous multiattribute group decision making with Atanassov's intuitionistic fuzzy truth degrees. IEEE Trans. Fuzzy Syst. **22**(2), 300–312 (2014)
16. Xu, Z., Cai, X.: Recent advances in intuitionistic fuzzy information aggregation. Fuzzy Optim. Decis. Making **9**(4), 359–381 (2010)
17. Zhao, X.F., Wei, G.W.: Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making. Knowl.-Based Syst. **37**, 472–479 (2013)
18. Wang, J.-Q., Zhang, H.-Y.: Multicriteria decision-making approach based on atanassov's intuitionistic fuzzy sets with incomplete certain information on weights. IEEE Trans. Fuzzy Syst. **21**(3), 510–515 (2013)
19. Wang, J.-Q., Han, Z.-Q., Zhang, H.-Y.: Multi-criteria group decision-making method based on intuitionistic interval fuzzy information. Group Decis. Negot. **23**, 715–733 (2014)
20. Tan, C.: Generalized intuitionistic fuzzy geometric aggregation operator and its application to multi-criteria group decision making. Soft Comput. **15**(5), 867–876 (2011)
21. Wang, J.-Q., Nie, R.-R., Zhang, H.-Y. and Chen, X.-H.: New operators on triangular intuitionistic fuzzy numbers and their applications in system fault analysis. Inf. Sci. **251**, 79–95 (2013)
22. Wang, J.-Q., Li, H.-B.: Multi-criteria decision-making method based on aggregation operators for intuitionistic linguistic fuzzy numbers. Control Decis. **25**(10), 1571–1574, 1584 (2010)
23. Zhang, Y., Ma, H., Liu, B., Liu, J.: Group decision making with 2-tuple intuitionistic fuzzy linguistic preference relations. Soft Comput. **16**(8), 1439–1446 (2012)
24. Liu, P.: Some generalized dependent aggregation operators with intuitionistic linguistic numbers and their application to group decision making. J. Comput. Syst. Sci. **79**(1), 131–143 (2013)
25. Wang, X.-F., Wang, J.-Q., Yang, W.-E.: Multi-criteria group decision making method based on intuitionistic linguistic aggregation operators. J. Intell. Fuzzy Syst. **26**(1), 115–125 (2014)
26. Su, W., Li, W., Zeng, S., Zhang, C.: Atanassov's intuitionistic linguistic ordered weighted averaging distance operator and its application to decision making. J. Intell. Fuzzy Syst. **26**(3), 1491–1502 (2014)
27. Wang, J.-Q., Wang, P.: Intuitionistic linguistic fuzzy multi-criteria decision-making method based on intuitionistic fuzzy entropy. Control Decis. **27**(11), 1694–1698 (2012)
28. Wang, J.Q., Wang, P., Wang, J., Zhang, H.Y., Chen, X.H.: Atanassov's interval-valued intuitionistic linguistic multi-criteria group decision-making method based on the trapezium cloud model. IEEE Trans. Fuzzy Syst. **23**(3), 542–554 (2015)
29. Xu, Z.S.: A direct approach to group decision making with uncertain additive linguistic preference relations. Fuzzy Optim. Decis. Making **5**(1), 21–32 (2006)
30. Lan, J.B., Sun, Q., Chen, Q.M., Wang, Z.X.: Group decision making based on induced uncertain linguistic OWA operators. Decis. Support Syst. **55**(1), 296–303 (2013)

31. Liu, P.-D., Zhang, X.: Intuitionistic uncertain linguistic aggregation operators and their application to group decision making. Syst. Eng. Theor. Pract. **32**(12), 2704–2711 (2012)
32. Liu, P., Jin, F.: Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making. Inf. Sci. **205**(1), 58–71 (2012)
33. Herrera, F., Herrera-Viedma, E., Verdegay, J.L.: A model of consensus in group decision making under linguistic assessments. Fuzzy Sets Syst. **78**(1), 73–87 (1996)
34. Xu, Z.S.: Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. Inf. Sci. **168**(1–4), 171–184 (2004)
35. Xu, Z.S., Da, Q.L.: The uncertain OWA operator. Int. J. Intell. Syst. **17**(6), 569–575 (2002)

Statistical Diagnostic of Center Fuzzy Linear Regression Model Based on Fuzzy Decentring Degree

Jia-wei Xu and Ruo-ning Xu

Abstract Aimed at the center fuzzy linear regression model with real input and fuzzy output, the model parameter estimation is discussed on data deletion of fuzzy decentring degree. Statistical diagnostic quantity is established to test outliers or strong influential points in the observed data. And the step is given to examine with the statistical diagnostic quantity. At last the example suggests that the method can improve the accuracy of the computational results effectively.

Keywords Center fuzzy linear regression model · Case deletion model · Fuzzy decentring degree

1 Introduction

Tanakan et al. [1] puts forward the fuzzy linear regression model with numerical input and fuzzy output first, the computational complexity and the accuracy in which the model are low; and then Diamond gives the least squares model fitting in with triangular fuzzy number, the model prediction's accuracy is improved, but its computational complexity is increased, too. In the model building process, due to the interference of gross error, random error and so on [2, 3], once mixed with outliers, the methods will face critical challenges, and even get wrong conclusions. Above all, it's necessary to detect outliers or strong influential points of the observed data while we reduce the computational complexity of the model.

So in order to improve the predictive accuracy of the model, it's necessary to find out the data point that influences the model much more when we build up the model according to the original data. In contrast to the past, we discuss the model

J. Xu · R. Xu (✉)
School of Mathematics and Information Sciences, Guangzhou University,
Guangzhou 510006, China
e-mail: ruxu@163.com

J. Xu
e-mail: 15626155596@163.com

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parameter estimation on data deletion while the model is the center fuzzy linear regression model with real input and fuzzy output. Fuzzy decentring degree is the statistical diagnostic quantity which is established to test outliers or strong influential points in the observed data by using the least-squares method in this paper. At last the example suggests the statistic constructed of the center fuzzy linear regression model in this paper that is effective.

2 Preliminaries

In this section, some basic concepts and results on fuzzy numbers are reviewed, which are necessary for developing our improved method.

Let R be the real number set $R^+ = \{x|x \in R, x > 0\}$.

Definition 2.1 [4] Let $\tilde{A}$ be a fuzzy set on R. And

(1) If $\tilde{A}(x_0) = 1$, when $\exists x_0 \in R$;
(2) If $\forall\lambda \in (0, 1]$, when $A_\lambda = \{x|\tilde{A}(x) \geq \lambda\}$ is a closed convex set;

then $\tilde{A}$ is called the fuzzy number on R. $\tilde{R}$ is denoted all the fuzzy numbers on R.

Definition 2.2 [4] On R, if the membership function has the following form:

$$\tilde{A}(x) = \begin{cases} \frac{1}{a}x - \frac{m-a}{a}, & m-a \leq x \leq m, \\ -\frac{1}{b}x + \frac{m+b}{b}, & m < x \leq m+b, \ (a, b > 0), \\ 0, \text{otherwise}, \end{cases}$$

the fuzzy number $\tilde{A}$ is called triangular fuzzy number, where m is the most possible value of fuzzy number $\tilde{A}$, a and b are the left and right spread of $\tilde{A}$ respectively. We denote it as $\tilde{A} = (m, a, b)$. If $a = b$, then we say that $\tilde{A}$ is symmetric, and write it as $\tilde{A} = (m, a)$.

3 Statistical Diagnostic of Center Fuzzy Linear Regression Model Based on Fuzzy Decentring Degree

We always discussed the fuzzy linear regression model with real input and fuzzy output. About the fuzzy linear regression model, in order to evaluate the ith datum's effect and influence in regression analysis, we can compare the statistical inference results' changes of statistical inference results before and after deleting the ith datum. Thus we can know whether the point is abnormal or strong influence. The model that deleted the ith datum is called the data deletion fuzzy linear regression model. In this paper, we assume a set of fuzzy number as $\{(x_i, \tilde{y}_i)\,|\tilde{y}_i = (y_i, p_i, q_i), i = 1, 2, \ldots, n\}$, and the barycenter of the fuzzy set is

$(\bar{x}, \bar{\tilde{y}})$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{\tilde{y}} = (\frac{1}{n}\sum_{i=1}^{n} y_i, \frac{1}{n}\sum_{i=1}^{n} p_i, \frac{1}{n}\sum_{i=1}^{n} q_i)$. Then exchange the ith datum of the fuzzy set to the barycenter coordinate. So the fuzzy linear regression model $Z_i (i = 1, 2, \ldots, n)$ which is obtained from the processed data is called the ith data deleted center fuzzy linear regression equation. Its form is the following:

$$\tilde{y}_{[i]} = a_{[i]} + \tilde{b}_{[i]} x, a_{[i]} \in R, \tilde{b}_{[i]} \in \tilde{R}. \tag{1}$$

According to the least-squares method, we have the following result for the parameters above the question.

Theorem 3.1 *For* $\tilde{y}_{[i]} = a_{[i]} + \tilde{b}_{[i]} x, a_{[i]} \in R, \tilde{b}_{[i]} \in \tilde{R}$ *, if* $(x_m, \tilde{y}_m)(m \neq i, m = 1, 2, \ldots n)$ *is a set of observed data, and* $\tilde{y}_m = (y_m, l_m, r_m)$ *is triangular fuzzy number, then:*

$$\begin{aligned} a_{[i]} &= \frac{\sum\limits_{m \neq i} x_m^2 \sum\limits_{m \neq i} y_m - \sum\limits_{m \neq i} x_m \sum\limits_{m \neq i} x_m y_m - \frac{1}{4} \sum\limits_{m \neq i} x_m^2 (l_m - r_m) + \sum\limits_{m \neq i} x_m \sum\limits_{m \neq i} x_m (l_m - r_m)}{(n-1) \sum\limits_{m \neq i} x_m^2 - (\sum\limits_{m \neq i} x_m)^2}, \\ \tilde{b}_{[i]} &= \left(\frac{\sum\limits_{m \neq i} x_m y_m - \hat{a}_{[i]} \sum\limits_{m \neq i} x_m}{\sum\limits_{m \neq i} x_m^2}, \frac{\sum\limits_{m \neq i} x_m l_m}{\sum\limits_{m \neq i} x_m^2}, \frac{\sum\limits_{m \neq i} x_m r_m}{\sum\limits_{m \neq i} x_m^2} \right). \end{aligned} \tag{2}$$

Fuzzy center value reflects the value of fuzzy number, and fuzzy decentring degree is an index to evaluate the value of fuzzy number concentration degree [4]. So in order to improve the prediction accuracy of the model and detect the much more relative anomaly or influential data points about the set of data, we combine fuzzy decentring degree to analyze the prediction data point of the center fuzzy linear regression model before and after deleting the data in this paper.

Definition 3.1 [4]. For $\forall \tilde{A} \in \tilde{R}$, we record supp $\tilde{A} = (a, b)$ and

(1) When $a = b$, let $C(\tilde{A}) = a$.
(2) When $a \neq b$, let

$$C(\tilde{A}) = \frac{\int_a^b x\tilde{A}(x)dx}{\int_a^b \tilde{A}(x)dx}. \tag{3}$$

Then $C(\tilde{A})$ is called the central value of $\tilde{A}$.

Definition 3.2 [4]. For $\forall \tilde{A} \in \tilde{R}$, we record supp $\tilde{A} = (a, b)$, $C(\tilde{A})$ as the central value of $\tilde{A}$, and

(1) When $a = b$, let $D(\tilde{A}) = 0$.
(2) When $a \neq b$, let

$$D(\tilde{A}) = \left\{ \frac{\int_a^b \left[x - C(\tilde{A})\right]^2 \tilde{A}(x)dx}{\int_a^b \tilde{A}(x)dx} \right\}^{\frac{1}{2}} . \tag{4}$$

Then $D(\tilde{A})$ is called the fuzzy decentring degree of $\tilde{A}$.

According to the nature of fuzzy decentring degree, if the fuzzy number $\tilde{A}$ is triangular fuzzy number, we have the following function:

$$D^2(\tilde{A}) = \frac{1}{18}(p^2 + q^2 + pq) \quad . \tag{5}$$

In order to determine theoretically whether the ith datum is outlier or strong influential point. The next we introduce the evaluation method.

Definition 3.3 [5]. Assume a choose optimal set as $U = \{u_1, u_2, \ldots, u_m\}$, give n kinds of ranking advises that are $L_1, L_2, \ldots, L_n$. According to the objective conditions, give the corresponding weights that are $a_i (i = 1, 2, \ldots, n)$. And $B_i(u_i)$ is the element number which is positioned after u_i in L_i. Then we called the following $B(u_i)$ is the weighted *Borda* number of u_i:

$$B(u_i) = \sum_{i=1}^{m} a_i B_i(u_i). \tag{6}$$

The scoring method [5] is now presented as follows:

(1) Calculate the weighted *Borda* number of each element in the collection.
(2) Sort the elements according to the size of the weighted *Borda* number.

The different model is considered as a element of a big set in practical application, and according to the scoring method, we can actually compare the model after assigning values.

When we make analysis of data,we operate on the following steps:

- Calculate the original fuzzy linear regression equation Z;

(2) Calculate the barycenter of the fuzzy set and exchange the ith datum, and then calculate the center fuzzy linear regression equation $Z_i (i = 1, 2, \ldots, n)$ after deleting the ith datum;
(3) Calculate the ith fuzzy output datum of Z, and its fuzzy decentring degree D_i $(i = 1, 2, \ldots, n)$, calculate the ith fuzzy output datum of Z_i, and its fuzzy decentring degree $D_i' (i = 1, 2, \ldots, n)$;
(4) Compare the size of each fuzzy decentring degree on step (3), and calculate the data points' fuzzy center value of each model that we get from step (1) and (2);
(5) According to the scoring method, calculate the weighted *Borda* number of the data deleted model, and rank them. The larger is the weighted *Borda* number, the better is the model. The result indicates that the deleted data point corresponding to the model is outlier or strong influential point.

Certainly, we should consider to combine the data actual background with the meaning of the outlier or strong influential point during we diagnose the specific data points.

4 Example Analysis

Through making a survey on an information company's sales data [6] in recent years, we get the following Table 1 (The volume $\tilde{y}_i$ is a triangular fuzzy number). Analyze the data by the two models presented earlier in this article, and then discuss the outliers.

For the convenience of calculation, transform in the following at first:

$$\kappa_i = x_i - 1986, i = 1, 2, \ldots, n. \tag{7}$$

Set $\tilde{y} = a + \tilde{b}\kappa_i$, and then the fuzzy regression equation is:

$$\tilde{y} = 221.169 + (6.436, 0.294, 0.320)\kappa.$$

We can acquire the ith datum deleted center fuzzy linear regression equation by Eqs. (2) and (7), the following Table 2 shows the equations:

So we can obtain different fuzzy output values corresponding to the *ith* datum from the equation, according to Eq. (5) to calculate the fuzzy decentring degree, we can get the following data in Table 3:

The data in Table 3 show that the eighth output datum fuzzy decentring degree is the smallest in these equations. So maybe the eighth datum is outlier or strong influential point compare with other data. According to Eq. (3), we can calculate the fuzzy center value and then we can get the following data in Table 4:

Through the scoring method (The observed data can decide the model while the product is on sales, and actually, each year sales data effect of the model is the same. The weights value are $1/n$ respectively.) we can calculate the value of $B(u_i)$ (It's the weighted *Borda* number of x.):

Table 1 The sale of a company's information

Year x_i	Sales volume (million) $\tilde{y}_i = (y_i, p_i, q_i)$	Year x_i	Sales volume (million) $\tilde{y}_i = (y_i, p_i, q_i)$
1987	(230,2,1)	1992	(257,2,1)
1988	(236,3,2)	1993	(262,3,2)
1989	(241,2,3)	1994	(276,3,3)
1990	(246,1,2)	1995	(281,2,3)
1991	(252,2,2)	1996	(286,1,2)

Table 2 The center fuzzy linear regression equation

ith datum	The ith datum deleted center fuzzy linear regression equation Z_i	ith datum	The ith datum deleted center fuzzy linear regression equation Z_i
1	$\tilde{y}_1 = 219.669 + (6.647, 0.296, 0.234)\kappa$	6	$\tilde{y}_6 = 220.968 + (6.510, 0.297, 0.339)\kappa$
2	$\tilde{y}_2 = 213.411 + (7.532, 0.288, 0.317)\kappa$	7	$\tilde{y}_7 = 220.672 + (6.591, 0.283, 0.329)\kappa$
3	$\tilde{y}_3 = 221.253 + (6.422, 0.292, 0.309)\kappa$	8	$\tilde{y}_8 = 221.467 + (6.319, 0.286, 0.315)\kappa$
4	$\tilde{y}_4 = 221.664 + (6.399, 0.302, 0.317)\kappa$	9	$\tilde{y}_9 = 221.130 + (6.238, 0.319, 0.322)\kappa$
5	$\tilde{y}_5 = 221.370 + (6.426, 0.294, 0.319)\kappa$	10	$\tilde{y}_{10} = 220.042 + (6.283, 0.363, 0.363)\kappa$

$$B(u_1) = 0.1 \times (10 + 10 + 10 + 10 + 10 + 8 + 6 + 3 + 3 + 3) = 7.3.$$

At the same time, we can get the other weighted *Borda* numbers of $u_i (i = 2, 3, \ldots, n)$ are:

$$B(u_2) = 6.2, B(u_3) = 5.0, B(u_4) = 3.0, B(u_5) = 2.9, B(u_6) = 2.6, B(u_7) = 2.0, B(u_8) = 7.4, B(u_9) = 5.5, B(u_{10}) = 4.3.$$

Obviously, the value of $B(I_8)$ is the largest, namely the fuzzy linear regression model which is obtained by deleting the eighth datum is the best model. In summary, we makes a theoretical analysis from the above two aspects and the analysis suggests that the eighth datum is outlier or influential point. If we remove the eighth datum when we are building the model, the accuracy of the model can be improved much more.

On the other hand, different fuzzy linear regression can be fitted according to different observation data, so that the prediction conclusion varies with model changes. When we use the model to predict unknown data in practical application, it's hard to consider the conclusion accuracy. Therefore, in order to improve the accuracy of the forecasting results, we should establish the data deletion model after finding out the datum which influence is relatively large. At the same time, use the above data in Table 2 to forecast the sales volume of 1997, and then calculate the square of the corresponding fuzzy decentring degree. We can get the result as follow:

$$D_1^2 = 1.9406, D_2^2 = 1.8511, D_3^2 = 1.8233, D_4^2 = 1.9315, D_5^2 = 1.8938,$$
$$D_6^2 = 2.0140, D_7^2 = 1.8907, D_8^2 = 1.8224, D_9^2 = 2.0690, D_{10}^2 = 2.6632.$$

Table 3 The *i*th predictions' fuzzy decentring degree of the data deletion center fuzzy linear regression equation

Index	D_1^2	D_2^2	D_3^2	D_4^2	D_5^2	D_6^2	D_7^2	D_8^2	D_9^2	D_{10}^2
Equation Z_1	0.0160	0.0641	0.1443	0.2566	0.4010	0.5774	0.7859	1.0265	1.2991	1.6038
Equation Z_2	0.0153	0.0612	0.1377	0.2448	0.3825	0.5507	0.7496	0.9791	1.2391	1.5298
Equation Z_3	0.0152	0.0603	0.1358	0.2411	0.3766	0.5425	0.7382	0.9640	1.2111	1.5062
Equation Z_4	0.0160	0.0639	0.1437	0.2554	0.3991	0.5747	0.7822	1.0216	1.2930	1.5963
Equation Z_6	0.0157	0.0626	0.1409	0.2504	0.3913	0.5634	0.7669	1.0017	1.2677	1.5651
Equation Z_6	0.0169	0.0675	0.1518	0.2699	0.4217	0.6073	0.8265	1.0796	1.3663	1.6868
Equation Z_7	0.0156	0.0625	0.1406	0.2500	0.3906	0.5625	0.7657	1.0000	1.2657	1.5626
Equation Z_8	0.0151	0.0602	0.1356	0.2410	0.3765	0.5422	0.7380	0.9639	1.2110	1.5061
Equation Z_9	0.0171	0.0684	0.1539	0.2736	0.4275	0.6156	0.8378	1.0943	1.385	1.7099
Equation Z_{10}	0.0220	0.0880	0.1981	0.3522	0.5502	0.7924	1.0785	1.4086	1.7828	2.2010

(D_i^2 is the square of fuzzy decentring degree)

Table 4 Fuzzy center value of center fuzzy liner regression equation after deleting *i*th datum

Index	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	Weight
Z_1	226.325	232.982	239.638	246.294	252.951	259.607	266.263	272.920	279.576	286.232	0.1
Z_2	220.953	228.494	236.036	243.578	251.119	258.661	266.203	273.744	281.286	288.828	0.1
Z_3	227.681	234.108	240.536	246.964	253.391	259.819	266.247	272.674	279.102	285.530	0.1
Z_4	228.068	234.472	240.876	247.28	253.684	260.088	266.492	272.896	279.300	285.704	0.1
Z_5	227.804	234.239	240.673	247.107	253.542	259.976	266.410	272.845	279.279	285.713	0.1
Z_6	227.492	234.016	240.540	247.064	253.588	260.112	266.636	273.160	279.684	286.208	0.1
Z_7	227.278	233.885	240.491	247.097	253.704	260.310	266.916	273.523	280.129	286.735	0.1
Z_8	227.796	234.124	240.453	246.782	253.110	259.439	265.768	272.096	278.425	284.754	0.1
Z_9	226.369	232.608	238.847	245.086	251.325	257.564	263.803	270.042	276.281	282.520	0.1
Z_{10}	226.325	232.608	238.891	245.174	251.457	257.740	264.023	270.306	276.589	282.872	0.1

The smaller is the fuzzy decentring degree of the prediction point, and the higher is its accuracy. Obviously the data show that the fuzzy decentring degree of the prediction value after deleting the eighth datum from the application is the smallest, so its accuracy is the highest. From the application of the model, we can know that when we are building the model, the effect of the eighth datum is the largest. Thus it has a further sign that the statistical diagnostics about the fuzzy decentring degree is effective in the paper. And through the scoring method combined with fuzzy center value we can improve the prediction accuracy of model much better.

5 Conclusion

Based on data deletion, we obtain the parameter estimation of center fuzzy linear regression model when the data are deleted. We structure a statistical diagnosis that is the fuzzy decentring degree to test the outliers or strong influential points in the paper. Firstly, we present the steps of testing the outliers or strong influential points in the data, and then study the actual history data and distinguish the outliers or strong influential points. At last, it shows that the statistical diagnosis can improve the accuracy of the model's predictive ability and it has practical significance to improve the prediction accuracy of the model in our life.

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References

1. Tanaka, H., Uejima, S., Asai, K.: Linear regression analysis with fuzzy model. IEEE Trans. **12**, 903–907 (1982)
2. He, X.Q., Liu, W.Q.: Applied Regression Analysis, 3rd edn, pp. 5–13, 122–123. China Renmi University Press Co. Ltd., Beijing (2011)
3. Wei, B.C., Lin, J.G., Xie, F.C.: Statistical Inference, pp. 19–44. Higher Education Press, Beijing (2009)
4. Xu, R.N.: Fuzzy Mathematics Method in the Analysis of Economic Information, vol. 41, 58, pp. 94–107, 222–224. South China University of Technology Press, Guangzhou (1997)
5. Chen, S.L., Li, J.G., Wang, X.G.: Fuzzy Set Theory and Application, pp. 252–257. Science Press, Beijing (2005)
6. Xu, R.N.: A linear regression model in fuzzy environment. Adv. Model. Simul. **27**, 31–40 (1991)

A New Method of Multiple Attributes Evaluation and Selection in Fuzzy Environment

Xiao-yan Zhai

Abstract In a complex evaluation system, not only it is often necessary to deal with multiple attributes decision problems, but also it is unavoidable to deal with the uncertainness and fuzziness of people's judgment. This paper considers the multiple attributes and uncertainness decision problems in the complex decision systems, a fuzzy multiple attributes evaluation and selection model is built, a new method is developed for ranking fuzzy priorities from the model, and calculation formulae for fuzzy priorities are derived. Finally, an applied example on suppliers' selection is given for demonstrating the method.

Keywords Fuzzy multiple criteria decision · Evaluation and selection model · Fuzzy priority

1 Introduction

People often need to consider many factors or multiple attributes in complex decision systems and it usually need to give judgment by experts. Therefore it is unavoidable to consider the uncertainness and fuzziness of people's thinking and judgment in the multiple attributes decision problems. Jahanshahloo et al. [1], Wang and Parkan [2] considered the fuzziness of people's judgment and built the fuzzy decision models by using the triangular fuzzy numbers and fuzzy preference relation. It is effective method to deal with the complex decision problems, but the problems of the construction of the fuzzy numbers and their objectivity should be researched further. In the paper [3], we present a method that describes people's fuzzy judgment with interval fuzzy number on continuous judgment scale. Based on the research on paper [3], we developed several fuzzy multiple attributes decision methods [4–8]. In this paper, we will build a fuzzy multiple attributes evaluation and selection model based

X. Zhai (✉)
South China Business College, Guangdong University of Foreign Studies,
Guangzhou 510420, China
e-mail: zhaixiaoy@163.com

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on interval fuzzy number, and present a new ranking method to extract the fuzzy priorities for multiple criteria decision making. We also apply the model for supplier selection in the end.

2 Building a Fuzzy Multiple Attributes Evaluation and Selection Model

2.1 Operation Properties of Positive Bounded Closed Fuzzy Numbers

Professor Saaty presents an quantificational approach of people's judgment and describes people's judgment according to the 1–9 scale in paper [9], that is, 1, 3, 5, 7, 9 represented equal importance, moderate importance, strong importance, very strong importance, extreme importance respectively. In paper [3], we consider the uncertainness of people's judgment and extend the 1–9 scale of the importance comparison to continuous scale interval (0, 10], and we can used other point $x, x \in (0, 10]$, to present the intermediate state, the bigger the value x is, the more obvious the importance comparison is. So scale interval (0, 10] can be used to describe the uncertain judgment of important comparison. We can extend scale interval (0, 10] to positive real number area and give some relevant definition and conclusions in the following.

Definition 1 Let R^+ is a positive real number set. A closed interval $\xi = [a, b] = \{x|(0 < a \leq x \leq b\}$ on R^+ is said to be a judgment interval number if it is evaluated area given out by decision-maker according to some judgment scale or evaluation criterion.

Definition 2 ([10]). $F(R^+)$ denotes the set of all positive bounded closed fuzzy numbers, namely $\tilde{A} \in F(R^+)$ iff for every $t \in (0, 1]$ the t-level set $A_t = \{x \in R : \tilde{A}(x) \geq t\}$ is a non-empty bounded closed interval contained in R^+ (where $R^+ = \{x : x > 0, x \in R\}$).

Clearly, the judgment interval number $\xi = [a, b]$ is a real number when a = b, and judgment interval numbers and positive real numbers all are especial positive bounded closed fuzzy numbers. In the paper [4], we proved $A_t = \{x \in R : \tilde{A}(x) \geq t\}$ could be represented as a positive closed interval, and gave the operations of positive bounded closed fuzzy numbers as follows:

Theorem 1 *Let* $\tilde{A}, \tilde{B} \in F(R^+)$, $A_t = [s_t, r_t]$, $B_t = [p_t, q_t]$, $e > 0$, *then*

(1) $\tilde{A} + \tilde{B} = \bigcup_{0<t\leq 1} t\,(A_t + B_t) = \bigcup_{0<t\leq 1} t[s_t + p_t\,,\ r_t + q_t]$,

(2) $\tilde{A} \cdot \tilde{B} = \bigcup_{0<t\leq 1} t\,(A_t \cdot B_t) = \bigcup_{0<t\leq 1} t[s_t \cdot p_t\,,\ r_t \cdot q_t]$,

(3) $\tilde{A}/\tilde{B} = \bigcup_{0<t\leq 1} t\,(A_t/B_t) = \bigcup_{0<t\leq 1} t[s_t/q_t\,,\ r_t/p_t]$,

(4) $e \cdot \tilde{A} = \bigcup_{0<t\le 1} t\,(e \cdot A)_t = \bigcup_{0<t\le 1} t[e \cdot s_t\,,\ e \cdot r_t],$

(where $e \cdot \tilde{A}$ is derived from function $f(x) = e \cdot x$ by extension principle [11])

(5) $1/\tilde{A} = \bigcup_{0<t\le 1} t(1/A_t) = \bigcup_{0<t\le 1} t[1/r_t,\, 1/s_t].$

Here $t[a, b]$ is a fuzzy set on R^+ and its membership function is defined as

$$t\,[a, b](u) = t\Lambda[a, b](u) = \begin{cases} t, & u \in [a, b], \\ 0, & u \notin [a, b], \end{cases} \quad u \in R^+.$$

And this notation will be used in the sequel.

In a general way, people's views or judgment about same problem are often different because of the difference of their knowledge and experience. Suppose that there are m decision-makers in the group, every decision-maker make the uncertainness judgment independently according to some scale. Denoted $\xi^{(k)} = [a^{(k)}, b^{(k)}](k = 1, \ldots, m)$ as the judgment interval number, which is given out by the kth decision-maker. If $\xi^{(k)} \bigcap \xi^{(l)} \ne \phi$ for any $k, l \in \{1, \ldots, m\}$, it is easy to prove $\bigcap_{k=1}^{m} \xi^{(k)} \ne \phi$. It is possible to exist the judgment interval numbers $\xi^{(k_0)}$ and $\xi^{(l_0)}$ such that $\xi^{(k_0)} \bigcap \xi^{(l_0)} = \phi$ when the judgment $\xi^{(k_0)}$ of the k_0th decision-maker and the judgment $\xi^{(l_0)}$ of the l_0th decision-maker are completely different ($k_0, l_0 \in \{1, \ldots, m\}$). For this case, the results can be adjusted by exchanging their opinions between the k_0th and l_0th decision-maker, and the result:

$$\xi^{(k)} \bigcap \xi^{(l)} \ne \phi \text{ for any } k, l \in \{1, \ldots, m\} \tag{1}$$

will be reached finally. We call formula (1) as the approximate consistency condition of the group's judgments. We get the following conclusion under this condition.

Theorem 2 *Let $\xi^{(k)} = [a^{(k)}, b^{(k)}](k = 1, \ldots, m)$ be a judgment interval number, $\xi^{(k)} \bigcap \xi^{(l)} \ne \phi$ (for any k, $l \in \{1, \ldots, m\}$). Then*

$$\tilde{a}(x) = \frac{1}{m} \sum_{k=1}^{m} \chi_{\xi^{(k)}}(x)x \in R^+ \tag{2}$$

is a closed fuzzy number on R^+ (here $\chi_{\xi^{(k)}}$ is $\xi^{(k)'}s$ characteristic function).

Proof Let $a^{(k_1)}, a^{(k_2)}, \ldots, a^{(k_m)}$ be the order from small to large of the numbers $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$, where $k_1, \mathrm{k}_2, \ldots, k_m$ is a permutation of $1, \ldots, m$. We can prove that $\forall k, l \in \{1, \ldots, m\}, a^{(k)} \le b^{(l)}$ just based on $\xi^{(k)} \bigcap \xi^{(l)} \ne \phi$. So the order from small to large of the 2 m numbers $a^{(1)}, a^{(2)}, \ldots, a^{(m)}, b^{(1)}, b^{(2)}, \ldots, b^{(m)}$, can be expressed as $a^{(k_1)}, a^{(k_2)}, \ldots, a^{(k_m)}, b^{(l_m)}, b^{(l_{m-1})}, \ldots, b^{(l_1)}$, where $l_m, l_{m-1}, \ldots, l_1$ is also a permutation of m, $m-1$, …, 1. We can obtain formula (3), there $\tilde{a}(x)$ is a monotone increasing right continuous ladder function in interval $(0, b^{(l_m)})$. Similarly, we can prove that $\tilde{a}(x)$ is a monotone decreasing left continuous ladder function in $[b^{(l_m)}, b^{(l_1)}]$. Thus $\tilde{a}(x)$ is a positive bounded closed fuzzy numbers in step form according to the definition of closed fuzzy number.

$$\tilde{a}(x) = \begin{cases} 0, & when\ 0 \le x < a^{(k_1)} \text{ or } b^{(l_m)} < x \\ \sum_{n=1}^{t} \frac{1}{m}, & when\ a^{(k_t)} \le x < a^{(k_{t+1})}, (t = 1, \ldots, m-1); \\ \sum_{n=1}^{t} \frac{1}{m}, & when\ b^{(l_{t+1})} \le x < b^{(l_t)}, (t = 1, \ldots, m-1); \\ 1, & when\ a^{(k_m)} \le x \le b^{(l_m)}. \end{cases} \tag{3}$$

2.2 Building of a Fuzzy Multiple Attributes Evaluation and Selection Model

A evaluation system usually involves many different factors, which have different attributes and have some relationship to each other. It is difficult to consider these factors, attributes, evaluation criterions and their relationship in evaluation process. In the section, we shall consider the multiple attributes of evaluation problems and fuzziness of people's judgment, and extend classical multiple attributes decision making methods to fuzzy environment, and build a fuzzy multiple attributes evaluation and selection model based on positive bounded closed fuzzy numbers in step form.

Suppose that there are k projects $u_1, \ldots, u_k$ waiting for selection, and there are n attributes $v_1, \ldots, v_n$ in the evaluation index system. If the decision-making group have persons m, every decision-maker give out a range on interval to represent his uncertainness judgment independently according to some scale. Denoted $\xi_{ij}^{(l)} = [a_{ij}^{(l)}, b_{ij}^{(l)}](l = 1, \ldots, m)$ as ith project judgment interval number under jth attribute $(i = 1, \ldots, k, j = 1, \ldots, n)$, which is given out by the lth decision-maker. Let $\xi_{ij}^{(l)} = [a_{ij}^{(l)}, b_{ij}^{(l)}]\ (l = 1, \ldots, m)$ satisfy the approximate consistency condition of the group's judgments because the results can be adjusted by harmonizing the group's views when the group's judgments are inconsistent, that is we have $\xi_{ij}^{(p)} \bigcap \xi_{ij}^{(q)} \neq \phi$ for any $p, q \in \{1, \ldots, m\}(i = 1, \ldots, k, j = 1, \ldots, n)$. So we can use the formula (2) to aggregate the group's uncertainness judgments as a positive bounded closed fuzzy numbers $\tilde{a}_{ij}(x)$ on R^+ when the group's judgments satisfy the approximate consistency condition, that is

$$\tilde{a}_{ij}(x) = \frac{1}{m} \sum_{l=1}^{m} \chi_{\xi_{ij}^{(l)}}(x)\ x \in R^+ \tag{4}$$

Table 1 Evaluation table

Projects	Attributes			
	v_1	v_2	…	v_n
u_1	$\tilde{a}_{11}$	$\tilde{a}_{12}$		$\tilde{a}_{1k}$
u_2	$\tilde{a}_{21}$	$\tilde{a}_{22}$		$\tilde{a}_{2k}$
…	…	…	…	…
u_k	$\tilde{a}_{k1}$	$\tilde{a}_{k2}$		$\tilde{a}_{kn}$

and $\tilde{a}_{ij}(x)$ represents a group's judgment or view for ith project under jth attribute. Denoted $\tilde{a}_{ij}$ as $\tilde{a}_{ij}(x)$, the evaluation table of k projects under n attributes can be established, as shown in Table 1.

Based on the above analysis, a fuzzy multiple attributes decision matrix can be constructed as follows:

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{a}_{k1} & \tilde{a}_{k2} & \cdots & \tilde{a}_{kn} \end{bmatrix}. \tag{5}$$

We can take the formula (5) as a fuzzy multiple attributes evaluation and selection model, and then we shall discuss the problem of the fuzzy priorities from the model in next section.

3 A Method of Calculating Fuzzy Priorities

3.1 The Normative Method of Fuzzy Multiple Attributes Decision Matrix

Since the dimension or scale of some attribute values may be different in a fuzzy multiple attributes evaluation and selection model, it is necessary to be normalized before deriving fuzzy priorities from the fuzzy multiple attributes decision matrix. Usually the attribute can be parted tow types: benefit type and cost type. According to the proportion method in traditional multiple attributes decision-making, we have

$$z_{ij} = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}}, \quad \text{(when attribute } v_j \text{ is benefit type)} \tag{6}$$

$$z_{ij} = \frac{1/a_{ij}}{\sum_{i=1}^{n} \left(1/a_{ij}\right)}. \quad \text{(when attribute } v_j \text{ is cost type)} \tag{7}$$

Here a_{ij} represents a attribute value for ith project under jth attribute in non-fuzzy environment. We can obtain formulae (8) and (9) by form (6), (7) and extension principle [11]:

$$\tilde{z}_{ij} = \frac{\tilde{a}_{ij}}{\sum_{i=1}^{n} \tilde{a}_{ij}}, \quad \text{(when attribute } v_j \text{ is benefit type)} \tag{8}$$

$$\tilde{z}_{ij} = \frac{1/\tilde{a}_{ij}}{\sum_{i=1}^{n} \left(1/\tilde{a}_{ij}\right)}, \quad \text{(when attribute } v_j \text{ is cost type).} \tag{9}$$

There $\tilde{a}_{ij}$ a positive bounded closed fuzzy numbers in step. $\forall \lambda > 0$, the λ-level set of $\tilde{a}_{ij}$ is a closed interval. Denoted $(\tilde{a}_{ij})_\lambda = [a_{ij}(\lambda), b_{ij}(\lambda)]$. When attribute v_j is benefit type, we get formulae (10) by Theorem 1:

$$\begin{aligned} \tilde{z}_{ij} &= \bigcup_{0<\lambda\le 1} \lambda \left(\frac{(\tilde{a}_{ij})_\lambda}{\sum_{i=1}^{n} (\tilde{a}_{ij})_\lambda} \right) \\ &= \bigcup_{0<\lambda\le 1} \lambda \left(\frac{[a_{ij}(\lambda), b_{ij}(\lambda)]}{\sum_{i=1}^{n} [a_{ij}(\lambda), b_{ij}(\lambda)]} \right) \\ &= \bigcup_{0<\lambda\le 1} \lambda \left[\frac{a_{ij}(\lambda)}{\sum_{i=1}^{n} b_{ij}(\lambda)}, \frac{b_{ij}(\lambda)}{\sum_{i=1}^{n} a_{ij}(\lambda)} \right]. \end{aligned} \tag{10}$$

And we also get formulae (12) by Theorem 1 when attribute v_j is cost type:

$$\begin{aligned} \tilde{z}_{ij} &= \bigcup_{0<\lambda\le 1} \lambda \left(\frac{1/(\tilde{a}_{ij})_\lambda}{\sum_{i=1}^{n} \left(1/(\tilde{a}_{ij})_\lambda\right)} \right) \\ &= \bigcup_{0<\lambda\le 1} \lambda \left(\frac{\frac{1}{[a_{ij}(\lambda), b_{ij}(\lambda)]}}{\sum_{i=1}^{n} \left(\frac{1}{[a_{ij}(\lambda), b_{ij}(\lambda)]} \right)} \right) \\ &= \bigcup_{0<\lambda\le 1} \lambda \left[\frac{1}{b_{ij}(\lambda) \sum_{i=1}^{n} \frac{1}{a_{ij}(\lambda)}}, \frac{1}{a_{ij}(\lambda) \sum_{i=1}^{n} \frac{1}{b_{ij}(\lambda)}} \right]. \end{aligned} \tag{11}$$

The fuzzy multiple attributes decision matrix $\tilde{A} = (\tilde{a}_{ij})$ in (5) can be normalized based on the above discussions, and we can denoted the normalized fuzzy decision matrix as $\tilde{Z} = (\tilde{z}_{ij})$.

3.2 A Method for Calculating Weitgh of Multiple Attributes

It is difficult to determine weights of attributes in the evaluation system because their relationship is complex. In the analytic hierarchy process presented by Professor Saaty [9], the judgment matrix can be constructed by pairwise comparisons, and the priorities can be derived from the judgment matrix. Suppose that $v_1, \ldots, v_n$ are n attributes, for any $v_i, v_j \in V(V = \{v_1, v_2, \ldots, v_n\})$, let every decision-maker make out his judgment of importance comparison independently according to continuous scale interval (0, 10] mentioned above. Denote $h_{ij}^{(l)} = [c_{ij}^{(l)}, d_{ij}^{(l)}] \quad (l = 1, \ldots, m)$ as a judgment interval by the lth decision-maker given out for comparing the importance of v_i and $v_j (i < j)$. We can suppose $h_{ij}^{(l)} = [c_{ij}^{(l)}, d_{ij}^{(l)}] \, (l = 1, \ldots, m)$ satisfy the approximate consistency condition (1) according to described above. Therefore the fuzzy judgment $\tilde{a}'_{ij}(x)(i < j)$ for the group will be gotten by Theorem 2. Go a step further, we can obtain a fuzzy judgment matrix $\tilde{A}' = (\tilde{a}'_{ij})_{n \times n}$ based on the above discussions, where $\tilde{a}'_{ij}(i < j)$ are positive bounded closed fuzzy numbers in step form on (0, 10]. Since the diagonal elements represent the importance comparison of v_i with itself, we have that the elements in diagonal all equal to 1, namely $\tilde{a}'_{ii} = 1(i = 1, \ldots, n)$, and $\tilde{a}'_{ji} = 1/\tilde{a}'_{ij}$ $(i \neq j)$ by (5) in Theorem 1. By applying fuzzy extension principle [11] to the row addition normalized method in classical analytic hierarchy process [12], we can derive

$$\tilde{w}_i = \frac{\sum_{j=1}^{n} \tilde{a}'_{ij}}{\sum_{l=1}^{n} \sum_{j=1}^{n} \tilde{a}'_{lj}}, (i = 1, \ldots, n), \tag{12}$$

here $\tilde{w}_1, \ldots, \tilde{w}_n$ are the fuzzy wights derived by fuzzy judgment matrix $\tilde{A}'$. In details let $(\tilde{a}'_{ij})_\lambda = [c_{ij}(\lambda), d_{ij}(\lambda)]$ since $\tilde{a}'_{ij}$ are positive bounded closed fuzzy numbers $(i, j = 1, \ldots, n)$. By the operation properties of Theorem 1 about positive bounded closed fuzzy numbers, we easily obtain

$$\begin{aligned} \tilde{w}_i &= \bigcup_{0<\lambda\leq 1} \lambda\Big(\frac{\sum_{j=1}^{n} (\tilde{a}'_{ij})_\lambda}{\sum_{l=1}^{n} \sum_{j=1}^{n} (\tilde{a}'_{lj})_\lambda}\Big) \\ &= \bigcup_{0<\lambda\leq 1} \lambda\Big(\frac{\sum_{j=1}^{n} [c_{ij}(\lambda), d_{ij}(\lambda)]}{\sum_{l=1}^{n} \sum_{j=1}^{n} [c_{lj}(\lambda), d_{lj}(\lambda)]}\Big) \\ &= \bigcup_{0<\lambda\leq 1} \lambda\Big[\frac{\sum_{j=1}^{n} c_{ij}(\lambda)}{\sum_{l=1}^{n} \sum_{j=1}^{n} d_{lj}(\lambda)}, \frac{\sum_{j=1}^{n} d_{ij}(\lambda)}{\sum_{l=1}^{n} \sum_{j=1}^{n} c_{lj}(\lambda)}\Big] \, (i = 1, \ldots, n), \end{aligned} \tag{13}$$

namely,

$$\tilde{w}_i = \bigcup_{0<t\leq 1} \lambda \, [(L_j)_\lambda, (R_j)_\lambda] \, (i = 1, \ldots, n), \tag{14}$$

where

$$(L_j)_\lambda = \frac{\sum_{j=1}^{n} c_{ij}(\lambda)}{\sum_{l=1}^{n} \sum_{j=1}^{n} d_{lj}(\lambda)}, (R_j)_\lambda = \frac{\sum_{j=1}^{n} d_{ij}(\lambda)}{\sum_{l=1}^{n} \sum_{j=1}^{n} c_{lj}(\lambda)}.$$

3.3 Synthesis of Priorities for Projects

In following discussion, we will consider the problem to calculate the priorities $\tilde{r}_i$ of i projects u_i with respect to n different attributes $v_1, \ldots, v_n$. Suppose that the weight of attribute v_j is $\tilde{w}_j (j = 1, \ldots, n)$, and $\tilde{z}_{ij}$ is normalized attribute value of the project $u_i(i = 1, \ldots, k)$ under attribute v_j. Then the overall fuzzy priorities of the project u_i can be computed according to

$$\tilde{r}_i = \sum_{k=1}^{n} \tilde{w}_k \tilde{z}_{ik}, (i = 1, \ldots, k) \tag{15}$$

Let

$$(\tilde{w}_j)_\lambda = [p_j(\lambda),\ q_j(\lambda)], (j = 1, \ldots, n),$$
$$(\tilde{z}_{ij})_\lambda = [\alpha_{ij}(\lambda),\ \beta_{ij}(\lambda)], (i = 1, \ldots, k;\ j = 1, \ldots, n).$$

Then

$$\tilde{r}_i = \bigcup_{0<\lambda\leq 1} \lambda \left[\sum_{j=1}^{m} p_j(\lambda)\alpha_{ij}(\lambda), \sum_{j=1}^{m} q_j(\lambda)\beta_{ij}(\lambda)\right](i = 1, \ldots, k).$$

Similarly, for a multi-level decision problem, we can process the calculation by the same procedure from the top to bottom.

4 An Example on Supplier Selection

It is very important to select suppliers for a manufactory. Many factors are considered on supplier selection, such as credit standing, the price, service, etc. Let us consider the case of the MD&S Electronics Company, which produces electronic components for a computer manufacturer. There are three different suppliers SP, BLE and HUD (In order to discuss conveniently, we can denote SP, BLE, HUD as s_1, s_2, s_3 separately) can be chosen for MD&S. In order to choose the best supplier MD&S committee to make this decision according to following attribute: c_1: credit standing; c_2: the price; c_3: service. In Table 2, the uncertain judgments $\xi_{ij}^{(l)}$ given by the lth decision-maker are pairwise importance comparison about attribute c_i and c_j according to continuous scale interval $(0,\ 10]$.

Table 2 Judgment for importance comparison

	c_1	c_2	c_3
c_1	1	$\xi_{12}^{(1)} = [3.0, 3.9]$	$\xi_{13}^{(1)} = [1.0, 2.0]$
		$\xi_{12}^{(2)} = [3.0, 4.5]$	$\xi_{13}^{(2)} = [1.5, 2.8]$
		$\xi_{12}^{(3)} = [3.5, 4.5]$	$\xi_{13}^{(3)} = [1.8, 3.0]$
c_2		1	$\xi_{23}^{(1)} = [5.0, 5.5]$
			$\xi_{23}^{(2)} = [5.0, 6.0]$
			$\xi_{23}^{(3)} = [4.5, 6.5]$
c_3			1

We can get the fuzzy judgment matrix (see Table 3) of attributes according to Table 2.

We can calculate fuzzy wight $\tilde{w}_j$ of attribute $c_j (j = 1, 2, 3)$ by formula (13) (see Table 4). Suppose the price be gaved by s_1, s_2, s_3 are 1, 0.9 and 0.8 million respectively and the reciprocal of the price is the corresponding priority. Then after normalizing the priorities are 0.29, 0.33, 0.35. The fuzzy decision matrix under the attributes c_i, c_2 , c_3 are as follows (Table 5).

We can derive the fuzzy priorities $\tilde{r}_i$ about suppliers s_i under the attributes c_j by using formula (10) and (14) (see Table 6).

Table 3 Tuzzy judgment matrix table

	λ	c_1	c_2	c_3
c_1	1	1	[3.5, 3.9]	[1.8, 2.0]
	2/3		[3.0, 4.5]	[1.5, 2.8]
	1/3		[3.0, 4.5]	[1.0, 3.0]
c_2	1	[0.256, 0.286]	1	[5.0, 5.5]
	2/3	[0.222, 0.333]		[5.0, 6.0]
	1/3	[0.222, 0.333]		[4.5, 6.5]
c_3	1	[0.50, 0.556]	[0.182, 0.20]	1
	2/3	[0.357, 0.667]	[0.167, 0.20]	
	1/3	[0.333, 1.0]	[0.154, 0.222]	

Table 4 Calculate fuzzy wight

λ	$\tilde{w}_1$	$\tilde{w}_2$	$\tilde{w}_3$
1	[0.51, 0.54]	[0.28, 0.29]	[0.19, 0.20]
2/3	[0.49, 0.58]	[0.26, 0.32]	[0.18, 0.22]
1/3	[0.46, 0.60]	[0.25, 0.33]	[0.17, 0.23]

Table 5 The fuzzy decision matrix

	λ	c_1	c_2	c_3
s_1	1	[4.0, 5.0]	0.29	[2.7, 3.2]
	2/3	[3.7, 5.3]		[2.5, 3.8]
	1/3	[3.5, 5.5]		[2.0, 4.0]
s_2	1	[5.0, 5.5]	0.33	[3.0, 3.5]
	2/3	[4.7, 5.8]		[3.5, 4.0]
	1/3	[3.8, 6.3]		[3.5, 4.5]
s_3	1	[6.0, 6.8]	0.35	[5.8, 6.5]
	2/3	[5.7, 7.1]		[5.5, 6.8]
	1/3	[5.3, 7.5]		[5.0, 7.0]

Table 6 The fuzzy priorities $\tilde{r}_i$

λ	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$
1	[0.32, 0.34]	[0.38, 0.39]	[0.43, 0.50]
2/3	[0.28, 0.36]	[0.36, 0.40]	[0.36, 0.53]
1/3	[0.26, 0.42]	[0.35, 0.43]	[0.33, 0.58]

Obviously, the ranking of fuzzy priorities is $\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_1$, the decision committee should choose the brand u_1 from the results above.

According to demonstration of our example, it is easily to see that the method presented above is rather simple and convenient for operation and can be used for dealing with actual multicriteria decision problems in complex systems.

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References

1. Jahanshahloo, G.R., Hosseinzadeh, F., Lotfi, M.: Izadikhah: extension of the TOPSIS method for decision-making problems with fuzzy data. Appl. Math. Comput. **181**, 1544–1551 (2006)
2. Wang, Y., Parkan, C.: Multiple attribute decision making based on fuzzy preference information on alternatives: ranking and weighting. Fuzzy Sets Syst. **153**, 331–346 (2005)
3. Zhai, X.: A weighting multicriteria group decision-making model in fuzzy environment, 2nd WRI global congress on intelligent systems. GCIS 2010 **1**, 187–190 (2010)
4. Zhai, X., Xu, R.: A multiple criteria decision method based on uncertain judgment. Sixth International Conference on Fuzzy Systems and Knowledge Discovery (2009)
5. Zhai, X.: The aggregation methods of the interval number judgment matrixes and the calculation of the weights in the group decision-making. Syst. Eng. **23**(9), 103–107 (2005)
6. Zhai, X., Zhang, X., Xiao, A.: A new fuzzy multiple attributive decision making method dynamics of continuous. Discrete Impulsive Syst. **13**(Series A), 1429–1432 (2007)
7. Zhai, X., Zhang, X.: A ranking and aggregating method of fuzzy judgment matrix and its application in the water quality plan dynamics of continuous. Discrete Impulsive Syst. **13**, 1177–1181 (2006)
8. Zhai, X., Zhang, X.: A fuzzy group decision method based on uncertain judgments dynamics of continuous. Discrete Impulsive Syst. **13**, 1789–1796 (2006)
9. Saaty, T.L.: The analytical hierarchy process, 3rd edn. RWS Publications, Pittsburgh (2005)
10. Xu, R., Zhai, X.: Extension of the analytic hierarchy process in fuzzy environment. Fuzzy Sets Syst. **3**, 251–257 (1992)
11. Luo, C.: Extension principle and fuzzy number. Fuzzy Math. **4**(3), 109–116 (1984)
12. Wang, L.F.: Computing method of priorities in AHP. Syst. Eng. Theor. Pract. **2**, 31–37 (1987)

Part IV
Operations Research and Management and Its Applications

A Discrete-Time Geo/G/1 Retrial Queue with Balking Customer, Second Optional Service, Bernoulli Vacation and General Retrial Time

Yan Chen, Li Cai and Cai-min Wei

Abstract In this paper, we study a discrete-time Geo/G/1 retrial queue with a balking customer and a second optional service where retrial time follows a general distribution. If an arriving customer finds the server free, he begins the service immediately. If an arriving customer finds the server busy or to be on vacation, then he will consider whether he enters into the system. After a customer accepts his first essential service, he may leave the system or ask for a second service. In this model, we assume that the server, after each service completion, may begin a vacation or wait to serve the following customer. This paper studies the Markov chain underlying this model, we establish the probability generating functions of the orbit size and system size. Finally, some performance measures and some numerical examples are presented.

Keywords Retrial queue · Balking customer · Optional service · Bernoulli vacation

1 Introduction

The retrial queue firstly proposed by Cohen was received extensive attention of scholars in the past of thirty years. Retrial queue has the characteristics as follows: If arriving customers find all servers busy, then they leave the server temporarily and join the retrial group (which is called orbit) to try their luck again after a random time period, otherwise they will accept the service. Note that they are always in orbit no matter accepting the service or not. Queues in which customers are allowed to retry

Y. Chen
Department of Natural Sciences, Shantou Polytechnic, Shantou 515078,
People's Republic of China
e-mail: cylxq331@126.com

C. Wei (✉)
Guangdong Provincial Key Laboratory of Digital Signal and Image Processing Techniques,
Shantou University, Shantou 515063, People's Republic of China
e-mail: cmwei@stu.edu.cn

Y. Chen · L. Cai · C. Wei
Department of Mathematics, Shantou University, Shantou 515063,
People's Republic of China

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have been extensively used to model many problems in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processor unit. Now there is a large number of related literature on retrial queues and a good survey of the results and fundamental methods of retrial queues. For detailed overviews of the main results on this topic can be found in the paper by Artalejo [1, 2], Falin and Templeton [3] and Gómez Corral [4].

In most cases, the standard models in classical queueing theory are systems operating in continuous time. However, in practice, discrete-time queues are more appropriate than their continuous-time counterparts for modelling computer and telecommunication systems in which time is slotted. The more detailed discussion and applications of discrete-time queues can be found in the books by Takagi [5], Woodward [6] and Tian et al. [7].

In lots of queueing systems, the arriving customers won't leave the system until the completion of his service. But in our real life, many arriving customers often don't do so. Consequently, the resulting problem, i.e., a balking customer is pay attention to by more and more learners. Recently, the problem has been extended to discrete-time retrial queues. Aboubl-Hassan and other scholars [8, 9] studied a discrete-time Geom/G/1 retrial queue with balking customers.

In addition, now there are two or more chances for the customers to give in the system. For example, after accepting the first essential service in the system, all customers have an alternative choice: one is that some of them require a second service, another is that some of them leave the the system. Atencia and Moreno [10] studied a discrete-time Geom/G/1 retrial queue with a second optional service. Wang and Zhao [11] considered a discrete-time Geom/G/1 retrial queue with starting failures and a second optional service. We note that the server may stop running temporarily after some time (perhaps some servers need a vacation or some machines need repairing etc.). So Wang [12] studied a discrete-time Geo/G/1 retrial queues with general retrial time and Bernoulli vacation. Chen et al. [13] considered a discrete-time Geo/G/1 retrial queue with Bernoulli vacation and a second optional service. Wei et al. [14] considered a discrete-time Geom/G/1 retrial queue with balking customers and a second optional service. But, there is no work about a general retrial time Geom/G/1 retrial queue with balking customers, a second optional service and Bernoulli vacation. That is our motivation to study this model.

2 Model Description and Markov Chain

In this paper, we consider a discrete-time Geom/G/1 retrial queue with a balking customer, a second optional service, Bernoulli vacation and general retrial time in the early arrival system (EAS, see Hunter for details [5]). The time slots be marked by $0, 1, \ldots, m, \ldots$ in order. Either the customers leave or vacation ends occurs in (m^-, m), the customer arrives or retries in (m, m^+).

It is assumed that customers arrive according to a Bernoulli arrival process with parameter p, and there is no waiting space in front of the server. Therefore, if the customer finds the server busy state or vacation state when he arrives at the system, he either joins the orbit and waits to attempt service again later with probability θ or leaves the system forever with probability $\bar{\theta} = 1 - \theta$; if the customer finds the server idle when he arrives at the system, the arriving customer begins his service immediately. After completion of the first basic service, the customer may leave system with probability $\bar{\alpha} = 1 - \alpha$ or choose a second optional service with probability α. Here, we assume that if two events which customers who complete their basic service decide to receive an extra service and new arriving customers (external or repeated) arrive at the system occur at the same time and find the server idle, then customers of the first case begin their service immediately. Customers of the second case regard the system as busy and consider whether join the orbit to wait to attempt service again. Therefore, we say that the customers who decide to receive second optional service have priority than new arriving customers.

When there is no customer in the orbit, the server may not be coming vacation, only after the server completed the customer's service, he can attempt a vacation, that is, after the customers in the two periods accept their services, the server will wait for the next customer's service by probability q or choose a vacation by probability $\bar{q} = 1 - q$.

Suppose that the tow periods of service time are independent and identically distributed with general distribution $\{s_{1,i}\}_{i=1}^{\infty}$, $\{s_{2,i}\}_{i=1}^{\infty}$, and their probability generating functions $S_1(x) = \sum_{i=1}^{\infty} s_{1,i}x^i$, and $S_2(x) = \sum_{i=1}^{\infty} s_{2,i}x^i$ and their corresponding nth factorial moments $\beta_{1,n}$ and $\beta_{2,n}$, respectively. Vacation time is assumed to follow a general distribution $\{s_{3,i}\}_{i=1}^{\infty}$ with probability generating function $S_3(x) = \sum_{i=1}^{\infty} s_{3,i}x^i$ and nth factorial moments $\beta_{3,n}$. Retrial time also follows a general distribution $\{a_i\}_{i=0}^{\infty}$ with probability generating function $A(x) = \sum_{i=0}^{\infty} a_i x^i$.

The inter-arrival time, the service time, the retrial time and vacation time are all assumed to be mutually independent. In order to avoid trivial cases, it is also supposed $0 < p < 1$, $0 < \theta \leq 1$, $0 \leq \alpha < 1$, $0 < q \leq 1$, let $\rho_1 = p\beta_{1,1}$, $\rho_2 = p\alpha\beta_{2,1}$, $\rho_3 = p\bar{q}\beta_{3,1}$, the intensity of traffic system is $\rho = \rho_1 + \rho_2 + \rho_3$.

At time m^+, the system can be described by the process $Y_m = (C_m, \xi_{i,m}, N_m)$, where

$$C_m = \begin{cases} 0, & \text{the server is idle,} \\ 1, & \text{the server is busy caused by a first essential service,} \\ 2, & \text{the server is busy caused by a second optional service,} \\ 3, & \text{the server is vacation,} \end{cases}$$

and N_m is the number of repeated customers in the orbit. When $C_m = 0$ and $N_m > 0$, $\xi_{0,m}$ represents the remaining retrial time; when $C_m = 1$, $\xi_{1,m}$ represents the remaining service time of the first essential service; when $C_{2,m} = 2$, ξ_m represents the remaining service time of the second optional service; $C_m = 3$, $\xi_{3,m}$ represents the remaining vacation time.

It can be shown that $\{Y_m, m \geq 1\}$ is the Markov chain of our queueing system, whose state space is $\{(0, 0); (0, i, k) : i \geq 1, k \geq 1; (1, i, k) : i \geq 1, k \geq 0; (2, i, k) : i \geq 1, k \geq 0; (3, i, k) : i \geq 1, k \geq 0\}$. Next our goal is to find the stationary distribution of the Markov chain $\{Y_m, m \geq 1\}$ which is defined as follows

$$\pi_{0,0} = \lim_{m\to\infty} P\{C_m = 0, N_m = 0\},$$
$$\pi_{0,i,k} = \lim_{m\to\infty} P\{C_m = 0, \xi_{j,m} = i, N_m = k\}; \quad i \geq 1, k \geq 1,$$
$$\pi_{j,i,k} = \lim_{m\to\infty} P\{C_m = j, \xi_{j,m} = i, N_m = k\}, \quad i \geq 1, k \geq 0, j = 1, 2, 3,$$

where $i \geq 1, k \geq 0, j = 1, 2$.

The one-step transition probabilities $P_{yy'} = P\{Y_{m+1} = y'|Y_m = y\}$ are given by:

(1) When $k = 0$, we have

$$P_{(0,0)(0,0)} = \bar{p},\ \ P_{(1,1,0)(0,0)} = \bar{\alpha}\bar{p}q,\qquad P_{(2,1,0)(0,0)} = \bar{p}q,\ \ P_{(3,1,0)(0,0)} = \bar{p}.$$

(2) When $i \geq 1, k \geq 1$, we have

$$P_{(0,i+1,k)(0,i,k)} = \bar{p},\ P_{(1,1,k)(0,i,k)} = \bar{p}\bar{\alpha}qa_i P_{(2,1,k)(0,i,k)} = \bar{p}qa_i,$$
$$P_{(3,1,k)(0,i,k)} = \bar{p}a_i.$$

(3) When $i \geq 1, k \geq 0$, we have

$$P_{(0,0)(1,i,0)} = ps_{1,i},\ \ (k = 0),\ P_{(0,1,k+1)(1,i,k)} = \bar{p}s_{1,i},$$
$$P_{(0,j,k)(1,i,k)} = ps_{1,i};\ j \geq 1,\ (k \geq 1),\ P_{(1,1,k)(1,i,k)} = \bar{\alpha}pqs_{1,i},$$
$$P_{(1,1,k+1)(1,i,k)} = \bar{\alpha}\bar{p}qa_0s_{1,i},\ P_{(1,i+1,k-1)(1,i,k)} = p\theta,\ (k \geq 1),$$
$$P_{(1,i+1,k)(1,i,k)} = \bar{p} + p\bar{\theta},\ P_{(2,1,k)(1,i,k)} = pqs_{1,i},\ P_{(2,1,k+1)(1,i,k)} = \bar{p}qa_0s_{1,i},$$
$$P_{(3,1,k)(1,i,k)} = ps_{1,i},\ P_{(3,1,k+1)(1,i,k)} = \bar{p}a_0s_{1,i}.$$

(4) When $i \geq 1, k \geq 0$, we have

$$P_{(1,1,k-1)(2,i,k)} = p\alpha\theta s_{2,i},\ (k \geq 1),\ P_{(1,1,k)(2,i,k)} = \alpha(\bar{p} + p\bar{\theta})s_{2,i},$$
$$P_{(2,i+1,k-1)(2,i,k)} = p\theta,\ (k \geq 1),\ P_{(2,i+1,k)(2,i,k)} = \bar{p} + p\bar{\theta}.$$

(5) When $i \geq 1, k \geq 0$, we have

$$P_{(1,1,k-1)(3,i,k)} = \bar{\alpha}\bar{q}p\theta s_{3,i},\ (k \geq 1),\ P_{(1,1,k)(3,i,k)} = \bar{\alpha}\bar{q}(\bar{p} + p\bar{\theta})s_{3,i},$$
$$P_{(2,1,k-1)(3,i,k)} = p\bar{q}\theta s_{3,i},\ (k \geq 1),\ P_{(2,1,k)(3,i,k)} = (\bar{p} + p\bar{\theta})\bar{q}s_{3,i},$$
$$P_{(3,i+1,k-1)(3,i,k)} = p\theta,\ (k \geq 1),\ P_{(3,i+1,k)(3,i,k)} = \bar{p} + p\bar{\theta},$$

where $\delta_{0,k}$ denotes the Kronecker's symbol.

3 Measure Performances of System

In this section, we will derive the measure performances of state system. From above stationary distribution of the Markov chain $\{Y_m, m \geq 1\}$, we obtain the Kolmogorov equations under balance state as follows

$$\pi_{0,0} = \bar{p}\pi_{0,0} + \bar{\alpha}\bar{p}q\pi_{1,1,0} + \bar{p}q\pi_{2,1,0} + \bar{p}\pi_{3,1,0}, \tag{1}$$

$$\pi_{0,i,k} = \bar{p}\pi_{0,i+1,k} + \bar{\alpha}\bar{p}qa_i\pi_{1,1,k} + \bar{p}qa_i\pi_{2,1,k} + \bar{p}a_i\pi_{3,1,k}, \quad i \geq 1, k \geq 1, \tag{2}$$

$$\begin{aligned}\pi_{1,i,k} &= \delta_{0,k}ps_{1,i}\pi_{0,0} + \bar{p}s_{1,i}\pi_{0,1,k+1} + (1-\delta_{0,k})ps_{1,i}\sum_{j=1}^{\infty}\pi_{0,j,k} + \bar{\alpha}pqs_{1,i}\pi_{1,1,k} \\ &+ \bar{\alpha}\bar{p}qa_0s_{1,i}\pi_{1,1,k+1} + (1-\delta_{0,k})p\theta\pi_{1,i+1,k-1} + (\bar{p}+p\bar{\theta})\pi_{1,i+1,k} \\ &+ pqs_{1,i}\pi_{2,1,k} + \bar{p}qa_0s_{1,i}\pi_{2,1,k+1} + ps_{1,i}\pi_{3,1,k} + \bar{p}a_0s_{1,i}\pi_{3,1,k+1}, \quad i \geq 1, k \geq 0,\end{aligned} \tag{3}$$

$$\begin{aligned}\pi_{2,i,k} &= (1-\delta_{0,k})p\alpha\theta s_{2,i}\pi_{1,1,k-1} + \alpha(\bar{p}+p\bar{\theta})s_{2,i}\pi_{1,1,k} \\ &+ (1-\delta_{0,k})p\theta\pi_{2,i+1,k-1} + (\bar{p}+p\bar{\theta})\pi_{2,i+1,k}, \quad i \geq 1, k \geq 0,\end{aligned} \tag{4}$$

$$\begin{aligned}\pi_{3,i,k} &= (1-\delta_{0,k})p\bar{\alpha}\bar{q}\theta s_{3,i}\pi_{1,1,k-1} + \bar{\alpha}(\bar{p}+p\bar{\theta})\bar{q}s_{3,i}\pi_{1,1,k} \\ &+ (1-\delta_{0,k})p\bar{q}\theta s_{3,i}\pi_{2,1,k-1} + (\bar{p}+p\bar{\theta})\bar{q}s_{3,i}\pi_{2,1,k} \\ &+ (1-\delta_{0,k})p\theta\pi_{3,i+1,k-1} + (\bar{p}+p\bar{\theta})\pi_{3,i+1,k}, \quad i \geq 1, k \geq 0.\end{aligned} \tag{5}$$

The normalization condition is

$$\pi_{0,0} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k} + \sum_{j=1}^{3}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{j,i,k} = 1.$$

To solve Eqs. (1)–(5), let us define the following auxiliary function

$$\Phi_{0,i}(z) = \sum_{k=1}^{\infty}\pi_{0,i,k}z^k, \quad \Phi_{j,i}(z) = \sum_{k=0}^{\infty}\pi_{j,i,k}z^k, \quad i \geq 1,\ j = 1, 2, 3.$$

In addition, we introduce the following generating functions

$$\Phi_0(x,z) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k}z^k x^i, \ i \geq 1, \quad \Phi_j(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{j,i,k}z^k x^i, \ i \geq 1,\ j = 1, 2, 3.$$

From Eq. (1), we have

$$p\pi_{0,0} = \bar{\alpha}\bar{p}q\pi_{1,1,0} + \bar{p}q\pi_{2,1,0} + \bar{p}\pi_{3,1,0}. \tag{6}$$

Multiplying both sides of Eqs. (2)–(5) by z^k, and summing over k with Eq. (6), we get

$$\Phi_{0,i}(z) = \bar{p}\Phi_{0,i+1}(z) + \bar{\alpha}\bar{p}qa_i\Phi_{1,1}(z) + \bar{p}qa_i\Phi_{2,1}(z) + \bar{p}a_i\Phi_{3,1}(z) - pa_i\pi_{0,0}. \tag{7}$$

$$\begin{aligned}\Phi_{1,i}(z) &= (\bar{p}+p\bar{\theta}+p\theta z)\Phi_{1,i+1}(z) + \frac{\bar{p}}{z}s_{1,i}\Phi_{0,1}(z) + ps_{1,i}\Phi_0(1,z) \\ &+ \frac{\bar{p}a_0+pz}{z}s_{1,i}[\bar{\alpha}q\Phi_{1,1}(z) + q\Phi_{2,1}(z) + \Phi_{3,1}(z)] + \frac{z-a_0}{z}ps_{1,i}\pi_{0,0}.\end{aligned} \tag{8}$$

$$\Phi_{2,i}(z) = \alpha(\bar{p} + p\bar{\theta} + p\theta z)\Phi_{1,1}(z) + (\bar{p} + p\bar{\theta} + p\theta z)\Phi_{2,i+1}(z). \tag{9}$$

$$\Phi_{3,i}(z) = \bar{\alpha}(\bar{p} + p\bar{\theta} + p\theta z)\bar{q}s_{3,i}\Phi_{1,1}(z) + (\bar{p} + p\bar{\theta} + p\theta z)\bar{q}s_{3,i}\Phi_{2,1}(z)(\bar{p} + p\bar{\theta} + p\theta z)\Phi_{3,i+1}(z). \tag{10}$$

Multiplying both sides of Eqs. (7)–(10) by x^i, and summing over i yields

$$\frac{x-\bar{p}}{x}\Phi_0(x,z) = \bar{\alpha}\bar{p}q\Phi_{1,1}(z)[A(x)-a_0] + \bar{p}q\Phi_{2,1}(z)[A(x)-a_0] + \bar{p}\Phi_{3,1}(z)[A(x)-a_0] - p[A(x)-a_0]\pi_{0,0} - \bar{p}\Phi_{0,1}(z). \tag{11}$$

Let $x = 1$ in Eq. (11). We obtain

$$p\Phi_0(1,z) = \bar{\alpha}\bar{p}q\Phi_{1,1}(z)(1-a_0) + \bar{p}q\Phi_{2,1}(z)(1-a_0) + \bar{p}\Phi_{3,1}(z)(1-a_0) - p(1-a_0)\pi_{0,0} - \bar{p}\Phi_{0,1}(z). \tag{12}$$

Let $R(z) = \bar{p} + p\bar{\theta} + p\theta z$. We have

$$\frac{x-R(z)}{x}\Phi_1(x,z) = \frac{\bar{p}}{z}\Phi_{0,1}(z)S_1(x) + p\Phi_0(1,z)S_1(x) + [\frac{\bar{p}a_0+pz}{z}\bar{\alpha}qS_1(x) - R(z)]\Phi_{1,1}(z) + \frac{\bar{p}a_0+pz}{z}S_1(x)[q\Phi_{2,1}(z) + \Phi_{3,1}(z)] + \frac{z-a_0}{z}pS_1(x)\pi_{0,0}. \tag{13}$$

$$\frac{x-R(z)}{x}\Phi_2(x,z) = \alpha R(z)\Phi_{1,1}(z)S_2(x) - R(z)\Phi_{2,1}(z). \tag{14}$$

$$\frac{x-R(z)}{x}\Phi_3(x,z) = \bar{\alpha}\bar{q}R(z)\Phi_{1,1}(z)S_3(x) + R(z)\bar{q}\Phi_{2,1}(z)S_3(x) - R(z)\Phi_{3,1}(z). \tag{15}$$

Let $x = \bar{p}$ in Eq. (11). We obtain

$$p[A(\bar{p})-a_0]\pi_{0,0} = \bar{\alpha}\bar{p}q\Phi_{1,1}(z)[A(\bar{p})-a_0] + \bar{p}q\Phi_{2,1}(z)[A(\bar{p})-a_0] + \bar{p}\Phi_{3,1}(z)[A(\bar{p})-a_0] - \bar{p}\Phi_{0,1}(z). \tag{16}$$

Let $x = R(z)$. Substituting it into Eqs. (13)–(15) with Eq. (12), we have

$$R(z)\Phi_{2,1}(z) = \alpha R(z)\Phi_{1,1}(z)S_2(R(z)). \tag{17}$$

$$\Phi_{3,1}(z) = \bar{\alpha}\bar{q}\Phi_{1,1}(z)S_3(R(z)) + \bar{q}\Phi_{2,1}(z)S_3(R(z)). \tag{18}$$

From the above equations, we obtain

$$pa_0(1-z)S_1(R(z))\pi_{0,0} = (1-z)\bar{p}S_1(R(z))\Phi_{0,1}(z) + \{[z + (1-z)\bar{p}a_0]S_1(R(z))B(z) - R(z)z\}\Phi_{1,1}(z), \tag{19}$$

where

$$B(z) = \bar{\alpha}q + \alpha q S_2(R(z)) + [\bar{\alpha} + \alpha S_2(R(z))]\bar{q} S_3(R(z)).$$

From Eqs. (16)–(19), we obtain

$$\Phi_{0,1}(z) = \frac{pz[A(\bar{p}) - a_0][R(z) - B(z)S_1(R(z))]}{[z + (1-z)\bar{p}A(\bar{p})]B(z)S_1(R(z)) - R(z)z} \cdot \frac{\pi_{0,0}}{\bar{p}}. \quad (20)$$

$$\Phi_{1,1}(z) = \frac{pA(\bar{p})(1-z)S_1(R(z))}{[z + (1-z)\bar{p}A(\bar{p})]B(z)S_1(R(z)) - R(z)z} \cdot \pi_{0,0}. \quad (21)$$

$$\Phi_{2,1}(z) = \frac{pA(\bar{p})(1-z)\alpha S_1(R(z))S_2(R(z))}{[z + (1-z)\bar{p}A(\bar{p})]B(z)S_1(R(z)) - R(z)z} \cdot \pi_{0,0}. \quad (3.22)$$

$$\Phi_{3,1}(z) = \frac{pA(\bar{p})(1-z)[\bar{\alpha} + \alpha S_2(R(z))]\bar{q} S_1(R(z))S_3(R(z))}{[z + (1-z)\bar{p}A(\bar{p})]B(z)S_1(R(z)) - R(z)z} \cdot \pi_{0,0}. \quad (22)$$

In order to obtain the the probability generating function of steady distribution of Markov chain $\{Y_m, m \geq 1\}$, we will introduce the following Lemma, let

$$\Lambda(z) = [z + (1-z)\bar{p}A(\bar{p})]S_1(R(z))[\bar{\alpha}q + \alpha q S_2(R(z)) + [\bar{\alpha} + \alpha S_2(R(z))]\bar{q} S_3(R(z))] - R(z)z.$$

In order to find the analytical solution for our system, we give two necessary lemmas as follows:

Lemma 1 [14] If $0 \leq x \leq 1$, then inequalities $S_1(x) \leq x$ and $S_2(x) \leq x$ are established.

Lemma 2 [14] If $\rho = \rho_1 + \rho_2 < 1$, then $\Lambda(z) > 0$ holds for $0 \leq z < 1$. Under this condition, the following limits exist

$$\lim_{z \to 1} \frac{1-z}{\Lambda(z)} = \frac{1}{1 - \bar{\theta}p - \theta\rho},$$

where

$$\Lambda(z) = (\bar{p} + pz)S_1(\bar{p} + p\bar{\theta} + p\theta z)[\bar{\alpha} + \alpha S_2(\bar{p} + p\bar{\theta} + p\theta z)] - z(\bar{p} + p\bar{\theta} + p\theta z).$$

So we can obtain the limits exist based on the above inequalities and L'Hospitials's rule, then the proof completed.

Theorem 1 *If $\theta\rho < \theta p + \bar{p}A(\bar{p})$, then the Markov chain $\{Y_m, m \geq 1\}$ has the state distribution, its probability generating functions are follows*

$$\Phi_0(x,z) = \frac{A(x) - A(\bar{p})}{x - \bar{p}} \cdot \frac{pxz[R(z) - B(z)S_1(R(z))]}{\Lambda(z)} \cdot \pi_{0,0}.$$
$$\Phi_1(x,z) = \frac{S_1(x) - S_1(R(z))}{x - R(z)} \cdot \frac{px(1-z)A(\bar{p})R(z)}{\Lambda(z)} \cdot \pi_{0,0}.$$
$$\Phi_2(x,z) = \frac{S_2(x) - S_2(R(z))}{x - R(z)} \cdot \frac{\alpha px(1-z)A(\bar{p})R(z)S_1(R(z))}{\Lambda(z)} \cdot \pi_{0,0}.$$
$$\Phi_3(x,z) = \frac{S_3(x) - S_3(R(z))}{x - R(z)} \cdot \frac{px(1-z)A(\bar{p})\bar{q}[\bar{\alpha} + \alpha S_2(R(z))]R(z)S_1(R(z))}{\Lambda(z)} \cdot \pi_{0,0},$$

where

$$\pi_{0,0} = \frac{\bar{p}A(\bar{p}) + \theta p - \theta\rho}{A(\bar{p})(1 - \bar{\theta}p - \bar{\theta}\rho)}.$$

Proof From Eqs. (20)–(22) and Eqs. (11)–(15), we can give $\Phi_0(x,z)$, $\Phi_1(x,z)$, $\Phi_2(x,z)$, $\Phi_3(x,z)$. By the regular condition $\pi_{0,0}+\Phi_0(1,1)+\Phi_1(1,1)+\Phi_2(1,1)+\Phi_3(1,1) = 1$, we obtain $\pi_{0,0}$. When $\pi_{0,0} > 0$, we have $\theta\rho < \theta p + \bar{p}A(\bar{p})$, therefore $\theta\rho < \theta p + \bar{p}A(\bar{p})$ is an ergodic conditions of Markov chain $\{Y_m, m \geq 1\}$.

From the above conclusion, we can obtain the following:

Corollary 1 *(1) When the server is idle, the probability generating function of the number of customers in the orbit is given by*

$$\pi_{0,0} + \Phi_0(1,z) = \frac{A(\bar{p})[(\bar{p} + pz)B(z)S_1(R(z)) - zR(z)]}{\Lambda(z)} \cdot \pi_{0,0}.$$

(2) When the server is busy, the probability generating function of the number of customers in the orbit is given by

$$\Phi_1(1,z) + \Phi_2(1,z) = \frac{A(\bar{p})R(z)}{\theta} \cdot \frac{1 - S_1(R(z)) + \alpha S_1(R(z))[1 - S_2(R(z))]}{\Lambda(z)} \cdot \pi_{0,0}.$$

(3) When the server is vacation, the probability generating function of the number of customers in the orbit is given by

$$\Phi_3(1,z) = \frac{\bar{q}A(\bar{p})R(z)}{\theta} \cdot \frac{[1 - S_3(R(z))][\bar{\alpha} + \alpha S_2(R(z))]S_1(R(z))}{\Lambda(z)} \cdot \pi_{0,0}.$$

(4) The probability generating function of the number of customers in the orbit is given by

$$\begin{aligned} N(z) &= \pi_{0,0} + \Phi_0(1,z) + \Phi_1(1,z) + \Phi_2(1,z) + \Phi_3(1,z) \\ &= \frac{A(\bar{p})}{\theta} \cdot \frac{(1-\theta z)R(z) - \bar{\theta}B(z)S_1(R(z))}{\Lambda(z)} \cdot \pi_{0,0}. \end{aligned}$$

(5) The probability generating function of the number of customers in the system is given by:

$$L(z) = \pi_{0,0} + \Phi_0(1, z) + z\Phi_1(1, z) + z\Phi_2(1, z) + \Phi_3(1, z)$$
$$= \frac{A(\bar{p})}{\theta} \cdot \frac{\bar{\theta}[R(z)z - B(z)S_1(R(z))] + (1 - z)R(z)[\bar{\alpha} + \alpha S_2(R(z))]S_1(R(z))]}{\Lambda(z)} \cdot \pi_{0,0}.$$

Corollary 2 *(1) The probability which the system is in idle period is by*

$$\pi_{0,0} = \frac{\bar{p}A(\bar{p}) + \theta p - \theta\rho}{A(\bar{p})(1 - \bar{\theta}p + \bar{\theta}\rho)}.$$

(2) The probability which the system is in busy is by

$$\Phi_0(1, 1) + \Phi_1(1, 1) + \Phi_2(1, 1) + \Phi_3(1, 1) = \frac{A(\bar{p})(\theta p - \bar{\theta}\rho) - \theta(p - \rho)}{A(\bar{p})(1 - \bar{\theta}p + \bar{\theta}\rho)}.$$

(3) The probability which the server is in idle period is by

$$\pi_{0,0} + \Phi_0(1, 1) = \frac{1 - \bar{\theta}p - \theta\rho}{1 - \bar{\theta}p + \bar{\theta}\rho}.$$

(4) The probability which the server is in busy period is by

$$\Phi_1(1, 1) + \Phi_2(1, 1) = \frac{\rho_1 + \rho_2}{1 - \bar{\theta}p + \bar{\theta}\rho}.$$

(5) The probability which the server is in vacation period is by

$$\Phi_3(1, 1) = \frac{\rho_3}{1 - \bar{\theta}p + \bar{\theta}\rho}.$$

(6) The number of customers is in the orbit is given by

$$E(N) = \lim_{z\to 1} N'(z)$$
$$= \frac{2\theta(p - \rho)[\theta p + \bar{p}A(\bar{p}) + \bar{\theta}\bar{p}\rho A(\bar{p})] + \theta(\bar{\theta}\bar{p}A(\bar{p}) + \theta)[p^2 E(\rho) + 2F(\rho)]}{2(\bar{p}A(\bar{p}) + \theta p - \theta\rho)(1 - \bar{\theta}p + \bar{\theta}\rho)}.$$

(7) The mean customer number of system is

$$E(L) = E(N) + \theta\frac{\rho_1 + \rho_2}{1 - \bar{\theta}p + \bar{\theta}\rho}.$$

where $E(\rho) = \beta_{1,2} + \alpha\beta_{2,2} + \bar{q}\beta_{3,2}, \quad F(\rho) = \rho_1\rho_2 + \rho_2\rho_3 + \rho_1\rho_3.$

Remark 1 Two special models.
(a) When $\alpha = 0, q = 1$, our model can change to the discrete-time Geo/G/1retrial queue system with a balking customer and general retrial time, we have the probability generating function of the system size as follows:

$$L(z) = \frac{A(\bar{p})}{\theta} \cdot \frac{\theta \bar{p}(1-z)S_1(R(z)) - \theta z(R(z) - S_1(R(z))) + z(1 - S_1(R(z)))R(z)}{\Lambda(z)} \pi_{0,0},$$

where

$$\Lambda(z) = \bar{p}A(\bar{p})(1-z)S_1(R(z)) - z[R(z) - S_1(R(z))],$$

its result coincides with the result in [5].

(b) When $\alpha = 0, \theta = 1$, our model can become a discrete time Geo/G/1 retrial queue system with Bernoulli vacation and general retrial time, so we have

$$\begin{aligned} \Lambda(z) = \bar{p}A(\bar{p})(1 - z)S_1(\bar{p} + pz)[q + \bar{q}S_3(\bar{p} + pz)] \\ - z[(\bar{p} + pz) - S_1(\bar{p} + pz)(q + \bar{q}S_3(\bar{p} + pz))], \end{aligned}$$

therefore, its probability generating function of system size is follows:

$$L(z) = \frac{A(\bar{p})(1-z)(\bar{p} + pz)S_1(\bar{p} + pz)}{\Lambda(z)} \pi_{0,0},$$

where

$$\pi_{0,0} = \frac{p + \bar{p}A(\bar{p}) - \rho_1 - \rho_3}{A(\bar{p})},$$

its result coincides with the result in [6].

(c) When $\alpha = 0, \theta = 1, q = 1$, our model can become a discrete time Geo/G/1retrial queue system with general retrial time, then obtain

$$\Lambda(z) = \bar{p}A(\bar{p})(1-z)S_1(\bar{p} + pz) - z[(\bar{p} + pz) - S_1(\bar{p} + pz)],$$

therefore, its probability generating function of system size is follows:

$$L(z) = \frac{A(\bar{p})(1-z)(\bar{p} + pz)S_1(\bar{p} + pz)}{\Lambda(z)} \pi_{0,0},$$

where

$$\pi_{0,0} = \frac{p + \bar{p}A(\bar{p}) - \rho_1}{A(\bar{p})}.$$

It coincides with the result in [7].

(d) Let $a_0 = 1$ in the result of case (c), we have $A(\bar{p}) = 1, \pi_{0,0} = 1 - \rho$, our model can become a discrete-time $Geo/G/1/\infty$ retrial queue system with random service policy and general retrial time, its probability generating function of system size is follows:

$$L(z) = \frac{(1-\rho_1)(1-z)S_1(\bar{p}+pz)}{S_1(\bar{p}+pz)-z}\pi_{0,0}.$$

It is the probability generating function of a discrete time Geom/G/1 retrial queue which coincides with the result in [8].

Remark 2 Let $a_0 = 1$ (it is $(A(\bar{p}) = 1)$, the the probability generating function of the system size is in our model as follows:

$$L(z) = \frac{\bar{\theta}[R(z)z - B(z)S_1(R(z))] + (1-z)R(z)[\bar{\alpha} + \alpha S_2(R(z))]S_1(R(z))}{\theta(\bar{p}+pz)B(z)S_1(R(z)) - zR(z)} \cdot \frac{1-\bar{\theta}p-\theta\rho}{1-\bar{\theta}p+\bar{\theta}\rho},$$

when the server is idle, the first customer immediately accepts his service in the orbit.

4 Stochastic Decomposition

In this section, the stochastic decomposition of the probability generating function of system size can be expressed as follows

$$L(z) = \frac{\pi_{0,0} + \Phi_0(1,z)}{\pi_{0,0} + \Phi_0(1,1)} \times \frac{\bar{\theta}[R(z)z - B(z)S_1(R(z))] + (1-z)R(z)[\bar{\alpha} + \alpha S_2(R(z))]S_1(R(z))}{\theta(\bar{p}+pz)B(z)S_1(R(z)) - zR(z)} \cdot \frac{1-\bar{\theta}p-\theta\rho}{1-\bar{\theta}p+\bar{\theta}\rho}.$$

Theorem 2 *The total number of customers in the system L can be decomposed as the sum of two independent random variables, one of which is the number L' of customers in the orbit when the server is idle, and the other one is the number L'' in the Geom/G/1/∞ queue with the balking customer, the second optional service and Bernoulli vacation. That is, $L = L' + L''$,*

$$L(z) = L'(z) \times L''(z),$$

where

$$L''(z) = \frac{\bar{\theta}[R(z)z - B(z)S_1(R(z))] + (1-z)R(z)[\bar{\alpha} + \alpha S_2(R(z))]S_1(R(z))}{\theta(\bar{p}+pz)B(z)S_1(R(z)) - zR(z)} \times \frac{1-\bar{\theta}p-\theta\rho}{1-\bar{\theta}p+\bar{\theta}\rho},$$

$$L'(z) = \frac{\pi_{0,0} + \Phi_0(1,z)}{\pi_{0,0} + \Phi_0(1,1)}.$$

We note that $L''(z)$ is the probability generating function of the number of customers in the classical Geo/G/1 queue system, it is consistent with the results of Remark 2.

5 Conclusion

Based on Geom/G/1 retrial queue with the balking customer and the second optional service, this paper considers further the Bernoulli vacation strategy and retrial time following a general distribution. Using the supplementary variable technique to analyze the Markov chain, we derive the probability generating functions of the system state distribution.

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References

1. Artalejo, J.: A classified bibliography of research on retrial queues: progress in 1990–1999. Top **7**(2), 187–211 (1999)
2. Artalejo, J.: Accessible bibliography on retrial queues: progress in 2000–2009. Math. Comput. Modell. **51**, 1071–1081 (2010)
3. Falin, G., Templeton, J.: Retrial Queues. Chapman and Hall, London (1997)
4. GómezC Corral, A.: A bibliographical guide to analysis of retrial queues through matrix analytic techniques. Ann. Oper. Res. **141**, 163–191 (2006)
5. Takagi, H.: Queueing analysis: a foundation of performance evaluation. Discrete-time systems, Vol. 3. North-Holland, Amsterdam (1993)
6. Woodward, M.: Communication and computer networks: modelling with discrete-time queues. IEEE Computer Society Press, Los Alamitos, California (1994)
7. Tian, N., Xu, X., Ma, Z.: Discrete-time queueing theory. Science Press, Beijing (2008)
8. Aboul-Hassan, A., Rabia, S., Kadry, A.: A recursive approach for analyzing a discrete-time retrial queue with balking customers and early arrival scheme. Alexandria Eng. J. **44**, 919–925 (2005)
9. Aboul-Hassan, A., Rabia, S., Taboly, F.: A discrete time Geo/G/1 retrial queue with general retrial times and balking customers. Journal of the Korean Statistical Society **37**, 335–48 (2008)
10. Atencia, I., Morena, P.: A discrete-time retrial queue with optional service. In: Proceeding of the Fifth International Workshop On Retrial Queues, Seoul, vol. 9, pp. 117–121 (2004)
11. Wang, J., Zhao, Q.: A discrete-time Geo/G/1 retrial queue with starting failures and second optional service. Comput. Math. Appl. **53**, 115–127 (2007)
12. Wang, J.: Discrete-time Geo/G/1 retrial queues with general retrial time and Bernoulli vacation. J. Syst. Sci. Complex. **25**(3), 504–513 (2012)
13. Chen, P., Zhu, Y., Chen, Y.: A discrete-time Geo/G/1 retrial queue with Bernoulli vacation and Second optional service. Math. Pract. Theory **41**(3), 121–128 (2011)
14. Wei, C., Cai, L., Wang, J.: A discrete-time Geom/G/1 retrial queue with balking customers and second optional service. OPSEARCH, revision (2014)

Controlled Remote Information Concentration via GHZ-type States

Jin-wei Wang, Lan Shu, Zhi-wen Mo and Zhi-hua Zhang

Abstract In this paper, we propose two schemes of controlled remote information concentration via maximally entangled GHZ-type states and non-maximally entangled GHZ-type states. The necessary measurements and operations are given detailedly.

Keywords Remote information concentration · Qubit · Telecloning · GHZ state

1 Introduction

Quantum entangled states play a pivotal role in quantum information theory. Since Hillery et al. [1] proposed an original scheme of quantum teleportation, in which Alice transmitted an unknown quantum state to Bob via maximally entangled three-qubit GHZ states, different schemes have been investigated with differently entangled states as the channel, such as Bell states [1–3], GHZ states [4–8], W states [9, 10] and other entangled states [11, 12]. The remote quantum state cloning is the important one of quantum tasks.

Although we all know that an unknown quantum state can not be perfectly copied because of the quantum no-cloning theory [13], approximate or probabilistic quantum cloning has attracted much attention to its potential applications in the quantum information science [14–16]. For realizing remote quantum cloning, Muro et al. introduced the concept of telecloning [17, 18], which is the combination of quantum cloning and quantum teleportation. As the reverse process of telecloning, remote quantum information concentration was first introduced by Murao and Vedral [17]. From then on, More and more people devote themselves to studying in this field [19–21]. Recently, Wang and Tang have proposed a scheme for the reversal of ancilla-free phase-covariant telecloning via W states [22]. Based on this study of Wang, Bai

J. Wang (✉) · L. Shu · Z. Mo · Z. Zhang
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chendu 611731, Sichuan, People's Republic of China
e-mail: jinweiwang888@gmail.com

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et al., have proposed some schemes of remote quantum information concentration via GHZ-type states [23]. Inspired by those papers, we try to study this reversal of symmetrical $1 \to 2$ telecloning scheme by using some GHZ-type states.

This paper is organized as follows. We firstly investigate a scheme of controlled quantum information concentration by using GHZ-type states as quantum channel in Sect. 2. In Sect. 3, we have study a controlled quantum information concentration scheme via the partially entangled GHZ-type states. Finally, discussions and conclusions are given in Sect. 4.

2 Controlled Information Concentration via Maximally GHZ-type Entangled States

Let us start to introduce our scheme of controlled information concentration by using maximally entangled GHZ-type states as the channel. We suppose the information of an unknown equatorial state

$$|\gamma\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{ia}|1\rangle)$$

is diluted to a composite system consisting of qubits Alice and Bob, which was held by two separate parties. And assume that the two qubits are in a symmetrical cloning state $|\phi\rangle_{A'B'}$

$$|\gamma\rangle_{A'B'} = \frac{1}{\sqrt{2}}(|00\rangle + e^{ia}|11\rangle)\ here\ a \in [0, 2\pi). \tag{1}$$

Now our goal is to concentrate the quantum information $|\gamma\rangle_{A'B'} = \frac{1}{\sqrt{2}}(|00\rangle + e^{ia}|11\rangle)$ to a single state hold by David, $|\gamma\rangle_D = \frac{1}{\sqrt{2}}(|0\rangle + e^{ia}|1\rangle)$, i.e., $|\gamma\rangle_{A'B'} \longrightarrow |\gamma\rangle_D$. That is to say that we want to recover the equatorial state $|\gamma\rangle$ from the two-qubit state $|\gamma\rangle_{A'B'}$.

For achieving this quantum task, we choice a four qubits maximally entangled GHZ-type entangled state $|\phi_2\rangle$ as the channel in our scheme. This maximally entangled GHZ-type states are expressed as

$$\begin{aligned}|\phi_2\rangle = \frac{1}{\sqrt{8}}(&|0000\rangle + |0010\rangle + |0101\rangle + |0111\rangle \\ &+|1000\rangle - |1010\rangle - |1101\rangle + |1111\rangle),\end{aligned} \tag{2}$$

which can be transformed from the GHZ states

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle). \tag{3}$$

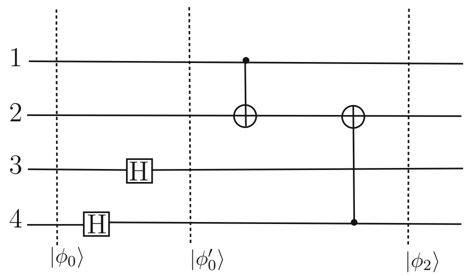

Fig. 1 A quantum circuit to generate the 4-qubit GHZ-type states

As showed in Fig. 1, by implemented two Hadamard operations on the fourth qubit and the fifth qubit in Eq. 3, the GHZ state $|\phi_0\rangle$ is transformed into the following state

$$|\phi_0'\rangle = \frac{1}{\sqrt{8}}(|0000\rangle+|0010\rangle+|0001\rangle+|0011\rangle+|1100\rangle-|1110\rangle-|1101\rangle+|1111\rangle).$$

Through two CNOT gates, we obtain the maximally entangled GHZ state $|\phi_2\rangle$ in the end.

Now the total physical system states, which are hold by Alice, Bob, Charlie and David, can be expressed as follows

$$|\psi\rangle_{A'AB'BCD} = |\gamma\rangle_{A'B'} \otimes |\phi\rangle_{ABCD}$$
$$= \frac{1}{4}(|000000\rangle + |000010\rangle + |000101\rangle + |000111\rangle + |010000\rangle - |010010\rangle - |010101\rangle + |010111\rangle + e^{ia}|101000\rangle + e^{ia}|101010\rangle + e^{ia}|101101\rangle + e^{ia}|101111\rangle + e^{ia}|111000\rangle - e^{ia}|111010\rangle - e^{ia}|111101\rangle + e^{ia}|111111\rangle)_{A'AB'BCD}.$$

where the qubits $A'A$ belong to Alice, qubits $B'B$ to Bob, and qubits C, D to Charlie and David respectively.

Firstly, Alice takes a Bell measurement on the two qubits AA' hold by herself, and broadcast her outcomes to the others on a classic channel. The total system states can be rewritten as

$$|\psi\rangle = \frac{1}{\sqrt{32}}|\Phi^+\rangle_{A'A}(|0000\rangle+|0010\rangle+|0101\rangle+|0111\rangle+e^{ia}|1000\rangle-e^{ia}|1010\rangle-e^{ia}|1101\rangle+e^{ia}|1111\rangle)$$
$$+\frac{1}{\sqrt{32}}|\Phi^-\rangle_{A'A}(|0000\rangle+|0010\rangle+|0101\rangle+|0111\rangle-e^{ia}|1000\rangle+e^{ia}|1010\rangle+e^{ia}|1101\rangle-e^{ia}|1111\rangle)$$
$$+\frac{1}{\sqrt{32}}|\Psi^+\rangle_{A'A}(|0000\rangle-|0010\rangle-|0101\rangle+|0111\rangle+e^{ia}|1000\rangle+e^{ia}|1010\rangle+e^{ia}|1101\rangle+e^{ia}|1111\rangle)$$
$$+\frac{1}{\sqrt{32}}|\Psi^-\rangle_{A'A}(|0000\rangle-|0010\rangle-|0101\rangle+|0111\rangle-e^{ia}|1000\rangle-e^{ia}|1010\rangle-e^{ia}|1101\rangle-e^{ia}|1111\rangle),$$

where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

Meanwhile, Bob also measures her qubits BB' on the basis $\{|\Phi^\pm\rangle, \ |\Psi^\pm\rangle\}$ and broadcasts her measuring results. Thus, after Bob have finished her measurement, the initial states of total physical system can be rewritten as

$|\psi\rangle = \frac{1}{8}|\Phi^{+}\rangle(|\Phi^{+}\rangle(|00\rangle + |10\rangle - e^{ia}|01\rangle + e^{ia}|11\rangle) + |\Phi^{-}\rangle(|00\rangle + |10\rangle + e^{ia}|01\rangle - e^{ia}|11\rangle) + |\Psi^{+}\rangle(|01\rangle + |11\rangle + e^{ia}|00\rangle - e^{ia}|10\rangle) + |\Psi^{-}\rangle(|01\rangle + |11\rangle - e^{ia}|00\rangle + e^{ia}|10\rangle))$
$+ \frac{1}{8}|\Phi^{-}\rangle(|\Phi^{+}\rangle(|00\rangle + |10\rangle + e^{ia}|01\rangle - e^{ia}|11\rangle) + |\Phi^{-}\rangle(|00\rangle + |10\rangle - e^{ia}|01\rangle + e^{ia}|11\rangle) + |\Psi^{+}\rangle(|01\rangle + |11\rangle - e^{ia}|00\rangle + e^{ia}|10\rangle) + |\Psi^{-}\rangle(|01\rangle + |11\rangle + e^{ia}|00\rangle - e^{ia}|10\rangle))$
$+ \frac{1}{8}|\Psi^{+}\rangle(|\Phi^{+}\rangle(|00\rangle - |10\rangle + e^{ia}|01\rangle + e^{ia}|11\rangle) + |\Phi^{-}\rangle(|00\rangle - |10\rangle - e^{ia}|01\rangle - e^{ia}|11\rangle) + |\Psi^{+}\rangle(-|01\rangle + |11\rangle + e^{ia}|00\rangle + e^{ia}|10\rangle) + |\Psi^{-}\rangle(-|01\rangle + |11\rangle - e^{ia}|00\rangle + e^{ia}|10\rangle))$
$+ \frac{1}{8}|\Psi^{-}\rangle(|\Phi^{+}\rangle(|00\rangle - |10\rangle - e^{ia}|01\rangle - e^{ia}|11\rangle) + |\Phi^{-}\rangle(|00\rangle - |10\rangle + e^{ia}|01\rangle + e^{ia}|11\rangle) + |\Psi^{+}\rangle(-|01\rangle + |11\rangle - e^{ia}|00\rangle - e^{ia}|10\rangle) + |\Psi^{-}\rangle(-|01\rangle + |11\rangle + e^{ia}|00\rangle + e^{ia}|10\rangle)).$

In term of Alice and Bob's measuring results, David can recover the right quantum qubit $|\gamma\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{ia}|1\rangle)$ by taking an appropriate Pauli operation on her qubit D with Charlie's help. For example, if Alice and Bob get the results $|\Phi^{-}\rangle|\Psi^{+}\rangle$ after taking their measurement respectively, the quantum states hold by Charlie and David will be

$$|\psi\rangle_{CD} = \frac{1}{2}(|00\rangle - |10\rangle - e^{ia}|01\rangle - e^{ia}|11\rangle).$$

If Charlie do not allow David to get this quantum information coming from Alice and Bob, she do nothing about her qubit C. Otherwise, she need to take a classical measurement on her qubit C with the basis {0, 1} and broadcast her outcome to the others via classical channel. Now the quantum states hold in Charlie and David can be rewritten as follows

$$|\psi\rangle_{CD} = \frac{1}{2}(|0\rangle(|0\rangle - e^{ia}|1\rangle) - |1\rangle(|0\rangle + e^{ia}|1\rangle)).$$

According to all of the measuring results informed by Alice, Bob and Charlie, David can recover the quantum information after taking some Pauli operations, which can be expressed as $\sigma_0 = |0\rangle\langle 0| + |1\rangle\langle 1|,\ \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|,\ \sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|,\ \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, on the qubit D hold by herself. Let back to the above instance. If the measuring results of Charlie is $|0\rangle$, David can take the operation σ_x on the qubit D to get the qubit $|\gamma\rangle$. On the other hand, if the result of Charlie' measuring is $|1\rangle$, David can also get the qubit $|\gamma\rangle$ by taking the Pauli operation σ_0 on the qubit D.

Moreover, when Alice and Bob get any other measuring results, David can also effectively recover the state $|\gamma\rangle$ after taking some appropriate Pauli operations on the qubit D hold by herself with the agreement of Charlie.

3 Controlled Information Concentration Via Partially Entangled GHZ-type States

In this section, we study a quantum information concentration scheme via a partially entangled GHZ-type sates channel. Just as above scheme, we also suppose that the symmetrical two-qubit states hold by Alice and Bob are $|\gamma\rangle_{A'B'}$ (Eq. 1), and our goal

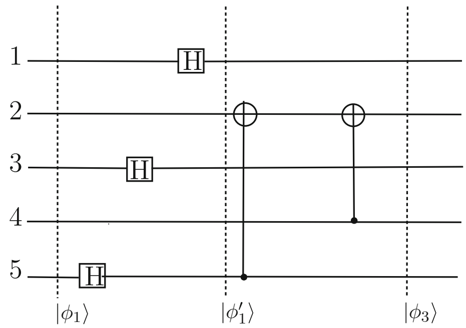

Fig. 2 A quantum circuit to generate the 5-qubit GHZ-type states

is to concentrate this quantum information to a single state $|\gamma\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{ia}|1\rangle)$. However, in this scheme, we replace the maximally entangled GHZ-type states $|\phi_2\rangle$, which are work as the quantum channel in the first scheme, with a partially entangled GHZ-type states $|\phi_3\rangle$. The five-qubit partially entangled GHZ-type states $|\phi_3\rangle$ are expressed as

$$\begin{aligned}|\phi_3\rangle = \frac{1}{\sqrt{8}}(&sin\theta|00000\rangle + sin\theta|00100\rangle + sin\theta|10000\rangle + sin\theta|10100\rangle\\ &+ sin\theta|01001\rangle + sin\theta|01101\rangle + sin\theta|11001\rangle + sin\theta|11101\rangle\\ &+ cos\theta|00010\rangle - cos\theta|00110\rangle - cos\theta|10010\rangle + cos\theta|10110\rangle\\ &cos\theta|01011\rangle - cos\theta|01111\rangle + cos\theta|11011\rangle - cos\theta|11111\rangle). \end{aligned} \tag{4}$$

Similar to the quantum states $|\phi_2\rangle$, this partially entangled GHZ-type states can also be transformed by a partially entangled GHZ state

$$|\phi_1\rangle = sin\theta|00000\rangle + cos\theta|11111\rangle,\ \theta \in [0, 2\pi). \tag{5}$$

As showed in Fig. 2, by implemented three Hadamard operations on the first qubit, the third qubit and the fifth qubit in Eq. 5, the five-qubit GHZ-type states $|\phi_1\rangle$ is transformed into the following states

$$\begin{aligned}|\phi_1'\rangle = \frac{1}{\sqrt{8}}(&sin\theta|00000\rangle + sin\theta|00100\rangle + sin\theta|10000\rangle + sin\theta|10100\rangle + sin\theta|00001\rangle\\ &+ sin\theta|00101\rangle + sin\theta|10001\rangle + sin\theta|10101\rangle + cos\theta|01010\rangle - cos\theta|01110\rangle\\ &- cos\theta|11010\rangle + cos\theta|11110\rangle - cos\theta|01011\rangle + cos\theta|01111\rangle + cos\theta|11011\rangle\\ &- cos\theta|11111\rangle), \end{aligned} \tag{6}$$

Performing two C-NOT gates onto the second qubit of the $|\phi_1\rangle'$ in Eq. 6, we obtain the partially entangled GHZ-type states $|\phi_3\rangle$ in the end. So that the total system states, which are hold by Alice, Bob, Charlie and David, can be expressed as the follows

$$\begin{aligned}
&|\Psi\rangle_{A'AB'BC'CD} = |\gamma\rangle_{A'B'} \otimes |\phi_3\rangle_{ABC'CD} \\
&= \tfrac{1}{4}(sin\theta|0000000\rangle + sin\theta|0000100\rangle + sin\theta|0100000\rangle + sin\theta|0100100\rangle \\
&\quad + sin\theta|0001001\rangle + sin\theta|0001101\rangle + sin\theta|0101001\rangle + sin\theta|0101101\rangle \\
&\quad + cos\theta|0000010\rangle - cos\theta|0000110\rangle - cos\theta|0100010\rangle + cos\theta|0100110\rangle \\
&\quad - cos\theta|0001011\rangle + cos\theta|0001111\rangle + cos\theta|0101011\rangle - cos\theta|0101111\rangle \\
&\quad + e^{ia}sin\theta|1010100\rangle + e^{ia}sin\theta|1010000\rangle + e^{ia}sin\theta|1110000\rangle + e^{ia}sin\theta|1110100\rangle \\
&\quad + e^{ia}sin\theta|1011001\rangle + e^{ia}sin\theta|1011101\rangle + e^{ia}sin\theta|1111001\rangle + e^{ia}sin\theta|1111101\rangle \\
&\quad + e^{ia}cos\theta|1010010\rangle - e^{ia}cos\theta|1010110\rangle - e^{ia}cos\theta|1110010\rangle + e^{ia}cos\theta|1110110\rangle \\
&\quad - e^{ia}cos\theta|1011011\rangle + e^{ia}cos\theta|1011111\rangle + e^{ia}cos\theta|1111011\rangle - e^{ia}cos\theta|1111111\rangle),
\end{aligned}$$

where the qubits $A'A$ belong to Alice, qubits $B'B$ to Bob, qubits $C'C$ to Charlie and qubit D to David respectively. Now the total quantum system states,which are hold by Alice, Bob, Charlie and David, can be rewritten as

$$\begin{aligned}
|\psi\rangle = &\tfrac{1}{8}|\Phi^+\rangle_{A'A}(|\Phi^+\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle + cos\theta|010\rangle - cos\theta|110\rangle \\
&\quad + e^{ia}sin\theta|001\rangle + e^{ia}sin\theta|101\rangle + e^{ia}cos\theta|011\rangle - e^{ia}cos\theta|111\rangle) \\
&\quad + |\Phi^-\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle + cos\theta|010\rangle - cos\theta|110\rangle \\
&\quad - e^{ia}sin\theta|001\rangle - e^{ia}sin\theta|101\rangle - e^{ia}cos\theta|011\rangle + e^{ia}cos\theta|111\rangle) \\
&\quad + |\Psi^+\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|011\rangle + cos\theta|111\rangle \\
&\quad + e^{ia}sin\theta|000\rangle + e^{ia}sin\theta|100\rangle - e^{ia}cos\theta|010\rangle + e^{ia}cos\theta|110\rangle) \\
&\quad + |\Psi^-\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|011\rangle + cos\theta|111\rangle \\
&\quad - e^{ia}sin\theta|000\rangle - e^{ia}sin\theta|100\rangle + e^{ia}cos\theta|010\rangle - e^{ia}cos\theta|110\rangle)) \\
&+ \tfrac{1}{8}|\Phi^-\rangle_{A'A}(|\Phi^+\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle + cos\theta|010\rangle - cos\theta|110\rangle \\
&\quad - e^{ia}sin\theta|001\rangle - e^{ia}sin\theta|101\rangle - e^{ia}cos\theta|011\rangle + e^{ia}cos\theta|111\rangle) \\
&\quad + |\Phi^-\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle + cos\theta|010\rangle - cos\theta|110\rangle \\
&\quad + ve^{ia}sin\theta|001\rangle + e^{ia}sin\theta|101\rangle + e^{ia}cos\theta|011\rangle - e^{ia}cos\theta|111\rangle) \\
&\quad + |\Psi^+\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|011\rangle + cos\theta|111\rangle \\
&\quad - e^{ia}sin\theta|000\rangle - e^{ia}sin\theta|100\rangle + e^{ia}cos\theta|010\rangle - e^{ia}cos\theta|110\rangle) \\
&\quad + |\Psi^-\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|011\rangle + cos\theta|111\rangle \\
&\quad + e^{ia}sin\theta|000\rangle + e^{ia}sin\theta|100\rangle - e^{ia}cos\theta|010\rangle + e^{ia}cos\theta|110\rangle)) \\
&+ \tfrac{1}{8}|\Psi^+\rangle_{A'A}(|\Phi^+\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle - cos\theta|010\rangle + cos\theta|110\rangle \\
&\quad + e^{ia}sin\theta|001\rangle + e^{ia}sin\theta|101\rangle - e^{ia}cos\theta|011\rangle + e^{ia}cos\theta|111\rangle) \\
&\quad + |\Phi^-\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle - cos\theta|010\rangle + cos\theta|110\rangle \\
&\quad - e^{ia}sin\theta|001\rangle - e^{ia}sin\theta|101\rangle + e^{ia}cos\theta|011\rangle - e^{ia}cos\theta|111\rangle) \\
&\quad + |\Psi^+\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|111\rangle + cos\theta|011\rangle \\
&\quad + e^{ia}sin\theta|000\rangle + e^{ia}sin\theta|100\rangle + e^{ia}cos\theta|010\rangle - e^{ia}cos\theta|110\rangle) \\
&\quad + |\Psi^-\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|111\rangle + cos\theta|011\rangle \\
&\quad - e^{ia}sin\theta|000\rangle - e^{ia}sin\theta|100\rangle - e^{ia}cos\theta|010\rangle + e^{ia}cos\theta|110\rangle)) \\
&+ \tfrac{1}{8}|\Psi^-\rangle_{A'A}(|\Phi^+\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle - cos\theta|010\rangle + cos\theta|110\rangle \\
&\quad - e^{ia}sin\theta|001\rangle - e^{ia}sin\theta|101\rangle + e^{ia}cos\theta|011\rangle - e^{ia}cos\theta|111\rangle) \\
&\quad + |\Phi^-\rangle_{B'B}(sin\theta|000\rangle + sin\theta|100\rangle - cos\theta|010\rangle + cos\theta|110\rangle \\
&\quad + e^{ia}sin\theta|001\rangle + e^{ia}sin\theta|101\rangle - e^{ia}cos\theta|011\rangle + e^{ia}cos\theta|111\rangle) \\
&\quad + |\Psi^+\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|111\rangle + cos\theta|011\rangle \\
&\quad - e^{ia}sin\theta|000\rangle - e^{ia}sin\theta|100\rangle - e^{ia}cos\theta|010\rangle + e^{ia}cos\theta|110\rangle)
\end{aligned}$$

$$+|\Psi^-\rangle_{B'B}(sin\theta|001\rangle + sin\theta|101\rangle - cos\theta|111\rangle + cos\theta|011\rangle$$
$$+e^{ia}sin\theta|000\rangle + e^{ia}sin\theta|100\rangle + e^{ia}cos\theta|010\rangle - e^{ia}cos\theta|110\rangle)).$$

Just as the first scheme, Alice takes a Bell measurement on her qubits $A'A$ and Bob takes a bell measurement on her qubits $B'B$ respectively. Then, they all broadcast their measuring results on a classical channel. Then Charlie, as the controller, will do nothing on her qubits if she can not allow David to recover the quantum information coming from Alice and Bob. Otherwise she must take a classical measurement on the qubits $C'C$ hold by herself under the basis {00, 01, 10, 11} one by one and inform David of her measuring results. Taking some appropriate Pauli operations on the qubit D, David can get the qubit $|\gamma\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{ia}|1\rangle)$. For example, if the measuring results of Alice, Bob and Charlie are $|\Psi^-\rangle_{A'A}|\Phi^+\rangle_{B'B}|01\rangle_{C'C}$, the qubit states hold in David will be

$$|\psi\rangle_D = \frac{1}{\sqrt{2}}(cos\theta|1\rangle - e^{ia}cos\theta|0\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - e^{ia}|0\rangle).$$

Then, by taking two Pauli operations $\sigma_z\sigma_x$ on the qubit D, David can recover the qubit$|\gamma\rangle$ coming from Alice and Bob. That is $|\gamma\rangle = \sigma_z\sigma_x|\psi\rangle_D$. Furthermore, if Alice, Bob and Charlie get other result when they have taken their measured on the qubit hold by themselves, David can recover the quantum information $|\gamma\rangle$ by taking some appropriate Pauli operations.

4 Conclusion

In this work, we firstly introduce a optimal remote quantum information concentration scheme by using maximally entangled GHZ-type states, and then extend the scheme to the case with a five qubits partially entangled GHZ-type state as the channel. Contrast to other schemes [23], All of the two schemes are implemented deterministically and need no auxiliary qubit in the last step.

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References

1. Hillery, M., Bužuk, V., Berthiaume, A.: Quantum secre sharing. Phys. Rev. A **59**(3) 1829–1834 (1999)
2. Lance, A.M., Symul, T., Bowen, W.P., Sanders, B.C., Sanders, B.C., Lam, P.K.: Tripartite quantum state sharing. Phys. Rev. Lett. **92**, 177903 (2004)
3. Deng, F.G., Li, X., Li, H., Zhou, P., Zhou, H.Y.: Multiparty quantum-state sharing of an arbitrary two-particle state with Einstein-Podolsky-Rosen pairs. Phys. Rev. A **72**, 044301 (2005)
4. Li, X.H., Zhou, P., Li, C.Y., Zhou, H.Y., Deng, F.G.: Efficient symmetric multiparty quantum state sharing of an arbitrary m-qubit state. J. Phys. B **39**(8) 1975–1984 (2006)
5. Wang, M.Y., Yan, F.L.: Probabilistic chain teleportation of a qutrit-state. Commun. Theor. Phys. **54**, 263–268 (2010)
6. Karlsson, A., Koashi, M., Imoto, N.: Quantum entanglement for secret sharing and secret splitting. Phys. Rev. A **59**(1), 162–168 (1999)

7. Tsai, C.W., Tzonelih, H.: Teleportation of a pure EPR state via GHZ-like state. Int. J. Theor. Phys. **49**, 1969–1975 (2010)
8. Hu, M.L.: Disentanglement, Bell-nonlocality violation and teleportation capacity of the decaying tripartite states. Ann. Phys. **327**(9), 2332–2342 (2012)
9. Zheng, S.B.: Splitting quantum information via W states. Phys. Rev. A **74**, 054303 (2006)
10. Zhang, Z.H., Shu, L., MO, Z.W.: Quantum teleportation and superdense coding through the composite W-Bell channel. Quantum Inf. Process. **12**, 1957–1967 (2013)
11. Luo, M.X., Deng, Y.: Quantum splitting an arbitrary three-qubit state with x-state. Quantum Inf. Process. **12**(2), 773–784 (2013)
12. Zhang, Y.L., Wang, Y.N., Xiao, X.R., Jing, L., Mu, L.Z., Korepin, V.E., Fan, H.: Quantum network teleportation for quantum information distribution and concentration. Phys. Rev. A **87**, 022302 (2013)
13. Wootters, W.K., Zurek, W.H.: Asingle quantum cannot be cloned. Nature **299**, 802–803 (1982)
14. Bužuk, V., Hillery, M: Quantum copying: beyond the no-cloning theorem. Phys. Rev. A **54**, 1844 (1996)
15. Galvao, E.F., Hardy, L.: Cloning and quantum computation. Phys. Rev. A **62**, 022301 (2000)
16. Ricci, M., Sciarrino, F., Cerf, N.J., Filip, R., Fiurasek, De Martini, F.: Separating the classical and quantum information via quantum cloning. Phys. Rev. Lett. **95**, 090504 (2005)
17. Murao, M., Plenio, M.B., Vedral, V.: Quantum-information distribution via entanglement. Phys. Rev. A **61**(3), 032311 (2000)
18. Murao, M., Vedral, V.: Remote information concentration using a boud entangled state. Phys. Rev. Lett. **86**(2), 353–355 (2001)
19. Yu, Y.F., Feng, J., Zhan, M.S.: Remote information concentration by a Greenberger-Horne-Zeilinger state and by a bound Entangled state. Phys. Rev. A **68**, 024303 (2003)
20. Peng, J.Y., Bai, M.Q., Mo, Z.W.: Remote information concentration via W stat:reverse of ancilla-free phase-covariant telecloning. Quantum Inf. Process. **12**, 3511–3525 (2013)
21. Peng, J.Y., Bai, M.Q., Mo, Z.W.: Remote information concentration by W state. Int. J. Modern Phys. B **27**(26) (2013)
22. Wang, X.W., Tang, S.Q.: Qemote quantum-information concentration: reversal of ancilla-free phase-covariant telecloning. Open J. Microphysics **3**,18–21 (2013)
23. Bai, M.Q., Peng, J.Y., Mo, Z.W.: Three schemes of remote information concentration based on ancilla-free phase-covariant telecloning. Quantum Inf. Process. **13**(5) (2014)

A New Precipitable Water Vapor STARMA Model Based on Newton's Method

Zhihui Li and Zhihong Miao

Abstract The STARMA (space-time autoregressive moving average) model is introduced in 2002–2008 monthly PWV fitting and forecast. To enhance the model's ability of dealing with satellite remote sensing raster data, this article extends the parameter estimation process in the STARMA model by augmenting the Newton's method to high dimensions for solving system of nonlinear equations, and the process of parameter estimation is elaborated. This operation is validated by real data experiment results. The confirmation results of this method reveals that the STARMA model has good accuracy in both fitting and predicting.

Keywords Space-time series modeling · STARMA model · Newton's method · Precipitable water vapour (PWV)

1 Introduction

Everything is in constant movement and development, and during the process of exploring the regularity of a variety of real world phenomena, we often find that a lot of observations and measurements have closely relationship with last moment. And there are a lot of observations change synchronized at neighboring regions. For example, the wheat harvest of a certain country during this year can be affected by the harvest during the previous year at the same place and at the neighboring country.

Therefore, STARMA (Space-Time Autoregressive Moving Average) model has attracted much attention [1–15]. STARMA model has three STAR models characterized by linear dependencies in both space and time.

Z. Li (✉)
Key Laboratory of Firefighting and Rescue Technology, MPS,
Langfang Hebei 065000, China
e-mail: lizhihuifanqie@163.com

Z. Miao
Fire Engineering Department of Chinese People's Armed Police Forces Academy,
Langfang 065000, China

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Precipitable water vapour (PWV) [16] is defined as the equivalent to the height of liquid precipitation that would result from condensing all the water vapour in a specified column of atmosphere from the surface to the top of the atmosphere, which is a critical indicator of atmospheric water vapor and water resource of the atmosphere. This paper mainly focuses on parameter estimation process of STARMA model for PWV spatial and temporal trend prediction.

Crespo [17] tried to give a short-term forecast for satellite images with STAR model firstly. But he didn't refer to how to solve the parameter estimation nonlinear in STARMA model. Hence, the Newton's method is proposed in this paper in order to apply STARMA model in satellite images sequence prediction.

2 STARMA Model

STARMA model is the extension of ARMA model into spatial-temporal domain. Suppose $Z_{i,t}$ is the observation of the space-time random variable at site i $(i = 1, 2, \ldots, N)$ and time t. Let Z_t be a multivariate time series of N components. $\mathrm{W}^{(l)}$ has elements $\mathrm{w}^{(l)}$ (i, j) weighting sequence of N $\times$ N spatial contiguity matrices. With elements w $(i, j) > 0$ if locations i and j are contiguous at spatial lag l, and w $(i, j) = 0$ elsewhere.

Then the general form of the STARMA model is defined as [18]:

$$Z_t = \sum_{k=1}^{p}\sum_{l=0}^{r}\phi_{kl}W^{(l)}Z_{t-k} - \sum_{k=1}^{q}\sum_{l=0}^{s}\theta_{kl}W^{(l)}\varepsilon_{t-k} + \varepsilon_t, \tag{1}$$

where p and r are the maximum autoregressive temporal and spatial orders, respectively; q and s are the maximum moving average temporal and spatial orders, respectively. ϕ_{kl} and θ_{kl} are parameters of the autoregressive and moving average at temporal lag $k$and spatial lag l, respectively, and $\varepsilon_t = [\varepsilon_{1,t}, \varepsilon_{2,t}, \ldots, \varepsilon_{N,t}]'$ is normally independently random noise vector at time t.

The STARMA model has two special subclasses. When models that contain no autoregressive term ($p = 0$) are considered as Space Time Moving Average (STMA) model. When $q = 0$ the class is considered as Space Time Auto Regressive (STAR) model.

3 Parameter Estimation Based on Extensive Newton's Iteration Method

Basically, the estimation process of parameters ϕ_{kl} and θ_{kl} from the STARMA model can be conducted by minimizing the residual sum of squares, also known as maximum likelihood method:

$$s(\hat{\beta}) = \sum_{t=1}^{T} (Z_t - \sum_{k=1}^{p} \sum_{l=0}^{\lambda_k} \hat{\phi}_{kl} W^{(l)} Z_{t-k} + \sum_{k=1}^{q} \sum_{l=0}^{m_k} \hat{\theta}_{kl} W^{(l)} \varepsilon_{t-k})^2, \tag{2}$$

where T is the total number of observations in time, estimation of the parameters $\hat{\beta} = [\hat{\phi}_{10}, \hat{\phi}_{11}, \ldots, \hat{\phi}_{p\lambda}, \hat{\theta}_{10}, \hat{\theta}_{11}, \ldots, \hat{\theta}_{qm}]'$ are what we want to be determined. This is a quadratic nonlinear optimization problem.

For linear problem, maximum likelihood estimation can be solved with linear estimation method. For example, in the estimation process of STAR model, all of the observations can be simplified into a linear model $Z = X\Phi + E$. Then the best linear unbiased estimation can be calculated by $\hat{\Phi} = (X'X)^{-1}X'Z$, where $Z = [Z_1, Z_2, \ldots, Z_T]'$ represents the observations, model parameters $\Phi = [\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}]'$, random error terms $E = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T]'$, and X is the max matrix.

However, the STMA model and STARMA model are nonlinear, so the above-mentioned linear estimation method may not be useful. The error terms ε_t is white noise, therefore its average comes to zero and variance matrix comes to $\sigma^2 I$. The likelihood function for nonlinear spatial-temporal model can be acquired from standard normal joint distribution of the error terms [19]:

$$\begin{aligned} L(\varepsilon|\phi, \theta, \sigma^2) &= (2\pi)^{-TN/2} |\sigma^2|^{-TN/2} \exp(-\tfrac{1}{2\sigma^2}\varepsilon' I \varepsilon) \\ &= (2\pi)^{-TN/2} |\sigma^2|^{-TN/2} \exp(-\tfrac{s(\hat{\beta})}{2\sigma^2}). \end{aligned} \tag{3}$$

Its logarithmic form of above equation:

$$\ln L = -(TN/2)\ln(2\pi) - (TN/2)\ln(\sigma^2) - (1/2\sigma^2)s(\hat{\beta}), \tag{4}$$

where $s(\hat{\beta})$ is the squared sum of error terms. If we want to get the maximum of log-likelihood function, that is to get the minimum of $s(\hat{\beta})$. Then the parameters ϕ, θ and σ^2 can be obtained by computing the partial derivative vector of above equation ($\partial L/\partial\phi = 0, \partial L/\partial\theta = 0, \partial L/\partial\sigma^2 = 0$).

Actually the likelihood equations constitute by $p + q + 1$ transcendental equations [20], so we can only get analytical solutions of these equations. But the analytical solutions is obviously very difficult in spatial-temporal model, especially when we predict the spatial-temporal trend of image or picture. Therefore, in this article we consider expanding the Newton iteration method into high dimensions in order to obtain the approximate solutions.

When exact solutions become very difficult, or sometimes impossible, Newton's method is a very simple and effective method to solving equations with approximate solutions.

The spatial-temporal extension of Newton's method has four steps.

(1) First of all, according to the linear parameter estimation method, we get the partial correlation coefficient $\hat{\eta}_{kl}$ from high order STAR model. The process can solve with Yule-Walker function,

$$Z_t^* = \sum_{k=1}^{u} \sum_{l=0}^{v} \hat{\eta}_{kl} W^{(l)} Z_{t-k}^* + \varepsilon_t^*, \tag{5}$$

where u is the maximum temporal lag, v is the maximum spatial, and they are all higher than defined from spatial-temporal autocorrelation function (STACF) and spatial-temporal partial correlation function (STPACF). Z_t^* means the sample at time t.

(2) After we get the parameter $\hat{\eta}_{kl}$, we can use the following equation to estimate ε_t^*

$$\varepsilon_t^* = Z_t^* - \sum_{k=1}^{u} \sum_{l=0}^{v} \hat{\eta}_{kl} W^{(l)} Z_{t-k}^*. \tag{6}$$

(3) For the time being, we introduce linear estimation method to obtain $\hat{\alpha}$ which represents the initial value of $\hat{\beta}$:

$$s(\hat{\alpha}) = \sum_{t=m+1}^{T} (Z_t - \sum_{k=1}^{p} \sum_{l=0}^{\lambda_k} \hat{\phi}_{kl} W^{(l)} Z_{t-k} + \sum_{k=1}^{q} \sum_{l=0}^{m_k} \hat{\theta}_{kl} W^{(l)} \varepsilon_{t-k})^2, \tag{7}$$

then:

$$\hat{\alpha} = (X'X)^{-1} X' \Omega, \tag{8}$$

where $\Omega = [Z_{m+1}, Z_{m+2}, \ldots, Z_T]'$ the value of m is a very important factor in deciding whether matrix X is reversible, and $t = m$+1, ..., T. According to the least square method, X can be expressed as:

$$X = \left[\begin{array}{ccccccc} Z_m & W^{(1)} Z_m & \cdots & W^{(r)} Z_m & Z_{m-1} & \cdots & W^{(r)} Z_{m-1} \\ Z_{m+1} & W^{(1)} Z_{m+1} & \cdots & W^{(r)} Z_{m+1} & Z_m & \cdots & W^{(r)} Z_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Z_{T-1} & W^{(1)} Z_{T-1} & \cdots & W^{(r)} Z_{T-1} & Z_{T-2} & \cdots & W^{(r)} Z_{T-2} \end{array} \right.$$

$$\vdots\vdots\vdots$$

$$\vdots\vdots\vdots$$

$$\vdots\vdots\vdots$$

$$\left. \begin{array}{cccccccc} \hat{\varepsilon}_m & W^{(1)} \hat{\varepsilon}_m & \cdots & W^{(r)} \hat{\varepsilon}_m & \hat{\varepsilon}_{m-1} & \cdots & W^{(r)} \hat{\varepsilon}_{m-1} & \cdots \\ \hat{\varepsilon}_{m+1} & W^{(1)} \hat{\varepsilon}_{m+1} & \cdots & W^{(r)} \hat{\varepsilon}_{m+1} & \hat{\varepsilon}_m & \cdots & W^{(r)} \hat{\varepsilon}_m & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{\varepsilon}_{T-1} & W^{(1)} \hat{\varepsilon}_{T-1} & \cdots & W^{(r)} \hat{\varepsilon}_{T-1} & \hat{\varepsilon}_{T-2} & \cdots & W^{(r)} \hat{\varepsilon}_{T-2} & \cdots \end{array} \right],$$

(4) Based on Newton's iterative method, $\hat{\beta}$ can be approximately reach from $\hat{\alpha}$.

According to Newton's method, Gradient operator $\nabla f(x^{(k)})$ and Laplace operator $\nabla^2 f(x^{(k)})$ should be calculated respectively. That means we need to calculate $\nabla s(x)$ and $\nabla^2 s(x)$ in STARMA model.

Suppose $F_t(x) = Z_t - \sum_{k=1}^{p}\sum_{l=0}^{\lambda_k}\hat{\phi}_{kl}W^{(l)}Z_{t-k} + \sum_{k=1}^{q}\sum_{l=0}^{m_k}\hat{\theta}_{kl}W^{(l)}\varepsilon_{t-k}$, then $\nabla s(x)$ can be expressed as:

$$\nabla s(x) = 2\sum_{t=1}^{T} F_t^T(x)\frac{\partial F}{\partial x}, \tag{9}$$

where x represents $\hat{\beta} = [\hat{\phi}_{10}, \hat{\phi}_{11}, \ldots, \hat{\phi}_{p\lambda}, \hat{\theta}_{10}, \hat{\theta}_{11}, \ldots, \hat{\theta}_{qm}]'$. Then we use column vector to show $\nabla s(x)$:

$$\nabla s(x) = \begin{bmatrix} \vdots \\ 2\sum_{t=1}^{T} F_t^T(:,t)\frac{\partial F}{\partial \phi_{kl}} \\ \vdots \\ ----------- \\ \vdots \\ 2\sum_{t=1}^{T} F_t^T(:,t)\frac{\partial F}{\partial \theta_{kl}} \\ \vdots \end{bmatrix}, \tag{10}$$

where the upper half have $p(\lambda + 1)$ elements, and the lower half have $q(m + 1)$ elements. Suppose $J_{t,i} = \frac{\partial F_t}{\partial x_i}$, then $\nabla^2 s(x)$ could be written as:

$$\nabla^2 s(x) = \begin{bmatrix} \vdots & | & \vdots \\ 2\sum_{t=1}^{T} J_{t,\phi}J_{t,\phi} & | & 2\sum_{t=1}^{T} J_{t,\phi}J_{t,\theta} \\ \vdots & | & \vdots \\ ------- & | & ------- \\ \vdots & | & \vdots \\ 2\sum_{t=1}^{T} J_{t,\theta}J_{t,\phi} & | & 2\sum_{t=1}^{T} J_{t,\theta}J_{t,\theta} \\ \vdots & | & \vdots \end{bmatrix}. \tag{11}$$

After several iterations, we can terminate the iteration process when $\left|x^{(k+1)} - x^k\right| \leq \varepsilon$. At this moment, x_{k+1} is the approximate optimal solution.

4 Case Study on PWV STARMA Model

In this section, a case study on PWV STARMA model is presented to demonstrate the feasibility and effectiveness of the proposed approach. The details are described below, beginning with PWV data collection.

4.1 PWV Data Collection

The original data of this section come from MODIS MOD08 Gridded Atmospheric Product (https://wist.echo.nasa.gov). Its spatial resolution is 1 km with daily, ten days and monthly composite data. Average monthly PWV data was used in this case (Fig. 1).

A total number of 84 views of MODIS monthly data were chosen from 2002 to 2008. Where the first 72 were used to construct the model, and the last 12 monthly data were specially used to predict. Figure 2 shows the monthly PWV data in 2007.

4.2 Model Construction

The main point of this article is to introduce the estimation method, so the computation of STACF and STPACF could refer to Pfeifer [7, 8].

According to the STACF and STPACF in Tables 1 and 2, the model can be defined into STARMA $(2, 2)$. That means $p = 2$ and $q = 2$. So the STARMA model is defined as:

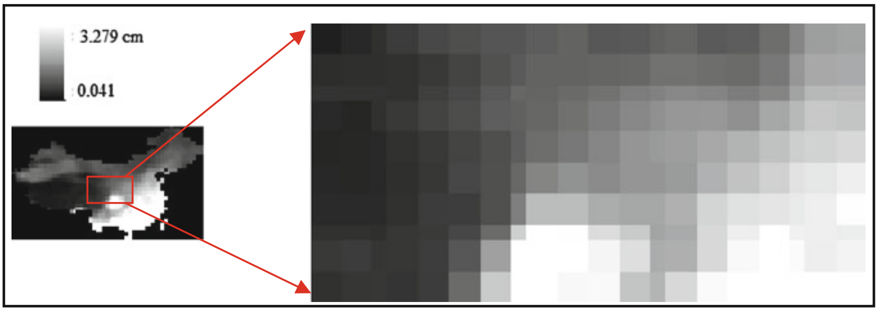

Fig. 1 MOD08 PWV distribution in China

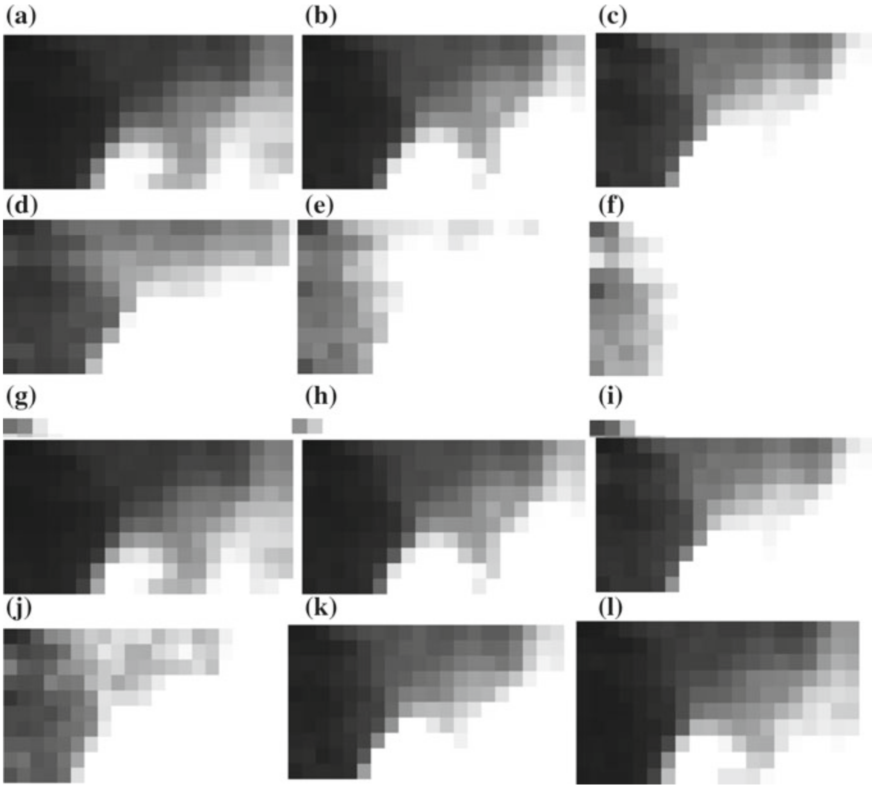

Fig. 2 Monthly PWV data in 2007

Table 1 Spatial-temporal autocorrelation functions for STARMA model

Spatial lag (h) Temporal lag (k)	0	1	2
1	–0.1809	–0.1603	–0.1615
2	–0.3167	–0.2794	–0.2847
3	–0.0794	–0.0735	–0.0752
4	0.0038	0.00253	0.0264
5	–0.0015	–0.0013	–0.0013
6	0.0012	0.0016	0.0095

Table 2 Spatial-temporal partial autocorrelation functions for STARMA model

Spatial lag (h) Temporal lag (k)	0	1	2
1	–0.5694	–0.1201	0.1142
2	–0.7715	–0.1249	0.0864
3	–0.0930	–0.0246	0.0882
4	0.0726	–0.0471	0.0717
5	0.0411	–0.0454	–0.0404
6	–0.0305	–0.0407	0.0249

$$
\begin{aligned}
Z(t) &= \varphi_{10}(t-1) - \varphi_{11} W^{(1)} Z(t-1) - \varphi_{12} W^{(2)} Z(t-1) \\
&- \varphi_{20} Z(t-2) - \varphi_{21} W^{(1)} Z(t-2) - \varphi_{22} W^{(2)} Z(t-2) \\
&+ \varepsilon(t) - \theta_{10}\varepsilon(t-1) - \theta_{11} W^{(1)}\varepsilon(t-1) - \theta_{12} W^{(2)}\varepsilon(t-1) \\
&- \theta_{20}\varepsilon(t-2) - \theta_{21} W^{(1)}\varepsilon(t-2) - \theta_{22} W^{(2)}\varepsilon(t-2)
\end{aligned} \tag{12}
$$

There are 12 parameters need to be specified, $\hat{\beta} = [\hat{\phi}_{10}, \hat{\phi}_{11}, \ldots, \hat{\phi}_{22}, \hat{\theta}_{10}, \hat{\theta}_{11}, \ldots, \hat{\theta}_{22}]'$.

Then extensive Newton's iteration method is applied in the estimation process.

4.3 Model Validation

The results of these 12 parameters and their evaluation are shown in Table 3. In order to make the model more simplified and effective, significant tests are required. If a

Table 3 STARMA parameters estimation and significant tests

Parameter estimation	Value	T test	P value
φ_{10}	–2.6578	41.3269	<0.0001
φ_{11}	–0.2337	2.3058	<0.01
φ_{12}	0.0989	1.3866	0.085
φ_{20}	–1.3542	38.4497	<0.0001
φ_{21}	–0.3133	10.0545	<0.0001
φ_{22}	–0.095	2.3757	<0.01
θ_{10}	0.0807	0.4954	0.311
θ_{11}	0.0316	0.2099	0.418
θ_{12}	0.0095	0.0870	0.466
θ_{20}	0.0889	0.9494	0.173
θ_{21}	–0.0324	0.3224	0.374
θ_{22}	0.0245	0.3126	0.378

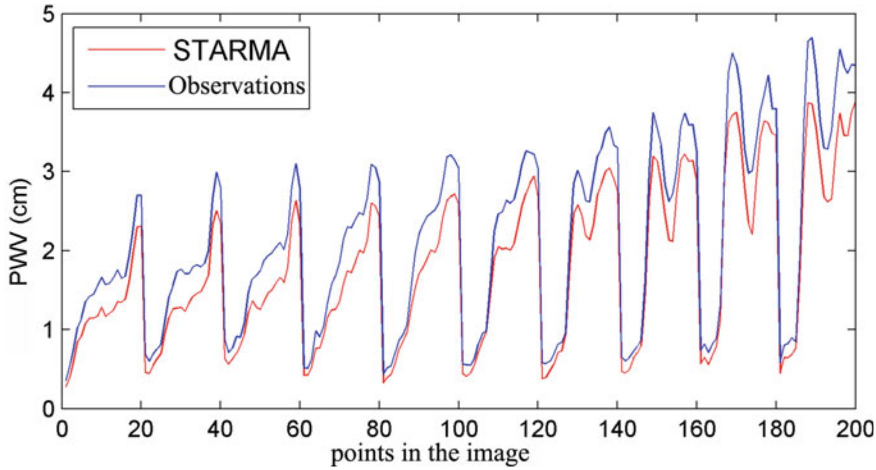

Fig. 3 PWV fitting results from STARMA model in June, 2003

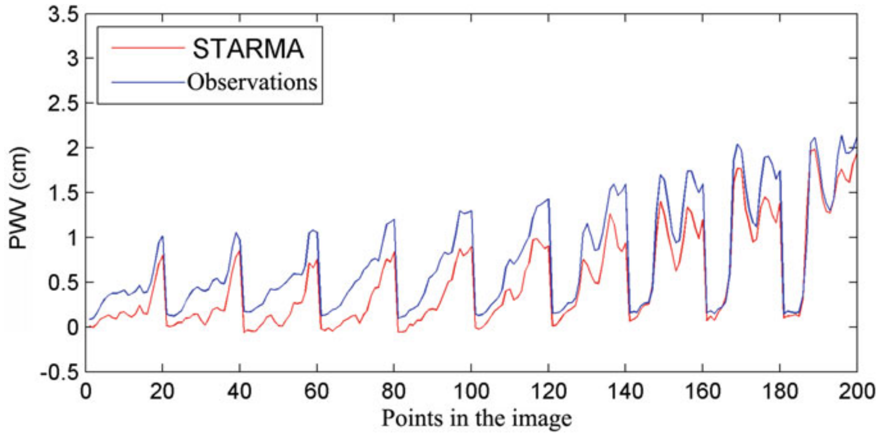

Fig. 4 PWV fitting results from STARMA model in November, 2003

parameter is not significant, it means that the independent variable corresponding to this parameter is not obvious, thus the variable can be deleted from the model.

When P is 0.05, there is 95 % probability to refuse the assumption that the variable is uncorrelated. In statistics, generally $P < 0.05$ was considered significantly efficient, and $P < 0.01$ was considered very significant.

From Table 3, it is obviously found that only some parameters are significantly efficient, which are φ_{10}, φ_{11}, φ_{12}, φ_{20}, φ_{21}, φ_{22}. These parameters θ_{10}, θ_{11}, θ_{12}, θ_{20}, θ_{21}, θ_{22} are not significant, so they can be deleted from the model.

Therefore, this model could be further optimized into STAR(2, 0).

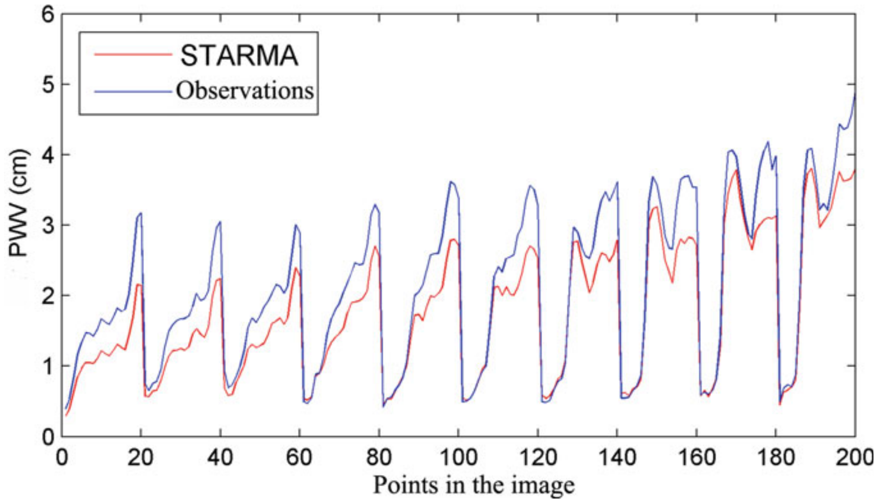

Fig. 5 PWV prediction results from STARMA model in June, 2008

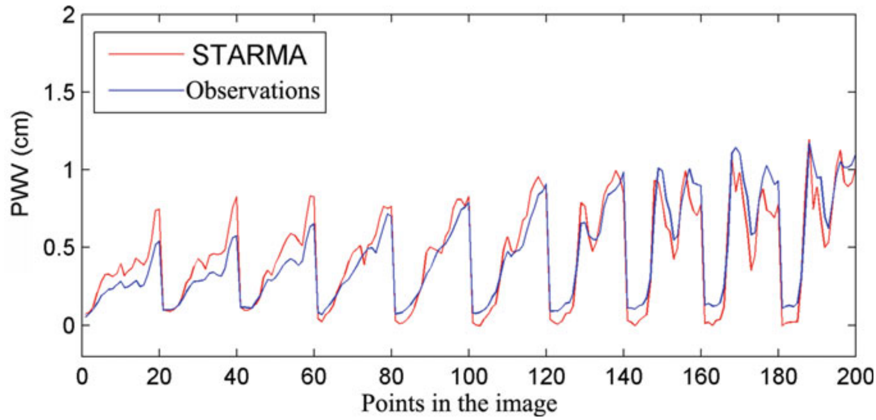

Fig. 6 PWV prediction results from STARMA model in December, 2008

Figures 3 and 4 show the fitting results of PWV calculated by model constructed with the latest model in June and November, 2003, respectively. And Figs. 5 and 6 show the prediction results of PWV in June and December, 2008, respectively. The red line represents the computing results by STARMA model, and the blue line represents the real data. For comparison, the pixel value of the image was arranged in a row on the horizontal line from 0 to 200 which represents one image with 10*20 grids. As is shown in these results, the trend in STARMA model is similar to real data.

4.4 Evaluation of the Model Performance

The correlation coefficient (R), root mean square error (RMSE), mean relative error (MRE) and the mean absolute error (MAE) were implemented to evaluate the overall performance of this model. Table 4 shows the fitting accuracy in 2003. Table 5 shows the prediction accuracy in 2008.

As shown in Tables 4 and 5, the correlation coefficient is very high. Hence, the model could be considered as an efficient and robust method for predicting PWV. And the parameters estimation process is successful.

Table 4 Fitting accuracy of STARMA model

Month	R	RMSE (cm)	MRE	MAE (cm)
Jan, 2003	0.9806	0.1803	0.5117	0.1624
Feb, 2003	0.9950	0.1252	0.2714	0.1114
Mar, 2003	0.9861	0.2043	0.3632	0.1756
Apr, 2003	0.9950	0.1055	0.0782	0.0790
May, 2003	0.9971	0.1876	0.0878	0.1459
Jun, 2003	0.9943	0.4108	0.1778	0.3611
Jul, 2003	0.9611	0.6577	0.1454	0.4790
Aug, 2003	0.9786	0.3774	0.0899	0.4790
Sep, 2003	0.9948	0.4743	0.1777	0.3806
Oct, 2003	0.9888	0.2391	0.1658	0.1827
Nov, 2003	0.9678	0.2938	0.4961	0.2581
Dec, 2003	0.9900	0.1921	0.2691	0.1451

Table 5 Prediction accuracy of STARMA model

Month	R	RMSE (cm)	MRE	MAE (cm)
Jan, 2008	0.9763	0.1394	0.351	0.1183
Feb, 2008	0.9905	0.3074	0.0131	0.2789
Mar, 2008	0.9919	0.0776	0.0796	0.0527
Apr, 2008	0.9964	0.0647	0.0431	0.0414
May, 2008	0.9932	0.2481	0.1656	0.2135
Jun, 2008	0.9798	0.4715	0.1624	0.3764
Jul, 2008	0.989	0.6694	0.148	0.5126
Aug, 2008	0.9861	0.369	0.1028	0.2664
Sep, 2008	0.9837	0.2973	0.1019	0.2258
Oct, 2008	0.9857	0.2017	0.17	0.158
Nov, 2008	0.9875	0.441	0.3144	0.2808
Dec, 2008	0.936	0.1121	0.3021	0.0937

5 Conclusion

The STARMA model has attracted wide attention in prediction applications. To optimize overall performance, the estimation process must be very critical. In the past, the estimation process is very difficult to solve and sometimes could not be solved at all. In this paper, a new approach based on Newton's method is put forward to give out the approximate solutions. The proposed approach was illustrated by a case study on monthly average PWV from 2002 to 2008. The confirmation results of this method revealed that the STARMA model has good accuracy in both fitting and predicting. From the value of RMSE, MRE and MAE, there are a few differences between the STARMA model and the real data. For example, though the correlation coefficient is very high in June and July, 2008, the RMSE and MAE are not reasonable. This is mainly due to the great changes of PWV in summer. That is to say, PWV STARMA model has lower accuracy in summer than in winter. Consequently, further study will be carried out to improve the performance caused by the seasonal differences.

References

1. Cliff, A.D., Ord, J.K.: Space-time modelling with an application to regional forecasting. Trans. Inst. Br. Geogr. **64**, 119–128 (1975)
2. Martin, R.L., Oeppen, J.E.: The identification of regional forecasting models using space-time correlation functions. Trans. Inst. Br. Geogr. **66**, 95–118 (1975)
3. Aroian, L.A.: Time series in m-dimensions: definition, problems and prospects. Commun. Stat.-Simul. Comput. **5**(B9), 453–465 (1980)
4. Aroian, L.A.: Time series in m-dimensions: autoregressive models. Commun. Stat.-Simul. Comput. **5**(B9), 491–513 (1980)
5. Oprian, C.A., Taneja, V.S., Voss, D.A., et al.: General considerations and interrelationships between MA and AR models, time series in m dimensions, the ARMA model. Commun. Stat.-Simul. Comput. **5**(B9), 515–532 (1980)
6. Voss, D.A., Oprian, C.A., Aroian, L.A.: Moving average models time series in m-dimensions. Commun. Stat.-Simul. Comput. **5**(B9), 467–489 (1980)
7. Pfeifer, P.E., Deutsch, S.J.: A comparison of estimation procedures for the parameters of the STAR model. Commun. Stat.-Simul. Comput. **3**(B9), 255–270 (1980)
8. Pfeifer, P.E., Deutsch, S.J.: Stationarity and invertibility regions for low order STARMA models. Commun. Stat.-Simul. Comput. **5**(B9), 551–562 (1980)
9. Epperson, B.K.: Spatial and space-time correlations in systems of subpopulations with genetic drift and migration. Genetics **133**(3), 711–727 (1993)
10. Cressie, N., Majure, J.J.: Spatio-temporal statistical modeling of livestock waste in streams. J. Agric. Biol. Environ. Stat. **2**(1), 24–47 (1997)
11. Giacomini, R., Granger, C.W.J.: Aggregation of space-time processes. J. Econometrics **118**, 7–26 (2004)
12. Sartoris, A.: STARMA Models for Crime in the City of Sao Paulo, Kiel, Germany. Kiel Institute for World Economics (2005)
13. Hernández-Murillo, R., Owyang, M.T.: The information content of regional employment data for forecasting aggregate conditions. Econ. Lett. **90**(3), 335–339 (2006)
14. Garrido, R.A.: Spatial interaction between the truck flows through the Mexico-Texas border. Transp. Res. A: Policy Pract. **34**(1), 23–33 (2000)

15. Kamarianakis, Y., Prastacos, P.: Space-time modeling of traffic flow. Comput. Geosci. **31**(2), 119–133 (2005)
16. Bevis, M., Businger, S., Herring, T.A., et al.: GPS meteorology: remote sensing of atmospheric water vapor using the global positioning system. J. Geophys. Res.: Atmos. **97**(D14), 15787–15801 (1992)
17. Crespo, J.L., Zorrilla, M., Bernardos, P., et al.: A new image prediction model based on spatio-temporal techniques. Visual Comput. **23**(6), 419–431 (2007)
18. Lee, C.: Space-Time Modeling and Application to Emerging Infectious Diseases. Michigan State University (2005)
19. Miu, W.J., Ming, D., Tao, C., et al.: Space and Time Series Data Analysis and Modeling. Science Press, Beijing (2012)
20. Yan, W.: Application of Time Series Analysis (2008)

Comparison of Candidate-Well Selection Mathematical Models for Hydraulic Fracturing

Ting Yu, Xiang-jun Xie, Ling-na Li and Wen-bin Wu

Abstract The selection of a target well and formation is considered as the first step in hydraulic fracturing (HF) and naturally regarded as a critical decision-making throughout process of HF treatment. The candidate-well selection process for HF is taken as a complex, nonlinear and uncertainty system. Modern mathematical methods, such as Artificial Intelligence (AI), offer the opportunities to examine the sample data, clarify the relationships among effect factors, in other ways to maximize the concealed potential. However, the performance of these methods is not specified in the certain application of candidate-well selection in gas field. This paper aims to provide a comparison of three candidate-well selection techniques, including BP-ANN, GA-based FNN, and SVM, as well as make clear the most effective one among them to pick a target well for HF. The application result of X gas field shows that BP-ANN is not as effective as GA-based FNN. Despite the advantage of more simple and intuitive evaluation, SVM has its own limitation in the uncertain system.

Keywords Hydraulic fracturing · Candidate-well selection · Nonlinear and uncertainty system · BP-ANN FNN · SVM

1 Introduction

Due to the continuous growth in hydrocarbons demand, the oil and gas industry has spared no efforts to increase production in oil and gas reservoirs. Hydraulic fracturing (HF) is not only applied to enhance ultimate recovery for these wells with lower production in conventional reservoirs, but also to obtain production for these wells in unconventional reservoirs, such as tight gas, shale gas, etc. [1, 2]. For these reasons, HF is one of the most common completion operations in oil and gas wells today [2, 3].

T. Yu (✉) · X. Xie · L. Li · W. Wu
School of Science, Southwest Petroleum University, Chengdu 610500, China
e-mail: yuting1123@126.com

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Choosing a target well and formation is considered as the first stage in HF and of course regarded as a critical decision-making in the entire process of HF treatment. However, the oilfield operation shows that the candidate-well selection is not direct process and up to now, there has not been a standard method for universally selecting candidate wells through different geological areas. Accurate candidate-well selection for HF treatment not only saves money and time but also eliminates possible failures for HF treatment, so it is very important.

The methods, applied in HF candidate-well selection design, in the past two decades, can be divided into two parts, containing: (1) conventional techniques; and (2) advanced methods. The former mainly deals with engineering, geological, etc. aspects in decision making process. The later mainly fill the gap for classification and manipulation of the parameters and mainly employs Artificial Intelligence (AI) methods that offers the opportunities to examine the data, illustrate their relationships, and in other ways to maximize the concealed potential [4]. The candidate-well selecting is a nonlinear and uncertain system, for the data set used is natural ambiguous and imprecise. So the advanced methods have get more attention in recent years in the candidate-well selection design. There are many advanced technologies, such as Artificial Neural Network (ANN) [5, 6], Fuzzy Neural Network (FNN) [7], Support Vector Machine (SVM) [8], Pattern Recognition (PR) [9], Gray Relative Analysis (GRA) [10, 11], Fuzzy Method (FM) [12–14], etc. However, the performance of these methods is not specified in the certain application of candidate-well selection for HF in gas field. This paper aims to provide a comparison of three candidate-well selection techniques, including BP-ANN, FNN, and SVM, as well as make clear the most effective one among these methods when coping with this problem.

The three models all allow a prediction for production response after fracturing treatment on the basis of historical measured data of fractured wells to be designed in a fast manner, offering an effective method to evaluate and select fractured wells for fracturing treatment.

In this paper, three-layer BP-ANN of candidate-well selection is build due to its complexity and strong nonlinear characteristic. FNN integrates the fuzzy logic with neural network, which not only has the advantages of knowledge representation and fuzzy reasoning, as well as the good self-learning capacity of neural network, but also the ability of adjusting the rules and the parameters of controller according to the variation for control subject parameters and the environment. Genetic algorithm (GA)-based identification method is used to specify FNN model parameters such that the defined error function is minimized. It is a global optimization algorithm with good robust stability, conducing to better performance compared to others, especially for the nonlinear systems like FNN. Economic, observation and control technology, etc., limit the sample data scale obtained from geological data, logging data and fracturing treatment materials. Support Vector Machine (SVM) is a powerful tool to deal with this situation with small sample, providing a way to determine the grade of potential production of gas wells according to the post-production standard.

2 Three Advanced Methods of Selecting Candidate Wells

The aim of this section is to provide detailed description of three candidate-well selection designs, including BP-ANN, FNN, and SVM.

2.1 Artificial Neural Network

Artificial neural networks (ANN) have been widely used in many application areas, such as forecast, pattern recognition and intelligent control, particularly for back-propagation artificial neural network (BP-ANN), which is a multi-layer perceptron (MLP) ANN with the error back-propagation training algorithm. The BP-ANN consists of one input layer, one or more hidden layers and one output layer. A set of nodes compose each layer. Each node in each layer receives one input signal from all nodes of its previous layer and sends one output signal to all nodes in its next layer, but the nodes of the same layer are unconnected. Information from the input layer is multiplied by the corresponding weights which show the strength of that input during transmitting to the hidden layer, and one input result of each node in the hidden layer is then computed through an activation function, as the input information in the next layer. After that, information is processed from the upper layer to the lower layer in a similar way. The output result goes into an activation function [15].

In our study, three-layer BP-ANN of candidate-well selection for HF is build. The input layer consists of all the information that influence the effect of post-fracture response. Its node number is equal to the dimension of feature parameters of candidate-well selection obtained by the feature extraction technology. The following is determination the node number of the output layer, that is, the dimension of output vector, which depends on how to set the target output. Test production, reflecting post-fracture response very clearly, is set as target output. A schematic description of proposed ANN is given in Fig. 1.

In order to develop an appropriate BP-ANN of candidate-well selection, two main works have to be done. First, a good network structure, the number of nodes in the hidden layer and activation function for each layer, should be specified. The second task is to use the error back-propagation training algorithm to modify parameters of the network (i.e. connective weights and thresholds) to make total error smaller than a given value [16].

2.2 Fuzzy Neural Network

Compared with ANN, fuzzy neural network (FNN) is capable of the description and processing of uncertain information, expert knowledge expression, as well as lower requirement for the sample, and it is also well suited for the complex, uncertain,

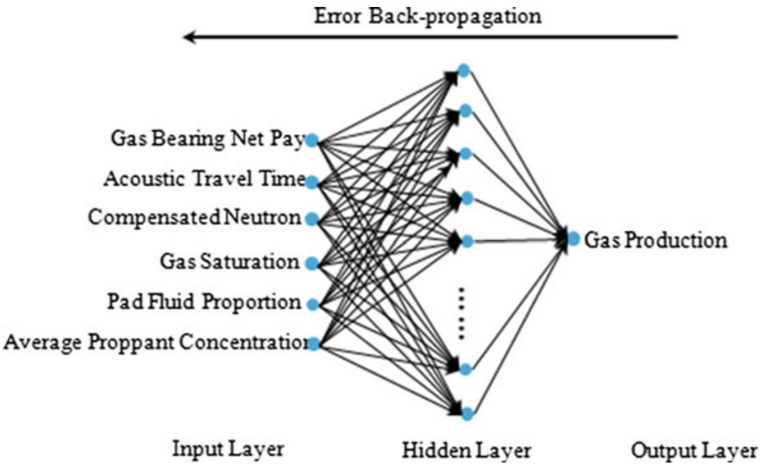

Fig. 1 Structure of three-layer BP-ANN

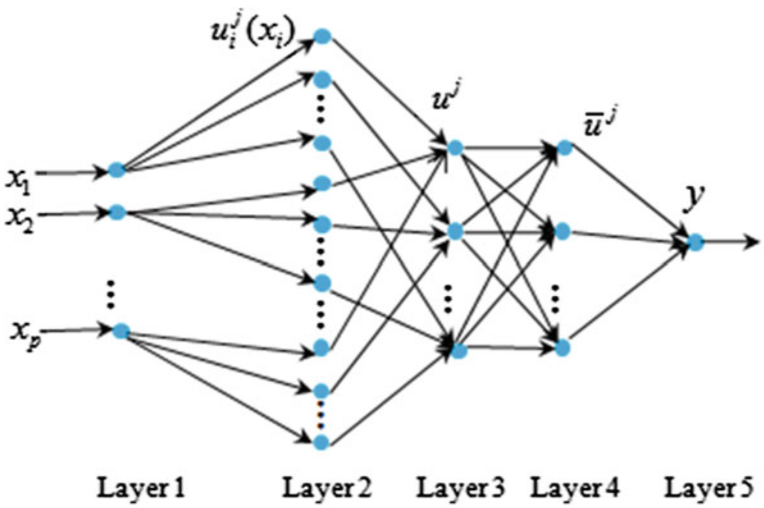

Fig. 2 Structure of five-layer FNN

and nonlinear systems, like candidate-well selection for HF [17]. In this paper, a candidate-well selection model based on FNN is designed. As is shown in Fig. 2, its structure has five layers and the discussion of each layer is as below [18].

Layer 1 (Input Layer): The nodes in this layer, like ANN, represent all the parameters ($x_i, i = 1, \ldots, p$) that influence the effect of post-fracture response.

Layer 2 (Fuzzification Layer): All parameters are mapped to the fuzzy set in this layers. Membership function for each fuzzy set is in the form of a Gaussian function:

$$u_i^j(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i - m_i^j}{\sigma_i^j}\right)^2\right] (j = 1, \ldots, l_i), \tag{1}$$

where l_i is the number of fuzzy partitions of x_i, m_i^j and σ_i^j are the Gaussian function's center and width respectively.

Layer 3 (Rule Layer): The number of nodes equals the total IF-THEN fuzzy rules that describe the input-output relationship of FNN. Each rule has the following form:

$$\begin{aligned} &\text{Rule } j : \text{IF } x_1 \text{ is } A_1^j, \ldots, x_p \text{ is } A_p^j \\ &\quad\text{THEN } y^j \text{ is } c_0^j + c_1^j x_1 + \cdots + c_p^j x_p \end{aligned} \tag{2}$$

Product or minimum inference is used to obtain the fitness of each rule (u^j), that is the weight of each rule. The calculation is as follows:

$$u^j = \prod_{i=1}^{p} u_i^j(x_i) \text{ or } u^j = \min\nolimits_{1 \le i \le p} u_i^j(x_i)\ (j = 1, \ldots, l), \tag{3}$$

where $j\,(j \le \prod_{i=1}^{p} l_i)$ is the total rules.

Layer 4 (Normalized Layer): Node number is equal to that in the Rule Layer. This layer can realize normalized calculation which is represented as:

$$\bar{u}^j = \frac{u_j}{\sum_{j=1}^{l} u_j}. \tag{4}$$

Layer 5 (Output Layer): The function of this layer is to get the final output by defuzzification. As designed in ANN, test production is set as target output. The final output can then be calculated according to the following formula:

$$y = \sum_{j=1}^{l} \bar{u}^j y^j = \sum_{j=1}^{l} \bar{u}^j \left(c_0^j + c_1^j x_1 + \cdots + c_p^j x_p\right). \tag{5}$$

After the structure construction of FNN, the parameter identification method must be determined. All parameters contain two parts: the parameters m_i^j and σ_i^j of these membership functions, as well as consequent coefficients c_p^j of these rules.

Given an input-output training pair $(x^k, y^k)(k = 1, \ldots, N)$, we wish to design a method for minimization of the following error function:

$$e^k = \frac{1}{2}\left[f\left(x^k\right) - y^k\right]^2, \tag{6}$$

where $f(x^k)$ and y^k are defined as the predicted and measured output, respectively. In this paper, genetic algorithm (GA)-based identification method is established to specify m_i^j, σ_i^j and c_p^j such that error function is minimized [19].

2.3 Support Vector Machine

Economic, observation and control technology, etc. aspects, limit the sample data scale obtained from geological data, logging data and fracturing treatment materials. Support Vector Machine (SVM) is a kind of new, complete and well-formed machine learning theory and method on the basis of statistical learning theory, and as a powerful tool for pattern recognition against the limited sample. In this paper, SVM was used to determine the grade of potential production of gas well according to the post-production standard.

The real issue of SVM is to solve quadratic programming problem by the aid of Lagrange multipliers, thereby finding the optimum separating hyperplane, which is able to classify data points as much as possible and to separate them again into two classification points as much as possible [20].

Given several input-output training sets$(x^i, y^i)(i = 1, \ldots, N)$, here, the input vector $x^i \in R^n$, the desired output $y^i \in \{+1, -1\}$ (+1 and -1 are two kinds of class identifiers). The algorithm tries to find a decision function, classifying data as accurately as possible. The original problem is described as follows [21]:

$$\begin{aligned} &\min \tfrac{1}{2} w^T w + C \textstyle\sum_{i=1}^N \xi_i \\ &s.t.\ \ y^i(w^T x^i + b) \geq 1 - \xi_i,\ \ \xi_i \geq 0 \end{aligned} \tag{7}$$

where C denotes the penalty parameter between the error term and the margin of hyperplane which balances margin maximization with classification violation. When we are trying to solve original problem by Lagrange multipliers, the dual form is obtained as the following form:

$$\begin{aligned} &\min \tfrac{1}{2}\alpha^T A\alpha - e^T\alpha \\ &s.t.\ \ 0 \leq \alpha_i \leq C, \\ &\qquad y^T\alpha = 0, \end{aligned} \tag{8}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)^{\mathrm{T}}$, $e = (1, \ldots, 1)^{\mathrm{T}}$, $y = \left(y^1, \ldots, y^N\right)$ and $A_{ij} = y^i y^j K\left(x^i, x^j\right)$, α_i are all zeros except for those training patterns close to these separating planes which are called support vectors. A_{ij} is the kernel function which converts nonlinear input data sets into high dimensional linear feature space. According to the Lagrange multipliers, the decision function is specified by

$$f(x) = \operatorname{sgn}\left\{ \sum_{i=1}^{N} \alpha_i y_i K\left(x^i, x\right) + b \right\}. \tag{9}$$

Each step of support vector machine classification algorithm is description in detail as follows:

Step1: choice the right kind of kernel function;
Step2: solve the optimization function and obtain support vector, namely, relevant Lagrange operators;
Step3: write down the function of optimum separating hyperplane;
Step4: according to the value of sign function, output the grade.

3 Methods Comparison and Results

This section is to application of the above methods, including BP-ANN, FNN based GA, as well as SVM to construct the complex, uncertain, and non-linear model candidate-well selection for HF in X gas field. We then compare the results of the three different methods and display the overall results.

3.1 Artificial Neural Network Forecaster

With BP-ANN, we model a forecaster of the gas production in X gas field, then verify this model through 10 others gas wells of this field, after that, use the average error function $AE = \sum_{i=1}^{N} \left| p_{predicted} - p_{measured} \right| / N$ to evaluate the performance of BP-ANN-based candidate-well selection for HF model, in which the hidden layer number as100, learning rate as 0.2, and the others as defaults. We calculate the average error *AE* = 0.0226, which is dimensionless value (dimensional value as 0.7746). The predicted outputs of the BP-ANN-based candidate-well selection forecaster, measured output, and fitting precision are given in Fig. 3. and Table 1.

3.2 Fuzzy Neural Network Forecaster

When we use 12 Gaussian membership functions for FNN candidate-well selection for HF forecaster, the total identification parameters are 224 by GA, including 72 means and standard deviations of Gaussian membership functions, respectively, as well as 84 consequent coefficients. The final results based on GA-FNN forecaster are given in Fig. 4. and Table 2.

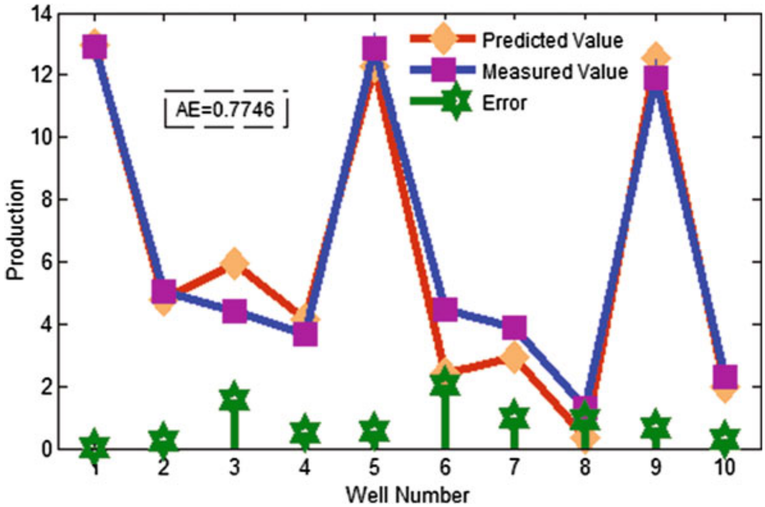

Fig. 3 Comparison of predicted and measured results of BP-ANN forecaster

Table 1 Outputs and error of the BP-ANN forecaster

Well number	*H*	*SDT*	*CN*	S_g	*P*	*C*	Q_m	Q_p	*Atol*
1	25.6	85	13	84	13	437	12.92	12.95	0.03
2	14.2	70	16	42	28	409	5.06	4.80	0.26
3	14.8	69	15	50	23	421	4.4	5.95	1.55
4	30	65	13	49	21	365	3.68	4.17	0.49
5	40	77	13	60	18	402	12.84	12.30	0.54
6	25	72	12.5	48	21	428	4.45	2.41	2.04
7	13.2	71	13.4	54	20	441	3.89	2.94	0.95
8	15.8	66	11	72.5	13	519	1.3	0.36	0.94
9	36	76	13	59	18	380	11.93	12.56	0.63
10	24.5	66	12	44	21	433	2.29	1.99	0.30

Note
H = Gas bearing net pay, m; *SDT* = Acoustic travel time, μs/ft
CN = Compensated neutron, PU; S_g = Gas saturation, %
P = Pad fluid proportion, that is the ratio of pad fluid in total fracturing fluid, %
C = Average of proppant concentration, Kg/m^3
Q_m = Measured production, 10^4 m^3/d; Q_p = Predicted production, 10^4 m^3/d
Atol = The absolute error between measured production and prediction production, 10^4 m^3/d

3.3 Support Vector Machine Classifier

We use SVM of four classifiers to construct the candidate-well selection for HF design according to post-production standard obtained by mathematical statistics

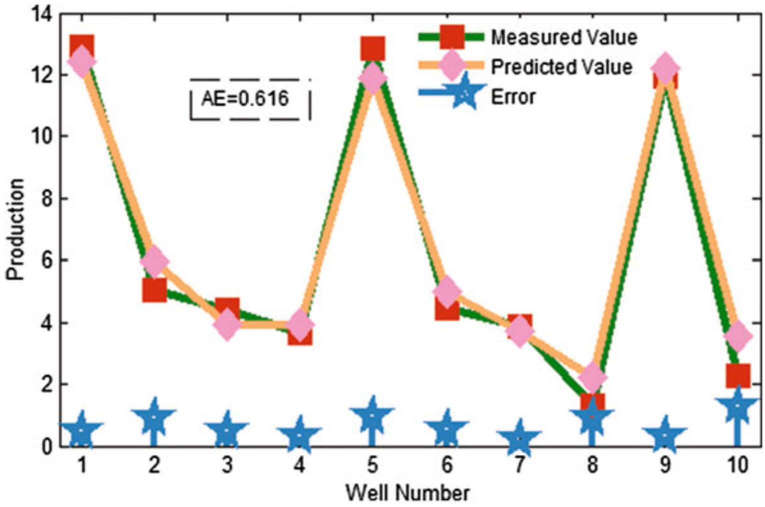

Fig. 4 Comparison of predicted and measured results of GA-FNN forecaster according to the value of average error *AE*, BP-ANN cannot compare with GA-FNN in the application of candidate-well selection for HF: Complex, uncertain and nonlinear system

Table 2 Outputs and error of the GA-FNN forecaster

Well number	H	SDT	CN	S_g	P	C	Q_m	Q_p	$Atol$
1	25.6	85	13	84	13	437	12.92	12.44	0.48
2	14.2	70	16	42	28	409	5.06	5.94	0.88
3	14.8	69	15	50	23	421	4.4	3.94	0.46
4	30	65	13	49	21	365	3.68	3.95	0.27
5	40	77	13	60	18	402	12.84	11.9	0.94
6	25	72	12.5	48	21	428	4.45	4.97	0.52
7	13.2	71	13.4	54	20	441	3.89	3.71	0.18
8	15.8	66	11	72.5	13	519	1.3	2.2	0.9
9	36	76	13	59	18	380	11.93	12.2	0.27
10	24.5	66	12	44	21	433	2.29	3.55	1.26

analysis and expert advice, in which the inputs of SVM classifier are the same as BP-ANN and GA-FNN forester, and the output is the evaluation grade. The higher grade the gas well calculated, the higher potential production it has. The final classification results of 10 wells are given in Table 3.

SVM classifier method can estimate the potential production of candidate fracturing wells simply and intuitively. Post-production standard of candidate fracturing wells is characterized by their measured production, so from this aspect and the results of Table 3, we get that: the evaluation is not ideal, and grades of Well 3, 7 and

Table 3 Output grades of SVM classifier

Well number	1	2	3	4	5	6	7	8	9	10
Grade	I	II	I	II	I	II	IV	III	II	III

9 are unreasonable due to the unmatched between classification grade obtained by SVM and measured production.

3.4 Results

GA-based FNN has an advantage over BP-ANN forecaster when designing the complex, uncertain and nonlinear candidate-well selection for HF system. SVM classifier method is more direct and compact, but the final classification results shows that it cannot deal with fuzzy information well because all kinds of uncertainties existed in the measured data can not characterize a well into a certain category. GA-based FNN introduces membership function to process the sample data which makes up for lack of ability to handle the uncertain information for SVM.

4 Conclusion

BP-ANN cannot compare with GA-based FNN in the application of candidate-well selection for HF in gas field: complex, uncertain and nonlinear system. Despite the advantage of more simple and intuitive evaluation, SVM has its own limitation in the uncertain system.

In the future, we wish to make a much more comprehensive analysis for all existed advanced candidate-well selection methods.

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References

1. Rushing, J.A., Sullivan, R.B.: Evaluation of a hybrid water-frac stimulation technology in the Bossier tight gas sand play SPE annual technical conference and exhibition. Society of Petroleum Engineers (2003)
2. King, G.E.: Thirty years of gas shale fracturing: what have we learned? SPE annual technical conference and exhibition. Society of Petroleum Engineers (2010)

3. Daneshy, A.: Hydraulic fracturing to improve production. Way Ahead: Tech. **101**(6), 3 (2010)
4. Samsuri, A., Zoveidavianpoor, M., Shadizadeh, S.R.: A review on conventional candidate-well selection for hydraulic fracturing in oil and gas wells. Int. J. Eng. Technol. **2**(1), 51–60 (2012)
5. Saeedi, A., Camarda, K.V., Liang, J.: Using neural networks for candidate selection and well performance prediction in water-shutoff treatments using polymer gels-a field-case study. SPE Prod. Oper. **22**(04), 417–424 (2007)
6. Ouenes, A.: Practical application of fuzzy logic and neural networks to fractured reservoir characterization. Comput. Geosci-Uk **26**(8), 953–962 (2000)
7. Mohaghegh, S.: Virtual intelligence and its applications in petroleum engineering. J. Pet. Technol. Distinguished Author Series (2000)
8. Yang, W., Tian, F., Gong, S.: Optimized determination of fractured well layers based on support vector machine. Comput. Eng. **7**, 87 (2006)
9. Reeves, S.R., Hill, D.G., Hopkins, C.W., Conway, M.W., Tiner, R.L., Mohaghegh, S.: Restimulation technology for tight gas sand wells SPE annual technical conference (1999)
10. Cheng, K., Wei, Y., Wu, W., Holditch, S.A.: A novel optimization model for analyzing production DataSPE western regional meeting. Society of Petroleum Engineers (2010)
11. Guo, J., Xiao, Y., Zhu, H.: A new method for fracturing wells reservoir evaluation in fractured gas reservoir. Mathematical Problems in Engineering (2014)
12. Xiang-jun, X., Ting, Y.: Application of TS fuzzy model in candidate-well selection for hydraulic fracturing fuzzy information and engineering and operations research and management, pp. 563–570. Springer, New York (2014)
13. Zoveidavianpoor, M., Samsuri, A., Shadizadeh, S.R.: Hydraulic fracturing candidate-well selection by interval type-2 fuzzy set and system IPTC 2013: international petroleum technology conference (2013)
14. Zoveidavianpoor, M., Samsuri, A., Shadizadeh, S.R.: Fuzzy logic in candidate-well selection for hydraulic fracturing in oil and gas wells: a critical review. Int. J. Phys. Sci. **7**(26), 4049–4060 (2012)
15. Esmaeili, R., Dashtbayazi, M.R.: Modeling and optimization for microstructural properties of Al/SiC nanocomposite by artificial neural network and genetic algorithm. Expert Syst. Appl. **41**(13), 5817–5831 (2014)
16. Zettler, A.H., Poisel, R., Reichl, I., Stadler, G.: Pressure sensitive grouting (PSG) using an artifical neural network combined with fuzzy logic. Int. J. Rock Mech. Min. **34**(3), 351–358 (1997)
17. Zhang, M., Liu, X.: A soft sensor based on adaptive fuzzy neural network and support vector regression for industrial melt index prediction. Chemometr. Intell. Lab. **126**, 83–90 (2013)
18. Zhang, J., Qin, W., Wu, L.H., Zhai, W.B.: Fuzzy neural network-based rescheduling decision mechanism for semiconductor manufacturing. Comput. Ind. **65**(8), 1115–1125 (2014)
19. Benamrane, N., Aribi, A., Kraoula, L.: Fuzzy neural networks and genetic algorithms for medical images interpretation. In: IEEE Geometric Modeling And Imaging-new Trends, 2006, pp. 259–264 (1993)
20. Güraksın, G.E., Haklı, H., Uğuz, H.: Support vector machines classification based on particle swarm optimization for bone age determination. Appl. Soft Comput. **24**, 597–602 (2014)
21. Liu, P., Sun, J., Han, L., Wang, B.: Research on the construction of macro assets price index based on support vector machine. Procedia Comput. Sci. **29**, 1801–1815 (2014)

Analysis of Graphic Method to Fuzzy Linear Programming and Its Application in Science Diet

Bing-Yuan Cao, Shu-Quan Lu, Pei-Hua Wang and Gen-Tao Zhang

Abstract In this paper, a simple graphic method is mentioned and then used to solve the problem of two-dimensional fuzzy linear programming. Besides, we discuss the set of feasible solution constituted by constraint function. Based on this analysis, some regularities are obtained for an optimal solution. At last, an applied model that can be solved with the proposed method is found in science diet and health care.

Keywords Fuzzy LP · Fuzzy graphic method · Vegetarian problems · Science diet · Application

1 Introduction

In some practical problems, the objective function and constraint conditions are often not absolute. It has some elasticity. Especially, the two dimensional fuzzy linear programming model can be considered as follows.

$$z^{\widetilde{\max}} = c_1x_1 + c_2x_2.$$
$$\text{s.t.} \begin{cases} a_{11}x_1 + a_{12}x_2 \lesssim (\text{or} \gtrsim) b_1, \\ a_{21}x_1 + a_{22}x_2 \lesssim (\text{or} \gtrsim) b_2, \\ x_1 \geq 0, x_2 \geq 0, \end{cases} \tag{1.1}$$

B.-Y. Cao · S.-Q. Lu · G.-T. Zhang (✉)
School of Mathematics and Information Sciences, Guangzhou University Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes (Guangzhou University), Guangzhou 510006, China
e-mail: 948980702@qq.com

P.-H. Wang
Guangzhou Vocational College of Science and Technology, Guangzhou 510550, China

B.-Y. Cao
Guangzhou University, Guangzhou 510550, China

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the symbol of "$\lesssim$" or "$\gtrsim$" indicates that some approximately equal to or less than (or greater than) another and the meaning of "$\widetilde{\max}$" signifies fuzzy maximum for a target function. It shows that the constraints are fuzzy concept.

At present, there are two main methods to solve the FLP model, they are iterative method and Zimmermann method [1, 2]. According to Zimmermann's idea, he defines different membership function for fuzzy constraints and fuzzy goal, respective. Finally, an FLP model can be converted into an LP problem. However it is not intuitive for us to analyze the changes of the feasible region, because of the machinery of computer programming. Therefore, this paper firstly provides a simple method to the problem of two-dimensional fuzzy linear programming. Secondly, basted on the intuitive method, the effect of fuzzy extremely produced by the change of the feasible region is discussed in detail. Finally, we use a variety of methods to compare and verify its accuracy in science diet, and find some regularity.

2 A Brief Introduction for Graphic Method

2.1 Basic Explanation

Suppose the objective function is $z = c_1x_1 + c_2x_2$, the corresponding fuzzy goal is $\widetilde{G}$. If a telescopic value of an objective function is selected as $d_0 = z_0^{'} - z_0$, its membership function can be defined as follows.

$$\begin{aligned}\widetilde{G}(x) &= g(cx)\\ &= g(z) = \begin{cases} 0, & cx \le z_0,\\ \frac{(cx-z_0)}{d_0}, & z_0 < cx \le z_0 + d_0,\\ 1, & z_0 + d_0 < cx.\end{cases}\end{aligned} \tag{2.1}$$

Similarly, we also give the telescopic value for a constraint function $d_0 = z_0^{'} - z_0$. And then its membership function $\widetilde{D}(x)$ is given below.

$$\begin{aligned}\widetilde{D}(x) &= f(ax)\\ &= f(y) = \begin{cases} 1, & a_ix \le b_i,\\ 1 - \frac{(a_{i1}x_1 + a_{i2}x_2 - b_i)}{d_i}, & b_i < a_ix \le b_i + d_i,\\ 0, & b_i + d_i < a_ix, (i = 1, 2, \ldots, n).\end{cases}\end{aligned} \tag{2.2}$$

In addition, we define the fuzzy judgment $\widetilde{D}_f = \widetilde{D} \cap \widetilde{G}$. If x^* is an optimum solution to FLP (1.1), it satisfies

$$\begin{aligned}\widetilde{D}_f &= \max_{x\ge 0} \widetilde{D}_f(x) = \max_{x\ge 0}\{\widetilde{G}(x) \wedge \widetilde{D}(x)\}\\ &= \max_{x\ge 0}\{\widetilde{G}(x) \wedge \widetilde{D}_1(x) \wedge \widetilde{D}_2(x), \ldots, \widetilde{D}_n(x)\}.\end{aligned} \tag{2.3}$$

The following images are given to show the relationship defined above (Figs. 1 and 2).

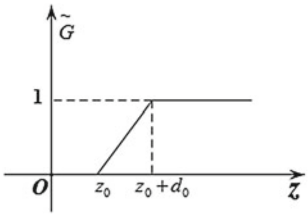


Fig. 1 Relationship between $\tilde{G}$ and z

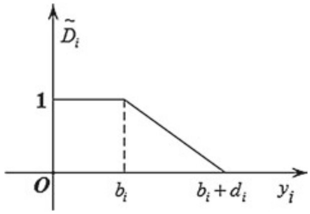


Fig. 2 Relationship between $\tilde{D}_i$ and y_i

2.2 An Example

We use the following example to illustrate the FLP graphic method [3].

$$z^{\widetilde{\max}} = 7x + 12y$$
$$\text{s.t.} \begin{cases} 9x + 4y \underset{\sim}{<} 360, \\ 3x + 10y \underset{\sim}{<} 300, \\ x \geq 0,\, y \geq 0. \end{cases} \tag{2.4}$$

(1) Let the first corresponding classical LP problem as follows and then solve it by using the LP graphical method.

$$z^{\max} = 7x + 12y$$
$$\text{s.t.} \begin{cases} 9x + 4y \leq 360, \\ 3x + 10y \leq 300, \\ x \geq 0,\, y \geq 0. \end{cases} \tag{2.5}$$

In Fig. 3, the convex quadrilateral *OABC* is the feasible region of LP (2.5). The line l is a contour line of the objective function, it intersects at point B, which is one of the apex in feasible region. Easy to know that point B is an optimum, where coordinate is $x = 30.77$, $y = 20.77$ and $z_0^{\max} = 464.63$.

(2) Let the constraint function expand to maximum range. We solve the following new programming using the LP graphical method.

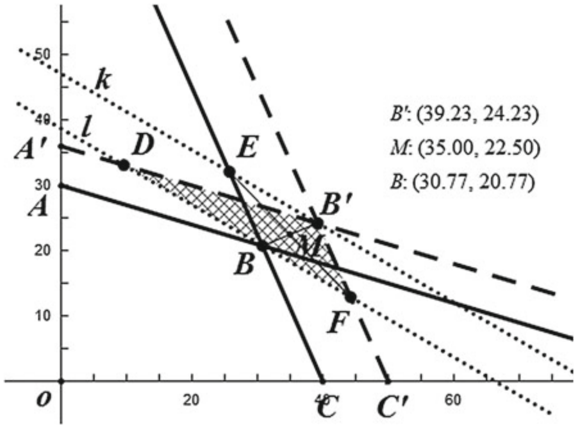


Fig. 3 Example for FLP graphic method

$$z^{\max} = 7x + 12y$$
$$\text{s.t.} \begin{cases} 9x + 4y \leq 360 + 90, \\ 3x + 10y \leq 300 + 60, \\ x \geq 0, y \geq 0. \end{cases} \tag{2.6}$$

In Fig. 3, the feasible region expands from quadrilateral $OABC$ to $OA'B'C'$. A contour line k of the objective function intersect at point B', where coordinate is $x = 39.23$, $y = 24.23$ and $z_1^{\max} = 565.37$. Besides, it is obvious that the number $d_0 = z_1^{\max} - z_0^{\max} = 100.74$ can be regarded as a telescopic value for a target function.

(3) When the feasible region enlarges, the optimal solution moves from point B to B'. Meanwhile, it is easy to verify that one or two constraint conditions are violated in this process. That is to say, membership function $\tilde{D}_1(x)$ and $\tilde{D}_2(x)$ gradually become less than before. At the same time, the membership function $\tilde{G}(x)$ will slowly become greater and the optimal point B closes to the line k, which is the maximum contour line of objective function. The question now is to find a suitable point so that the fuzzy judgment reaches maximum.

Consequently, when the objection function z changes in interval [464.63, 565.37], the change of membership function $\tilde{D}(x)$ and $\tilde{G}(x)$ is shown in Fig. 4. We can get that the midpoint M of the line BB' just is an optimal solution to the FLP (2.1), where the coordinates is $x = 35$, $y = 22.5$ and $z^{\max} = 515$.

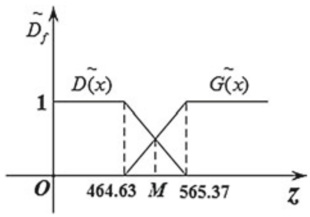


Fig. 4 Relationship between $\tilde{D}_f$ and Z

2.3 Definition and Method

In order to explain this graphic method, we firstly give some definition to it. Next, we summarize the steps of the proposed method. This method can be used to solve the two-dimensional FLP (1.1).

Definition 2.1 A region is said to be a telescopic region, if the telescopic range for each constraint conditions and the target is the area between two parallel lines, which was or is expanded before or after.

Definition 2.2 A region is said to be the telescopic parallelogram, if two different telescopic region intersections is a parallelogram.

Definition 2.3 A triangle, constituted by three lines which membership grade is number "0" in telescopic region, is said to be a telescopic triangle, if one of vertex of the triangle is a solution to the fuzzy constraint expanded to a maximum range.

Definition 2.4 A line in telescopic triangle is said to be the diving angle line of an equal membership grade if the line is divided into one of the corner. The mapping method is that connect diagonal containing this corner in the telescopic parallelogram.

Hence, the FLP graphic method is as follows.

Step 1. Compute the optimum solution that the constraint conditions are not stretched by classical LP graphical method.
Step 2. Calculate the optimum solution that the constraint conditions expands to the maximum range using a classical LP graphical method.
Step 3. Make two diving angle lines of equal membership grade in a telescopic triangle, then their intersection is an optimal solution (like the point M in example).

3 A Two-Dimensional FLP Model in Vegetarian Diet

Vegetarian Diet. If vegetarians can only be limited to eat vegetables or fruit juices, he/she purchase x units vegetables each week, and y units fruit juices. According to nutrition experts, each person consumes vitamin should be limited in a certain scope, but this scope is vague. As a result, the vegetarians have a fuzzy constraint in selection food. In this way, he will choose about 40 g vitamin A, 50 g vitamin B, 70 g of vitamin C, 10 g of vitamin D and 60 g of vitamin E. Besides, the vitamin content in each units of vegetables or juice can be obtained by measuring. They are shown as follows. As we know, the price of each units vegetables and fruits juice is about 0.02

and 0.03 dollars. Consequently, how to determine the quantity for using vegetables and fruits juice, let him/her not only meet the needs of the body, but also spends less.
(1) Write the known date in the table below.

Location	A	B	C	D	E	Price $
Vegetables	0.1	0.2	0.04	0.1	0.06	0.02
Fruits juice	0.05	0.15	0.2	0.1	0.1	0.03
Vitamins	$40 - d_A$	$50 - d_B$	$70 - d_c$	$10 - d_D$	$60 - d_E$	

(2) Establish the two-dimensional FLP model in vegetarian diet.
Suppose vegetarians eat x units vegetable every week and y units fruits juice. And then, he/she had to buy x units vegetables and y unites juice. If letter c repress of the target function, he/she will pay the total money $c = 0.02x + 0.03y$.

The problem is that looking for two numbers x and y, which makes fuzzy objective function $\tilde{C}^{\min}$ minimum, and they must meet the following conditions.

$$\begin{aligned} &A: 0.1x + 0.05y \gtrsim 40, \\ &B: 0.2x + 0.15y \gtrsim 50, \\ &C: 0.04x + 0.2y \gtrsim 70, \\ &D: 0.1x + 0.1y \gtrsim 10, \\ &E: 0.06x + 0.1y \gtrsim 60, \\ &x \geq 0, y \geq 0, \end{aligned} \tag{3.1}$$

where the numbers on the right side of inequality is this.

$40 - d_A = 39.8, 50 - d_B = 59.8, 70 - d_C = 69.5, 10 - d_D = 9.8, 60 - d_E = 59.8$, respectively.

4 Graphical Method Is Used to Solve the Vegetarian Diet Model

Step 1. Draw a feasible region of the following LP (4.1) and then solve it by using an ordinary graphical method as shown in Fig. 5.

$$\begin{aligned} &\min = 0.02x + 0.03y \\ &st. \begin{cases} 0.1x + 0.05y \geq 40, \\ 0.2x + 0.15y \geq 50, \\ 0.04x + 0.2y \geq 70, \\ 0.1x + 0.1y \geq 10, \\ 0.06x + 0.1y \geq 60, \\ x \geq 0, y \geq 0. \end{cases} \end{aligned} \tag{4.1}$$

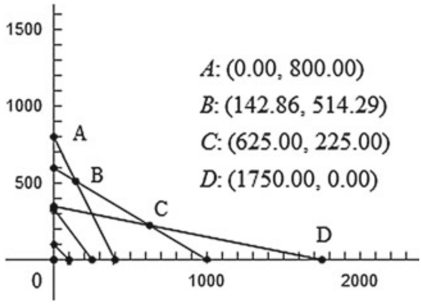


Fig. 5 Constraint functions of programming 4.1

The feasible solution in linear programming [4] is convex, so its optimal solution can be achieved in one of the vertex. As a result, we can put the four vertices into the objective function and compare their values. Finally, we get point B to be an optimum point, where its coordinate is $x = 142.86$, $y = 514.29$ and $z^{\min} = 18.2859$.

Step 2. Draw another feasible region of following LP (4.2), where the numbers on right side of inequality become smaller and solve it, which is shown in Fig. 6.

$$\min = 0.02x + 0.03y$$
$$st. \begin{cases} 0.1x + 0.05y \geq 39.8, \\ 0.2x + 0.15y \geq 49.5, \\ 0.04x + 0.2y \geq 69.5, \\ 0.1x + 0.1y \geq 9.8, \\ 0.06x + 0.1y \geq 59.8, \\ x \geq 0, y \geq 0. \end{cases} \tag{4.2}$$

Similarly, we compare the function values of some vertex points and found that point F (141.43, 513.14) is the optimal point, where its value of target function is $z^{\min} = 18.2228$.

Step 3. In Fig. 7, draw two diving angle lines of equal membership grade in the telescopic triangle. According to the fuzzy graphic method mentioned above, the intersection M of two straight lines is an optimal solution.

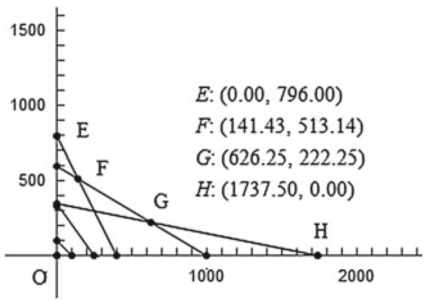
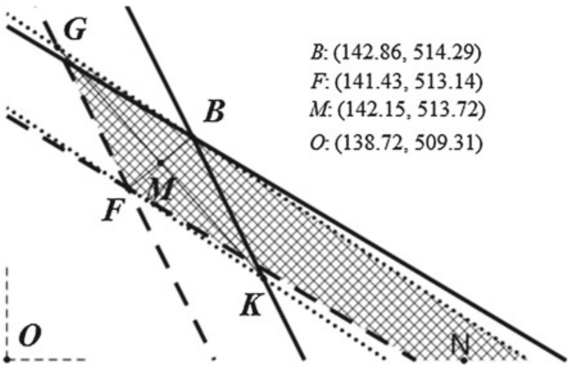


Fig. 6 Constraint functions of programming 4.2

Fig. 7 FLP graphic method in vegetarian diet model



In order to draw a more accurate picture, we make a few specification as follows.
1. For programming 4.1, only inequalities $\begin{cases} 0.1x + 0.05y \geq 40 \\ 0.06x + 0.1y \geq 60 \end{cases}$ is an active constraint, they intersect at a point $B(142.82, 514.29)$.
2. For programming 4.2, only inequalities $\begin{cases} 0.1x + 0.05 \geq 39.2 \\ 0.06x + 0.1y \geq 59.8 \end{cases}$ is active constraint, they intersect at a point $F(142.43, 513.14)$.
3. We only draw the contour line of objective function and active lines because of the complex line in feasible region. Besides, the value of telescopic value $d_A = -0.2$, $d_B = d_C = -0.5$, $d_E = d_F = -0.2$ is too small to clearly distinguish two sets of line between $\begin{cases} 0.1x + 0.05y = 40 \\ 0.1x + 0.05y = 39.8 \end{cases}$ and $\begin{cases} 0.06x + 0.1y = 60 \\ 0.06x + 0.1y = 59.8 \end{cases}$, so image is magnified.

Result: According to the fuzzy graphic method, a part of the triangle constituted by 3 lines, where membership grade is "0", is drawn as the shaded area shown in Fig. 7. When the numbers on right side of inequality is reduced, the feasible region will become large. If the optimum point gradually moves from point B to point F, its membership grape also changes from number "1" to number "0". Making two diving angle lines of the telescopic triangle, we find that their intersection is just the intersection of two diagonals in parallelogram. Therefore, point M must be the midpoint of the line BF. It is to say that putting the coordinate into the objective function can get optimal value $z^{\min} = 18.2546$.

5 An Analysis and Comparison of Fuzzy Graphic Method

5.1 Analysis

If the numbers of vitamin in model 3 which human body needs were changed into this: $40 + d_A = 40.2, 50 + d_B = 50.5, 70 - d_c = 69.5, 10 + d_D = 10.2, 60 + d_E = 60.2$, and corresponding target function $d_Z = 0.5$. The final fuzzy optimal values become this.

$$\lambda = 1, x = 142.8571, y = 514.2857, z^{\min} = 18.28571.$$

But with the result $x = 144.2857, y = 515.4286, z^{\min} = 18.34857$ which the constrain function is expanded, is still a certain gap. Or, to put it another way, it is not right for reality that the optimum solution of the programming increasing the telescopic value can not yet be compared with the original. In fact, the primary reason for the result was a smaller feasible region when the telescopic value is increased, which means that we take a wrong telescopic value.

5.2 Comparison

(1) The result as shown in Fig. 8 solved by Excel method are consistent with the graphical method, among them

$$x = 142.1430, y = 513.7144, z^{\min} = 18.2543.$$

(2) Turn it into a standard type as follows, and then solve it using a simple algorithm providing by literature [5].

$$\min z = 0.02x_1 + 0.03x_2$$
$$s.t. \begin{cases} 0.1x_1 + 0.05x_2 - x_3 = 40, \\ 0.2x_1 + 0.15x_2 - x_4 = 50, \\ 0.04x_1 + 0.2x_2 - x_5 = 70, \\ 0.1x_1 + 0.1x_2 - x_6 = 10, \\ 0.06x_1 + 0.1x_2 - x_7 = 60, \\ x_1, \ldots, x_7 \geq 0. \end{cases} \tag{5.1}$$

Fig. 8 Final result in excel method

	A	B	C
1	Variables	Values	
2	x	142.1430209	
3	y	513.7144144	
4	lamada	0.500113342	
5			
6	Subject to	39.80000014	39.8
7		105.2357097	49.5
8		108.178547	69.5
9		65.48572086	9.8
10		59.80000003	59.8
11		18.28569997	19.2857
12			
13	Max lamada	0.500113342	
14	Min z	18.25429285	

Calculate the basis matrix

$$B^{-1} = \begin{bmatrix} 14.2857 & 0 & 0 & 0 & -7.1400 \\ 0.5714 & 0 & 0 & -1 & 0.7140 \\ 1.5714 & -1 & 0 & 0 & 0.7138 \\ -8.5729 & 0 & 0 & 0 & 14.2799 \\ -1.429 & 0 & -1 & 0 & 2.5712 \end{bmatrix}.$$

We can get

$$B^{-1}(b+d) = B^{-1}\begin{bmatrix} 39.8 \\ 49.5 \\ 69.5 \\ 9.8 \\ 59.8 \end{bmatrix} = \begin{bmatrix} 141.5989 \\ 55.6389 \\ 55.7270 \\ 512.7366 \\ 38.7703 \end{bmatrix} > 0.$$

So the optimum point is

$$x^* = \frac{1}{2}(142.9284 + 141.5989, 514.1084 + 512.7366) = (142.2636, 513.4225).$$

where coordinate is $x = 142.2636$, $y = 513.4225$ and $z^{\min} = 18.2479$. This result also consists with the graphical method.

6 Conclusion

In this paper a two-dimensional FLP model is established about vegetarian problem, providing a simple and intuitive method-graphical, to analyze and discuss the changes of fuzzy extreme when the telescopic value change. The study founds its way to the idea that it will directly lead to the change of a feasible region when telescopic values become larger or smaller. Specifically, suppose the feasible region becomes smaller after increasing telescopic value. It will result in the outcome $\lambda = 1$, which measures the fuzzy judgment. This shows that, when the optimum solution to the programming increases the telescopic, we take a wrong value. Still another condition is an enlarged feasible region when its telescopic value is increased. The optimum point in FLP model becomes the midpoint between two optimums which increase the telescopic value and the original, if this two points are corresponding in location. This will be the content we discuss further.

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References

1. Cao, B.Y.: Application of Fuzzy Mathematics and Systems, pp. 119–131. Science Press, Beijing (2005)
2. Cao, B.Y.: Optimal Models and Methods with Fuzzy Quantity. Springer-Verlag, Berlin (2010)
3. Zhang, Z.K.: Graphic method of fuzzy linear programming. J. Tsinghua Univ. (Sci. Technol.) **37**(9), 6–9 (1997)
4. Benjamin, L.: Introduction to Mathematics Programming. Edward Arnold, Elsevier North Holland Inc., (1982)
5. Pan, R.F.: A simple solution for fuzzy linear programming. J. Xiang Tan Univ. (Sci. Technol.) **3**, 29–36 (1987)

Part V
Others

A Fixed-Length Source Coding Theorem on Quasi-Probability Space

Yang Yang, Lin-qing Gao, Chao Wang and Ming-hu Ha

Abstract The existing source coding theorems are established on probability measure space or Sugeno measure space. It is difficult to deal with the source coding problems on quasi-probability space which is an extension of probability measure space and Sugeno measure space. In order to overcome the limitation, fixed-length source coding problems on quasi-probability space are discussed. Based on the definition and properties of information entropy on quasi-probability space, an asymptotic equipartition property of discrete memoryless information source on quasi-probability space is proved. Then, a fixed-length source coding theorem for discrete memoryless information source on quasi-probability space is provided.

Keywords Fixed-length source coding theorem · Sugeno measure · Quasi-probability space · Information entropy

1 Introduction

Information theory developed by Shannon [1] is an intersection of physics (statistical mechanics), mathematics (probability theory), electrical engineering (communication theory) and computer science (algorithmic complexity) [2]. Source coding theory, an important branch of information theory, has been applied in encoding, compression and encryption of different information source such as image, audio and video [3–8]. The source coding theorem indicates that the expected length of the optimal code is bounded below by the Shannon entropy of the source, which is regarded as a sophisticated central part of source coding theory [9].

Y. Yang · L. Gao
Department of Management, Hebei University, Baoding 071002,
Hebei, People's Republic of China

C. Wang · M. Ha (✉)
School of Economics and Management, Hebei University of Engineering,
Handan 056038, Hebei, People's Republic of China
e-mail: mhha@hbu.edu.cn

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However, the classic source coding theorem is established on probability measure space where the uncertainty of information is described by probability measure on probability measure space whereas probability measure is a set function which satisfies additivity or countable additivity [10]. It is well known that in many situations, the condition of additivity is so strict that it might lead to the failure of probability measure in describing the uncertainty of some information. Inspired by this, many non-additive measures are proposed by substituting the additivity requirement of probability measures with other requirements [11–18]. For example, Sugeno [12] proposed a class of non-additive measures, Sugeno measures, by replacing the additivity with $\sigma - \lambda$ rule [17].

Motivated by the generalization of additive measure into non-additive measure, information theory on probability measure space was extended to the non-additive measure space. For example, on belief measure space and plausibility measure space, Klir [19] proposed generalized information theory to make the treatment of uncertainty-based information more realistic; on possibility space, Sgarro [20, 21] introduced the possibilistic entropy and gave a model for information sources and transmission channels which is possibilistic rather than probabilistic; on Sugeno measure space, Zhang and Yang [22] defined the information entropy based on Sugeno measure and proved a source coding theorem for discrete information source with g_λ distribution.

In this paper, we try to incorporate quasi-probability [13] into source coding theory and discuss the fixed-length source coding problems on quasi-probability space. The concept of quasi-probability is introduced and discussed by Wang and Klir [13, 14, 17] as a normalized (or regular) quasi-measure. Every quasi-probability is connected to an additive measure by a transformation function (T-function), which is a significant specialty of quasi-probability. It is well known that one of the main difficulties for applying non-additive measures is that they are set functions defined on a class of subsets of a given universal set and thus they require a large number of parameters. For example, to get a fuzzy measure defined on the power set of a given universal set X, usually we must set $2^{|X|}$ parameters. Quasi-probability can overcome the difficulty because any quasi-probability μ can be obtained by $\mu = \theta^{-1}(\theta \circ \mu)$, where θ^{-1} is the inverse function of T-function θ, $\theta \circ \mu$ is an additive measure. In this sense, quasi-probability is more usable than some other non-additive measures in practice. Furthermore, it has been proved that quasi-probability is an extension of Sugeno measure and probability measure [17], and therefore, the investigation of source coding theorem on quasi-probability space is significant.

The rest of the paper is organized as follows: In Sect. 2, some notations of quasi-probability and information entropy are reviewed. In Sect. 3, main results of this paper are presented.

2 Preliminaries

Let X be a non-empty set. F is a $\sigma-$algebra of subsets of X, and C is any class of subsets of X.

Definition 1 ([17]). Let $a \in (0, \infty]$. An extended real function $\theta : [0, a] \to [0, \infty]$ is called a T-function iff θ is continuous, strictly increasing, and such that $\theta(0) = 0$ and $\theta^{-1}(\{\infty\}) = \varnothing$ or $\{\infty\}$, according to a being finite or not.

Obviously, θ^{-1} is continuous, strictly increasing function when θ is continuous, strictly function.

Definition 2 ([17]). μ is called a quasi-measure iff there exists a T-function θ such that $\theta \circ \mu$ is an additive measure on C, where $(\theta \circ \mu)(E) = \theta(\mu(E)), \forall E \in C$. A normalized quasi-measure μ is called a quasi-probability.

As the relation of the discrete random information source and discrete random variable on probability measure space, the discrete information source on quasi-probability space can be depicted by discrete quasi-random variable [23].

Definition 3 ([23]). Let ξ be a discrete quasi-random variable taking values x_i $(i = 1, 2, \ldots)$. The expected value of ξ is defined by

$$E_\mu[\xi] = \sum_{i=1}^{\infty} x_i \mu_i,$$

if $\sum_{i=1}^{\infty} |x_i|\mu_i < \infty$ where $\mu_i = \mu\{\xi = x_i\}, i = 1, 2, \ldots$.

Definition 4 ([24]). The self-information of the event $\xi = x_i$ on quasi-probability space (X, F, μ) is defined as $I(x_i) = -\log_a(\theta \circ \mu(x_i))$.

The base of the logarithm a is not specified in the definition. Unless otherwise specified, we will take all logarithms to base 2, and take $\log x$ instead of $\log_2 x$.

Definition 5 ([24]). The information entropy of a quasi-random variable ξ is defined as $H(\xi) = E[I(x_i)] = \sum_{i=1}^{p} I(x_i) \cdot (\theta \circ \mu(x_i))$.

Proposition 1 ([23]). *Suppose that ξ is a discrete memoryless information source on quasi-probability space (X, F, μ), and $H(\mu)$ is the information entropy of ξ. Then the following properties of $H(\mu)$ is obtained:*

(i) $H(\mu_1, \mu_2, \ldots, \mu_p) = H(\mu_2, \mu_1, \ldots, \mu_p) = \cdots = H(\mu_p, \mu_1, \ldots, \mu_{p-1})$.
(ii) $H(\mu, 0) = H(\mu, 0, 0) = H(\mu, 0, 0, 0) = \cdots = H(\mu, 0, \ldots, 0) = 0$, $\mu > 0$.
(iii) $H(\mu_1, \mu_2, \ldots, \mu_p) \geq 0$;
(iv) $H(\mu_1, \mu_2, \ldots, \mu_p) \leq \log p$.

3 Main Results

Assume ξ is a discrete memoryless information source with quasi-probability distribution

$$\begin{bmatrix} \xi \\ \mu(\xi) \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_i & \cdots & x_p \\ \mu_1 & \cdots & \mu_i & \cdots & \mu_p \end{bmatrix}.$$

Then, the quasi-probability distribution of its Nth extension ζ is given as following

$$\begin{bmatrix} \zeta \\ \mu(\zeta) \end{bmatrix} = \begin{bmatrix} y_1 & \cdots & y_j & \cdots & y_{p^{\mathrm{N}}} \\ \mu(y_1) & \cdots & \mu(y_j) & \cdots & \mu(y_{p^{\mathrm{N}}}) \end{bmatrix},$$

where $y_j = (x_{j_1}, x_{j_2}, \ldots, x_{j_N})$, $j = 1, 2, \ldots, p^N$, and $x_{j_1}, x_{j_2}, \ldots, x_{j_N} \in \{x_1, x_2, \ldots, x_p\}$.

Theorem 1 *Suppose ξ is a discrete memoryless information source on quasi-probability space, and $\zeta = (\xi_1, \xi_2, \ldots, \xi_N)$ is the N-$thextension$ of ξ, where $\xi_1, \xi_2, \ldots, \xi_N$ are independent and identically distributed quasi-random variables. Then the sequence $\frac{1}{N} I(y_j)$ is convergent in quasi-probability to $H(\xi)$.*

Proof It follows from Definition 5 that $E[I(x_{j_k})] = H(\xi)$. Thus $\frac{1}{N}\sum_{k=1}^{N} I(x_{j_k})$ is convergent in probability to $H(\xi)$ according to the law of large numbers on probability measure space.

Since

$$\frac{I(y_j)}{N} = \frac{-\log\left(\prod_{k=1}^{N} \theta \circ \mu_{j_k}\right)}{N} = \frac{\sum_{k=1}^{N} I(x_{j_k})}{N},$$

we can obtain that $\frac{I(y_j)}{N}$ is convergent in probability to $H(\xi)$, i.e. ,

$$\theta \circ \mu\left(\left\{\left|\frac{I(y_j)}{N} - H(\xi)\right| \geq \varepsilon\right\}\right) < \theta(\delta), \forall \varepsilon > 0, \forall \delta > 0, \exists N_0,$$

when $N \geq N_0$.

It follows from Definition 2 that

$$\mu\left(\left\{\left|\frac{I(y_j)}{N} - H(\xi)\right| \geq \varepsilon\right\}\right) < \theta^{-1} \circ \theta(\delta) \leq \delta.$$

Hence, the sequence $\frac{1}{N} I(y_j)$ is convergent in quasi-probability to $H(\xi)$.

Theorem 1 enables us to divide the set of all sequences into two sets, the typical set, where the mean of self-information is close to information entropy $H(\xi)$, and the nontypical set, which contains the other sequences. In other words, Theorem 1 shows an asymptotic equipartition property (AEP).

Definition 6 Let $y_j = \left(x_{j_1}, x_{j_2}, \ldots, x_{j_N}\right)$ denote sequence of length N, where $x_{j_1}, x_{j_2}, \ldots, x_{j_N} \in \left\{x_1, x_2, \ldots, x_p\right\}$. If $\forall \varepsilon > 0$, $\left|\frac{I(y_j)}{N} - H\left(\xi\right)\right| < \varepsilon$, then y_j is said to be a typical sequence, otherwise, y_j is said to be a nontypical sequence. In this paper, the set of typical sequences is denoted by $G_{\varepsilon N}$, and the set of nontypical sequences is denoted by $\bar{G}_{\varepsilon N}$.

Lemma 1 *If* $y_j = \left(x_{j_1}, x_{j_2}, \ldots, x_{j_N}\right) \in G_{\varepsilon N}$, *then*

$$\theta^{-1}\left(2^{-N[H(\xi)+\varepsilon]}\right) < \mu\left(y_j\right) < \theta^{-1}\left(2^{-N[H(\xi)-\varepsilon]}\right).$$

Proof It follows from $y_j = \left(x_{j_1}, x_{j_2}, \ldots, x_{j_N}\right) \in G_{\varepsilon N}$ that

$$-\varepsilon < \frac{I\left(y_j\right)}{N} - H\left(\xi\right) < \varepsilon,$$

and therefore

$$N\left(H(\xi) - \varepsilon\right) < I(y_j) < N\left(H(\xi) + \varepsilon\right).$$

According to Definition 4, we have

$$2^{-N(H(\xi)+\varepsilon)} < \theta \circ \mu(y_j) < 2^{-N(H(\xi)-\varepsilon)}.$$

Hence, $\theta^{-1}\left(2^{-N(H(\xi)+\varepsilon)}\right) < \mu\left(y_j\right) < \theta^{-1}\left(2^{-N(H(\xi)-\varepsilon)}\right)$.

Lemma 2 *Let* $\|G_{\varepsilon N}\|$ *denote the number of elements in the set* $G_{\varepsilon N}$. *Then*

$$\left(1 - \theta\left(\delta\right)\right) 2^{N(H(\xi)-\varepsilon)} \leq \|G_{\varepsilon N}\| \leq 2^{N[H(\xi)+\varepsilon]}.$$

Proof The lower bound can be derived as follows:

It follows from Theorem 1 that

$$\mu\left(\left\{\left|\frac{I(y_j)}{N} - H(\xi)\right| \geq \varepsilon\right\}\right) < \delta.$$

Hence, $\theta \circ \mu\left(\left\{\left|\frac{I(y_j)}{N} - H(\xi)\right| \geq \varepsilon\right\}\right) < \theta\left(\delta\right)$ and $\left(1 - \theta\left(\delta\right)\right) < \theta \circ \mu(G_{\varepsilon N})$.

Then we can get

$$\begin{aligned}(1-\theta\,(\delta)) &< \theta\circ\mu(G_{\varepsilon N})\\ &\le \|G_{\varepsilon N}\|\cdot\max_{y_j\in G_{\varepsilon N}}\theta\circ\mu(y_j)\\ &\le \|G_{\varepsilon N}\|\cdot 2^{-N(H(\xi)-\varepsilon)},\end{aligned}$$

when N is sufficiently large.
The upper bound can be derived as follows:

$$\begin{aligned}1 &= \sum_{j=1}^{p^N}\theta\circ\mu\left(y_j\right)\\ &\ge \sum_{y_j\in G_{\varepsilon N}}\theta\circ\mu\left(y_j\right)\\ &\ge \sum_{y_j\in G_{\varepsilon N}}2^{-N(H(\xi)+\varepsilon)} = \|G_{\varepsilon N}\|\cdot 2^{-N(H(\xi)+\varepsilon)}.\end{aligned}$$

Hence, $\|G_{\varepsilon N}\|\le 2^{N(H(\xi)+\varepsilon)}$. The theorem is proved.

As a consequence of Lemma 2, we have

$$\eta=\frac{\|G_{\varepsilon N}\|}{p^N}\le\frac{2^{N[H(\xi)+\varepsilon]}}{p^N}=2^{-N[\log p-H(\xi)-\varepsilon]}.$$

Since $H\,(\xi) < \log p$ by the property of information entropy of quasi-random variable, we have $\log p - H\,(\xi) - \varepsilon > 0$, and therefore, $\eta\to 0$ when $N\to\infty$, that is $\left\|\bar{G}_{\varepsilon N}\right\| > \|G_{\varepsilon N}\|$.

Theorem 2 (A Fixed-length source coding theorem) *Let ξ be a discrete memoryless information source on quasi-probability space, and $\zeta=(\xi_1,\xi_2,\ldots,\xi_N)$ be the N-th extension of ξ, where $\xi_1,\xi_2,\ldots,\xi_N$ are independent and identically distributed quasi-random variables. Suppose we encode sequence y_i from ζ into a codeword with length l using code alphabet $A=\{a_1,a_2,\ldots,a_r\}$. Then the quasi-probability of decoding failure can be made arbitrarily small when N is sufficiently large if $\frac{l}{N}\ge\frac{H(\xi)+\varepsilon}{\log r}$, $\forall\varepsilon>0$.*

Proof According to Theorem 1, we have that sequences y_j from ζ can be divided into two sets: typical sequences $G_{\varepsilon N}$ and nontypical sequences $\bar{G}_{\varepsilon N}$. Notice that $\mu(\bar{G}_{\varepsilon N}) < \delta$ and $\theta^{-1}\,[(1-\theta\,(\delta))] < \mu(G_{\varepsilon N})$, then $\mu(\bar{G}_{\varepsilon N})\to 0$, $\mu(G_{\varepsilon N})\to 1$ when N is sufficiently large.

It follows from Lemma 2 that $\eta = \frac{\|G_{\varepsilon N}\|}{p^N} \to 0$ when N is sufficiently large, so that our attention can be restricted to these typical sequences. Consequently, to prove the theorem, we need to prove that the number r^l of codeword satisfies $r^l \geq \|G_{\varepsilon N}\|$.

If the length of codeword l satisfies $\frac{l}{N} \geq \frac{H(\xi)+\varepsilon}{\log r}$, then $\log r^l \geq N\left(H(\xi) + \varepsilon\right)$, so $r^l \geq 2^{N(H(\xi)+\varepsilon)}$.

It follows from Lemma 2 that $2^{N[H(\xi)+\varepsilon]} \geq \|G_{\varepsilon N}\|$.

Hence, $r^l \geq \|G_{\varepsilon N}\|$, which implies that every typical sequence $y_j \in G_{\varepsilon N}$ has its own codeword. Certainly, there exists encode failure since nontypical sequences aren't under consideration, however, the quasi-probability of sequence y_j from ζ being nontypical sequence tends to 0 when N is sufficiently large.

4 Conclusion

A fixed-length source coding theorem on quasi-probability space is initially derived in this paper to investigate the source coding problems on quasi-probability space. The proposed fixed-length source coding theorem can be considered as a generalization of the source coding theorems on probability measure space and Sugeno measure space.

Topics for future research may include the variable-length source coding theorem and the source coding theory with continuous quasi-random information source, which aim to lay a theoretical foundation for source coding problems on quasi-probability space.

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References

1. Shannon, C.E.: A mathematical theory of communication. Bell Syst. Tech. J. *27*(3&4), 379–423, 623–656 (1948)
2. Cover, T.M., Thomas, J.A.: Elements of Information Theory, 2nd edn. Wiley, New York (2006)
3. Noll, P.: Digital audio coding for visual communications. Proc. IEEE **83**(6), 925–943 (1995)
4. Atallah, M., Genin, Y., Szpankowski, W.: Pattern matching image compression: algorithmic and empirical results. IEEE Trans. Pattern Anal. Mach. Intell. **21**(7), 614–627 (1999)
5. Girod, B., Margot, A., Rane, S., Rebollo-Monedero, D.: Distributed video coding. Proc. IEEE **93**(1), 71–83 (2005)
6. Schonberg, D., Draper, S.C., Yeo, C.H., Ramchandran, K.: Toward compression of encrypted images and video sequences. IEEE Trans. Inf. Forensic. Secur. **3**(4), 749–762 (2008)
7. Klinc, D., Hazay, C., Jagmohan, A., Krawczyk, H., Rabin, T.: On compression of data encrypted with block ciphers. IEEE Trans. Inf. Theory **58**(11), 6989–7001 (2012)
8. Toto-Zarasoa, V., Roumy, A., Guillemot, C.: Source modeling for distributed video coding. IEEE Trans. Circ. Syst. Video Technol. **22**(2), 174–187 (2012)

9. Bercher, J.F.: Source coding with escort distributions and Rényi entropy bounds. Phys. Lett. A **373**, 3235–3238 (2009)
10. Ha, M.H., Feng, Z.F., Song, S.J., Gao, L.Q.: The key theorem and the bounds on the rate of uniform convergence of statistical learning theory on quasi-probability spaces. Chin. J. Comput. **31**(3), 476–485 (2008)
11. Choquet, G.: Theory of capacities. Ann. de l'institut Fourier **5**, 131–295 (1954)
12. Sugeno, M.: Theory of fuzzy integrals and its applications. Ph.D. thesis, Tokyo institute of Technology, Tokyo (1974)
13. Wang, Z.Y., Klir, G.J.: Fuzzy Measure Theory. Plenum Press, New York (1992)
14. Wang, Z.Y.: Une classe de mesures floues–les quasi-mesures. BUSEFAL **6**, 28–37 (1981)
15. Aumann, R.J., Shapley, L.S.: Values of Non-atomic Games. Princeton University Press, Princeton (1974)
16. Pap, E.: Null-additive Set Functions. Kluwer, Boston (1995)
17. Wang, Z.Y., Klir, G.J.: Generalized Measure Theory. Springer, New York (2008)
18. Ha, M.H., Yang, L.Z., Wu, C.X.: Introduction to Generalized Fuzzy Set-Valued Measure. Science Press, Beijing (2009)
19. Klir, G.J.: Generalized information theory. Fuzzy Sets Syst. **40**(1), 127–142 (1991)
20. Sgarro, A.: Possibilistic information theory: a coding-theoretic approach. Fuzzy Sets Syst. **132**(1), 11–32 (2002)
21. Sgarro, A.: An axiomatic derivation of the coding-theoretic possibilistic entropy. Fuzzy Sets Syst. **143**(3), 335–353 (2004)
22. Zhang, J.L., Yang, Q.: Information entropy of discrete information source with g_λ distribution and their application. Fuzzy Syst. Math. **23**(1), 52–58 (2009)
23. Ha, M.H., Wang, C., Zhang, Z.M., Tian, D.Z.: Uncertainty Statistical Learning Theory. Science Press, Beijing (2010)
24. Yang, Y., Wang, C., Ha, M.H.: Information entropy of discrete quasi-random variables and its properties. In: The 5th International Conference on Fuzzy Information and Engineering, Cheng Du, China, pp. 165–172 (2011)

The Distribution of Zeros of Solutions of Differential Equations with a Variable Delay

Dong-hai Peng and Liu-wei Zhang

Abstract This paper is concerned with the distribution of zeros of solutions of the first order linear differential equations with a variable delay of the form

$$x'(t) + P(t)x(\tau(t)) = 0, \quad t \geq t_0,$$

where $P, \tau \in C([t_0, \infty), [0, \infty))$, $\tau(t) \leq t$, $\tau(t)$ is nondecreasing, and $\lim_{t\to+\infty} \tau(t) = +\infty$. By introducing a class of new series, we are able to derive sharper upper bounds on the distance between zeros of solutions of the above delay differential equations. Some examples and a table are given to support our accomplishment.

Keywords Distribution of zeros · Oscillation · Variable delay

1 Introduction

In this paper, we investigate the upper bound for the distance between adjacent zeros of solutions of the first order differential equations with a variable delay of the form

$$x'(t) + P(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $P, \tau \in C([t_0, \infty), [0, \infty))$, $\tau(t) \leq t$, $\tau(t)$ is nondecreasing, and $\lim_{t\to\infty} \tau(t) = \infty$. For simplicity of notations, we use the notation $\tau^0(t) = t$ and inductively define the iterates of τ^{-1} by

D. Peng (✉) · L. Zhang
Department of Fundamental Courses, Zhongshan Polytechnic,
Zhongshan 528404, Guangdong, Peoples Republic of China
e-mail: 22998066@qq.com

L. Zhang
Sino-French Institute of Nuclear Engineering and Technology,
Sun Yat-sen University, Zhuhai 519082, Peoples Republic of China

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$$\tau^{-i}(t) = \tau^{-1}\left(\tau^{-(i-1)}(t)\right), \quad i = 1, 2, \ldots,$$

where τ^{-1} is the inverse of τ.

In general, let $C = C([\tau^{-1}(t_0),\ t_0],\ \mathbb{R})$ denote the Banach space consisting of all continuous function from $[\tau^{-1}(t_0),\ t_0]$ to $\mathbb{R}$, which is also called Phase Space of Eq. (1). Using the method of steps, it follows that for every continuous function $\psi \in C([\tau^{-1}(t_0),\ t_0],\ \mathbb{R})$, there exists a unique solution of Eq. (1) valid for $t \geq t_0$. For further questions on existence, uniqueness and continuous dependence, see Hale [1] and Erbe [2]. As is customary, a solution $x(t)$ of Eq. (1) is said to be an eventually positive solution if $x(t) > 0$ when t is large enough. A solution of Eq. (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory.

We note that most of the existing results have been established for constant delay equations [3–6]. In [7], Zhang and Zhou firstly considered the distribution of zeros of the solutions of differential equations with a variable delay, where the following two sequences defined by

$$\begin{cases} f_0(\rho) = 1, \quad f_{n+1}(\rho) = e^{\rho f_n(\rho)}, \\ g_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \quad g_{m+1}(\rho) = 2\,(1-\rho)\,/\left(\rho^2 + \frac{2}{g_m^2(\rho)}\right), \end{cases} \tag{2}$$

with $0 < \rho < 1$, play an decided important role. After that, by constructing different sequences, many researcher established more precise results about the distribution of zeros of solutions to Eq. (1). In [8], Tang and Yu use only an increasing sequence $\{r_n(\rho)\}_{n=0}^{\infty}$ defined by

$$r_0(\rho) = 1, \quad r_1(\rho) = \frac{1}{1-\rho}, \quad r_{n+2}(\rho) = \frac{r_n(\rho)}{r_n(\rho) + 1 - e^{\rho r_n(\rho)}}. \tag{3}$$

It is easy to see that the increasing rate of the sequences defined above is better than the corresponding sequences in (2). On the base of the sequences in (3), Wu and Xu [9] give a pair of sequences defined by

$$\begin{cases} f_0(\rho) = 1, \quad f_1(\rho) = \frac{1}{1-\rho}, \quad f_{n+2}(\rho) = \frac{f_n(\rho)}{f_n(\rho)+1-e^{\rho f_n(\rho)}}, \\ \varphi_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \quad \varphi_{m+1}(\rho) = 2\left(1-\rho-\frac{1}{\varphi_m(\rho)}\right)/\rho^2. \end{cases} \tag{4}$$

More results about this topic, one can refer to [10].

In this paper, by constructing more elaborate sequences, we obtain a better upper bound estimate for the interval-length successive zeros of the solutions of Eq. (1).

To simplify the description of sequences, we define by (4)

$$\theta(\rho,\ 1) = \frac{\rho^2}{2}, \quad \theta(\rho,\ n) = \frac{e^{\rho f_{n-2}(\rho)} - 1 - \rho f_{n-2}(\rho)}{f_{n-2}^2(\rho)}, \quad n = 2, 3, \ldots.$$

In this paper, instead of above sequences, we define the following sequences

$$\begin{cases} f_0(\rho) = 1, \quad f_1(\rho) = \frac{1}{1-\rho}, \quad f_{n+1}(\rho) = \frac{1}{1-\rho-\theta(\rho,n) f_n(\rho)} \\ \varphi_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \quad \varphi_{m+1}(\rho) = 2\left(1-\rho-\frac{1}{\varphi_m(\rho)}\right)/\rho^2 \end{cases} \tag{5}$$

and

$$\begin{cases} f_0(\rho) = 1, \quad f_1(\rho) = \frac{1}{1-\rho}, \quad f_{n+1}(\rho) = \frac{1}{1-\rho-\theta(\rho,n) f_n(\rho)}, \\ \phi_1(\rho,n) = \frac{1-\rho}{\theta(\rho,n)}, \quad \phi_{m+1}(\rho,\, n) = \frac{1}{\theta(\rho,\, n)}\left(1-\rho-\frac{1}{\phi_m(\rho,\, n)}\right). \end{cases} \tag{6}$$

The properties of $\{f_n(\rho)\}$ have been discussed in [8–10]. Roughly, if $\rho > 1/e$, then either $f_n(\rho)$ is nondecreasing and $\lim_{n\to\infty} f_n(\rho) = \infty$ or $f_n(\rho)$ is negative or approaches to ∞ after a finite numbers of terms. However, for $0 < \rho \le 1/e$, we have $1 \le f_n(\rho) \le f_{n+2}(\rho) \le e$ for $n = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} f_n(\rho) = f(\rho) \in [1, e]$, where $f(\rho)$ is a real root of the equation

$$f(\rho) = e^{\rho f(\rho)}. \tag{7}$$

Moreover, if $\phi_m(\rho,\, n) > 0$ for $m = 1, 2, \ldots,$ then $\phi_{m+1}(\rho,\, n) < \phi_m(\rho,\, n)$. So we get $\phi_1(\rho, n) > (1-\rho)/(2\theta(\rho,\, n))$ when $(1-\rho)^2 \ge 4\theta(\rho,\, n)$, and

$$\begin{aligned} \phi_{m+1}(\rho,\, n) &= \frac{1}{\theta(\rho,\, n)}\left(1-\rho-\frac{1}{\phi_m(\rho,\, n)}\right) \\ &> \frac{1-\rho}{\theta(\rho,\, n)} - \frac{2}{1-\rho} \ge \frac{1-\rho}{2\theta(\rho,\, n)} \end{aligned}$$

for $m \ge 1$. Namely, the sequence $\{\phi_m\}$ is decreasing and bounded below. Thus there exists a function $\phi(\rho,\, n)$ such that $\lim_{m\to\infty} \phi_m(\rho,\, n) = \phi(\rho,\, n)$ and $\phi(\rho,\, n)$ satisfies

$$\phi(\rho,\, n) = \frac{2}{\theta(\rho,\, n)}\left(1-\rho-\frac{1}{\phi(\rho,\, n)}\right).$$

This implies

$$\phi(\rho,\, n) = \frac{1-\rho+\sqrt{(1-\rho)^2-4\theta(\rho,\, n)}}{2\theta(\rho,\, n)}. \tag{8}$$

2 Some Lemmas

In what follows, we shall assume that there exist $t_1 \ge t_0$ and positive constant ρ such that

(A_1) : $\int_{\tau(t)}^{t} P(s)ds \ge \rho$, for $t \ge t_1$.

$(A_2): \int_{\tau(u)}^{u} P(s)ds \geq \int_{\tau(t)}^{t} P(s)ds$, for $\tau(t) \leq u \leq t$.

In order to prove our main results, we present some lemmas.

Lemma 1 *Assume that* (A_1) *holds. If there exist* $T_0 \geq t_1$ *and a solution* $x(t)$ *of Eq. (1) such that* $x(t)$ *is positive on* $[T_0,\ \tau^{-N}(T_0)]$ *for some* $N \geq 3$*. Then, for some* $m \leq N-3$,

$$\frac{x(\tau(t))}{x(t)} < \varphi_m(\rho), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-m)}(T_0)],$$

where $\varphi_m(\rho)$ *is defined by (5).*

Proof From (A_1), we know that

$$\int_t^{\tau(t)} P(s)ds \geq \rho.$$

Observe that $F(\lambda) = \int_t^{\lambda} P(s)ds$ is a continuous function, $F(\tau^{-1}(t)) \geq \rho$, and $F(t) = 0$. Thus, there exists a λ_t such that $\int_t^{\lambda_t} P(s)ds = \rho$, where $t < \lambda_t \leq \tau^{-1}(t)$. When $\tau^{-3}(T_0) \leq t \leq \tau^{-(N-1)}(T_0)$, integrating both sides of Eq. (1) from t to λ_t gives

$$x(t) - x(\lambda_t) = \int_t^{\lambda_t} P(s)x(\tau(s))ds. \tag{9}$$

Since $t \leq s < \lambda_t \leq \tau^{-1}(t)$, we get that $\tau^{-2}(T_0) \leq \tau(t) \leq \tau(s) \leq \tau(\lambda_t) \leq t$. Integrating both sides of Eq. (1) from $\tau(s)$ to t gives

$$x(\tau(s)) - x(t) = \int_{\tau(s)}^{t} P(u)x(\tau(u))du. \tag{10}$$

On the other hand, from Eq. (1), we obtain

$$x'(t) = -P(t)x(\tau(t)) \leq 0,\ t \in [\tau^{-1}(T_0),\ \tau^{-N}(T_0)], \tag{11}$$

which implies that $x(t)$ is nonincreasing on $[\tau^{-1}(T_0),\ \tau^{-N}(T_0)]$. Thus, $x(\tau(u))$ is nonincreasing on $\tau^{-2}(T_0) \leq \tau(s) \leq u \leq t$. It is easy to see that

$$\begin{aligned} x\left(\tau(s)\right) &= x(t) + x\left(\tau(t)\right) \int_{\tau(s)}^{t} P(u)du \\ &= x(t) + x\left(\tau(t)\right) \left\{ \int_{\tau(s)}^{s} P(u)du - \int_t^{s} P(u)du \right\} \\ &\geq x(t) + \rho x\left(\tau(t)\right) - x\left(\tau(t)\right) \int_t^{s} P(u)du. \end{aligned} \tag{12}$$

From (9) and (12), for $t \in [\tau^{-3}(T_0),\ \tau^{-(N-1)}(T_0)]$, we get

$$\begin{aligned} x(t) &= x(\lambda_t) + \int_t^{\lambda_t} P(s)x(\tau(s))ds \\ &\geq x(\lambda_t) + \int_t^{\lambda_t} P(s)\left\{x(t) + \rho x\left(\tau(t)\right) - x\left(\tau(t)\right)\int_t^s P(u)du\right\}ds \\ &= x(\lambda_t) + \rho x(t) + \rho^2 x(\tau(t)) - x(\tau(t))\int_t^{\lambda_t} P(s)\int_t^s P(u)duds. \end{aligned} \tag{13}$$

Changing of variable for double integration gives

$$\int_t^{\lambda_t}\int_t^s P(s)P(u)duds = \int_t^{\lambda_t}\int_u^{\lambda_t} P(s)P(u)dsdu.$$

If we exchange the variable notation of integration s and u, the above equality becomes

$$\int_t^{\lambda_t} ds\int_t^s P(s)P(u)du = \int_t^{\lambda_t} ds\int_s^{\lambda_t} P(u)P(s)ds,$$

which implies

$$\begin{aligned} \int_t^{\lambda_t} ds\int_t^s P(s)P(u)du &= \frac{1}{2}\int_t^{\lambda_t} P(s)\left\{\int_t^s P(u)du + \int_s^{\lambda_t} P(u)du\right\}ds \\ &= \frac{1}{2}\int_t^{\lambda_t}\int_t^{\lambda_t} P(u)P(s)duds = \frac{\rho^2}{2}. \end{aligned}$$

Substituting this into (13), we have

$$x(t) \geq x(\lambda_t) + \rho x(t) + \frac{\rho^2}{2}x(\tau(t)) \tag{14}$$

$$\geq \rho x(t) + \frac{\rho^2}{2}x(\tau(t)), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-1)}(T_0)]. \tag{15}$$

By (15), we obtain

$$\frac{x(\tau(t))}{x(t)} < \frac{2(1-\rho)}{\rho^2} = \varphi_1(\rho), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-1)}(T_0)]. \tag{16}$$

When $\tau^{-3}(T_0) \leq t \leq \tau^{-(N-2)}(T_0)$, we easily see that $\tau^{-3}(T_0) \leq t \leq \lambda_t \leq \tau^{-1}(t) \leq \tau^{-(N-1)}(T_0)$. Thus, by (16), we have

$$x(\lambda_t) > \frac{1}{\varphi_1(\rho)}x(\tau(\lambda_t)), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-2)}(T_0)]. \tag{17}$$

Since $x(t)$ is nonincreasing on $[\tau^{-1}(T_0), \tau^{-N}(T_0)]$ and $\tau^{-2}(T_0) \leq \tau(\lambda_t) < t < \lambda_t < \tau^{-(N-1)}(T_0)$, from (17), we obtain $x(\lambda_t) > \frac{1}{\varphi_1(\rho)} x(t)$. Substituting this into (14), we get

$$x(t) > \frac{1}{\varphi_1(\rho)} x(t) + \rho x(t) + \frac{\rho^2}{2} x(\tau(t)), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-2)}(T_0)].$$

Therefore

$$\frac{x(\tau(t))}{x(t)} < \frac{2\left(1 - \rho - \frac{1}{\varphi_1(\rho)}\right)}{\rho^2} = \varphi_2(\rho), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-2)}(T_0)].$$

Repeating the procedure just described gives

$$\frac{x(\tau(t))}{x(t)} < \frac{2\left(1 - \rho - \frac{1}{\varphi_{m-1}(\rho)}\right)}{\rho^2} = \varphi_m(\rho), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-m)}(T_0)].$$

The proof is complete.

Lemma 2 *Assume that (A_1) and (A_2) hold. Further assume $x(t)$ be a solution of Eq. (1) on $[t_0, \ \infty)$, and that there exists $T_0 \geq t_1$ such that $x(t)$ is positive on $[T_0, \ \tau^{-N}(T_0)]$ for some $N \geq 3$. Then for some $N - 2 \geq n \geq 1$,*

$$\frac{x(\tau(t))}{x(t)} \geq f_n(\rho) > 0, \quad t \in [\tau^{-(2+n)}(T_0),\ \tau^{-N}(T_0)],$$

where $f_n(\rho)$ is defined by (5).

Proof From (11), we know that $x(t)$ is nonincreasing on $[\tau^{-1}(T_0),\ \tau^{-N}(T_0)]$. It follow that

$$\frac{x(\tau(t))}{x(t)} \geq 1 = f_0(\rho) \qquad \text{for} \ \ t \in [\tau^{-2}(T_0),\ \tau^{-N}(T_0)]. \tag{18}$$

When $\tau^{-3}(T_0) \leq t \leq T$, integrating Eq. (1) from $\tau(t)$ to t we get

$$\begin{aligned} x(\tau(t)) &= x(t) + \int_{\tau(t)}^{t} P(s) x(\tau(s)) ds \\ &\geq x(t) + \rho x(\tau(t)). \end{aligned}$$

So

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{1 - \rho} = f_1(\rho) > 0, \quad t \in [\tau^{-3}(T_0),\ \tau^{-N}(T_0)]. \tag{19}$$

Next we proof that

$$\frac{x(\tau(t))}{x(t)} \geq f_2(\rho) > 0, \quad t \in [\tau^{-4}(T_0),\ \tau^{-N}(T_0)].$$

From (11), (14), and (19) we obtain

$$\begin{aligned} x(t) &\geq x\left(\tau^{-1}(t)\right) + \rho x(t) + \frac{\rho^2}{2} x(\tau(t)) \\ &\geq x\left(\tau^{-1}(t)\right) + \rho x(t) + \frac{\rho^2}{2(1-\rho)} x(t) \\ &= x\left(\tau^{-1}(t)\right) + \rho x(t) + \theta(\rho,\ 1) f_1(\rho) x(t), \quad t \in [\tau^{-3}(T_0),\ \tau^{-(N-1)}(T_0)], \end{aligned}$$

which shows that

$$0 < x\left(\tau^{-1}(t)\right) \leq \frac{2 - 4\rho + \rho^2}{2(1-\rho)} x(t) = (1 - \rho - \theta(\rho,\ 1) f_1(\rho)) x(t).$$

Therefore, $0 < x(t) \leq (1-\rho-\theta(\rho,\ 1) f_1(\rho)) x(\tau(t))$, for $t \in [\tau^{-4}(T_0),\ \tau^{-N}(T_0)]$. That is, for $t \in [\tau^{-4}(T_0),\ \tau^{-N}(T_0)]$

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{1 - \rho - \theta(\rho,\ 1) f_1(\rho)} = f_2(\rho) > 0. \tag{20}$$

Finally, we prove that

$$\frac{x(\tau(t))}{x(t)} \geq f_3(\rho) > 0, \quad t \in [\tau^{-5}(T_0),\ \tau^{-N}(T_0)].$$

When $\tau^{-4}(T_0) \leq t \leq \tau^{-(N-1)}(T_0)$, integrating both sides of Eq. (1) from t to λ_t gives

$$x(t) - x(\lambda_t) = \int_t^{\lambda_t} P(s) x(\tau(s)) ds. \tag{21}$$

Since $t \leq s < \lambda_t \leq \tau^{-1}(t)$, we get that $\tau^{-3}(T_0) \leq \tau(t) \leq \tau(s) \leq \tau(\lambda_t) \leq t$. Integrating both sides of Eq. (1) from $\tau(s)$ to $\tau(t)$ gives

$$x(\tau(s)) - x(t) = \int_{\tau(s)}^{t} P(u) x(\tau(u)) du. \tag{22}$$

From (21) and (22), for $t \in [\tau^{-4}(T_0),\ \tau^{-(N-1)}(T_0)]$, we have

$$
\begin{aligned}
x(t) &= x(\lambda_t) + \int_t^{\lambda_t} P(s)x(\tau(s))ds \\
&= x(\lambda_t) + \int_t^{\lambda_t} P(s)\left\{x(t) + \int_{\tau(s)}^{t} P(u)x(\tau(u))du\right\}ds \\
&= x(\lambda_t) + \int_t^{\lambda_t} P(s)\left\{x(t) + x(\tau(t))\int_{\tau(s)}^{t} P(u)\frac{x(\tau(u))}{x(\tau(t))}du\right\}ds. \quad (23)
\end{aligned}
$$

Dividing both sides of Eq. (1) by $x(t)$ and integrating from $\tau(u)$ to $\tau(t)$, we obtain

$$
\frac{x(\tau(u))}{x(\tau(t))} = \exp\left(\int_{\tau(u)}^{\tau(t)} P(\eta)\frac{x(\tau(\eta))}{x(\eta)}d\eta\right), \quad \tau(s) \le u \le t. \quad (24)
$$

Since $\tau(\tau(t)) \le \eta \le \tau(t)$, from (18) and (24), we get

$$
\frac{x(\tau(u))}{x(\tau(t))} \ge \exp\left(f_0(\rho)\int_{\tau(u)}^{\tau(t)} P(\eta)d\eta\right), \quad t \in [\tau^{-4}(T_0),\ \tau^{-(N-1)}(T_0)]. \quad (25)
$$

On the other hand, from condition (A_2) and (25), it is easy to see that

$$
\begin{aligned}
&\int_{\tau(s)}^{t} P(u)\frac{x(\tau(u))}{x(\tau(t))}du \\
&\ge \int_{\tau(s)}^{t} P(u)\exp\left(f_0(\rho)\int_{\tau(u)}^{\tau(t)} P(\eta)d\eta\right)du \\
&= \int_{\tau(s)}^{t} P(u)\exp\left[f_0(\rho)\left(\int_{\tau(u)}^{u} P(\eta) - \int_{\tau(t)}^{u} P(\eta)\right)d\eta\right]du \\
&\ge \exp\left(f_0(\rho)\left(\int_{\tau(t)}^{t} P(\eta)\right)\right)\int_{\tau(s)}^{t} P(u)\exp\left(-f_0(\rho)\int_{\tau(t)}^{u} P(\eta)d\eta\right)du \quad \text{(by } (A_2)\text{)} \\
&= \exp\left(f_0(\rho)\int_{\tau(t)}^{t} P(\eta)d\eta\right)\frac{1}{f_0(\rho)} \\
&\quad \times\left[\exp\left(-f_0(\rho)\int_{\tau(t)}^{\tau(s)} P(\eta)d\eta\right) - \exp\left(-f_0(\rho)\int_{\tau(t)}^{t} P(\eta)d\eta\right)\right] \text{(by Integration by Parts)} \\
&= \frac{1}{f_0(\rho)}\exp\left(f_0(\rho)\int_{\tau(u)}^{u} P(\eta)d\eta\right)\exp\left(-f_0(\rho)\int_{\tau(t)}^{t} P(\eta)d\eta\right) \\
&\quad \times\left[\exp\left(-f_0(\rho)\int_{t}^{\tau(s)} P(\eta)d\eta\right) - 1\right] \\
&\ge \frac{1}{f_0(\rho)}\left[e^{\rho f_0(\rho)}\exp\left(-f_0(\rho)\int_{t}^{s} P(\eta)d\eta\right) - 1\right].
\end{aligned}
$$

By the above inequality, it follows from (23) that

$$\begin{aligned} x(t) &\geq x(\lambda_t) + \rho x(t) - \frac{\rho}{f_0(\rho)} x\left(\tau(t)\right) \\ &\quad + \frac{e^{\rho f_0(\rho)}}{f_0(\rho)} \int_t^{\lambda_t} P(s) \exp\left(-f_0(\rho) \int_t^s P(\eta) d\eta\right) ds \, x\left(\tau(t)\right) \\ &\geq x(\lambda_t) + \rho x(t) + \frac{1}{f^2{}_0(\rho)} \left[e^{\rho f_0(\rho)} - 1 - \rho f_0(\rho)\right] x\left(\tau(t)\right). \end{aligned} \tag{26}$$

By $\lambda_t \leq \tau^{-1}(t)$ and (20), we obtain that for $t \in [\tau^{-4}(T_0),\ \tau^{-(N-1)}(T_0)]$,

$$x(t) \geq x(\tau^{-1}(t)) + \rho x(t) + \frac{f_2(\rho)}{f^2{}_0(\rho)} \left[e^{\rho f_0(\rho)} - 1 - \rho f_0(\rho)\right] x(t).$$

So

$$\frac{x(t)}{x(\tau^{-1}(t))} \geq \frac{1}{1 - \rho - \theta(\rho,\ 2) f_2(\rho)}, \quad t \in [\tau^{-4}(T_0),\ \tau^{-(N-1)}(T_0)],$$

which implies that

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{1 - \rho - \theta(\rho,\ 2) f_2(\rho)} = f_3(\rho) > 0, \quad t \in [\tau^{-5}(T_0),\ \tau^{-N}(T_0)].$$

Repeating the above procedure, we get for $t \in [\tau^{-(2+n)}(T_0),\ \tau^{-N}(T_0)]$

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{1 - \rho - \theta(\rho,\ n-1) f_{n-1}(\rho)} = f_n(\rho) > 0,$$

where $n = 3, 4, 5, \ldots$. The proof is complete.

Lemma 3 *Assume that* (A_2) *and* (A_2) *hold. If there exist* $T_0 \geq t_1$ *and a solution* $x(t)$ *of Eq. (1) that is positive on* $[T_0, \tau^{-N}(T_0)]$ *for some* $N \geq 5$*, then, for some* $n \geq 2$ *and* $m \geq 1$ *satisfying* $2 + n + m \leq N$,

$$\frac{x(\tau(t))}{x(t)} < \phi_m(\rho, n), \quad t \in [\tau^{-(n+2)}(T_0),\ \tau^{-(N-m)}(T_0)],$$

where $\phi_m(\rho, n)$ *is defined by (6)*

Proof As in the proof of Lemma 2, we get that (23), (24), and

$$\frac{x(\tau(t))}{x(t)} \geq f_n(\rho),\ t \in [\tau^{-(2+n)}(T_0), \tau^{-N}(T_0)],\ n = 1, 2, \ldots. \tag{27}$$

Since $\tau^2(t) \le \eta \le \tau(t)$, from (24) and (27), we get

$$\frac{x(\tau(u))}{x(\tau(t))} \ge \exp\left(f_{n-2}(\rho) \int_{\tau(u)}^{\tau(t)} P(\eta)d\eta \right), \quad t \in [\tau^{-(2+n)}(T_0),\ T]. \tag{28}$$

Similar to the proof of Lemma 2, from (A_2) and (28), it is easy to see that

$$\int_{\tau(s)}^{t} P(u)\frac{x(\tau(u))}{x(\tau(t))}du \quad \ge \frac{1}{f_{n-2}(\rho)}\left[e^{\rho f_{n-2}(\rho)} \exp\left(-f_{n-2}(\rho)\int_t^s P(\eta)d\eta\right) - 1\right].$$

From (23), we get

$$x(t) \ge x(\lambda_t) + \rho x(t) + \frac{x\,(\tau(t))}{f^2{}_{n-2}(\rho)}\left[e^{\rho f_{n-2}(\rho)} - 1 - \rho f_{n-2}(\rho)\right]. \tag{29}$$

Thus

$$x(t) > \rho x(t) + \frac{x\,(\tau(t))}{f^2{}_{n-2}(\rho)}\left[e^{\rho f_{n-2}(\rho)} - 1 - \rho f_{n-2}(\rho)\right],$$

which implies that, for $t \in [\tau^{-(2+n)}(T_0), \tau^{-(N-1)}(T_0)]$,

$$\frac{x(\tau(t))}{x(t)} < \frac{(1-\rho)f_{n-2}^2(\rho)}{e^{\rho f_{n-2}(\rho)} - 1 - \rho f_{n-2}(\rho)} = \phi_1(\rho, n).$$

When $\tau^{-(2+n)}(T_0) \le t \le \tau^{-(N-2)}(T_0)$, it is easy to see that $\tau^{-(2+n)}(T_0) \le t \le \lambda_t \le \tau^{-1}(t) \le \tau^{-(N-1)}(T_0)$, and therefore, we have

$$x(\lambda_t) > \frac{1}{\phi_1(\rho, n)}x(\tau(\lambda_t)), \quad t \in [\tau^{-(2+n)}(T_0),\ \tau^{-(N-2)}(T_0)].$$

Since $x(t)$ is nonincreasing on $[\tau^{-1}(T_0), \tau^{-N}(T_0)]$ and $\tau^{-(1+n)}(T_0) \le \tau(\lambda_t) < t < \lambda_t < \tau^{-(N-1)}(T_0)$, we obtain

$$x(\lambda_t) > \frac{1}{\phi_1(\rho, n)}x(t). \tag{30}$$

Substituting (30) into (29), for $t \in [\tau^{-(2+n)}(T_0),\ \tau^{-(N-2)}(T_0)]$, we have

$$x(t) > \frac{1}{\phi_1(\rho, n)}x(t) + \rho x(t) + \frac{x\,(\tau(t))}{f^2{}_{n-2}(\rho)}\left[e^{\rho f_{n-2}(\rho)} - 1 - \rho f_{n-2}(\rho)\right].$$

By the above inequality, it follows that for $t \in [\tau^{-(2+n)}(T_0),\ \tau^{-(N-2)}(T_0)]$,

$$\frac{x(\tau(t))}{x(t)} < \frac{\left(1-\rho-\frac{1}{\phi_1(\rho,n)}\right) f_{n-2}^2(\rho)}{e^{\rho f_{n-2}(\rho)} - 1 - \rho f_{n-2}(\rho)} = \phi_2(\rho, n).$$

Repeating the procedure just described gives

$$\frac{x(\tau(t))}{x(t)} < \frac{\left(1-\rho-\frac{1}{\phi_{m-1}(\rho,n)}\right) f_{n-2}^2(\rho)}{e^{\rho f_{n-2}(\rho)}-1-\rho f_{n-2}(\rho)} = \phi_m(\rho,n),$$

where $t \in [\tau^{-(2+n)}(T_0),\ \tau^{-(N-m)}(T_0)]$. The proof is complete.

3 Main Results

Our first result is the following.

Theorem 1 *Assume that* (A_1) *with* $\rho > 1/e$ *and* (A_2) *holds. Then, for any* $T \geq t_1$, *every solution of Eq. (1) has at least one zero on* $[T,\ \tau^{-k}(T)]$, *where*

$$k = \min\{\alpha,\ \beta\} \text{ and } \begin{cases} \alpha = 2 + \min\limits_{n\geq 1, m\geq 1} \{n+m | f_n(\rho) \geq \varphi_m(\rho)\}, \\ \beta = 2 + \min\limits_{n\geq 1}\{n | f_n(\rho) < 0 \ \text{ or } \ f_n(\rho) = \infty\}, \end{cases} \tag{31}$$

where $f_n(\rho)$ *and* $\phi_m(\rho,\ n)$ *are defined by (5).*

Proof Without loss of generality, we might assume that $x(t)$ is a solution to Eq. (1) for $t \in [T, \tau^{-k}(T)]$. Suppose to the contrary that $x(t) > 0$ for $T \leq t \leq \tau^{-k}(T)$(This situation is the same when $x(t) < 0$, since $y(t) = -x(t)$ is also a solution of Eq. (1)). When $\rho > 1/e$, let $k = 2 + m^* + n^*$ satisfy

$$\varphi_{m^*}(\rho) \leq f_{n^*}(\rho), \quad t \in [T,\ \tau^{-k}(T)], \tag{32}$$

where $m^*,\ n^* \in \{1, 2, 3, \ldots\}$. From the definitions of φ_m and f_n, we find that m^* and n^* must exist. By Lemma 1, we obtain

$$\frac{x(\tau(t))}{x(t)} < \varphi_{m^*}(\rho), \quad t \in [\tau^{-3}(T),\ \tau^{-(k-m^*)}(T)]. \tag{33}$$

On the other hand, by Lemma 2, we have

$$\frac{x(\tau(t))}{x(t)} \geq f_{n^*}(\rho), \quad t \in [\tau^{-(2+n^*)}(T),\ \tau^{-k}(T)]. \tag{34}$$

Setting $t = \tau^{-(2+n^*)}(T) = \tau^{-(k-m^*)}(T)$ in (33) and (34) gives

$$f_{n^*}(\rho) \leq \frac{x(\tau^{-(1+n^*)}(T))}{x(\tau^{-(2+n^*)}(T))} < \varphi_{m^*}(\rho),$$

which contradicts (32). So $k \le \alpha = 2 + \min_{n\ge1,m\ge1} \{n + m | f_n(\rho) \ge \varphi_m(\rho)\}$. Since $f_n(\rho) > 0$, it is easy to see that $k \le \beta = 2 + \min_{n\ge1}\{n | f_n(\rho) < 0 \text{ or } f_n(\rho) = \infty\}$. The proof is complete.

Corollary 1 *In addition to the assumption of Theorem 1, further, assume that there exist constants M_i such that $\tau^{-i}(T) - T \le M_i$, $i = 1, 2, \ldots$, for any $t \ge t_1$. Then the distance between the adjacent zeros of every solution of Eq. (1) on $[t_1, \infty]$ are less than M_k, where k is defined by (31).*

In a manner similar to the proof of Theorem 1 just completed, we can prove the following theorems and corollaries.

Theorem 2 *Assume that (A_1) and (A_2) hold with $\rho > 1/e$. Then, for any $T \ge t_1$, every solution of Eq. (1) has at least one zero in $[T, \tau^{-k}(T)]$, where*

$$k = \min\{\alpha, \beta, \gamma\} \text{ and } \begin{cases} \alpha = 2 + \min\limits_{n\ge2,\, m\ge1} \{n + m | f_n(\rho) \ge \phi_m(\rho, n)\}, \\ \beta = 2 + \min\limits_{n\ge1}\{n | f_n(\rho) < 0 \text{ or } f_n(\rho) = \infty\}, \\ \gamma = 2 + \min\limits_{n\ge2,\, m\ge1} \{n + m | \phi_m(\rho, n) < 0\}, \end{cases} \tag{35}$$

where $f_n(\rho)$ and $\phi_m(\rho, n)$ are defined by (6).

Corollary 2 *In addition to the assumption of Theorem 2, further, assume that there exist constants M_i such that $\tau^{-i}(T) - T \le M_i$, $i = 1, 2, \ldots$, for any $t \ge t_1$. Then the distance between the adjacent zeros of every solution of Eq. (1) on $[t_1, \infty]$ are less than M_k, where k is defined by (35).*

When $0 < \rho \le 1/e$ in (A_1), we have following results.

Theorem 3 *Assume that (A_1) holds with $0 < \rho \le 1/e$ and (A_2) holds. Further, assume that there exists a sequence $\{T_i\}$, $T_i \to \infty$ as $i \to \infty$, such that*

$$\int_{\tau(T_i)}^{T_i} P(s)ds \ge L > \frac{1 + \ln f(\rho)}{f(\rho)} - \frac{1 - \rho - \sqrt{1 - 2\rho - \rho^2}}{2}, \tag{36}$$

where $f(\rho)$ is a real root of (7) on $[1, e]$. Then every solution of Eq. (1) has at least a zero on $[\tau^{(2+n^*)}(T_i), \tau^{-m^*}(T_i)]$, where $T_i \ge \tau^{-(2+n^*)}(t_1)$ and

$$n^* + m^* = \min_{n\ge1,\, m\ge1} \left\{ n + m | L > \frac{1 + \ln f_{n-1}(\rho)}{f_{n-1}(\rho)} - \frac{1}{\varphi_m(\rho)} \right\}, \tag{37}$$

where $\varphi_m(\rho)$ is defined by (5) and $f(\rho)$ is a real root of (7) on $[1, e]$.

Proof Without loss of generality, we might assume that $x(t)$ is a solution to Eq. (1) satisfying $x(t) > 0$ for $t \in [\tau^{(2+n^*)}(T_i), \tau^{-m^*}(T_i)]$, where $T_i \ge \tau^{-(2+n^*)}(t_1)$. By Lemma 1 and Lemma 2, we obtain

$$\frac{x(\tau(t))}{x(t)} \geq f_{n^*}(\rho), \quad t \in [T_i, \tau^{-m^*}(T_i)], \tag{38}$$

$$\frac{x(\tau(t))}{x(t)} \geq f_{n^*-1}(\rho), \quad t \in [\tau(T_i), \tau^{-m^*}(T_i)], \tag{39}$$

and

$$\frac{x(\tau(t))}{x(t)} < \phi_{m^*}(\rho, n), \quad t \in [\tau^{n^*-1}(T_i), T_i]. \tag{40}$$

From Eq. (1), it is easy to see that $x(t)$ is nonincreasing on $[\tau^{(1+n^*)}(T_i), \tau^{-m^*}(T_i)]$ and $\tau^{-(1+n)}(T_0) \leq \tau(\lambda_t) < t < \lambda_t < \tau^{-(N-1)}(T_0)$. By (37), we obtain

$$L > \frac{1 + \ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)} - \frac{1}{\varphi_m^*(\rho)}. \tag{41}$$

Since $\{f_n(\rho)\}$ is increasing, by (38), we have

$$\frac{x(\tau(T_i))}{x(T_i)} > f_{n^*-1}(\rho). \tag{42}$$

So there exists a $t_i^* \in [\tau(T_i), T_i]$ such that

$$\frac{x(\tau(T_i))}{x(t_i^*)} = f_{n^*-1}(\rho). \tag{43}$$

Integrating Eq. (1) from t_i^* to T_i, and noting that $\tau(T_i) \leq t_i^* \leq s \leq T_i$, by (39) and (40), we obtain that

$$x(t_i^*) - x(T_i) = \int_{t_i^*}^{T_i} P(s)x(\tau(s))ds \geq x(\tau(T_i)) \int_{t_i^*}^{T_i} P(s)x(\tau(s))ds,$$

which implies

$$\int_{t_i^*}^{T_i} P(s)ds \leq \frac{x(t_i^*)}{x(\tau(T_i))} - \frac{x(T_i)}{x(\tau(T_i))}. \tag{44}$$

From (40), (43) and (44), we get

$$\int_{t_i^*}^{T_i} P(s)ds \leq \frac{1}{f_{n^*-1}(\rho)} - \frac{1}{\phi_{m^*}(\rho, n)}. \tag{45}$$

Dividing both sides of Eq. (1) by $x(t)$ and integrating from $\tau(T_i)$ to t_i^*, we obtain

$$\int_{\tau(T_i)}^{t_i^*} \frac{x'(s)}{x(s)} ds = -\int_{\tau(T_i)}^{t_i^*} P(s)\frac{x(\tau(s))}{x(s)} ds \leq -f_{n^*-1}(\rho)\int_{\tau(T_i)}^{t_i^*} P(s)ds,$$

which implies

$$\int_{\tau(T_i)}^{t_i^*} P(s)ds \leq -\frac{1}{f_{n^*-1}(\rho)}\int_{\tau(T_i)}^{t_i^*} P(s)\frac{x'(s)}{x(s)} ds = \frac{\ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)}. \tag{46}$$

From (41), (45) and (46), we have

$$\int_{\tau(T_i)}^{t_i^*} P(s)ds \leq \frac{1+\ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)} - \frac{1}{\phi_{m^*}(\rho, n)} < L,$$

which contradicts (36) and completes the proof.

Corollary 3 *Assume that* (A_1) *with* $0 < \rho \leq 1/e$ *and* (A_2) *hold, and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} P(s)ds \geq L > \frac{1+\ln f(\rho)}{f(\rho)} - \frac{1-\rho-\sqrt{1-2\rho-\rho^2}}{2},$$

where $f(\rho)$ *is defined by (7). Then every solution of Eq. (1) oscillates.*

Theorem 4 *Assume that* (A_1) *holds with* $0 < \rho \leq 1/e$ *and* (A_2) *holds. Further, assume that there exists a sequence* $\{T_i\}$, $T_i \to \infty$ *as* $i \to \infty$, *such that*

$$\int_{\tau(T_i)}^{T_i} P ds \geq L > \frac{1+\ln f(\rho)}{f(\rho)} - \phi(\rho,\ n),$$

where $f(\rho)$ *is a real root of (7) on* $[1, e]$. *Then every solution of Eq. (1) has at least a zero on* $[\tau^{(2+n^*)}(T_i), \tau^{-m^*}(T_i)]$, *where* $T_i \geq \tau^{-(2+n^*)}(t_1)$ *and*

$$n^* + m^* = \min_{n\geq 1,\ m\geq 1} \left\{ n+m | L > \frac{1+\ln f_{n-1}(\rho)}{f_{n-1}(\rho)} - \frac{1}{\phi_m(\rho,\ n)} \right\},$$

where $f_n(\rho)$ *and* $\phi_m(\rho,\ n)$ *are defined by (6), and* $\phi(\rho,\ n)$ *is defined by (8).*

Corollary 4 *Assume that* (A_1) *holds with* $0 < \rho \leq 1/e$, *and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} P ds \geq L > \frac{1+\ln f(\rho)}{f(\rho)} - \phi(\rho,\ n),$$

where $f(\rho)$ *is a real root of (7) on* $[1, e]$ *and* $\phi(\rho,\ n)$ *is defined by (8). Then every solution of Eq. (1) oscillates.*

4 Some Examples

In this section, we will show the application of our estimate for the distance between adjacent zeros of oscillatory solutions by two examples and a table.

Example 4.1 Consider the delay differential equation as follows:

$$x'(t) + \frac{10+9t}{20t}x(t-1) = 0.$$

It is easy to see choosing $\rho = 0.45$, by Theorem 3.1 in [9], we have

$$\begin{cases} \varphi_1 = 5.4320\cdots, \ \varphi_2 = 3.6138\cdots, \varphi_3 = 2.6991\cdots, \\ f_1 = 1.8181\cdots, \ f_2 = 2.3164\cdots, \ f_3 = 3.2949\cdots. \end{cases}$$

Hence $f_3 = 3.2949\cdots \geq \varphi_3 = 2.6991\cdots$, we get $k = 2+3+3 = 8$. On the other hand, by Theorem 1 in this paper, we get

$$\begin{cases} \varphi_1 = 5.4320\cdots, \ \varphi_2 = 3.6138\cdots, \varphi_3 = 2.6991\cdots, \\ f_1 = 1.8181\cdots, \ f_2 = 2.7329\cdots, \ f_3 = 3.2949\cdots. \end{cases}$$

Obviously, $f_2 = 2.7329\cdots \geq \varphi_3 = 2.6991\cdots$, we get $k = 2+2+3 = 7$. Our results improve the corresponding theorems in [3–9].

Example 4.2 Consider the delay differential equation as follows:

$$x'(t) + 0.4x(t-1) = 0.$$

Choosing $\rho = 0.4$, by Theorem 3.1 in [9], we have

$$\begin{cases} \varphi_1 = 7.5000\cdots, \ \varphi_2 = 5.8333\cdots, \varphi_4 = 5.1667\cdots, \ \varphi_5 = 5.0806\cdots. \\ f_1 = 1.6667\cdots, \ f_7 = 4.1839\cdots, \ f_8 = 5.8467\cdots, \ f_9 = -28.4195\cdots. \end{cases}$$

Hence $f_9 = -28.4195\cdots < 0$, $f_8 = 5.8467\cdots \geq \varphi_2 = 5.8333\cdots$, we get $\alpha_1 = 2+9 = 11$, $\alpha_2 = 2+8+2 = 12$, and therefore $k = 11$. By Theorem 3.1 in [10], let $n = 6$ in ϕ_m. We have

$$\begin{cases} \phi_1 = 4.5992\cdots, \ \phi_3 = 1.9854\cdots, \phi_4 = 0.7383\cdots, \ \phi_5 = -5.7820\cdots. \\ f_1 = 1.6667\cdots, \ f_7 = 4.1839\cdots, \ f_8 = 5.8467\cdots, \ f_9 = -28.4195\cdots. \end{cases}$$

Obviously, $\phi_m < \varphi_m$. Hence $f_9 = -28.4195\cdots < 0$, $\quad f_8 = 5.8467\cdots \geq \varphi_1 = 4.5992\cdots$, we get $\alpha_1 = 2+9 = 11$, $\quad \alpha_2 = 2+8+1 = 11$, and therefore $k = 11$ also.

Table 1 The comparison of the ρ

$D(x) \leq$	3τ	4τ	5τ	6τ
In [5]	$1 \leq \rho \leq \infty$	$0.6295 \leq \rho < 1$	$0.5495 \leq \rho < 0.6295$	$0.5159 \leq \rho < 0.5495$
In [7]	$1 \leq \rho \leq \infty$	$0.6290 \leq \rho < 1$	$0.5483 \leq \rho < 0.6290$	$0.5140 \leq \rho < 0.5483$
In [8]	$1 \leq \rho \leq \infty$	$0.5858 \leq \rho < 1$	$0.5307 \leq \rho < 0.5858$	$0.4778 \leq \rho < 0.5307$
In [9]	$1 \leq \rho \leq \infty$	$0.5858 \leq \rho < 1$	$0.5307 \leq \rho < 0.5858$	$0.4778 \leq \rho < 0.5307$
Here	$1 \leq \rho \leq \infty$	$0.5858 \leq \rho < 1$	$0.4890 \leq \rho < 0.5858$	$0.4446 \leq \rho < 0.4890$

Finally, in our paper, by Theorem 3.2, let $n = 6$ in ϕ_m, we have

$$\begin{cases} \phi_1 = 5.0386\cdots, \ \phi_2 = 3.3719\cdots, \ \phi_3 = 2.5481\cdots, \ \phi_4 = 1.7430\cdots. \\ f_1 = 1.6666\cdots, \ f_2 = 2.7247\cdots, \ f_6 = 3.6292\cdots, \ f_8 = 10.1133\cdots. \end{cases}$$

Hence $f_6 = 3.6292\cdots \geq \varphi_2 = 3.3719\cdots$, we get $\alpha = 2 + 6 + 2 = 10$, and therefore $k = 10$. That is to say, our results improve the Theorem 3.1 in [10].

Below is a table, the results of the Theorem 1 compared with the previous corresponding results of the literature (Table 1).

References

1. Hale, J.K.: Theory of Functionai Differential Equations. Springer, New York (1977)
2. Erbe, L.H., Kong, Q., Zhang, B.G.: Oscillation Theory for Functional Differential Equations. Marcel Dekker, New York (1995)
3. Liang, F.X.: The distribution of zeros of solution of first order delay differential equations. J. Math. Anal. Appl. **186**, 383–392 (1994)
4. Domshlsak, Y., Stvroulakis, I.P.: Oscillation of first order differential equations in a critical state. Appl. Anal. **61**, 359–379 (1996)
5. Zhou, Y., Liu, Z.R., Yu, Y.H.: An estimate for distance between adjacent zeros of solutions of neutral delay differential equations. Acta Math. Applicatae Sinica **21**(4), 505–512 (1998)
6. Zhou, Y.: The distribution of zeros of solutions of first order functional differential equations. Bull. Austral. Math. Soc. **59**, 305–314 (1999)
7. Zhang, B.G., Zhou, Y.: The distribution of zeros of solutions of differential equations with a variable delay. J. Math. Anal. Appl. **256**, 216–228 (2001)
8. Tang, X.H., Yu, J.S.: The maximum existence interval of positive solutions of first order delay differential inequalities with applications. Math. Pract. Theory **30**(4), 447–452 (2000)
9. Wu, H.W., Xu, Y.T.: The distribution of zeros of solutions of neutral differential equations. Appl. Math. Comput. **156**(3), 665–677 (2004)
10. Wu, H.W., Cheng, S.S., Wang, Q.R.: The distribution of zeros of solutions of functional differential equations. Appl. Math. Comput. **193**(3), 154–161 (2007)

An Approach to Dual Hesitant Fuzzy Soft Set Based on Decision Making

Yan-ping He

Abstract By combining the dual hesitant fuzzy set and soft set models, the purpose of this paper is to introduce the concept of dual hesitant fuzzy soft sets. Then, we present an adjustable approach to dual hesitant fuzzy soft sets based on decision making and some numerical examples are provided to illustrate the developed approach.

Keywords Soft set · Dual hesitant fuzzy soft set · Level soft set

1 Introduction

Since existing uncertainty theories such as fuzzy sets [1], intuitionistic fuzzy sets [2, 3] and interval-valued fuzzy sets [4] have their inherent difficulties, the soft set theory, originally initiated by Molodtsov [5] as a new mathematical tool for dealing with uncertainties, is free from the inadequacy of the parameterization tools of those theories. Recently, research works on soft sets are very active and progressing rapidly and many fruitful results have been achieved in the theoretical aspect by combining soft sets with other mathematical structures [6–11].

Furthermore, Zhu et al. [12] proposed dual hesitant fuzzy sets [13, 14], which encompass fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets, and fuzzy multisets [15] as special cases, and investigated the basic operations and properties of DHFSs, and then they gave the application of DHFSs in group forecasting. The DHFS is a comprehensive set encompassing several existing sets, and whose membership degrees and nonmembership degrees are represented by two sets of possible values. Therefore, it has the desirable characteristics and advantages of its own and appears to be a more flexible method to be valued in multifold ways according to the practical

Y. He (✉)
School of Electrical Engineering, Northwest University for Nationalities,
Lanzhou 730030, China
e-mail: he_yanping@126.com

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demands than the existing fuzzy sets, taking into account much more information given by decision makers.

The purpose of the present paper is to present the new hybrid model called dual hesitant fuzzy soft sets by combining dual hesitant fuzzy set and soft set, and attempt to build up a general framework of the decision methodology based on dual hesitant fuzzy soft set theory. Because the new hybrid model includes both ingredients of dual hesitant fuzzy set and soft set, it is more flexible and effective to cope with imperfect and imprecise information than dual hesitant fuzzy set and soft set.

2 Preliminaries

2.1 Soft Sets and Fuzzy Soft Sets

The concept of soft sets is defined as follows:

Definition 1 ([5]) Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a soft set over U if $F : E \to P(U)$, where $P(U)$ is the set of all subsets of U.

Combining fuzzy sets and soft sets, Maji et al. [7] initiated the following hybrid model called fuzzy soft sets, which can be seen as an extension of both fuzzy sets and crisp soft sets.

Definition 2 ([7]) Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a fuzzy soft set over U if $F : E \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.

2.2 Dual Hesitant Fuzzy Sets

Definition 3 ([12]) Let U be a fixed set. A dual hesitant fuzzy (DHF, for short) set $\mathbb{D}$ on U is described as:

$$\mathbb{D} = \{\langle x, h_{\mathbb{D}}(x), g_{\mathbb{D}}(x)\rangle | x \in U\},$$

in which $h_{\mathbb{D}}(x)$ and $g_{\mathbb{D}}(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and non-membership degrees of the element $x \in U$ to the set $\mathbb{D}$ respectively, with the conditions: $0 \le \gamma, \eta \le 1$ and $0 \le \gamma^+ + \eta^+ \le 1$, where for all $x \in U$, $\gamma \in h_{\mathbb{D}}(x)$, $\eta \in g_{\mathbb{D}}(x)$, $\gamma^+ \in h^+_{\mathbb{D}}(x) = \cup_{\gamma \in h_{\mathbb{D}}(x)} max\{\gamma\}$, $\eta^+ \in g^+_{\mathbb{D}}(x) = \cup_{\eta \in g_{\mathbb{D}}(x)} max\{\eta\}$.

For convenience, the pair $d(x) = (h_{\mathbb{D}}(x), g_{\mathbb{D}}(x))$ is called a DHF element denoted by $d = (h, g)$. The set of all DHF sets on U is denoted by $DHF(U)$.

Remark 1 From Definition 3, we can see that it consists of two parts, that is, the membership hesitancy function and the nonmembership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and can handle two kinds of hesitancy in this situation. The existing sets, including fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets, and fuzzy multisets, can be regarded as special cases of DHF sets.

Example 1 Let $U = \{x_1, x_2\}$ be a reference set. $d(x_1) = (h_{\mathbb{D}}(x_1), g_{\mathbb{D}}(x_1)) = (\{0.3, 0.4\}, \{0.7\})$ and $d(x_2) = (h_{\mathbb{D}}(x_2), g_{\mathbb{D}}(x_2)) = (\{0.1, 0.5\}, \{0.3, 0.8\})$ be the DHF elements of $x_i (i = 1, 2)$ to a set $\mathbb{D}$, respectively. Then $\mathbb{D}$ can be considered as a DHF set, that is,

$$\mathbb{D} = \{\langle x_1, \{0.3, 0.4\}, \{0.7\}\rangle, \langle x_2, \{0.1, 0.5\}, \{0.3, 0.8\}\rangle\}.$$

3 Dual Hesitant Fuzzy Soft Sets

In the section, we first propose the concept of dual hesitant fuzzy soft sets by integrating dual hesitant fuzzy sets with soft sets.

Definition 4 Let (U, E) be a soft universe and $A \subseteq E$. A pair $\mathfrak{S} = (\tilde{F}, A)$ is called a dual hesitant fuzzy soft set over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : A \to DHF(U)$.

In general, $\tilde{F}(e)$ can be written as

$$\tilde{F}(e) = \{\langle x, h_{\tilde{F}(e)}(x), g_{\tilde{F}(e)}(x)\rangle | x \in U\},$$

where $h_{\tilde{F}(e)}(x)$ and $g_{\tilde{F}(e)}(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and non-membership degrees that object x holds on parameter e, respectively.

Remark 2 From Remark 1, we know that fuzzy sets, intuitionistic fuzzy sets and hesitant fuzzy sets can be regarded as special cases of dual hesitant fuzzy sets. Therefore, under certain conditions a dual hesitant fuzzy soft set can degenerate to fuzzy soft sets, intuitionistic fuzzy soft sets and hesitant fuzzy soft sets, respectively.

For convenience, we denote $\tilde{F}$ as $(\tilde{F}, E)$. If $A \subseteq E$, we can also denote a dual hesitant fuzzy soft set by $(\tilde{F}, A)$.

Example 2 Let U be a set of four participants performing dance programme, which is denoted by $U = \{x_1, x_2, x_3, x_4\}$. Let E be a parameter set, where $E = \{e_1, e_2, e_3\} = \{$confident; creative; graceful$\}$. Suppose that there are three judges who are invited to evaluate the membership degrees and non-membership degrees of a candidate x_j to a parameter e_i with several possible values in [0, 1]. Then dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$ defined as below gives the evaluation of the performance of candidates by three judges.

Table 1 Dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$

U	e_1	e_2	e_3
x_1	{0.6,0.7,0.8},{0.3,0.2,0.1}	{0.5,0.6,0.4},{0.4,0.3,0.2}	{0.4,0.4,0.3},{0.7,0.6,0.6}
x_2	{0.4,0.5,0.6},{0.3,0.2,0.1}	{0.5,0.4,0.3},{0.5,0.3,0.3}	{0.5,0.7,0.7},{0.3,0.2,0.2}
x_3	{0.8,0.7,0.7},{0.2,0.1,0.1}	{0.7,0.8,0.8},{0.2,0.2,0.1}	{0.5,0.6,0.7},{0.3,0.2,0.1}
x_4	{0.3,0.4,0.4},{0.6,0.5,0.4}	{0.5,0.6,0.6},{0.4,0.3,0.2}	{0.7,0.6,0.8},{0.2,0.1,0.1}

$$\begin{aligned}\tilde{F}(e_1) =&\{\langle x_1, \{0.6, 0.7, 0.8\}, \{0.3, 0.2, 0.1\}\rangle, \langle x_2, \{0.4, 0.5, 0.6\}, \{0.3, 0.2, 0.1\}\rangle,\\ &\langle x_3, \{0.8, 0.7, 0.7\}, \{0.2, 0.1, 0.1\}\rangle, \langle x_4, \{0.3, 0.4, 0.4\}, \{0.6, 0.5, 0.4\}\rangle\},\end{aligned}$$

$$\begin{aligned}\tilde{F}(e_2) =&\{\langle x_1, \{0.5, 0.6, 0.4\}, \{0.4, 0.3, 0.2\}\rangle, \langle x_2, \{0.5, 0.4, 0.3\}, \{0.5, 0.3, 0.3\}\rangle,\\ &\langle x_3, \{0.7, 0.8, 0.8\}, \{0.2, 0.2, 0.1\}\rangle, \langle x_4, \{0.5, 0.6, 0.6\}, \{0.4, 0.3, 0.2\}\rangle\},\end{aligned}$$

$$\begin{aligned}\tilde{F}(e_3) =&\{\langle x_1, \{0.4, 0.4, 0.3\}, \{0.7, 0.6, 0.6\}\rangle, \langle x_2, \{0.5, 0.7, 0.7\}, \{0.3, 0.2, 0.2\}\rangle,\\ &\langle x_3, \{0.5, 0.6, 0.7\}, \{0.3, 0.2, 0.1\}\rangle, \langle x_4, \{0.7, 0.6, 0.8\}, \{0.2, 0.1, 0.1\}\rangle\}.\end{aligned}$$

The tabular representation of $\mathfrak{S} = (\tilde{F}, A)$ is shown in Table 1.

All the available information on these participants performing dance programme can be characterized by a dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$. In Table 1, we can see that the precise evaluation for an alternative to satisfy a criterion is unknown while possible membership degrees and non-membership degrees of such an evaluation are given. For example, we cannot present the precise membership degree and non-membership degree of how confident the candidate x_1 performing dance programme is, but we have a certain hesitancy in providing the membership degree and non-membership degree of the candidate x_1 performing dance programme with respect to the parameter confident e_1. In other words, we provide three possible values 0.6, 0.7 and 0.8 to depict the degree of how confident the candidate x_1 is and three possible values 0.3, 0.2 and 0.1 to depict the degree of how confident the candidate x_1 is not.

4 An Adjustable Approach to Dual Hesitant Fuzzy Soft Sets Based on Decision Making

Since DHF sets were introduced by Zhu [12], DHF set theory has been applied in dealing with uncertainty decision making problems [16–18]. In the current section, we shall present an adjustable approach to dual hesitant fuzzy soft sets based on decision making problems.

Firstly, we introduce the score function of HF elements given by Xia and Xu to compare two HF elements.

Definition 5 ([19]) For a HF element $h_{\mathbb{A}}(x)$,

$$s(h_{\mathbb{A}}(x)) = \frac{\sum_{\gamma \in h_{\mathbb{A}}(x)} \gamma}{l(h_{\mathbb{A}}(x))}$$

is called the score function of $h_{\mathbb{A}}(x)$, where $l(h_{\mathbb{A}}(x))$ is the number of the elements in $h_{\mathbb{A}}(x)$.

On the basis of the score function in Definition 5, we will further introduce the concept of level soft sets of dual hesitant fuzzy soft sets.

Definition 6 Let $\mathfrak{S} = (\tilde{F}, A)$ be the dual hesitant fuzzy soft set over U, where $A \subseteq E$ and E is the parameter set. For $(\alpha, \beta) \in L^*$, the (α, β)-level soft set of $\mathfrak{S}$ is a crisp soft set $L(\mathfrak{S}; \alpha, \beta) = (\tilde{F}_{(\alpha,\beta)}, A)$ defined by

$$\begin{aligned} \tilde{F}_{(\alpha,\beta)}(e) = L(\tilde{F}(e); \alpha, \beta) &= \{x \in U | (s(h_{\tilde{F}(e)}(x)), s(g_{\tilde{F}(e)}(x))) \geq_{L^*} (\alpha, \beta)\} \\ &= \{x \in U | s(h_{\tilde{F}(e)}(x)) \geq \alpha, s(g_{\tilde{F}(e)}(x)) \leq \beta\} \end{aligned}$$

for all $e \in A$.

To illustrate Definition 6, let us consider the dual hesitant fuzzy soft set $(\tilde{F}, A)$ shown in Example 2.

Example 3 (The (0.5, 0.2)-level soft set of the dual hesitant fuzzy soft set $\mathfrak{S}$) Reconsider Example 2. Now taking $\alpha = 0.5$ and $\beta = 0.2$, then by Definitions 5 and 6, we can obtain the following (0.5, 0.2)-level soft sets:

$$\begin{aligned} \tilde{F}_{(0.5,0.2)}(e_1) &= L(\tilde{F}(e_1); (0.5, 0.2)) \\ &= \{x \in U | s(h_{\tilde{F}(e_1)}(x)) \geq 0.5, s(g_{\tilde{F}(e_1)}(x)) \leq 0.2\} = \{x_1, x_2, x_3\}, \end{aligned}$$

which implies that $x_1, x_2,$ and x_3 are confident on the degree more than 0.5, and $x_1, x_2,$ and x_3 are not confident on the degree less than 0.2.

$$\begin{aligned} \tilde{F}_{(0.5,0.2)}(e_2) &= L(\tilde{F}(e_2); (0.5, 0.2)) \\ &= \{x \in U | s(h_{\tilde{F}(e_2)}(x)) \geq 0.5, s(g_{\tilde{F}(e_2)}(x)) \leq 0.2\} = \{x_3\}, \end{aligned}$$

$$\begin{aligned} \tilde{F}_{(0.5,0.2)}(e_3) &= L(\tilde{F}(e_3); (0.5, 0.2)) \\ &= \{x \in U | s(h_{\tilde{F}(e_3)}(x)) \geq 0.5, s(g_{\tilde{F}(e_3)}(x)) \leq 0.2\} = \{x_3, x_4\}. \end{aligned}$$

In Definition 6, the level pair (or threshold pair) assigned to each parameter is always the constant value pair $(\alpha, \beta) \in L^*$. However, in some practical applications, decision makers need to impose different thresholds on different parameters. To address this issue, we replace a constant value pair by a function as the thresholds on membership values and non-membership values, respectively.

Definition 7 Let $(\tilde{F}, A)$ be the dual hesitant fuzzy soft set over U, where $A \subseteq E$ and E is the parameter set. Let $\lambda : A \to L^*$ be an intuitionistic fuzzy set in A which is called a threshold intuitionistic fuzzy set. The level soft set of $\mathfrak{S}$ with respect to λ is a crisp soft set $L(\mathfrak{S}; \lambda) = (\tilde{F}_\lambda, A)$ defined by

$$\begin{aligned}\tilde{F}_\lambda(e) &= L(\tilde{F}(e); \lambda(e)) = \{x \in U | (s(h_{\tilde{F}(e)}(x)), s(g_{\tilde{F}(e)}(x))) \geq_{L^*} (\mu_\lambda(e), \nu_\lambda(e))\} \\ &= \{x \in U | s(h_{\tilde{F}(e)}(x)) \geq \mu_\lambda(e), s(g_{\tilde{F}(e)}(x)) \leq \nu_\lambda(e)\}\end{aligned}$$

for all $e \in A$.

In order to better understand the above definition, let us consider the following examples.

Example 4 (The mid-level soft set of dual hesitant fuzzy soft set) Let $\mathfrak{S} = (\tilde{F}, A)$ be a dual hesitant fuzzy soft sets over U, where $A \subseteq E$ and E is the parameter set. Based on dual fuzzy soft set $(\tilde{F}, A)$, we can define an intuitionistic fuzzy set $mid_{\mathfrak{S}} : A \to L^*$ by

$$\mu_{mid_{\mathfrak{S}}}(e) = \frac{1}{|U|} \sum_{x \in U} s(h_{\tilde{F}(e)}(x)), \nu_{mid_{\mathfrak{S}}}(e) = \frac{1}{|U|} \sum_{x \in U} s(g_{\tilde{F}(e)}(x)),$$

for all $e \in A$. The intuitionistic fuzzy set $mid_{\mathfrak{S}}$ is called the mid-threshold of the dual hesitant fuzzy soft set $\mathfrak{S}$. In addition, the level soft set of the dual hesitant fuzzy soft set $\mathfrak{S}$ with respect to the mid-threshold intuitionistic fuzzy set $mid_{\mathfrak{S}}$, namely $L(\mathfrak{S}; mid_{\mathfrak{S}})$ is called the mid-level soft set of $\mathfrak{S}$ and simply denoted $L(\mathfrak{S}; mid)$. In the following discussions, the mid-level decision rule will mean using the mid-threshold and considering the mid-level soft set in dual hesitant fuzzy soft sets based decision making.

Let us reconsider the dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$ with its tabular representation shown in Table 1. The mid-threshold $mid_{\mathfrak{S}}$ of $\mathfrak{S}$ is an intuitionistic fuzzy set and can be calculated as follows:

$$\mu_{mid_{\mathfrak{S}}}(e_1) = \frac{1}{|U|} \sum_{x \in U} s(h_{\tilde{F}(e_1)}(x)) = \frac{1}{4}(0.7 + 0.5 + 0.73 + 0.37) = 0.58,$$

$$\nu_{mid_{\mathfrak{S}}}(e_1) = \frac{1}{|U|} \sum_{x \in U} s(g_{\tilde{F}(e_1)}(x)) = \frac{1}{4}(0.2 + 0.2 + 0.13 + 0.5) = 0.26,$$

$$\mu_{mid_{\mathfrak{S}}}(e_2) = \frac{1}{|U|} \sum_{x \in U} s(h_{\tilde{F}(e_2)}(x)) = \frac{1}{4}(0.5 + 0.4 + 0.77 + 0.57) = 0.56,$$

$$\nu_{mid_{\mathfrak{S}}}(e_2) = \frac{1}{|U|} \sum_{x \in U} s(g_{\tilde{F}(e_2)}(x)) = \frac{1}{4}(0.3 + 0.37 + 0.17 + 0.3) = 0.29,$$

$$\mu_{mid_{\mathfrak{S}}}(e_3) = \frac{1}{|U|} \sum_{x \in U} s(h_{\tilde{F}(e_3)}(x)) = \frac{1}{4}(0.37 + 0.63 + 0.6 + 0.7) = 0.58,$$

Table 2 Tabular representation of $L(\mathfrak{S}; mid)$

U	e_1	e_2	e_3	Choice value (c_j)
x_1	1	0	0	$c_1 = 1$
x_2	0	0	1	$c_2 = 1$
x_3	1	1	1	$c_3 = 3$
x_4	0	0	1	$c_4 = 1$

$$\nu_{mid_{\mathfrak{S}}}(e_3) = \frac{1}{|U|}\sum_{x\in U} s(g_{\tilde{F}(e_3)}(x)) = \frac{1}{4}(0.63 + 0.23 + 0.2 + 0.13) = 0.30.$$

Therefore, we have

$$mid_{\mathfrak{S}} = \{\langle e_1, (0.58, 0.26)\rangle, \langle e_2, (0.56, 0.29)\rangle, \langle e_3, (0.58, 0.30)\rangle\}.$$

The mid-level soft set of $\mathfrak{S}$ is a soft set $L(\mathfrak{S}; mid) = (\tilde{F}_{mid_{\mathfrak{S}}}, A)$ and it can be calculated as follows:

$$\begin{aligned}\tilde{F}_{(0.58,0.26)}(e_1) &= L(\tilde{F}(e_1); (0.58, 0.26)) \\ &= \{x \in U | s(h_{\tilde{F}(e_1)}(x)) \geq 0.58, s(g_{\tilde{F}(e_1)}(x)) \leq 0.26\} = \{x_1, x_3\},\end{aligned}$$

$$\begin{aligned}\tilde{F}_{(0.56,0.29)}(e_2) &= L(\tilde{F}(e_2); (0.56, 0.29)) \\ &= \{x \in U | s(h_{\tilde{F}(e_2)}(x)) \geq 0.56, s(g_{\tilde{F}(e_2)}(x)) \leq 0.29\} = \{x_3\},\end{aligned}$$

$$\begin{aligned}\tilde{F}_{(0.58,0.30)}(e_3) &= L(\tilde{F}(e_3); (0.58, 0.30)) \\ &= \{x \in U | s(h_{\tilde{F}(e_3)}(x)) \geq 0.58, s(g_{\tilde{F}(e_3)}(x)) \leq 0.30\} = \{x_2, x_3, x_4\}.\end{aligned}$$

Table 2 gives the tabular representation of the mid-level soft set $L(\mathfrak{S}; mid)$. If $x_i \in \tilde{F}_{mid_{\mathfrak{S}}}(e_j)$, then $x_{ij} = 1$, otherwise $x_{ij} = 0$, where x_{ij} are the entries in Table 2.

Example 5 (The top-bottom-level soft set and bottom-bottom-level soft set of dual hesitant fuzzy soft set) Let $\mathfrak{S} = (\tilde{F}, A)$ be a dual hesitant fuzzy soft sets over U, where $A \subseteq E$ and E is the parameter set. Based on the dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$, we can define an intuitionistic fuzzy set $top - bottom_{\mathfrak{S}} : A \to L^*$ by

$$\mu_{top-bottom_{\mathfrak{S}}}(e) = \max_{x\in U} s(h_{\tilde{F}(e)}(x)), \nu_{top-bottom_{\mathfrak{S}}}(e) = \min_{x\in U} s(g_{\tilde{F}(e)}(x)),$$

for all $e \in A$. We can also define an intuitionistic fuzzy set $bottom - bottom_{\mathfrak{S}} : A \to L^*$ by

$$\mu_{bottom-bottom_{\mathfrak{S}}}(e) = \min_{x\in U} s(h_{\tilde{F}(e)}(x)), \nu_{bottom-bottom_{\mathfrak{S}}}(e) = \min_{x\in U} s(g_{\tilde{F}(e)}(x)),$$

for all $e \in A$.

The intuitionistic fuzzy set $top - bottom_{\mathfrak{S}}$ is called the top-bottom-threshold of the dual hesitant fuzzy soft set $\mathfrak{S}$. In addition, the level soft set of the dual hesitant fuzzy soft set $\mathfrak{S}$ with respect to the top-bottom-threshold intuitionistic fuzzy set $top - bottom_{\mathfrak{S}}$, namely $L(\mathfrak{S}; top - bottom_{\mathfrak{S}})$ is called the top-bottom-level soft set of $\mathfrak{S}$ and simply denoted $L(\mathfrak{S}; top - bottom)$. In the following discussions, the top-bottom-level decision rule will mean using the top-bottom-threshold and considering the top-bottom-level soft set in dual hesitant fuzzy soft sets based on decision making.

The intuitionistic fuzzy set $bottom - bottom_{\mathfrak{S}}$ is called the bottom-bottom-threshold of the dual hesitant fuzzy soft set $\mathfrak{S}$. In addition, the level soft set of the dual hesitant fuzzy soft set $\mathfrak{S}$ with respect to the bottom-bottom-threshold intuitionistic fuzzy set $bottom - bottom_{\mathfrak{S}}$, namely $L(\mathfrak{S}; bottom - bottom_{\mathfrak{S}})$ is called the bottom-bottom-level soft set of $\mathfrak{S}$ and simply denoted $L(\mathfrak{S}; bottom - bottom)$. The bottom-bottom-level decision rule will mean using the bottom-bottom-threshold and considering the bottom-bottom-level soft set in dual hesitant fuzzy soft sets based on decision making.

We reconsider the dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$ with its tabular representation shown in Table 1. The top-bottom-threshold $top - bottom_{\mathfrak{S}}$ of $\mathfrak{S}$ is an intuitionistic fuzzy set and can be given as follows:

$$top - bottom_{\mathfrak{S}} = \{\langle e_1, (0.73, 0.13)\rangle, \langle e_2, (0.77, 0.17)\rangle, \langle e_3, (0.7, 0.13)\rangle\}.$$

The top-bottom-level soft set of $\mathfrak{S}$ is a soft set $L(\mathfrak{S}; top-bottom) = (\tilde{F}_{top-bottom_{\mathfrak{S}}}, A)$ and it can be calculated as follows:

$$\begin{aligned}\tilde{F}_{top-bottom_{\mathfrak{S}}}(e_1) &= L(\tilde{F}(e_1); (0.73, 0.13)) \\ &= \{x \in U | s(h_{\tilde{F}(e_1)}(x)) \geq 0.73, s(g_{\tilde{F}(e_1)}(x)) \leq 0.13\} = \{x_3\},\end{aligned}$$

$$\begin{aligned}\tilde{F}_{top-bottom_{\mathfrak{S}}}(e_2) &= L(\tilde{F}(e_2); (0.77, 0.17)) \\ &= \{x \in U | s(h_{\tilde{F}(e_2)}(x)) \geq 0.77, s(g_{\tilde{F}(e_2)}(x)) \leq 0.17\} = \{x_3\},\end{aligned}$$

$$\begin{aligned}\tilde{F}_{top-bottom_{\mathfrak{S}}}(e_3) &= L(\tilde{F}(e_3); (0.7, 0.13)) \\ &= \{x \in U | s(h_{\tilde{F}(e_3)}(x)) \geq 0.7, s(g_{\tilde{F}(e_3)}(x)) \leq 0.13\} = \{x_4\}.\end{aligned}$$

Table 3 gives the tabular representation of the top-bottom-level soft set $L(\mathfrak{S}; top - bottom)$. If $x_i \in \tilde{F}_{top-bottom_{\mathfrak{S}}}(e_j)$, then $x_{ij} = 1$, otherwise $x_{ij} = 0$, where x_{ij} are the entries in Table 3.

Similarly, the bottom-bottom-threshold $bottom - bottom_{\mathfrak{S}}$ of $\mathfrak{S}$ is an intuitionistic fuzzy set and can be given as follows:

$$bottom - bottom_{\mathfrak{S}} = \{\langle e_1, (0.37, 0.13)\rangle, \langle e_2, (0.4, 0.17)\rangle, \langle e_3, (0.37, 0.13)\rangle\}.$$

Table 3 Tabular representation of $L(\mathfrak{S}; top - bottom)$

U	e_1	e_2	e_3	Choice value (c_j)
x_1	0	0	0	$c_1 = 0$
x_2	0	0	0	$c_2 = 0$
x_3	1	1	0	$c_3 = 2$
x_4	0	0	1	$c_4 = 1$

Table 4 Tabular representation of $L(\mathfrak{S}; bottom - bottom)$

U	e_1	e_2	e_3	Choice value (c_j)
x_1	0	0	0	$c_1 = 0$
x_2	0	0	0	$c_2 = 0$
x_3	1	1	0	$c_3 = 2$
x_4	0	0	1	$c_4 = 1$

The bottom-bottom-level soft set of $\mathfrak{S}$ is a soft set $L(\mathfrak{S}; bottom - bottom) = (\tilde{F}_{bottom-bottom_{\mathfrak{S}}}, A)$ and it can be calculated as follows:

$$\tilde{F}_{bottom-bottom_{\mathfrak{S}}}(e_1) = L(\tilde{F}(e_1); (0.37, 0.13)) = \{x_3\},$$
$$\tilde{F}_{bottom-bottom_{\mathfrak{S}}}(e_2) = L(\tilde{F}(e_2); (0.4, 0.17)) = \{x_3\},$$
$$\tilde{F}_{bottom-bottom_{\mathfrak{S}}}(e_3) = L(\tilde{F}(e_3); (0.37, 0.13)) = \{x_4\}.$$

Table 4 gives the tabular representation of the bottom-bottom-level soft set $L(\mathfrak{S}; bottom - bottom)$. If $x_i \in \tilde{F}_{bottom-bottom_{\mathfrak{S}}}(e_j)$, then $x_{ij} = 1$, otherwise $x_{ij} = 0$, where x_{ij} are the entries in Table 4.

In what follows we will establish an adjustable approach to dual hesitant fuzzy soft sets based on decision making with six steps:

Algorithm 1

Step 1. Input a dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$.

Step 2. Input a threshold intuitionistic fuzzy set $\lambda : A \to L^*$ (or give a threshold value pair $(\alpha, \beta) \in L^*$; or choose the mid-level decision rule; or choose the top-bottom-level decision rule; or choose the bottom-bottom-level decision rule) for decision making.

Step 3. Compute the level soft set $L(\mathfrak{S}; \lambda)$ of $\mathfrak{S}$ with respect to the threshold intuitionistic fuzzy set λ (or the (α, β)-level soft set $L(\mathfrak{S}; (\alpha, \beta))$; or the mid-level soft set $L(\mathfrak{S}; mid)$; or the top-bottom soft set $L(\mathfrak{S}; top - bottom)$; or the bottom-bottom soft set $L(\mathfrak{S}; bottom - bottom)$).

Step 4. Present the level soft set $L(\mathfrak{S}; \lambda)$ (or $L(\mathfrak{S}; (\alpha, \beta))$; or $L(\mathfrak{S}; mid)$; or $L(\mathfrak{S}; top - bottom)$; or $L(\mathfrak{S}; bottom - bottom)$) in tabular form. For any $x_i \in U$, compute the choice value c_i of x_i.

Step 5. The optimal decision is to select x_k if $c_k = \max_i c_i$.

Step 6. If k has more than one value then any one of x_k may be chosen.

It should be noted that if there are too many optimal choices in Step 6, we may go back to the second step and change the threshold (or decision rule) such that only one optimal choice remains in the end.

Remark 3 The primary motivation for designing Algorithm 1 is to solve dual hesitant fuzzy soft set based decision making problem by using level soft sets initiated in this study. By choosing certain thresholds or decision strategies such as the mid-level or the top-bottom-level decision rules, we can convert a complex dual hesitant fuzzy soft set into a crisp soft set called the level soft set. Thus we need not treat dual hesitant fuzzy soft sets directly in decision making but only deal with the level soft sets derived from them, which provides us great convenience to decision making.

To illustrate the basic idea of Algorithm 1, we give the following example.

Example 6 Let us reconsider the decision making problem that involves the dual hesitant fuzzy soft set $\mathfrak{S} = (\tilde{F}, A)$ with tabular representation as in Table 1. Suppose that we deal with the decision making problem by mid-level decision rule. In Example 4, we have obtained the mid-level soft set $L(\mathfrak{S}; mid)$ of $\mathfrak{S}$ with respect to the threshold intuitionistic fuzzy set $mid_{\mathfrak{S}}$ with its tabular representation given by Table 2.

In Table 2, we can calculate choice values as follows:

$$\begin{aligned} c_1 &= 1 + 0 + 0 = 1, \\ c_2 &= 0 + 0 + 1 = 1, \\ c_3 &= 1 + 1 + 1 = 3, \\ c_4 &= 0 + 0 + 1 = 1. \end{aligned}$$

From the choice value listed in Table 2, we can conclude that the ranking of the four candidates is $x_3 > x_1 = x_2 = x_4$, and the best choice is x_3.

Next, we use the top-bottom-level decision rule to handle the decision making problem. Table 3 gives the tabular representation of the level soft set $L(\mathfrak{S}; top - bottom)$ of $\mathfrak{S}$. From Table 3, it follows that the ranking of the four candidates is $x_3 > x_4 > x_1 = x_2$, and the optimal decision is still to select x_3.

5 Conclusion

In this study, we propose the concept of dual hesitant fuzzy soft sets. Meanwhile, an adjustable approach to dual hesitant fuzzy soft sets based on decision making is proposed by using level soft sets and we also illustrate this novel method with some concrete examples. This new proposal is proved to be not only more suitable but more feasible for some real-life applications of decision making in an imprecise environment.

In the future, we can further investigate level soft set approach to decision making based on other extensions of hesitant fuzzy soft set theory, such as interval-valued hesitant fuzzy soft set theory and dual interval-valued fuzzy soft set theory.

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References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 352–378 (1965)
2. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**(1), 87–96 (1986)
3. Atanassov, K.: Intuitionistic Fuzzy Sets: Theory and Applications. Physica-Verlag, Heidelberg (1999)
4. Deschrijver, G., Kerre, E.E.: Implicators based on binary aggregation operators in interval-valued fuzzy set theory. Fuzzy Sets Syst. **153**, 229–248 (2005)
5. Molodtsov, D.: Soft set theory-first results. Comput. Math. Appl. **37**, 19–31 (1999)
6. Jiang, Y., Tang, Y., Chen, Q., Liu, H., Tang, J.: Interval-valued intuitionistic fuzzy soft sets and their properties. Comput. Math. Appl. **60**, 906–918 (2010)
7. Maji, P.K., Biswas, R., Roy, A.R.: Fuzzy soft set. J. Fuzzy Math. **9**(3), 589–602 (2001)
8. Maji, P.K.: More on intuitionistic fuzzy soft sets. Lect. Notes Comput. Sci. **5908**, 231–240 (2009)
9. Xu, W., Ma, J., Wang, S.Y., Hao, G.: Vague soft sets and their properties. Comput. Math. Appl. **59**, 787–794 (2010)
10. Yang, X.B., Lin, T.Y., Yang, J.Y., Li, Y., Yu, D.: Combination of interval-valued fuzzy set and soft set. Comput. Math. Appl. **58**(3), 521–527 (2009)
11. Yang, Y., Tan, X., Meng, C.C.: The multi-fuzzy soft set and its application in decision making. Appl. Math. Model. **37**, 4915–4923 (2013)
12. Zhu, B., Xu, Z.S., Xia, M.M.: Dual hesitant fuzzy sets. J. Appl. Math. **2012**, Article ID 879629, 13 (2012)
13. Torra, V., Narukawa, Y.: On hesitant fuzzy sets and decision. In: The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, pp. 1378–1382 (2009)
14. Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**, 529–539 (2010)
15. Miyamoto, S.: Multisets and fuzzy multisets. In: Liu, Z.Q., Miyamoto, S. (eds.) Soft Computing and Human-Centered Machines, pp. 9–33. Springer, Berlin (2000)
16. Chen, Y.F., Peng, X.D., Guan, G.H., Jiang, H.D.: Approaches to multiple attribute decision making based on the correlation coefficient with dual hesitant fuzzy information. J. Intell. Fuzzy Syst. **26**, 2547–2556 (2014)
17. Farhadinia, B.: Correlation for dual hesitant fuzzy sets and dual interval-valued hesitant fuzzy Sets. Int. J. Intell. Syst. **29**, 184–205 (2014)
18. Ye, J.: Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making. Appl. Math. Model. **38**, 659–666 (2014)
19. Xia, M.M., Xu, Z.S.: Hesitant fuzzy information aggregation in decision making. Int. J. Approx. Reason. **52**, 395–407 (2011)

Extremal Trees of Terminal Wiener Index

Dan-fang Chen, Fu-yi Wei, Hong-ying Zhu, Yang Wu and Shan-zhang Nong

Abstract The generalized Terminal Wiener Index is the sum of distances between each pair of vertices of degree k. In this paper, we proved the conjecture of reference [1]: when $k \geq 4$, the generalized terminal Wiener index of trees reaches its maximum value at $T_{n,k,p}$ caterpillars. We also determined the corresponding p value of the extremal graph.

Keywords Wiener index · Generalized terminal wiener index · Maximum · Extremal graph · Caterpillar

1 Introduction

Let $G = (V, E)$ be a connected simple graph with vertices set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_n\}$. The number of edges connected with vertex v is called the degree of vertex v, denoted by (v). The vertex with its degree being one is called a pendent vertex. The distance $d(u, v)$ is defined as the shortest path between u and v in G. If, for each pair of vertices u and v in G, the path $u - v$ exists, then graph G is called a connected graph. A tree, denoted by T, is a connected simple graph with no circle. Deleting an edge e from T but retaining its vertex, is denoted by $T - e$ while $T + e$ stands for adding an edge between vertices u and v on T. The caterpillar can be obtained by connecting the pendent vertices $n_i (n_i \geq 0, i = 1, 2, \ldots, s - 1)$ to the vertices of the path $= v_0 v_1 \ldots v_s$, denoted by $T(n, s; n_1 n_2 \ldots n_{s-1})$ [2]. To k subdivide an edge e is to replace the edge e with a path P_{k+1}.

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D. Chen · F. Wei (✉) · H. Zhu · Y. Wu · S. Nong
College of Science, South China Agricultural University,
Guangzhou 510642, China
e-mail: weifuyi@scau.edu.cn

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Chemical graph theory is closely related to the chemical molecular graphs. In a simple undirected graph, if every vertex represents one atom of the molecule while each edge represents the chemical bond between atoms, then such graphs are defined as molecular graphs. A topology index of molecule graph is a mapping form molecule graph to the real number set. The topology index of molecule played an important role in of QSPR [1, 3, 4] researches. Further more, with the application of graph theory in Cryptography, communication system and Internet, the topology index has also been more and more applied in other subjects.

In 1947 [5], when Wiener was studing the boiling point of the alkane molecule, he defined the Wiener Index: $W(G) = \sum_{u,v \in V(G)} d(u,v)$; Gutman and his coworkers introduced the terminal Wiener index $TW(G) = \sum_{\substack{u,v \in V(G) \\ deg(u) = deg(v) = 1}} d(u,v)$ in their paper published in 2009 [3]. Ilić and Ilić extended the terminal Wiener index to generalized terminal Wiener index $TW_k(G) = \sum_{\substack{u,v \in V(G) \\ deg(u) = deg(v) = k}} d(u,v)$ in their paper published in 2013 [1] and at the same time, both the upper bound together with the extremal graph of the generalized terminal Wiener index on n-vertex tree when $k = 3$ and a conjecture when $k > 3$ were given.

This paper studied the mathematical properties of the generalized terminal Wiener index of $T_{n,k,p}$ caterpillars, proved the conjecture of article [1], that is, when $k \geq 4$, the generalized terminal Wiener index reaches its maximum value at $T_{n,k,p}$ caterpillars. Additionally, we have figured out the corresponding p value when TW_k reaches its maximum value and have worked out the extremal graph.

2 Preliminary Study and the Proof to the Conjecture

When there is only one k-degree vertex in tree, $TW_k \equiv 0$. In this paper, we assume, there are 2 k-degree vertexes at least. We say a tree T is k-bounded if all vertices of T have degree less than or equal to k.

The lemmas below are to provide theoretical support for the subsequent research and then enable us to prove the conjecture proposed in article [1].

Lemma 2.1 ([1]) *The number of vertices of degree k ($k \geq 2$) in a tree with n vertices is no more than*

$$m = \lfloor \frac{n-2}{k-1} \rfloor$$

$T_{n,k,p}$ represents the caterpillar created by attaching pendent vertices to p vertices of path P from either end of the path to the middle successively and symmetrically, $T_{n,k,p} \cong T(n, s; \underbrace{k-2, \ldots, k-2}_{[p/2]}, \underbrace{2, \ldots, 2}_{s-p-1}, \underbrace{k-2, \ldots, k-2}_{[p/2]})$, as shown in Fig. 1.

Lemma 2.2 ([1]) *The generalized terminal Wiener index of $T_{n,k,p}$ is calculated as:*

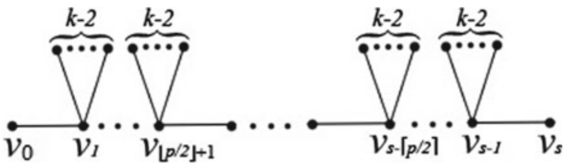

Fig. 1 The $T_{n,k,p}$ caterpillar

$$TW_k\left(T_{n,k,p}\right) = \begin{cases} \frac{1}{12}p\left(3np + 5p^2 - 3kp^2 - 2 - 6p\right), & if\ p\ is\ even \\ \frac{1}{12}\left(p+1\right)\left(p-1\right)\left(3n + 5p - 3kp - 6\right), & if\ p\ is\ odd \end{cases}$$

Lemma 2.3 ([1]) *Let T be a tree with n vertices (n > 4). Then*

$$TW_3\left(T\right) \leq TW_3(T_{n,3,[n/2]-1})$$

with the equality holds if and only if $T \cong T_{n,3,[n/2]-1}$.

Lemma 2.3 demonstrates the extremal tree of n-vertex tree when $k = 3$ and the next lemma will prove that the generalized terminal Wiener index reaches its maximum value at $T_{n,k,p}$ caterpillars when $k \geq 4$.

Theorem 2.1 For $k \geq 4$, the generalized terminal Wiener index reaches its maximum value at $T_{n,k,p}$ caterpillars.

Proof We assume that the number of vertices of degree k is greater than two.

If T^* is not a k-bounded caterpillar, consider a branching vertex ω such that in the subtree under ω there are only k-bounded caterpillars attached at ω (it can happen that there are only pendent vertices attached to ω). Let T_1 and T_2 be two caterpillars attached at ω, such that T_1 has p vertices $v_1v_2\ldots v_p$ of degree k, and T_2 has q vertices $u_1u_2\ldots u_q$ of degree k, subgraph G has r vertices of degree k, with $r \geq 2$ and $r \geq p \geq q$, as shown in Fig. 2 on the left.

Let $a_1, a_2, \ldots, a_p$ be the distances from the vertex ω to the vertices $v_1, v_2, \ldots, v_p$ and $b_1, b_2, \ldots, b_q$ be the distance from the vertex ω to the vertices $u_1, u_2, \ldots, u_q$ respectively.

Let $D(\omega)$ be the sum of distances from the vertex ω to each vertex of degree k in the subgraph G. s denotes the distances between v_s and ω, as shown in Fig. 2 on the left.

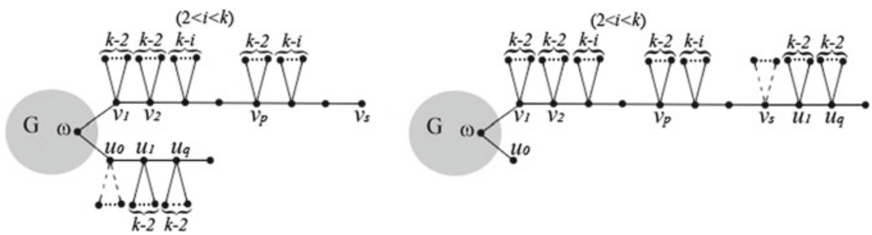

Fig. 2 T^* before the operation (*left*) and T^* after the operation (*right*)

Operation A: Attach caterpillar T_2 to the end of caterpillar T_1 while keep the degree of ω unchanged, as shown in Fig. 2 on the right. If the degree of vertex u_0 in T_2 which is adjacent to ω is greater than 2, then move pendent vertices $u_{01}, u_{02}, \ldots, u_{0x}$ $(0 \leq$ x $\leq$ k $- 2)$ attached to u_0 to the pendent vertex v_s of T_1 which is most distanced from ω, as shown in Fig. 2.

The operation above can be denoted by

$$T^{'} = T^{*} - u_0u_1 + v_su_1 - (u_0u_{01} + u_0u_{02} + \cdots + u_0u_{0x}) + (v_su_{01} + v_su_{02} + \cdots + v_su_{0x})$$

The generalized terminal Wiener index of T^* and T' after Operation A can be calculated as follow:

$$TW_k\left(T^*\right) = TW_k(G) + (p+q)D(\omega) + r\left(\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j\right) + \sum_{i<j}(a_j - a_i) + \sum_{i<j}(b_j - b_i) + q\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j$$

$$TW_k\left(T^{'}\right) = TW_k(G) + (p+q)D(\omega) + r\sum_{i=1}^{p} a_i + r\left[q(s-1) + \sum_{j=1}^{q} b_j\right] + \sum_{i<j}(a_j - a_i) + \sum_{i<j}(b_j - b_i) + q\left[p(s-1) - \sum_{i=1}^{p} a_i\right] + p\sum_{j=1}^{q} b_j$$

By computation, we get

$$TW_k\left(T^{'}\right) - TW_k\left(T^*\right) = rq(s-1) + pq(s-1) - 2q\sum_{i=1}^{p} a_i$$

Since $(s-1) \geq a_p \geq a_i$ $(i = 1, 2, \ldots, p)$ and $\geq p$, we have

$$TW_k\left(T^{'}\right) - TW_k\left(T^*\right) = q\left[(s-1)(r+p) - 2\sum_{i=1}^{p} a_i\right] \geq 2\mathrm{q}\sum_{i=0}^{p}((s-1) - a_i) \geq 0$$

With the equality holds if and only if $r = p = q = 1$ and $(s-1) = a_p$, or $q = 0$. Therefore, such operations will not reduce the value of the generalized terminal Wiener index.

Operation B: If $d(v_0) > k$, then we remove $d(v_0) -$ k pendent vertices attached to v_0 and use these vertices to subdivide v_0v_1.

Apparently the generalized terminal Wiener index will remain increase after Operate B.

By applying Operation A finite times, we can transform T into a caterpillar and it will further turned into a k-bounded caterpillar by operating Operation B for finite times.

Otherwise, we can remove the edges of the pendent path one by one, and use them to subdivide some edge e, so that both sides of e contain k-degree vertices. Or we can attach those removed pendent vertices to some non-pendent vertex on whose degree is less than k and turn them into k-degree vertices. In this way, TW_k will be increased. By this way, we can transform the vertices of T^* that is greater than 2 but less than k into2-degree vertices or k-degree vertices, which increase TW_k further. The above two results both contradict the choosing of T^*.

After applied all the operations above, assume there are $p(p \geq 2)$ k-degree vertexes and $h(h \geq 0)$ 2-degree vertexes on T^*, then we rearrange those 2-degree vertices so that there are no k-degree lie between them, and there are x k-degree vertices and $p - x$ k-degree vertices lie on each side of the path. Then the generalized terminal Wiener Index of T^* is calculated as:

$$\begin{aligned} TW_k\left(T^*\right) = &\frac{1}{6}(x-1)x(x+1) + \frac{1}{6}(p-x-1)(p-x)(p-x+1) \\ &+ x(p-x)(h+1) + \frac{1}{2}x(x-1)\cdot(p-x) \\ &+ \frac{1}{2}(p-x)(p-x-1)\cdot x \\ = &-h\left(x-\frac{p}{2}\right)^2 + \frac{1}{6}p^3 + \frac{1}{4}hp^2 - p, \end{aligned}$$

As can be seen from the formula, $TW_k(T^*)$ reaches its maximum when $x = [p/2]$ or $x = [p/2]$, namely, when k-degree vertices are equally distributed in the two sides of path, the generalized terminal Wiener Index of T^* achieve its maximum. In conclusion, the maximum TW_k of the generalized terminal Wiener index is achieved on $T_{n,k,p}$ caterpillar. □

3 The Generalized Terminal Wiener Index of Trees

Based on Theorem 2.1, for any given n, k, the maximum value of the generalized terminal Wiener index and its extremal graph can be determined as the corresponding value of p of $TW_k\left(T_{n,k,p}\right)$ is determined.

Suppose $y = f(p) = TW_k\left(T_{n,k,p}\right)$, we now determine the value of p when $y = f(p)$ reaches its maximum value, for even and odd p respectively.

For the function of $TW_k\left(T_{n,k,p}\right)$ vary from an even p to an odd p, when $n \geq 4$, they must be discussed separately as follow:

(i) If p is even, according to Lemma 2.2, we have

$$TW_k\left(T_{n,k,p}\right) = f\left(p\right) = \frac{1}{12}p\left(3np + 5p^2 - 3kp^2 - 2 - 6p\right)$$

Apparently, the curve of $y = f(p)$ goes through point (0, 0). In addition, let $3np + 5p^2 - 3kp^2 - 2 - 6p = 0$, then $\Delta = 9(n-2)^2 + 8(5-3k)$. For $n \geq 2k$, thus $\Delta > 0$. Therefore, there are 2 distinct real roots to this equation, denote them by p_1, p_2, by Vieta theorem and $p_1 p_2 > 0$, $p_1 + p_2 > 0$, we know that $p_1 > 0$, $p_2 > 0$.

Therefore, there are 3 zero-points: 0, p_1 and p_2 and of $y = f(p)$ and since the coefficient of the cubic term of $-(3\text{k} - 5) < 0$, as shown in Fig. 3 on the left.

Take a derivative of $y = f(p)$, we have

$$f'\left(p\right) = \frac{1}{12}\left[-\left(9k - 15\right)p^2 + \left(6n - 12\right)p - 2\right]$$

Let $f'(p) = 0$, and we have

$$p_0 = \frac{\sqrt{9(n-2)^2 + 6(5-3k)} + 3(n-2)}{9k - 15}$$

(ii) If p is odd, by Lemma 2.2 we have

$$TW_k\left(T_{n,k,p}\right) = f\left(p\right) = \frac{1}{12}\left(p+1\right)\left(p-1\right)\left(3n + 5p - 3kp - 6\right)$$

Apparently, there are 3 zero-points: -1, 1, $\frac{3n-6}{3k-5} > 0$ of $y = f(p)$. Since the coefficient of the cubic term of $-(3\text{k} - 5) < 0$, as shown in Fig. 3 on the right.

Take a derivative of $y = f(p)$, we have

$$f'(p) = \frac{1}{12}[-(9k - 15)p^2 + (6n - 12)p + 3k - 5],$$

Let $f'(p) = 0$, we have arrest point

$$p_0 = \frac{\sqrt{9(n-2)^2 + 3(3k - 15)^2} + 3(n-2)}{9k - 15}.$$

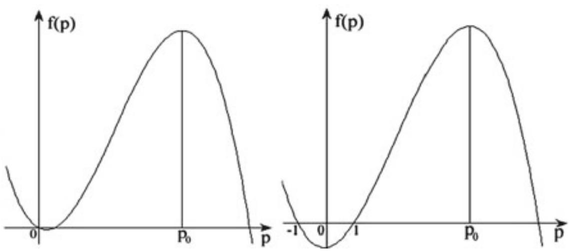

Fig. 3 y = f (p), if p is even (*left*), if p is odd (*right*)

Obviously $y = f(p) > 0$ when $p > 0$. As can be seen from the figure, $f(p)$ is monotone increasing when $0 < p < p_0$, vice versa.

According to Lemma 2.1, the maximum value of p is $m = \left\lfloor \frac{n-2}{k-1} \right\rfloor$. Hence to find out the maximum value of $f(p)$, we should first determine which of m and p_0 is the larger one.

By direct computation we have

Lemma 3.1 *For $k \geq 4$, $n \geq 2k$, $\frac{n-2}{k-1} > p_0$.*

Lemma 3.2 *For $k \geq 4$, $f(p_0 - 1) > f(p_0 + 1)$.*

Proof If p is even:

Since $f'(p_0) = \frac{1}{12}\left(15p_0^2 - 9kp_0^2 + 6np_0 - 12p_0 - 2\right) = 0$, we have

$$
\begin{aligned}
& f(p_0 - 1) - f(p_0 + 1) \\
&= \frac{1}{12}\left[3n(p_0 - 1)^2 + 5(p_0 - 1)^3 - 3k(p_0 - 1)^3 - 2(p_0 - 1) - 6(p_0 - 1)^2\right] \\
&- \frac{1}{12}\left[3n(p_0 + 1)^2 + 5(p_0 + 1)^3 - 3k(p_0 + 1)^3 - 2(p_0 + 1) - 6(p_0 + 1)^2\right] \\
&= -2 \times \frac{1}{12}\left(15p_0^2 - 9kp_0^2 + 6np_0 - 12p_0 - 2\right) + \frac{1}{6}(3k - 5), \\
&= -2f'(p_0) + \frac{1}{6}(3k - 5) = \frac{1}{6}(3k - 5) > 0,
\end{aligned}
$$

Thus $f(p_0 - 1) > f(p_0 + 1)$ always holds when $k \geq 4$.

Similarly, the same conclusion can be drawn if p is odd. □

Suppose that P^*_{even} makes $TW_k\left(T_{n,k,p}\right)$ reaches its maximum value when p is even, P^*_{odd} makes $TW_k\left(T_{n,k,p}\right)$ reaches its maximum value when p is odd.

Theorem 3.1 For positive integer $k \geq 4$, $n \geq 2k$, suppose $P_t^* \in \{P^*_{even}, P^*_{odd}\}$, then

(1) when m and p have the same parity, or m and p have different parity but $\frac{n-2}{k-1} - 1 > p_0$

$$
p_t^* = \begin{cases} \lfloor p_0 \rfloor, & \lfloor p_0 \rfloor \text{ and } p \text{ have the same parity} \\ \lfloor p_0 \rfloor - 1 \text{或} \lceil p_0 \rceil, & \lfloor p_0 \rfloor \text{ and } p \text{ have different parity} \end{cases}
$$

(2) when m and p have different parity but, $\frac{n-2}{k-1} - 1 \leq p_0$, $P_t^* = \left\lfloor \frac{n-2}{k-1} \right\rfloor - 1$.

where $p_0 = \frac{\sqrt{9(n-2)^2+6(5-3k)}+3(n-2)}{9k-15}$ if p is even and $p_0 = \frac{\sqrt{9(n-2)^2+3(3k-5)^2}+3(n-2)}{9k-15}$ if p is odd.

Proof If p is even

(1) If m is even, the upper bound of P is $p_{max} = m = \left\lfloor \frac{n-2}{k-1} \right\rfloor$.

For $\frac{n-2}{k-1} > p_0$ always holds (see Lemma 3.1), $p_{max} \geq \lfloor p_0 \rfloor$.
Therefore, the range of p_{max} is

$$p_{max} = \lfloor p_0 \rfloor (\lfloor p_0 \rfloor \text{ is even or } p_{max} > p_0)$$

As shown in Fig. 3 on the left, when $p_{max} = \lfloor p_0 \rfloor$, $f(p)$ is monotone increasing, so $P^*_{even} = p_{max} = \lfloor p_0 \rfloor (\lfloor p_0 \rfloor$ is even). If $p_{max} > p_0$, $f(p)$ reaches its maximum value at $p = p_0$, so under this circumstance, $TW_k\left(T_{n,k,p}\right)$ reaches its maximum value at either of the 2 evens near p_0, or at p_0 if p_0 is an even integer.

Therefore, P^*_{even} is bound to be the even that is closest to p_0 and if p_0 happens to be even, then P^*_{even} is p_0.

(i) If $\lfloor p_0 \rfloor$ is even, $P^*_{even} = \lfloor p_0 \rfloor$

When $\lfloor p_0 \rfloor = p_0$, apparently it's true that $P^*_{even} = p_0 = \lfloor p_0 \rfloor$.

When $\lfloor p_0 \rfloor \neq p_0$, $\lfloor p_0 \rfloor \neq p_0$ and $\lfloor p_0 \rfloor + 1$ are the two even that are closest to p_0.

Since according to Lemma 3.2, $f(p_0 - 1) > f(p_0 + 1)$ always holds

$$f(\lfloor p_0 \rfloor) > f(p_0 - 1) > f(p_0 + 1) > f(\lceil p_0 \rceil + 1), P^*_{even} = \lfloor p_0 \rfloor$$

(ii) If $\lfloor p_0 \rfloor$ is odd, $P^*_{even} = \lfloor p_0 \rfloor - 1$ or $\lceil p_0 \rceil$

For $\lfloor p_0 \rfloor = p_0$, $p_0 - 1$ and $p_0 + 1$ are the two even that are closest to p_0 and for $f(p_0 - 1) > f(p_0 + 1)$ always holds, $P^*_{even} = p_0 - 1 = \lfloor p_0 \rfloor - 1$.

When $\lfloor p_0 \rfloor \neq p_0$, $\lfloor p_0 \rfloor - 1$ and $\lceil p_0 \rceil$ are the two even that are closet to p_0. So when $(\lfloor p_0 \rfloor - 1) > f(\lceil p_0 \rceil)$, $P^*_{even} = \lfloor p_0 \rfloor - 1$; when $f(\lfloor p_0 \rfloor - 1) = f(\lceil p_0 \rceil)$, $P^*_{even} = \lfloor p_0 \rfloor - 1$ or $\lceil p_0 \rceil$; when $f(\lfloor p_0 \rfloor - 1) < f(\lceil p_0 \rceil)$, $P^*_{even} = \lceil p_0 \rceil$.

(2) If m is odd, the maximum of p is

$$p_{max} = m - 1 = \left\lfloor \frac{n-2}{k-1} \right\rfloor - 1 = \left\lfloor \frac{n-2}{k-1} - 1 \right\rfloor$$

(i) For $\frac{n-2}{k-1} - 1 \leq p_0$, $p_{max} \leq \lfloor p_0 \rfloor$.

As can be seen from Fig. 3 on the left, under such circumstances, $f(p)$ is monotone increasing, so it reaches its maximum value when $p = p_{max}$. Therefore,

$$P^*_{even} = p_{max} = \left\lfloor \frac{n-2}{k-1} \right\rfloor - 1$$

(ii) When $\frac{n-2}{k-1} - 1 > p_0$, $p_{max} \geq \lfloor p_0 \rfloor$, we have

If $\lfloor p_0 \rfloor$ is an even, $P^*_{even} = \lfloor p_0 \rfloor$, if $\lfloor p_0 \rfloor$ is an odd, $P^*_{even} = \lfloor p_0 \rfloor - 1$ or $\lceil p_0 \rceil$.

In conclusion, when p is an even, the value of P^*_{even} is determined as follow:

if m is even, or m is odd, but $\frac{n-2}{k-1} - 1 > p_0$,

$$p^*_{even} = \begin{cases} \lfloor p_0 \rfloor, & \text{if } \lfloor p_0 \rfloor \text{ is even} \\ \lfloor p_0 \rfloor - 1 \ or \ \lceil p_0 \rceil, & \text{if } \lfloor p_0 \rfloor \text{ is odd} \end{cases}$$

if m is odd and $\frac{n-2}{k-1} - 1 \leq p_0$,

$$p^*_{even} = \left\lfloor \frac{n-2}{k-1} \right\rfloor - 1$$

Similarly, when p is an odd, the value of P^*_{odd} can be deduced.
Combining the two cases mentioned above, we may arrive at Theorem 3.1. □

Theorem 3.2 can be drawn from Lemma 3.1 and Theorem 3.1.

Theorem 3.2 For $k \geq 4$, $n \geq 2k$, suppose that T is an n-vertex tree, then

$$TW_k(T) \leq max\left\{TW_k\left(T_{n,k,P^*_{even}}\right), TW_k\left(T_{n,k,P^*_{odd}}\right)\right\}$$

where P^*_{even} and P^*_{odd} is determined by Theorem 3.1.

Conclusions of this paper are applied and we can follow the following steps to work out the maximum value of the generalized terminal Wiener index of n-vertex $(n \geq 2k)$ tree:

(1) Determine the value of $\mathrm{P}^*_{\mathrm{even}}$ and $\mathrm{P}^*_{\mathrm{odd}}$ by Theorem 3.1.

(2) Determine which of $\mathrm{TW_k}\left(\mathrm{T_{n,k,P^*_{even}}}\right)$ and $\mathrm{TW_k}\left(\mathrm{T_{n,k,P^*_{odd}}}\right)$ is the greater one, and given the maximum value.

(3) Determine the extremal graph.

4 Conclusion

This paper proved the conjecture of reference [1]: when k $\geq$ 4, the generalized terminal Wiener index reaches its maximum value at $T_{n,k,p}$ caterpillars. Furthermore, we have worked out the maximum value of the generalized terminal Wiener index of n-vertex tree and its extremal graph.

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References

1. Ilić, A., Ilić, M.: Generalizations of Wiener polarity index and terminal Wiener index. Graphs and Combinatorics **29**, 1–10 (2013)
2. Zhen, J., et al.: The variable wiener index of trees with given maximum degree. In: International Conference on Computer, Communications and Information Technology (CCIT), pp. 36–37 (2014)

3. Gutman, I., Furtula, B., Petrović, M.: Terminal Wiener index. J. Math. Chem. **46**, 522–531 (2009)
4. Hosoya, H.: Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons. Bull. Chem. Soc. Jpn. **4**, 2332–2339 (1971)
5. Wiener, H.: Structural determination of paraffin boiling points. J. Am. Chem. Soc. **69**, 17–20 (1947)

A Time-Sensitive Targets System Simulation Model Based on Extendsim

Tan Kai-jia and Wang Rui

Abstract Precision strike of time-sensitive targets requires high-efficiency systems to perform surveillance, reconnaissance, tracking, command, control, striking and effect evaluation in limited time accurately. This paper analyzes architecture and operational process of time-sensitive targets systems, and presents a discrete-event simulation model of time-sensitive targets system. Within the model, target generating and evolving, reconnaissance plane, command and control, fire strike shooter are modeled based on ExtendSim. Simulation rationale and simulation model in typical scenario is built. The joint military planner can assess mission success probability and average length of kill chain, and study variations and sensitivities in a controlled manner among different plan of weapon resources by the simulation model.

Keywords Time-Sensitive Targets · Simulation · Evaluation · Extendsim

1 Introduction

Time-sensitive targets (TST) are that which have the actual threat, potential threat or very highly value. TST occur randomly and its striking opportunity is restrained by time window. TST usually include transient maneuver warfare target (such as armored forces, surface ships, submarines, etc.), mobile warfare facilities (such as mobile missile launchers, mobile warning, communications and command facilities, vehicles for weapons of mass destruction) and the facility with time limited strike (such as buildings in which terrorists hide or bridges enemy tactical units expected to pass, etc.) [1].

T. Kai-jia · W. Rui (✉)
Department of Military Logistics and Armament, National Defense University,
Beijing 100091, China
e-mail: ru01@sina.com

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Precision strike of TST requires to be completed in a very short period of time, and needs to combine sensors, command and control (C2) and fire strike weapons to construct system of systems to rapid accurate detection, rapid scientific decision-making and rapid precision strikes in close coordination and cooperation. As the improvement of information technology, networking, systematic and precise level for weapon equipment, the system of systems for striking TST is continuous improved, and that provides a solid material foundation for performing a task of striking TST. However, how to make the plan of weapon to construct system of systems scientifically according to operational missions, battle environment and operational targets becomes a major issue for joint operation commander to resolve.

A lot of research on system of systems modeling, weapon allocation and management of targets were conducted. Woitalla [2] described the system of systems architecture and studied a joint approach for target management for the precision strike of TST. William [3] analyzed the relationship among sensors, C2 and fire strike weapons with TCT matrix, and built TST relationship model. Luo [4] established TST evaluation mathematical models. However, the above research mainly studied decision-making problem of TST striking weapon system based on analytical method and cannot descript TST complex and dynamic mission process accurately. Johnson [5] studied the application of common relevant operational picture (CROP) on description of architecture and evaluation of effectiveness for striking TST, and established TST system model based on CROP by event graph method. But event graph method is not strong on time characteristic modeling, and the operational time node must be determined before modeling. So, event graph method cannot describe operation coordination, cooperate, synchronization of operational process of striking TST accurately. This paper analyzes architecture and operational process of TST system, and designs a discrete-event simulation model for TST system. Mission success probability and average length of kill chain to TST is given as TST operational effectiveness index. The simulation rational is provided and the evaluating effectiveness simulation model for typical striking TST is built. That helps to improve effectiveness of striking TST by allocating weapon resources properly based on the simulation.

2 Analysis of Architecture and Operational Process of Striking TST

2.1 Architecture of Striking TST

System of systems for striking TST is mainly composed of Sensor node, C2 node, fire strike shooter node and each node is integrated for interconnected network via data link to complete surveillance, reconnaissance, identification, decision-making, tracking, and evaluation for TST.

Sensors include reconnaissance satellites, unmanned reconnaissance aircraft, battlefield surveillance aircraft, etc. and the main function is surveillance, reconnaissance in detail, tracking and effectiveness evaluation. C2 includes ground command center, airborne warning and control system, etc. and the main function is to command and control sensor node and fire strike node, achieve information fusion of each sensor, analysis and identify targets and use of sensor node and fire strike node jointly and collaboratively according to targets, battlefield environment and the performance and dynamic quantity of fire strike weapons. Fire strike shooter include air-based, sea-based and land-based long-range missiles, unmanned attack aircraft, attack fighters and bombers, etc. and the main function is to strike against TST rapidly and precisely via the guidance instructions of sensor network and decision-making of C2.

2.2 Operational Process of Striking TST

The operational process of striking TST is composed of intelligence, surveillance and reconnaissance, C2, fire strike and impact assessment. Intelligence, surveillance and reconnaissance for battlefield is completed by reconnaissance satellites, reconnaissance aircraft and others sensors. Real-time battlefield awareness, navigation and intelligence information are shared for other operational process via data link. As suspected TST are detected, C2 node determines TST according to details of TST by reconnaissance aircraft, and allocates fire strike shooter to strike TST precisely under instructions guide of intelligence, surveillance and reconnaissance according to battlefield conditions, target property and fire strike shooter capability. Then TST impacted information by sensors is feedback to C2 node for effect assessment and further decisions. The operation process of striking TST is shown in Fig. 1.

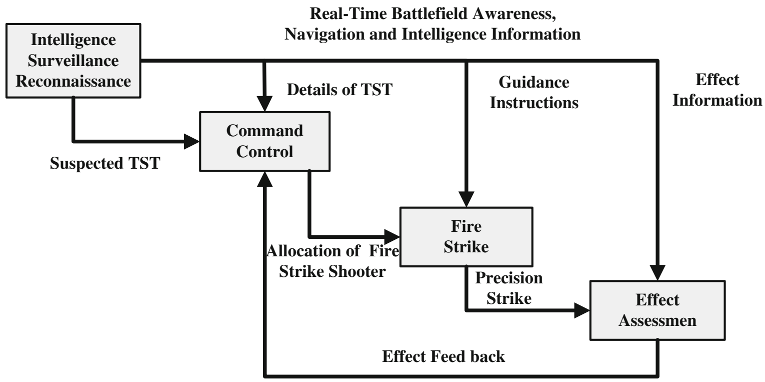

Fig. 1 The operation process of striking TST

3 Model Design

3.1 TST Model

TST is the drive of the striking TST operation process. TST model describes target type, target length, target threat degree, generating time, exposure time window and striking time window. Target types includes maneuver warfare target, mobile warfare facilities and the facility with time limited strike. Target threat degree, as the priority decision basis to allocate target to fire strike shooter for C2, is divided into three levels: serious, heavier and generally. Generating time is the time that TST exposes. Exposure time window is the period from generation to departure or hide for TST. TST may be only detected by overhead reconnaissance satellites. Otherwise, TST will be missed and the striking task for TST fails. Striking time window is the period from being detected to departure or hide for TST. Striking TST has to be finished during striking time window, otherwise TST will be missed and the striking task for TST fails. So, whether to complete tasks during exposure time window and striking time window are the key restricting factors for striking TST task. Generating time, exposure time window and striking time window of different TST are different and random, and the random distribution and distributed parameter can be estimated based on combat experience.

3.2 Reconnaissance Aircraft Model

Reconnaissance aircraft model describes reconnaissance aircraft type, dynamic quantity, reconnaissance capability, time to fly to the TST area and time to assess striking effect of TST. Reconnaissance aircraft is divided into two types: manned reconnaissance aircraft and unmanned reconnaissance aircraft. It is assumed that one reconnaissance aircraft only conducts reconnaissance for one TST and reconnaissance ends when striking time window closes or assessment over. The reconnaissance capability is the probability that the reconnaissance aircraft can reconnoiter TST. Reconnaissance capability among different reconnaissance aircraft to different TST is also different. The probability that high resolution aircrafts reconnoiter small length TST is high, otherwise, the probability that lower resolution reconnoiter is low. The time for reconnaissance aircraft flying from standby area to TST is random, and the random distribution and distributed parameter can be estimated based on combat experience. The reconnaissance capability is also modeled by the binomial distribution. The number of reconnaissance aircraft is limited. When no reconnaissance aircraft available, lower priority TST needs to be queued for reconnaissance aircraft in an available state, and higher priority TST can be reconnoitered in a priority.

Table 1 Part of reconnaissance aircraft allocation rules

Allocation rule	Description
Rule1	If there is a TST detected and there is reconnaissance aircrafts available
	Then allocate a reconnaissance aircraft to the TST
Rule2	If there is a TST detected and TST's priority is low and there is no reconnaissance aircrafts available
	Then the TST queues for a reconnaissance aircraft in available
Rule3	If there is a TST detected and TST's priority is high and there is no reconnaissance aircrafts available
	Then stop reconnaissance for the TST with lower priority, allocate the reconnaissance aircraft to the TST with high priority, and the TST with lower priority queues for a reconnaissance aircraft in available

3.3 C2 Model

C2 model is the core of the whole system of systems, and include reconnaissance aircraft allocation decision model, TST analysis model, fire strike shooter allocation decision model and effect analysis model. Reconnaissance aircraft allocation decision model describes allocation rule considering reconnaissance capability, priority of TST, matching relationship between reconnaissance aircraft and TST and available state of reconnaissance aircrafts. Part of rules is shown in Table 1. Fire strike shooter allocation decision model describes allocation rule considering fire strike capability, priority of TST, matching relationship between weapon and TST and available state of fire strike shooter. TST analysis model describes time in TST analysis and the probability of correct target identification. Effect analysis model describes time in effect analysis and the probability of correct effect identification. The time in TST analysis and the time in effect analysis are random, and the random distribution and distributed parameter can be estimated based on combat experience. The probability of correct target identification and the probability of correct effect identification are modeled by the binomial distribution.

3.4 Fire Strike Shooter Model

Fire strike shooter model describes type of fire strike shooter, dynamic quantity, fire strike capability and strike duration. Fire strike shooter is divided in two types:

long-range strike weapons and their platforms (such as destroyers and bombers equipped cruise missiles) and close-distance strike weapons and their platforms (such as bombers and attack aircrafts equipped precision-guided munitions). Fire strike capability is related to strike accuracy, kill probability, firing range and TST's size, mobility and feature. So, different fire strike capability to different TST is different. Strike duration of long-range strike weapons includes platform preparing duration that from the time receiving command to the time launching and weapons flight duration that from the time launching to the time TST stroked. Because of the short duration from the time launching to the time TST stroked, it can be neglected. Then strike duration of close-distance strike weapons equals to platform preparing duration. Platform preparing duration and weapons flight duration are random, and the random distribution and distributed parameter can be estimated based on combat experience. The number of fire strike shooter is limited. When no fire strike shooter available, lower priority TST needs to be queued for fire strike shooter in an available state, and higher priority TST can be allocated in a priority.

3.5 Effectiveness Index Model

Mission success probability and average length of kill chain to TST is given as TST operational effectiveness index for evaluating effectiveness of striking TST system. Mission success probability reflects comprehensive combat effectiveness of system of systems, and can be represented by the following formula:

$$P_M = \frac{\sum_{j=1}^{J} \sum_{i=1}^{I} N_{i,j}}{\sum_{j=1}^{J} \sum_{i=1}^{I} M_{i,j}} \tag{1}$$

P_M is Mission success probability. $N_{i,j}$ is the quantity of TST i which is destroyed in striking time window of TST i in the jth simulation run. I is the quantity of TST, and J is run number of simulation.

Average length of kill chain to TST reflects rapid response ability of system of systems can be represented by the following formula:

$$T_{kc} = \frac{\sum_{j=1}^{J} \sum_{i=1}^{I} T_{i,j}^{a} + T_{i,j}^{b} + T_{i,j}^{c} + T_{i,j}^{d} + T_{i,j}^{e} + T_{i,j}^{f}}{J \cdot I} \tag{2}$$

T_{kc} is average length of kill chain. $T_{i,j}^{a}$ is the time to fly to TST i in the jth simulation times. $T_{i,j}^{b}$ is the time to assess striking effect of TST i in the jth simulation times. $T_{i,j}^{c}$ is the striking duration to TST i in the jth simulation times. $T_{i,j}^{d}$ is the time in TST i analysis in the jth simulation times. $T_{i,j}^{e}$ is the time in effect analysis for TST i in the jth simulation times. $T_{i,j}^{f}$ is the time in deciding allocation for TST i in the jth simulation times. I is the quantity of TST, and J is times of simulation.

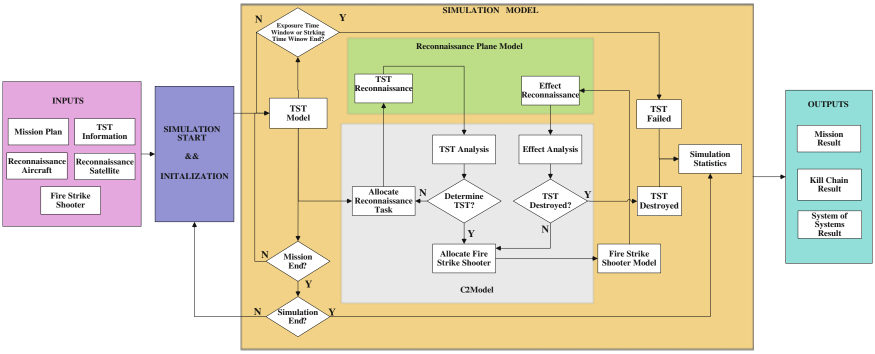

Fig. 2 Simulation rationale

3.6 Simulation Rationale

Having built the model of TST model, reconnaissance aircraft model, C2 model, fire strike shooter model and effectiveness index model, the main simulation rationale is designed, shown in Fig. 2. In this paper, Monte Carlo simulation is applied to perform the simulation and the desired result is taken as an average over the number of observations.

Before the model begins, the inputs which include mission plan, TST information, reconnaissance aircraft, and fire strike shooter are setup. Then the simulation

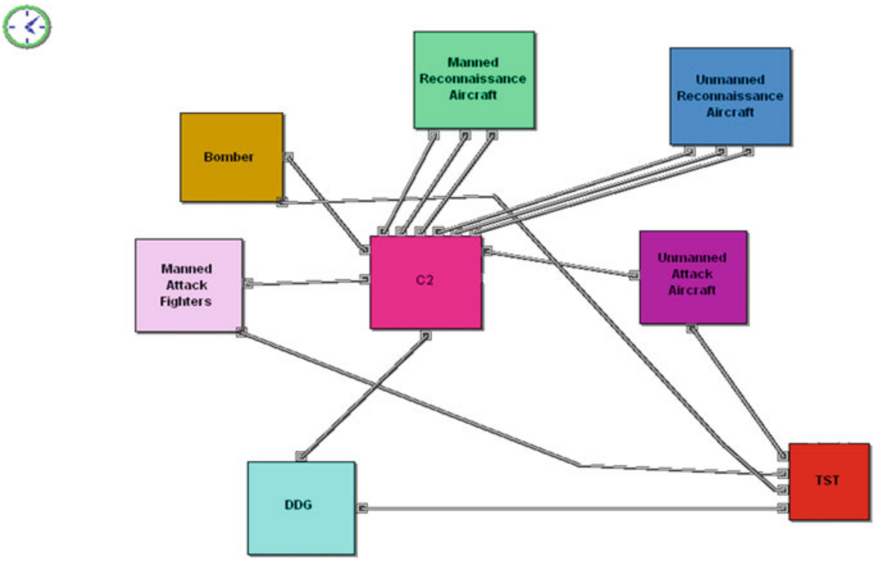

Fig. 3 Hierarchical view of the simulation model for typical scenario in ExtendSim

starts and is initialized. The simulation process continues until the mission end. The simulation is repeated many times and the average value of all trials then gives the probability of mission success and average length of kill chain. The system of systems results are also calculated for further manipulation and analysis.

Based on the model design and simulation rationale, the simulation model is implemented based on ExtendSim. Hierarchical view of the simulation model for typical scenario in ExtendSim is shown in Fig. 3. In the interior of the block, TST model, reconnaissance aircraft model, C2 model, fire strike shooter model and effectiveness index model are modeled by assembling the blocks the software provides, which improves efficiency of modeling. The model inputs are setup within a Microsoft Excel and are integrated into the ExtendSim model via the use of hot links [6]. The outputs are extracted into Excel from the simulation model by the same way.

4 Conclusion

We have shown in this paper a simulation model used for evaluating system of systems effectiveness for striking time-sensitive targets. The simulation environment allowed the military planner to understand the model to visualize any problems. The mode can help scientific planning weapon and is great significance for improving efficiency and rapid response ability for time-sensitive targets.

References

1. Nash, G.C.: Commander's Handbook for Joint Time-Sensitive Targeting. United States Joint Forces Command Joint Warfighting Center, Washington D.C. (2002)
2. Woitalla, M.J., Beary, S.M.: Joint Approach for Target Management for the Precision Strike of Time Sensitive Targets. Raytheon Network Centric systems Inc. Texas (2006)
3. William, B., Danskine, C.: Time-Sensitive Targeting Model. Air Command and Staff College of Air University (2001)
4. Luo, W.T., Xu, Y.S., Xiao, B.S.: Operation strategies and effectiveness analysis of system for destroying time-sensitive target. Fire Control Command Control **38**(9), 67–70 (2013)
5. Johnson, C.: Common Relevant Operational Picture: An Analysis of Effects on the Prosecution of Time-Critical Targets. Naval Postgraduate School Monterey, California (2002)
6. Krahl, D.: The extend simulation environment. In: Proceedings of the 2001 Winter Simulation Conference. Institute of Electrical and Electronics Engineers, Piscataway, New Jersey (2001)

Study of the Solvability of the Fuzzy Error Matrix Set Equation in Equality Form of Type II 4

Kai-zhong Guo, Ran Li and Jian-xin Li

Abstract The concept of error matrix is presented in this paper, and the types of the fuzzy error matrix equation are presented too. The paper is aimed at researching the solvability of the fuzzy error matrix equation which elements are sets and solutions to it. Theorems about the necessary and sufficient condition for the solvability of the substitution transformation matrix equation of Type II as XA = B are obtained in the paper. And an example of solving this equation would be given in the last part of the paper.

Keywords Fuzzy · Error matrix · Set equation · Equation solving

1 Introduction

Throughout the ages, people glorify scientists and inventors by praising their performance and wise, focusing on their success, thought process and way of thinking, but rarely to track their footprint of failures and mistakes, especially lack of multidimensional studies and comprehensive analysis about their error. It is really a pity.

In fact, the error plays an important role in the history of understanding. Error and truth always be the guide of scientist and explorers, which are both in the unity of opposites. The latter one always enlightens people that "rebound lute", "back to find the shore". Joule believed in the theory of perpetual motion machine, buried in studying it, after many failures, he began to think from the opposite side of the heat and work in the motion of a machine, and finally discovered the law of conservation

K. Guo · R. Li (✉) · J. Li
Guangzhou Vocational College of Science and Technology, Guangzhou 510550, China
e-mail: ranli2006@126.com

K. Guo · J. Li
Guang Dong University of Technology, Guangzhou 510006, China

K. Guo · R. Li · J. Li
School of Management, Harbin Institute of Technology, Harbin 150001, China

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and transformation of energy. Edison had failed fifty thousand times when he testing battery, but he said “my harvest is great, I know that more than fifty thousand kinds of method doesn’t work.”

Indeed, humans always like correct, hope be correct, however, to be correct we must avoid and eliminate errors, and in order to avoid and eliminate errors, you have to study the cause and the law of errors. And it is necessary to find out the cause of errors and the methods for eliminating them when errors happened. And in this paper we will discuss how to utilize the error matrix equation to eliminate errors.

2 The Concept of Error Matrix

Definition 1 Suppose

$$\begin{pmatrix} \left(\left(U_{111}\ U_{112}\cdots U_{11k}\right), X_{11}\right) & \left(\left(U_{121}\ U_{122}\cdots U_{12k}\right), X_{12}\right) & \cdots & \left(\left(U_{1n1}\ U_{1n2}\cdots U_{1nk}\right), X_{1n}\right) \\ \left(\left(U_{211}\ U_{212}\cdots U_{21k}\right), X_{21}\right) & \left(\left(U_{221}\ U_{222}\cdots U_{22k}\right), X_{22}\right) & \cdots & \left(\left(U_{2n1}\ U_{2n2}\cdots U_{2nk}\right), X_{2n}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \left(\left(U_{m11}\ U_{m12}\cdots U_{m1k}\right), X_{m1}\right) & \left(\left(U_{m21}\ U_{m22}\cdots U_{m2k}\right), X_{m2}\right) & \cdots & \left(\left(U_{mn1}\ U_{mn2}\cdots U_{mnk}\right), X_{mn}\right) \end{pmatrix}$$

be an error matrix with m $\times$ n order of K elements.

Definition 2 Suppose

$$\begin{pmatrix} U_{20} & S_{20}(t) & \vec{P}_{20(x_1,x_2,\ldots,x_n)} & T_{20}(t) & L_{20}(t) & y_{20}(t) = f_{20}((u(t),\vec{P}_{20}), G_{U20}(t)) & G_{U20}(t) \\ U_{21} & S_{21}(t) & \vec{P}_{21(x_1,x_2,\ldots,x_n)} & T_{21}(t) & L_{21}(t) & y_{21}(t) = f_{21}((u(t),\vec{P}_{21}), G_{U21}(t)) & G_{U21}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ U_{2t} & S_{2t}(t) & \vec{P}_{2t(x_1,x_2,\ldots,x_n)} & T_{2t}(t) & L_{2t}(t) & y_{2t}(t) = f_{2t}((u(t),\vec{P}_{2t}), G_{U2t}(t)) & G_{U2t}(t) \end{pmatrix}$$

be an error matrix with (t + 1)×7 or m $\times$ 7 order. The element of this error matrix is set.

Definition 3 The set relationship which containing the unknown set is called set equations.

Definition 4 Suppose X, A′ and B be m $\times$ 7 order error matrix, so XA′ $\supseteq$ (or other relational operators) B is called TPYE I set (matrix) equations.

3 Error Matrix Equations

Error matrix equation:

TypeI AX=B A●X=B A▲X=B A∨X=B A∧X=B
Type II XA=B X●A=B X▲A=B X∨A=B X∧A=B

4 The Solution of Error Matrix Equation

The solution of Type II 1 XA = B

$$X = (u, x) = (U_1, S_1(t), \vec{P}_1, T_1(t), L_1(t), x(t) = f((u(t), \vec{P}_1), G_{U1}(t)))$$

$$= \left(\begin{matrix} U_{10x} & S_{10x}(t) & \vec{P}_{10x(x_1,x_2,\ldots,x_n)} & T_{10x}(t) & L_{10x}(t) & y_{10x}(t) \\ U_{11x} & S_{11x}(t) & \vec{P}_{11x(x_1,x_2,\ldots,x_n)} & T_{11x}(t) & L_{11x}(t) & y_{11x}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ U_{1tx} & S_{1tx}(t) & \vec{P}_{1tx(x_1,x_2,\ldots,x_n)} & T_{1tx}(t) & L_{1tx}(t) & y_{1tx}(t) \end{matrix}\right.$$

$$\left.\begin{matrix} = f_{10x}((u(t), \vec{P}_{10x}), G_{U10x}(t)) & G_{U10x}(t) \\ = f_{11x}((u(t), \vec{P}_{11x}), G_{U11x}(t)) & G_{U11x}(t) \\ \cdots & \cdots \\ = f_{1tx}((u(t), \vec{P}_{1tx}), G_{U1tx}(t)) & G_{U1tx}(t) \end{matrix}\right)$$

$$A = (U_2, S_2(t), \vec{P}_2, T_2(t), L_2(t), x_2(t) = f_2((u(t), \vec{P}_2), G_{U2}(t)))$$

$$= \left(\begin{matrix} U_{20} & S_{20}(t) & \vec{P}_{20(x_1,x_2,\ldots,x_n)} & T_{20}(t) & L_{20}(t) & y_{20}(t) \\ U_{21} & S_{21}(t) & \vec{P}_{21(x_1,x_2,\ldots,x_n)} & T_{21}(t) & L_{21}(t) & y_{21}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ U_{2t} & S_{2t}(t) & \vec{P}_{2t(x_1,x_2,\ldots,x_n)} & T_{2t}(t) & L_{2t}(t) & y_{2t}(t) \end{matrix}\right.$$

$$\left.\begin{matrix} = f_{20}((u(t), \vec{P}_{20}), G_{U20}(t)) & G_{U20}(t) \\ = f_{21}((u(t), \vec{P}_{21}), G_{U21}(t)) & G_{U21}(t) \\ \cdots & \cdots \\ = f_{2t}((u(t), \vec{P}_{2t}), G_{U2t}(t)) & G_{U2t}(t) \end{matrix}\right)$$

$$B = \begin{pmatrix} V_{201} & S_{v201}(t) & \vec{P}_{v201(x_1,x_2,\ldots,x_n)} & T_{v201}(t) & L_{v201}(t) & y_{v201}(t) \\ V_{21j} & S_{v21j}(t) & \vec{P}_{v21j(x_1,x_2,\ldots,x_n)} & T_{v21j}(t) & L_{21j}(t) & y_{v21j}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ V_{2m2m1} & S_{v2m2m1}(t) & \vec{P}_{v2m2m1(x_1,x_2,\ldots,x_n)} & T_{v2m2m1}(t) & L_{v2m2m1}(t) & y_{v2m2m1}(t) \end{pmatrix.}$$

$$\begin{matrix} = f_{v201}((u(t), \vec{P}_{v201}), G_{v201}(t)) & G_{v201}(t) \\ = f_{v21j}((u(t), \vec{P}_{v21j}), G_{v21j}(t)) & G_{v21j}(t) \\ \cdots & \cdots \\ = f_{v2m2m1}((u(t), \vec{P}_{v2m2m1}), G_{v2m2m1}(t)) & G_{v2m2m1}(t) \end{matrix} \Bigg)$$

Definition 4.1 Let

$$XA' = \begin{pmatrix} (W_{11}, Z_{11}) & (W_{1m}, Z_{12}) & \cdots & (W_{1m1}, Z_{1m1}) \\ (W_{21}, Z_{21}) & (W_{22}, Z_{mm}) & \cdots & (W_{2m2}, Z_{2m2}) \\ \cdots & \cdots & \cdots & \cdots \\ (W_{m21}, Z_{m21}) & (W_{m21}, Z_{m21}) & \cdots & (W_{m2m1}, Z_{m2m1}) \end{pmatrix}$$

$$= \Bigg(\begin{matrix} V_{201} & S_{v201}(t) & \vec{P}_{v201(x_1,x_2,\ldots,x_n)} & T_{v201}(t) & L_{v201}(t) & y_{v201}(t) \\ V_{21j} & S_{v21j}(t) & \vec{P}_{v21j(x_1,x_2,\ldots,x_n)} & T_{v21j}(t) & L_{21j}(t) & y_{v21j}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ V_{2m2m1} & S_{v2m2m1}(t) & \vec{P}_{v2m2m1(x_1,x_2,\ldots,x_n)} & T_{v2m2m1}(t) & L_{v2m2m1}(t) & y_{v2m2m1}(t) \end{matrix}$$

$$\begin{matrix} = f_{v201}((v(t), \vec{P}_{v201}), G_{v201}(t)) & G_{v101}(t) \\ = f_{v21j}((v(t), \vec{P}_{v21j}), G_{v21j}(t)) & G_{v11j}(t) \\ \cdots & \cdots \\ = f_{v2m2m1}((v(t), \vec{P}_{v2m2m1}), G_{v2m2m1}(t)) & G_{v1m2m1}(t) \end{matrix} \Bigg)$$

There into,

$$\left(W_{ij}, Z_{ij}\right) = U_{1ix} \wedge U_{2j}, S_{1ix}(t) \wedge S_{2j}(t), \vec{P}_{1ix(x_1,x_2,\ldots,x_n)} \wedge \vec{P}_{2j}, T_{1ix}(t) \wedge T_{2j}(t),$$

$$L_{1ix}(t) \wedge L_{2j}(t), x_{1ix}(t) \wedge y_{2j}(t), G_{1ix}(t) \wedge G_{2j}(t)$$

There into,

$$\begin{pmatrix} U_{10x}\wedge U_{20} & S_{10x}(t)\wedge S_{20}(t) & \vec{P}_{10x(x_1,x_2,\ldots,x_n)}\wedge \vec{P}_{20} & T_{10x}(t)\wedge T_{20}(t) \\ U_{11x}\wedge U_{21} & S_{11x}(t)\wedge S_{21}(t) & \vec{P}_{11x(x_1,x_2,\ldots,x_n)}\wedge \vec{P}_{21} & T_{11x}(t)\wedge T_{21}(t) \\ \cdots & \cdots & \cdots & \cdots \\ U_{1tx}\wedge U_{2t} & S_{1tx}(t)\wedge S_{2t}(t) & \vec{P}_{1tx(x_1,x_2,\ldots,x_n)}\wedge \vec{P}_{2t} & T_{1tx}(t)\wedge T_{2t}(t) \end{pmatrix.}$$

$$\begin{matrix} L_{10x}(t)\wedge L_{20}(t) & x_{10x}(t)\wedge y_{20}(t) & G_{u10x}(t)\wedge G_{u20}(t) \\ L_{11x}(t)\wedge L_{21}(t) & x_{11x}(t)\wedge y_{21}(t) & G_{u11x}(t)\wedge G_{u21}(t) \\ \cdots & \cdots & \cdots \\ L_{1tx}(t)\wedge L_{2t}(t) & x_{1tx}(t)\wedge y_{2t}(t) & G_{u1tx}(t)\wedge G_{u2t}(t) \end{matrix}\right)$$

$$= \left(\begin{matrix} V_{201} & S_{v201}(t) & \vec{P}_{v201(x_1,x_2,\ldots,x_n)} & T_{v201}(t) & L_{v201}(t) & y_{v201}(t) \\ V_{21j} & S_{v21j}(t) & \vec{P}_{v21j(x_1,x_2,\ldots,x_n)} & T_{v21j}(t) & L_{21j}(t) & y_{v21j}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ V_{2m2m1} & S_{v2m2m1}(t) & \vec{P}_{v2m2m1(x_1,x_2,\ldots,x_n)} & T_{v2m2m1}(t) & L_{v2m2m1}(t) & y_{v2m2m1}(t) \end{matrix}\right.$$

$$\left.\begin{matrix} = f_{v201}((v(t),\vec{P}_{v201}),G_{v201}(t)) & G_{v101}(t) \\ = f_{v21j}((v(t),\vec{P}_{v21j}),G_{v21j}(t)) & G_{v11j}(t) \\ \cdots & \cdots \\ = f_{v2m2m1}((v(t),\vec{P}_{v2m2m1}),G_{v2m2m1}(t)) & G_{v1m2m1}(t) \end{matrix}\right)$$

By the definition of equal matrices: if two matrices contain each other, so corresponding elements in both matrices contain each other. Namely, $\left(W_{ij}, Z_{ij}\right) = (b_{ij}, y_{ij})$ so,

$$U_{1ix}\wedge U_{2j}, S_{1ix}(t)\wedge S_{2j}(t), \vec{P}_{1ix(x_1,x_2,\ldots,x_n)}\wedge \vec{P}_{2j}, T_{1ix}(t)\wedge T_{2j}(t),$$

$$L_{1ix}(t)\wedge L_{2j}(t), x_{1ix}(t)\wedge y_{2j}(t), G_{1ix}(t)\wedge G_{2j}(t)$$

$$= (b_{ij}, y_{ij}) = \big(V_{2ij}\ S_{v2ij}(t)\ \vec{P}_{v2ij(x_1,x_2,\ldots,x_n)}\ T_{v2ij}(t)\ L_{v2ij}(t)\ y_{v2ij}(t)$$

$$= f_{v2ij}(v(t),\vec{P}_{v2ij}),G_{v2ij}(t))\ G_{v2ij}(t)\big)$$

So the following set equations are obtained:

$$U_{10x}\wedge U_{20} = V_{v20}$$

$$S_{10x}(t)\wedge S_{20}(t) = S_{v20}(t)$$

$$\vec{P}_{10x(x_1,x_2,\ldots,x_n)}\wedge \vec{P}_{20} = \vec{P}_{v20(x_1,x_2,\ldots,x_n)}$$

$$T_{10x}(t) \wedge T_{20}(t) = T_{v20}(t)$$

$$L_{10x}(t) \wedge L_{20}(t) = L_{v20}(t)$$

$$x_{10x}(t) = f_{10x}(u(t), \vec{P}_{10x}), G_{u10x}(t)) \wedge y_{20}(t) = f_{v20}(u(t), \vec{P}_{v20}), G_{v20}(t))$$

$$G_{u10x}(t) \wedge G_{u20}(t) = G_{v20}(t)$$

$$\cdots\cdots$$

$$U_{1ix} \wedge U_{2j} = V_{v2j}$$

$$S_{1ix}(t) \wedge S_{2j}(t) = S_{v2j}(t)$$

$$\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} \wedge \vec{P}_{2j} = \vec{P}_{v2j(x_1,x_2,\ldots,x_n)}$$

$$T_{1ix}(t) \wedge T_{2j}(t) = T_{v2j}(t)$$

$$L_{1ix}(t) \wedge L_{2j}(t) = L_{v2j}(t)$$

$$x_{1ix}(t) = f_{1ix}(u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge y_{2j}(t) = f_{v2j}(u(t), \vec{P}_{v2j}), G_{v2j}(t))$$

$$G_{u1jx}(t) \wedge G_{u2j}(t) = G_{v2j}(t)$$

$$\cdots\cdots$$

$$U_{ttx} \wedge U_{2t} = V_{v2t}$$

$$S_{ttx}(t) \wedge S_{2t}(t) = S_{v2t}(t)$$

$$\vec{P}_{ttx(x_1,x_2,\ldots,x_n)} \wedge \vec{P}_{2t} = \vec{P}_{v2t(x_1,x_2,\ldots,x_n)}$$

$$T_{ttx}(t) \wedge T_{2t}(t) = T_{v2t}(t)$$

$$L_{ttx}(t) \wedge L_{2t}(t) = L_{v2t}(t)$$

$$x_{ttx}(t) = f_{ttx}(u(t), \vec{P}_{ttx}), G_{uttx}(t)) \wedge y_{2t}(t) = y_{v2t}(t) = f_{v2t}(u(t)\vec{P}_{v2t}), G_{v2t}(t))$$

$$G_{uttx}(t) \wedge G_{u2t}(t) = G_{v2t}(t)$$

About operation symbol "$\wedge$", if both sides of the equation are sets, then it means "intersection", if both sides of the equation are numbers, then it means "minimum".

As for

$$\left(U_{1ix} \wedge U_{2j}\right) h_1 \left(S_{1ix}(t) \wedge S_{2j}(t)\right) h_2(\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} \wedge \vec{P}_{2j}) \vee h_3 \left(T_{1ix}(t) \wedge T_{2j}(t)\right) h_4(L_{1ix}(t) \wedge L_{2j}(t)) h_5(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge y_{2j}(t)) h_6(G_{u1ix}(t) \wedge G_{u2j}(t)),$$

namely "h$_i$, i = 1, 2, … 6" operation of matrix elements, it means that elements have been computed could "compose" a complete matrix element (proposition). The mode of combination depends on different situations. One way is to constitute a new set of error elements or error logic proposition by parameter that operated. And this way is called the multiplication of m × 7 order error matrix.

Now to analyze the error metric equation XA′ = B. If let X = (x_1, x_2 ,…, x_{m2}), B = (B_1, B_2,…, B_{m2}) and divide XA′ = B into x_i A′ = B_i, i = (1, 2, …, m2), then there is one unknown noted x_i in the equation x_iA′ = B_i. So we can get the solution to XA′ = B by solving x_iA′ = B_i, i = (1, 2, …, m2) respectively. Thereout,

$$X_i A' = B_i = \big(U_{10x}\ S_{10x}(t)\ \vec{P}_{10x(x_1,x_2,\ldots x_n)}\ T_{10x}(t)\ L_{10x}(t) x_{10x}(t)$$

$$= f_{10x}((u(t), \vec{P}_{10x}), G_{u10x}(t))\ G_{u10x}(t)\big)\ A'$$

$$= \big((b_{i1}, y_{i1})\ (b_{ii}, y_{i2})\ \ldots\ (b_{im2}.y_{im2}) = (U_{10x} \wedge U_{20}) \vee (S_{10x}(t) \wedge S_{20}(t))\big)$$

$$\vee \left(\vec{P}_{10x(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{20}\right) \vee (T_{10x}(t) \wedge T_{20}(t)) \vee (L_{10x}(t) \wedge L_{20}(t)) \vee \left(x_{10x}(t) = f_{10x}((u(t), \vec{P}_{10x}), G_{u10x}(t)) \wedge y_{20}(t)\right) \vee (G_{u10x}(t) \wedge G_{u20}(t))$$

$$= \big(V_{20}\ S_{v20}(t)\ \vec{P}_{v20(x_1,x_2,\ldots x_n)}\ T_{v20}(t)\ L_{v20}(t) y_{v20}(t)$$

$$= f_{v20}((v(t), \vec{P}_{v20}), G_{v20}(t))\ G_{v20}(t)\big) \ldots\ldots$$

$$(U_{10x} \wedge U_{2j}) \vee \left(S_{10x}(t) \wedge S_{2j}(t)\right) \vee \left(\vec{P}_{10x(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2j}\right) \vee \left(T_{10x}(t) \wedge T_{2j}(t)\right) \vee \left(L_{10x}(t) \wedge L_{2j}(t)\right) \vee \left(x_{10x}(t) = f_{10x}((u(t), \vec{P}_{10x}), G_{u10x}(t)) \wedge y_{2j}(t)\right) \vee \left(G_{u10x}(t) \wedge G_{u2j}(t)\right)$$

$$= \big(V_{2j}\ S_{v2j}(t)\ \vec{P}_{v2j(x_1,x_2,\ldots x_n)}\ T_{v2j}(t)\ L_{v2j}(t) y_{v2j}(t)$$

$$= f_{v2j}((v(t), \vec{P}_{v2j}), G_{v2j}(t))\ G_{v2j}(t)\big)$$

$$\ldots\ldots$$

$$(U_{10x} \wedge U_{2t}) \vee (S_{10x}(t) \wedge S_{2t}(t)) \vee \left(\vec{P}_{10x(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2t}\right) \vee (T_{10x}(t) \wedge T_{2t}(t)) \vee (L_{10x}(t) \wedge L_{2t}(t)) \vee \left(x_{10x}(t) = f_{10x}((u(t), \vec{P}_{10x}), G_{u10x}(t)) \wedge y_{2t}(t)\right) \vee (G_{u10x}(t) \wedge G_{u2t}(t))$$

$$= \left(V_{2t}\ S_{v2t}(t)\ \vec{P}_{v2t(x_1,x_2,\ldots x_n)}\ T_{v2t}(t)\ L_{v2t}(t)\ y_{v2t}(t) = f_{v2t}((v(t), \vec{P}_{v2t}), G_{v2t}(t))\ G_{v2t}(t) \right)$$

$$\cdots\cdots\cdots$$

$$X_i A' = B_i = \left(U_{1ix}\ S_{1ix}(t)\ \vec{P}_{1ix(x_1,x_2,\ldots x_n)}\ T_{1ix}(t)\ L_{1ix}(t) x_{1ix}(t) \right.$$

$$\left. = f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t))\ G_{u1ix}(t) \right) A'$$

$$= \left((b_{i1}, y_{i1})\ (b_{ii}, y_{i2})\ \ldots\ (b_{im2}.y_{im2}) = (U_{1ix} \wedge U_{2j}) \vee \left(S_{1ix}(t) \wedge S_{2j}(t) \right) \right)$$

$$\vee \left(\vec{P}_{1ix(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2j} \right) \vee \left(T_{1ix}(t) \wedge T_{2j}(t) \right) \vee \left(L_{1ix}(t) \wedge L_{2j}(t) \right)$$
$$\vee \left(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge y_{2j}(t) \right) \vee \left(G_{u1ix}(t) \wedge G_{u2j}(t) \right)$$

$$= \left(V_{2j}\ S_{v2j}(t)\ \vec{P}_{v2j(x_1,x_2,\ldots x_n)}\ T_{v2j}(t)\ L_{v2j}(t)\ y_{v2j}(t) = f_{v2j}((v(t), \vec{P}_{v2j}), G_{v2j}(t))\ G_{v2j}(t) \right)$$

$$\cdots\cdots$$

$$(U_{10x} \wedge U_{2j}) \vee \left(S_{10x}(t) \wedge S_{2j}(t) \right) \vee \left(\vec{P}_{10x(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2j} \right)$$
$$\vee \left(T_{10x}(t) \wedge T_{2j}(t) \right) \vee \left(L_{10x}(t) \wedge L_{2j}(t) \right)$$
$$\vee \left(x_{10x}(t) = f_{10x}((u(t), \vec{P}_{10x}), G_{u10x}(t)) \wedge y_{2j}(t) \right) \vee \left(G_{u10x}(t) \wedge G_{u2j}(t) \right)$$

$$= \left(V_{2j}\ S_{v2j}(t)\ \vec{P}_{v2j(x_1,x_2,\ldots x_n)}\ T_{v2j}(t)\ L_{v2j}(t)\ y_{v2j}(t) \right.$$

$$\left. = f_{v2j}((v(t), \vec{P}_{v2j}), G_{v2j}(t))\ G_{v2j}(t) \right) \ldots\ldots$$

$$(U_{1tx} \wedge U_{2t}) \vee (S_{1tx}(t) \wedge S_{2t}(t)) \vee \left(\vec{P}_{1tx(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2t} \right)$$
$$\vee (T_{1tx}(t) \wedge T_{2t}(t)) \vee (L_{1tx}(t) \wedge L_{2t}(t))$$
$$\vee \left(x_{1tx}(t) = f_{1tx}((u(t), \vec{P}_{1tx}), G_{u1tx}(t)) \wedge y_{2t}(t) \right) \vee (G_{utx}(t) \wedge G_{u2t}(t))$$

$$= \left(V_{2t}\ S_{v2t}(t)\ \vec{P}_{v2t(x_1,x_2,\ldots x_n)}\ T_{v2t}(t)\ L_{v2t}(t)\ y_{v2t}(t) \right.$$

$$\left. = f_{v2t}((v(t), \vec{P}_{v2t}), G_{v2t}(t))\ G_{v2t}(t) \right)$$

$$\cdots\cdots$$

$$X_{m2} A' = B_{m2} = \left(U_{1m2x}\ S_{1m2x}(t)\ \vec{P}_{1m2x(x_1,x_2,\ldots x_n)}\ T_{1m2x}(t)\ L_{1m2x}(t)\ x_{1m2x}(t) \right.$$

$$\left. = f_{1m2x}((u(t), \vec{P}_{1m2x}), G_{u1m2x}(t))\ G_{u1m2x}(t) \right) A'$$

$$= \left((b_{m21}, y_{m21})\ (b_{m2i}, y_{m22})\ \ldots\ (b_{im2}, y_{im1}) \right)$$

$$= (U_{1m2x} \wedge U_{2m1}) \vee (S_{1m2x}(t) \wedge S_{2m1}(t)) \vee \left(\vec{P}_{1m2x(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2m1}\right)$$
$$\vee (T_{1m2x}(t) \wedge T_{2m1}(t)) \vee (L_{1m2x}(t) \wedge L_{2m1}(t))$$
$$\vee \left(x_{1m2x}(t) = f_{1m2x}((u(t), \vec{P}_{1m2x}), G_{u1m2x}(t)) \wedge y_{2m1}(t)\right)$$
$$\vee (G_{u1m2x}(t) \wedge G_{u2m1}(t))$$

$$= \big(V_{2m1}\ S_{v2m1}(t)\ \vec{P}_{v2j(x_1,x_2,\ldots x_n)}\ T_{v2m1}(t)\ L_{v2m1}(t) y_{v2m1}(t)$$

$$= f_{v2m1}((v(t), \vec{P}_{v2j}), G_{v2m1}(t))\ G_{v2m1}(t) \big)$$

……

$$(U_{10x} \wedge U_{2m1}) \vee (S_{10x}(t) \wedge S_{2m1}(t)) \vee \left(\vec{P}_{10x(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2j}\right)$$
$$\vee (T_{10x}(t) \wedge T_{2m1}(t)) \vee (L_{10x}(t) \wedge L_{2m1}(t))$$
$$\vee \left(x_{10x}(t) = f_{10x}((u(t), \vec{P}_{1m2x}), G_{u10x}(t)) \wedge y_{2m1}(t)\right)$$
$$\vee (G_{u10x}(t) \wedge G_{u2m1}(t))$$

$$= (V_{2m1}, S_{v2m1}(t))\ \vec{P}_{v2j(x_1,x_2,\ldots x_n)}\ T_{v2m1}(t)\ L_{v2m1}(t)\ y_{v2m1}(t)$$

$$= f_{v2m1}((v(t), \vec{P}_{v2j}), G_{v2m1}(t))\ G_{v2m1}(t)$$

……

$$(U_{1tx} \wedge U_{2t}) \vee (S_{1tx}(t) \wedge S_{2t}(t)) \vee \left(\vec{P}_{1tx(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2t}\right) \vee (T_{1tx}(t) \wedge T_{2t}(t))$$
$$\vee (L_{1tx}(t) \wedge L_{2t}(t)) \vee \left(x_{1tx}(t) = f_{1tx}((u(t), \vec{P}_{1tx}), G_{u1tx}(t)) \wedge y_{2t}(t)\right)$$
$$\vee (G_{u1t}(V_{20}, S_{v20}(t)) \vec{P}_{v20(x_1,x_2,\ldots x_n)} T_{v2}(t) L_{v2}(t) y_{v20}(t)$$
$$= f_{v20}((v(t), \vec{P}_{v20}), G_v(t)) G_{v10}(t) x(t) \wedge G_{u2t}(t)$$

$$= (V_{2t}, S_{v2t}(t))\ \vec{P}_{v2t(x_1,x_2,\ldots x_n)}\ T_{v2t}(t)\ L_{v2t}(t)\ y_{v2t}(t) = f_{v2t}((v(t), \vec{P}_{v2t}), G_v(t))\ G_{v2t}(t)$$

Theorem 1 *The sufficient and necessary condition for the solvability of error matrix equation* $XA' = B$ *is the solvability of* $x_i A' = B_i$, $i = (1, 2, \ldots m2)$.

Proof if $XA' = B$ has solvability, it is can be known by the definitions of $XA' = B$ and $x_i\ A' = B_i$, i = (1, 2, …, m2) that they are the equivalent equations, so it is necessary for $x_i\ A' = B_i$, i = (1, 2, …,m2) has solvability; Otherwise, if the solvability of $x_i\ A' = B_i$, i = (1, 2, …, m2) exists, similarly it does for $XA' = B$.

Thereout, we use the method of discussing the solvability of $x_i\ A' = B_i$, i = (0, 1, 2, ……, m2) to discuss the solution of $XA' = B$.

Then for $X_i A' = B_i$, we can get

$$\big(U_{1ix}\ S_{1ix}(t)\ \vec{P}_{1ix(x_1,x_2,\ldots x_n)}\ T_{1ix}(t)\ L_{1ix}(t) x_{1ix}(t)$$

$$= f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t))\ G_{u1ix}(t)\big)\ A'$$

$$\begin{aligned}
&= (U_{1ix} \wedge U_{20}) \vee (S_{1ix}(t) \wedge S_{20}(t)) \vee \left(\vec{P}_{1ix(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{20}\right) \\
&\vee (T_{1ix}(t) \wedge T_{20}(t)) \vee (L_{1ix}(t) \wedge L_{20}(t)) \\
&\vee \left(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{10x}), G_{u1ix}(t)) \wedge x_{20}(t)\right) \\
&\vee (G_{u1ix}(t) \wedge G_{u20}(t))
\end{aligned}$$

$$\ldots\ldots$$

$$\begin{aligned}
&= (U_{1ix} \wedge U_{2j}) \vee \big(S_{1ix}(t) \wedge S_{2j}(t)\big) \vee \left(\vec{P}_{1ix(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2j}\right) \\
&\vee \big(T_{1ix}(t) \wedge T_{2j}(t)\big) \vee \big(L_{1ix}(t) \wedge L_{2j}(t)\big) \\
&\vee \left(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge x_{2j}(t)\right) \vee \big(G_{u1ix}(t) \wedge G_{u2j}(t)\big)
\end{aligned}$$

$$\ldots\ldots$$

$$\begin{aligned}
&= (U_{1ix} \wedge U_{2m1}) \vee (S_{1ix}(t) \wedge S_{2m1}(t)) \vee \left(\vec{P}_{1ix(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2m1}\right) \\
&\vee (T_{1ix}(t) \wedge T_{2m1}(t)) \vee (L_{1ix}(t) \wedge L_{2m1}(t)) \\
&\vee \left(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge x_{2m1}(t)\right) \vee (G_{u1ix}(t) \wedge G_{u2m1}(t)) \\
&= \big((b_{i1}, y_{i1})\ (b_{i2}, y_{i2}) \cdots (b_{im1}.y_{im1})\big)
\end{aligned}$$

That is,

$$\begin{aligned}
&(U_{1ix} \wedge U_{20}) \vee (S_{1ix}(t) \wedge S_{20}(t)) \vee \left(\vec{P}_{1ix(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{20}\right) \\
&\vee (T_{1ix}(t) \wedge T_{20}(t)) \vee (L_{1ix}(t) \wedge L_{20}(t)) \\
&\vee \left(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge x_{20}(t)\right) \vee (G_{u1ix}(t) \wedge G_{u20}(t))
\end{aligned}$$

$$= \big(V_{20}\ S_{v20}(t)\ \vec{P}_{v20(x_1,x_2,\ldots x_n)}\ T_{v20j}(t)\ L_{v20}(t) y_{v20}(t)$$

$$= f_{v20}((v(t), \vec{P}_{v20}), G_{2jv}(t))\ G_{v20}(t)\big)$$

$$\ldots\ldots$$

$$\begin{aligned}
&(U_{1ix} \wedge U_{2j}) \vee \big(S_{1ix}(t) \wedge S_{2j}(t)\big) \vee \left(\vec{P}_{1ix(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2j}\right) \\
&\vee \big(T_{1ix}(t) \wedge T_{2j}(t)\big) \vee \big(L_{1ix}(t) \wedge L_{2j}(t)\big) \\
&\vee \left(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge x_{2j}(t)\right) \vee \big(G_{u1ix}(t) \wedge G_{u2j}(t)\big)
\end{aligned}$$

$$= \big(V_{2j}\ S_{v2j}(t)\ \vec{P}_{v2j(x_1,x_2,\ldots x_n)}\ T_{v2j}(t)\ L_{v2j}(t) y_{v2j}(t)$$

$$= f_{v2j}((v(t), \vec{P}_{v2j}), G_{v2j}(t))\ G_{v2j}(t)\big)$$

$$\ldots\ldots$$

$$(U_{1tx} \wedge U_{2m1}) \vee (S_{1tx}(t) \wedge S_{2m1}(t)) \vee \left(\vec{P}_{1tx(x_1,x_2,\ldots x_n)} \wedge \vec{P}_{2m1}\right)$$
$$\vee (T_{1tx}(t) \wedge T_{2m1}(t)) \vee (L_{1tx}(t) \wedge L_{2m1}(t))$$
$$\vee \left(x_{1tx}(t) = f_{1tx}((u(t), \vec{P}_{1tx}), G_{u1tx}(t)) \wedge x_{2m1}(t)\right) \vee (G_{u1tx}(t) \wedge G_{um1}(t))$$
$$= (V_{2t}, S_{v2t}(t))\ \vec{P}_{v2t(x_1,x_2,\ldots x_n)}\ T_{v2t}(t)\ L_{v2t}(t)\ y_{v2t}(t) = f_{v2t}((v(t), \vec{P}_{v2t}), G_v(t))\ G_{v2t}(t)$$

A series of set equations is obtained,

$$(U_{1ix} \wedge U_{20}) = V_{20}$$

$$(S_{1ix}(t) \wedge S_{20}(t)) = S_{v20}(t)$$

$$(\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} \wedge \vec{P}_{20}) = \vec{P}_{v20(x_1,x_2,\ldots,x_n)}$$

$$(T_{1ix}(t) \wedge T_{20}(t)) = T_{v20}(t)$$

$$(L_{1ix}(t) \wedge L_{20}(t)) = L_{v20}(t)$$

$$(x_{1ix}(t) = f_{1ix}(u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge x_{20}(t)) = y_{v20}(t) = f_{v20}((u(t), \vec{P}_{v20}), G_{2jv}(t))$$

$$(G_{u1ix}(t) \wedge G_{u20}(t)) = G_{v20}(t)$$

$$\cdots\cdots$$

$$(U_{1ix} \wedge U_{2j}) = V_{2j}$$

$$(S_{1ix}(t) \wedge S_{2j}(t)) = S_{v2j}(t)$$

$$(\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} \wedge \vec{P}_{2j}) = \vec{P}_{v2j(x_1,x_2,\ldots,x_n)}$$

$$(T_{1ix}(t) \wedge T_{2j}(t)) = T_{v2j}(t)$$

$$(L_{1ix}(t) \wedge L_{2j}(t)) = L_{v2j}(t)$$

$$(x_{1ix}(t) = f_{1ix}(u(t), \vec{P}_{1ix}), G_{u1ix}(t)) \wedge x_{2i}(t)) = y_{v2j}(t) = f_{v2j}((u(t), \vec{P}_{v2j}), G_{2jv}(t))$$

$$(G_{u1ix}(t) \wedge G_{u2j}(t)) = G_{v2j}(t)$$

$$\cdots\cdots$$

$$(U_{1tx} \wedge U_{2m1}) = V_{2t}$$

$$(S_{1tx}(t) \wedge S_{2m1}(t)) = S_{v2t}(t)$$

$$(\vec{P}_{1tx(x_1,x_2,\ldots,x_n)} \wedge \vec{P}_{2m1}) = \vec{P}_{v2t(x_1,x_2,\ldots,x_n)}$$

$$(T_{1tx}(t) \wedge T_{2m1}(t)) = T_{v2t}(t)$$

$$(L_{1tx}(t) \wedge L_{2m1}(t)) = L_{v2t}(t)$$

$$(x_{1tx}(t) = f_{1tx}(u(t), \vec{P}_{1tx}), G_{u1tx}(t)) \wedge x_{2m1}(t)) = y_{v2t}(t) = f_{v2t}((u(t), \vec{P}_{v2t}), G_v(t))$$
$$(G_{u1tx}(t) \wedge G_{u2m1}(t)) = G_{v2t}(t)$$

Theorem 2 *The necessary condition for the solvability of $X_i A' = B_i$ is*

$$U_{20} \supseteq V_{20},$$

$$S_{20}(t) \supseteq S_{v20}(t),$$

$$\vec{P}_{20} \supseteq \vec{P}_{v20(x_1,x_2,\ldots,x_n)},$$

$$T_{20}(t) \supseteq T_{v20}(t),$$

$$L_{20}(t) \supseteq L_{v20}(t),$$

$$x_{20}(t) \geq y_{v20}(t) = f_{v20}\left((v(t), \vec{p}_{v20}), G_{2jv}(t)\right),$$

$$G_{u20}(t) \supseteq L_{v20}(t)$$

$$\cdots\cdots$$

$$U_{2j} \supseteq V_{2j},$$

$$S_{2j}(t) \supseteq S_{v2j}(t),$$

$$\vec{P}_{2j} \supseteq \vec{P}_{v2j(x_1,x_2,\ldots,x_n)},$$

$$T_{2j}(t) \supseteq T_{v2j}(t),$$

$$L_{2j}(t) \supseteq L_{v2j}(t),$$

$$x_{2i}(t) \geq y_{v2j}(t) = f_{v2j}\left(\left(v(t), \vec{p}_{v2j}\right), G_{2jv}(t)\right),$$

$$G_{u2j}(t) \supseteq L_{v2j}(t),$$

$$\cdots\cdots$$

$$U_{2m1} \supseteq V_{2t},$$

$$S_{2m1}(t) \supseteq S_{v2t}(t),$$

$$\vec{P}_{2m1} \supseteq \vec{P}_{v2t(x_1,x_2,\ldots,x_n)},$$

$$T_{2m1}(t) \supseteq T_{v2t}(t),$$

$$L_{2m1}(t) \geq L_{v2t}(t),$$

$$x_{2m1}(t) \geq y_{v2t}(t) = f_{v2t}\left((v(t), \vec{p}_{v2t}), G_v(t)\right),$$

$$G_{u2m1}(t) \supseteq L_{v2t}(t).$$

Proof If one of the conditions above is not meet, suppose $S_{2j}(t) \supseteq S_{v2j}(t)$is not meet, so no mater what value $S_{1ix}(t)$ is, we can not get $(S_{1ix}(t) \wedge S_{2j}(t)) = S_{v2j}(t)$.
Proved.

Theorem 3 *The sufficient condition for the solvability of* $X_i A' = B_i$ *is*

$$U_{20} = V_{20},$$

$$S_{20}(t) = S_{v20}(t),$$

$$\vec{P}_{20} = \vec{P}_{v20(x_1,x_2,\ldots,x_n)},$$

$$T_{20}(t) = T_{v20j}(t),$$

$$L_{20}(t) = L_{v20}(t),$$

$$x_{20}(t) \geq y_{v20}(t) = f_{v20}\left((v(t), \vec{p}_{v20}), G_{2jv}(t)\right),$$

$$G_{u20}(t) = L_{v20}(t),$$

$$\cdots\cdots$$

$$U_{2j} = V_{2j},$$

$$S_{2j}(t) = S_{v2j}(t),$$

$$\vec{P}_{2j} = \vec{P}_{v2j(x_1,x_2,\ldots,x_n)},$$

$$T_{2j}(t) = T_{v20j}(t),$$

$$L_{20}(t) = L_{v20}(t),$$

$$x_{2i}(t) = y_{v2j}(t) = f_{v2j}\left(\left(v(t), \vec{p}_{v2j}\right), G_{2jv}(t)\right),$$

$$G_{u2j}(t) = L_{v2j}(t),$$

$$\cdots\cdots$$

$$U_{2m1} = V_{2t},$$

$$S_{2m1}(t) = S_{v2t}(t),$$

$$\vec{P}_{2m1} = \vec{P}_{v2t(x_1,x_2,\ldots,x_n)},$$

$$T_{2m1}(t) = T_{v2t}(t),$$

$$L_{2m1}(t) = L_{v2t}(t),$$

$$x_{2m1}(t) = y_{v2t}(t) = f_{v2t}\left((v(t), \vec{p}_{v2t}), G_v(t)\right),$$

$$G_{u2m1}(t) = L_{v2t}(t),$$

Proof Because in the $X_i A' = B_i$, we should only take union operation of the corresponding element of A in X_i, that is

$$U_{1ix} = U_{20} \cup U_{21} \cup \cdots \cup U_{2j} \cup \cdots \cup U_{2t},$$

$$S_{1ix} = S_{20}(t) \cup S_{21}(t) \cup \cdots \cup S_{2j}(t) \cup \cdots \cup S_{2t}(t),$$

$$\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} = \vec{P}_{20} \cup \vec{P}_{21} \cup \cdots \cup \vec{P}_{2j} \cup \cdots \cup \vec{P}_{2t},$$

$$T_{1ix} = T_{20}(t) \cup T_{21}(t) \cup \cdots \cup T_{2j}(t) \cup \cdots \cup T_{2t}(t),$$

$$L_{1ix} = L_{20}(t) \cup L_{21}(t) \cup \cdots \cup L_{2j}(t) \cup \cdots \cup L_{2t}(t),$$

$$x_{1ix}(t) = f_{lix}((u(t), \vec{P}_{lix}), G_{ulix}(t)) = x_{20}(t) \cup x_{21} \cup \cdots \cup x_{2j} \cup \cdots \cup x_{2t},$$

$$G_{u1ix}(t) = G_{u20}(t) \cup G_{u21}(t) \cup \cdots \cup G_{u2j}(t) \cup \cdots \cup G_{u2t}(t),$$

proved.

From the discussion about Theorems 2 and 3 we can get that the solution to $X_i A' = B_i$ exists in the correspondence parameters of the corresponding elements of A and B_i.

Then we discuss all the solution of $X_i A' = B_i$ and $XA' = B$:

(1) When the conditions of Theorem 3 are meet, the solution of the equation set is:

$$U_{1ix} = U_{20} \cup U_{21} \cup \cdots \cup U_{2j} \cup \cdots \cup U_{2t},$$

$$S_{1ix}(t) = S_{20}(t) \cup S_{21} \cup \cdots \cup S_{2j} \cup \cdots \cup S_{2t},$$

$$\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} = \vec{P}_{20} \cup \vec{P}_{21} \cup \cdots \cup \vec{P}_{2j} \cup \cdots \cup \vec{P}_{2t},$$

$$T_{1ix}(t) = T_{20}(t) \cup T_{21} \cup \cdots \cup T_{2j} \cup \cdots \cup T_{2t},$$

$$L_{1ix}(t) = L_{20}(t) \cup L_{21} \cup \cdots \cup L_{2j} \cup \cdots \cup L_{2t},$$

$$x_{1ix}(t) = f_{lix}((u(t), \vec{P}_{lix}), G_{ulix}(t)) = x_{20}(t) \cup x_{21} \cup \cdots \cup x_{2j} \cup \cdots \cup x_{2t},$$

$$G_{u1ix}(t) = G_{u20}(t) \cup G_{u21}(t) \cup \cdots \cup G_{u2j}(t) \cup \cdots \cup G_{u2t}(t).$$

(2) When one condition of Theorem 3 turns from equality to inclusion relation, suppose $S_{2j}(t) \supseteq S_{v2j}(t)$, so it is must be demanded that $S_{lix}(t) = S_{v2j}(t)$ on account of $X_i A' = B_i$, so must be for

$$S_{2j}(t) = S_{20}(t) \cup S_{21} \cup \cdots \cup S_{2j} \cup \cdots \cup S_{2t},$$

then, $S_{1ix}(t) = S_{2j} = S_{20}(t) \cup S_{21} \cup \cdots \cup S_{2j} \cup \cdots \cup S_{2t} = S_{v20}(t)$, that is,

$$U_{1ix} = U_{20} \cup U_{21} \cup \cdots \cup U_{2j} \cup \cdots \cup U_{2t},$$

$$S_{1ix}(t) = S_{20}(t) \cup S_{21} \cup \cdots \cup S_{2j} \cup \cdots \cup S_{2t},$$

$$\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} = \vec{P}_{20} \cup \vec{P}_{21} \cup \cdots \cup \vec{P}_{2j} \cup \cdots \cup \vec{P}_{2t},$$

$$T_{1ix}(t) = T_{20}(t) \cup T_{21} \cup \cdots \cup T_{2j} \cup \cdots \cup T_{2t},$$

$$L_{1ix}(t) = L_{20}(t) \cup L_{21} \cup \cdots \cup L_{2j} \cup \cdots \cup L_{2t},$$

$$x_{1ix}(t) = f_{lix}((u(t), \vec{P}_{lix}), G_{ulix}(t)) = x_{20}(t) \cup x_{21} \cup \cdots \cup x_{2j} \cup \cdots \cup x_{2t},$$

$$G_{u1ix}(t) = G_{u20}(t) \cup G_{u21}(t) \cup \cdots \cup G_{u2j}(t) \cup \cdots \cup G_{u2t}(t).$$

(3) When two conditions of Theorem 3 turn from equality to inclusion relation, suppose $S_{2j}(t) \supseteq S_{v2j}(t)$, so it is must be demanded that $S_{lix}(t) = S_{v2i}(t) = S_{v2j}(t)$ on account of $X_i A' = B$, so must be for $S_{2i} = S_{2j} = S_{20}(t) \cup S_{21} \cup \cdots \cup S_{2j} \cup \cdots \cup S_{2t}$.
then, $S_{1ix}(t) = S_{2i} = S_{2j} = S_{20}(t) \cup S_{21} \cup \cdots \cup S_{2j} \cup \cdots \cup S_{2t} = S_{v20}(t)$.
that is,

$$U_{1ix} = U_{20} \cup U_{21} \cup \cdots \cup U_{2j} \cup \cdots \cup U_{2t},$$

$$S_{1ix}(t) = S_{20}(t) \cup S_{21} \cup \cdots \cup S_{2j} \cup \cdots \cup S_{2t},$$

$$\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} = \vec{P}_{20} \cup \vec{P}_{21} \cup \cdots \cup \vec{P}_{2j} \cup \cdots \cup \vec{P}_{2t},$$

$$T_{1ix}(t) = T_{20}(t) \cup T_{21} \cup \cdots \cup T_{2j} \cup \cdots \cup T_{2t},$$

$$L_{1ix}(t) = L_{20}(t) \cup L_{21} \cup \cdots \cup L_{2j} \cup \cdots \cup L_{2t},$$

$$x_{1ix}(t) = f_{lix}((u(t), \vec{P}_{lix}), G_{ulix}(t)) = x_{20}(t) \cup x_{21} \cup \cdots \cup x_{2j} \cup \cdots \cup x_{2t},$$

$$G_{u1ix}(t) = G_{u20}(t) \cup G_{u21}(t) \cup \cdots \cup G_{u2j}(t) \cup \cdots \cup G_{u2t}(t).$$

(4) When all conditions of Theorem 3 turn from equality to inclusion relation, so it is must be demanded that $S_{1ix}(t) = S_{v20}(t) = S_{v21}(t) = \cdots = S_{v2t}(t)$, so must be for $S_{2i}(t) = S_{v20}(t) = S_{v21}(t) = \cdots = S_{v2t}(t)$, then, $S_{1ix}(t) = S_{v20}(t) = S_{v21}(t) = \cdots = S_{v2t}(t)$.
That is,

$$U_{1ix} = U_{20} \cup U_{21} \cup \cdots \cup U_{2j} \cup \cdots \cup U_{2t}$$

$$S_{1ix}(t) = S_{v20}(t) = S_{v21}(t) = \cdots = S_{v2t}(t)$$

$$\vec{P}_{1ix(x_1,x_2,\ldots,x_n)} = \vec{P}_{20} \cup \vec{P}_{21} \cup \cdots \cup \vec{P}_{2j} \cup \cdots \cup \vec{P}_{2t}$$

$$T_{1ix}(t) = T_{20}(t) \cup T_{21} \cup \cdots \cup T_{2j} \cup \cdots \cup T_{2t}$$

$$L_{1ix}(t) = L_{20}(t) \cup L_{21} \cup \cdots \cup L_{2j} \cup \cdots \cup L_{2t}$$

$$x_{1ix}(t) = f_{lix}((u(t), \vec{P}_{lix}), G_{ulix}(t)) = x_{20}(t) \cup x_{21} \cup \cdots \cup x_{2j} \cup \cdots \cup x_{2t}$$

$$G_{u1ix}(t) = G_{u20}(t) \cup G_{u21}(t) \cup \cdots \cup G_{u2j}(t) \cup \cdots \cup G_{u2t}(t)$$

(5) When two conditions of Theorem 3 turn from equality to inclusion relation, suppose $S_{2i}(t) \supseteq S_{v2i}(t)$and $S_{2j}(t) \supseteq S_{v2j}(t)$, if $S_{v2i}(t) \neq S_{v2j}(t)$, so there is no solution to $X_i A' = B$, on account of $X_i A' = B$.

That is because: $S_{lix}(t) = S_{2i}$, and $S_{lix}(t) = S_{2j}$, can't meet at the same time. So

$$S_{1ix}(t) \wedge S_{20}(t) = S_{v20}(t);$$

$$S_{1ix}(t) \wedge S_{21}(t) = S_{v21}(t);$$

$$\cdots\cdots$$

$$S_{1ix}(t) \wedge S_{2t}(t) = S_{v2t}(t),$$

are false.

We can discuss other parameters in the same way.

5 The Example of Application of Error Matrix Equation

For the error matrix equation suppose

$$A = \begin{pmatrix} U_1 & S_1(t) & \vec{P}_{1(x_1,x_2,\ldots,x_n)} & T_1(t) & L_1(t) & y_1(t) = f_1((u(t), \vec{P}_1), G_{u1}(t)) & G_{u1}(t) \\ U_2 & S_2(t) & \vec{P}_{2(x_1,x_2,\ldots,x_n)} & T_2(t) & L_2(t) & y_2(t) = f_2((u(t), \vec{P}_2), G_{u2}(t)) & G_{u2}(t) \\ U_3 & S_3(t) & \vec{P}_{3(x_1,x_2,\ldots,x_n)} & T_3(t) & L_3(t) & y_3(t) = f_3((u(t), \vec{P}_3), G_{u3}(t)) & G_{u3}(t) \\ U_4 & S_4(t) & \vec{P}_{4(x_1,x_2,\ldots,x_n)} & T_4(t) & L_4(t) & y_4(t) = f_4((u(t), \vec{P}_4), G_{u4}(t)) & G_{u4}(t) \end{pmatrix}$$

$$U = \{u_i \mid u_i \in \{\text{all shares of SSE}\}\}$$

$$U_1 = U,$$

$$S_{u1}(t) = \{\text{China Union}\},$$

$$\vec{P}_{u1(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{u1}(t) = \{\text{PE Ratio}\},$$

$$L_{u1}(t) = \{21.7\},$$

$$Y_{u1}(t) = \begin{cases} 1 & \text{PE Ratio} \geq 10 \\ 0 & \text{PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{China Union, PE Ratio} = 21.7) = 1,$$

$$G_{u1}(t) = \{\text{PE Ratio} < 10\}.$$

$$U_2 = U,$$

$$S_{u2}(t) = \{\text{SINOPEC}\},$$

$$\vec{P}_{u2(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{u2}(t) = \{\text{PE Ratio}\},$$

$$L_{u2}(t) = \{8.05\},$$

$$Y_{u2}(t) = \begin{cases} 1 & \text{PE Ratio} \geq 10 \\ 0 & \text{PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{SINOPEC, PE Ratio} = 8.05) = 0,$$

$$G_{u2}(t) = \{\text{PE Ratio} < 10\}.$$

$$U_3 = U,$$

$$S_{u3}(t) = \{\text{CSCEC}\},$$

$$\vec{P}_{u3(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{u3}(t) = \{\text{PE Ratio}\},$$

$$L_{u3}(t) = \{5.29\},$$

$$Y_{u3}(t) = \begin{cases} 1 \text{ PE Ratio} \geq 10 \\ 0 \text{ PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{CSCEC, PE Ratio} = 5.29) = 0,$$

$$G_{u3}(t) = \{\text{PE Ratio} < 10\}.$$

$$U_4 = U,$$

$$S_{u4}(t) = \{\text{SINOPEC}\},$$

$$\vec{P}_{u4(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{u4}(t) = \{\text{PE Ratio}\},$$

$$L_{u4}(t) = \{11.99\},$$

$$Y_{u4}(t) = \begin{cases} 1 \text{ PE Ratio} \geq 10 \\ 0 \text{ PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{SINOPEC, PE Ratio} = 11.99) = 1,$$

$$G_{u4}(t) = \{\text{PE Ratio} < 10\}.$$

$$X = \begin{pmatrix} U_{1x} & S_{1x}(t) & \vec{P}_{1x(x_1,x_2,\ldots,x_n)} & T_{1x}(t) & L_{1x}(t) & x_{1x}(t) = f_{1x}((u(t), \vec{P}_{1x}), G_{u1x}(t)) & G_{u1x}(t) \\ U_{2x} & S_{2x}(t) & \vec{P}_{2x(x_1,x_2,\ldots,x_n)} & T_{2x}(t) & L_{2x}(t) & x_{2x}(t) = f_{2x}((u(t), \vec{P}_{2x}), G_{u2x}(t)) & G_{u2x}(t) \\ U_{3x} & S_{3x}(t) & \vec{P}_{3x(x_1,x_2,\ldots,x_n)} & T_{3x}(t) & L_{3x}(t) & x_{3x}(t) = f_{3x}((u(t), \vec{P}_{3x}), G_{u3x}(t)) & G_{u3x}(t) \\ U_{4x} & S_{4x}(t) & \vec{P}_{4x(x_1,x_2,\ldots,x_n)} & T_{4x}(t) & L_{4x}(t) & x_{4x}(t) = f_{4x}((u(t), \vec{P}_{4x}), G_{u4x}(t)) & G_{u4x}(t) \end{pmatrix}$$

$$B = \begin{pmatrix} (V_1, S_{v1}(t)) & \vec{P}_{v1(x_1,x_2,\ldots,x_n)} & T_{v1}(t) & L_{v1}(t) & y_{v1}(t) = f_{v1}((v(t), \vec{P}_{v1}), G_{v1}(t)) & G_{v1}(t) \\ (V_2, S_{v2}(t)) & \vec{P}_{v2(x_1,x_2,\ldots,x_n)} & T_{v2}(t) & L_{v2}(t) & y_{v2}(t) = f_{v2}((v(t), \vec{P}_{v2}), G_{v2}(t)) & G_{v2}(t) \\ (V_3, S_{v3}(t)) & \vec{P}_{v3(x_1,x_2,\ldots,x_n)} & T_{v3}(t) & L_{v3}(t) & y_{v3}(t) = f_{v3}((v(t), \vec{P}_{v3}), G_{v3}(t)) & G_{v3}(t) \\ (V_4, S_{v4}(t)) & \vec{P}_{v4(x_1,x_2,\ldots,x_n)} & T_{v4}(t) & L_{v4}(t) & y_{v4}(t) = f_{v4}((v(t), \vec{P}_{v4}), G_{v4}(t)) & G_{v4}(t) \end{pmatrix}$$

$$V_1 = U,$$

$$S_{v1}(t) = \{\text{China Union}\},$$

$$\vec{P}_{v1(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{v1}(t) = \{\text{PE Ratio}\},$$

$$L_{v1}(t) = \{21.7\},$$

$$Y_{v1}(t) = \begin{cases} 1 \text{ PE Ratio} \geq 10 \\ 0 \text{ PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{China Union, PE Ratio} = 21.7) = 1,$$

$$G_{v1}(t) = \{\text{PE Ratio} < 10\}.$$

$$V_2 = U,$$

$$S_{v2}(t) = \{\text{SINOPEC}\},$$

$$\vec{P}_{v2(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{v2}(t) = \{\text{PE Ratio}\},$$

$$L_{v2}(t) = \{8.05\},$$

$$Y_{v2}(t) = \begin{cases} 1 \text{ PE Ratio} \geq 10 \\ 0 \text{ PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{SINOPEC, PE Ratio} = 8.05) = 0$$

$$G_{v2}(t) = \{\text{PE Ratio} < 10\}.$$

$$V_3 = U,$$

$$S_{v3}(t) = \{\text{ CSCEC }\},$$

$$\vec{P}_{v3(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{v3}(t) = \{\text{PE Ratio}\},$$

$$L_{v3}(t) = \{5.29\},$$

$$Y_{v3}(t) = \begin{cases} 1 \text{ PE Ratio} \geq 10 \\ 0 \text{ PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{CSCEC, PE Ratio} = 5.29) = 0,$$

$$G_{v3}(t) = \{\text{PE Ratio} < 10\}.$$

$$V_4 = U,$$

$$S_{v4}(t) = \{\text{SINOPEC}\},$$

$$\vec{P}_{v3(x_1,x_2,\ldots,x_n)} = \{SSE\},$$

$$T_{v4}(t) = \{\text{PE Ratio}\},$$

$$L_{v4}(t) = \{11.99\},$$

$$Y_{v4}(t) = \begin{cases} 1 \text{ PE Ratio} \geq 10 \\ 0 \text{ PE Ratio} < 10 \end{cases} \Rightarrow f_1(\text{SINOPEC, PE Ratio} = 11.99) = 1,$$

$$G_{v4}(t) = \{\text{PE Ratio} < 10\}.$$

Suppose X = B, then X is the solution of $X \wedge A = B$ on the account of $A = B$ and Theorem 2.

6 Conclusion

We get the necessary condition and the necessary and sufficient condition for the solvability of the error matrix equation XA = B, and there are also get be proved by the case studies.

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Empirical Research on Efficiency Measure of Financial Investment in Education Based on SE-DEA

Yan-yan Yao and Ren-shou Zhang

Abstract This paper makes use of SE-DEA model to measure the performance of financial investment in education in Guangdong from 2000 to 2012, solves the problem that the traditional DEA model is unable to further distinguish the effective units, achieves the discrimination and sorting of the performance of the financial investment in education in Guangdong, and makes suggestions on a structural adjustment of the financial investment before making of a relevant policy in Guangdong by means of computation of looseness amount.

Keywords SE-DEA mode · Financial investment in education · Performance measurement

1 Introduction

Due to the "non-competitiveness" in the consumption and the "non-excludability" in the benefits of the educational resource, there are many subjects of educational investment, including government, social institution, and charity organizations. The education can not only improve the personal cultural accomplishment and knowledge level and enable the well-educated person to earn the higher income and social status, but also generate the spillover effect through the social value created by educated persons, increase social productivity, and improve the cultural and ethic level of the nationality; so, the educational resource is generally considered to be a kind of "mixed public article"[14]. Therefore, the investment in education falls within the scope of the national public investment, and the government often faces the realistic problem on how to improve the efficiency while guaranteeing the justice in the financial fund to be invested. By offering the educational resource, the government may provide relatively fair competition conditions, increase mobility of the social class, give vulnerable groups t opportunities of upgrading of social status, change a

Y. Yao (✉) · R. Zhang
Guangzhou Financial Bureau,Guangzhou University, Guangzhou 510006, Guangdong Province
e-mail: 826638040@qq.com

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survival environment, narrow gaps between the poor and the rich, and solve problems in the social class stratification at the source.

The financial investment in the education shall be increased on the basis of a continuous rise in the demand for social education and continuous enhancement of the government's ability to offer the education service. In accordance with an inverted U curve theory, there is a relationship in inverted U-shape between the financial input and output. That is to say that at the early stage of education development, there is a direct proportional relation between them, meaning the more the input is made, the more the output is produced; however, at the later stage of its development, the limited increase in the output will be brought from the increasing input. This paper makes an analysis of the financial allocation efficiency of the public education in Guangdong over the past years, and proposes an ideology for solution to the deficiency in the investment and output.

2 Review of Literatures

2.1 Research at Abroad

In 1950s, in the paper "Role of Government in Education", Fridman proposed that the fund invested in the education should be converted into the education vouchers, which should be distributed to the student families. Students could select their schools with the voucher at will; the schools could exchange the vouchers with the government for cash, so as to guarantee the investment in the educational undertaking, but also introduce the competition mechanism into an educational system in order to improve educational quality and achieve survival of the fittest. Such mechanism of "education voucher" reflects the ideology of assessment made on the performance of the education expenditure [5]. On the basis of the measurement analysis, Psacharo [3] discovered that in the developing countries, the average rate of return from the financial investment in education is lower than 13 %; in the developed countries, the rate of return from the investment in education is higher than 20 %, demonstrating that there is a large gap in efficiency of the financial investment in education among countries at the different levels of development, and indicating that the efficiency is increased by re-planning financial resource for education [3]. Taylor [6] measured the efficiency of financial investment in the education in England by the least square method; however, there is a popular belief that it is difficult to use the least square method, maximum likelihood estimation, and stochastic frontier analysis (i.e. SFA method) for a production function with more investments and more outputs; the investment in and output from the higher education is a situation with more investments and more outputs [2]. Jill [7] made an analysis of the investment and output efficiency of the higher education in England by a data envelopment analysis (DEA) and on the basis of the data collected from 100 colleges and universities in England from 2000 to 2001, and drew the conclusion that the technology efficiency and scale efficiency of such 100 colleges and universities in England were higher than the average levels in Europe [7]. Benson [4] made the overall assessment on the educational financial system from three perspectives including whether the supply of the education is sufficient, whether the allocation of the education resource is efficient, and whether the distribution of the education resource is fair as considered traditionally in the research literatures on educational finance. Barlow [2] made the study on the trend of the production efficiency of the college education in England by the method of time series analysis, and finally drew the conclusion that the marginal benefit from the financial investment in education in England indicated that it was possible to improve the utilization efficiency of the financial fund by the further optimization of the resource, so as to maximize the benefits. Adams [1] thought that an education performance assessment system should lay the stress on the following indicators: the resource investment quality, including the schoolbooks for students, teaching qualification, and the ratio of teachers to students, and other review items; the result output quality, including the academic record (i.e. test scores), acquisition of self-knowledge, progress, or passing rate of student; the education process quality, including the interaction between teachers and students, participation and level of learners, entertainment of study, and other research items; the education content quality, including the coverage rate of fundamental knowledge, coherence and portability of knowledge, and the integration level of the knowledge; the education reputation quality, including the historical image and public perception of educational institution; the value affiliation quality, including the influence of the education on the all-round development of the students.

2.1.1 Research at Home

When representing the financial investment with the operating expense of education within an annual average budget, representing the investment of human resource with the number of full-time teachers, and assessing the higher education resource allocation efficiency by DEA, Fu [10] discovered that the efficiency was not high in financial investment since 1999. Huang [12] made use of a principal component analysis to induce and count up a number of variables involved in the higher education investment and output and the principal component variables with the high economic significance, and analyzed the relation between the principal component of output and the principal component of investment by a regression analysis method, so as to draw a conclusion that the output from the colleges and universities in China was increased in an extensive manner . Proceeding from the principle of the educational economics, Xu [15] made an analysis of the principal components for the benefits of investment in the public higher education from 1995 to 2006, and discovered that there was an efficiency defect in the investment in the higher education, and there was an insignificant positive relationship between academic research output and total investment amount. On a basis of perspective of the governmental welfare, Yan [16] made some suggestions on construction of the assessment system by proposing the importance of the education expenditure performance assessment, such as pertinence of the assessment indicator, the comprehensiveness of the assessment method, and independence of the assessment supervision. Feng [9] proposed that an investment ratio of the public education expenditure should be higher than that of the higher education, and there was an unbalance development of the basic education in various regions and the resource allocation in the education system in China [9]. Zhang [18] made a research, considering that despite of a continuous increase in a financial investment in education in China and its increasing in the education, there was still a problem in the lack of educational funds, mainly because there are unreasonable factors of the financial education resource allocation structure among the different hierarchies of the education; therefore, it was important to optimize an expenditure structure and increase the financial education expenditure benefit while making more financial investments in the education. In accordance with the actual conditions of the public education expenditure in China, Yan [17] proposed an educational expenditure performance assessment indicator system in China, and defined the weights of the assessment indicators by the AHP method, and made use of data on the public education expenditure to reach a conclusion that the investment in the public education in China was lower compared to that in foreign countries, and a distribution structure of the public education fund was not in conformity with the requirements of an assessment indicator system; that is to say that the economy is not ideal, and the efficiency is declining continuously, but the effectiveness is better [17]. From the quantitative perspective of a research on the basic education expenditure performance assessment, on the basis of analysis on the quasi-public goods property of the basic education, Pei [8] expounded a theoretical base of the financial basic education expenditure performance assessment, reviewed the outcomes acquired from, problems and difficulties in the performance assessment in China, and guided with a basic idea established for the financial basic education performance assessment system in China.

On the basis of summarizing the literatures above, we discovered that the scholars at home and abroad made a great number of researches on the financial education expenditure performance, but less qualitative researches and more quantitative researches; some of scholars used DEA model, but it was impossible to solve the problem in the efficiency sorting in case of more effective samples. This paper intends to use SE-DEA model to assess performance of the financial investment in education in Guangdong from 2000 to 2012.

3 Selection of DEA Model

3.1 Profile of Traditional DEA Model

Being the most basic one, CCR model is used for measurement of the comprehensive technology efficiency (technology efficiency, TE), i.e. the ratio of the optimal investment to actual investment on the productive frontier with the unchanged returns to scale. The unchanged return to scale means that during process of the production, when the investment is increased or decreased in the equal proportion, the output is also increased or decreased in the same equal proportion. If TE $= 1$, it indicates that the investment and output reach state of the relatively best efficiency. If TE < 1, the decision-making unit is DEA invalidity, indicating the decision-making unit is non-technical valid or non-scale valid. Given that there are N decision-making units (Decision Making Units, hereinafter referred to as DMU), and each decision-making unit has M resource consumption investment variables and S output variables, x_{ij} denotes the investment made by the DMU No.j in the investment No.i, and y_{ij} denotes the output of the DMU No.j to the output No.i, and the investment and output of the DMU No.j should be:

$$x_j = [x_{1j}, x_{2j}.....x_{mj}]^T, j = 1, 2.....n$$

$$y_j = [y_{1j}, y_{2j}.....y_{mj}]^T, j = 1, 2.....n$$

Given that the weights of the input and output are

$$v = [v_1, v_2.....v_m]^T$$

$$u = [u_1, u_2.....u_s]^T$$

At this time, the efficiency indicator for assessment of the DMU No.j should be

$$h_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \qquad j = 1, 2.....n$$

Select appropriate weight coefficients v and u, so that they should satisfy the requirement of $h_j \le 1, \; j = 1, 2.....n$, then an optimization model should be the next case, by the linear transformation simplex method, the following solution is obtained

$$C^2R \begin{cases} \max \frac{\sum_{r=1}^{s} u_r y_{rj_0}}{\sum_{i=1}^{m} v_i x_{ij_0}} = h^*_{j_0} \\ \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, j = 1, 2...n \\ v = [v_1, v_2....v_m]^T \ge 0 \\ u = [u_1, u_2....u_s]^T \ge 0 \end{cases} \xrightarrow{\textit{Simple typelinear change, introduction}\text{ to}} \mathrm{C^2R} \begin{cases} \min \theta \\ \sum_{j=1}^{n} X_j \lambda_j \le \theta X_k \\ \sum_{j=1}^{n} Y_j \lambda_j \le Y_k \\ \lambda_j \ge 0, j = 1, 2......n \end{cases}$$

The traditional CCR model has an advantage in effectives of the assessment of the DMU with more investments and more output; however, the biggest disadvantage of such model contains the existence of more effective units. The traditional CCR model is unable to further distinguish the effective units. In order to make up this deficiency, Andersen & Perterson (1993) proposed the SE-DEA model based on the CCR model [13]:

3.2 SE-DEA Model

$$\left(D_{C^2R}\right)\begin{cases}\min\theta\\ \sum\limits_{j=1,j\neq j_o}^{n} X_j\lambda_j + s^- = \theta X_o\\ \sum\limits_{j=1,j\neq j_o}^{n} Y_o\lambda_j - s^+ = Y_o\\ \lambda_j \geq 0\\ s^+ \geq 0, s^- \geq 0\\ j = 1, 2.......n\end{cases}$$

The greatest advantage of SE-DEA model is that it is possible to sort the efficiency values of many effective samples. When assessing the performance of the financial investment in the education, the special DMU in the traditional DEA model is included in the set of all the DMUs, and the super-efficient DEA model keeps the special DMU to be assessed out of the set, and replace such DMU with the linear combination of investment and input in other DMUs. Generally, the restraint that the efficiency value must be 1 in case of the validity of the DMU is eliminated; therefore, the situation that the value is larger than 1 will appear in case of the valid DMU in the SE-DEA model. The SE-DEA model has no influence on the assessment of the valid unit but distinguishes a number of effective units for the efficiency. The idea of such model is indicated in Fig. 1. When calculating the performance value of the financial investment in education c in a year, for the invalid years, the efficiency value of such two models should be $OE'/OE < 1$; however, if the financial education expenditure in a year is valid, and the productive frontier is changed from *ABCD*(ordinary DEA) to *ABC*, the efficiency value should be $OC'/OC > 1$.

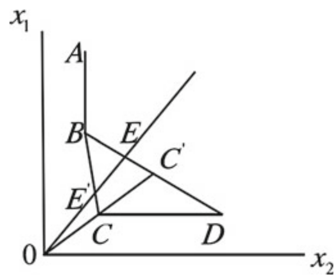

Fig. 1 The performance value of the financial investment in education c in a year

4 Empirical Analysis

4.1 Selection of Variables and Data Source

By review of the literatures, we discovers that when measuring the education investment and output efficiency, the average operating expense of education of the students in the schools at all levels shall be used as a variable of investment, and the gross enrollment rate of the schools at all levels shall be used as the output. In this paper, the average operating expense of education within the budget of elementary school students, the average operating expense of education within the budget of junior middle school students, the average operating expense of education within the budget of senior ones, and the average operating expense of education within the budget of students in higher education are used as the input variables, and the gross enrollment rate of the school-age children, the gross enrollment rate of the junior middle schools, and the gross enrollment rate of senior middle schools, and the gross enrollment rate of the higher education are used as the output variables, in order to make a super-efficiency data envelopment analysis. In this paper, the efficiency of the financial investment in the education is analyzed from four viewpoints including pure technology efficiency, scale efficiency, overall efficiency, and super efficiency, and the analysis is made on the current situation and the main problems in the financial investment in the education in Guangdong and the suggestions on improvement are made on the basis of data from 2000 to 2012. The data used herein is sourced from the Statistical Yearbook of Guangdong from 2001 to 2013 and the Statistical Bulletin of Implementation of Educational Funds in China from 2000 to 2012.

4.2 Calculation Results

The values of investment and output used in this paper are greater than zero, there is an obvious correlation between the variables of the investment and output, and the DMU and investment and output investors meet the requirements of SE-DEA model. DEA-SOLVER Pro 5.0 software is use to calculate the financial efficiency of education in Guangdong from 2000 to 2012, and the results therefore are shown in the Table 1. The value of the overall technology efficiency in it is calculated with an option of CCR-I (CCR includes the investment-oriented CCR-I and output-oriented CCR-O, respectively indicating that in case of the unchanged output, how to minimize the investment; or in case of the unchanged investment, how to maximize the output). The pure technology efficiency (PTE) is used to measure the ratio of an optimal investment to an actual investment of the DMU on the productive frontier with the changeable returns to scale, and the value of the pure technology efficiency is calculated with an option of BCC-I in software. The scale efficiency (SE) is used to measure the distance from a productive frontier in case of the changeable returns to scale and productive frontier of unchangeable returns to scale, and to indicate the degree of the inefficiency due to the impossible production with the unchangeable returns to scale. It is calculated with the formula: Scale efficiency = overall technology efficiency/pure technology efficiency; the value of the comprehensive super technology efficiency is calculated with an option of Super-CCR-I in the Super-Radial module (Super-CCR includes investment-oriented Super-CCR-I and output-oriented Super-CCR-O, respectively indicating that in case of unchangeable output, how to minimize the investment; or in case of the unchangeable investment, how to maximize the output).

4.3 Analysis of Results

From the viewpoint of the technology efficiency, technology efficiency means influence of the internal management level and technical level of the enterprise on the production efficiency. In this paper, the technology efficiency refers to the use efficiency of the financial investment in education at the given policy conditions and financial investment level. From Table 1, it is observed that there are twelve years from 2000 to 2012, in which the technology efficiency reaches the optimal level, and only in 2002, the optimal technology efficiency is not reached. It indicates that the technology efficiency of the financial investment in the education in Guangdong over the past years is in an ideal state.

The scale efficiency means the proportional relation between the investment scale and the output scale. In this paper, we use the ratio of the increase in the investment

Table 1 Performance of financial investment in education in Guangdong from 2000 to 2012

Year	Pure technology efficiency (%)	Scale efficiency (%)	Overall technology efficiency (%)	Overall super technology efficiency (%)	Returns to scale
2000	100.00	100.00	100.00	117.15	–
2001	100.00	100.00	100.00	106.86	–
2002	95.80	96.20	92.16	92.07	DRS
2003	100.00	93.10	93.10	93.12	DRS
2004	100.00	95.20	95.20	95.22	DRS
2005	100.00	100.00	100.00	109.76	–
2006	100.00	99.30	99.30	99.29	DRS
2007	100.00	99.20	99.20	99.17	DRS
2008	100.00	92.60	92.60	92.60	DRS
2009	100.00	95.60	95.60	95.61	DRS
2010	100.00	100.00	100.00	100.78	–
2011	100.00	99.20	99.20	99.16	DRS
2012	100.00	93.30	93.30	93.33	DRS

scale in the operating expense of the education within a budget to an increase in the gross enrollment rate scale at various stages to represent it. If this value is greater than 1, it indicates that the returns to scale exist in the financial investment in education; if the value is less than 1, it indicates the scale insufficiency exists in the financial investment in the education. From the viewpoint of the scale efficiency, during the 13 years from 2000 to 2012, the scale efficiency is lower in the financial investment in the education in Guangdong; it reaches the optimal state only within four years; in the remaining 8 years, the scale benefit is decreased progressively.

From the viewpoint of the overall efficiency, during the 13 years from 2000 to 2012, only within four years, the overall efficiency. (If the overall efficiency is valid, the technology efficiency and scale efficiency must be valid) of the financial investment in education in Guangdong reaches an optimal state, indicating that the financial investment in education in Guangdong fails to reach the productive frontier because of the lower efficiency in most years.

The super efficiency analysis is used to further evaluate the efficiency in the years with the valid DEA. On the basis of the super efficiency analysis, efficiency of the financial investment in the education in Guangdong from 2000 to 2012 shows a downward tendency. See Fig. 2 for details.

As seen from analysis above, the use efficiency is not very high; furthermore, on the basis of a super efficiency analysis, the efficiency shows a downward tendency. Overall, there are more years in which the technology efficiency is reached and less years in which the scale efficiency is reached. However, the financial investment in the education is not valid unless the technology efficiency, scale efficiency, and overall

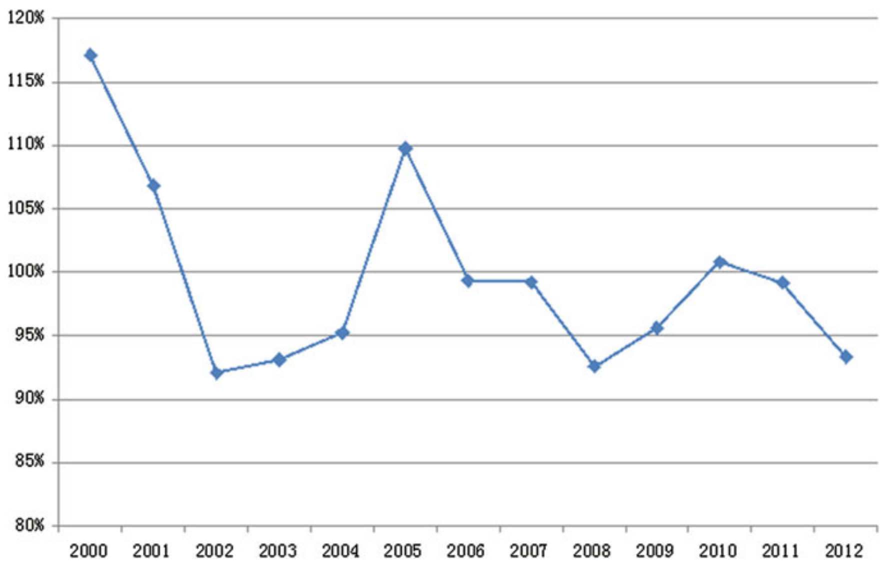

Fig. 2 Efficiency value of financial investment in education in Guangdong from 2000 to 2012

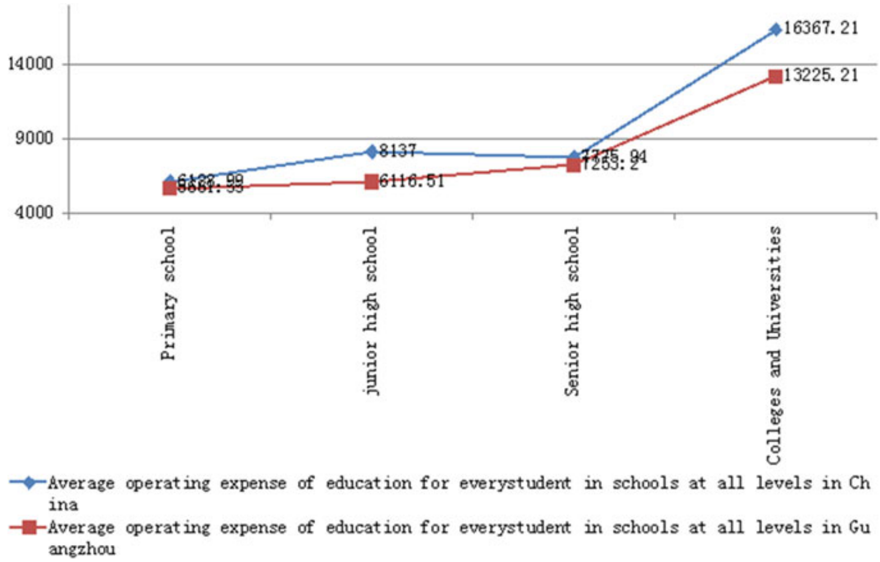

Fig. 3 Average operating expense of education for everystudent in schools at all levels in Guangzhou and China (Unit: RMB Yuan)

efficiency are valid. Specially, within 9 years in which the overall efficiency is not valid, it is a non-scale valid (progressive decrease in returns to scale), but basically the technically valid, indicating that the overall efficiency fails to reach the optimal state due to the mismatching between the scale and investment and output, so it is necessary to keep the existing investment proportion and management method, and increase the efficiency by means of the policy. Furthermore, in accordance with the data from the statistical bullet in of implementation of educational fund in China in 2012, despite of the larger total financial investment in the education in Guangdong, the average investment is lower for per student; in particular, compared to the national average level, the operating expense in education of students in the higher education is lower by RMB 3142 Yuan (See Fig. 3 for details); therefore, it is necessary to increase the financial investment in education.

From the view points of the looseness of investment, output and the improvement rate, the looseness of the average investment in an operating expense of education is larger for every student from elementary schools and junior ones, and that for the student in higher education is smaller; to senior middle schools, the looseness of the investment in the funds from 2000 to 2010 is smaller, but shows an increasing tend over the past two years. From the view point of the output looseness, the output looseness of enrollment rate of the school-age children is larger, and the enrolment rate of the primary graduate and the gross enrolment rate of the higher education are subject to the larger output looseness over the past two years. From the viewpoint of the DEA improvement rate, the improvement rate of the enrollment rate is higher in school-age children and gross enrollment rate of junior school before 2010, but declines over the past two years to a significant extent; the improvement rate of the gross enrolment rate of the higher education maintains a higher level over the past years (Table 2).

Here, it is assumed that the financial investment in the education is unchanged in order to make research on the output efficiency. Due to the dimensional difference between the output variables, in order to observe and analyze the extent to which the output is improved in a more clear way, DEA improvement rate is used to describe the improvement space. As seen from Table 3, all the improvement rates are the positive numbers, indicating that it is theoretically possible to increase the gross enrolment rate of the school at all levels from 2000 to 2012 from the viewpoint of efficiency of financial investment in education; specially, there is a bigger space for improvement of elementary and junior schools; however, it declines significantly over the past two years, and the improvement rate of senior school and higher education shows an upward trend.

Table 2 Analysis of looseness of DEA

	Input variables				Output variables			
	S1	S2	S3	S4	S5	S6	S7	S8
2000	–	–	–	–	–	–	–	–
2001	–	–	–	–	–	–	–	–
2002	56.72	41.77	0	988.31	6.61	6.66	0	0
2003	38.69	15.9	0	1007.41	1.08	0	0.59	0
2004	25.1	20.09	0	197.64	0.46	0	1.58	0
2005	–	–	–	–	–	–	–	–
2006	144.62	133.06	235.07	0	9.06	8.74	1.73	0
2007	517.58	617.22	0	1747.7	16.21	16.41	1.01	0
2008	683.44	755.29	0	0	29.86	29.57	0	0.05
2009	489.64	388.85	0	0	23.57	24.06	0	1.55
2010	–	–	–	–	–	–	–	–
2011	1037.13	756.83	796.8	0	4.79	4.99	0	1.34
2012	1459.61	1387.53	914.39	0	10.21	11.73	0	2.66

Note S1 denotes the operating expense of education for elementary school student within budget; *S2* denotes expense for junior school students within budget; *S3* denotes expense for senior school students within budget; *S4* denotes expense for students in higher education within budget; *S5* denotes the rate of the school-age children; *S6* denotes the rate of junior school; *S7* denotes the rate of senior schools snd *S8* denotes the rate of higher education

Table 3 Improvement rate of DEA output

	Gross enrolment rate of school-age students (%)	Gross enrolment rate of junior school (%)	Gross enrolment rate of senior school (%)	Gross enrolment rate of higher education (%)
2000	–	–	–	–
2001	–	–	–	–
2002	15.82	16.13	8.62	8.62
2003	8.56	7.39	8.72	7.39
2004	5.51	5.02	8.22	5.02
2005	–	–	–	–
2006	9.87	9.76	3.57	0.72
2007	17.22	17.95	2.38	0.84
2008	40.33	41.04	7.99	8.18
2009	29.27	30.88	4.59	10.50
2010	–	–	–	–
2011	5.67	6.13	0.84	5.69
2012	18.09	20.58	7.14	17.24

Note The improvement rate is the amount of looseness divided by actual value

5 Measures and Suggestions

The five suggestions next are made on the basis of results of SE-DEA model.

First, in order to solve a problem that a average investment for per student in Guangdong (e.g. average investment per student in higher education) is lower compared to the national level, the government departments shall increase the investment in its education practically. Overall, as the largest province in economy, the total amount of financial investment in education in Guangdong ranks the first place in China; however, the average investment in the operating expense of education for every student at all levels is lower than the national average level. For example, the average operating expense of education for every student in the schools at four levels is lower than the national level respectively by RMB 447.66 Yuan, RMB 2020.49 Yuan, RMB 522.74 Yuan, and RMB 3142 Yuan, indicating that the amount of its financial investment is mismatching with the economic aggregate in Guangdong. It is required to "make up a missed lesson" for its financial investment; the management at provincial and municipal tiers shall further increase the investment in the education fund, so that the residents in Guangdong will enjoy better education resource.

Besides, it is required to make the structural adjustment and policy guidance to the financial investment in education, and improve the use efficiency of the financial fund investment. From 2000 to 2012, the use efficiency of the financial investment in education in Guangdong is not very high; during this period, the productive frontier is reached only within four years; by analysis, the efficiency of its financial investment shows a downward trend. Specially, the technology efficiency of its investment reaches an optimal level, but the scale efficiency does not reach it (lower investment and output ratio), affecting value of the overall efficiency; furthermore, in the years when the optimal level is not reached, the returns to scale is decreasing progressively. If the structural adjustment and policy guidance of its financial investment iare not implemented, the marginal benefit from the increase in the enrolment rate of the school at all levels will decrease progressively. Therefore, the administrative departments in charge shall improve the use efficiency of the financial fund investment by adjusting policy, and the optimization of the financial resource allocation of schools at all levels.

Then, in order to embody the balanced education development, the government shall increase investment in the nine-year compulsory education in the underdeveloped areas such as the mountainous area East Guangdong, GD West GD, and North GD. From the looseness of the investment and output of the nine-year compulsory education and the improvement rate thereof, the looseness of the investment in its education is larger; if considering only from optimization of model, the investment in its operating expense for primary school and junior school within the budget shall be decreased, because both gross enrolment rate of the school-age children and that of the junior schools are higher than 95 %, and it is difficult to increase the output by increasing the investment. However, the investment in education falls within the "public goods" and the education in the underdeveloped areas is unbalanced, the compulsory education shall pay much attention to the fairness; particularly, the investment shall be increased in the nine-year compulsory education in the underdeveloped areas. From the viewpoints of the output looseness of the compulsory education and the improvement rate thereof, the education and the improvement rate thereof are larger. It is easy to discover that, the enrolment rate of the primary graduates in GD in 2012 is 93.5 %, lower than the national average level by 5 %, indicating that the great efforts shall be made to increase the investment in compulsory education, particularly the gross enrolment rate of the junior school.

Fourth, the gross enrolment rate of higher education in GD is lower than the national average level by 1.8 %, indicating that there is the great space of improvement. The looseness investment in education operating expense for senior school and higher education within the budget is at a lower level, indicating that the efficiency of the financial investment in them in GD is higher; however, the output looseness and the improvement rate thereof, the output looseness of higher education in GD is larger, showing an upward trend, and the improvement rate thereof is at the higher level, indicating that its gross enrolment rate of in GD is deficient, and there is a bigger space of improvement. Based on statistical data, it is not difficult to discover that the gross enrolment of the higher education in China is 30 %, and that in GD is 28.2 %, lower than that of the national. The occurrence of situation is related to the developed private economy in GD, in particular that a great number of the rural right-age students in specialized villages and towns with developed private economy will work in the household handicraft workshops after graduating from the senior schools, so as to reduce the enrolment rate. With regard to the above situations, the relevant functional departments shall strengthen propaganda and policy guidance, making the duly adjustment of specialties of provocation schools, colleges and universities in GD in order to attract the students to register for examination.

And finally, the use efficiency of its financial investment is subject to a greater impact on policy factors, and the efficiency of its financial investment in GD shows an obvious periodic change. As seen from Fig. 2, the change in the efficiency value presents wave in a period for every five years, reaching the extreme points in 2000, 2005, and 2010 respectively. It is worth noting that the above three years are the ending year of the "Ninth Five-Year" Plan, the "Tenth Five-Year" Plan, and the "Eleventh Five-Year" Plan respectively. The peak value of the efficiency occurring at the end of each five-year plan indicates that effectiveness of adjustment of the education policy in the "Five-Year Plan", fully indicating that the use of the financial fund in education is subject to the great impact on policy factors, making it possible to make the structural adjustment of the educational fund allocation and improve the use efficiency of the financial investment in the education.

References

1. Adams, E.K., odden, A.: Alterative wealth measures. In: Jordan, K.F., Cambron MeCabe, N.H., (Eds.) Perspectives in Programs. Cambridge, Ballinger (1987)
2. Barlow, R.: Efficiency accepts of local school finance, J. Polit. Econ. **78**(5), 1028–1040(1970)
3. Psacharo, P.C., Tan, G.: Financing Education in Developing Countries. Washington DC (1986)
4. Benson, S.C.: The Inequality Controversy:Schooling and Distributive Justice. Wiley, New York (1976)
5. Friedman, M.: The Role of Government in Education. Economics and the Public Interest (1995)
6. Johnes, J., Taylor.: Performance Indicators in High Education: UK universities Milton Keynes. Open University Press and The Society for Research into Higher Education (1990)
7. Jill, J.: Data envelopment analysis and its application to the measurement to efficiency in higher education. Eur. J. Educ. Rev. **25**, 273–288 (2006)
8. Pei, S.: Exploration on Construction of Performance Assessment System of Financial Expenditure for Elementary Education. Dongbei University of Finance and Economics (2005)
9. Feng, Y.: Research on Financial Expenditure for Elementary Education. Southwestern University of Finance and Economics (2006)
10. Fu, Y., Jia, Z.: Research on DEA-Based Assessment Model for Higher Education Resource Allocation. Heilongjiang Education (Version of higher education research and assessment)(2005)
11. Provincial Statistical Bureau of Guangdong, Statistical Yearbook of Guangdong China Statistics Press (2002–2013)
12. Huang, L.: Principal Component Analysis of Investment and Output of Higher Education. Journal of Finance and Economics (2005)
13. Andersen, P., Petersen, N. : A procedure for ranking efficient units in data envelopment analysis. Management Science, **39**(10), 1261–1264(1993)
14. Wu, K.: Educational Sociology. People's Education Press (1998)
15. Xu, C.: Study on Benefits from Financial Investment in Higher Education in China with Scale Expansion. Journal of South China Agricultural University (2009)
16. Yan, H.: Study on Financial Expenditure Performance Assessment Based on Perspective of Government Management., Economic Review, 4th issue (2010)
17. Yan, R.: Study on Assessment of Elementary Education Expenditure Performance. Henan Polytechnic University (2010)
18. Zhang, R., Fan, Y.: Optimization of Expenditure Structure, and Improvement of Use Benefit of Education Fund. Special Zone Economy (2004)