# Decomposability Conditions of Combinatorial Optimization Problems

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**Abstract.** Combinatorial Optimization Problems (COP) are generally complex and difficult to solve as a single monolithic problem. Thus, the process to solve the main initial COP may pass through solving intermediate problems and then combining the obtained partial solutions to find initial problem's global solutions. Such intermediate problems are supposed to be easier to handle than the initial problem. To be modeled using the hierarchical optimization framework, the master problem should satisfy a set of desirable conditions. These conditions are related to some characteristics of problems which are: multi-objectives problem, over constrained problems, conditions on data and problems with partial nested decisions. For each condition, we present supporting examples from the literature where it was applied. This paper aims to propose a new approach dealing with hard COPs particularly when the decomposition process leads to some well-known and canonical optimization sub-problems.

**Keywords:** Hierarchical optimization · Decomposability conditions · Complex problems · Problem with nested decisions · Large scale data sets

## 1 Introduction

Hierarchical optimization consists of dividing an optimization problem into two or more sub-problems; each sub-problem has its own objectives and constraints. These sub-problems are usually interconnected in a hierarchical structure where a sub-problem in level i coordinates with a sub-problem of level i-1. Hierarchical optimization can be viewed as an application of the divide and conquer strategy for handling complex and hard optimization problems. The final solution of the main problem is produced by combining in some way the solutions of different sub-problems through a set of links and relationships between sub-problems. The integration schema and the nature of links between sub-problems show simply the hierarchical structure proposed to represent the initial problem as a set of sub-problems. Modeling and solving above complex and large size problems as hierarchical optimization problem provide multiple benefits such as reducing the time, reducing the sub-problems search spaces, increasing the performance and reducing the implementation cost. Thus, two main questions should be answered; first when can a complex problem be decomposed into 'smaller' sub-problems? This question was addressed in the literature by Halford et al.


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[4]. Secondly, how to identify the required sub-problems? This question was addressed in the literature by Phillips [9]. Many characters aid us to precise if a particular problem can be considered as a hierarchical problem or not. We identified four characteristics of a particular complex problem to be modeled using the hierarchical optimization framework, which are:

Multi-objective problems.
Over-constrained problems.
Very large instances.
Problems with partial nested decisions.

We will detail the four decomposability conditions and argue them by a set of examples. First, multi-objectives optimization problems involve satisfying a set of objectives which facilitate its splitting into a set of interconnected sub-problems and modeled it hierarchically. The second condition to model a particular problem hierarchically is about the over constrained level characteristic of certain problems. The nature of data presents also a third decomposability condition based on two data characteristics; first the size of the date sets which, in special case, requires a data partitioning process to facilitate the solving of the whole data sets by iteratively applying the same solving process to the small data sets. Second, based on the categorical data characteristic, the large scale data sets are divided into a set of categories based on classes (patients/nurses, teacher/student, etc). Finally, Problem with nested decisions, require a certain number of decision makers, which needs a hierarchical structure to solve the complex model. All these conditions to model a particular problem hierarchically will be discussed in next sections. This paper will be organized as follows: the next section will be devoted to detail the necessary conditions to model an optimization problem hierarchically. The conditions presented in section 2 will be supported by a set of examples in section 3. The paper will then be concluded and some future research perspectives will be presented in the last section.

## 2 Decomposability Conditions of Optimization Problems

Problems' modeling is the main step in optimization problems handling process; it will help to prove the correct understanding and represent in a different form that facilitates its solving. In this work, we stipulate that a hierarchical decomposition of complex problems can yield to more effective solutions. However, some conditions shall be verified to model the problem using the hierarchical structure. Such conditions are problems' characteristics that will help to identify if a COP can be modeled hierarchically and they are detailed in the following subsections (see fig. 1)

### 2.1 Condition 1: Multi-objective Problem

Multi-objective Optimization Problems (MOOP) has been existing in many fields of science, including healthcare, economics, finance and logistics. Generally, multi-objective problems aim to realize multiple and often conflicting objectives to be

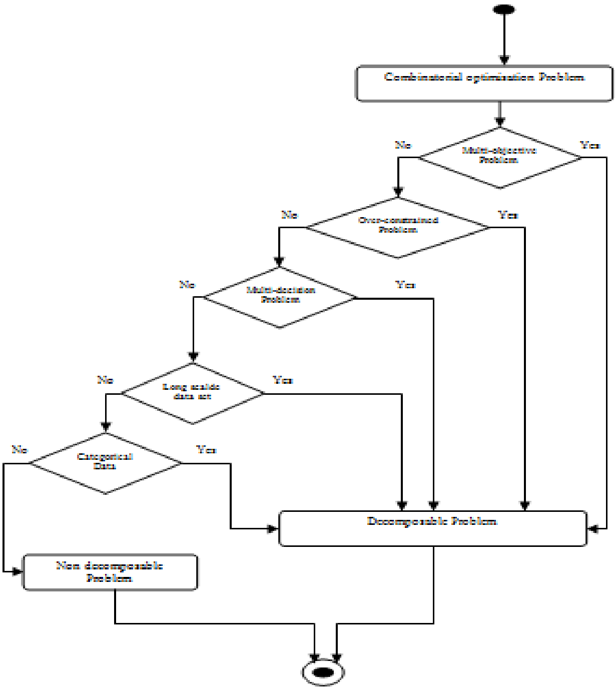


**Fig. 1.** Decomposability conditions framework

optimized. Consequently, a solving approach for a MOOP should provide a set of solutions with the best compromise between all required objectives. Optimizing with multiple different and conflicting objectives is an additional level of complexity of optimization problems. Then, it is possible to decompose the master problem into two or more sub-problems based on objectives; each sub-problem has its own objectives and constraints. To illustrate this idea, let's cite the work of Begur et al. [2] in which authors studied the Home Care Scheduling Problem (HCSP) and considered it like a multi-objectives problems that aims to satisfy three objectives to know:

The first objective is to assign patient visits to specific weekly time during a 16-or-so-week horizon,
The second objective is to allocate the visits planned for a given patient to a specific day of the week,
The third objective is to assign the patient visits scheduled for a given day to a particular nurse.

## 2.2 Condition 2: Over Constrained Problems

In many real-life applications (logistics, transportation, finance, etc) most optimization problems are highly constrained where different types of constraints have to be satisfied in the final solution. Over constrained problem are complex and difficult to solve like a single monolithic problem. To handle this problem, a constraint relaxation mechanism becomes necessary which consist to approximate of a difficult problem by nearby problems that are easier to solve by relaxing complicating constraints and recalling them after. Thus, the initial complex problem is modeled and solved in multiple levels, in each level a set of constraints will be satisfied until gratifying all required (hard) constraints and as much as possible satisfying preferential (soft) constraints. In this context, constraint hierarchies is a new concept proposed to describe high constrained problems by specifying constraints with hierarchical strengths or preferences, i.e. required and preferential constraints, most important and less important (preferences) constraints... Moreover, constraint hierarchies allow "relaxing" of constraints with the same strength by applying weighted-sum, least- squares or similar comparators [6] [1].

## 2.3 Condition 3: Conditions on Data

Two data sets characteristics can help to define the main problem to solve can be modeled using the proposed hierarchical framework. First, if the size of the data set of the instance is very large then it will be possible to divide it into two or more subsets based on particular criterion (geographic for example). Second, in some problems the data is classified into different types (level of proficiency of nurses, level of patients' illness, etc). For such problems the data sets can be partitioned following the defined data types. In the following we detail each condition and give some illustrative examples.

**Large Scale Data Sets**

Large scale data sets are collections of so large and complex data inputs that it becomes difficult to process traditionally as one batch. With very large data sets, experiments may face ambiguous situations and may not end. The data decomposition is the other primary form of breaking up monolithic processing into chunks that can be farmed out to multiple cores for parallel processing. The size of the problem space is one of the most obvious candidate measures of complexity which involve modeling a particular problem hierarchically. Problem difficulty was thought to vary with the size of the problem space. In COOP context, Hertz and Lahrichi [5] presented an illustration of data-partitioning decomposition strategy to model and solve the huge size of the problem space of the HCSP. Considering the very large size of the Canadian territory and to balance the work load of nurses while avoiding long travels to visit the clients, authors partitioned the Canadian territory into 6 districts.

**Categorical Data**

Another attribute of data sets of complex problem that can be modeled within the hierarchical framework is the existence of types and categories. Categorical data divide implicitly the input data into classes like types of customers in banking (VIP, Important, Ordinary) or type of employees following their skills (Expert, Skilled, Basic), etc. Such characteristics help to organize the main data set into smaller subsets following the proposed categories which define a set of sub-problems. The main problem will be consequently solved via solving each component and then merging the obtained partial solutions to form the main solution. Hertz and Lahrichi [5] propose to decompose the instance of the HCSP, following categories of patients and nurses. Similarly, Mullinax and Lawley [7] decompose the data set into three sub-groups based on the patient level of illness, into patients that require minimal care; patients that require close attention and critically ill patients.

### 2.4 Condition 4: Problems with Partial Nested Decisions

Generally, complex problems are multi-decision problems where some intermediate decisions must be taken to reach a final solution of the main problem. Multi-decisions problem solving process embeds the solving of sub-problems at different times by different decision makers at different levels. The final solution will be built by combining in some way the partial solutions of intermediate sub-problems. Consequently, the structure of the initial problem can be modeled as a particular combination of sub-problems. Such sub-problems are intuitively easier to handle and to solve than the main problem for different reasons: reduced search space and data sets, uncomplicated combinatorial structure, adapted solving approaches may be already known and solving tools (software) are available. For the above advantages, it is possible to represent multi-decisions problems by its components organized in such a way to fulfill the requirements of the initial problems. For instance, consider the HCSP where the question is about scheduling to serve patients at their home subject to different types of constraints and optimizing some objectives. Finding such a solution for the HCSP, passes through determining which nurse will serve which patient, then how their medical teams will be formed to move together and finally which routes will be followed to reach patients' homes in the transportation network [6] (see fig. 2).

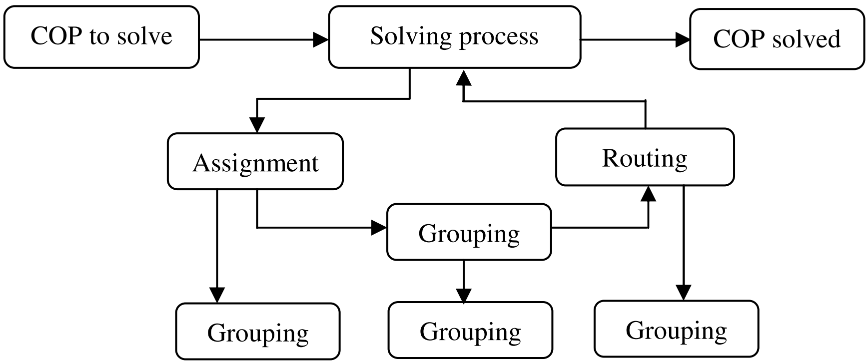


**Fig. 2.** Illustrative example of a solving process based on the multi-decision problem condition

## 3 Illustrative Examples

As mentioned in the introduction of this paper, it is necessary to validate our proposed approach through a set of research works from the literature. In the following section, a set of papers which modeled optimization problems using the hierarchically structure will be presented.

### 3.1 Multi-objectives Optimization Problems

The decomposition by objectives consists of dividing the basic problem into a set of sub-problems based on targets of the initial problem. For each problem at a given level, an optimization sub-problem is formulated to satisfy a set of constraints and achieve a certain objective function. In this context, Mutingi and Mbohwa [8] considered the initial problem as a multi-objective optimization problem with three conflicting management goals. The first sub-problem aims to minimize the schedule cost associated with the trips which is influenced by the nature of the routes assigned to healthcare workers to fulfill the demand requirements. The second component aims to maximize worker satisfaction which entails meeting the worker preferences to the highest degree possible and especially by ensuring the fairness in workload. The last sub-problem concerns the maximization of client satisfaction which can be expressed as a function of the violations of time windows preferred by the clients. The multi-objective formulation is achieved by optimizing the three objective functions jointly and the three components were solved in a parallel processing form.

### 3.2 Over-Constraints Optimization Problems

This approach consists of relaxing a set of complicating constraints in order to obtain a more tractable model. In this context, Bartk [1], thinks that constraint hierarchies allows to specify declaratively not only the constraints that are required to hold (hard constraints), but also weaker, and a finite hierarchy and order of satisfying. According to authors, weakening the strength of constraints helps to find a solution of previously over- constrained system of constraints. To illustrate the constraint hierarchy concept, we cite the work of Clark and Walker [3], in which authors considered the problem as an over constrained problem. They developed two models for the problem for comparison purposes. They called the first model “the individual days model” because the model assign individual shifts to each of the nurses and give more exibility over the schedules produced. The second one called the patterns model because it uses predefined weekly patterns which form a set covering problem. In the first model, authors try to satisfy simple (important ,hard) constraints in a first level like ensuring that every shift has the correct number of nurses assigned to it, calculating any shortage or surplus of nurses, obliging each nurse to work no more than shift per day and to have at least 2 days off every week. In the next levels of the hierarchy, to improve the quality of the solution, authors decided to add some new constraints preferred by nurses. This addition was made in two stages and in each stage the system became more complex. In the same context, Clark and Walker [3] added two constraints to the model to improve the perception of nurses of the shifts which in turn improves morale and has a positive effect on nursing staff. The first constraint insure that each nurse gets at least one complete week end off in every four weeks, the second one ensure that a nurse works no more than five days in a row. After adding these two constraints, the authors remark that the computational time remains reasonable and model complexity is not an issue for the user, so they decided to add more constraints to improve the model and its perceived fairness. The first added constraint ensures that night shifts and worked weekends are evenly distributed among nurses. The second one ensures that the day a night shift ends should be considered as a working day when calculating weekends off. The third constraint ensures working five consecutive days and isolated days off. In the second model, which named patterns model, authors used the same strategy to model the nurse scheduling problem using the constraints relaxing hierarchy. The patterns model selects from an input set of predefined shift patterns to identify shifts for each nurse. In the first stage they developed a simple model with two essential constraints, the first one ensure that every shift has the correct number of nurses assigned to it, calculating any shortage or surplus of nurses, and the second one makes sure that only one shift pattern per week is assigned to each nurse. Clark and Walker [3] remarked that they still need to additional constraints for week-linking because the patterns were one week long. Thus, they added four additional constraints to the model to checks whether 2 patterns can be adjacent and ensure that no nurse works more than 5 consecutive days. The second constraint ensures that each nurse gets at least one weekend off in 4. The two last constraints make sure that prohibited patterns are not assigned to nurses on non-full-time contracts.

### 3.3 Conditions on Data

In this subsection, we propose a set of illustrative examples of research works from the literature for illustrate both data decomposability conditions; the large scale data sets condition and the categorical data condition.

### 3.4 Large Scale Data Sets

The large scale data is a term used to describe a data collection of so large and complex that it becomes difficult to process traditionally (in a monolithic form). However, with very large data sets, experiments faced ambiguous situations and still not sure what to do. Thus, dividing the large scale data sets into smaller data sets can provide a good solution to reduce the complexity of the problem. In this context, the work of Hertz and Lahrichi [5] presented a perfect illustration of using the data-partitioning decomposition strategy to facilitate the modeling and the resolution of the huge size of the problem space of the HCSP. Given the large size of the Canadian territory, authors partitioned it into 6 districts (each one being constituted by several basic units) to balance the work load of nurses while avoiding long travels to visit the clients. According to authors [5], this data partitioning of the territory increased the efficiency in terms of client assignment, reduced transportation time, and therefore allowed for more time for direct patient care.

**Categorical Data**

Another data decomposability measure of hierarchy is to check if the data sets of the complex problem can be divided into small data sets based on its types (category, characteristics). Divided into small data sets, the complex problem will be recursively solved by iterating the solving approach into small sub-problem search space. Hertz and Lahrichi [5] used a data partitioning process based on categories of patients and nurses to simplify the main problem. The set of nurses was decomposed into three sets based on the level of patients' illness. The first sub-set was named case manager nurses (who typically hold a Bachelor's degree in nursing) which give care to patients requiring more complex experience like coordination of visits and ensuring links with doctors and specialists, organizing the activities of daily living. The second sub-set of nurses was named nurse technicians (who typically hold a community college degree in nursing) which give cares to the short-term clients or long-term clients needing punctual nursing care. The third sub-set of nurses present a surplus team that is not assigned neither any patient nor any district. Their role is to deliver specific nursing care treatments for the client. Authors cited a set of particular intervention of this group of nurses (handle nursing visits that the team nurses are unable to absorb visits that are needed outside regular working hours). It is important to mention here that these three types of groups are a part of the six multidisciplinary teams. The last data partitioning decomposition was based on patients' types which affect the work load of patients because the nurse volume of depends on the time needed to treat his/her specific patients. Thus, authors identified five category of clients based on the type of patients (see Fig. 3).

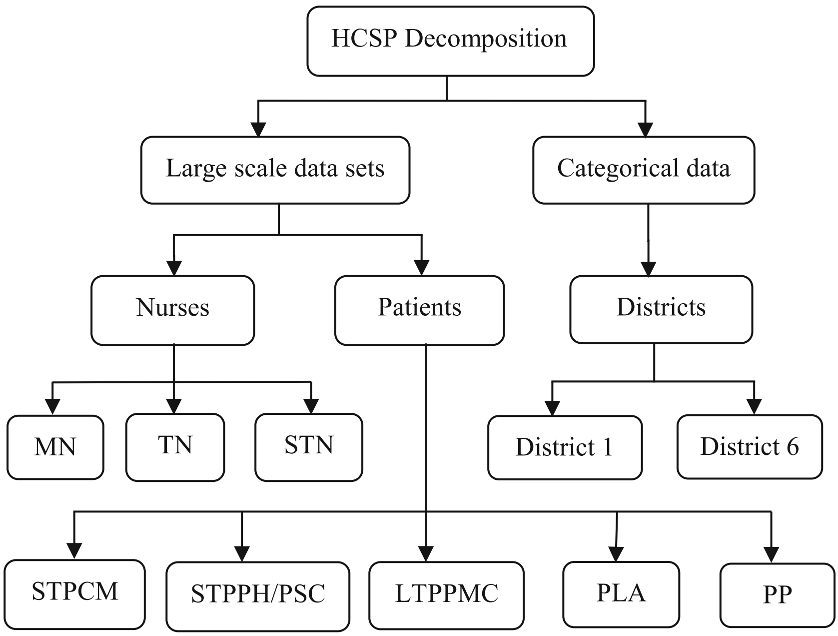


**Fig. 3.** Illustrative example of a solving process based on the multi-decision problem condition

The first category contains short-term clients that do not require case management (STPCM), the second category contain short-term clients that need post-hospitalization or post-surgery care (STPPH/ STPPPSC), the third category contain long-term clients needing punctual nursing care (LTPPMC), the fourth category contain clients with loss of autonomy (PLA) and the last category contain palliative patients (PP). A second example to cite here is the work of Mullinax and Lawley [7] in which authors decompose the data set (infants) into three sub-groups based on the patient level which means the type (state) of the patient. They grouped patients into three levels: level I patients that require minimal care; level II patients that require close attention, Level III care for critically ill patients. Then, a neonatal acuity system consists of 14 models contained scores were developed to precise to which level a newborn belong. The data partitioning strategy was also used in the second sub-problem (the assignment sub-problem) in which the newborn homes were divided into a number of physical zones. The following descriptive schema summarizes different data partitioning based on data conditions applied to the HCSP in the work of Hertz and Lahrichi [5].

### 3.5 Problems with Partial Nested Decisions

Generally, complex problems are multi-decision problems; in another words to reach the final solution of the problem, a set of decisions must be taken in a particular order to solve the principle problem. Multi-decision problems present a discipline of operations research (OR) that explicitly considers multiple decision-making in different level of decision in a particular problem. Generally, the targets to ensure in a particular problem are conflicted, thus, it is advised to decompose the whole problem based on decisions to establish a good hierarchy. To illustrate by an example, we cite here our previous work [6], in which we modeled the Home Care Scheduling Problem like a hierarchical optimization problem. We aimed to satisfy a set of conflicted objectives such minimizing travelling costs, maximizing satisfaction level of both patients and nurses... We decomposed the HCSP into three sub-problems in a hierarchical form based on the order of decisions to take. The three sub-problems defined were the assignment component, the grouping component and the routing component. Before modeling this problem we defined semantically a decision-making map (decision hierarchy) to facilitate the modeling process. We considered that first of all, the assignment task of patients to nurses must be modeled because this first component will affect the decisions made by the latter components. The choice of the next component to model depends on our decision making. Semantically, we think that the grouping component must be defined before the routing component to minimize fuel and by result minimizing costs because the last component depends of the routing component (it is so logic, that after defining the routes to be followed by nurses, sets of nurses with same routes will be travelled in the same groups). May be for other persons thinks that grouping nurses in clusters is more important that the routing component because for authors, it is important to form multidisciplinary group of nurses than the routing component. Thus the grouping component will be modeled in the second level and the routing component in the last level.

## 4 Conclusions and Perspectives

The hierarchical decomposition frameworks to model complex optimization problems are based on their decomposition into a set of interconnected sub-problems easier to handle. It's an application of the Divide and Conquer strategy to facilitate the handling of difficult problems. We detailed through this paper the different decomposability conditions to precise if a particular problem can be modeled hierarchically or not. The fourth measures were illustrated by a set of research papers from the literature. In the forthcoming project, we will attempt to develop a hierarchical decomposition strategy framework based on the four decomposability conditions detailed in this paper. It is important to mention here that the set of derived sub-problem should be linked and their partial solution should participate to build the final solution of the main initial problem. Thus, we aims in a future work to present the possible relationships that may combine the set of sub-problems to obtain a final solution for the global problem.

## References


1. Bartk, R.: Constraint hierarchy networks. In: Proceedings of the 3rd ERCIM/Compulog Workshop on Constraints, Lecture Notes in Computer Science. Springer (1998)
2. Begur, S., Miller, D., Weaver, J.: An integrated spatial DSS for scheduling and routing home-health-care nurses. Interfaces **27**(4), 35–48 (1997)
3. Clark, A., Walker, H.: Nurse rescheduling with shift preferences and minimal disruption. Journal of Applied Operational Research **3**(3), 148–162 (2011)
4. Halford, G., Wilson, W., Phillips, S.: Processing capacity defined by relational complexity: Implications for comparative, developmental, and cognitive psychology. Behavioral & Brain Sciences **21**(6), 803–864 (1998)
5. Hertz, A., Lahrichi, N.: A patient assignment algorithm for home care services. Journal of the Operational Research Society **60**, 481–495 (2009)
6. Jemai, J., Chaieb, M., Mellouli, K.: The home care scheduling problem: a modeling and solving issue. In: Proceedings of the 5th International Conference on Modeling, Simulation and Applied Optimization (2013)
7. Mullinax, C., Lawley, M.: Assigning patients to nurses in neonatal intensive care. Journal of the Operational Research Society **53**, 25–35 (2002)
8. Mutingi, M., Mbohwa, C.: A satisficing approach to home healthcare worker scheduling. In: International Conference on Law, Entrepreneurship and Industrial Engineering (2013)
9. Phillips, S.: Measuring relational complexity in oddity discrimination tasks. Noetica **3**(1), 1–14 (1997)