Dependence Factor as a Rule Evaluation Measure

Marzena Kryszkiewicz

Abstract Certainty factor and lift are known evaluation measures of association rules. Nevertheless, they do not guarantee accurate evaluation of the strength of dependence between rule's constituents. In particular, even if there is a strongest possible positive or negative dependence between rule's constituents X and Y, these measures may reach values quite close to the values indicating independence of X and Y. Recently, we have proposed a new measure called a dependence factor to overcome this drawback. Unlike in the case of the certainty factor, when defining the dependence factor, we took into account the fact that for a given rule $X \rightarrow Y$, the minimal conditional probability of the occurrence of Y given X may be greater than 0, while its maximal possible value may less than 1. In this paper, we first recall definitions and properties of all the three measures. Then, we examine the dependence factor from the point of view of an interestingness measure as well as we examine the relationship among the dependence factor for X and Y with those for $\bar{X}$ and Y, X and $\bar{Y}$, as well as $\bar{X}$ and $\bar{Y}$, respectively. As a result, we obtain a number of new properties of the dependence factor.

1 Introduction

Certainty factor and *lift* are known measures of association rules. The former measure was offered in the expert system Mycin [9], while the latter is widely implemented in both commercial and non-commercial data mining systems [2]. Nevertheless, they do not guarantee accurate evaluation of the strength of dependence between rule's constituents. In particular, even if there is a strongest possible positive or negative dependence between rule's constituents X and Y, these measures may reach values quite close to the values indicating independence of X and Y. This might suggest that one deals with a weak dependence, while in fact the dependence is strong. In [4],

M. Kryszkiewicz (✉)
Institute of Computer Science, Warsaw University of Technology,
Nowowiejska 15/19, 00-665 Warsaw, Poland
e-mail: mkr@ii.pw.edu.pl

S. Matwin and J. Mielniczuk (eds.), *Challenges in Computational Statistics and Data Mining*, Studies in Computational Intelligence 605,
DOI 10.1007/978-3-319-18781-5_12

we proposed a new measure called a *dependence factor* to overcome this drawback. Unlike in the case of the certainty factor, when defining the dependence factor, we took into account the fact that for a given rule $X \rightarrow Y$, the minimal conditional probability of the occurrence of Y given X may be greater than 0, while its maximal possible value may less than 1. The dependence factor always takes value 1 if a dependence is strongest possible positive, whereas for a strongest possible negative dependence, it always takes value –1; in the case of independence, it takes value 0.

In [4], we have focused on examining properties of the dependence factor as a measure of dependence between rule's constituents/events. Our new main contribution in this paper is: (1) the examination of the dependence factor as an interestingness measure with respect to the interestingness postulates formulated by Piatetsky-Shapiro in [7], and (2) the derivation of the relationship among the dependence factor for X and Y, with those for $\bar{X}$ and Y, X and $\bar{Y}$, as well as $\bar{X}$ and $\bar{Y}$, respectively.

Our paper has the following layout. In Sect. 2, we briefly recall basic notions of association rules, their basic measures (support, confidence) as well as lift and certainty factor. In Sect. 3, we recall maximal and minimal values of examined measures in the case when probabilities of rule's constituents are fixed, as shown in [4]. In Sect. 4, we recall the definition and properties of the dependence factor after [4]. Our new contribution is presented in Sects. 5 and 6. In Sect. 5, we examine the usefulness of the dependence factor as an interestingness measure, while in Sect. 6, we identify the relationship between the dependence factors for events and their complements. Section 7 concludes our work.

2 Basic Notions and Properties

In this section, we recall the notion of association rules after [1].

Definition 1 Let $I = \{i_1, i_2, \ldots, i_m\}$ be a set of distinct literals, called *items* (e.g. products, features, symptoms). Any subset X of the set I is called an *itemset*. A *transaction database* is denoted by $\mathscr{D}$ and is defined as a set of itemsets. Each itemset T in $\mathscr{D}$ is a *transaction*. An *association rule* is an expression associating two itemsets:

$$X \rightarrow Y, \text{ where } \emptyset \neq Y \subseteq I \text{ and } X \subseteq I \backslash Y.$$

Itemsets and association rules are typically characterized by *support* and *confidence*, which are simple statistical parameters.

Definition 2 *Support* of an itemset X is denoted by $sup(X)$ and is defined as the number of transactions in $\mathscr{D}$ that contain X; that is

$$sup(X) = |\{T \in \mathscr{D} | X \subseteq T\}|.$$

Support of a rule $X \rightarrow Y$ is denoted by $sup(X \rightarrow Y)$ and is defined as the support of $X \cup Y$; that is,

$$sup(X \rightarrow Y) = sup(X \cup Y).$$

Clearly, the probability of the event that itemset X occurs in a transaction equals $sup(X)/|\mathscr{D}|$, while the probability of the event that both X and Y occur in a transaction equals $sup(X \cup Y)/|\mathrm{D}|$. In the remainder, the former probability will be denoted by $P(X)$, while the latter by $P(XY)$.

Definition 3 The *confidence* of an association rule $X \rightarrow Y$ is denoted by $conf(X \rightarrow Y)$ and is defined as the conditional probability that Y occurs in a transaction provided X occurs in the transaction; that is:

$$conf(X \rightarrow Y) = \frac{sup(X \rightarrow Y)}{sup(X)} = \frac{P(XY)}{P(X)}.$$

A large amount of research was devoted to *strong association rules* understood as those association rules the supports and confidences of which exceed user-defined support threshold and confidence threshold, respectively. However, it has been argued in the literature that these two measures are not sufficient to express different interestingness, usefulness or unexpectedness aspects of association rules [3, 5–8, 10–12]. In fact, a number of such measures of association rules was proposed (see e.g. [3, 5–8, 10–12]. Among them very popular measures are *lift* [2] and *certainty factor* [9].

Definition 4 The *lift* of an association rule $X \rightarrow Y$ is denoted by $lift(X \rightarrow Y)$ and is defined as the ratio of the conditional probability of the occurrence of Y in a transaction given X occurs there to the probability of the occurrence of Y; that is:

$$lift(X \rightarrow Y) = \frac{conf(X \rightarrow Y)}{P(Y)}.$$

Lift may be also defined in an equivalent way in terms of probabilities only:

Property 1

$$lift(X \rightarrow Y) = \frac{P(XY)}{P(X) \times P(Y)}.$$

Definition 5 The *certainty factor* of an association rule $X \rightarrow Y$ is denoted by $cf(X \rightarrow Y)$ and is defined as the degree to which the probability of the occurrence of Y in a transaction can change when X occurs there as follows:

$$cf(X \rightarrow Y) = \begin{cases} \frac{conf(X \rightarrow Y) - P(Y)}{1 - P(Y)} & \text{if } conf(X \rightarrow Y) > P(Y), \\ 0 & \text{if } conf(X \rightarrow Y) = P(Y), \\ -\frac{P(Y) - conf(X \rightarrow Y)}{P(Y) - 0} & \text{if } conf(X \rightarrow Y) < P(Y). \end{cases}$$

The definition of the certainty factor is based on the assumption that the probability of the occurrence of Y in a transaction given X occurs there ($conf(X \rightarrow Y)$) can

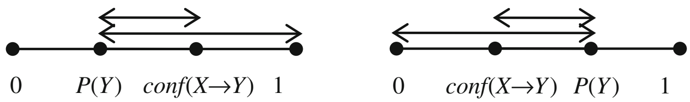

Fig. 1 Calculating the absolute value of the certainty factor as the ratio of the lengths of respective intervals when $conf(X \rightarrow Y) > P(Y)$ (on the *left-hand side*) and when $conf(X \rightarrow Y) < P(Y)$ (on the *right-hand side*)

be increased from $P(Y)$ up to 1 and decreased from $P(Y)$ down to 0. In Fig. 1, we visualize the meaning of the absolute value of the certainty factor as the ratio of the lengths of respective intervals.

As shown in Property 2, the certainty factor can be expressed equivalently in terms of unconditional probabilities (by multiplying the numerator and denominator of the formula in Definition 5 by $P(X)$) or lift (by dividing the numerator and denominator of the original *cf* formula by $P(Y)$).

Property 2

$$(a)\ \ cf(X \rightarrow Y) = \begin{cases} \frac{P(XY) - P(X) \times P(Y)}{P(X) - P(X) \times P(Y)} & \text{if } P(XY) > P(X) \times P(Y), \\ 0 & \text{if } P(XY) = P(X) \times P(Y), \\ -\frac{P(X) \times P(Y) - P(XY)}{P(X) \times P(Y) - 0} & \text{if } P(XY) < P(X) \times P(Y). \end{cases}$$

$$(b)\ \ cf(X \rightarrow Y) = \begin{cases} \frac{lift(X \rightarrow Y) - 1}{\frac{1}{P(Y)} - 1} & \text{if } lift(X \rightarrow Y) > 1, \\ 0 & \text{if } lift(X \rightarrow Y) = 1, \\ -\frac{1 - lift(X \rightarrow Y)}{1 - 0} & \text{if } lift(X \rightarrow Y) < 1. \end{cases}$$

Both lift and certainty factor are related to the notion of (in)dependence of events, where two events are treated as independent if the product of the probabilities of their occurrences equals the probability that the two events co-occur. Otherwise, they are regarded as dependent. Note that this notion of dependence does not indicate which event is a reason of the other. However, it allows formulating whether the dependence between events is positive or negative in the case when the events are dependent on each other.

Definition 6 X and Y are:

- *independent* if $P(XY) = P(X) \times P(Y)$,
- *dependent positively* if $P(XY) > P(X) \times P(Y)$,
- *dependent negatively* if $P(XY) < P(X) \times P(Y)$.

In Table 1, we provide equivalent conditions in terms of P, *conf*, *lift* and *cf* for independence, positive dependence and negative dependence, respectively, between two itemsets.

In general, one may distinguish between *symmetric* (*two direction*) *measures* of association rules and *asymmetric* (*one direction*) ones.

Table 1 Conditions for independence, positive dependence and negative dependence

(In)dependence	(In)dependence condition	Equivalent conditions in terms of measures for $X \rightarrow Y$	Equivalent conditions in terms of measures for $Y \rightarrow X$
Y and X are dependent positively	$P(XY) > P(X) \times P(Y)$	$conf(X \rightarrow Y) > P(Y)$ $lift(X \rightarrow Y) > 1$ $cf(X \rightarrow Y) > 0$	$conf(Y \rightarrow X) > P(X)$ $lift(Y \rightarrow X) > 1$ $cf(Y \rightarrow X) > 0$
Y and $X$are independent	$P(XY) = P(X) \times P(Y)$	$conf(X \rightarrow Y) = P(Y)$ $lift(X \rightarrow Y) = 1$ $cf(X \rightarrow Y) = 0$	$conf(Y \rightarrow X) = P(X)$ $lift(Y \rightarrow X) = 1$ $cf(Y \rightarrow X) = 0$
Y and $X$are dependent negatively	$P(XY) < P(X) \times P(Y)$	$conf(X \rightarrow Y) < P(Y)$ $lift(X \rightarrow Y) < 1$ $cf(X \rightarrow Y) < 0$	$conf(Y \rightarrow X) < P(X)$ $lift(Y \rightarrow X) < 1$ $cf(Y \rightarrow X) < 0$

Definition 7 A measure m is called *symmetric* (*two direction*) if $m(X \rightarrow Y) = m(Y \rightarrow X)$ for any X and Y. Otherwise, it is called an *asymmetric* (*one direction*) *measure*.

Property 3

(a) $conf(X \rightarrow Y) = conf(Y \rightarrow X)$ *is not guaranteed to hold.*
(b) $lift(X \rightarrow Y) = lift(Y \rightarrow X)$.
(c) $cf(X \rightarrow Y) = cf(Y \rightarrow X)$ *is not guaranteed to hold if* $conf(X \rightarrow Y) > P(Y)$.
(d) $cf(X \rightarrow Y) = cf(Y \rightarrow X)$ *if* $conf(X \rightarrow Y) \leq P(Y)$.

As follows from Property 3, *conf* is an asymmetric measure and *lift* is a symmetric measure. On the other hand, we observe that strangely *cf* has a mixed nature—asymmetric for positive dependences and symmetric for negative dependences and independences. This observation provoked us to revisit the definition of *cf* and to propose its modification in [4]. When defining the dependence factor there, we took into account the fact that in some circumstances it may be infeasible to increase the probability of the occurrence of Y in a transaction under the presence of X ($conf(X \rightarrow Y)$) from $P(Y)$ up to 1 as well as it may be infeasible to decrease it from $P(Y)$ down to 0.

3 Maximal and Minimal Values of Rule Measures

In this section, we first recall global maximal and minimal values of rule measures (Table 2). Next, following [4], we recall maximal and minimal values of rule measures for given values of $P(X)$ and $P(Y)$.

In the remainder of the paper, we denote *maximal probability* and *minimal probability* of the co-occurrence of X and Y given $P(X)$ and $P(Y)$ are fixed by $max\_P(XY|_{P(X),P(Y)})$ and $min\_P(XY|_{P(X),P(Y)})$, respectively. Analogously, *maximal confidence* and *minimal*

Table 2 Global maximal and minimal values of rule measures

Measure	Max	Min
$P(XY)$	1	0
$conf(X \rightarrow Y)$	1	0
$lift(X \rightarrow Y)$	∞	0
$cf(X \rightarrow Y)$	1 if Y depends on X positively	-1 if Y depends on X negatively

confidence (*maximal lift*, *minimal lift*, *maximal certainty factor*, *minimal certainty factor*) *of* $X \rightarrow Y$ given $P(X)$ and $P(Y)$ are fixed are denoted by $max\_conf(X \rightarrow Y|_{P(X),P(Y)})$ and $min\_conf(X \rightarrow Y|_{P(X),P(Y)})$ ($max\_lift(X \rightarrow Y|_{P(X),P(Y)})$, $min\_lift(X \rightarrow Y|_{P(X),P(Y)})$, $max\_cf(X \rightarrow Y|_{P(X),P(Y)})$, $min\_cf(X \rightarrow Y|_{P(X),P(Y)})$, respectively.

Property 4

(a) $max\_conf(X \rightarrow Y|_{P(X),P(Y)}) = \frac{max\_P(XY|_{P(X),P(Y)})}{P(X)}$

(b) $min\_conf(X \rightarrow Y|_{P(X),P(Y)}) = \frac{min\_P(XY|_{P(X),P(Y)})}{P(X)}$

(c) $max\_lift(X \rightarrow Y|_{P(X),P(Y)}) = \frac{max\_conf(XY|_{P(X),P(Y)})}{P(Y)} = \frac{max\_P(XY|_{P(X),P(Y)})}{P(X) \times P(Y)}$

(d) $min\_lift(X \rightarrow Y|_{P(X),P(Y)}) = \frac{min\_conf(XY|_{P(X),P(Y)})}{P(Y)} = \frac{min\_P(XY|_{P(X),P(Y)})}{P(X) \times P(Y)}$

(e) $max\_cf(X \rightarrow Y|_{P(X),P(Y)}) = \frac{max\_conf(X \rightarrow Y|_{P(X),P(Y)}) - P(Y)}{1 - P(Y)}$
$= \frac{max\_P(XY|_{P(X),P(Y)}) - P(X) \times P(Y)}{P(X) - P(X) \times P(Y)} = \frac{max\_lift(XY|_{P(X),P(Y)}) - 1}{\frac{1}{P(Y)} - 1}$

(f) $min\_cf(X \rightarrow Y|_{P(X),P(Y)}) = -\frac{P(Y) - min\_conf(X \rightarrow Y|_{P(X),P(Y)})}{P(Y) - 0}$
$= -\frac{P(X) \times P(Y) - min\_P(XY|_{P(X),P(Y)})}{P(X) \times P(Y) - 0} = -\frac{1 - min\_lift(XY|_{P(X),P(Y)})}{1 - 0}$

In Proposition 1, we show how to calculate $min\_P(XY|_{P(X),P(Y)})$ and $max\_P(XY|_{P(X),P(Y)})$. We note that neither $max\_P(XY|_{P(X),P(Y)})$ necessarily equals 1 nor $min\_P(XY|_{P(X),P(Y)})$ necessarily equals 0. Figure 2 illustrates this.

Proposition 1

(a) $max\_P(XY|_{P(X),P(Y)}) = \min\{P(X), P(Y)\}$

(b) $min\_P(XY|_{P(X),P(Y)}) = \begin{cases} 0 & \text{if } P(X) + P(Y) \leq 1 \\ P(X) + P(Y) - 1 & \text{if } P(X) + P(Y) > 1 \end{cases}$
$= \max\{0, P(X) + P(Y) - 1\}$

The next proposition follows from Property 4 and Proposition 1.

Proposition 2

(a) $max\_conf(X \rightarrow Y|_{P(X),P(Y)}) = \frac{\min\{P(X), P(Y)\}}{P(X)} =$
$\begin{cases} 1 & \text{if } P(X) \leq P(Y), \\ \frac{P(Y)}{P(X)} & \text{if } P(Y) < P(X). \end{cases}$

(a)

X	*Y*
x	x
x	x
x	

(b)

X	*Y*
x	
x	
x	
	x
	x

(c)

X	*Y*
x	
x	
x	x
x	x
x	x
	x

Fig. 2 **a** $max\_P(XY|_{P(X),P(Y)}) = \min\{P(X), P(Y)\} = \min\left\{\frac{3}{6}, \frac{2}{6}\right\} = \frac{2}{6}$. **b** $min\_P(XY|_{P(X),P(Y)}) = 0$ if $P(X) + P(Y) \leq 1$. **c** $min\_P(XY|_{P(X),P(Y)}) = P(X) + P(Y) - 1 = \frac{5}{6} + \frac{4}{6} - 1 = \frac{3}{6}$ if $P(X) + P(Y) > 1$

$$(b)\ \ min\_conf(X \rightarrow Y|_{P(X),P(Y)}) = \frac{\max\{0, P(X)+P(Y)-1\}}{P(X)}$$
$$= \begin{cases} 0 & \text{if } P(X) + P(Y) \leq 1, \\ \frac{P(X)+P(Y)-1}{P(X)} & \text{if } P(X) + P(Y) > 1. \end{cases}$$

$$(c)\ \ max\_lift(X \rightarrow Y|_{P(X),P(Y)}) = \frac{\min\{P(X),P(Y)\}}{P(X) \times P(Y)} = \frac{1}{max\{P(X),P(Y)\}}.$$

$$(d)\ \ min\_lift(X \rightarrow Y|_{P(X),P(Y)}) = \frac{\max\{0, P(X)+P(Y)-1\}}{P(X) \times P(Y)}$$
$$= \begin{cases} 0 & \text{if } P(X) + P(Y) \leq 1, \\ \frac{P(X)+P(Y)-1}{P(X) \times P(Y)} & \text{if } P(X) + P(Y) > 1. \end{cases}$$

$$(e)\ \ max\_cf(X \rightarrow Y|_{P(X),P(Y)}) = \frac{\min\{P(X),P(Y)\} - P(X) \times P(Y)}{P(X) - P(X) \times P(Y)}$$
$$= \frac{\frac{1}{\max\{P(X),P(Y)\}} - 1}{\frac{1}{P(Y)} - 1} = \begin{cases} 1 & \text{if } P(X) \leq P(Y), \\ \frac{\frac{1}{P(X)} - 1}{\frac{1}{P(Y)} - 1} & \text{if } P(X) > P(Y). \end{cases}$$

$$(f)\ \ min\_cf(X \rightarrow Y|_{P(X),P(Y)}) = -\frac{P(X) \times P(Y) - \max\{0, P(X)+P(Y)-1\}}{P(X) \times P(Y) - 0}$$
$$= \frac{\max\{0, P(X)+P(Y)-1\}}{P(X) \times P(Y)} - 1 = \begin{cases} -1 & \text{if } P(X) + P(Y) \leq 1 \\ \frac{P(X)+P(Y)-1}{P(X) \times P(Y)} - 1 & \text{if } P(X) + P(Y) > 1. \end{cases}$$

In Table 3, we summarize real achievable maximal and minimal values of $P(XY)$, $conf(X \rightarrow Y)$, $lift(X \rightarrow Y)$ and $cf(X \rightarrow Y)$ for given values of $P(X)$ and $P(Y)$.

4 Dependence Factor

In this section, we recall the definition of the *dependence factor* of a rule $X \rightarrow Y$, which we offered in [4] as a modification of the certainty factor. Unlike the certainty factor, it is based on real maximal and minimal values of $conf(X \rightarrow Y)$ for given values of $P(X)$ and $P(Y)$. Then we present the properties of this measure.

Table 3 Real achievable maximal and minimal values of $P(XY)$, $conf(X \to Y)$, $lift(X \to Y)$ and $cf(X \to Y)$ for given values of $P(X)$ and $P(Y)$

Measure	Max for given values of $P(X)$ and $P(Y)$	Min for given values of $P(X)$ and $P(Y)$
$P(XY)$	$\min\{P(X), P(Y)\}$	$\max\{0, P(X)+P(Y)-1\}$
$conf(X \to Y)$	$\frac{\min\{P(X),P(Y)\}}{P(X)}$	$\frac{\max\{0,P(X)+P(Y)-1\}}{P(X)}$
$lift(X \to Y)$	$\frac{\min\{P(X),P(Y)\}}{P(X)\times P(Y)}$	$\frac{\max\{0,P(X)+P(Y)-1\}}{P(X)\times P(Y)}$
$cf(X \to Y)$	$\frac{\min\{P(X),P(Y)\}-P(X)\times P(Y)}{P(X)-P(X)\times P(Y)}$ if Y depends on X positively	$-\frac{P(X)\times P(Y)-\max\{0,P(X)+P(Y)-1\}}{P(X)\times P(Y)-0}$ if Y depends on X negatively

Definition 8 The *dependence factor* of $X \to Y$ is denoted by $df(X \to Y)$ and is defined as the ratio of the actual change of the probability of the occurrence of Y in a transaction given X occurs there to its maximal feasible change as follows:

$$df(X \to Y) = \begin{cases} \frac{conf(X\to Y)-P(Y)}{max\_conf(X\to Y|_{P(X),\, P(Y)})-P(Y)} & \text{if } conf(X \to Y) > P(Y), \\ 0 & \text{if } conf(X \to Y) = P(Y), \\ -\frac{P(Y)-conf(X\to Y)}{P(Y)-min\_conf(X\to Y|_{P(X),\, P(Y)})} & \text{if } conf(X \to Y) < P(Y). \end{cases}$$

The dependence factor not only determines by how much the probability of the occurrence of Y in a transaction changes under the presence of X with respect to by how much it could have changed, but also it determines by how much the probability of the occurrence of X and Y in a transaction differs from the probability of their common occurrence under independence assumption with respect to by how much it could have been different (see Proposition 3a). In addition, the dependence factor determines by how much the value of the lift of a rule $X \to Y$ differs from the value 1 (that is, from the value indicating independence of rule's constituents in terms of the lift measure) with respect to by how much it could have been be different (see Proposition 3b).

Proposition 3

(a) $$df(X \to Y) = \begin{cases} \frac{P(XY)-P(X)\times P(Y)}{max\_P(XY|_{P(X),P(Y)})-P(X)\times P(Y)} & \text{if } P(XY) > P(X)\times P(Y), \\ 0 & \text{if } P(XY) = P(X)\times P(Y), \\ -\frac{P(X)\times P(Y)-P(XY)}{P(X)\times P(Y)-min\_P(XY|_{P(X),P(Y)})} & \text{if } P(XY) < P(X)\times P(Y). \end{cases}$$

(b) $$df(X \to Y) = \begin{cases} \frac{lift(X\to Y)-1}{max\_lift(X\to Y|_{P(X),P(Y)})-1} & \text{if } lift(X \to Y) > 1, \\ 0 & \text{if } lift(X \to Y) = 1, \\ -\frac{1-lift(X\to Y)}{1-min\_lift(X\to Y|_{P(X),P(Y)})} & \text{if } lift(X \to Y) < 1. \end{cases}$$

Theorem 1

(a) If $P(XY) > P(X) \times P(Y)$, then $df(X \to Y) \in (0, 1]$.
(b) If $P(XY) = P(X) \times P(Y)$, then $df(X \to Y) = 0$.
(c) If $P(XY) < P(X) \times P(Y)$, then $df(X \to Y) \in [-1, 0)$.

Proof Follows from Proposition 3a. □

Table 4 Maximal and minimal values of $df(X \to Y)$ for any given values of $P(X)$ and $P(Y)$

Measure	Max for any given values of $P(X)$ and $P(Y)$	Min for any given values of $P(X)$ and $P(Y)$
$df(X \to Y)$	1 if X and Y are dependent positively	–1 if X and Y are dependent negatively

As follows from Proposition 3a, the dependence factor is a symmetric measure.

Theorem 2 $df(X \to Y) = df(Y \to X)$.

Based on Proposition 1 and 3a, we will express the dependence factor $df(X \to Y)$ in terms of $P(XY)$, $P(X)$ and $P(Y)$, which will be useful for examining properties of this measure.

Theorem 3

$$df(X \to Y) = \begin{cases} \frac{P(XY) - P(X) \times P(Y)}{\min\{P(X), P(Y)\} - P(X) \times P(Y)} & \text{if } P(XY) > P(X) \times P(Y), \\ 0 & \text{if } P(XY) = P(X) \times P(Y), \\ -\frac{P(X) \times P(Y) - P(XY)}{P(X) \times P(Y) - \max\{0, P(X) + P(Y) - 1\}} & \text{if } P(XY) < P(X) \times P(Y). \end{cases}$$

One may easily note that $df(X \to Y)$ reaches 1 when $P(XY)$ is maximal for given values of $P(X)$ and $P(Y)$; that is, when $P(XY) = \min\{P(X), P(Y)\}$ or, in other words, when the dependence between X and Y is strongest possible positive for given values of $P(X)$ and $P(Y)$. Analogously, $df(X \to Y)$ reaches –1 when $P(XY)$ is minimal for given values of $P(X)$ and $P(Y)$; that is, when $P(XY) = \max\{0, P(X) + P(Y) - 1\}$ or, in other words, when the dependence between X and Y is strongest possible negative for these probability values (Table 4).

Based on Theorem 3 and Property 2a, one may derive relations between the dependence factor and the certainty factor as follows:

Theorem 4

(a) $df(X \to Y) \geq cf(X \to Y)$ if $P(XY) > P(X) \times P(Y)$,
(b) $df(X \to Y) = cf(X \to Y) = 0$ if $P(XY) = P(X) \times P(Y)$,
(c) $df(X \to Y) \leq cf(X \to Y)$ if $P(XY) < P(X) \times P(Y)$,
(d) $df(X \to Y) = \max\{cf(X \to Y), cf(Y \to X)\}$ if $P(XY) > P(X) \times P(Y)$,
(e) $df(X \to Y) = cf(X \to Y)$ if $P(XY) < P(X) \times P(Y)$ *and* $P(X) + P(Y) < 1$,
(f) $df(X \to Y) < cf(X \to Y)$ if $P(XY) < P(X) \times P(Y)$ *and* $P(X) + P(Y) > 1$.

Tables 5–6 illustrate the findings expressed as Theorem 4. In particular, Table 5 shows values of $lift(X \to Y)$, $cf(X \to Y)$ and $df(X \to Y)$ for $P(X) = 0.6$ and $P(Y) = 0.3$; that is, in the case when $P(X) + P(Y) \leq 1$. For these values of $P(X)$ and $P(Y)$, the maximal possible value for $P(XY)$ equals $\min\{P(X), P(Y)\} = 0.3$. The fact of

Table 5 Comparison of values of $lift(X \rightarrow Y)$, $cf(X \rightarrow Y)$ and $df(X \rightarrow Y)$ when $P(X) + P(Y) \leq 1$

$P(X)$	$P(Y)$	$P(XY)$	$P(X) \times P(Y)$	$lift(X \rightarrow Y)$	$cf(X \rightarrow Y)$	$cf(Y \rightarrow X)$	$df(X \rightarrow Y) = df(Y \rightarrow X)$
0.60	**0.30**	**0.30**	**0.18**	**1.67**	**0.29**	**1.00**	**1.00**
0.60	0.30	0.25	0.18	1.39	0.17	0.58	0.58
0.60	0.30	0.20	0.18	1.11	0.05	0.17	0.17
0.60	0.30	0.18	0.18	1.00	0.00	0.00	0.00
0.60	0.30	0.15	0.18	0.83	–0.17	–0.17	–0.17
0.60	0.30	0.10	0.18	0.56	–0.44	–0.44	–0.44
0.60	0.30	0.00	0.18	0.00	–1.00	–1.00	–1.00

Table 6 Comparison of values of $lift(X \rightarrow Y)$, $cf(X \rightarrow Y)$ and $df(X \rightarrow Y)$ when $P(X) + P(Y) > 1$

$P(X)$	$P(Y)$	$P(XY)$	$P(X) \times P(Y)$	$lift(X \rightarrow Y)$	$cf(X \rightarrow Y)$	$cf(Y \rightarrow X)$	$df(X \rightarrow Y) = df(Y \rightarrow X)$
0.80	0.60	0.60	0.48	1.25	0.38	1.00	1.00
0.80	0.60	0.55	0.48	1.15	0.22	0.58	0.58
0.80	0.60	0.50	0.48	1.04	0.06	0.17	0.17
0.80	0.60	0.48	0.48	1.00	0.00	0.00	0.00
0.80	0.60	0.45	0.48	0.94	–0.06	–0.06	–0.37
0.80	**0.60**	**0.40**	**0.48**	**0.83**	**–0.17**	**–0.17**	**–1.00**

reaching the maximal possible value by $P(XY)$ for the given values of $P(X)$ and $P(Y)$ is reflected by the value of $df(X \rightarrow Y) = 1$, which means that the dependence between X and Y is strongest possible positive. On the other hand, $cf(X \rightarrow Y)$ = 0.29 does not reflect this fact. In general, the real dependence of Y on X may be underestimated when expressed in terms of $cf(X \rightarrow Y)$. Also the value 1.67 of $lift(X \rightarrow Y)$ itself does not reflect the strong positive dependence between X and Y in the considered case in the view that the lift may reach very large values (close to infinity) in general.

Table 6 shows values of $lift(X \rightarrow Y)$, $cf(X \rightarrow Y)$ and $df(X \rightarrow Y)$ for $P(X) = 0.8$ and $P(Y) = 0.6$; that is, in the case when $P(X) + P(Y) > 1$. For these values of $P(X)$ and $P(Y)$, the minimal possible value of $P(XY)$ equals $P(X) + P(Y) - 1 =$ 0.4. Then the dependence between X and Y is strongest possible negative. This is reflected by the value of $df(X \rightarrow Y) = -1$. On the other hand, $cf(X \rightarrow Y)$ = –0.17 does not reflect this fact by itself. Also the value 0.83 of $lift(X \rightarrow Y)$ itself does not reflect the strong negative dependence between X and Y as it is positioned closer to the value 1 characteristic for independence rather than to the value 0.

5 Dependence Factor as an Interestingness Measure

In [7], Piatetsky-Shapiro postulated that a good interestingness measure of an association rules $X \rightarrow Y$ should fulfill the following conditions:

1. be equal to 0 if X and Y are independent; that is, if $P(XY) = P(X) \times P(Y)$,
2. be increasing with respect to $P(XY)$ given $P(X)$ and $P(Y)$ are fixed,
3. be decreasing with respect to $P(X)$ given $P(XY)$ and $P(Y)$ are fixed or be decreasing with respect to $P(Y)$ given $P(XY)$ and $P(X)$ are fixed.

According to [7], the following rule interest measure $ri(X \rightarrow Y) = |\mathscr{D}| \times [P(XY) - P(X) \times P(Y)]$ fulfills the above postulates. Nevertheless, we notice that this measure does not always satisfy the third postulate. Beneath we present the case in which the ri measure violates this postulate:

Let $P(Y) = 0$. Then, $P(XY) = 0$. In this case, $ri(X \rightarrow Y) = 0$ for each value of $P(X)$ in the interval [0, 1]. Thus, $ri(X \rightarrow Y)$ is not guaranteed to be decreasing with respect to $P(X)$ given $P(XY)$ and $P(Y)$ are fixed. Analogically, we would derive that $ri(X \rightarrow Y) = 0$ for each value of $P(Y)$ in the interval [0, 1] if $P(X) = 0$. So, $ri(X \rightarrow Y)$ is not guaranteed to be decreasing with respect to $P(X)$ given $P(XY)$ and $P(Y)$ are fixed. As a result, $ri(X \rightarrow Y)$ does not fulfill the third postulate if $P(X)$ or $P(Y)$ equals 0.

In fact, the $novelty(X \rightarrow Y)$ measure, which was defined in [5] as $[P(XY)–P(X) \times P(Y)]$, violates the third postulate in the same way as $ri(X \rightarrow Y)$.

Now, we will focus on examining if the dependence factor fulfills the postulates of rule interestingness. We start with formulating the properties of probabilities of events which will be useful in our examination.

Proposition 4

(a) If $P(X) = 0$ or $P(Y) = 0$ or $P(X) = 1$ or $P(Y) = 1$, then $P(XY) = P(X) \times P(Y)$.
(b) If $P(XY) \neq P(X) \times P(Y)$, then $P(X), P(Y) \in (0, 1)$.

Proof Ad (a) Trivial.

Ad (b) Follows from Proposition 4a. □

Theorem 5 *Let $X \rightarrow Y$ be an association rule.*

(a) $df(X \rightarrow Y) = 0$ iff $P(XY) = P(X) \times P(Y)$.
(b) df is increasing with respect to $P(XY)$ given $P(X)$ and $P(Y)$ are fixed.
(c) df is non-increasing with respect to $P(X)$ given $P(XY)$ and $P(Y)$ are fixed. In addition, df is decreasing with respect to $P(X)$ given $P(XY)$ and $P(Y)$ are fixed, $P(Y) \notin \{0, P(XY), 1\}$ and $P(XY) \neq 0$.
(d) df is non-increasing with respect to $P(Y)$ given $P(XY)$ and $P(X)$ are fixed. In addition, df is decreasing with respect to $P(Y)$ given $P(XY)$ and $P(X)$ are fixed, $P(X) \notin \{0, P(XY), 1\}$ and $P(XY) \neq 0$.

Proof Ad (a, b) Follow trivially from Theorems 1 and 3.

Ad (c) Let us first determine the derivative $df'(X \rightarrow Y)$ of $df(X \rightarrow Y)$ as a function of variable $P(X)$ based on Theorem 3 in all possible cases when $P(XY) \neq P(X) \times P(Y)$. We will use the fact that in such cases $P(X), P(Y) \in (0, 1)$ (by Proposition 4b).

Case $P(XY) > P(X) \times P(Y)$ and $\min\{P(X), P(Y)\} = P(X)$.
Then $P(XY) > P(X) \times P(Y) > 0$ and
$df'(X \rightarrow Y) = \frac{P(XY)\times(1-P(Y))}{(P(X)-P(X)\times P(Y))^2} < 0$.

Case $P(XY) > P(X) \times P(Y)$ and $\min\{P(X), P(Y)\} = P(Y)$.
Then
$df'(X \rightarrow Y) = \frac{P(Y)\times(P(XY)-P(Y))}{(P(Y)-P(X)\times P(Y))^2}$.
Hence:

- If $P(XY) = P(Y)$, then $df'(X \rightarrow Y) = 0$.
- If $P(XY) \neq P(Y)$, then $P(XY) < P(Y)$, so $df'(X \rightarrow Y) < 0$.

Case $P(XY) < P(X) \times P(Y)$ and $\max\{0, P(X) + P(Y) - 1\} = 0$.
Then
$df'(X \rightarrow Y) = \frac{PXY\times(P(Y)}{(P(X)\times P(Y))^2}$.
Hence:

- If $P(XY) = 0$, then $df'(X \rightarrow Y) = 0$.
- If $P(XY) \neq 0$, then $df'(X \rightarrow Y) < 0$.

Case $P(XY) < P(X) \times P(Y)$ and $\max\{0, P(X) + P(Y) - 1\} = P(X) + P(Y) - 1$.
Then
$df'(X \rightarrow Y) = \frac{(1-P(Y))\times(P(XY)-P(Y))}{(P(X)\times P(Y)-(P(X)+P(Y)-1))^2} = \frac{(1-P(Y))\times(P(XY)-P(Y))}{((1-P(X))\times(1-P(Y)))^2}$.
Hence:

- If $P(XY) = P(Y)$, then $df'(X \rightarrow Y) = 0$.
- If $P(XY) \neq P(Y)$, then $P(XY) < P(Y)$, so $df'(X \rightarrow Y) < 0$.

Now, let us consider the case when $P(XY) = P(X) \times P(Y)$ and $P(Y) \in (0, 1)$. Then $P(X)$ may take only one value, namely $\frac{P(XY)}{P(Y)}$.

Finally, we note that for $P(Y) = 0$ as well as for $P(Y) = 1$, $P(XY) = P(X) \times P(Y)$ (by Proposition 4a), and so, $df(X \rightarrow Y) = 0$ for each value of $P(X)$ in the interval $[0, 1]$.

Thus, *df* is a non-increasing function with respect to $P(X)$ given $P(XY)$ and $P(Y)$ are fixed. However, if $P(Y) \notin \{0, P(XY), 1\}$ and $P(XY) \neq 0$, then *df* is a decreasing function with respect to $P(X)$ given $P(XY)$ and $P(Y)$ are fixed.
Ad (d) Analogous to the proof of Theorem 5c. □

Corollary 1 *$df(X \rightarrow Y)$ fulfills the first and second Piatetsky-Shapiro postulates. In addition, it fulfills the third Piatetsky-Shapiro postulate if $P(Y) \notin \{0, P(XY), 1\}$ and $P(XY) \neq 0$* or if $P(X) \notin \{0, P(XY), 1\}$ and $P(XY) \neq 0$.

Proof By Theorem 5. □

6 Dependence Factors for Events and Their Complements

In this section, we examine the relationship between the dependence factors for events and their complements. We start with determining extreme values of joint probabilities of events and their complements. Next, we prove that the character of the (in)dependence between X and Y determines uniquely the character of the (in)dependence between $\bar{X}$ and Y, X and $\bar{Y}$, as well as $\bar{X}$ and $\bar{Y}$, respectively. Eventually, we derive the relationship among the dependence factor for X and Y, with those for $\bar{X}$ and Y, X and $\bar{Y}$, as well as $\bar{X}$ and $\bar{Y}$, respectively.

Proposition 5

(a) $max\_P(XY|_{P(X),P(Y)}) = 1$ *iff* $P(X) = P(Y) = 1$.
(b) $min\_P(XY|_{P(X),P(Y)}) = 0$ *iff* $P(X) + P(Y) \leq 1$.
(c) $P(X) + P(Y) \leq 1$ *iff* $(1 - P(X)) + (1 - P(Y)) \geq 1$ *iff* $P(\bar{X}) + P(\bar{Y}) \geq 1$.

Proof Ad (a) Follows from Proposition 1a.
Ad (b) Follows from Proposition 1b.
Ad (c) Trivial. □

Proposition 6

(a) $max\_P(\bar{X}\bar{Y}|_{P(\bar{X}),P(\bar{Y})}) = \min\{P(\bar{X}), P(\bar{Y})\} = \min\{1 - P(X), 1 - P(Y)\} = 1 - \max\{P(X), P(Y)\}$
(b) $min\_P(\bar{X}\bar{Y}|_{P(\bar{X}),P(\bar{Y})}) = \max\{0, P(\bar{X}) + P(\bar{Y}) - 1\} = \max\{0, (1 - P(X)) + (1 - P(Y)) - 1\} = \max\{0, 1 - P(X) - P(Y)\}$
(c) $max\_P(X\bar{Y}|_{P(X),P(\bar{Y})}) = \min\{P(X), P(\bar{Y})\} = min\{P(X), 1 - P(Y)\}$
(d) $min\_P(X\bar{Y}|_{P(X),P(\bar{Y})}) = \max\{0, P(X) + P(\bar{Y}) - 1\} = \max\{0, P(X) + (1 - P(Y)) - 1\} = \max\{0, P(X) - P(Y)\}$
(e) $max\_P(\bar{X}Y|_{P(\bar{X}),P(Y)}) = \min\{P(\bar{X}), P(Y)\} = \min\{1 - P(X), P(Y)\}$
(f) $min\_P(\bar{X}Y|_{P(\bar{X}),P(Y)}) = \max\{0, P(\bar{X}) + P(Y) - 1\} = \max\{0, (1 - P(X)) + P(Y) - 1\} = \max\{0, P(Y) - P(X)\}$

Proof Ad (a, c, e) Follows from Proposition 1a, saying that $max\_P(VZ|_{P(V),P(Z)}) = \min\{P(V), P(Z)\}$.
Ad (b, d, f) Follows Proposition 1b, saying that $min\_P(VZ|_{P(V),P(Z)}) = \max\{0, P(V) + P(Z) - 1\}$. □

Lemma 1

(a) $P(XY) > P(X) \times P(Y)$ *iff* $P(\bar{X}\bar{Y}) > P(\bar{X}) \times P(\bar{Y})$ *iff* $P(X\bar{Y}) < P(X) \times P(\bar{Y})$ *iff* $P(\bar{X}Y) < P(\bar{X}) \times P(Y)$.
(b) $P(XY) = P(X) \times P(Y)$ *iff* $P(\bar{X}\bar{Y}) = P(\bar{X}) \times P(\bar{Y})$ *iff* $P(X\bar{Y}) = P(X) \times P(\bar{Y})$ *iff* $P(\bar{X}Y) = P(\bar{X}) \times P(Y)$.
(c) $P(XY) < P(X) \times P(Y)$ *iff* $P(\bar{X}\bar{Y}) < P(\bar{X}) \times P(\bar{Y})$ *iff* $P(X\bar{Y}) > P(X) \times P(\bar{Y})$ *iff* $P(\bar{X}Y) > P(\bar{X}) \times P(Y)$.

Proof We will proof the proposition using the following equations:

- $P(\bar{X}) = 1 - P(X), \quad P(\bar{Y}) = 1 - P(Y)$,
- $P(\bar{X}Y) = P(Y) - P(XY)$, $P(X\bar{Y}) = P(X) - P(XY)$,
- $P(\bar{X}\bar{Y}) = P(\bar{X}) - P(\bar{X}Y) = 1 - P(X) - P(Y) + P(XY)$.

Ad (a)

- $P(\bar{X}\bar{Y}) > P(\bar{X}) \times P(\bar{Y})$ iff $1 - P(X) - P(Y) + P(XY) > (1 - P(X)) \times (1 - P(Y))$ iff $P(XY) > P(X) \times P(Y)$.
- $P(X\bar{Y}) < P(X) \times P(\bar{Y})$ iff $P(X) - P(XY) < P(X) \times (1 - P(Y))$ iff $P(XY) > P(X) \times P(Y)$.
- $P(\bar{X}Y) < P(\bar{X}) \times P(Y)$ iff $P(Y) - P(XY) < (1 - P(X)) \times P(Y)$ iff $P(XY) > P(X) \times P(Y)$.

Ad (b, c) Analogous to the proof of Lemma 1a. □

Proposition 7

(a) X and Y are dependent positively iff $\bar{X}$ and $\bar{Y}$ are dependent positively iff X and $\bar{Y}$ are dependent negatively iff $\bar{X}$ and Y are dependent negatively.
(b) X and Y are independent iff $\bar{X}$ and $\bar{Y}$ are independent iff X and $\bar{Y}$ are independent iff $\bar{X}$ and Y are independent.
(c) X and Y are dependent negatively iff $\bar{X}$ and $\bar{Y}$ are dependent negatively iff X and $\bar{Y}$ are dependent positively iff $\bar{X}$ and Y are dependent positively.

Proof Follows from Lemma 1. □

Lemma 2 (Proof in Appendix)

(a) $df(X \to Y) = df(\bar{X} \to \bar{Y})$
(b) $df(X \to \bar{Y}) = df(\bar{X} \to Y)$
(c) $df(X \to \bar{Y}) = -df(X \to Y)$

Theorem 6 follows immediately from Lemma 2.

Theorem 6

$$df(X \to Y) = df(\bar{X} \to \bar{Y}) = -df(X \to \bar{Y}) = -df(\bar{X} \to Y).$$

Corollary 2

(a) $df(X \to Y)$ reaches maximum iff $df(\bar{X} \to \bar{Y})$ reaches maximum iff $df(X \to \bar{Y})$ reaches minimum iff $df(\bar{X} \to Y)$ reaches minimum.
(b) $df(X \to Y)$ reaches minimum iff $df(\bar{X} \to \bar{Y})$ reaches minimum iff $df(X \to \bar{Y})$ reaches maximum iff $df(\bar{X} \to Y)$ reaches maximum.

7 Conclusions

In [4], we have offered the dependence factor as a new measure for evaluating the strength of dependence between rules' constituents. Unlike in the case of the certainty factor, when defining the dependence factor, we took into account the fact that for a given rule $X \rightarrow Y$, the minimal conditional probability of the occurrence of Y given X may be greater than 0, while its maximal possible value may less than 1. $df(X \rightarrow Y)$ always reaches 1 when the dependence between X and Y is strongest possible positive, –1 when the dependence between X and Y is strongest possible negative, and 0 if X and Y are independent. Unlike the dependence factor, the certainty factor itself as well as lift are misleading in expressing the strength of the dependence. In particular, if there is strongest possible positive dependence between X and Y, $cf(X \rightarrow Y)$ is not guaranteed to reach its global maximum value 1 (in fact, its value can be quite close to 0 that suggests independence). On the other hand, if there is strongest possible negative dependence between X and Y, $cf(X \rightarrow Y)$ is not guaranteed to reach its global minimum value –1 (in fact, its value can be quite close to 0). Similarly, lift may reach values close to the value 1 (that means independence in terms of this measure) even in the cases when the dependence between X and Y is strongest possible positive or strongest possible negative. Thus, we find the dependence factor more accurate measure of a rule constituents' dependence than the certainty factor and lift.

In this paper, we have: (1) examined the dependence factor as an interestingness measure with respect to the interestingness postulates formulated by Piatetsky-Shapiro in [7], and (2) derived the relationship among the dependence factor for X and Y with those for $\bar{X}$ and Y, X and $\bar{Y}$, as well as $\bar{X}$ and $\bar{Y}$, respectively. We have proved that the dependence factor $df(X \rightarrow Y)$ fulfills all Piatetsky-Shapiro interestingness postulates if $P(Y) \notin \{0, P(XY), 1\}$ and $P(XY) \neq 0$ or if $P(X) \notin \{0, P(XY), 1\}$ and $P(XY) \neq 0$. Otherwise, it fulfills the first two postulates entirely and the third postulate partially as $df(X \rightarrow Y)$ is a non-increasing function rather than decreasing with respect to the marginal probability of an event given the joint probability and the marginal probability of the other event are fixed. On the other hand, it can be observed that several interestingness measures of association rules proposed and/or discussed in the literature does not fulfill all interestingness postulates from [7], including the rule interest *ri* [7] and *novelty* [5], which violate the third postulate for zero marginal probabilities.

In this paper, we have found that the character of the (in)dependence between X and Y determines uniquely the character (positive/negative) of the (in)dependence between $\bar{X}$ and Y, X and $\bar{Y}$, as well as $\bar{X}$ and $\bar{Y}$, respectively. We have also found that the absolute value of the dependence factors is the same for events and their complements. We find this result justified as the marginal and joint probabilities of events and all their complements depend uniquely on the triple of the probabilities $\langle P(X), P(Y), P(XY)\rangle$.

Acknowledgments We wish to thank an anonymous reviewer for constructive comments, which influenced the final version of this paper positively.

Appendix

Proof of Lemma 2
In the proof, we will use the following equations:

- $P(\bar{X}) = 1 - P(X), \qquad P(\bar{Y}) = 1 - P(Y),$
- $P(\bar{X}Y) = P(Y) - P(XY), \quad P(X\bar{Y}) = P(X) - P(XY),$
- $P(\bar{X}\bar{Y}) = P(\bar{X}) - P(\bar{X}Y) = 1 - P(X) - P(Y) + P(XY).$

Ad (a)

Case $P(\bar{X}\bar{Y}) > P(\bar{X}) \times P(\bar{Y})$:
This case is equivalent to the case when $P(XY) > P(X) \times P(Y)$ (by Lemma 1a). Then:
$df(\bar{X} \rightarrow \bar{Y}) =$ /* by Proposition 3a */

$$= \frac{P(\bar{X}\bar{Y}) - P(\bar{X}) \times P(\bar{Y})}{max\_P(\bar{X}\bar{Y}|_{P(\bar{X}),P(\bar{Y})}) - P(\bar{X}) \times P(\bar{Y})} = \text{/* by Proposition 6a */}$$
$$= \frac{(1 - P(X) - P(Y) + P(XY)) - (1 - P(X)) \times (1 - P(Y))}{(1 - \max\{P(X), P(Y)\}) - (1 - P(X)) \times (1 - P(Y))}$$
$$= \frac{P(XY) - P(X) \times P(Y)}{\min\{P(X), P(Y)\} - P(X) \times P(Y)} = \text{/* by Theorem 3 */}$$
$$= df(X \rightarrow Y).$$

Case $P(\bar{X}\bar{Y}) = P(\bar{X}) \times P(\bar{Y})$:
This case is equivalent to the case when $P(XY) = P(X) \times P(Y)$ (by Lemma 1b). Then:
$df(\bar{X} \rightarrow \bar{Y}) =$ /* by Proposition 3a */
$= 0 =$ /* by Proposition 3a */
$= df(X \rightarrow Y).$

Case $P(\bar{X}\bar{Y}) < P(\bar{X}) \times P(\bar{Y})$ and $P(\bar{X}) + P(\bar{Y}) \leq 1$:
This case is equivalent to the case when $P(XY) < P(X) \times P(Y)$ (by Lemma 1c) and $P(X) + P(Y) \geq 1$ (by Proposition 5c). Then:
$df(\bar{X} \rightarrow \bar{Y}) =$ /* by Proposition 3a */

$$- \frac{P(\bar{X}) \times P(\bar{Y}) - P(\bar{X}\bar{Y})}{P(\bar{X}) \times P(\bar{Y}) - min\_P(\bar{X}\bar{Y}|_{P(\bar{X}),P(\bar{Y})})} = \text{/* by Proposition 6b */}$$
$$= - \frac{(1 - P(X)) \times (1 - P(Y)) - 1(-P(X) - P(Y) + P(XY))}{(1 - P(X)) \times (1 - P(Y)) - \max\{0, 1 - P(X), P(Y)\}}$$
$$= - \frac{P(X) \times P(Y) - P(XY)}{(1 - P(X) - P(Y) + P(X) \times P(Y)) - (0)}$$
$$= - \frac{P(X) \times P(Y) - P(XY)}{(P(X) \times P(Y) - (P(X) + P(Y) - 1)}$$

$$= -\frac{P(X) \times P(Y) - P(XY)}{P(X) \times P(Y) - \max\{0, P(X) + P(Y) - 1\}} = \text{ /* by Theorem 3 */}$$
$$= df(X \rightarrow Y).$$

Case $P(\bar{X}\bar{Y}) < P(\bar{X}) \times P(\bar{Y})$ and $P(\bar{X}) + P(\bar{Y}) > 1$:

This case is equivalent to the case when $P(XY) < P(X) \times P(Y)$ (by Lemma 1c) and $P(X) + P(Y) < 1$ (by Proposition 5c). Then:

$df(\bar{X} \rightarrow \bar{Y}) =$ /* by Proposition 3a */

$$= -\frac{P(\bar{X}) \times P(\bar{Y}) - P(\bar{X}\bar{Y})}{P(\bar{X}) \times P(\bar{Y}) - min\_P(\bar{X}\bar{Y}|_{P(\bar{X}),P(\bar{Y})})} \text{ /* by Proposition 6b */}$$
$$= -\frac{(1 - P(X)) \times (1 - P(Y)) - (1 - P(X) - P(Y) + P(XY))}{(1 - P(X)) \times (1 - P(Y)) - \max\{0, 1 - P(X), P(Y)\}}$$
$$= -\frac{P(X) \times P(Y) - P(XY)}{(1 - P(X) - P(Y) + P(X) \times P(Y)) - (1 - P(X) - P(Y))}$$
$$= -\frac{P(X) \times P(Y) - P(XY)}{(P(X) \times P(Y) - 0}$$
$$= -\frac{P(X) \times P(Y) - P(XY)}{(P(X) \times P(Y) - \max\{0, P(X) + P(Y) - 1\}} = \text{ /* by Theorem 3 */}$$
$$= df(X \rightarrow Y).$$

Ad (b)

The proof is analogous to the proof of Lemma 1a.

Ad (c)

Case $P(X\bar{Y}) > P(X) \times P(\bar{Y})$ and $P(X) \leq P(\bar{Y})$:

This case is equivalent to the case when $P(XY) < P(X) \times P(Y)$ (by Lemma 1c) and $P(X) \leq 1 - P(Y)$. Then:

$df(X \rightarrow \bar{Y}) =$ /* by Proposition 3a */

$$= \frac{P(X\bar{Y}) - P(X) \times P(\bar{Y})}{max\_P(X\bar{Y}|_{P(X),P(\bar{Y})}) - P(X) \times P(\bar{Y})} = \text{ /* by Proposition 6c */}$$
$$= \frac{(P(X) - P(XY)) - P(X) \times (1 - P(Y))}{\min\{P(X), 1 - P(Y)\} - P(X) \times (1 - P(Y))}$$
$$= \frac{P(X) \times P(Y) - P(XY)}{P(X) \times P(Y) - 0}$$
$$= \frac{P(X) \times P(Y) - P(XY)}{P(X) \times P(Y) - \max\{0, P(X) + P(Y) - 1\}} = \text{ /* by Theorem 3 */}$$
$$= -df(X \rightarrow Y).$$

Case $P(X\bar{Y}) > P(X) \times P(\bar{Y})$ and $P(X) > P(\bar{Y})$.

This case is equivalent to the case when $P(XY) < P(X) \times P(Y)$ (by Lemma 1c) and $P(X) > 1 - P(Y)$. Then:

$df(X \to \bar{Y}) =$ /* by Proposition 3a */

$$= \frac{P(X\bar{Y}) - P(X) \times P(\bar{Y})}{max\_P(X\bar{Y}|_{P(X),P(\bar{Y})}) - P(X) \times P(\bar{Y})} = \text{/* by Proposition 6c */}$$
$$= \frac{(P(X) - P(XY)) - P(X) \times (1 - P(Y))}{\min\{P(X), 1 - P(Y)\} - P(X) \times (1 - P(Y))}$$
$$= \frac{P(X) \times P(Y) - P(XY)}{(1 - P(Y)) - P(X) \times (1 - P(Y))}$$
$$= \frac{P(X) \times P(Y) - P(XY)}{P(X) \times P(Y) - \max\{0, P(X) + P(Y) - 1\}} = \text{/* by Theorem 3 */}$$
$$= -df(X \to Y).$$

Case $P(X\bar{Y}) = P(X) \times P(\bar{Y})$:

This case is equivalent to the case when $P(XY) = P(X) \times P(Y)$ (by Lemma 1b). Then:

$df(\bar{X} \to \bar{Y}) =$ /* by Proposition 3a */

$= 0 =$ /* by Proposition 3a */

$= -df(X \to Y)$.

Case $P(X\bar{Y}) < P(X) \times P(\bar{Y})$ and $P(X) + P(\bar{Y}) \leq 1$.

This case is equivalent to the case when $P(XY) > P(X) \times P(Y)$ (by Lemma 1a) and $P(X) \leq P(Y)$. Then:

$df(X \to \bar{Y}) =$ /* by Proposition 3a */

$$= -\frac{P(X) \times P(\bar{Y}) - P(X\bar{Y})}{P(X) \times P(\bar{Y}) - min\_P(X\bar{Y}|_{P(X),P(\bar{Y})})} = \text{/* by Proposition 6d */}$$
$$= -\frac{P(X) \times (1 - P(Y)) - (P(X) - P(XY))}{P(X) \times (1 - P(Y)) - \max\{0, P(X) - P(Y)\}}$$
$$= -\frac{P(XY) - P(X) \times P(Y)}{(P(X) - P(X) \times P(Y)) - (0)}$$
$$= -\frac{P(XY) - P(X) \times P(Y)}{\min\{P(X), P(Y)\} - P(X) \times P(Y)} = \text{/* by Theorem 3 */}$$
$$= -df(X \to Y).$$

Case $P(X\bar{Y}) < P(X) \times P(\bar{Y})$ and $P(X) + P(\bar{Y}) > 1$.

This case is equivalent to the case when $P(XY) > P(X) \times P(Y)$ (by Lemma 1a) and $P(X) > P(Y)$. Then:

$df(X \to \bar{Y}) =$ /* by Proposition 3a */

$$= -\frac{P(X) \times P(\bar{Y}) - P(X\bar{Y})}{P(X) \times P(\bar{Y}) - min\_P(X\bar{Y}|_{P(X),P(\bar{Y})})} = \text{/* by Proposition 6d */}$$
$$= -\frac{P(X) \times (1 - P(Y)) - (P(X) - P(XY))}{P(X) \times (1 - P(Y)) - \max\{0, P(X) - P(Y)\}}$$
$$= -\frac{P(XY) - P(X) \times P(Y)}{P(X) \times (1 - P(Y)) - (P(X) - P(Y))}$$
$$= -\frac{P(XY) - P(X) \times P(Y)}{P(Y) - P(X) \times P(Y)}$$
$$= -\frac{P(XY) - P(X) \times P(Y)}{\min\{P(X), P(Y)\} - P(X) \times P(Y)} = \text{/* by Theorem 3 */}$$
$$= -df(X \to Y). \qquad \square$$

References

1. Agrawal R, Imielinski T, Swami AN (1993) Mining association rules between sets of items in large databases. In: ACM SIGMOD international conference on management of data, pp 207–216
2. Brin S, Motwani R, Ullman JD, Tsur S (1997) Dynamic itemset counting and implication rules for market basket data. In: ACM SIGMOD 1997 international conference on management of data, pp 255–264
3. Hilderman RJ, Hamilton HJ (2001) Evaluation of interestingness measures for ranking discovered knowledge. LNCS 2035:247–259
4. Kryszkiewicz M (2015) Dependence factor for association rules. In: Proceedings of ACIIDS 2015, part II LNAI, vol 9012, pp 135–145, Springer, New York
5. Lavrac N, Flach P, Zupan B (1999) Rule evaluation measures: a unifying view. In: Proceedings of ILP-1999. LNAI, vol 1634, pp 174–185. Springer, New York
6. Lenca P, Meyer P, Vaillant B, Lallich S (2008) On selecting interestingness measures for association rules: user oriented description and multiple criteria decision aid. In: European journal of operational research, vol 184, pp 610–626. Elsevier, France
7. Piatetsky-Shapiro G (1991) Discovery, analysis, and presentation of strong rules. Knowledge discovery in databases, pp 229–248. AAAI/MIT Press, Cambridge
8. Sheikh LM, Tanveer B, Hamdani SMA (2004) Interesting measures for mining association rules. In: Proceedings of INMIC 2004, IEEE
9. Shortliffe E, Buchanan B (1975) A model of inexact reasoning in medicine. Math Biosci 23:351–379
10. Silberschatz A, Tuzhilin A (1995) On subjective measures of interestingness in knowledge discovery. Proc KDD 1995:275–281
11. Suzuki E (2008) Pitfalls for categorizations of objective interestingness measures for rule discovery. In: Statistical implicative analysis: theory and applications, pp 383–395. Springer, New York
12. Suzuki E (2009) Interestingness measures—limits, desiderata, and recent results. In: QIMIE/PAKDD