Studies in Fuzziness and Soft Computing

Rudolf Seising
Enric Trillas
Janusz Kacprzyk *Editors*

# Towards the Future of Fuzzy Logic

Springer

# Studies in Fuzziness and Soft Computing

Volume 325


**Series editor**

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ISSN 1434-9922 ISSN 1860-0808 (electronic)
Studies in Fuzziness and Soft Computing
ISBN 978-3-319-18749-5 ISBN 978-3-319-18750-1 (eBook)
DOI 10.1007/978-3-319-18750-1

Library of Congress Control Number: 2015939675

Springer Cham Heidelberg New York Dordrecht London


Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

# Preface

After the first fifty years from 1965, when Lotfi A. Zadeh published his famous and seminal paper 'Fuzzy Sets', an impressive path has been followed for the consolidation of the theories of fuzzy sets and fuzzy logic, as well as for their application to practical problems. In both the theoretical and the applied sides, and mainly in the second, relevant successes have been accomplished. An interesting characteristic shown by the new field, usually known as 'Fuzzy Logic', is that its study and practice has been developed in many countries of Europe, Asia, America and Oceania; it can be said that after these fifty years Fuzzy Logic is spread all over the Earth.

It was thanks to Fuzzy Logic that the new field of Soft Computing, in which fuzzy logic has a central role, appeared and developed as a new approach to problems that before the mixing of fuzzy logic with neural nets, genetic algorithms, probabilistic models, etc., could not be satisfactorily posed or solved. Concerning the future of Fuzzy Logic, it seems today that it lies in the new ideas arising from 'Computing with Words and Perceptions' (CwW).

Fuzzy logic, Soft computing and CwW were established by Lotfi A. Zadeh, who introduced a good deal of the seminal ideas on which their theoretical and practical development is grounded on. Zadeh, an engineer with a strong personality and of infinite courtesy, is one of the few people in the history of science and technology who, having introduced a new field of research, not only pushed its study and applications, but in his long life personally contemplated them. Today and happily, Zadeh continues seeing how fuzzy logic follows up its strengthening and actual penetration in the welfare of industries and people.

Along the first forty-five years forthcoming 1965, Zadeh traveled through all the continents to explain his new ideas in conferences and meetings. As a consequence of his efforts, many young researchers all over the world were compelled to work in or with fuzzy logic. Thanks to his sweet intellectual form of confronting the adverse opinions that arose from the very beginning of fuzzy set theory, this discipline survived and joined researchers with a nice sense of camaraderie as well as a lack of the typical internal or external academic fights. Fifty years later, there can be no doubt that it was, and it will be thanks to the drive of young researchers that fuzzy logic can evolve towards the challenging goals posed by CwW in the twenty-first century. Today, when very few people doubt about the importance of fuzziness and on the relevance of its study and applications, new frontiers of knowledge are waiting to be explored. It is for this reason that this book's editors asked their authors to write, from different disciplines and points of view, papers potentially able to motivate young people to devote efforts in the future development of fuzzy logic, fuzzy methodologies, fuzzy applications, etc.

As the editors we thank all authors for their contributions to this volume, for their willingness to write their chapters. We also thank Springer Verlag and in particular Dr. Thomas Ditzinger, Dr. Leontina Di Cecco and Holger Schäpe.

The editors of this book, in their own names and also in those of the authors contributing to it, would like to express their affectionate respect and admiration for Prof. Lotfi A. Zadeh and not only for the man, but also for his work.

Jena, Germany — Rudolf Seising
Mieres, Spain — Enric Trillas
Warsaw, Poland — Janusz Kacprzyk
February 2015

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# Chapter 1
# On Reasoning with Words and Perceptions

**Pere Julià**

**Abstract** Fuzzy theorizing has included references to human behavior in general, and to reasoning in particular, almost from its inception. Experience is fuzzy. The strong connections between fuzzy logic and control theory take over, however, when we rely on the machine as a conceptual blueprint, effectively leaving out what makes human behavior unique. Failure to make the proper distinctions is tantamount to letting the available technical means determine our conception of the subject matter instead of the other way around. We will concentrate on the empirical status of the words, perceptions, and rules on which reasoning builds. Language and perception are forms of activity and thus functionally related to the sets of base variables in the absence of which there would be no behavior to speak of. Simulation itself would be a flight of fancy. The fact that claims about the relevance of fuzzy thinking to human behavior continue to be supported through reference to spectacular achievements in engineering clearly calls for reflection.

## 1.1 Introduction

Reasoning is something human beings do. Few people would question that it is embedded in the broader human activity we usually call thinking. Just what their boundaries and overlap are is far from clear, as the overwhelming literature on the language/thinking/action trinity clearly attests. Some would argue that neither can be ultimately dissociated from practical or theoretical problem solving. Underlying it all is the traditional Olympian disregard for the multiplicity of factors in the absence of which we can speculate endlessly about what we do or say, and why. What should be clear by now, however, is the untenability of the lighthearted identification of reasoning with language alone, particularly in its written form.

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_1

Elsewhere I have argued (see, e.g., Julià [4–6]) that much of the prevailing confusion can be traced to (1) the reluctance to confront the hypostatical nature of the linguistic entities we usually operate with; and (2) the unwillingness to bring conative factors into the picture. (1) and (2) are jointly responsible for the systematic decoupling of the individual from the environment to which he/she must adapt to survive. Conative variables—very roughly speaking, motivational and emotional factors—fuel all forms of activity in real life; the study of cognition in abstraction from them is doomed to sterility, as the record shows. But we will not be concerned with conation here; we will concentrate instead on the methodological traps associated with (1).

Not surprisingly, the decoupling of the individual from the environment has become an integral part of systems discourse (theoretical or instrumental) where, by and large, "system" stands for the organism, human or otherwise, and for the machines presumably designed to simulate the former's behavior. It would appear that in this kind of narrative, all we need by way of explanation is a set of rules that could conceivably "cover" the facts. There is nothing wrong with this strategy, of course, provided that we keep clearly in mind that rules are only hypothetical statements—strings of words that we put together as experts ($E$). In the final analysis, their descriptive relevance depends on the reliability of the variables contemplated, the adequacy of the terms used, and the suitability of the operations we perform on the data.

Hypostasis cautions us against the peril that confronts every use of symbols, viz., that the symbol may become detached from reality and be cultivated as an end in itself (for a particularly powerful indictment, see Kantor [9–11]). Much of the opacity concerning the symbolic and referential value of language stems from the failure to keep words and their referents distinctly apart. The stage is then set for the confusion between events and the constructs we propose to deal with them—an old story in the philosophy of science, under different guises. We recall the admixture of ontological and epistemological culs-de-sac associated with the status of the "laws of nature"—tentative and always open to revision, as we all know, yet nature somehow managing to oblige and "obey" our formulations at every stage.

The risk of substituting constructs for events is as real as it is widespread, although the extent of the perils entailed varies with the different sciences. No appreciable harm seems to follow from speaking about the "behavior of particles" in physics, about cells "communicating" with one another, and the like; and the same presumably goes for the "behavior" of computers and other machines, even if we wonder whether "behavior" here does not illustrate the kind of opacity just alluded to.

But we cannot so easily ignore the consequences of confusing constructs and events where bona fide animal and human behavior is at issue. It is here that hypostatical traps take over; it is here that we can quickly lose track of what it is we are trying to account for. For obvious reasons, the probability of following false analytical leads is far greater where language itself is the object of study; the chances of getting trapped in our own representational devices are vastly increased when the events described (generically, words) and the means developed to deal with them share a point-to-point correspondence.

The upshot is lack of clarity or downright distortion of the concepts juggled, notably when an action (or its traces, as we shall see) is treated independently of the functional relations converging on its occurrence; or, coming closer to home, when, conflating granules and words, the posited rules inadvertently morph into processes "out there"—a clear case of imputation (some might call it projection) of *E*'s behavior to the behavior of the subject (*S*) or of the subject matter. Special mention must be made of the use of the machine as a conceptual template, effectively discarding what makes human behavior unique as so much psychological gossamer. The differences will become clear, I trust, as we proceed.

Sometimes this is done in the name of simplicity. Simplicity is always welcome, of course, but the price we pay is obviously too high when it commits us to this kind of procedural reductionism. The proliferation of formally motivated models in recent decades, along with the subsequent need to invoke initially extraneous or dubious notions to supply some kind of empirical footing for them, bears witness to the difficulties inherent in any approach that, starting from real or imagined final products, prefers to reconstruct or "reverse engineer" when, in many cases, it could equally appeal directly to the variables and processes responsible for those final products. Hypotheses can easily be upheld as facts.

The temptation to extrapolate from the machine blueprint to the complexities of human behavior is all the more understandable where natural language—as opposed to the artificially constructed symbolic languages used elsewhere—constitutes the cornerstone, as in fuzzy logic (FL) and soft computing (SC). Still, claims continue to be made on the basis of models seemingly more concerned with the accuracy of our guesses than with the effective integration of empirical data. To cite but one example: Smithson's "fuzzy set theory may provide models or metaphors which mirror the rules by which people manipulate fuzzy categories in thought and language" [21, p. 5] is as representative a statement as any. How can we be so sure about the categories invoked beforehand? What does "mirroring" translate into, in fact? It is claims like these that motivate the present essay.

If knowledge and reasoning are to be represented as part of our research strategy, we should be clear whose knowledge and reasoning we are talking about and where the relevant processes are grounded. Answers are more likely to come from an embodied approach to the raw facts than from abstract characterizations of what might be the case. The bottom line is actually the choice between a standard top-down versus an authentic bottom-up strategy.

We will not be concerned with reasoning proper at this point but with the empirical status of the words and perceptions on which reasoning builds. The insights and implications which FL brings to the contemporary scene are too deep, and the possibilities offered by SC too important, not to ensure that we "get it right," that is to say, that respect for the proprieties prevails. The disambiguation of expressions such as "computing with words" (CW) and "manipulating perceptions" (MP) seems like a good place to begin.

## 1.2 Words, Words, Words

### *1.2.1 Some Historical Background*

Writing is a very recent development in human cultural history, but when we talk and think about language we tend to do so in terms of words—in particular, written words. The Western tradition of language study is eminently textual. This reliance on the written word is only too understandable: speech occurs but once, and its acoustic effects quickly vanish into thin air; written records, by contrast, enjoy a kind of permanence, a staying power, as it were, which allows for the unlimited return to them for consideration and manipulation. Historical and comparative linguistics, from which modern theoretical research derives, proceed by reconstruction; they are at bottom more akin to archaeology or paleontology than to a science purporting to describe ongoing dynamic processes.

As a result, language itself is routinely conceptualized as a self-contained system of forms, signs, and symbols—independent, so to speak, of the speakers and listeners without whom there be would be no "language systems" for us to analyze. Saussure's [2] structuralist definition of language as "a system in which everything is coherent with everything else, knowing no order but its own" continues to underwrite contemporary formal research of all stripes. Language becomes *L*, a system of words and rules opaquely referred to as "a tool or instrument which we 'use' to communicate".

This kind of abstractionism comes at a price, and a heavy price it is: once we ignore the linguistic *activity* of speakers and listeners, and the circumstances under which it occurs, all we are left with are the *traces* or *products* of this activity—acoustic in the case of speech, graphic in the written case. The thoughts, ideas, and feelings said to be communicated, as well as their effects at the receiving end, remain unaccounted for.

The reluctance to distinguish between activity and product is at the root of most if not all semantic quibbles and contradictions, old and new. To assume that words are somehow imbued with intrinsic meaning is a well-known dead end: consider the non sequiturs that come up, for example, in connection with synonymy, polysemy or etymology, or the never-ending explanatory gyrations forced upon us by the subtleties of figurative language. Nor is this the only reason that words, sentences, propositions, and the like so sullenly resist a comprehensive and methodologically stable definition. Clearly, the oft-invoked notion of "context" proves too vague to help us track down the factors that make the just-noticeable difference.

We must realize that the linguistic and logical entities we usually deal with are no longer the original utterances but transcriptions thereof, which they resemble in point of form. As such they are, at best, quotations, "a certain anomalous feature which calls for special attention: from the standpoint of logical analysis each whole quotation must be regarded as a single word or sign, whose parts count for no more than serifs or syllables. A quotation is not a *description* but a *hieroglyph*; it designates its object not by describing it in terms of other objects, but by picturing it. The meaning of the whole does not depend on the meanings of the constituent words" (Quine [19, p. 26]).

Quine's quotation points to the essentially hypostatical nature of linguistic "facts" and the ease with which we can be misled by our notational devices, wherein lies a story of its own. The distinction between "use" and "mention" was an indirect acknowledgment of the largely arbitrary nature of the freestanding units that materialize in writing. To say that they conventionally "stand for" or "symbolize" or "mean" whatever it is that the speaker or writer may be trying to say raises more questions than it answers. Meaning successfully continues to elude us.

If pursued, the "use-mention" dichotomy could have provided valuable insight into the plasticity and variability characteristic of verbal units in the raw and a better appreciation of the dynamic nature of language—for instance, by exploring what lies behind "designating" as opposed to "picturing," or "mentioning" as opposed to "use". Who is doing the designating and picturing anyway? Surely not the words themselves.

If pursued, it might also have brought various time-honored analytical positions and the attendant philosophical impasses into a more realistic perspective. Alas, that this did not happen is another indication of the deep-seated fascination with the written word. It is no coincidence that we speak of "putting a reading" between the lines when we are in fact *listening* to someone. Different variables are involved in each case and different treatments are therefore required, as we shall see in the sequel.

### *1.2.2 On Speakers and Listeners, Writers and Readers*

The reification of language as *L* has a twofold effect: (1) the more or less explicit identification of speaking and understanding as two sides of the same coin, despite the overwhelming practical and theoretical evidence to the contrary; and (2) the disengagement of the vocal-auditory aspects of communicative behavior from the rest of the individual's activity—perceptual and motor—necessary for his/her adaptation to the surrounding world. Does language ever occur by itself?

(1) The *L*-narrative construes speaking as the concatenation of words and other constructs; understanding becomes a kind of reverse process of decomposition. Theoretically anyway, the descriptive rules proposed should thus account for both speaking and understanding. Respect for the raw facts tells a different story: for starters, much of the behavior of the listener qua listener is not verbal; this is particularly transparent when rules are followed rather than formulated. (It might be noted in passing that this overall outlook similarly vitiates information and computational models at large.)
It seems more reasonable to start at the beginning. To say that a given bit of behavior—any behavior or traces there of—occurs is not enough: a satisfactory scientific description requires the specification of the pertinent variables, too. We note that motor behavior (e.g., pressing on a bar or a pedal) is bound to

have a direct effect upon the environment; on the other hand, linguistic or verbal behavior depends upon the presence of a listener for its effects. This is especially obvious during the early phases of language acquisition, but the speaker-listener compact remains in place throughout life. We generally speak in the presence of others.

Faced with a green light, there are many things that an individual can do or say. For simplicity, we will concentrate on two minimal possibilities: the $S$ may (a) press a pedal (a motor response); or (b) she/he may say "green" (a linguistic or verbal response). The obvious answer to the question, Which will it be? is that under normal circumstances, say, sitting in a car waiting for the traffic light to change, "green" is more likely to be produced if someone else is present. We are interested in "green": it is the verbal response that lends itself to hypostasis and the one that has a direct bearing on linguistic variables.

Elaborating on Kantor's simplest schematization of linguistic events as adaptive behavior, we can represent this example as follows:

where $(A)R$ stands for $A$'s uttering "green," $S_{aj}$ (or adjustment stimulus) stands for the green light, and $S_{ax}$ (or auxiliary stimulus) stands for the listener. $R(B)$ ordinarily antecedes further perceptual, verbal and/or motor behavior.

This kind of configuration is called "bi-stimulational" on account of $S_{aj}$ and $S_{ax}$ acting upon the speaker ($A$), simultaneously or in quick succession. We have a similarly bi-stimulational situation for the listener too, but the role of $S_{aj}$, $S_{ax}$, and $B$ have changed accordingly.

Upon seeing the green light, $A$ is prompted to press the accelerator pedal but he could just as easily do something else, for example, adjust his glasses or shade his eyes against the sun. If $A$ says "green," the acoustic stimulus (the result of the speaker's action in Fig. 1.1b, becomes a variable for the listener: $B$, who may be looking in her handbag, can now respond to "green" *and* to the green light ($S_{ax}$ and $S_{aj}$, respectively) in a variety of ways: she may, for example, fasten her seat belt, make herself comfortable, or roll up her window, or she may say, "Finally!" or "Let's go!" or "So now, what where you saying?" inviting a conversation. What both $A$ and $B$ will do or say at a given time $t$ will naturally depend on several factors, but the unique status and contribution of "green" as a decisive factor for $B$ needs to be pointed out.

A good deal of speech serves to put the listener in touch, as it were, with the variables that prompted the speaker's response in the first place. These variables may be momentarily outside the listener's range of vision or hearing, or he/she may simply be unaware of them (we will come back to this point below). Another instance: $A$ looks at the clock on the wall, realizes that it is five o'clock and remarks "It's five o'clock," bringing $B$'s attention to the clock in turn. ($A$ may, of course, be simultaneously reminding $B$ of an impending task, or indicating to $B$ that he need not hurry yet, that she has already missed the bus, and so on.) Thus informed by $A$, $B$ may now look at the clock for confirmation and say, "Thanks for telling me!" (perhaps ironically) and bank on the most convenient interpretation before taking action, ignoring the remark altogether, and so forth.

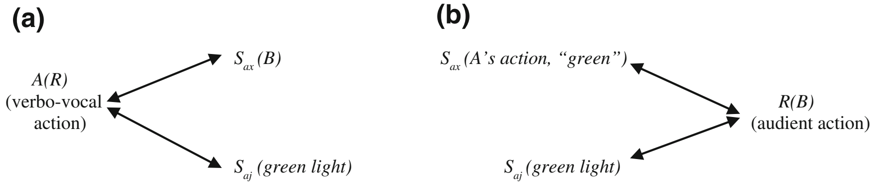


**Fig. 1.1** Schema of linguistic events as adaptive behavior

The resulting multiplicity of meanings for both speaker and listener should warn us against the temptation to conflate any one verbal stimulus with any one given referent. The variables at work and the specific activities involved are different for *A* and *B* and must be kept clearly apart if we are to avoid the snares of hypostasis. This is not to say that there is no overlap between speaking and understanding: there are indeed good behavioral reasons, not to mention abundant electromyographic and other physiological evidence, to argue against the traditional view of listening-cum-understanding as an essentially passive exercise.

The listener responds to verbal stimuli along with nonverbal ones. Different listeners bring different dispositions to bear on the stimulus complex at different times, hence the room for misunderstanding and disagreement. Indeed, the same individual may derive a different understanding from the same objective piece of information (linguistic or otherwise) on different occasions, as we have all experienced. Generically speaking, the ease and depth of understanding is a function of: (a) the intensity and/or clarity of the verbal stimulus; (b) the contribution of adjacent stimuli; and (c) the dispositions—nonverbal as well as verbal—brought by the listener or reader to the stimulus setting.

(2) The usual disengagement of speaker and listener from their surrounding world is complete when we fail to take into account the numerous and intricate connections prevailing between language and other forms of behavior intimately tied up with it—from simple cases of verbally supported or mediated perceiving to full-fledged reasoning, thinking, self-monitoring, and genuinely creative activity of all sorts. Of special importance are those self-descriptive repertoires that have the speaking individual himself or herself as a referent. Oddly enough, this subtle intertwining of verbal and nonverbal activity—which makes human language so radically different from any other "systems of communication"—ordinarily gets short shrift in the literature, if mentioned at all, despite the current flurry of research on consciousness.

The external sources of stimulation discussed so far (the green light, the clock) may be said to be located "out there"; they are, moreover, available to speaker and listener alike. Crucially, however, human individuals soon, yet gradually, learn to discriminate stimulation available only to themselves. This may require a word or two of explanation.

The referent is now one's own body: contrast, for example, "my arm/nose" with "your/her arm/nose," and/or various forms of stimulation arising from the nose, the position or movement of the arm, and so on. Similarly, different feelings are reported when we say, "I'm hungry," "My foot's asleep," "I'm happy/angry/ready/tired…" Or compare these with "I need/want…," "I dis/like…," "I prefer X/choose to…," and scores of others, which point to *both* internal dispositions and external stimulation, depending on how we fill in the blank.

We describe (or better, label) our behavior past, present or future: if "I'm walking" versus "I'm talking" seems too facile an example, it is good to remember the network of related discriminations that any thesaurus will offer to refer to different ways of walking and talking. The importance of proprioceptive stimulation when compelled to monitor our movements hardly needs mentioning. Nor must this be restricted to gross motor action: it is worth pondering what is behind the saying/thinking continuum, as in "Then I thought/said to myself …" and "That's when I decided…," or more conjecturally, "I must have thought/reasoned…" and the like.

Labeling slides into description when we include a reference to the variables that may have triggered the behavior: "I did *X* when I heard that…," "When I see *X*, I simply…," "I'm saying this because…." The young child ceases to merely "act tired" and knows that she is tired, and eventually why, when she can report how she feels as the cause or reason for her sluggishness—a never-ending learning exercise throughout life.

Self-referential behavior would appear to engage our senses in seemingly more intimate ways than referential responses to the "external world" but the simultaneous participation of external and internal sources is in reality a matter of degree. Language—self-referential as much as communicative—is a social phenomenon; it is within the relative parameters defined by our social milieu that we literally learn to *tell* the difference and get a sense of reality into the bargain, as the answer to "Am I cold or is it cold?" makes all too clear. External and internal feedback combine to provide a sense of reality, but also individuality, to members of a social group.

The relevance of self-responding to the study of important aspects of memory, anticipation and self-regulation—where rule-related behavior is bound to play an important role—should be immediately apparent. It is for this reason that at least a thumbnail presentation has been deemed necessary at this point (for a related discussion of how this comes about, see, e.g., Julià [8] and references cited therein).

### *1.2.3 On Reference and Linguistic Variables*

The speaker-listener distinction permits a closer look at LVs. As already noted, linguistic utterances or, more precisely, the written records we usually work with, are not the original referent (their denotation, if you like) but a substitute for it, and we may be led to believe that the referent postulated, and the associated labeling and reasoning, are equally valid for speaker and listener, which we know not to be the case (see, e.g., various contributions in [1]). Much confusion concerning the "symbolic" value of linguistic constructs stems from the failure to distinguish between unalloyed referential language and quotation. Harking back to Quine's observations above, we are entitled to wonder: What is the referent of a quotation? Consider [24]:

> In CW, a granule, $g$, which is the denotation of a word, $w$, is viewed as a fuzzy constraint on a variable. …As a simple illustration, consider the proposition Mary is young, which may be a linguistic characterization of a perception. In this case, young is the label of a granule young. (*Note that for simplicity the same symbol is used both for a word and its denotation.*) The fuzzy set young plays the role of a fuzzy constraint on the age of Mary" (emphasis added).

We are essentially asking, What kind of variable is a linguistic variable? How does a LV operate or, more exactly, how is it made to operate? One obvious approach is to examine the manner of its establishment by *E*.

In setting up a LV, *E* is acting as a speaker: like *S*, *E* responds to a given range of stimulation to which he/she can respond by saying or writing "green" or "tall," "young," "Mary," and so forth. Crucially, no nouns, verbs, phrases, rules, or propositions of any kind are part of the referent for either *S* or *E*. But the similarities end there.

*E* is doing much more than *S* and what he does has very little in common with what an ordinary speaker, or himself when not working as an *E*, does. He tentatively "fixes" a referent ($g$) and supplies a label ($w$) and does so for very specific purposes. "Young" plays the role of a fuzzy constraint on Mary's age, given certain concrete parameters, deliberately specified. Words stand for fuzzy sets; computation necessitates the membership functions associated with them. People do not go around setting up LVs. There is a world of difference between everyday discriminative behavior to different aspects of the environment and the deliberate labeling and targeted chunking here under consideration.

The tentativeness of the postulated granule will be resolved through subsequent operations, depending on how well the assigned quantitative values conform with the reality we are interested in; if necessary, *E* will write a revised set of rules or instructions for the machine to follow. And so on.

It proves helpful to backtrack and consider the following: if the speaker's PRESS (a motor response) were to function as a cue for the listener to act (e.g., by pressing another pedal in turn), we would still have the same $g$ but no $w$, and no risk of hypostatizing. The best we can do in such a case is to label the response ("a press" versus "a push" or "a shove") and adjust the pertinent values within a continuum (PRESSING, PUSHING, SHOVING). This would be a "linguistic characterization of a movement"—only a label, really, like "tall" or "green," and no less dignified for being about motor behavior than a "linguistic characterization of a perception" with its air of intangibility.

In short, neither a PRESS nor "green" can be considered *variables for the speaker*: they are his/her response to a given range of stimulation—the referent, denotation, or granule $g$—as a reaction to which "green" ($w$) is produced. It is here that conceptual confusion arises. There is at any rate no functional connection between the referent granule and $w$, both of which will be readjusted by *E* as need arises.

On close inspection, a LV plays the role of a *variable for the listener*, for example, by being included in rules of operation as a "linguistic characterization of a perception." When the speaker's response materializes (particularly in writing) as "green," "young," "Mary," and so on, it becomes an effective stimulus for subsequent "manipulation," a concept that demands a closer look (see below). Nor can "labeling" and "characterization" be taken as necessarily synonymous on every occasion.

For the speaker/writer, the fuzziness associated with "green" as a constraint on a variable involves the relation between his/her response and the range of stimulation that triggered "green," "young," "Mary," etc. For the listener/reader, it resides primarily within the range of possible responses to "green" and what that may additionally trigger in him/her, not only in terms of semantic clusters, presuppositions, and so forth but also in terms of nonverbal dispositions. It may be argued that the listener responding to "green" is in practice beginning to respond to the original stimulation as well. That is correct; that is what "green" is supposed to do. It promotes an indirect kind of reference, which will become full-fledged to the extent that the listener comes to respond more fully to the same variables as the speaker. We will meet a parallel situation in connection with rules later.

What may be ambiguous to the speaker need not be ambiguous to the listener (and vice versa) at different times and places, and that raises challenging questions about the source and nature of imprecision, vagueness, uncertainty, and their role in common sense or approximate reasoning. Where are they to be found? In the original physical referent? In the speaker's response to it? Or perhaps in the listener's response to either of those, or even to both?

Finally, the risk that the connection between $w$ and $g$ become increasingly remote and abstract, and their relation lost sight of, is real enough, particularly in theoretical research by different $E$s, who may take $w$ as an established fact and proceed from there. Zadeh's phraseology must be interpreted in this sense. He is obviously not equating words and their denotations, and no one working with empirical facts and getting back to them at every step, whether in science or engineering, would be inclined to think so. Zadeh is, in fact, recognizing that there are events (the granules defined as fuzzy constraints on variables) and constructs ($w$) and that we can conveniently represent them as we please. It cannot be too strongly emphasized, however, that things can go badly awry when we extrapolate directly to everyday activity, as the history of perceptual studies all too plainly shows. To repeat: people do not go around setting up LVs.

## 1.3 On Perception

Collectively, perception has to do with those processes that lend a measure of coherence and unity to sensory input; included here are physical, physiological, neurological, cognitive, and affective components. In the normal course of events, perceptual responses precede motor and verbal activity; just as naturally, however, perceptual responses can accompany, partially overlap, and eventually initiate or even become a substitute for the original stimuli. The literature is complex and often inconsistent, different approaches reflecting different emphases on different components and methodologies.

The bottom line is the relation between the stimulation impinging upon the behaving organism and its response to it, neither of which can be effectively discussed independently of the other. To do so is to perpetuate the most resilient conundrums that continue to plague the field. We are essentially concerned with the presence or absence of stimuli, the differences between them, and the relationships among them.

In our discussion of reference—or better, referential behavior—we de-emphasized, as it were, the physical stimulus complex that might trigger a response like "green." The present section aims to make the relation between the responses and their antecedent stimuli more explicit. The importance of doing so will become especially clear when we consider behavior in the absence of the requisite stimulus complex, the interlacing of verbal and perceptual responses, and the status of rules in human as opposed to machine, behavior.

What we are interested in at this point is amply covered by the permanent give-and-take inherent in the well-known processes of *generalization* and *discrimination*—both Pavlovian and operant, with both positively reinforcing and aversive stimuli—which routinely make up a sizable part of any textbook on the experimental analysis of behavior.

In a nutshell, generalization is the tendency of an organism—infrahuman or human—to respond to stimuli that lie along the same continuum as the objects, properties and events present when the response was first acquired. A behavioral event occurs but once and then passes into the history of the individual as a latent disposition. It is extremely doubtful whether we could survive very long if different bits of behavior had to be separately learned to meet all the variations and nuances of the environment.

Unchecked generalization, however, would backfire if it made no room for specificity. Discrimination provides the counterpart: we do want to move on a green light; failure to discriminate entails ineffective outcomes, or worse. In psychological parlance, effective responding in the presence of one stimulus and not in the presence of other stimuli along the same continuum is such as to cause the related response strength—usually treated in terms of probability of occurrence—to separate gradually.

The classic experiment [3] is actually quite simple, and the results could scarcely be clearer. The *X* axis is wavelength of light in perceptual terms (color); the *Y* axis shows percent of total responses.

Four different groups of pigeons were respectively conditioned to peck at a disc having the wavelength of one of the four arrows in the following figure. When the colored disc was on, the *S*s would be reinforced with corn on a time-interval schedule averaging one grain of corn per minute. When the disc color was off, they received no corn at all. The final rate of responding came to approximately 100 pecks per minute. Next, reinforcement (delivery of corn) was entirely discontinued and different colors were placed on the discs in order to observe the rate of pecking at these new wavelengths, taking points to the left and to the right of each of the arrows.

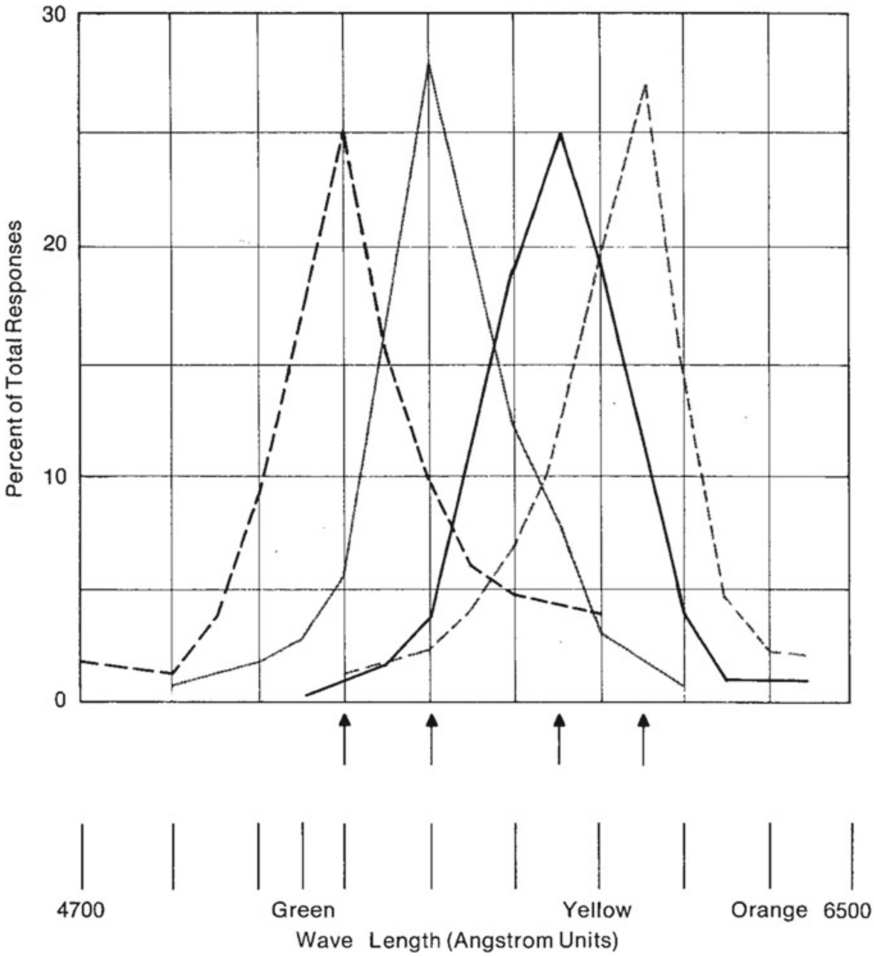


**Fig. 1.2** Pigeons' total response depending on the light's wavelength

As Fig. 1.2 shows, the animals continued to peck at the highest rate in the presence of the original wavelength; however, they also pecked at discs with wavelengths above and below the original wavelength, the new rates being proportional to the disparity between the new or test stimulus and the originally conditioned stimulus.

With the widening of the gap different bonds and dispositions develop. Enlarging on Notterman's [17, p. 119] discussion, this all seems so self-evident and sensible! What is not so obvious is that the available evidence, obtained from a large variety of species (ours included), behaviors, and sensory modalities, can be as orderly and consistent as it has been found to be.

Where generalization makes for behavioral stability (and labor saving), discrimination makes for the development of more finely defined subsets of behavioral fields of interaction between the *S* and the surrounding world. The resulting sets and subsets are permanently open to generalization or further discrimination in turn—as the environment may demand. This generalization-discrimination interplay provides the baseline for the development of more complex and subtle forms of abstraction and concept formation.

Ordinarily there is no need to specify what shade of green we are dealing with, even though degrees of greenness can be stated precisely to the micromillimeter of a wavelength. And there's the rub: the very idea that the range of the stimulus setting must be specified or precisely stated points directly to the interests and procedures of *E* qua *E*, who can be a psychologist performing a fine-grain psychophysical study or someone engaged in the design of a program in SC. Ordinary citizens have no use for this kind of detail: faced, for example, with a green ball, they merely respond according to its momentary relevance or possible use; the relative salience of its physical features and other current incidental parameters, along with their history of interaction (call it "experience") with green objects and colored balls, will do.

A priori speculation about degrees of greenness, lightness, size, etc. would be meaningless. The humble hyphen in "pattern re-cognition" just about sums up the story behind numerous historically deep-rooted dualisms (say, ontology versus epistemology), which even the more solid sectors of psychophysics still seem unable to do away with. This is not a matter of logic or mathematics but of behavioral variables and the physical properties of the responses generated. Mathematics proves useful when we know what to be mathematical about.

It would be surprising if the study of similarity, complexity, familiarity, contrast, distinguishability, and the like, could not profit much in the future from the data and principles gathered under rigorous experimental control, without having to drag in the usual plethora of theoretical constructs to "explain" the strength of the resulting stimulus and response bonds.

We have been concerned with the speaker so far. Although we have zeroed in on the joint presence of a referent state of affairs—a green light or a clock prompting the speaker to say "green" or "It's five o'clock" in the presence of a listener—a multiplicity of variables is always at work.

What is in it for the listener, though? We meet, of course, a different configuration. The specific verbal stimulus, say, "It's five o'clock," does become focal for the listener, and even more so when he/she does not have direct access to the clock. It puts the listener automatically in touch with ("in mind of") the state of affairs that prompted the speaker's response. Much perceptual behavior and "transfer of knowledge" takes place "out of context," wholly or in part. How is this to be understood?

Verbal stimulation, possibly even more than nonverbal, conjures up events in a life, the meaning of which is continually revealed and enriched as more connections make themselves felt or become recognized. As already noted, the effect and extent of this overlap between speaker and listener is at the basis of *understanding* and constitutes one of the greatest challenges in the study of perception and experimental psycholinguistics. Progress in this area can go a long way toward clarifying the structure of knowledge, the dynamics of thinking and the place of reasoning within it.

We get an entering wedge when we consider a special property that verbal and perceptual behavior have in common: where motor behavior requires a physical environment with specific characteristics for its execution—you need an actual closed door if you are to open it—perceptual and verbal activity can and often does occur "in the absence of the stimulus." We can speak about the past, the future, hypothetical cases, alternative ways of handling specific situations, and so forth. In such situations we are likely to find ourselves behaving both verbally and perceptually (seeing, hearing, smelling, and so on) to different extents dictated by the nature of the task itself and our discriminative history. Try *not* to see an elephant (or anything else) when instructed not to think of one! [15].

Responding in the absence of the stimulus (sometimes significantly referred to as "the mind's eye") enlarges the human individual's range of adaptive possibilities immeasurably and lies at the core of what makes human behavior the enormously complex phenomenon that it is. This is the world of problem solving—where imagination, intuition, contextualization, and such, are given free rein and must indeed be kept in check, lest they become misperception, idle speculation, or wishful thinking. Intentionality and rationality (or better, intentional states and rational behavior) must balance each other out in useful reconstructive memory, anticipatory behavior, choice and decision making, and numerous other cognitive tasks.

Different sensory modalities are simultaneously at work at all times, though we focus on some more than others according to circumstance. The plot thickens in the human case because, given the different physiological structures involved, perceptual and verbal responses can become interlaced with each other and with other responses of the individual.

The availability of "green" may help sharpen the perceptual response (hence also the subsequent action, as in choosing among different objects according to color) in a momentarily ambiguous situation. Similarly, hefting and balancing a ball allows for greater certainty before declaring it "light," "heavy," "hollow," and degrees thereof, and thus suitable for one purpose rather than another. The central role of internal cues when one or more external sources of stimulation fail—e.g., in groping our way through a familiar but suddenly darkened room, as opposed to doing so in a relatively or completely unfamiliar one—scarcely needs pointing out.

We take the wrong turn when we assume that the relative or absolute absence of one or more (internal or external) stimuli *from our vantage point as analysts* similarly implies their absence for the *S*, and then proceed to set up models based on such glaringly insufficient grounds. Whether *S* is aware of it or not, private stimulation is part and parcel of any adjustments to the incoming stimulation at any given time.

As adumbrated above, active self-responding becomes all-important, for example, in self-reference, self-regulation, and complex forms of delayed mediated responding, of which rule-related behavior is a good example.

### *1.3.1 An Interpolation on Rules*

The need for RUs is universally taken for granted; although exceptions can be cited, their status is rarely questioned. Different fields share different sets of assumptions, different manipulative response sets, and different schemes of inference. Membership in the guild presupposes the adoption of these assumptions, sets, and schemes, which soon become unchallenged givens for most practitioners.

As with LVs, confusion arises when we take RUs out of context and deal with them hypostatically as objects in themselves—strings of words, actually—forgetting that they are the product of someone's activity. RUs do not come out of the blue: they are linguistic constructs put together by someone under specific circumstances, generally so as to facilitate someone's response—often after a lapse of time—to the state of affairs that led to their formulation to begin with. Sometimes they include a specification of the behavior that they direct the listener or reader to perform. In a very real sense, they can be conceived as an extended version of LVs.

We circumvent the attendant hypostatical traps by zeroing in not on the RUs themselves but on RU-related behavior: generically, RU-formulation, on the one hand, and RU-following, on the other.

The individual engaged in a hands-on task—likely to be a mixture of perceptual, verbal and manipulative behaviors—provides a useful basic scenario:

Harking back to Fig. 1.1b, the referent for *A*'s behavior may not be sufficiently simple or unambiguous to automatically elicit "green" and A may have to figure out the adequacy of the requisite response before she can reliably state the prevailing relations. Having "put two and two together," A sums up a state of affairs for herself: "Oh, I see, if *X* is *A*, then *Y* is *B*," "If/When X, then *Y*," "If *X*, what I must do is …," and changes her ongoing course of action accordingly. She may also make a mental note: "I must remember that if/when *X*, *Y* happens"; or "If/When I do *X*, *Y* happens," "If I say '*x*' they are bound to understand '*y*'; therefore I'd better qualify my statement." Few people would deny that these are forms of thinking or problem solving involving a measure of self-awareness. By the same token, *A* may jot down these observations for future reference, her own, or someone else's. Writing bestows upon such memoranda the sort of durability helpful to any kind of delayed action.

By definition, future reference or "use" implies a different situation. These examples have to do with *A* acting as a speaker or writer. *A* may, of course, be also instructing *B* in situ, where both can respond perceptually and verbally to the relevant referent. But the green light, the clock, and so on may be out of the immediate range of *B*, as we saw earlier. We face a similar situation when we follow a piece of advice, a warning, an order, a set of instructions, conjectures, etc. It is because of this "sight unseen" nature of much RU-related behavior that rules were alluded to earlier under the auspices of private stimulation and the absence of the stimulus.

Although by no means to be equated with reasoning and thinking (as is often supposed), rules certainly play a major role in them and we can conveniently talk about RUs tout court so long as we bear in mind that:

(a) RU-related behavior is both referential and relational: in putting two and two together, the speaker responds to different aspects of the world, perhaps for the first time, and summarizes the relation in verbal form, say, in the standard IF/THEN frame. Crucially again, nouns, verbs, phrases, propositions, etc. do not enter the speaker's stimulus configuration when he fills in the frame, but they do become central as a variable for *B*. Alas, this is a fundamental fact all too often forgotten in the heat of battle, as we already saw in connection with LVs.
Assuming a measure of shared experience with the speaker, the listener who understands the IF/THEN frame can begin to respond to those aspects of the world referred to by the speaker, perhaps hesitantly at first. The uncertainty diminishes with effective action, and we can eventually claim knowledge (intuitive knowledge, as it is often called) of a new place, a procedure, and so on when the behavior becomes automatic and we can dispense with the RU and, in time, forget it altogether. We gratefully acknowledge that most knowledge is intuitive, or automatic, to begin with.

(b) Knowledge of a RU as such (or any verbal construct, for that matter) is a meager kind of knowledge and we further distort reality when we assume that knowledge of the RU automatically implies the ability of carrying out the behavior that it is supposed to direct. This is a widespread misperception, unabashedly strengthened by the computer metaphor, of which much is sometimes explicitly made (compare, e.g., [14, 22]) in line with mainstream artificial intelligence, cognitive science, and even systems thinking. Despite obvious differences, this applies equally to cognitive tasks, say, logical and mathematical procedures, and to activities directly related to the physical world. Consider the differences in ease of retention and execution of a set of instructions (e.g., following a recipe), depending on how familiar we are with the terms employed, on the one hand, and on the amount of culinary experience we bring to the task, on the other. Ease of retention and execution point again to the relevant, ever active give-and-take of the pertinent generalization/discrimination gradients.

Not all RU-related behaviors lend themselves equally well to an IF/THEN schematization. We need not stray too far to acknowledge, for example, that there is a fundamental difference between what we might call (1) "observational" RUs, which point to stimulation "out there" and (2) "manipulative" RUs, which call for specific nonverbal action following the initial perception. Unraveling the similarities and differences is a tall order indeed and we ignore the subject at our own peril whenever, taking the machine as a template, we expect the solutions we concoct for the latter to provide an account of human reasoning as well.

### *1.3.2 Perception Resumed*

Experience is fuzzy. The numerous factors that make it up—both behavioral and neurophysiological—are continuous and their possible combinations and quantifications open-ended. It stands to reason that in the absence of a disciplined appraisal, any attempt "to incorporate the 'experience' of a human controller" must remain metaphorical or so much guesswork. Indeed, "the main advantages of this approach seem to be the possibility of implementing 'rules of thumb' experience, intuition, heuristics, and the fact that it does not need a model of the process" [16].

More recently, Zadeh [25, p. 18] states: "Perceptions are intrinsically imprecise, reflecting the bounded ability of human sensory organs and ultimately the brain, to resolve detail and store information. Imprecision of perceptions is passed on to natural languages".

We can only agree that perceptions are intrinsically imprecise, but we must take exception to the following: "A natural language is basically a system for describing perceptions," for it can easily lead to a biased view of what is being represented, computed, and manipulated. Such a statement suggests a relapse to the old conception of language as a self-contained entity of some sort, detached from people and circumstance. It betrays the standpoint of *E* qua *E*, who needs to describe, or at least label, concrete aspects of the subject matter for his own purposes, generally with machine programming in mind.

As we have seen, the relation between perception and language is clearly far more complex than is suggested here. Their mutual interpenetration, as well as their joint interlacing with motor activity, puts them squarely at the center of virtually every branch of psychology, culminating in the extremely complex and subtle activities that become embodied in self-responding and self-regulation.

The recurring methodological problem is always the same and it should be easy to spot at this point: we can transcribe (from Latin *trans-scribere*) "green," "tall," "Mary," and so on but we cannot transcribe a color, a height, Mary, a clock, or any object or property, or the perceptual and motor responses they evoke—let alone the internal stimulation and feeling that accompany them. We "capture" them by "putting them into words," as we say, i.e., by labeling or by describing them: "When the light turns green, *S* presses the accelerator pedal," "If you feel cold, put on a sweater or ask someone to close the window or turn up the thermostat." We are left with strings of words open to all the vagaries of hypostasis.

We are operating on LVs and RUs, which are then tested and modified as many times as needed—a far cry from trying to account for the variables responsible for someone pressing a pedal, putting on a sweater, or turning up the thermostat. As a rule, machines are calibrated to detect external stimulation, to do so when we "tell" them, and to do it the way we want it done. The sensors are specifically designed to that end. The rules themselves are *orders/commands* to be executed if other well-specified variables and constraints concur. But what about the less stringent RU-like behaviors—from simple advice to shared conjectures—and all those complex situations calling for further reflection? Mathematical and technological improvements serve to make manipulative implementation as efficient as possible. The expression "manipulation of a perception" must be understood in this restricted sense.

There is no MP, as such, in the behavior of either *S* or of *E* when the latter is describing the former. When, analogizing, we speak of humans "rounding off" in the manner of CW, the computational connotations and implications are difficult to resist. This is quite in accordance with Mamdani's suggested incorporation of the "experience of a human controller" in fuzzy systems, assuming this to be feasible. We must remember, however, that this is based primarily on *E*'s analytic behavior. It is but one easy step for the postulated relations to be taken as actual processes in the *S*. In contradistinction to the machine, the *S*'s range of perceptual stimulation remains unaffected throughout, no matter how many changes *E* might make over $g$ or $w$. There is, after all, no intrinsic relation between them.

Staying with Zadeh's quotation, whatever "storage" and "information" may ultimately turn out to be in neurophysiological terms, resolution of detail—"how imprecision of perception is passed on to natural languages"—must appeal to the actual complex sets of variables combining in the behavior of *S*, without whom we would not be discussing language, perception, or any other activity of behaving organisms. The very notion of simulation would be a flight of fancy—never mind the accompanying speculation about sensory organs and the brain (see, e.g., [7, 18, 20]).

## 1.4 Envoi

We generally take words for granted, oblivious of the fact that language, like perception, is a form of behavior and that both must be treated as such despite their peculiarities vis-à-vis motor activity. No matter how you shuffle the cards, perceptual, motor, and verbal behaviors participate, to different extents and in different ways, in the human individual's interaction with the environment—reasoning being one major part of this interaction.

In view of the centrality of natural language in FL in general and in SC in particular, we have concentrated on LVs and RUs, partly for the light they shed on computing with words and manipulating perceptions. We have found that in their hypostatical, unrevised form, both LVs and RUs reflect (for lack of a better term) the activity of the *E* that came up with them. On closer inspection, LVs and RUs lack the semantic symmetry routinely assigned to them in the machine context, on the assumption that speaking and understanding are two sides of the same coin. In the nature of things, the speaker/writer supplies the LVs and the RUs; the listener acts upon them in a variety of ways. Speaker and listener cannot be treated as one and the same.

Such considerations clearly pose no problem in engineering, but we face a very different situation when these and other concepts, procedures, and problems are simply extrapolated to human behavior at large. Efficiency alone will not justify, for instance, the frequently encountered identification of knowledge with knowledge of RUs, their usefulness in the machine notwithstanding. There is in all this a sense of déjà vu: recall the tortured teleological speculations of cybernetics and the random theorizing in much systems thinking (including its recent revival of constructivism), not to mention the widespread physiologizing in standard AI research, psycholinguistics, and cognitive science. Failure to make the proper distinctions is tantamount to letting the available technical means determine our conception of the subject matter instead of the other way around.

FL and SC could follow in the same tradition. Paradigmatically, their strong roots in control theory seep through, for example, in Zadeh's [23] insightful appraisal of operant behavior, which he tellingly reduces to behavior *modification*; not surprisingly, Zadeh concludes that a system-theoretic formulation would facilitate *computer* analysis. Claims about their relevance to human behavior in general, and to reasoning in particular—what we might call their philosophical and epistemological implications—by and large continue to be supported through reference to spectacular achievements taken from engineering. Smithson's observation that "…claims are open to debate since fuzzy set applications in the human sciences are still in infancy. However, the concepts, evidence and arguments in themselves are creatively stimulating for both researchers and theorists" ([21], ibid.) remains as relevant today as when it was issued in 1987.

We can make machines smarter (some speak of raising their IQ!), and, in the process, we become smarter ourselves about machines but not about *S*, whose behavior the machine is supposed to simulate. To come to grips with *S*, *E* has to confront *S*, his/her relevant variables and the resulting behaviors. Whatever a machine does, it cannot even be said to go "through the same motions" as either *S* or *E*. That is partly why "one of the greatest drawbacks of modern robotic devices is that they lack … flexible coordinative ability" (see Kelso [12, p. 30ff] for a forceful discussion).

The fundamental fact remains: machines just do not have the right nerves going to the right places. In their case, we do not have to worry about all-too-human factors like frustration, fatigue, self-doubt, cognitive dissonance, absent stimulation, delayed action in the sense described, the perceptual/verbal/motor triad, not to speak of the automatic/non-automatic dichotomy that proves pivotal in human self-knowledge and self-regulation. Behind them all are those inescapable conative factors mentioned at the beginning of this paper, which have been deliberately ignored throughout this discussion in the interest of brevity of exposition. Metaphors have their place also in science but, as elsewhere, they can be extended only so far.

Another look seems needed. In their survey of possible fields of application, Klir and Yuan [13, p. 463] go as far as to say: "Psychology is not only a field in which it is reasonable to anticipate profound applications of fuzzy set theory, but also one that is very important to the development of fuzzy set theory itself." This is definitely not a task for seekers of immediate gratification, but the rewards beckoning are too promising to ignore. The present essay has been written in this spirit and, one hopes, not in greater detail or contrapuntal reiteration of the same underlying themes than necessary.

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# Chapter 2
# Language, Fuzzy Logic, Metalogic

**Josep-Maria Terricabras**

**Abstract** This article envisages two main goals: (1) It stresses the importance of language not just as an instrument but above all as an atmosphere, an ambience, where we grow up, we learn, we love and hate, we have pain and we pray, we think, we make mistakes and we finally die. That's why language, though it is not governed by arbitrary rules, is not either a rigid and inflexible instrument, since it has to express the situations, emotions and experiences of our daily life. (2) It points out that logic has always tried to establish the fundamental rules of language and reasoning. But logic has not been able, till very recent times, to recognize its own possibility conditions, which build up its sense and strength. There is a dogmatic demand of determinacy of sense which should be avoided. Everyday language very often expresses doubt or disagreement, hesitance, ambiguity or vagueness. Indeed, the ideal of exactness is over. Fuzzy logic makes a fundamental contribution to the understanding of this fact. Any sentence can have an exact or a vague meaning depending on its context and purpose. Context is given by ordinary life which imposes conditions to sense and understanding. Fuzzy logic takes into account the specific conditions governing exactness and vagueness in the use of words.

## 2.1 Language

I will start with something commonly known, namely that language is the most decisive element in shaping human thoughts and consequently human personality and culture. Through language we communicate and express our ideas and opinions, with language we build our inner world, our beliefs, feelings and hopes. Hence, each language is the privileged place for the most personal and intimate life, and at the same time a means of communication and social achievement, in favour of dialogue, coexistence and peace.

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_2

Our world is a real cluster of languages. Some people resent it and complain about this fact, probably because multiplicity of languages makes their lives a bit more difficult, especially when they go around as tourists. Fortunately there are also people and institutions which defend linguistic pluralism and try to preserve language diversity, even if some individuals or groups are much more concerned with the protection of their own languages than with the protection of all languages without discrimination.

So we can affirm both at once: that linguistic pluralism is a fact, a simple and undeniable fact; and, at the same time, that this fact is not experienced as a great opportunity for building strong and rich cultures, but is merely seen as (perhaps) an interesting cultural fruit of past geographical and political divisions. A strange sense for globalization leads many people to believe that a linguistic unity would be better than linguistic diversity. Those people think in terms of practical management of public life, not in terms of real appreciation of goods and values. Those who want to make some contribution to a better understanding among human beings should not be so naïve as to think that they will be better off if they speak finally the same language. Well-being depends on the capacity of expounding and increasing thoughts and experiences. This is what humans do when they exploit all the possibilities offered to them by the atmosphere of their native or acquired languages.

From what I have said I can draw the following conclusion: linguistic capacity is a literally basic, fundamental capacity, without which humans would not be humans anymore, since they would loose their own way of having thoughts and emotions, of expressing ethical judgments, of making projects, of imagining and of communicating with each other. Each language does all these things in its own way, with all the extraordinary possibilities and amazing means that history and tradition have slowly developed. This richness must be preserved as a fundamental heritage of the human species, which shows, in the dialogue of languages, how unity can be better preserved through an active respect and consideration for linguistic plurality.

In that sense, languages should not be diminished to the level of political weapons, but should be considered at their real human value. I am afraid that there is now a very general opinion according to which a language is just a means (an instrument) for human understanding. And certainly it is. But it is not just this. If language were just a means for understanding, this would justify the attempt to achieve the superiority of one language over the others. In that case, even if many languages were to disappear, the mere existence of one language known by all would be enough guarantee to preserve understanding in human relationships. The idea that language is just an instrument for communication is the idea that lies beneath the lack of respect for language diversity. If we only need to communicate with each other, than we do not need many languages but a single one known by everybody.

But the truth is that language is more than a ladder, more than an instrument, more than a means designed to pass messages from one person to the other. Language is not like a phone, which in case of being out of order, can be substituted by other means like a letter, an e-mail or a carrier pigeon. Language is not just a means but a medium, a milieu, not just an instrument but an atmosphere, an ambiance, where we grow up, we learn, we love and hate, we have pain and we pray, we think, we make mistakes and we finally die. Language embraces and produces our different forms of life. That is why languages cannot be simply substituted for each other. Each language shows a vision of the world—perhaps even a *Weltanschauung*,—each one produces different ways of interpreting and understanding not just what speakers say but also what they do, how they react, what they feel. Each language is a world, a real world. Of course there are connections between those worlds -that is why we translate texts and books-, but we should not forget that translation is an art, and this means that there are not automatic bridges which connect languages. (Translators know better than anyone else how difficult it is to translate poetry properly, for instance.)

Those who pay attention to these very basic facts will be able to understand two complementary things: (a) that languages are not totally arbitrary devices, since they serve not only to unit communities but also to interconnect different linguistic groups with each other; (b) that languages cannot be rigid but that they are rather very flexible structures able to give way to all sort of situations, emotions and realities. Point (a) explains why mankind has always been interested in logic; point (b) justifies the creation and existence of something called ≪fuzzy logic≫.

## 2.2 Logic

Nineteenth century logic was still a very much philosophico-hegelian science, a philosophical discipline which moved between metaphysics and psychology. A new logic, based on strict calculus (sometimes called ≪mathematical logic≫), appears firmly in the twentieth century. Nevertheless, if we look at the classical logic—the one going from Aristotelian logic to nineteenth century logic—and compare it with the new logic of the twentieth century, we easily realize that both, despite their technical differences, have been cultivating extensional logic, i.e. both have being concerned with the exactness of logical predicates they used. Some authors—as Frege and Russell—tried to find exactness through the creation of ideal languages. According to them, science feels uncomfortable with vagueness and needs absolute precision in the use of its terms. Since ordinary language is not useful for the purpose of exactness, it is necessary to build up ideal languages to ensure it.

Other authors, however, looked after exactness through the analysis of natural language. They considered that the surface of natural (everyday) language is vague, but that after closer analysis of our sentences we will eventually discover that exactness is embedded in their deep structure. This is, in fact, the idea proposed by Wittgenstein in his *Tractatus*. His main objective was to isolate all logic vagueness, ambiguity and inexactness. In fact, the idea of exactness has always been part of the bedrock of theoretical knowledge considered as higher knowledge. That is why sciences -including logic and mathematics- have until recently focused their attention on the investigation of fixed entities, permanent, unchanging and almost unchangeable realities.

Predicate logic has also been speaking of ≪classical (or fregean) predicates≫ to refer to predicates which only admit two truth values: true and false (0 and 1), with no degrees or shades of any kind. This means that, for any object whatsoever, it is always decidable whether the object falls or does not fall under the concept expressed by the predicate, as it is shown in examples like ≪being a Berlin resident≫ or ≪being twenty-one years old≫. This is clearly a binary logic, one of ≪yes≫ or ≪no≫; any third possibility is totally excluded from it.

During the twentieth century, mathematics, thanks in part to statistics and probability theory, has expanded its traditional concern and has finally applied to humanities and social sciences. Concern for accuracy has been losing ground in front of the idea of approximation. Mathematical results are no longer measured by their alleged certainty but by their degree of probability. It was also during the twentieth century that logics have been developed accepting more than two truth-values: that was the reason of the appearance of trivalent, tetravalent or, generally, multivalued logics. However, not even those logics can adequately cope with such a basic and general phenomenon as vagueness, which fills ordinary language, our expressions and our reasoning.

## 2.3 Fuzzy Logic

≪Vague≫ or ≪fuzzy≫ predicates do not admit a crisp classification of objects, as it is seen in examples like ≪being bald≫, ≪being rich≫, ≪being healthy≫. Frege hated these predicates because, as already said, he was seeking rigorous exactness without which, according to him, science would not work. In fact, a vague predicate opens an interval of infinite gradation (infinitely divisible between 1 and 0). Ordinary language reflects these gradations, these nuances, and logic has to cope with them if it wants to give real answers to the challenges of everyday life. Nevertheless, when logic does so, the expression ≪fuzzy logic≫ may lead to confusion, because it is not logic which is fuzzy but the use of language analyzed by logic. What fuzzy logic is precisely about to demonstrate is that fuzzy predicates can be treated with mathematical rigor if you quantify them correctly. ≪High≫ is a vague predicate, but people are 1.70, 1.80, 1.90 m. high, which are very precise heights. Fuzzy logic has to specify the degree of highness of those people since it evaluates each case within the range between 0 and 1.

Here I would like to mention again Wittgenstein's position, since he had a preeminent saying in analytic philosophy when fuzzy logic was officially born. Indeed, he showed an own view on vagueness both in the *Tractatus* (1921) and in the *Philosophical Investigations* (1953). These works are often seen as equally maintaining that vagueness is an essential feature of language. According to this interpretation, Wittgenstein's work became an inspiration for some attempts to construct or justify a logic of vagueness. But this was not exactly Wittgenstein's position, since he did not want to construct *a logic* of anything, neither of exactness nor of vagueness. We have seen that the *Tractatus* tried to supersede vagueness by a proper analysis of sentences. Later on, Wittgenstein's turn to everyday language did not mean the acceptance of ordinary language without more ado. Wittgenstein doesn't want to *promote* vagueness; he merely resists the dogmatic demand of determinacy of sense, that is, he tries to resist the insistence that the *possibility* of doubt or disagreement about the application of an expression must be eliminated. This is a crucial point in Wittgenstein's approach to vagueness and to philosophy. In fact, he tries to distort the ideal of exactness. On this line we can establish the following points:

(a) There is no single ideal of exactness. The contrast between exact and inexact is relative to a context and a purpose (e.g., whether we are measuring our distance to the sun or the length of a table). An inexact definition is not one which fails to meet the elusive ideal of determinacy, but one which fails to meet the requirements of understanding in a given context.
(b) No explanation could avert all possibility of indeterminacy, since no system of rules can budget for the countless bizarre possibilities in advance.[1]
(c) Even if vagueness is considered a defect, a proposition with a vague sense still has a sense, just as a vague boundary is still a boundary. If there is only one gap in an enclosure, it is determined that there is only one way out (a fly-bottle is a trap, although there is a way out). For a concept to be useful, all that is required is that it is defined for some cases, so that some things would definitely fall under it, and others definitely would not.
(d) One might respond in the spirit of the *Tractatus* that although the rules may allow a certain degree of elasticity, that degree must itself be determinate: there may be borderline cases, but it must be exactly determined what counts as such a case. However, this idea leads to a vicious regress. If we try to make the limits of an area more precise by drawing a line, that line has a breadth. If we try to avoid this by using the colour-edge of the line, the only way of determining what counts as overlapping this exact boundary is to draw another line, etc.[2]

As we can see, Wittgenstein is not working on a fuzzy logic but acts as an objective ally since he accepts limits to exactness and evaluates their importance. Exactness and inexactness (or vagueness) give sense to each other. Wittgenstein's example is important not just for his enormous influence in the history of thought but also because it shows that the future of fuzzy logic does not depend only on its technical improvements but also on the alliances and on the new ranges and applications it can win. These applications are very important when they belong to the domains of linguistics, psychology, literary criticism, ethics or political science.

## 2.4 Metalogic

Let's now ask the following question: when we talk about ≪fuzzy logic≫, are we entering into a new field, into some kind of second level order, from which we will be able to clarify logical problems and prepare new routes? To put it in a more already classical terminology: can we accept the existence of something

[1]Cf. Ludwig Wittgenstein: *Philosophical Investigations*, §§80, 84–87.
[2]Cf. id. *Philosophical Investigations*, §88; Zettel, §§441–442.

called ≪metalogic≫? Carnap's *Logical Syntax of Language* (1937)[3] ascribes the origin of the term ≪metalogic≫ to the Warsaw logicians. At present, this term is used to refer to second-order reflections about logic (e.g., to proofs of soundness and completeness). In agreement with the position of Wittgenstein on this matter, I would oppose to the necessity of accepting metalogic. We refer to our logical work using terms which can be considered ≪formal concepts≫, like ≪proposition≫, ≪name≫, ≪predicate≫, ≪function≫, etc. They are used to explain what we do and the terms we use. But this doesn't mean that they have an extra-ordinary status within our logical system. They are simply terms or concepts which help us to explain things, but they do not belong to the signs system. They are of a different order (since strictly speaking we don't use ≪propositions≫ but ≪p≫, ≪q≫, etc.), but they are not of a second-order as if they were of a higher order.

A reflection is a reflection, no matter which sequential order they take. There is no need to accept a second-order logic which allegedly would be more comprehensive and better articulated then the previous one. We can expect no gain derived from such an acceptance. Logic determines what is necessary, but there is no metalogic which makes logic necessary to control previous levels. In fact, all concepts which philosophy uses in describing ordinary language are themselves ordinary. The same happens with logical concepts, which are, all of them, at the same level.

> When I talk about language (words, sentences, etc.) I must speak the language of every day. Is this language somehow too coarse and material for what we want to say? Then *how is another one to be constructed?*—And how strange that we should be able to do anything at all with the one we have! In giving explanations I already have to use language full-blown (not some sort of preparatory, provisional one); this by itself shows that I can adduce only exterior facts about language.[4]

This does not exclude that fuzzy logic—or logic *tout court*—may offer many interesting aspects of discussion and reflection. In many cases logic uses terms and concepts which are also important in other areas of knowledge. This is the case, for instance of ≪ambiguity≫ (≪good life depends on a liver≫), that affects semantics, pragmatics and particularly rhetoric, or ≪vagueness≫ (≪John is bald≫), that affects also semantics, but it is less related to pragmatics and more linked to syntax and logic; for that reason it is an absolutely central concept for fuzzy logic.

There is still another term which deserves a major attention, but which in the shortness of this text it will only be treated—as it has been done with other terms—in a rather informal way. I am referring to ≪uncertainty≫ or ≪uncertain≫. Language is a field without walls or doors. Indeed, the borders between the uses of language are always tenuous and vague, but we can also say that they are uncertain. In fact, we often live in the uncertainty (which is not simply a doubt), for instance when we do not know if we make good use or misuse of some terms. This indicates that there are at least two major uses of the word ≪uncertain≫: our uses of terms may not only be vague but also uncertain, and we may of course feel uncertain about things. If I say ≪the epidemic outbreak caused by contaminated food is uncertain≫, I'm

[3]Translation from *Logische Syntax der Sprache*, 1934.

[4]Ludwig Wittgenstein: *Philosophical Investigations*, §120.

saying it fails to show clear profiles. This would be ≪objective uncertainty≫. But I can also say that I am uncertain about some situation, for example when I say ≪my involvement in the business is uncertain≫, which means that it is not safe, that I did not yet take a final decision on it. This might be called ≪subjective uncertainty≫.

And still we can make a third use of the term ≪uncertain≫ which I will call ≪amphibological≫, which consists in having both at once (that a fact is uncertain and that I feel uncertain about a fact), that is, when the objective and subjective uncertainty match. This happens, for example, when some experts involved in a political debate say that ≪the impact of the broadcasted debate is very uncertain≫. They mean the impact itself is staggering, unequal, that it depends on small, unpredictable, almost imperceptible details, and yet they also want to say that they are unsure about what they do, about what they should do to positively influence the electorate. (This amphibological character is also a special case of ambiguity: indeed, an expression is amphibological when it has different meanings at once, that is when it cannot simply have them and can play with them. Typically, the ambiguity is unique and demands an alternative interpretation, as seen in the example given above. The term ≪uncertain≫, however, admits an amphibological use, i.e. it allows a simultaneous interpretation of its ambiguity, but doesn't impose it.)

The subjective component of ≪uncertain≫ has surely provoked that it had little interest in fuzzy logic. But the very fact that uncertainty has sometimes a psychological component of subjective nature confirms the interest shown in its analysis by scholars of the humanities and social sciences. Indeed, it seems to me uncontroversial that in the future it will be crucial to give to psychological uncertainty the importance it has been given so far to vagueness in physics and biology.

I'm not saying that logic returns to psychology. I'm only proposing that we try to model and analyze our psychological capacities without having to accept the behaviorist model, which relies on the analysis of behavior without sufficiently examining the linguistic terms of its analysis. Terms like ≪certain≫ and ≪uncertain≫ (which don't imply respectively ≪knowledge≫ or ≪ignorance≫) are crucial to that purpose.

This brings me to say that, in this twenty-first century, logic researchers are able to make some connected statements that surely they could not have made some years ago. I will organize those statements around two important points:

1. *Language, considered in abstract terms, is neither exact nor vague.* Any sentence can have an exact or a vague meaning depending on its context and purpose. Vagueness is not an exception in language but a common possibility. I do not mean that there is an essence of vagueness in language but that any term can be used or interpreted in a vague sense, not previously foreseen by yourself or others. This doesn't bring us to paralysis. Vagueness works because we constantly try to narrow it for understanding purposes. Whenever we fail or are wrong, we have self-correcting mechanisms. That's why logic and analysis of language must be increasingly flexible and subtle.
2. *Exactness is the limit of vagueness.* Vagueness, on its turn, has also its limits. That is why we often accept many expressions as adequately accurate even if we

know that they could be interpreted vaguely. This means that in our everyday life we pragmatically accept the limits of the interval between 0 and 1.
It is, therefore, important to reverse the classical assumption according to which vagueness is merely the limit of exactness. This is an important change of perspective which allows me to point out to some consequences which are relevant for contemporary culture and which go far beyond the field of mathematics and logic. I will briefly mention only six of those consequences:

(a) The theoretical shift towards understanding inexactness means a radical change of perspective in the way humans look at reality. Therefore, we are not facing a new theory, a new hypothesis or a new work project. This is rather a Copernican turnaround that has already begun to have huge consequences for industry and technology, but also for domestic life, education, medicine and the rest of our lives.
(b) The shift towards understanding vagueness and inexactness redefines the epistemological optimism of many modern scientists and thinkers. They thought they might arrive to exact knowledge of everything, being able to capture everything with precision and accuracy. Sometimes they even assumed that what they could not understand with total precision had no interest or was worthless. Knowledge has now become cautious. In the past exactness had been linked with knowledge of reality. Inexactness and vagueness are now only linked with our way of speaking about reality, without any ontological claim, without any attempt to know if reality is accurate or inaccurate, simply because we just talk about it. This looks like a more humble position but, paradoxically, it is more ≪realistic≫.
(c) The shift towards understanding inexactness also redefines the concept of description. What does it mean now the attempt ≪to describe reality≫? Over a hundred years ago nonfigurative art started this debate and questioned the notion of plain description. Indeed, facts accept multiple descriptions. It is precisely the kind of descriptions people use what shows us whether they are describing the same thing in a different way or whether they are talking about completely different things.
(d) The binomial true/false has lost its static and abstract character. Strictly speaking, we should only refer to truth or falsity in terms of degree or range. To assign a degree allows us to be both flexible and rigorous, because in assigning a degree of truth we are also assigning a degree of falsity or of ignorance. This is not a defense of relativism—which only makes sense when opposed to absolutism-, but it highlights the relational nature of truth, since nothing is true or false in itself or in an abstract way, but always with respect to some criterion and within some system or framework. A new problem arises when we ask whether we can compare different frames of reference with each other. This is a capital issue in social sciences, when they are about to make, for example, intercultural statements.
(e) No ideal of exactness can be laid down. Such an ideal have only been supported by the belief that what is not predictable is irrelevant. But an

inexact predicate is not useless; on the contrary we can obtain from it a lot of very relevant information. Consider, for example, weather forecasts which, though often very inaccurate, provide valuable information. After all, we establish, not always in a very accurate way, our ideals of exactness depending on time, subject, people and age. We should always ask: ≪exactness what for?≫ The lack of exactness cannot be used for blame. The lack of rigor should certainly be, because you can be very rigorous in your treatment of inexactness.

In the field of ethics this is very important. Let's see what happens in a hospital: a doctor has no longer to ask what can be done to save a life. Now doctors have to ask what should be done, because they have many resources available. Now you must choose and have preferences and priorities. Usually this has not been the case before.

(f) The shift towards understanding inexactness makes explicit the approach of science to society. Fortunately science has abandoned its selfish dogmatism. Now we are confronted with enormous possibilities which remain open to analysis and treatment in large areas of knowledge. This happens in a far more vivid form in those areas which, due to its elasticity and flexibility, had been virtually marginalized by a too narrow logico-mathematical perspective.

Indeed, nothing is negligible in principle. In a parody of Pascal we might say that, if the heart has reasons that our head does not understand, then our head has to do a greater effort. Or you can say with Goethe: ≪For those who think they know everything, almost everything is ridiculous. For those who are reasonable, almost nothing is.≫ Leibniz expresses the same thought schematically, i.e., in a sentence which is clear and vague at the same time: ≪Je ne méprise presque rien.≫

# Chapter 3
# On What I Still Hope from Fuzzy Logic

**Enric Trillas**

**Abstract** This paper just tries to look towards a particular aspect of fuzzy logic's future. Namely, that concerning to the form in which its theoretic evolution should be followed up to approach, in a scientific manner, one of the greats desiderata in the way to Zadeh's Computing with Words: the renewal of the old thought of making interact Fuzzy Logic and Natural Language Common Sense Reasoning. For such a way, and from the perspective of its author, it is argued that fuzzy logic deserves to move towards a new experimental science of linguistic imprecision and non-random uncertainty. A goal for which a critical review of some still open questions should be done, but not only taking as models those of logic and mathematics. The author thinks that a better model is that of physics, joining observation, controlled experimentation and mathematical models in the ground of language and ordinary reasoning, with the extensive use of computational technology and its applications to practical problems.

## 3.1 Introduction

I met with fuzzy logic in the summer of 1974, a few months after I got my first position as a university professor, and thanks to a news in a French Newspaper. The news referred to the first of the several books professor Arnold Kauffman published on fuzzy sets, and it called my attention to the point that I bought the book, 'Théorie des sous-ensembles flous' [1]. After trying to read it, and although if the book basically deceived me, the idea of a fuzzy set left me worried since I linked it with an old writing in French by Menger [2] on what, in English, he called Hazy Sets to translate the French name 'ensembles flous' he coined to use in his paper, and that I knew by being then working in Menger's Statistical Metrics. Thus, I decided to have a look on the paper that introduced the idea, the 1965s 'Fuzzy Sets' by Zadeh [3].

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_3

Before reading Zadeh's paper I was unable to imagine a fuzzy set as something different of a probabilistic-driven idea. In fact, Menger's hazy sets were defined as something similar to the function given by the probability that an element in a ground set belongs to a given subset of it. Zadeh's first paper greatly surprised me since I immediately realized that it referred to a wider concept, essentially linked with the use of predicates in natural language. The problem appeared to me related with the meaning of words, and not with random experiments. If Menger arrived at hazy sets by some worries regarding quantum physics, Zadeh arrived at fuzzy sets by worries on problems coming from a different, and actually distant, field, that of Cybernetics. Since in that time, and by the intellectual interests of a colleague of mine, an engineer working on Cybernetics, I was a little bit concerned with the idea of the 'Analogical Computer', I decided to try to work on the theory of fuzzy sets that seemed to me closer to the analogical than to the digital. Namely, I began with some work on the measures of fuzziness introduced in 1972 by Luca and Termini [4]; the concept of fuzziness as a measurable quantity seemed me both new and interesting. I saw to 'fuzziness' as a scientific, although restricted, alternative for the philosophical term 'vagueness' that seemed me too vague and on which I read the famous Russell's paper [5]; namely, as something close to non-boolean and that, contrarily to vague, can be measured once the linguistic terms are represented by fuzzy sets.

The idea of mathematically modeling the linguistic use of an imprecise predicate, was completely new for me and I noticed it as something mathematicians and mathematical logicians refused to consider from, at least, the work of Frege. It was my personal boringness towards the strong 'bourbakism' then driving both the teaching and the research in mathematics in Spain, jointly with my own concerns on the 'set extremism' in the European pedagogical movement of the so-called 'Modern Mathematics', what impelled me towards fuzzy sets. I not only felt an interesting challenge in the new concept, that of introducing mathematical models in language, but, and perhaps in a too naïve form, I linked this study with one of my heroes, George Boole, of whose two books on logic [6, 7].

I was an enthusiastic reader during my graduate student years. My thought quickly moved from the old Boole's mathematical analysis of logic, towards a larger and new mathematical analysis of language. My scientific horizon was enriched with a new, and unexpected, point of view, and in the next years I step-by-step changed my research on Generalized Metric Spaces to that in Fuzzy Logic. I must confess that I was then just thinking in the Leibniz's 'practical dictum' *Calculemus!*; a dictum that required the previous establishment of some kind of 'calculus' for language of which Boole's is a part, since it allows to represent some 'large' crisp statements by formulas that, managed through it, help to better understand their meaning by either enlarging or shortening them.

From my old field of research on the Menger [8], Wald [9], Schweizer and Sklar's [10], first Statistical and later on Probabilistic Metric Spaces, to which I did look at coming from Fréchet's abstract distances ('écarts'), and Blumenthal's Boolean metrics, I preserved two concepts and a methodology that later on resulted of some fertility in fuzzy logic. The concepts were those of t-norm/t-conorm, and Indistinguishability Relations, and the methodology of trying to pose some problems by means of functional equations. A lot of clarifications came from them between 1979 and 2010, the first of which was the result concerning the structure of strong negation functions [11], and the last one the analysis of some classical schemes of inference in fuzzy logic [12], in short, the analysis of the Aristotelian Forms in fuzzy logic. It also appeared the fact, unknown in classical logic, that linguistic connectives cannot have universal representations, as well as that the Aristotle's principles of Non-contradiction and Excluded-Middle can be posed in ways different from the usual one [13] (either with the metaphysical predicates 'false' and 'true' or with classical sets and its complement), and with which both principles universally hold with fuzzy sets.

This paper just tries to briefly reflect, far of any hard mathematical formalism, on what I yet expect from the evolution of fuzzy logic towards the Zadeh's 'Computing with Words and Perceptions' [14]. Analogously to what Satosi Watanabe said [15], perhaps, this paper contents could be inscribed in a new and wider field called *Epistemometrics*.

## 3.2 Predicates, Collectives, and Their States

The concept of 'meaning' is something actually enigmatic; everybody understands what can be captured by 'the meaning of this word', but no definition of the meaning concept is actually and universally accepted. The problem of meaning, basic for knowledge, and on which there is a big number of philosophical writings, well deserves to be posed in mathematical terms. What follows is nothing else than a trial towards it, by departing from the Wittgenstein's statement [16], 'The meaning of a word is its use in the language', that seems to be implicitly accepted in fuzzy logic where the meaning of a statement clearly depends on context and is driven by the purpose of its use.

### 3.2.1

Basic to arrive at the concept of a fuzzy set is the analysis of the use, or behavior, of a predicate $P$ in a universe of discourse $X$, that is, the study of the meaning of $P$ in $X$ through the elemental statements '$x$ is $P$', for all $x$ in $X$. In fuzzy logic $X$ is always taken as a classical set.

A predicate $P$ alone has no meaning; its meaning only appears when $P$, naming a property $p$ that can be shown by the $X$'s elements, is actually predicated of them. The pair $(X, p)$ is the starting point for the concept of the fuzzy set labeled $P$. This is, in my view, the deep sense of the Wittgenstein's statement. Without a context and a purpose for its use, predicates are just something metaphysical, but not quantifiable, that is, whose study can be conducted by means of a numerical quantity. Numerical quantities are essential not only for all kind of scientific studies, but also for the technological applications [17].

A first observation related to what concerns the use of predicates is that they and their names can change over time with the change of the universe of discourse. For instance, the first meaning of $P$ = high, did possibly appear with $X_1$ = Forests, then changed to $X_2$ = Lions and Giraffes, to $X_3$ = Mountain chains, to $X_4$ = Buildings, and perhaps finally passed to $X_5$ = People, with the new name $P^*$ = tall. In these change-processes the cultural advancement produced by the use of numerical scales of measurement, should have an important role that is not independent of the apparition of modern science, one of whose characteristics is not only measurement but the interpretation of qualitative concepts by means of quantities. In a good part, modern science is a movement towards the quantitative analysis of concepts of which technology can benefit; for instance, it is difficult that the control of an inverted pendulum can be done by a fuzzy controller without previously representing predicates like 'a little bit ahead' by a numerical function reflecting its meaning by a fuzzy set whose values are in the unit interval.

A second observation relates with the empirical fact that the use of a predicate $P$ in a ground set $X$, generates a 'collective', let's call it $\boldsymbol{P}$, among the $X$'s elements. For instance, $P^*$ = tall, in $X$ = Inhabitants of New York City, immediately generates the idea given by the 'collective of the tall inhabitants of NYC'. The collective $\boldsymbol{P}$, is a classical set whenever the use of $P$ classifies the elements in $X$ in just one or two classes, that of the elements $x$ that show the property $p$ without any doubt, and the class of those that without any doubt do not show it, and that in some cases can be reduced to one of them by the emptiness of the other. For instance, if $P$ = greater than or equal to 10 in $X = [0, 100]$, then it is $\boldsymbol{P} = [10, 100]$, and $[0, 10)$ is the set of the elements in $X$ that are not at all greater than or equal to 10. Notice that $\boldsymbol{P}$, is a classical set whenever the use of $P$ in $X$ $P$ is known by means of an 'if and only if' definition. The use or meaning of most of the predicates appearing in Natural Language is not 'definable' in this way, but is only 'describable' by means of some empirical rules of use. For instance, once accepted that there is a threshold $\boldsymbol{t}$ = approximately $t_0$ centimeters (or whatsoever unity of length perceptively sensible to the naked eye), the qualitative use of 'tall' can be described by some empirically perceived and imprecise rules like,

1. If a person $x$ seems to be below $\boldsymbol{t}$, it can be asserted that $x$ is not at all tall.
2. If a person $y$ seems to be above $\boldsymbol{t}$, it can be asserted that $x$ is tall.
3. If $x$ allows to assert '$x$ is tall', and $y$ seems to be further of $\boldsymbol{t}$ than $x$, then it can be also asserted '$y$ is tall',

three rules that do not allow to say that the collective $P$ generates is a classical set, as it can be easily shown by means of the so-called 'Sorites method' [18].

Once the hypothesis that the use of $P$ in $X$ generates a collective $\boldsymbol{P}$ is, even in a provisory form, accepted, there is the problem of describing and mathematically representing $\boldsymbol{P}$. Such hypothesis is well supported by language; people usually talk of 'big houses', 'tall people', 'small apartments', 'small birds', 'impressing trees', 'many hundred euros', etc.

### 3.2.2

To show a more or less natural way by which the idea of a collective $\boldsymbol{P}$ could be approached when $P$ is acting on a universe $X$, let us begin by considering the case in which $\boldsymbol{P}$ is a classical set, that is, in which the elements in $X$ either completely show $p$, or do not show it at all. Thus, it can be said that all the elements in each one of the mentioned two classes show $P$ equally, but that of two elements whatsoever and each one in a different class, one shows $P$ strictly less than the other. That is, one of the two classes just consists of minimals for what corresponds the showing of $p$, and the other consists of maximals.

Changing to say '$x$ is less $P$ than $y$' if and only if $x$ shows $p$ less or equal than $y$, the set $X$ is endowed with the binary, and empirically generated relation,

$$x \leq_P y \Leftrightarrow x \text{ is less } P \text{ than } y, \tag{3.1}$$

with which a graph $(X, \leq_P)$ is added to $X$. Since the relation defined by, $x =_P y \Leftrightarrow x \leq_P y \,\&\, y \leq_P x$, is an equivalence if and only if $\leq_P$ enjoys the reflexive and transitive properties, that is, if $\leq_P$ is a preorder, the quotient-set $X/ =_P$ consists of the two before mentioned classes among which it can be defined the relation $[x] \leq_P^* [y] \Leftrightarrow x \leq_P y$, that is a partial order. Then one of the classes contains the minimals for $\leq_P$, and the other the maximals, something that can be asserted without using the terms 'false' and 'true', and permits to describe meaning in a Truth-free way.

It is thanks to this chain of denotations that it is possible to define a unique numerical function $t : X \to \{0, 1\}$, such that all $x$ in the class of the minimals has the value $t(x) = 0$, besides for all $y$ in the class of maximals is $t(y) = 1$. Hence, the two classes are, respectively, the sets $t^{-1}(0)$ and $t^{-1}(1)$, and the first corresponds to the statements '$x$ is $P$' of which it is commonly said that are 'false', and the second to those that are said to be 'true'. In this form, the use of $P$ in $X$, that is, its meaning, the collective $\boldsymbol{P}$, is represented by the unique triplet $(X, \leq_P, t)$ with the mapping $t$ satisfying the properties,

(1) If $x \leq_P y$, then $t(x) \leq t(y)$
(2) If $x$ is minimal relatively to $\leq_P$, then $t(x) = 0$
(3) if $x$ is maximal, then $t(y) = 1$.

Such a unique triplet is nothing else than a form of expressing the so-called axiom of specification of a precise predicate $P$, under which $P$ specifies a unique (classical) subset $\boldsymbol{P} = t^{-1}(1)$ of $X$, characterized by $x \in \boldsymbol{P} \Leftrightarrow$ '$x$ is $P$' is true $\Leftrightarrow t(x) = 1$.

What happens when $\boldsymbol{P}$ is not a classical set? That is, when $P$ does not classify $X$ in just one or two classes modulo $=_P$?, or, what happens when the statements 'x is P' are not only claimed to be either 'false' or 'true'? In short, what happens when the use of the predicate $P$ is not precise, rigid, but imprecise, flexible, in $X$? Last comments show a somehow natural way to afford this question [19].

### 3.2.3

If the use of $P$ in $X$ is not precise, but imprecise, there are the following three possibilities,

(a) A relation $\leq_P$ can be clearly established and accepted by a big enough group of people. That is, the use of $P$ allows recognizing where it is, or it is not, '$x$ is less $P$ than $y$' or, equivalently '$y$ is more $P$ than $x$': $\leq_P$ is not empty. It is said that $P$ is **potentially measurable** in $X$, and that the graph $(X, \leq_P)$ represents the **primary meaning or use** of $P$ in $X$.

(b) No relation $\leq_P$ can be either established, or imagined. It is said that the use of $P$ is **metaphysical** in $X$.

(c) As an intermediate case between (a) and (b), it can be $\leq_P = \emptyset$. It is said that, in principle, $P$ is **meaningless** in $X$.

Instances of these cases are: (a) $P =$ Big, in $X = [0, 10]$; (b) $P =$ Eternal, in $X =$ The Heavens (notice that this $X$ is not known to be a set), and (c) $P =$ Green, in $X =$ The set of London's inhabitants. Of course, and without neglecting the interest some metaphysical predicates could have, and supposed a single one actually exists, the predicates that mostly interest to science and technology are those that are potentially measurable. How they can be **actually measured**?

In principle, the range of applicability of predicates $P$ is any kind of universes of discourse $X$, either if they are previously embedded with some structure, or not, but if potentially measurable, they 'organize' or 'structure' $X$ by a graph $(X, \leq_P)$. This corresponds to the more general and intuitive idea that, with a 'rational discourse' on something, it is tried to 'order', or 'structure', what is considered; that rational talking tries to semantically order the objects that are considered.

Like in happens in the case of length, surface, volume, current intensity, random events, etc., it is difficult to imagine how a coherent numerical evaluation, or measure of the elements in play, can be introduced without previously knowing how these elements grow, their conceptual dynamism. In the case of predicates, the arrows in $(X, \leq_P)$, that is, the idea that '$y$ follows $x$' whenever $x \leq_P y$, indicates an elemental, but basic, form of growing. Like it also happens with random events once they are represented by subsets $A, B, \ldots$ in a Boolean algebra $\Omega$, and it is accepted that 'The event represented by $A$ in $\Omega$ is less-random than that represented by $B' \Leftrightarrow A \subseteq B$, or also with the length of sticks after accepting that a stick is shorter than another one when a movement can superpose the first into a part of the second.

Once a primary meaning $(X, \leq_P)$ is known, we will say that

$$t_P : X \to [0, 1] \tag{3.2}$$

is a measure of the extent up to which each $x$ in $X$ is $P$ or, for short a *measure of* $P$, provided [19],

1. $x \leq_P y \Rightarrow t_P(x) \leq t_P(y)$ in the total order $\leq$ of the unit interval,
2. If $x$ is a maximal for $\leq_P$, then $t_P(x) = 1$, and
3. If $y$ is a minimal for $\leq_P$, then $t_P(y) = 0$.

Notice that it is not said that If $z$ is neither a maximal, nor a minimal, it should necessarily be $t_P(z) \in (0, 1)$, and that it is also not said that the function $t$ should be strictly non-decreasing. In general, neither $t^{-1}(0)$ is just the set of minimals, nor is $t^{-1}(1)$ that of maximals.

Then, each quantity $(X, \leq_P, t_P)$ can be considered to represent a **meaning** of $P$ in $X$ and, obviously, properties 1, 2, and 3, do not allow the specification of a single measure $t_P$. To specify one of them, to measure the extent up to which the $x$'s are $P$, more information is needed on the particular use of $P$ than that only furnished by $\leq_P$; more information on, for instance, the 'shape' of $t_P$. For each primary meaning, several meanings are possible.

**Example** *If $P = Big$ in $X = [0, 10]$, it is easily acceptable that $\leq_{Big}$ is coincidental with the total order $\leq$ of real numbers, and then there is a single maximal, 10, and a single minimal, 0. Hence, the corresponding measures $t_{Big} = t$, are those functions $t : [0, 10] \to [0, 1]$, such that:*

1. $x \leq y \Rightarrow t(x) \leq t(y)$*: t is non-decreasing*
2. $t(10) = 1$
3. $t(0) = 0$,

*of which there are uncountable many. If, for instance, it is known that the growing of t should be linear, the only measure that can be specified is $t(x) = x/10$, but if it should be quadratic, there are all of those $t(x) = ax^2 + (0.1 - 10a)x$, that are non-decreasing, with values in the unit interval and depend on the parameter a.*

*Provided it were additionally known that $t(1) = 0.2$, it will be $a = -0.01$, and a single quadratic t will be specified. Provided it were known that t is not necessarily continuous, another possible function t can be*

$$t(x) = 0, \text{ if } 0 \leq x < 7, \text{ and } t(x) = 1, \text{ if } 7 \leq x \leq 10, \tag{3.3}$$

*that is, the classical set $[7, 10]$. The specification of each t depends on the total available information on the behavior of P in X, and thus not only a single meaning can be attributed to Big. With more information on it, less number of measures t can be actually defined.*

*In the case with $t(x) = 0$, if $x \in [0, 3]$; $t(x) = 1$, if $x \in [7, 10]$, and $t(x) = (x - 3)/4$, if $x \in (3, 7)$, the set $t^{-1}(0) = [0, 3]$ contains the minimum, and the set $t^{-1}(1) = [7, 10]$ the maximum.*

It is a similar problem than that happening with probabilities. For instance, in the experiment of throwing a dice, no probability for the random event 'getting a 5' can be established without some information on the dice and the surface on which it will fall in. Without it, and supposing a perfect landing surface, the only answer is one of the many probabilities verifying

$$\begin{gathered} Prob(\text{'5'}) = a_5 = 1 - (a_1 + a_2 + a_3 + a_4 + a_6), \\ \text{with} a_i = Prob(\text{'}i\text{'}) \in [0, 1], \text{ and} \sum ai = 1. \end{gathered} \tag{3.4}$$

With the information 'the dice is not loaded at all', or $a_i = 1/6$ $(i = 1, 2, 3, 4, 6)$, it is $Prob(\text{'5'}) = 1/6$, but with the information 'the dice is loaded in such a way that $a_6 = 0.6$', many solutions $Prob(\text{'5'}) = 0.4 - (a1 + a2 + a3 + a4)$ are yet possible.

## *Notes*

(1) Most of Science would be almost nothing without numerical magnitudes, and this often requires representing the concepts by quantities like the use of $P$ is represented by the quantities $(X, \leq_P, t_P)$. Notwithstanding, there are cases in which it is very difficult to reach a complete knowledge of the relation $\leq_P$ as it is, for instance, in those where $X$ is with many elements. In such cases it could happen that only some increasing parts of $\leq_P$ can be sequentially known: $\leq_P^1 \subseteq \leq_P^2 \subseteq \ldots \leq_P$.

(2) Each measure $t_P$ is just the 'idea' of the fuzzy set with linguistic label $P$ and membership function $\mu_P = t_P$. But it is important to notice that in the practice of fuzzy logic, the membership functions $\mu_P$ are often (and, sometimes, reasonably) designed with not too much information on the behavior of $P$ and, hence, they should be taken as some kind of 'approximations' to the measures. For this reason, the designers of the membership functions should be aware that as more information on the behavior of $P$ in $X$ they can collect and use to design $\mu_P$, better will approximate this membership function to an ideal measure of $P$ in $X$. The design of all the fuzzy terms involved in a problem must be always done very carefully [20]; design's processes should be rational processes of thinking.

(3) It should be noticed that $\leq_P$ cannot always enjoy a total character like it shows the ordering of the unit interval; that is, there are often pairs $x$, $y$ such that it is neither $x \leq_P y$, nor $y \leq_P x$. A good example is supplied by $P$ = Around 4, in $X = [0, 10]$, where it can be defined by : $x \leq_P y \Leftrightarrow 0 \leq x \leq y \leq 4$, or $4 \leq y \leq 10$, a relation for which, for instance, it is neither $3 \leq_P 6$, nor $6 \leq_P 3$.

(4) The case where **P** is a classical subset is well subsumed in the precedent model since it simply corresponds to the particular case in which it is $t_P(X) \subseteq \{0, 1\} \subseteq [0, 1]$. In this 'crisp' case it could happen that one of the two classes $t^{-1}(0)$ or $t^{-1}(1)$ is empty, and that respectively corresponds to either $t_P = \mu_0$, or $t_P = \mu_1$, the characteristic functions of the empty set Ø and the full set $X$, respectively.
With the definition $\mu_r(x) = r$ for all $x$ in $X$, and $r$ in [0, 1], the constant fuzzy sets, it is also possible to have cases in which the classes given (whenever $\leq_P$ is a preorder) by the equivalence $=_P = \leq_P \cap \leq_P^{-1}$, representing 'equal $P$ as', are in a finite number, and in each one of them $t_P$ is nothing else than the restriction of some $\mu_r$; then, the corresponding predicate can be called quasi-imprecise.

(5) In general, the quotient set $X/=_P$, consisting in the classes $[y] = \{x \in X; x =_P y\}$, has at most the same number of classes than different values has $t_P$ since it is,

$$x \in [y] \Leftrightarrow x =_P y \Leftrightarrow x \leq_P y \;\&\; y \leq_P x \Rightarrow t_P(x) = t_P(y), \tag{3.5}$$

and the measure $t_P$ is constant in each class. Thus, it can be defined the mapping $t_P^* : X/=_P \to [0, 1]$, $t_P^*([x]) = t_P(x)$, and the meaning of $P$ in $X$ could be also represented by the quantity $(X/=_P, \leq_P^*, t_P^*)$, with $[x] \leq_P^* [y] \Leftrightarrow x \leq_P y$.

When: $t_P^*([x]) = t_P^*([y]) \Leftrightarrow [x] = [y] \Leftrightarrow x =_P y \Leftrightarrow t_P(x) = t_P(y)$, or, equivalently,

$$t_P(x) \leq t_P(y) \Leftrightarrow x \leq_P y, \tag{3.6}$$

it can be said that $t_P$ **perfectly reflects** the primary meaning of $P$ in $X$. Since the order $\leq$ is a total one, it can only happen when the relation $\leq_P$ is also a total one. Nevertheless and since this relation is not always a preorder, the character of perfectly reflecting the meaning can be, in general, simply defined by (3.6) that does not refer to $\leq_P^*$.

In the example of 'Big' in [0, 10] it is clear that all continuous measures $t_{Big}$ will perfectly reflect the meaning of the predicate, but in the case of the specification made by the interval [7, 10] in which the measure is discontinuous at the point $x = 7$, the representation is not perfect. It is only perfect provided 'Big' is reduced to 'Bigger or equal than 7'. Different is the case of 'Around four' where $\leq_P$ is not total, and no measure will allow to perfectly reflecting its primary meaning.

### 3.2.4

Once a primary meaning of $P$ in $X$ is fixed, the collective $\boldsymbol{P}$ is manifested through their **information-states** $t_P$, and is coincidental with just one of them only if $P$ is precise, has a rigid behavior in $X$ manifested by a single use given by means of a 'if and only if' definition concerning the elemental statements 'x is P'. But if the predicate is imprecise, its behavior in $X$ is flexible and such flexibility is manifested by the several states in which it can appear, or being used, or of being described what is understood by 'x is P'. Each piece of information, either on $\leq_P$, or on the characteristics of $t_P$, will be like a point of light allowing to throw a mathematical 'shadow' of $\boldsymbol{P}$; like the different shadows in the Earth's surface a cloud throws depending on the solar light, and a shadow always allows to suspect that there is something 'material' producing it. In principle, collectives can only be considered by looking at their mathematical shadows; the existence of collectives is a hypothesis that, nevertheless, is well grounded in the language.

Consequently, the collective $\boldsymbol{P}$ can be considered as an 'idea' generated by something like a 'conceptual envelope' of all the possible primary meanings $\leq_P$, and all the information-states $t_P$; an 'envelope' that, coming from the family,

$$\{(X, \leq_P, t_P) \text{ ; for all possible } \leq_P, \text{and all possible } t_P\}, \tag{3.7}$$

throws some 'shadows' that are nothing else than each one of its quantity's elements. In this sense, the collective could appear as a cloudy or nebulous entity that, notwithstanding, can be individualized through an identity defined by:

$\boldsymbol{P} = \boldsymbol{Q} \Leftrightarrow$ For all possible primary meaning $\leq_P$:
Every state $t_P$ is coincidental with a state $t_Q$, and reciprocally.

that is, $P$ and $Q$ are always used in $X$ in forms that are actually indistinguishable. Perhaps and by such nebulous idea, the collectives could be called hazy sets as a way to recover, even in a more general form, the old Menger's concept. Anyway, it yet lacks to give a mathematical form to the 'conceptual envelope' that the collective seems to be; if this mathematical formalization is someday reached, what the full meaning of measurable predicates is will be actually better understood.

## *Notes*

1. If the relation $\leq_P$ 'grows', that is, more information on the primary use of $P$ is available, then the corresponding measure should be consequently extended. Provided it is $\leq^1_P \subseteq \leq^2_P$, and a quantity $(X, \leq^1_P, t^1_P)$ is known, then to have a refined quantity $(X, \leq^2_P, t^2_P)$, $t^1_P$ is only useful (monotonic, verifies property 1) for those pairs $(x, y)$ in $\leq^1_P$, but not necessarily for the pairs in the difference set $\leq^2_P - \leq^1_P$. This problem appears in the design of membership functions, since only a part of $\leq_P$ is often available to the designer who, sometimes, needs to quickly know a first 'shape' of $\mu_P$, and only later on could need to refine such function. Usually, in the practice, design processes are done step-by-step and a first approximation must be subsequently corrected.
2. Once a measure $t$ is specified, the designer counts with the new relation $\leq_t$, defined by $x \leq_t y \Leftrightarrow t(x) \leq t(y)$, that is larger than $\leq_P$:

$$x \leq_P y \Rightarrow t(x) \leq t(y) \Leftrightarrow x \leq_t y, \text{ or } \leq_P \subseteq \leq_t . \tag{3.8}$$

Provided the difference-set $\Delta = \leq_t - \leq_P$ is not empty, the let's call it *working meaning* $\leq_t$, adds to the primary meaning what is in $\Delta$, and this enlargement is only not done when $t$ allows to perfectly reflect the primary meaning, that is, when $\Delta = \emptyset$. The measure $t$ adds information to the primary meaning, and this fact should be taken into account by the designer [19, 21]. The act of measuring enlarges the 'meaning' of $P$ in $X$.

Of course, a way for $\leq_t$ not being 'total' and, hence, having more possibilities for its coincidence with $\leq_P$, is by defining $t$ as a mapping into a poset as it is, for instance, the complex unit square [22]. In addition, when the quotient-set $X/=_P$ exists and is partially ordered by $\leq^*_P$, the mapping $t(x) = [x]$ is a kind of natural 'qualitative' measure perfectly reflecting the primary meaning [19].

## 3.3 The Case of Singular Statements/Events

For capturing what does it mean a quantity $(X, \leq_P, t_P)$ it could be helpful to consider the case in which the universe of discourse $X$ is just a singleton.

In a universe of discourse $X$ that is a singleton, $X = \{x\}$, if it can be considered a predicate $P$ there is the single atomic statement 'x is P'. The problem for capturing its meaning lies necessarily in external relevant information on the $P$'s character of $x$, like it is the case with $x =$ God, and $P =$ Eternal, or with $x =$ Robert and

$P$ = Rich. The reason for this dependence of external information comes from the following argumentation.

I. In the case of a "singular statement", coming from a universe reduced to a singleton, the empirical relation $\leq_P$ (less $P$ than) is either reduced to the relation $=_P$ (equally $P$ than), or is empty and then $P$ is meaningless in $P$. This happens depending on the verification or not of the single expression $x =_P x$. Provided the situation is the first, the elemental meaning of $P$ in $X$ is $(\{x\}, =_P)$, and all measures $t_P : X \to [0, 1]$, are given by a single real number $t_P(x) \in [0, 1]$ that will 'suffer' from the fact that being $x$ unique and $\leq_P = =_P$, there is no way of saying if it is maximal, minimal or intermediate, since the quotient set $X/ =_P$ is just reduced to the single class $[x] = \{x\} = X$. That is, there is nothing internal allowing to assert if it is $t_P(x) = 0$, or $t_P(x) = 1$, or $0 < t_P(x) < 1$, and the problem to find this number can only rest on the availability of some sources of external information on $x$, $P$, and '$x$ is $P$'. This is the case when $X$ is just a singleton, and $P$ is either the predicate probable, or the predicate possible.

II. Let $S$ be the statement '$x$ is $P$'. Provided $P$ is known, it can be immediately considered its negation not $S = S'$ (God is not eternal, Robert is not rich); on the contrary, nothing can follow since it will mean that $P$ is not understood. With that, the set of the four statements $S = \{x$ is $P$, $x$ is not $P$, $x$ is $P$ and not $P$, $x$ is $P$ or not $P\} = \left\{S, S', S \text{ and } S', S \text{ or } S'\right\}$, can be taken into account. Then, provided the statement $S$ can be represented by a crisp or fuzzy (sub)set $A$ in $X$, the set $\mathbf{A} = \left\{A, A', A \cap A', A \cup A'\right\}$ can be considered, and there are only two possible cases that can be supposed,

   a. $S$ or $S'$ is true (call it the Boolean case).
   b. $S$ or $S'$ is not true (call it the non-Boolean case),

   that is, either $A \cup A' = X$, or $A \cup A' \neq X$. Let us analyze these cases by previously noticing that nothing in both Kolmogorov's probability theory, and Zadeh's possibility theory, is against the existence of a probability of the statement $S$, and against the possibilities of both statements $S$, and $S'$. This provided in the first case (a) $A$ can be identified with a Boolean algebra, and in the second (b) with a De Morgan one. As it is usually done, we will consider the probabilities and possibilities of the sets in $A$ instead of those statements in $S$.

   1. In case (a), a probability of $A$ is defined by just a number $p(A) \in [0, 1]$, such that $p(A') = 1 - p(A)$. Then, provided the statement "$S$ and $S'$" is false ($A \cap A' = \emptyset$), it is $p(A \cup A') = p(A) + p(A') = 1$, and the problem just lies in determining the number $p(A)$.
      Provided the statement "$S$ and $S'$" is not false, $\mathbf{A}$ is not identifiable with a Boolean algebra and no probability $p$ can be defined.
   2. Provided $\mathbf{A}$ is identifiable with a De Morgan algebra, a possibility $P$ of $A$ is defined by just a number $P(A) \in [0, 1]$, such that $P(A \cup A') = z = max(P(A), P(A'))$, and then it follows $P(A) = z$, or $P(A') = z$. Determining the number $z$ is the problem.

Since possibility measures are also definable in Boolean algebras, in such case and from $A \cup A' = X$, follows $z = 1$.

III. In general, how the numbers $p(A)$ and $z$ can be determined? Notice that determining $z$ is equivalent to determine either $P(A)$ or $P(A')$. Both values $p(A)$ and $z$, can only be known from some previous, external and relevant information on $A$. Then,

(i) Provided that in 2, the statement "$S$ or $S'$" can be considered true, then it is $z = 1$, and $P(A) = 1$, or $P(A') = 1$.
(ii) Provided that in 1, no information is available on $A$, then it can be supposed $p(A) = p(A') = 0.5$, reflecting a state of total ignorance.
(iii) Concerning 2, and also provided no information on $A$ is available, it can be supposed $z = 0.5$, and then it is $P(A) = 0.5$, or $P(A') = 0.5$, also reflecting a state of total ignorance.
(iv) In 1, the state of full faith does correspond to $p(A) = 1$, or with $p(A') = 0$. The state of full incredibility does correspond to $p(A) = 0$, or with $p(A') = 1$.
(v) In 2, the state of full faith does correspond to $z = 1$, that is, with $P(A) = 1$, or $P(A') = 1$. The state of full incredibility does correspond with $z = 0$, that is, with $P(A) = 0$, or $P(A') = 0$. Nevertheless, all these cases correspond to just imaginary situations.
(vi) Intermediate cases between those of full faith and full incredibility do correspond with $0 < p(A) < 1$, and to $0 < z < 1$, and the only way to take a value strictly between 0 and 1, depends on external information.

IV. What can it be external relevant information? It is some background knowledge on the element $x$ (God, Robert, ...) as, for instance, how it is seen by people, or what is recorded on it in some knowledge-base, etc., as well as on how $P$ acts in other universes of discourse $Y \neq X$, etc.
For instance, if $x =$ Cacamboo, and $P =$ Generous, the statement $S =$ 'Cacamboo is generous' means nothing except if some knowledge on Cacamboo is available. Were Cacamboo a nickname for my friend Robert, then surely I can have good reasons to state $S$, and perhaps also to assign a numerical value to $t_P$ (Cacamboo).
Where such information could come from? Usually, it comes from some set of attributes on $S$, some of which are relevant for what concerns the applicability of $P$ to $x$. From this last subset of attributes is from the value $t_P(x)$ can come from through, for instance, an aggregation (a function *Aggreg*) of the values for the attributes. If the attributes are $A_1, \ldots, A_n$, and $t_k$= degree up to which $x$ verifies $A_k$, it can be taken $t_P(x) = Aggreg\ (t_1, \ldots, t_n)$.

V. What happens if $X$ has just two elements, $X = \{x, y\}$, and $P$ is such that it is not $x =_P y$? Then it should be either $x \leq_P y$, or $y \leq_P x$, with the sign $\leq_P$ taken strictly, or $x NC_P y$.
Provided the first case holds, all measures $t_P$ are determined by the two numbers $0 \leq t_P(x) \leq t_P(y) \leq 1$, and analogously in the second case. Notice,

notwithstanding, that $x$ is the minimum and $y$ is the maximum in the graph $(\{x\}, \leq_P)$ and, hence, it should necessarily be $t_P(x) = 0$ and $t_P(y) = 1$. With respect to the third case (let us call it a 'rare case), to fix a value for $t_P$ it will be strictly necessary to also take into account external information.

VI. It seems clear enough that these kinds of problems only have interest when $X$ is with a big enough number of different elements, and $\leq_P$ is not reduced to $=_P$. Were it just a single stick in all the world, the concept of length and its measure would not exist.

## 3.4 Towards an Algebraic Structure for the Analysis of Ordinary Reasoning

What is often understood by 'inference' is just formal deduction, but people only scarcely reason in a formal deductive way, and rarely by using an artificial language in a formal framework like it is the case of mathematics and mainly of the theorem's proving processes. People reason with and within a Natural Language and the reached conclusions are only conjectures or refutations, and, hence, not all of them are safe at all. In addition, the information on which the reasoning is based in (their premises) is often uncertain, imprecise, incomplete, etc. In the way of fuzzy logic towards Computing with Words and Perceptions, it seems that an approach to the forms in which humans do reason is, at least, of some interest. At the end, good reasoning cannot be done without refuting (falsifying either an information, or a conclusion), abducing (searching for hypotheses, explanations), and speculating (searching ahead for new ideas). Reasoning is not only for 'safely proving something', but mostly for either refuting or explaining information, and especially for going farther from what is known.

Let us call *conjecturing* to deducing (formally and informally), abducing and speculating, and using *guessing* for only denoting the two last forms of conjecturing. At the end, computers will not reason like people up to when they will be able to conjecture and, especially, to speculate, to launch nets into the ocean of the unknown and recognizing what appears in them; that is, to formulate questions and to obtain answers. In a good part, scientific research can be said to be an art of systematically controlled conjecturing, and specially of guessing. Mathematics, by its part, is the art of guessing and of deductively proving, or disproving, the obtained guesses.

### 3.4.1

All people do reason by following similar patterns, and it is convenient to quote what the brilliant scientist and Nobel Laureate, Sir Peter B. Medawar, called the Law of Conservation of Information [23] that, perhaps, could be better called of 'the formal conservation of information':

*'No process of logical reasoning can enlarge the information content of the axioms and premises or observation statements from which it proceeds'.*

Medawar's law expresses that the clothes of current logic are too short, or even too large, for a good dressing of reasoning; that deductive logic is not sufficient to mathematically model all processes of reasoning, as it can be suspected by what is said at the beginning of this Sect. 3.3. And it is less sufficient again when imprecise predicates play some role. Hence, it seems important or, at least interesting, to look for some answer to the question:

How can it be described in mathematical terms what is a conjecture from a set of premises in a general enough form, able to capture as many as possible kinds of reasoning?

In this question, 'kinds of reasoning' refer to those modeled by classical logic, quantum logic, and fuzzy logic, all of them theoretically build up from a partially ordered set with minimum, maximum and jointly with a negation. That is, and respectively, Boolean and De Morgan algebras, Orthomodular lattices, and Standard algebras of fuzzy sets, that are but instances of the more general structure of a *Basic Flexible Algebra* (BFA) [24] where only the essential properties commonly needed are preserved. For instance, in BFAs neither the connectives functional expressibility, their commutativity, nor associativity, nor distributivity, nor duality, nor the double-negation law, etc., are presumed, although either they or some restrictions of them can be added to reach some conclusions.

**Definition** *A BFA is a seven-tuple* $\Omega = (X, \leq, 0, 1; +, \cdot, ')$, *where* $(X, \leq)$ *is a poset with minimum* $0$ *and maximum* $1$, $+$ *and* $\cdot$ *mappings* $X \times X \to X$, *and* $'$ *a mapping* $X \to X$, *such that:*

*(1)* $a \cdot 1 = 1 \cdot a = a \;\;\&\;\; a \cdot 0 = 0 \cdot a = 0 \;\;\&\;\; a + 1 = 1 + a = 1 \;\;\&\;\; a + 0 = 0 + a = a$
*(2)* $a \leq b \Rightarrow a \cdot c \leq b \cdot c \;\;\&\;\; c \cdot a \leq c \cdot b \;\;\&\;\; a + c \leq b + c \;\;\&\;\; c + a \leq c + b$
*(3)* $a \leq b \Rightarrow b' \leq a'$, *and* $0' = 1$, $1' = 0$, *for all* $a, b$ *in* $X$.
*(4) There exists a subset* $X_0$, $\{0, 1\} \subseteq X_0 \subseteq X$, *such that with the restriction to it of* $\leq, +, \cdot$, *and* $'$, *it is* $\Omega_0 = (X_0, \leq, 0, 1; +, \cdot; ')$ *a Boolean algebra.*

Of course, $X_0 = \{0, 1\}$ generates the smallest Boolean algebra contained in $\Omega$, and obviously Boolean algebras, De Morgan algebras, Ortholattices, and Standard algebras of fuzzy sets, are but instances of a BFA.

Because of the few and weak laws a BFA owns, only the following few properties can be proven,

(I) $a \cdot b \leq a \leq a + b \;\;\&\;\; a \cdot b \leq b \leq a + b$, for all $a, b$ in $X$. Hence, provided $(X, \leq)$ were also a lattice with operations $min$, and $max$ (and with which axioms 1 to 3 are verified), it will be: $a \cdot b \leq min(a, b) \leq max(a, b) \leq a + b$. That is, in lattices $min$ is the greatest operation $\cdot$, and $max$ is the lowest $+$. In particular, it is $a \cdot a \leq a \leq a + a$, for all $a$ in $X$: Law of sub-idempotency.

(II) If $\Omega$ is a distributive lattice, and $'$ is involutive or a strong one , that is, $a'' = a$, for all $a$ in $X$, $\Omega$ is a De Morgan algebra.

(III) If $X = [0, 1]^Y$, $\Omega$ cannot be an Ortholattice, nor less an Orthomodular lattice, and less again a Boolean algebra. In this case, if $\cdot$ and $+$ are functionally expressible by a t-norm and a t-conorm, respectively, and $'$ by a strong negation function, $\Omega$ is a BFA that only if the t-norm is the greatest one, $min$, and the t-conorm is the lowest one, $max$, is a lattice and, namely, a De Morgan algebra when the negation $'$ is a strong one. In addition, if this algebra verifies the property, $a \cdot a' \leq b + b'$, for all $a, b$ in $X$; it is a Kleene's algebra.

(IV) $a \leq b \,\&\, c \leq d \Rightarrow a \cdot c \leq b \cdot d \,\&\, a + c \leq b + d$, for all $c, d$ in $X$: Laws of monotony.

(V) $a' \cdot b' \leq (a \cdot b)' \,\&\, (a + b)' \leq a' + b'$, for all $a, b$ in $X$.

(VI) If $+ = max$, regardless of $\cdot$ and $'$, it is $(a + b)' \leq a' \cdot b'$: First law of semi-duality.

(VII) If $\cdot = min$, regardless of $+$ and $'$, it is $(a + b)' \leq a' \cdot b'$: Second law of semi-duality.

(VIII) With $a \wedge b = (a' + b')'$, and $a \vee b = (a'.b')'$, also $\Omega^* = (X, \leq, 0, 1; \wedge, \vee;')$ is a BFA called the dual of $\Omega$, and it inherits all additional properties enjoyed by $\Omega$.

**3.4.2**

Since BFAs are defined by just a few and weak axioms, only allowing very simple calculations, what can be proven in their framework is but of a very general validity, and will be preserved by the addition of new axioms to the list of the four that define BFAs. One of the weaknesses of Boolean and De Morgan algebras, as well as of Ortholattices, for the study of Commonsense Reasoning, lies in the big amount of laws they enjoy and make such structures too rigid to afford the flexibility shown by both Natural Language and Ordinary Reasoning. They own laws that, like the commutative law of $\cdot$, not always appear in the language where, for instance, 'time' often causes the failure of such property for the conjunction 'and'. At this respect, the Standard algebras of fuzzy sets show a lot of cases in which some Aristotelian forms are valid in one but not in another algebra; for instance, the classical law of duality is broken when the t-conorm is not the dual of the corresponding t-norm [25]. It is also the case that the von Neumann's law of perfect repartition [26], $a = a \cdot b + a \cdot b'$, only can hold provided the t-norm (functionally representing $\cdot$), and the t-conorm (functionally representing $+$) are not dual. The worlds of language and ordinary reasoning are very different from the world of artificial languages and formal deductive reasoning.

In the words of Watanabe [15], a new land can be perhaps offered to young researchers, provided 'a direct contact with the world of common sense which is the mother earth of all knowledge' is not disdained at all.

## 3.5 Conjectures and Refutations

People often say something like 'from this I deduce that', and they, if questioned, distinguish such expression from another like 'from this I guess that'. But the fact is that when 'deducing' they don't conduct the corresponding reasoning with neither an artificial language, as it is that of mathematics, nor in a formal framework, nor do it step-by-step but with some jumps even if they are convinced that these jumps could be refilled with more steps, and that all the process starts from some premises (this) and ends with the conclusion (that). Then, it seems that a distinction between the formal and the informal idea of 'deduction' is in order, once accepted that the first is well captured by Tarski's operators of consequence.

A first thing that should be considered, even if usually people do not take special care of it, is if among the premises, or statements asserting the available information captured on the subject under scrutiny, any kind of contradiction is hidden, since, were it recognized a single one, everybody would refuse the conclusion even if the process of reasoning, that going from the premises to the conclusion, were not questionable in itself. Let us understand by contradiction between two statements that there is agreement on asserting 'If this, then not-that', provided the form of negating by 'not' is well understood.

To such an end, it will be supposed that all the statements are in a BFA, $\Omega$, whose operations model the linguistic *and* by $\cdot$, the linguistic *or* by $+$, the linguistic *not* by $'$, and the linguistic conditionals *if/then* by $\leq$. In this framework, a set of premises $P = \{p_1, \ldots, p_n\}$ is said to be *contradiction-free* provided no single formula $p_i \leq p'_j$ holds for any $i, j$ between 1 and $n$; that is, for all $i, j \in \{1, 2, \ldots, n\}$, it is $p_i \nleq p'_j$. Once assumed that $P$ is contradiction-free, let us consider as the *résumé* of $P$ the element, $p = p_1 \cdot (p_2 \cdot (\cdots) \cdot p_n)$ in $X$. With $p$, $P$ will be called *information-consistent*, provided it is $p \nleq p'$, and $p' \nleq p$; that is, if $p$ is not at all self-contradictory. Notice that if it is $p = 0$ since $0 < 1 = 0' = p'$, information-consistency implies $p \neq 0$.

With all that, let us define that $q$ in $X$ is *informally deduced* from $P$ [27], whenever there is a path under the partial order $\leq$ going from $p$ to $q$ like, for instance, it is $p \leq a \leq b \leq c \leq q$, with $a$, $b$, and $c$ in $X$, or it is simply $p \leq q$. Then, since $\leq$ is supposed to be a transitive relation, the set of the *informal consequences* of $P$ can be defined as the set

$$\boldsymbol{IC}(P) = \{q \in X; \, p \leq q\}, \tag{3.9}$$

in which there is not 0, but there are 1 and $p$ and, hence, is never empty. Notice that it is $\boldsymbol{IC}(P) = \boldsymbol{IC}(\{p\})$. Provided $\boldsymbol{IC}(P)$ enjoys the properties of a Tarski's operator of consequences, that is,

(1) $P \subseteq \boldsymbol{IC}(P)$: $\boldsymbol{IC}$ is extensive;

(2) If $P$ and $Q$ are contradiction-free and information-consistent, and if $P \subseteq Q$, then $\boldsymbol{IC}(P) \subseteq \boldsymbol{IC}(Q)$: $\boldsymbol{IC}$ is monotonic;

(3) If $q \in \boldsymbol{IC}(P)$, then $q' \notin \boldsymbol{IC}(P)$: $\boldsymbol{IC}$ is consistent, and

(4) If $\boldsymbol{IC}(P)$ is contradiction-free and information-consistent, then $\boldsymbol{IC}(\boldsymbol{IC}(P)) = \boldsymbol{IC}^2(P) = \boldsymbol{IC}(P)$: $\boldsymbol{IC}$ is a closure,

it will be said that the elements in $\boldsymbol{IC}P)$ are the *formal consequences* of $P$. Which of these properties do always hold out of formal deduction?

Of course, the first holds always since, regardless of the way of numbering the premises, it is $p = p_1 \cdot (\ldots) \leq p_1$, $p \leq p_2 \cdot (\ldots) \leq p_2$, etc., that is, all $p_i$ belong to $\boldsymbol{IC}(P)$ and, hence, $P \subseteq \boldsymbol{IC}(P)$.

With respect to the second property, *provided* the *operation* $\cdot$ is *associative*, then, since $P \subseteq Q$ means that it is $Q = \{p_1, \ldots, p_n, p_{n+1}, \ldots, p_m\}$, and it is sure that $q = p_1 \cdots p_n \cdot p_{n+1} \cdots p_m \leq p_1 \cdots p_n = p$, with which if $p \leq r$ it is also $q \leq r$, it follows $\boldsymbol{IC}(P) \subseteq \boldsymbol{IC}(Q)$. Hence, property 2 can hold provided, for instance, that the associative law of $\cdot$ can hold in $\Omega$.

Concerning property 3, if it is $p \leq q$ and $p \leq q'$, from $q' \leq p'$ it will follow $p \leq p'$, that is absurd; hence it should be $p \nleq q'$, and property 3 always holds.

With respect to the fourth property, for applying the 'operator' $\boldsymbol{IC}$ to the set $\boldsymbol{IC}(P)$, this set should be contradiction-free and information-consistent, something that can be *difficult to assert* when, for instance, $X$ is not clearly a finite set. *Provided* $\boldsymbol{IC}(P)$ were not finite, then the first problem lies in defining its r*é*sum*é*, and a good option for this is to take as it, and *provided* it exists, $\boldsymbol{Inf}(P)$. In such case, is $\boldsymbol{IC}(P) = \{q \in X; \boldsymbol{Inf}(P) \leq q\}$, and at least, when $P$ is finite it is $\boldsymbol{Inf}(P) = p$. Thus, and since $\boldsymbol{IC}\ (\boldsymbol{IC}(P)) = \{q \in X; \boldsymbol{Inf}\ \boldsymbol{IC}(P) \leq q\}$, it is $\boldsymbol{Inf}\ \boldsymbol{IC}(P) = \boldsymbol{Inf}\ \{q; \boldsymbol{Inf}\, P \leq q\} = \boldsymbol{Inf}(P)$, *provided* $\boldsymbol{Inf}(P) \nleq (\boldsymbol{Inf}(P))'$ and $(\boldsymbol{Inf}(P))' \nleq \boldsymbol{Inf}(P)$, it is $\boldsymbol{IC}(\boldsymbol{IC}(P)) = \boldsymbol{IC}(P)$, and property 3 holds.

It seems clear enough that it is difficult to state that $\boldsymbol{IC}$ is a closure in ordinary reasoning.

In summary, the set $\boldsymbol{IC}(P)$ of the informal consequences is not generally a set of formal consequences, unless some additional properties can be supposed in $\Omega$ as, for instance, that the operation $\cdot$ is associative, that the sets of premises $P$ and $\boldsymbol{IC}(P)$ are finite, contradiction-free, and information-consistent, or that $X$ is finite, etc. All this can easily happen when $\Omega$ is an Ortholattice or a De Morgan algebra, and in particular if it is a Boolean algebra, as well as if $\Omega$ is a Standard algebra of fuzzy sets. Nevertheless, and in general, $\boldsymbol{IC}$ is just an extensive and consistent operator although, in ordinary reasoning as it is made by people, it is just consistency what is taken into account; to refuse a 'deductive' conclusion it is usually tried to just show that there is some 'consequence' whose negation is also so. This is just the idea of 'falsifying' the reasoning.

Once the concept of an informal consequence is established, a conjecture from $P$ [27] will be defined as an element in $X$ that is not contradictory with the r*é*sum*é* $p : p \leq q'$. That is, that its negation cannot be informally deduced from $P$, that the reasoning is not falsifiable, or that $q$ is not refutable. Then the set of conjectures from $P$ can be defined by,

$$\boldsymbol{Conj}(P) = \left\{q \in X;\, p \nleq q'\right\} = \left\{q \in X;\, p \leq q'\right\}^c. \tag{3.10}$$

Since $p \leq q'$ reflects that 'If $p$, then not-$q$' can be asserted, that $q$ refutes $p$, it is reasonable to call *refutations* of $P$ to the elements in the set

$$\boldsymbol{Ref}(P) = \{q \in X; p \leq q'\} = \{q \in X; q' \in \boldsymbol{IC}(P)\}. \quad (3.11)$$

The refutations are those elements whose negation is an informal consequence of $P$, that is, whose negation can be informally deduced from the premises. Is a definition that seems to capture well the idea of 'refuting'.

Notice that,

$$X = \boldsymbol{Ref}(P) \cup \boldsymbol{Ref}(P)^c = \boldsymbol{Ref}(P) \cup \boldsymbol{Conj}(P), \quad (3.12)$$

shows that, once $P$ is given, there are no more elements in $X$ than either conjectures from $P$ or refutations of $P$.

Obviously, $\boldsymbol{IC}(P)$ is not a part of $\boldsymbol{Ref}(P)$, since the coexistence of $p \leq q$ (implying $q' < p'$) and $p \leq q'$ conducts to the absurd $p \leq p'$. Is it a part of $\boldsymbol{Conj}(P)$? That is: Does $p \leq q$ imply $p \not\leq q'$? The same argument proves it and, hence, it is $\boldsymbol{IC}(P) \subseteq \boldsymbol{Conj}(P)$: informal consequences are a particular type of conjectures. Then, what about those conjectures that are not informal consequences:

$$\begin{aligned}\boldsymbol{Conj}(P) - \boldsymbol{IC}(P) &= \{q \in X; p \not\leq q' \;\;\&\;\; p \not\leq q\} \\ &= \{q \in X; p \not\leq q' \;\;\&\;\; q < p\} \cup \{q \in X; p \not\leq q' \;\;\&\;\; p \,\mathbf{nc}\, q\}?\end{aligned} \quad (3.13)$$

where **nc** shortens 'not comparable under $\leq$'. The first subset in this union consists of those conjectures $q$ from which, in the informal deductive form, it follows the r*é*sum*é* $p$ and, hence, all the premises; that is, that $q$ 'explains' the premises. This is just the idea of what a hypothesis is, and, for this reason, we will call this subset that of the *hypotheses* for $P$, and denote it by $\boldsymbol{Hyp}(P)$.

The second subset consists of those conjectures that are not comparable with the r*é*sum*é*, and that can be called the *speculations* from $P$, and its set is,

$$\boldsymbol{Sp}(P) = \{q; q' < p \;\;\&\;\; p \,\mathbf{nc}\, q\} \cup \{q; p \,\mathbf{nc}\, q' \;\;\&\;\; p \,\mathbf{nc}\, q\}, \quad (3.14)$$

a partition in which it appears that, in the first part, the equivalence between $q' < p$ and $p' < q$ (provided the negation verifies the law $a'' \leq a$ for all $a$ in $X$), shows that the r*é*sum*é* $p$ is informally deducible from $q'$, and that $q$ is not comparable with $p$. The second part is, nevertheless, that of the absolutely non-deductible speculations. Let us denote, respectively, by $\boldsymbol{Sp}^1(P)$ and $\boldsymbol{Sp}^2(P)$ these two sets.

Since every $q$ in $\boldsymbol{Hyp}(P)$ is a conjecture from which, as it is $0 < q < p$, $p$ is informally deductible from $q$, the conjectures in $\boldsymbol{Sp}^2(P)$ are the only that are actually never reachable through a either forwards or backwards deductive process, provided the negation verifies $a'' \leq a$; let's call them *creative* conjectures. At the end, a complete classification of conjectures is given by the partition,

$$\boldsymbol{Conj}(P) = \boldsymbol{IC}(P) \cup \boldsymbol{Hyp}(P) \cup \boldsymbol{Sp}^1(P) \cup \boldsymbol{Sp}^2(P). \quad (3.15)$$

## Notes

1. Provided $\Omega$ is a Boolean algebra, it is $p \leq q' \Leftrightarrow p \cdot q = 0$, thus $q$ is in $\boldsymbol{Conj}(P) \Leftrightarrow p \cdot q \neq 0$. Thus, in Boolean algebras the conjectures are just those elements that are not disjoint with the r*é*sum*é* of the premises.
2. If $q \in \boldsymbol{Conj}(P)$ it cannot be $q = 0$ since, from $p \leq 1 = 0'$, it follows that $q$ is not a conjecture. On the contrary, it is $1 \in \boldsymbol{Conj}(P)$, since $p \leq 1' = 0$ implies the absurd $p = 0$.
3. If $P$ and $Q$ are contradiction-free and information-consistent, if it is $P \subseteq Q$, since it is $q \leq p$, provided $r \in \boldsymbol{Conj}(Q)$, or $q \nleq r'$, if it were $p \leq r'$ it will follow the absurd $q \leq r'$. Hence $r \in \boldsymbol{Conj}(P)$, and $\boldsymbol{Conj}(Q) \subseteq \boldsymbol{Conj}(P)$. That is, *the operator* ***Conj*** is *anti-monotonic* [27].
4. With the last supposition, if $r \in \boldsymbol{Hyp}(Q)$, or $r < q$, it follows $r < p$, and $r \in \boldsymbol{Hyp}(P)$. Hence, $\boldsymbol{Hyp}(Q) \subseteq \boldsymbol{Hyp}(P)$, and *the operator* ***Hyp*** is *anti-monotonic* [27].
5. If $\Omega$ is an Ortholattice, $p \leq q'$ implies $p \cdot q = 0$. Hence, if $q$ is a hypothesis, from $q < p$ follows $q = p \cdot q$, and $q = 0$ that is absurd since 0 is not a conjecture. Hence, in ortholattices it is simply $\boldsymbol{Hyp}(P) = \{q; 0 < q < p\}$, since the conjecture's condition of hypotheses, $p \nleq q'$, can be avoided [28].
6. If it is $p \nleq p'$, it could be $p' < p$, against the information-consistent character of $P$; it is absurd. Hence, it is not difficult to prove that if $P$ and $Q$ are contradiction-free and information-consistent, if $P \subseteq Q$, then $\boldsymbol{Sp}^1(Q) \subseteq \boldsymbol{Sp}^1(P)$, that is, *the operator* $\boldsymbol{Sp}^1$ is *anti-monotonic*.
7. It is easy to find examples in Boolean algebras showing that $\boldsymbol{Sp}^2$ is neither monotonic, nor anti-monotonic. It is enough to take a power set with two subsets $P \subseteq Q$, and show a type-two speculation from $P$ that is not one from $Q$, and reciprocally. Among conjectures the ones in $\boldsymbol{Sp}^2(P)$ (type-two speculations) are very special; there is no law for its growing when the number of premises grows. They are neither monotonic, nor anti-monotonic and, actually, they are the conjectures that can be properly called *non-monotonic*.
   Since, like conjectures in general, and hypotheses and type-1 speculations in particular, also the set of type-2 speculations is not necessarily consistent, the newly coined name 'creative conjectures' seems to be well attributed to them. They are non-monotonic and there is no path to reach them deductively neither from $p$, nor from $p'$, neither forwards, nor backwards, and their suitability as 'new' ideas, that is, ideas not contained in the premises, is open and should be externally checked. They deserve more analysis.
8. Anyway, to show the relevance of speculations for reasoning, if $q$ is any speculation such that $0 \neq p \cdot q \neq p$, then $p \cdot q$ is a hypothesis, and $p + q$ is a consequence; hence, and in particular, type-two speculations are useful to either find 'explanations' for the premises, or 'continuations' of them [27, 28].
   If $q$ is in $\boldsymbol{Sp}^2(P)$ it can be interesting to analyze what can be done with the elements in some special sets generated by $q$ as, for instance, $\{r \in X; q < r$, or $r < q\}$.

9. How to find creative conjectures or speculations and, mainly, type-2 ones? Those of type-1 can, in principle, be searched among the $q$ in $X$ that are not comparable with $p$, by going step-by-step and forwards, from $p'$ to $q$, since $q' < p \Rightarrow p' < q'' \leq q$, provided $'$ verifies the law $a'' \leq a$; that is, there are cases in which it is possible to find $q$ deductively. But this is not the case for type-2, creative conjectures. Thus, what? A typical way followed in ordinary reasoning is to tentatively reach q by analogy with a similar former case in which a problem was well or, at least, satisfactorily solved. This is another question deserving analysis.
10. In the scientific mode of reasoning it sometimes appears the problem called the 'falsification' of a presumed hypothesis that, the model here presented, allows to formalize [27, 28].
    If it is presumed that $h \in \boldsymbol{Hyp}(P)$, from $r \in \boldsymbol{IC}(P)$, or $p \leq r$, and from $h < p \leq r$, follows $h < r$, and $r \in \boldsymbol{IC}(h)$. That is $\boldsymbol{IC}(P) \subseteq \boldsymbol{IC}(\{h\})$: all the consequences of $P$ are also consequences of the hypothesis. Hence, to show that $h$ is not actually a hypothesis for $P$, it suffices to find a consequence of $P$ that cannot deductively follow from $h$; a current way of falsifying a hypothesis by 'reductio ad absurdum'.

## 3.6 An Important Open Problem

### 3.6.1

There is in course an interesting debate concerning the concepts o(which are in themselves obscure but so often appearing in both language and science) of uncertainty, possibility, and probability, as well as the relations between them. that they deserve attention. Attempting to look at them from the point of view presented in **section 2**, that of a non-formalized language, a first question concerns on how are commonly used the three mother-predicates $U$ = uncertain, $P$ = probable, and $\Pi$ = possible, from which the three concepts do come; that is, which quantities $(X, \leq, t)$ can reflect their respective meanings. In principle, it seems widely accepted that is

$$\text{`}x\text{ is probable'} \Rightarrow \text{` x is possible'} \Rightarrow \text{`}x\text{ is uncertain'}, \tag{3.16}$$

but this chain of implications is not, as far as I know, actually checked by experimental means. Nor are the inclusions [29] $\leq_P; \subseteq; \leq_U$ and $\leq_\Pi; \subseteq; \leq_U$, even if it seems accepted that the probability measures $(Prob)$ coherent with possibility measures $(Poss)$ for allowing to simultaneously consider the measures of the statements '$x$ is $P$' and '$x$ is $\Pi$', should verify $Prob \leq Poss$. The problem concerns reasoning, since those three statements appear in both the premises, and in the conclusions of reasoning; that is, that $x$ should be either a conjecture, or a refutation. For instance, it can be accepted that $q$ is possible or probable under the premises $P$, just means that $q$ is either a conjecture from $P$ or a refutation of $P$. In any case, the above chains of implications and inclusions can easily be context-dependent; something that should conduct to experiments in several domains.

Notwithstanding, and at least in the scientific field, the elements $x$ are supposed to be managed after they are represented in Boolean algebras (sets) in the case of probability, and in De Morgan algebras (fuzzy and crisp sets) in that of possibility, and it seems that it is accepted by both probabilist' and 'possibilist' people, that $\leq_P$ is taken as coincidental with the respective partial orders $\leq$ in these algebras, even if the only that is fully acceptable 'a priori' is that these partial orders are just part of the relations $\leq_P$ and $\leq_\Pi$, respectively. That is, it lacks to experimentally check if it is actually $\leq \subseteq \leq_P \subseteq \leq_U$, and $\leq \subseteq \leq_\Pi \leq \leq_U$, respectively. Before doing that, it is risky to accept that both probability and possibility can actually measure the uncertainty carried by the elements $x$; it lacks to know when the difference-sets $\leq_U - \leq$, for the partial orders of Boolean and De Morgan algebras, are truly empty, or not. If they are not empty, it is not possible, for instance, to be sure that if it is $x \leq_P y \Rightarrow x \leq y$, but it is not reciprocally, then $Prob(x)$ truly measures to which extent an element $x$ is uncertain.

In the case of the probability measures, and for instance, in the random game of throwing a dice, the universe of discourse is identified with the Boolean algebra with the six atoms in the set $X = \{1, 2, 3, 4, 5, 6\}$, that is, the results that can be obtained, and the r*é*sum*é* is $X$ itself. Then the conjectures are the events $q$ such that $X \nleq q^c$, or $q \nleq X^c = \emptyset$, that is, $\boldsymbol{Conj}(P) = \{q; q \neq \emptyset\}$, the consequences are those $q$ such that $X \leq q$, that is $\boldsymbol{IC}(P) = \{X\}$, and the hypotheses are $\boldsymbol{Hyp}(P) = \{q; \emptyset < q < X\}$. Thus, in this case there are no speculations, but only hypotheses and just a unique consequence. As the set of refutations is $\boldsymbol{Ref}(P) = \left\{q; X \leq q'\right\} = \{\emptyset\}$, excluding the sure event (to which, in addition, no betting is allowed), the measure of a probability will be just applied to hypotheses, and verifying $Prob(\emptyset) = 0$, $Prob(X) = 1$. In general, if $Prob(p) = 1$, the probability of all consequences will be 1.

To illustrate the case of the possibility measures, let's consider the following example: A variable $Z$ takes its values in [0, 10], and it is only available on Z the partial information that it has values in [3, 6], but not in $[0, 2] \cup [8, 10]$. The question is to measure the possibility that $Z$ could take values in the semi-open interval $q = [6.1, 8)$. Since the set of premises is $P = \{[3, 6], ([0, 2] \cup [8, 10])^c\}$, the r*é*sum*é* is $p = [3, 6] \cap ([0, 2] \cup [8, 10])^c = [3, 6] \cap (2, 10] \cap [0, 2) = [3, 6]$, that is not empty, and verifies $p \subseteq q^c = [6.1, 8)^c = [0, 6.1) \cup [8, 10]$; hence, $q$ is a refutation. Of course, like in the case of probabilities, if $Poss(p) = 1$, all consequences have also possibility 1.

In both examples, it is doubtful that in general $Prob(q)$, and $Poss(q)$, can be always taken as the more suitable measures of the uncertainty carried by $q$ [29].

### 3.6.2

The set of decidable elements, those $q$ for which either $q$ or not $q$ is deductible from the given $P$, is

$$\boldsymbol{Dec}(P) = \left\{q; p \leq q \text{ or } p \leq q'\right\} = \boldsymbol{IC}(P) \cup \boldsymbol{Ref}(P). \tag{3.17}$$

Thus, the un-decidable elements constitute the set [27, 28],

$$\boldsymbol{Un}(P) = X - \boldsymbol{Dec}(P) = \boldsymbol{Hyp}(P) \cup Sp(P). \tag{3.18}$$

In the case of consequences and hypotheses, and provided it is $Prob(p) > 0$, from $p \leq q$ and $0 < q < p$, respectively, it is always $0 < Prob(q)$ if $q$ is a consequence, but, depending on the character of the measure $Prob$, it can be $0 = Prob(q) < Prob(p)$. In the case of a refutation, from $p \leq q'$ follows $0 < Prob(p) \leq 1 - Prob(q)$, that is $Prob(q) < 1$. Similarly, in the case of the $Poss$ measures analogous inequalities hold, except for the case of refutations, where it is not $Poss(q') = 1 - Poss(q)$, since $Poss(q')$ is not functionally expressible.

Notice that since the only conjectures that are actually non-reachable by some deductive process are the speculations of the second type, perhaps and in front of ordinary reasoning, the concept of decidable statement deserves to be modified by including consequences, hypotheses and the first type of speculations.

### *Note*

A problem that remains open is that of knowing the basic properties a measure of uncertainty should verify, and to clarify when probabilities and possibilities can be taken as measuring uncertainty [29]. At least it is doubtful that they should be always additive like with probabilities, or sub-additive like with possibilities; like it is doubtful that the measure of not-$q$ should be functionally expressible from the measure of $q$. It is difficult to believe that uncertainty, in general, is not involved with losses of information, and that its measures should right believe like those of length, surface, or volume. At least, the additive law of probabilities seems to be 'rare' in a world where losses and gains of information are more the norm than the exception.

## 3.7 Conclusion

As it can be seen in the preceding sections, a lot of questions are yet waiting to be answered. Many, if not all of them, are not just of a purely theoretical character, since their possible answers do require to be checked with the practice of natural language and ordinary reasoning before being accepted as such, and even in a provisory form. There is yet another question that I judge as a relevant one:

Provided those answers do come from a mathematical model, can we expect that such model will be found by just coming from purely mathematical thinking? or, should it come from some abstractions on the information furnished by the corresponding reality? that is, from data captured thanks to a scientific typically controlled processes of observation and experimentation within language and reasoning? Notice that the concept of a fuzzy set, here re-introduced as an information-state of the behavior of its linguistic label, seems to show that a fuzzy set is more a scientific 'object' than a purely mathematical or logical one; and scientific objects are those that are typically analyzed by science.

Which should be the best working's methodology to follow up with: just pure theoretic thinking, like that of mathematicians, often based on second order abstractions, and at once responding to a theoretic interest on the mathematics of the subject, or mixed experimental-and-theoretic thinking like that of physicists where theoretic models should be tested against the observed physical reality? Is it possible to conduct a mathematical analysis of both language and ordinary reasoning, without previously considering the nuances and variability of the 'natural matter' for such study? Is not language and reasoning such natural matter [30]?

In my view, and, at least to actually become a science of language and reasoning, fuzzy logic needs to evolve towards an experimental science with a methodology analogous to that of physics. On the contrary, its evolution towards Zadeh's Computing with Words and Perceptions will only result in a more or less disjointed collection of computational techniques, based on the current fuzzy logic, and able to be used at each particular problem. Let me clearly say that this is not bad at all, and that without any doubt its research should be continued [31], but I seriously doubt it can be sufficient enough for a scientific enlightening on the actual 'functioning' of language and reasoning.

A scientific-like modeling of language and reasoning is, in my view, and jointly with that on how the human brain actually works, that is, what is and how it is 'thinking' reached, one of the big challenges for science in the XXI Century, and the starting point for the end of a very large path that begun many centuries ago. Two challenges between which I think there are many bridges communicating one with the other. At the end, before having a good understanding of language and reasoning, I keep many doubts concerning the possibility that technology can arrive at machines able to actually reason like people do.

This new way of studying imprecision and uncertainty in natural language and ordinary reasoning, is what I yet hope, and wish, it will come from the old fuzzy logic [31]. At the end fuzzy logic is where mathematical analysis actually begun into use for representing both language and inference like it happened before in natural sciences. Would young researchers devote their efforts to it!

**Acknowledgments** This paper is partially funded by the Foundation for the Advancement of Soft Computing, and by the Spain's Government project MICIIN/TIN $2011 - 29827 - C02 - 01$.

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# Chapter 4
# Fuzzy Logic and Modern Economics

**Francesc Trillas**

**Abstract** Fuzzy Logic has made two important contributions to economic analysis: a theory of fuzzy preferences, and the development of empirical techniques based on fuzzy sets. However, modern areas of economic research are not sufficiently influenced by these ideas. Behavioral and institutional economics, among other fields in modern economics, would benefit from insights from fuzzy logic.

## 4.1 Introduction

A typical PhD economist can obtain her degree and have a decent career without ever having to understand what fuzzy logic is. However, fuzzy logic has made very important contributions and obtained big success in other related branches of knowledge, including computer science and engineering.

Fuzzy logic has nevertheless two prominent applications in economics. First, as an expansion of preference and choice theory. And second as an empirical method to take into account degrees of membership in economic or social categories, and to establish set-theoretic relationships between these degrees of membership. Both have to do with the measurement of the complexity of behaviour of human individuals or institutions.

Fuzzy preferences represent vague preferences, relationships between alternatives where one alternative is not necessarily clearly preferred to another, although the agent can to some extent compare the alternatives (hence the preference relation is not incomplete). If a conventional economist is first introduced to fuzzy preferences, he or she, if familiar with behavioural economics (broadly speaking, the intersection between economics and psychology), would say that fuzzy preferences are an example of non-standard preferences (as compared to the standard of neo-classical economics, the economics usually taught at universities). Non-standard preferences are one set of predictable "anomalies" that psychology has uncovered in human

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_4

behavior, and that make human behaviour different from that assumed in traditional (neoclassical) economics, namely rational, meaning with well defined, exogenous and individualistic binary preferences.

The role of "non-standard" preferences in behavioral economics is crucial: it acknowledges that we are guided less by a correct exact knowledge of our self-interest than by a socially learned, evolved, intuitive grasp derived from mental shortcuts (frames, reference points, envy, addiction, temptation, fairness). In general, behavioral economics amounts to a critique of economic thought as it has been transmitted during the last six decades. This economic thought is well represented by neoclassical economics, with its assumption that consumers, firms and economic agents in general are rational and self-interested (individualistic); and by public choice. The latter criticized neoclassical economics because this concluded that in those cases where the self-interest of rational agents was not conducive to desirable social results (e.g. efficiency) a social planner (government) had a potential role to improve upon the so-called *market failures*. The public choice critique was that the assumption of self-interested and rational agents should be extended to people in the political process (citizens, politicians and bureaucrats, to use the words of public choice scholars), and therefore that the potential to improve upon market failures was more limited than originally expected. A new emphasis was born on *government failures*, but both the neoclassical approach and the public choice approach shared the assumption of rationality and self-interest of the economic agents. Actually, what the public choice school did was to expand the applicability of the assumption.

Of course, fuzzy logic can also shed light on other sets of predictable "anomalies" highlighted by behavioural economics, such as imperfect optimization and lack of self-control, to the extent that these types of anomalies point to changing preferences, and preferences could experience changes in the degree, measure or type of fuzziness depending on visibility, framing or the location of the individual relative to some reference point.

The potential of fuzzy logic to contribute to empirical work in economics and social science in general is well represented by Ragin [16] and the applications of his proposed methodology. This constitutes an empirical method to take into account degrees of membership in economic or social categories, and to establish set-theoretic relationships between these degrees of membership. Together with fuzzy preference theory, fuzzy set empirical methodologies contribute to measuring the complex behaviour of human individuals or institutions.

There is no doubt that an important component of this need to tackle social complexity is the fact that human judgement and decision making departs from the simplistic representation of rationality used by traditional economics. The first papers of fuzzy preference theory clearly related fuzzy preferences to degrees and forms of rationality. But the subsequent explosion of behavioural economics paid scarce attention to the theory of fuzzy preferences or to fuzzy logic in general. The 1984s paper by Basu on degrees of rationality that is reviewed below, for example, had no impact on the subsequent explosion of research on bounded rationality.

Basu himself in a later book[1] argued that his fuzzy adventure had been somehow sterile. He devoted a chapter to the difficulties of preference theory in traditional economics, and argued (p. 32) that "tucked away in the woodwork of our models, these assumptions—that each person has a utility function, and that this is unchanging—go unexamined, but if we do examine them we will find that they are strong assumptions and quite unacceptable in lots of contexts. Often, we have no well-defined preferences, and make choices from a vague and ill-defined idea of what we want. This leads to vacillation and changes in our ordering over different goods." But then in a footnote in the same page (footnote 11) he writes that "one way to capture ill-definedness is to assume that human preferences or utility functions are "fuzzy," in the technical sense of the term. This approach, which was in vogue at one point in time, does make for some leeway, but as is now evident, it brings about no fundamental change."

And still ideas such as the Eubulides or sorites paradox that Basu used in his 1984 paper to illustrate the relevance of fuzzy concepts in social sciences and economics seem as relevant as ever: if $n$ grains make a heap we must agree that $n - 1$ grains also make a heap, but it is very difficult then to say what amount of grains does not make a heap, although we will always agree that one grain does not make a heap.

In this brief chapter, an attempt is made to relate fuzzy logic to modern applied economics, without emphasizing a crisp split between applied and theoretical economics, quite the opposite. In Sect. 4.2, fuzzy preference theory is presented. Section 4.3 describes empirical techniques based on fuzzy sets. Section 4.4 suggests some fields of research in modern economics that could fruitfully be illuminated by a closer connection with fuzzy logic (and viceversa). And conclusions are briefly presented in Sect. 4.5.

## 4.2 Fuzzy Preferences and Rationality

Many of the important debates in modern economics have to do with the degree and nature of rationality of agents and with the role, nature and implications of individual preferences. This section summarizes the early contribution of fuzzy logic to these debates.

Following Basu [3], let $U$ be the universal (unfuzzy) set. $A$ is a *fuzzy* subset of $U$, or simply a *fuzzy* set, implies $A : U \to [0, 1]$. For all $x \in U$, $A(x)$ is interpreted as the extent to which $x$ belongs to $A$. In economics and social sciences, many categories are fuzzy, for instance the set of "developing countries".

Fuzzy sets can be applied to relations between alternatives. Again following Basu [3], let $X$ ($\#X < \infty$) be an exact set of alternatives, $R$ is a fuzzy binary relation (FBR) on $X$ means $R : X \times X \to [0, 1]$, i.e. $R$ is a fuzzy subset of $X \times X$.

In economics, a preference relation is a possible application of a fuzzy relation. As Aliev [1] argues, the conventional preference frameworks miss a very important feature of human preferences: these are vague. He supposes for example that Robert

[1] See Basu [5], published just before he became chief economist of the World Bank.

wants to decide among two possible jobs, $a_1$ and $a_2$, based on the following criteria: salary, excitement and travel time. If for Robert salary is more important than the time issues and slightly more important than excitement then it may be difficult to him to compare these alternatives. The relevant information is too vague for Robert to clearly give preference to any of the alternatives. He may think that to some degree job $a_1$ is as good as job $a_2$ and, at the same time, that to some degree job $a_2$ is as good as job $a_1$.

I may clearly prefer FC Barcelona to R. Madrid, but I may only prefer Arsenal to Manchester City to some degree, and this degree may depend on context. We clearly need a theory of preference that encompasses both possibilities: clear preference and vague preference. In many cases, human preferences are instances of indeterminacy or vagueness.[2] Under some conditions, preferences can be represented by utility functions. If one adheres to the idea that individual preferences represent individual welfare (something that modern behavioural economics questions) these preferences can be aggregated to represent social welfare functions, representing criteria for social choice, giving different weights and relating in different ways the individual utilities.

Given two fuzzy sets, one can define and compute the difference between two fuzzy sets. Given that an *ordinary* or *exact* set (one where members either belong or do not belong to the set, without intermediate degrees of membership) is an extreme example of a fuzzy set, one can compute the distance between a fuzzy set and an exact set. That is, one can compute the degree of fuzziness of a set or relation.

In neo-classical economics, a preference relation rationalizes individual actions (that is, an economic agent is rational) if individual choices are consistent with an exact preference relation. According to Basu [3], a fuzzy revealed preference theory has the advantage that it allows us to think in terms of human beings as possessed of different degrees of rationality instead of the harsher notion of a partition of rational and irrational persons. Taking into account the possibility that preferences are fuzzy, a person can be described as rational if his choice function, $C(\cdot)$, could be thought of as the expression of an underlying fuzzy preference.

Basu [3] suggests that if an individual has a (fuzzy or exact) preference ordering but he fails to adhere to it, then we could always conceive of another person who has a fuzzier preference ordering and adheres to it and whose behaviour is identical to that of the first. The possibility exists, according to Basu, of thinking of individuals as possessing different degrees of rationality: if $C(\cdot)$ and $\widehat{C}(\cdot)$ are such that $R$ and $\widehat{R}$ are the least-fuzzy orderings (respectively) which rationalize them and $R$ is less fuzzy than $\widehat{R}$, then $C(\cdot)$ could be thought as more rational than $\widehat{C}(\cdot)$. Of course the step from degrees of fuzziness to degrees of rationality could presumably be done in ways different from those considered by Basu. But to some extent, this argument establishes that the measurement of fuzziness can contribute to the measurement of rationality. And although fuzzy preference theory (it is perhaps my wrong view that) has evolved more towards the analysis of normative issues (such as how to aggregate individual fuzzy preferences into aggregate social preferences), the measurement of

[2] See Piggins and Salles [15].

degrees and forms of rationality seems very promising from a more positive point of view.

It is thus important to think about potential positive applications of these measures of rationality. Thus the measurement of the distance to rationality can be used for example to choose a model in the table by Munro [12] in page 135, where he distinguishes models where either citizens or rulers can be fully rational or boundedly rational, and when they are boundedly rational they can be aware or unaware of it. Or one could even better think about an extended version of this table where a continuum of degrees of rationality is allowed. This table shows the implications for the choice of model to explain public policy depending on the degree of rationality of citizens and rulers, but where the degree of rationality is constrained to be of three discrete types: fully rational, boundedly rational with awareness and boundedly rational without awareness.

The two dimensionality of Munro's table prevents us from seeing the potential for expanding the insight to the existence of a richer set of agents, not only citizens and rulers but also passive citizens, active citizens, civil society, politicians, civil servants, technocrats… Also, citizens could be thought of as more or less rational depending on whether they are considered individually or collectively (using a variety of mechanisms): rational individuals can have irrational (i.e. non-transitive) preferences, and biased individuals may make unbiased judgments when considered together under some conditions (wisdom of the crowds). Combining different possibilities, it is possible to think of models where individuals with different preferences delegate ones in others or where it is possible to study phenomena such as the manipulation of preferences in behavioural political economy (the interaction of politics and economics).

## 4.3 Empirical Techniques

Ragin [16] explains how fuzzy logic can be applied to improve upon traditional variable-oriented methodologies in empirical social sciences. His proposed methodology consists of measuring degrees of belonging to social categories and then exploring the relationship between categories using these measures. For example, generosity of the welfare states as related to presence of left parties, trade unions and homogeneous populations.

Many social science concepts do not involve simple dichotomous judgments, but complex categories with partial membership. These categories can then be related to explore necessity: the outcome is then a sub-set of the cause. Or to explore sufficiency: the cause is a sub-set of the outcome. Necessity and sufficiency are not distinguished with variable oriented techniques, but they are important for policy intervention. If you remove something necessary you avoid the outcome.

García Castro et al. [8] apply Ragin's methodology to the analysis of the relationship between corporate governance and firm performance. The important academic debate on this issue has centered on the possibility of different bundles of governance features being conducive to high performance in different societies. It is part of what has been called the "varieties of capitalism" literature. In this field, the notion of *complementarities* has an important role. Following García Castro et al. [8], complementarities are defined as situations where the difference in utility between two alternative practices $U(X') - U(X'')$ increases for all actors in the domain $X$, when the practice of $Z'$ prevails over practice $Z''$. That is, for complementarities to arise, the value of practice $X$ is higher when practice $Z$ is also present in a given domain. However, bundles of governance practices are selective in the sense that there is a point of decreasing marginal returns in organizational homogeneity.

García Castro et al. [8] demonstrate that there are multiple bundles of firm-level corporate governance practices leading to high firm performance. Governance practices in these bundles do not relate to each other in a monotonic and cumulative fashion because this would lead to higher costs and over-governance, and bundles may have hybrid characteristics conducive to high performance that had not been uncovered using conventional empirical methods.

The advantages of fuzzy empirical methods over variable-oriented methods include that the latter are unable to distinguish which independent variables are necessary, sufficient or redundant in affecting the dependent variable. García Castro et al. [8] add to this that fuzzy set techniques are a better measuring tool than conventional index construction techniques such as factor analysis because these present the limitation of treating each configuration as a black box, and hardly make explicit how each of the elements in a configuration combines with the others to achieve the outcome.

Ragin [16] claims that the use of fuzzy empirical techniques facilitates a better dialogue between theory and data, because fuzzy techniques do not work with ready made data but they construct their own data sets by using substantial and previous knowledge to measure the degrees of membership to categories. Fuzzy sets are better as tools of discovery than ready-made variables in conventional data-sets.

## 4.4 Relationship with Modern Economics

In this section I provide some vague thoughts on how fuzzy logic could contribute to some modern branches of the economics literature. Following most of the guide of Nicholas Stern,[3] a prominent economist involved both in academia and in public policy, in his speech as president of the European Economic Association a few years ago, I identify three fields where economics is making a lot of progress in the recent past and which depart in significant ways from traditional economic thinking. These fields are: behavioural economics, the economics of institutions and new theories of justice. I add one further field, not considered by Stern, which is agent-based

[3] See Stern [20]. He is one of the most prominent scholars in the study of the economics of climate change.

computational economics, because it seems complementary of some of these approaches and also has important connections with fuzzy logic.

### *4.4.1 Behavioral Economics*

As explained in the Introduction above, behavioural economics replaces the assumptions of rational and exogenously individualistic behaviour by an assumption that individual behaviour is presided by the mental shortcuts and predictable biases that real individuals show, according to research in empirical and social psychology. Decisions depend on framing, preferences are reference dependent, and the communication based on natural language (intuitions, vagueness, fast thinking) matters. Fuzzy logic seems complementary to this approach, to the extent that it provides a more realistic framework on which to base assumptions on preferences. In addition to this, to the extent that behavioral economics separates the measurement of individual *preferences* from the measurement of individual *welfare* (I may prefer something that is not good for me), fuzzy logic could help in providing a more realistic metric for both of them, and therefore analyze with more precision the relationship between them. Public policies could then be designed to fight the cognitive problems that prevent voters from having a real sense of the importance of phenomena such as climate change and other global challenges.

### *4.4.2 Institutional Economics*

Formal and informal rules that constrain and structure our social behaviour (institutions) are recognized as important in attempting to solve social dilemmas (cooperation and coordination problems) in our society. Aoki [2] for example stresses the role that social norms based on cognitive salience play in the consolidation of social architecture. Our behaviour in repeated games depends on cognitive representations, where languages are instrumental in a variety of domains: firm, market, politics, communities. Sugden [22] claims that institutions (such as property rights) make use of pre-existing and psychologically salient relations (such as first possession, or long possession). Salience is key in the emergence of conventions. The coordination of identities develops into interactive sense-making for example in Hermann-Pillath [10], where fuzzy property rights as a state (etics) result of the sequence elements, patterns, processes, creating a new element, which will create a new pattern via the sense-making activities of individual cognitive schemes (emics). Fuzzy logic could play an important role in better describing the function of natural language in cognitively structuring representations in our social games.

### *4.4.3 New Theories of Justice*

Given the difficulties of deriving welfare conclusions from individual preferences, Sen proposes not to use these preferences as a metric for welfare, and instead replace them by the measurement of capabilities that make possible a free life: health, education, social structures. The measurement of these components of a minimum living standard could be done using fuzzy empirical techniques, as suggested by Basu [4]. More in general, fuzzy logic could contribute to the development of new measurement tools to replace traditional metrics such as GDP (gross domestic product) as suggested by Stiglitz [21]. Traditional metrics have been criticized for not giving an accurate picture of the human welfare of the majority of populations. Relatedly, fuzzy logic can contribute to improving the measurement of qualitative features that are difficult to measure using conventional techniques, but that play an important role in the theory of incentives or in statistics (see Silver [19], for example). Sen ([17, 18]) are full of insights of how realistic concepts of justice and identities need to be more complex than the ones assumed in traditional social sciences.

### *4.4.4 Complexity and Agent-Based Models*

The economy as a complex adaptive system can be analyzed (Beinhocker [6]) using agent-based models where the dynamics of social systems is described by endowing agents with simple behavioural rules that combine with uncertainty. Agent-based models using fuzzy preferences have been used for example to analyze penalty kicks in soccer by Vu et al. [23], although again in this case with no communication (for example, no cross reference) to empirically successful attempts to analyze the same phenomenon using more traditional tools such as mixed-strategy Nash equilibrium of games with two pure strategies as in Palacios-Huerta [14].

In all these fields fuzzy logic could help not only with decisions, but also with strategic interactions: coordination/cooperation games, conventions for solving problems, games where communication plays some role. In all these fields models that incorporate fuzzy logic could:

- Test the robustness of traditional insights or tools to fuzzy assumptions, as has been done with the Gini coefficient in inequality by Basu [4].
- Explain behaviour that seems difficult to reconcile with binary relations, such as solutions to adverse selection problems (Georgescu [9] presents this and other related applications).

## 4.5 Conclusions

This paper has suggested some fields where a better dialogue between fuzzy logic and positive economics could be fruitful. Kahneman [11] argues that most of our thinking takes place with a cognitive system that is fast and intuitive, instead of slow and systematic. Similarly, Aliev [1] argues that the prevailing amount of relevant information is carried not at a level of objective measurements but at a level of subjective perceptions which are intrisically imprecise and are often described in natural language.

Merging behavioural sciences or at least improving the dialogue across disciplines we could take advantage of useful insights that have been developed quite independently but that point into a similar direction. Bowles [7] explains how we know very little about how to achieve coordination and cooperation (although we humans are not the only animals that try to achieve them), which are key to solve many of the complex social dilemmas of the XXI century. Ostrom [13] reflected on how the analysis of these dilemmas needs to go beyond the use of discrete categories such as market or state and rational versus irrational.

In the last century economics has made progress by explaining partially the workings of very simplistic market economies. The next step should be to analyze the dynamics of socio-political systems with boundedly rational players that navigate with fuzzy categories, where citizens as voters and policy makers interact with citizens as they behave in the domains of markets and communities.

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# Chapter 5
# Linguistic Summaries of Time Series: A Powerful and Prospective Tool for Discovering Knowledge on Time Varying Processes and Systems

**Janusz Kacprzyk and Sławomir Zadrożny**

**Abstract** We provide a critical state of the art survey of linguistic data summarization in its fuzzy logic based version, meant as a process for a comprehensive description of big and complex data sets via short statements in natural language. These statements are represented by protoforms in the form of linguistically quantified propositions dealt with using tools and techniques of fuzzy logic to grasp an inherent imprecision of natural language that is very difficult, if not impossible, for traditional natural language generation related approaches to linguistic summarization. Such linguistic data summaries can provide a human user, whose only natural means of articulation and communication is natural language, with a simple yet effective and efficient means for the representation and manipulation of knowledge about processes and systems. We concentrate on the linguistic summarization of dynamic processes and systems, dealing with data represented as time series. We extend the basic, static data oriented concept of a linguistic data summary to the case of time series data, present various possible protoforms of linguistic summaries, and an analysis of their properties and ways of generation. We show two our own real applications of the new tools of linguistic summarization of time series, for the summarization of quotations of an investment (mutual) fund, and of Web server logs, to show the power of the tool. We also mention some other applications known from the literature. We conclude with some remarks on the strength of the linguistic summarization for broadly perceived data mining and knowledge discovery, emphasize its potentials, and outline some possible further research directions, being strongly convinced that the fuzzy logic based approach to linguistic summarization of time series is one of more important areas in which fuzzy logic can play a crucial role in the years to come.

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_5

**Keywords** Linguistic summarization · Natural language · Fuzzy logic · Linguistic quantifiers · Data mining · Knowledge discovery · Big data sets

## 5.1 Introduction

The very purpose of this important volume is to present various opinions on what is, or will be, a promising fuzzy logic based solution, technology, application, etc. in the next decades to come. Many different views are here possible, and the main distinguishing factor is the very field or area of science or technology in which the particular authors are interested. It can be said without exaggeration that most people who are active in broadly perceived fuzzy logic work in applications, and most of their areas of interest and activities concern some systems and processes. They can be meant in various ways but all definitions involve some sort of a structural characterization, exemplified by components, their interconnections, architecture, processes involved, etc. and a characterization of behavior, both in terms of particular components and the particular processes involved, or even the system as a whole.

An overwhelming majority of real processes and problems which occur in nontrivial, real world systems, are dynamic and should be analyzed by taking into account their time varying behavior. This also concerns all kinds of approaches aimed at discovering information and/or knowledge about the processes and problems considered, as well as the whole systems.

As a result of the recent development of technology that concerns more and more widely used and accessible systems, a human being is more and more often a crucial element of virtually all complex processes and problem solving tasks as the research interest moves towards more and more complex systems. This implies a wider and wider gap between the human being, who is much better than the computer in sophisticated tasks but much worse than the computer in sheer number crunching. A human-centric systems paradigm initiated by Dertouzos [12] and developed by, e.g., Pedrycz and Gomide [31] has therefore been proposed to find a way to manage in such difficult environments of systems with a crucial human element. Basically, the very idea of this approach, or maybe better to say, philosophy, is that an artificial interface between the computer and the human being should be removed. In our context this would boil down to a possibly wide use of natural language for the representation of the information/knowledge extracted, conclusions, drawn, etc. The reason is simple, namely for the humans natural language is the only fully natural means of communication and articulation.

There may be many possible ways of using natural language in all kinds of descriptions, analyses, etc. of various systems and processes, both related to their structure and architecture, and to their behavior, that is, processes that occur within them. In this paper we will deal with *linguistic data summaries* in the sense of Yager [37], in their more implementable form proposed by Kacprzyk and Yager [20] and Kacprzyk et al. [21] in which, in the static case, we have a (relational) database that is too large to be comprehended by the human being and therefore we wish to find a short and comprehensible linguistic summary of its contents as a short linguistically quantified proposition in the sense of Zadeh [42]. For instance, in the case of a personnel database, a linguistic summary with respect to “age” and “salary” can be *most of young employees earn low salaries*. Notice that an explicit use of a (relational) database does provide universality of the approach as, in practice, all kinds of data on real processes and systems are almost always stored in relational database because this is a mature technology so that the data mentioned can be handled in an effective and efficient way.

This basic concept of a static linguistic summary mentioned above has been extended to the dynamic case, specifically to *linguistic summaries of time series* by Kacprzyk et al. [16]. By first performing a segmentation of the time series data into trends, i.e., parts of the time series exhibiting a uniform behavior, we obtained linguistic summaries like “most of slowly decreasing trends have a large variability”, “almost all of trends with a high variability are sharply decreasing”, etc. This approach was extended in, e.g., Kacprzyk et al. [18, 19], showing an application to investment (mutual) fund quotations, and in Zadrożny and Kacprzyk [43, 44] for Web log analyses. Then, many other approaches have been proposed as, e.g., Alvarez et al. [2], Batyrshin and Sheremetov [6, 7], Castillo et al. [9, 10], Anderson et al. [3], Ros et al. [32] for various application areas as, e.g., elderly care, traffic analyses, etc. Notice that this extension is extremely relevant because dynamics is what is crucial in virtually all analyses of processes and systems.

Obviously, the generation of linguistic summaries, even static, not to mention dynamic ones, can be a serious problem and has rarely been addressed except for Kacprzyk and Zadrożny [22] and their subsequent papers in which the generation is considered in the context of fuzzy database querying, Kacprzyk and Zadrożny’s [23] use of Zadeh’s protoforms representing linguistic summaries, Kacprzyk and Zadrożny’s [26] proposal to derive (generate) linguistic summaries via association rule mining, Kacprzyk and Zadrożny’s [25] proposal to generate linguistic summaries using natural language generation (NLG), and Kacprzyk and Zadrożny’s [24] proposal to generate linguistic summaries by using some elements of systemic functional linguistics (SFL).

As we can see, linguistic summaries, notably of time series data, are very powerful and can operationally be derived. We strongly believe that they will play an increasing role in the area of applications of fuzzy logic, and they should be included in this volume the purpose of which is, among others, to show some prospective research directions and solutions.

We will now briefly present the concept of a linguistic data summary, first of static and then dynamic (time series) data. Then, we will mention two examples of applications, for the summarization of investment (mutual) fund quotations and Web logs.

## 5.2 Linguistic Data Summaries: A Static and Dynamic Case

In its very essence, Yager's [37] source concept of a *linguistic data summary* concerns: $Y = \{y_1, y_2, \ldots, y_n\}$, the set of objects (records) in the database $D$ as, e.g., a set of employees; $A = \{A_1, A_2, \ldots, A_m\}$, the set of attributes (features) characterizing the objects from $Y$ as, e.g., a salary or age. A linguistic (data) summary includes:

- a summarizer $P$, i.e. an attribute together with a linguistic value (fuzzy predicate) defined on the domain of attribute $A_j$ (e.g., *low* for the attribute *salary*);
- a quantity in agreement $Q$, i.e. a linguistic quantifier (e.g., *most*);
- a truth value (validity) $\mathcal{T}$ of the summary, i.e. a number from [0, 1] yielding the truth (validity) of the summary (e.g., 0.7);
- optionally, a qualifier $R$, i.e. another attribute together with a linguistic value (fuzzy predicate) defined on the domain of attribute $A_k$ determining a (fuzzy) subset of $Y$ (e.g., *young* for attribute *age*).

A linguistic data summary in that sense may be exemplified by the simple form

$$\mathcal{T}(most \text{ of employees earn } low \text{ salary }) = 0.7 \tag{5.1}$$

or by an extended form, including a qualifier (e.g., *young*), by

$$\mathcal{T}(most \text{ of } young \text{ employees earn } low \text{ salary}) = 0.82 \tag{5.2}$$

The core of a linguistic summary is therefore a linguistically quantified proposition in the sense of Zadeh [42] which for (5.1) may be written as

$$Qy's \text{ are } P \tag{5.3}$$

and for (5.2) may be written as

$$QRy's \text{ are } P \tag{5.4}$$

The truth value (validity), $\mathcal{T}$, of the above simple and extended summary, (5.3) and (5.4), are then, respectively:

$$\mathcal{T}(Qy's \text{ are } P) = \mu_Q\left(\frac{1}{n}\sum_{i=1}^{n}\mu_P(y_i)\right) \tag{5.5}$$

$$\mathcal{T}(QRy's \text{ are } P) = \mu_Q\left(\frac{\sum_{i=1}^{n}\mu_P(y_i)\wedge\mu_R(y_i)}{\sum_{i=1}^{n}\mu_R(y_i)}\right) \tag{5.6}$$

where $\wedge$, the minimum operation, may be replaced by, e.g., a $t$-norm, if needed, and $Q$ is a fuzzy set representing the fuzzy linguistic quantifier like, for instance,

$$\mu_Q(x) = \begin{cases} 1 & \text{for } x \geq 0.8 \\ 2x - 0.6 & \text{for } 0.3 < x < 0.8 \\ 0 & \text{for } x \leq 0.3 \end{cases} \tag{5.7}$$

Other methods of calculating $\mathcal{T}$ can also be used, notably those based on the OWA (ordered weighted averaging) operators (cf. Yager [38, 39], Yager and Kacprzyk [40], and Yager et al. [41]), and the Sugeno and Choquet integrals (cf. Bosc et al. [8] or Grabisch [14]).

It is easy to see that the above concept of a linguistic (data) summary does provide an extremely powerful, yet simple, tool to represent many real world relationships in a form that is easily comprehensible to the human being.

As we have already mentioned, in virtually all cases of real world processes and systems dynamics is what really matters. The above approach can easily be extended to the dynamic case, i.e., to the summarization of time series, as proposed by Kacprzyk et al. [16], and then considerably developed in, e.g., Kacprzyk et al. [18, 19], to name a few. Basically, in that approach the linguistic summarization of a time series is performed as the linguistic summarization of the trends (segments) extracted from the time series. First, we assume a piecewise linear representation of time series data, and we extract segments, i.e., the constituent straight lines that represent an uniform behavior of the data. This can be done by using, for instance, on-line (sliding window) algorithms, bottom-up or top-down strategies (cf. Keogh et al. [28, 29]) or, as in our works, using a modification of the Sklansky and Gonzalez [34] algorithm.

The following basic features of trends in the time series are considered:

1. dynamics of change,
2. duration, and
3. variability,

meant as: the *dynamics of change* is the speed of change of the consecutive values of the time series which may be described by the slope of a line representing the trend, then represented by a linguistic variable, the *duration* is the length of a single trend, also represented by a linguistic variable and the *variability* describes how "spread out" a group of data within a segment is. The use of a small set of granulated linguistic labels as, e.g.: quickly increasing, increasing, slowly increasing, constant, slowly decreasing, decreasing, quickly decreasing, equated with fuzzy sets, is employed.

Then, we basically employ *protoforms* (abstract prototypes, or templates, of a linguistically quantified propositions) of linguistic summaries as proposed by Kacprzyk and Zadrożny [23], exemplified by

- for the simple form:

$$\text{Among all segments, } Q \text{ are } P \tag{5.8}$$

  e.g.: "Among all segments, *most* are *slowly increasing*".
- for the extended form:

$$\text{Among all } R \text{ segments, } Q \text{ are } P \tag{5.9}$$

  e.g.: "Among all *short* segments, *most* are *slowly increasing*".

Moreover, we can further enhance the extended protoforms given in (5.8) and (5.9) by adding a temporal expression, $E_T$, like: "recently", "initially", "in the very beginning","in the early Spring of 2010", etc., which yields the *temporal protoforms*:

- for the case of the simple protoform:

$$E_T \text{ among all segments, } Q \text{ are } P \tag{5.10}$$

  e.g.: "*Recently*, among all segments, *most* are *slowly increasing*".
- for the case of the extended protoform:

$$E_T \text{ among all } R \text{ segments, } Q \text{ are } P \tag{5.11}$$

  e.g.: "*Initially*, among all *short* segments, *most* are *slowly increasing*".

The truth (validity) of those linguistic summaries is calculated similarly as in the static case, and we obtain for the simple and extended protoform, respectively:

$$\mathcal{T}(\text{Among all } y\text{'s, } Q \text{ are } P) = \mu_Q\left(\frac{1}{n}\sum_{i=1}^{n}\mu_P(y_i)\right) \tag{5.12}$$

$$\mathcal{T}(\text{Among all } Ry\text{'s, } Q \text{ are } P) = \mu_Q\left(\frac{\sum_{i=1}^{n}\mu_R(y_i)\wedge\mu_P(y_i)}{\sum_{i=1}^{n}\mu_R(y_i)}\right) \tag{5.13}$$

where $\wedge$ is the minimum operation or, more generally, a $t$-norm.

The computation of the truth values of temporal summaries is very similar to the previous case, and—basically—we obtain for the simple temporal protoform summary (5.10) and for the extended temporal protoform summary (5.11), respectively:

$$\mathcal{T}(E_T \text{ among all } y\text{'s, } Q \text{ are } P) = \mu_Q\left(\frac{\sum_{i=1}^{n}\mu_{E_T}(y_i)\wedge\mu_P(y_i)}{\sum_{i=1}^{n}\mu_{E_T}(y_i)}\right) \tag{5.14}$$

where $\mu_{E_T}(y_i)$ is the degree to which a trend (segment) occurs during the time span described by $E_T$;

$$\mathcal{T}(E_T \text{ among all } Ry\text{'s}, Q \text{ are } P) = \mu_Q \left( \frac{\sum_{i=1}^{n} \mu_{E_T}(y_i) \wedge \mu_R(y_i) \wedge \mu_P(y_i)}{\sum_{i=1}^{n} \mu_{E_T}(y_i) \wedge \mu_R(y_i)} \right) \tag{5.15}$$

and for technicalities we refer the reader to our papers cited above.

Naturally, this is not the only way to calculate that degree. For instance, we may also use the OWA (ordered weighted averaging) operators (cf. Kacprzyk et al. [17]). One can also employ some other aggregation methods exemplified by those using the Sugeno integral (cf. Bosc and Lietard [8]) or the Choquet integral (cf. Grabisch [14]).

Moreover, it should be noticed that the degree of truth has been assumed as the single quality criterion in the first pioneering paper on linguistic data summaries by Yager [37] just for simplicity which is crucial for a paper that is meant to propose a novel concept. In next years in the papers by Kacprzyk and Yager [20] and Kacprzyk et al. [21] some new quality criteria have been proposed. Obviously, the analysis and, above all, the generation of such linguistic summaries becomes much more difficult but we obtain a much more adequate representation by a "best" summary (with respect to many quality criteria, not only its truth value) of the very essence of relationships occuring in data.

The use of multiple quality criteria for linguistic summaries, notably the degrees of: imprecision, specificity, fuzziness, covering, focus, appropriateness, informativeness, and the length of the summary, proposed conceptually in earlier works by Kacprzyk and Yager [20], Kacprzyk et al. [21], and then extended into the time series summarization, e.g., in Kacprzyk and Wilbik [15], is very important but needs much more research, and also implies much more difficult a process of summary generation because this should proceed with respect to multiple criteria.

## 5.3 Examples of Applications

Now, we will show two examples of our works on the linguistic summarization of: (1) daily quotations of a mutual (investment) fund that is to invest at least 66 % assets in shares listed at the Warsaw, Zagreb and Moscow Stock Exchanges, and (2) Web logs of the institute server. As we will see, different protoforms of linguistic summaries have been employed which is implied by the specificity of the problem. These examples will, first, show the very essence of the linguistic data summaries in the form considered here, and second, clearly indicate their prospective potentials making them an important candidate for a new set of prospective future tools and techniques in what might be called fuzzy technology.

### 5.3.1 Linguistic Summarization of Investment (Mutual) Fund Quotations

In the first example, the time series of daily quotations of an investment (mutual) fund was considered over the period from January 1, 2002 to the December 31, 2010; notice that in the period considered there were some economic disturbances in Europe. The value of one share was PLN 12.06 in the beginning and PLN 40.52 at the end of the time span (PLN stands for the Polish Zloty). The minimal value recorded was PLN 9.35 while the maximal one during this period was PLN 57.85. The biggest daily increase was PLN 2.32, while the biggest daily decrease was 3.46. Using the piecewise linear segmentation we obtained 100 segments from 1 to 191 days long. Almost 50 % of segments were shorter than 10 days, with very few segments longer than 50 days.

We used the granulation of 5 labels for each attribute, like very short, short, medium and long, very long for the duration, and similarly for other criteria.

Many linguistic summaries have been generated, taking into account, for instance, what a particular user has been interested in, and what has been comprehensible to him/her. Some examples of the linguistic summaries obtained were:

- the simple summaries:
  - Among all slowly decreasing segments, most are short,
  - Among all long and constant segments, most are of very high variability,
  - Among all very long segments, most are constant,
  - Among all of moderate variability segments, most are short,
  - Among all very long segments, most are constant and of very high variability,
- the summaries which reflect a temporal aspect:
  - Recently, among all slowly decreasing segments, most are short,
  - Recently among all long and of very high variability segments, most are constant,
  - Recently among all short and of very high variability segments, most are slowly decreasing,
  - Recently among all very long and of very high variability segments, most are constant,
  - Recently among all very long segments, most are constant and of very high variability,
  - Recently among all slowly increasing segments, most are of very high variability.

### 5.3.2 Linguistic Summarization of Web Server Logs

The second application concerns the linguistic summarization of Web server logs and was proposed by Zadrożny and Kacprzyk [43, 44]. That application was motivated by the fact that Web servers are crucial elements of virtually all IT systems in all kinds of institutions, organizations, companies, etc. and it may be very important to use some advanced tools and techniques for their design and running. Broadly perceived soft computing, or better to say, computational intelligence tools and techniques have been employed for that purpose, cf. [1, 4, 5, 11, 30, 35, 36]. However, the approaches cited, as well as virtually all other ones, have not explicitly used any natural language based method as we have proposed in our approach.

The idea of our approach, from the viewpoint of soft computing, is similar to Abraham [1, 35, 36] who deals, using fuzzy clustering, evolutionary algorithm, neural networks and the Takagi-Sugeno type fuzzy systems. On the other hand, their access trends analysis is similar in spirit to the dynamic linguistic summaries used by us though not explicitly natural language focused. Moreover, Shiu and Wong [33] approach is also somehow related to our approach due to their use of the fuzzy association rules which are employed for the generation of a subclass of linguistic summaries (cf., our work on that topic [27]). However, none of those works uses a human consistent natural language based approach.

Basically, each request to a Web server is recorded in one or more log files that contain the following main fields: the requesting computer name or IP address, the username of the user triggering the request, the user authentication data, the date and time of the request, the HTTP command related to the request which includes the path to the requested file, the status of the request, the number of bytes transferred as a result of the request, the software used to issue the request.

One may use an extended format with more fields but it is beyond the scope of this paper.

By using similar linguistic summarization algorithms, we obtained both static and dynamic summaries, which may be briefly exemplified as follows. First, for the static case:

- for the case of the *simple summaries*:
  - *Most* of the requests come from the Firefox browser
  - *Almost all* requested files are *small*
- for the case of the *extended summaries*:
  - *Almost all* failures concern files with an extension “ppt”
  - *Most* of the requests concerning *large* files happen in the *evening*.

What concerns the dynamic summaries, i.e., concerning time series of requests (their trends), we can quote the following examples:

- for the case of the *simple summaries*:
  - *Most* of the trends concerning the number of requests are *decreasing*
- for the case of the *simple summaries* which account for the duration:
  - Trends concerning the number of requests that took *most time* are *slowly increasing*

- for the case of the *extended summaries* which account for the frequency of event occurrence:
  - *Most* of *increasing* trends concerning the number of requests are of *high* variability
- for the case of the *extended summaries* which account for the duration of event occurrence:
  - *Increasing* trends concerning the total size of requested files, that took *most* of the time, are *very long*

It is easy to see that in both examples of real applications the linguistic summaries derived have provided much insight into the very essence of the set of data considered. They can be of a valuable help in making decisions.

## 5.4 Concluding Remarks

To respond to the very purpose of this volume, that is, a wide coverage of various possible directions, tools, techniques, areas of application, etc. of fuzzy logic which can be considered to be promising and prospective in the decades to come, we have presented the concept of a linguistic data summary of both static and dynamic (time series) data. This concept and its related tools and techniques have, in our opinion, an extremely high potential to become of top application examples of fuzzy logic, maybe one of "killer apps", due to its generality, power and an extreme human consistency. We briefly presented the concept of a linguistic summary of numeric data which boils down to a linguistically quantified proposition that is dealt with using fuzzy logic tools and techniques. This makes it possible, first, to provide a short and comprehensive, yet very useful representation of information and knowledge exhibited by the (large set of) data in questions in a short natural language form. Second, via the use of fuzzy logic, an inherent imprecision of natural language can be effectively and efficiently handled. This all is clearly a considerable step towards human consistent and human centric computing that can help bridge an inherent gap between the human being and the computer. We also believe that as an important topic for a further in depth research in that area should be an explicit inclusion in the very definition, analysis and generation of linguistic summaries powerful tools and techniques of natural language generation (NLG) and systemic functional linguistics (SFL), as outlined in our papers—cf. Kacprzyk and Zadrożny [24, 25], respectively.

Other directions of research, which have been already initiated, include further studying the inference and dependencies between summaries, handling sets of summaries or designing employing other protoforms of linguistic summaries leading to, e.g., bipolar summaries introduced by Dziedzic et al. [13].

We showed two real applications of linguistic summaries of time series data, for daily quotations of an investment mutual fund, and for Web server logs, listing some more interesting linguistic summaries of various types (protoforms) that could provide much insight and information which might be useful for the analysis of the problems considered.

**Acknowledgments** This work was supported by the National Centre of Science under Grant No. UMO-2012/05/B/ST6/03068.

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# Chapter 6
# Granular Geometry

**Gwendolin Wilke**

**Abstract** Many approaches have been proposed in the fuzzy logic research community to fuzzifying classical geometries. From the field of geographic information science (GIScience) arises the need for yet another approach, where geometric points and lines have granularity: Instead of being "infinitely precise", points and lines can have size. With the introduction of size as an additional parameter, the classical bivalent geometric predicates such as equality, incidence, parallelity or duality become graduated, i.e., fuzzy. The chapter introduces the Granular Geometry Framework (GGF) as an approach to establishing axiomatic theories of geometries that allow for sound, i.e., *reliable*, geometric reasoning with points and lines that have size. Following Lakoff's and Núñez' cognitive science of mathematics, the proposed framework is built upon the central assumption that classical geometry is an idealized abstraction of geometric relations between granular entities in the real world. In a granular world, an ideal classical geometric statement is sometimes wrong, but can be "more or less true", depending on the relative sizes and distances of the involved granular points and lines. The GGF augments every classical geometric axiom with a degree of similarity to the truth that indicates its reliability in the presence of granularity. The resulting fuzzy set of axioms is called a granular geometry, if all truthlikeness degrees are greater than zero. As a background logic, Łukasiewicz Fuzzy Logic with Evaluated Syntax is used, and its deduction apparatus allows for deducing the reliability of derived statements. The GGF assigns truthlikeness degrees to axioms in order to embed information about the intended granular model of the world in the syntax of the logical theory. As a result, a granular geometry in the sense of the framework is sound by design. The GGF allows for interpreting positional granules by different modalities of uncertainty (e.g. possibilistic or veristic). We elaborate the framework for possibilistic positional granules and exemplify it's application using the equality axioms and Euclid's First Postulate.

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_6

## 6.1 Introduction

Many approaches have been proposed in the fuzzy logic research community to fuzzify classical geometries. From the field of geographic information science (GIScience) arises the need for yet another approach, where geometric points and lines have granularity: Instead of being "infinitely precise", points and lines can have size. With the introduction of size as an additional parameter, the classical bivalent geometric predicates such as equality, incidence, parallelity or duality become graduated, i.e., fuzzy.

The chapter introduces the Granular Geometry Framework as an approach to establishing an axiomatic theory of geometry that allows for *reliable* geometric reasoning with points and lines that have size: In the presence of granularity, classical geometric statements are often wrong, and and it depends on the specific geometric configuration "how wrong" they are [35, 72, 73]. A granular geometry in the sense of the framework augments every classical geometric statement with a degree of reliability, and the reliability degree is propagated in geometric reasoning. The Granular Geometry Framework extends the work of Roberts [51] and Katz [32] by using Łukasiewicz Fuzzy Logic with Evaluated Syntax [41], $FL_{ev}$(Ł∀), as a background logic for representing and propagating the degree of reliability. $FL_{ev}$(Ł∀) allows for treating the reliability degree as an intrinsic part of the syntax of a logical formula and guarantees that reasoning in granular geometry is sound.

Following L. Zadeh's Restriction Centered Theory of Truth and Meaning [83] (RCT), points and lines with size can be seen as positional restrictions of the geographic space, and, in the style of Zadeh's theory of Z-valuations [82], a granular geometry can be seen as a precisiation of a reliability calculus for positional restrictions. In RCT, restrictions can have different modalities of uncertainty, and this is also the case for granular geometries. For instance, a "positional restriction of type point" may be precisiated by a possibility distribution, a probability distribution, or a verity distribution, and the Granular Geometry Framework provides a guideline for precisiating the geometric theory accordingly.

As a first step towards a *possibilistic* granular geometry, we apply the Granular Geometry Framework to the equality axioms and to Euclid's first postulate. We show exemplarily how the introduction of size in classical geometries causes a fuzzification of classic geometric predicates. Borrowing a terminology commonly used in the GIScience literature, we call the resulting theory an *Approximate Tolerance Geometry* [73–75].

The chapter is structured as follows: Sect. 6.2 gives a motivation for the introduction of granular geometries for geographic information systems (GIS) and reviews related work from the GIScience community and related fields. Section 6.3 introduces similarity logic as the underlying idea and main tool for incorporating a reliability measure in classical geometry, and—on this basis—develops the Granular Geometry Framework. Section 6.4 applies a possibilistic instantiation of the Granular Geometry Framework to the equality axioms and to Euclid's First Postulate, thus elaborating a first step towards an Approximate Tolerance Geometry. In Sect. 6.5, we briefly discuss the Granular Geometry Framework in the context of Zadeh's RCT. Section 6.6 finally gives a summery of results and discusses current limitations of the framework. We also give an outlook to future directions and further work.

## 6.2 Background

In Sect. 6.2.1, we briefly discuss the rationale of introducing a granular geometry, which stems from the area of geographic information science (GIScience) and gives simple examples of granular geometric configurations. Section 6.2.2 reviews existing approaches to handle granular geometric objects withing the GISience community and related research fields. Section 6.2.3 reviews related work from computer science and mathematics, with a special focus on axiomatic approaches.

### *6.2.1 Motivation*

As a motivation for the introduction of granular geometries for spatial analysis in geographic information systems (GIS), Fig. 6.1 illustrates three typical geographic scenarios: Part (a) shows a boat floating on a river. While the boat may be seen as a granular point, the river may be conceptualized as a granular line. We can say that the granular point approximately "sits on" the granular line. Using a geometric terminology, the granular point is *more or less incident* with the granular line. Part (b) shows a similar configuration. Here, a metro station building is shown, which is *more or less incident* with railway tracks. Part (c) shows a highway and a highlighted area that indicates an increased freuquency of car accidents in this sector of the

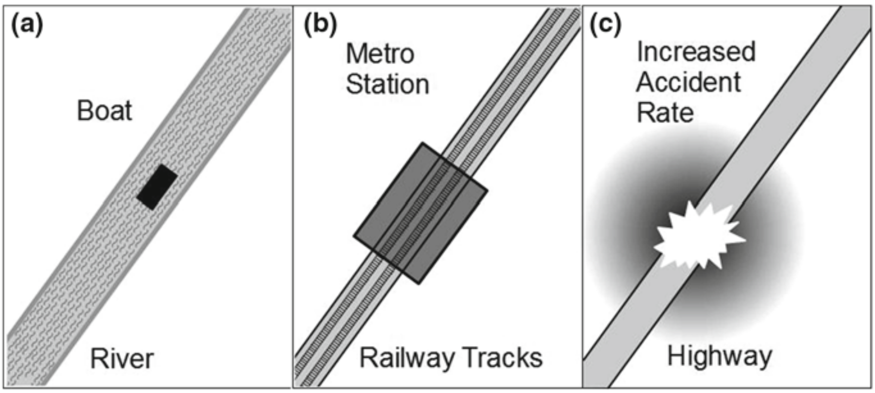


**Fig. 6.1** An example of **a** a veristic, **b** a possibilistic, and **c** a probabilistic interpretation of a granular point that is more or less incident with a granular line ((**a**) after [77])

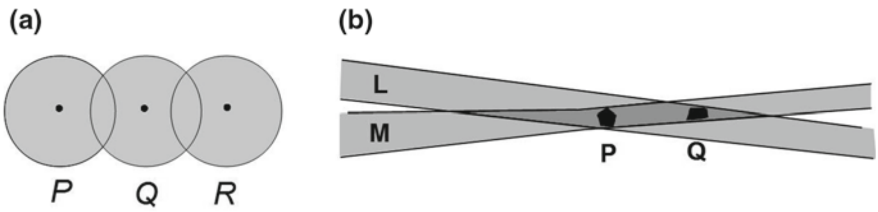


**Fig. 6.2** **a** Poincaré Paradox, **b** approximate direction

road. All three scenarios can be interpreted geometrically, namely as approximate incidence of a (granular) point with a (granular) line. The difference between the them is the modality of imperfection with which the granular features are modeled: The statement "The boat is on the river" refers to the boat as an indivisible entity, and we can model this by a (crisp) *verity* distribution. Here, every coordinate pair in the black rectangle is considered part of the boat with a degree of 1. In contrast, when I say "I am at the metro station", the metro station in (b) indicates the (crisp) set of coordinate pairs that *possibly* coincide with my true exact location at the time of speaking. Finally, in (c), the shaded area provides a color indication of the *probability* that I encounter a car accident when driving through this section of the highway.

The need for geometric reasoning with extended objects in GIS stems from a paradigm change that took place in GIScience in the recent years. While traditionally, imprecision in geographic data was considered an error, the proliferation of location based applications and crowd-sourced geographic data[1] raised the general awareness for the value of human-centered and perception based interaction with GIS applications. Today, it is considered a feature to offer GIS functionality for imprecise input. Since geometric reasoning lies at the core of the GIS vector data model and thereby also at the core of all derived spatial analysis capabilities, it is desirable to provide a geometric calculus for granular features in GIS. While most current approaches in the community are based on heuristics that where originally designed to deal with small imprecisions in traditional, well-maintained datasets, the proposed Granular Geometry Framework aims at providing a reliable granular geometric calculus that is reliable, even if the position granules involved are potentially very large.

Existing heuristic algorithms in GIS often map classical geometric predicates to the approximate geometric relations between granular points and lines. Yet, when classical geometric reasoning is applied to position granules, i.e., to points and/or lines with size, the result may be wrong. In other words, geometric reasoning with position granules is not reliable. A classical example of this fact is the *Poincaré Paradox* [46], which states that the equality of sensations and measurements in the physical continuum is not transitive. An instance of the Poincaré Paradox is illustrated in Fig. 6.2a.

[1] An well-known example for the use of crowed-sourced geographic data in participatory mapping is OpenStreetMap (www.openstreetmap.org).

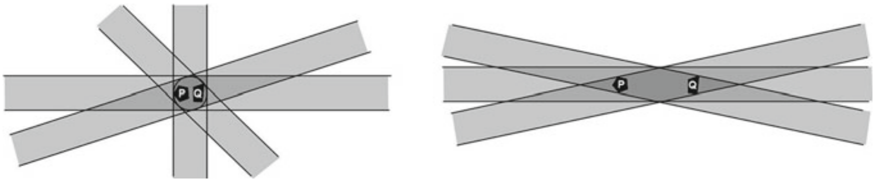


**Fig. 6.3** The "degree of uniqueness" of an approximate connection depends on the distance between the involved granular points [73]

A test person whose skin is stimulated in the spots $P$ and $Q$, e.g. by a sharp object, may not be able to distinguish the two approximate locations on their skin. In other words, in the person's perception, the two spots are *equal*. For the same reason, the person will also not be able to distinguish $Q$ and $R$, and therefore, they will also consider them as *equal*. Yet, since the spots $P$ and $R$ are not very close together, they can be distinguished easily, and the person will identify the relation between $P$ and $R$ as *not equal*. In other words, perceived equality is not always transitive. When we model the relation of perceived equality (i.e., of indistinguishability) by the classical (transitive) geometric equality relation, the result of the query "$P = R$?" may be wrong. In geographic information systems, these kind of problems arise as a result of, e.g., measurement inaccuracy, mapping error or overlay errors.[2] To the knowledge of the author a generic solution to the problem of sound, i.e., reliable, geometric reasoning with position granules in GIS has not been found as of yet.

With the introduction of size to classical geometries, classical crisp geometric notions become fuzzy. As an example, Fig. 6.2b shows two granular points $P$ and $Q$. The granular line $L$ is approximately incident with both, $P$ and $Q$, and so is the granular line $M$. What we observe is that, in contrast to classical geometry, "the" connecting line is not unique, and the direction specified by the two granular points $P$ and $Q$ is only an *approximate direction*.

Figure 6.3 illustrates exemplarily that the "degree of uniqueness" of "the" connecting granular line depends on the distance between the involved granular points. Figure 6.4 illustrates that it also depends on their size.

Notice that the fuzziness of geometric predicates is a result of the fact that the size of objects varies: If granular points have fixed size, there is a bijection from granular to exact geometry. The Granular Geometry Framework assumes variabel size of positional granules. It is an approach to parametrizing the fuzziness of classical geometric statements as a function of size and distance parameters.

Figure 6.5 illustrates exemplarily the fuzzification of the classic geometric notion of duality in granular geometries: In classical logic, every point can be seen as a pencil of lines, and every line can be interpreted as a range of points. In granular geometry, every (approximate) range of granular points specifies an approximate direction, i.e., it is more or less (granular) linear. Conversely, every (approximate)

[2] cf., e.g., [48].

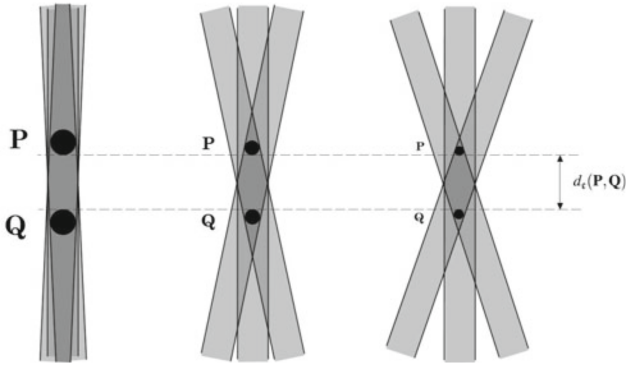


**Fig. 6.4** The "degree of uniqueness" of the approximate connection depends on the sizes of the involved granular points [73]

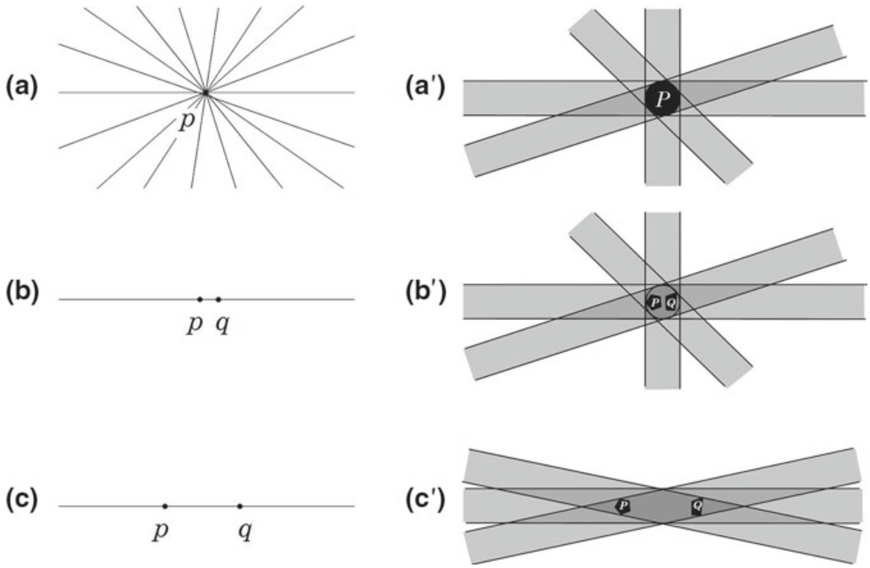


**Fig. 6.5** **a**–**c** Classical duality, **a**′–**c**′ Fuzzyfied duality [73]

pencil of granular lines specifies a an approximate location, which is more or less (granular) punctual.

## *6.2.2 Granulated Space in Geographic Information Science*

We discuss the increasing need for granular geometries in GIScience and give a brief overview over existing approaches from the community that relate to the problem.

### GIScience as a Mapping Science

The representation and propagation of error and uncertainty in measurement sciences like surveying is based on a well-developed mathematical foundation in measurement theory [23, 64]. Here, measurement inaccuracy can often be disregarded, "because it has been possible to devise increasingly more careful or refined methods of comparison for properties like weight and distance" [65, p. 299f.]. As opposed to surveying, which deals with physical measurements, geographic information science is not a measurement science, but a mapping science [25]. Mapping errors that arise, e.g., from geocoding [84], and the superposition of different data layers with different lineage and quality leads to geometrical discordance: different data layers represent what is meant to refer to the same entity in reality by different coordinate representations, thus introducing positional uncertainty of coordinate points and lines. The resulting uncertainty can often only be subsumed as possibilistic uncertainty and cannot be propagated with established statistical methods of Gaussian error propagation.

The need to deal with large positional uncertainties has been drastically increasing in the last years as lay users became prosumers by collecting geographic data (known as "volunteered geographic data", VGI [25]) in participatory GIS projects such as OpenStreetMap (OSM[3]). Here, the strict quality control of traditional mapping agencies does not apply, and the expert use of scale as a means to balance the granularity of representation with the sampling frequency of the available measurement points can not be taken for granted. Still, it is paramount to provide positional information that is sufficiently reliable for the end users, e.g., of OSM-based car navigation systems. To guarantee that the positional information that is derived by geometric construction from the original VGI measurement data is reliable enough for the respective application, a *sound* error calculus for possibilistic uncertainty is needed.

Besides possibilistic positional uncertainty, veristic uncertainty also becomes an issue in GI Science as the need for geometric reasoning with extended objects increases: A verity distribution [82] specifies the set of all positions, where a certain property applies, such as, e.g., the property of belonging to a certain registered place such as Picadilli Circus. With the proliferation of mobile computing, cheap GPS sensors, and natural language user interfaces such as Siri,[4] it is considered a feature to allow users of location based services to specify location on an appropriate level of detail instead of forcing them to be as precise and accurate as possible. An example is the natural language statement "My house is in the middle between Picadilli Circus and Leicester Square". Here, the geographic information system should be able to derive the approximate center point of the approximate line segment that connects Picadilli Circus and Leicester Square, including the corresponding uncertainty.

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[3] www.openstreetmap.org.

[4] www.apple.com/ios/siri.

In the following, we describe approaches from the GI Science community towards handling positional uncertainty in geometric reasoning. Most of these approaches are heuristics that are not geared towards handling large uncertainties and/or soundness of reasoning. When used in longer sequences of geometric constructions, they may produce highly unreliable or even completely wrong results.

### Models of Positional Granularity in GIScience

One of the earliest error models for representing point and line features with positional uncertainty is the epsilon band model introduced by J. Perkal [43, 44]. Here, every feature is associated with a zone of width epsilon around it. The zone can be associated with different modalities of imperfections, e.g., it may refer to the area in which the true feature possibly sits, or to a probability distribution for the position of the true feature with standard error epsilon. The epsilon band model gave rise to a huge amount of research in this direction, which today mainly focuses on statistical treatment of positional imperfection.[5] Another classical error model for line features is Peucker's 'Theory of the Cartographic Line' [45], which postulates thickness as an intrinsic characteristic of cartographic lines. The theory derives from research on the generalization of line features by D. Douglas and T.K. Peucker [15]. The Douglas-Peucker algorithm, which was independently suggested by U. Ramer [49], is still in use today.

In the GIS community, the representation of geographic regions such as mountains or informally defined places in a city as fuzzy sets is usually referred to by the term *vague regions* or *regions with broad boundaries*.[6] An approaches to reasoning with vague regions have been proposed, e.g., by A. Dilo [14]. Her work is based on fuzzy sets theory and defines topological and metrical operations for vague regions. In terms of geometric notions, these approaches use vague regions as point-like objects that have a fuzzy size, and extend point-operations to them. E. Clementini [13] extends the existing models for vague regions to linear (line-like) features that have extension. He defines the boundary of a line feature (i.e., a line segment) as its zero-dimensional boundary in $\mathbb{R}$, and broadens it by assigning to it a two dimensional extent in $\mathbb{R}^2$. Clementini's approach is not limited to fuzziness, but integrates different modalities of imperfections of positional information. From the field of remote sensing stems the approach of S. Heuel. His work is concerned with augmenting Grassmann-Cayley algebra (an algebraic approach to projective geometry that is widely used in remote sensing) with Gaussian probability density functions (pdfs). Heuel's calculus provides an error-propagation calculus for probabilistic uncertainty, yet, it does not represent the incomplete knowledge about geometric relations as an intrinsic part of the calculus, and soundness can not be shown directly.

[5] cf., e.g., [11, 36, 60–63].

[6] cf., e.g., [10].

Research on the quality of volunteered geographic information has not gained much interest yet, and the research on positional accuracy of VGI is even more sparse. One of the first papers that carried out a systematic analysis of VGI data quality is [28]. It is partly based on the work of N. Zulfiqar [86] on the positional accuracy of motorways in the UK using a buffer overlap analysis. C. Amelunxen [1] investigates the positional accuracy of OpenStreetMap data for the purpose of geocoding. M. Haklay [29] show that the positional accuracy of Open Street Map data increases with the number of contributors to the data set.

### *6.2.3 Fuzzy and Granular Geometries*

We review related literature from other research communities, in particular from computer science and mathematics. Here, we put particular emphasis on axiomatic approaches. We briefly introduce Robert's *tolerance geometry* and Katz' *inexact geometry* that both provide the basis for approximate tolerance geometry and the generic Granular Geometry Framework.

#### Digital Geometry

In the field of computer science, a predominant source of granulation of positional information is the discretization error that stems from the discrete representation and finite precision arithmetic used in digital image processing. From this problem field stems the research area of digital geometry. Digital geometry deals with the representation of geometric configurations as subsets of the discrete digital 2D and 3D space, and devises according geometric constructions and tests. It's main areas of application is digital image analysis and digital image processing. A prominent approach in digital geometry is the cell complex model of the digital plane that defines "digital points" as equivalence classes of points in $\mathbb{R}^2$. Derived from digital points are "digital lines", and problems of unsound reasoning arise from it: "The intersection of two digitized lines is not necessarily a digital point, and two digital points do not define a unique digital straight line, unless we introduce additional criteria to select such a line" [71, p. 100]. Based on the cell complex model, P. Veelaert [71] provides a mathematical framework that addresses the affine geometric relations of parallelism, collinearity, and concurrency in the digital plane. In contrast to the Granular Geometry Framework introduced here, Veelaert does not use an axiomatic approach, but instead shows that the digitized versions of affine geometric relations "can be verified by constructions that are still purely geometric, though slightly more complicated [...]" [71, p. 100]. He replaces the geometric relations by Helly-type properties, whose characteristic is to be not in general transitive and introduces a notion of *thickness* of digital points and lines to parametrize positional imprecision. Veelaert's approach differs from the one presented here in two main points: First, granular geometries are defined axiomatically and therefore meta-mathematical properties like soundness can be investigated. Second, Veelaert's approach is tailored to digital image processing, and consequently only applicable to positional error that results from discretization. In contrast to that, positional error in GIS stems from a large variety of sources, and it is necessary to provide a more general approach, where points and lines with size are not restricted to the constraints imposed by the structure of the digital plane. Yet another approach to capture a notion of granularity in geometric reasoning from the field of digital geometry is the 'Epsilon Geometry' proposed by D. Salesin, J. Stolfi, and L. Guibas.[7] Epsilon geometry is not only a model for error representation, but implements geometric constructions and tests.

### Fuzzy Geometry as a Part of Fuzzy Mathematics

In the mathematical literature, different approaches to fuzzifying classical geometry can be found, and we list some of them. One prominent field is *fuzzy geometry* as a subfield of *fuzzy mathematics*. Much of the early work in this area has been done by K. C. Gupta [26], A. Rosenfeld [53–56] J. J. Buckley and E. Eslami [8, 9], S.-C. Cheng and J. N. Mordeson [12]. Later work in this line is, e.g., H. Liu's and G. M. Coghill's fuzzy trigonometry [37]. An approach to propagating incomplete positional information in geometric reasoning that is based on fuzzy sets theory has been proposed by S. Dutta [16]. Following Zadeh's approach [79] of defining mathematical and logical operators on the semantic level, all these approaches have the disadvantage that metamathematical properties such as soundness of reasoning cannot be verified easily.

### Axiomatic Approaches to Fuzzy Geometry

In contrast to that, axiomatic approaches also exist that define a geometric theory on the syntactic level and provide corresponding models on the semantic level. An example is L. Kuijken's and H. Van Maldeghem's "fibered" projective geometry [33, 34]. Here, every point $x$ in $\mathbb{R}^2$ is a base point for several uncertain points. T. Topaloglou [68] axiomatizes one and two dimensional discrete space "with a built-in concept for imprecision" in a first order language. The axiom system is based on two sorts, *point* and *scale*, and two predicates, *haze* and *precedence*, where the haze relation is a reflexive and symmetric indistinguishability relation that is not necessarily transitive. In two dimensions, a theory of haze rectangles is constructed: A *haze rectangle* is a pair of *haze points*, where "haze points refer to points of space which are surrounded by a haze area, the smallest distinguishable quantity in the representation" [68, p. 47f.]. In 1971 T. Poston published his PhD thesis under the title of "Fuzzy Geometry" [47]. Yet, the term "geometry" in the title of Poston's thesis mostly refers to topological and not to geometric notions. The word "fuzzy" in the title of Poston's thesis can be misleading as well: Poston replaces the terms

[7] cf. [58].

"physical continuum" and "tolerance space", with the term *fuzzy space*, but the term "fuzzy" in his work is not directly related to notion of "fuzzy set" as introduced by L. A. Zadeh [79].

Y. Rylov discusses axiomatic approaches to multivaraint geometries in physics, which he defines as geometries with an "equivalence" relation that is not transitive in general. This definition corresponds to the relation of "possible equality " in Approximate Tolerance Geometry (cf. Sect. 6.4), which is an instance of granular geometry as introduced in this chapter. Rylov discusses that multivaraint geometries are granular geometries in the sense that they are partly continuous and partly discrete.

Region based geometries, or point-free geometries, are axiomatic theories of geometry that—instead of using the abstract concept of point as a primitive object—are based on the region primitive. Similar to granular geometries, region based geometries are motivated by the fact that the interpretation of a point as an infinitely small entity is counter-intuitive, whereas regions have extension and are cognitively more adequate. The difference between the two approaches is that granular geometries axiomatize a new form of geometric reasoning, namely *geometric reasoning with regions*, while region based geometries axiomatize classical geometric reasoning, *with exact points*, only using a different object primitive, namely the region primitive, to do that. As in granular geometry, the regions in region based geometries are considered to be location constraints for exact points. The literature on region-based geometries has a long history, starting with N. I. Lobacevskij [38]. A historical overview is given by G. Gerla [18], and recent results are summarized by D. Vakarelov [70]. Many newer approaches build upon A. Tarski's Geometry of Solids [67]. In particular, the work of Gerla and co-workers [6, 17, 21, 22], is relevant for granular geometry, because some of Gerla's results are used as a basis for it; The work of B. Bennett et al. [2–5], relates to the field of GIS.

The Granular Geometry Framework introduced in this chapter and, in particular, its possibilistic version, Approximate Tolerance Geometry, is based on F. S. Roberts' *tolerance geometry* [51], and M. Katz' *inexact geometry* [32]. The basis for Roberts' tolerance geometry was laid with the work of Zeeman [85] and Roberts and Suppes [52] on the granular nature of visual perception. Zeeman described as a distinguishing property of the geometry of visual perception the fact that the *identity* relation is reflexive and symmetric, but not in general transitive, and he calls such a relation a *tolerance relation*. Tolerance relations are often also called indistinguishability relations [40, 46, 47, 50, 69, 85]. Other examples of tolerance relations are the nearness relation and the relation of possible equality, as we use it in Approximate Tolerance Geometry. The observation that transitivity is often violated when dealing with identity in the physical continuum is usually attributed to H. Poincaré [46], and is also known as the *Poincaré Paradox*. Amongst others, it motivated the work of K. Menger [39] on *ensemble flou* (probabilistic equivalence relations) and was further continued by L. Zadeh [80] with the introduction of *fuzzy similarity relations* (fuzzy equivalence relations).

Roberts tolerance geometry is an axiomatic theory of one dimensional Euclidean geometry, where the notion of identity (i.e., equality or equivalence) is replaced by tolerance, i.e., by a tolerance relation. He derives his theory of tolerance geometry in an iterative process: First, he chooses an existing axiomatization of a classical geometry. Second, he chooses a specific model of the axiom system, i.e., an interpretation that complies with the axioms. Third, in this model, he substitutes the equality relation with an interpretation of the tolerance relation, namely with the *closeness* relation. In other words, he defines an "intended interpretation" of the equality relation. As a consequence of the changed interpretation, the interpretation does not comply with all of the axioms of the theory. In a fourth step, he modifies the respective axioms in such a way that the intended interpretation is a model again. Robert's iterative approach is the basis for the generic Granular Geometry Framework introduced in Sect. 6.3, where an "intended interpretation" of the geometric primitives *point* and *line* as position granules and their relations is the starting point for the fuzzy extension of an axiom system of classical geometry.

Roberts interprets the tolerance relation as a closeness relation with an arbitrary, but fixed distance threshold $\varepsilon$: Here, two points, $p$ and $q$, are considered close to each other if their distance is smaller than or equal to $\varepsilon$. In other words, the granularity of the position granules is fixed. Katz [32] shows that modeling *variable* granularity (i.e., variable size of position granules) requires a graduated, i.e., fuzzy version of the tolerance relation. As a consequence, he uses a fuzzy logical system for the axiomatization of his theory of inexact geometry instead of the classical crisp logical system used by Roberts. More specifically, Katz uses Łukasiewicz fuzzy predicate logic Ł∀, and Approximate Tolerance Geometry adopts his choice. A big drawback of Katz' theory is the fact that his definition of graduated tolerance is in fact not an extension of the crisp tolerance relation, but an extension of a crisp equivalence relation (i.e., its kernel is transitive). It was only in 2008—18 years after Katz' introduction of inexact geometry—that Gerla [20] introduced the notion of an approximate fuzzy equivalence relation, which is a true fuzzy extension of both, tolerance and equivalence relations. Approximate fuzzy equivalence relations lay the basis for approximate tolerance geometry as introduced in this chapter.

## 6.3 The Granular Geometry Framework

Before introducing the Granular Geometry Framework itself in Sects. 6.3.2 and 6.3.1 introduces similarity logic as the basic idea and main tool for axiomatizing geometric reasoning with granular points and lines as proposed in the framework.

### *6.3.1 Similarity Logic*

#### Soundness of Geometric Reasoning

Granular geometry is intended to provide a calculus for geometric reasoning with position granules that is *sound*, i.e., *reliable*. A logical theory is sound, if all formulas that can be derived from the theory with its inference rules are valid with respect to its semantics. In other words, the rules of a corresponding calculus do not produce wrong or erroneous output. This is clearly not the case for heuristic approaches towards geometric reasoning with position granules. For this reason, the Granular Geometry Framework adopts a logical and axiomatic approach to the problem, where soundness is provable. Yet, the classical logical definition of soundness only applies in the context of perfect information (i.e., information that is precise, complete and certain). Outside this context, we cannot speak of soundness or unsoundness. To solve this problem, the Granular Geometry Framework uses a fuzzy logical system in the narrow sense, i.e., a graduated logical system that allows for treating imperfections of data as an integral part of the logical theory, allowing for the definition of soundness despite the presence of imperfections. More specifically, we use Łukasiewicz Fuzzy Logic with Evaluated Syntax as a similarity logic.

### Similarity Logic

"In classical logic, falsity entails any statement. But, in many occasions, we may want to use 'false' theories, for instance, Newton's Gravitational Theory." [24, p. 80] These are theories that are not self-contradictory, but empirically or factually false. The problem can be solved by attaching to statements degrees of approximate truth, measured as proximity, or closeness, to the truth. The proximity of a statement $\Phi$ to the truth is measured as a "distance", or, dually, a similarity, between models of $\Phi$ and models of "reality". Logical formalisms that represent and reason with proximity to truth are usually subsumed under the name *similarity logics* or *similarity based reasoning*. They aim at "studying which kinds of logical consequence relations make sense when taking into account that some propositions may be closer to be true than others. A typical kind of inference which is in the scope of similarity-based reasoning responds to the form 'if $\Phi$ is true then $\Psi$ is close to be true', in the sense that, although $\Psi$ may be false (or not provable), knowing that $\Phi$ is true leads to infer that $\Psi$ is semantically close (or similar) to some other proposition which is indeed true [24, p. 83]."

Research in similarity-based reasoning is mainly divided in two major approaches: The first approach uses similarity relations between models (or "worlds") of a logical statement or theory. I.e., it defines similarity *semantically*. From the similarity relation a notion of approximate semantic entailment is derived which allows to draw approximate conclusions from approximate premises. The main reference in this area is Ruspini [57]. The second approach was proposed by Ying [78]. It uses similarity relations between formulas, i.e. it defines similarity *syntactically*. A notion of approximate proof is developed "by allowing the antecedent clause of a rule to match its premise only approximately" [78, p. 830], which again allows for drawing conclusions in an approximate setting. As shown by Biacino and Gerla [7], a generalization of Ying's apparatus can be reduced to Rational Pavelka Logic [42], which extends Łukasiewicz fuzzy predicate logic ($FL_{ev}$(Ł∀)) by adding additional truth constants [27]. Rational Pavelka Logic has been further generalized to Łukasiewicz fuzzy predicate Logic with Evaluated Syntax ($FL_{ev}$(Ł∀)) by V. Novák and I. Perfilieva [41] and G. Gerla [19]. $FŁ_{ev}$ is the main tool used in the Granular Geometry Framework to define an approximate geometric calculus.

## Godo's and Rodríguez' Ontology of Imperfections

L. Godo and R. O. Rodríguez [24] propose an ontology of imperfections that is based on a formal logical viewpoint and allows for assigning appropriate classes of approximate reasoning tools. They distinguish three types of imperfections: *vagueness, uncertainty and truthlikeness* (cf. Fig. 6.6).

In classical logics, *vagueness* can not be expressed, since a classical logical interpretation function as proposed by A. Tarski [66] is bivalent. Fuzzy logic solves the problem. *Uncertainty* stems from the problem of incomplete information: In classical logic, there is no means of deciding the truth value of a statement in case neither the statement nor it's negation can be deduced. The problem can be solved by attaching belief degrees to the truth values of statements, and examples of corresponding approximate reasoning tools are, e.g., probability theory, possibility theory or Dempster-Shafer theory of evidence. Finally, *truthlikeness* refers to the fact that classical logic has no means of expressing how close a false theory or statement comes to being true, and the problem can be solved by similarity based reasoning.

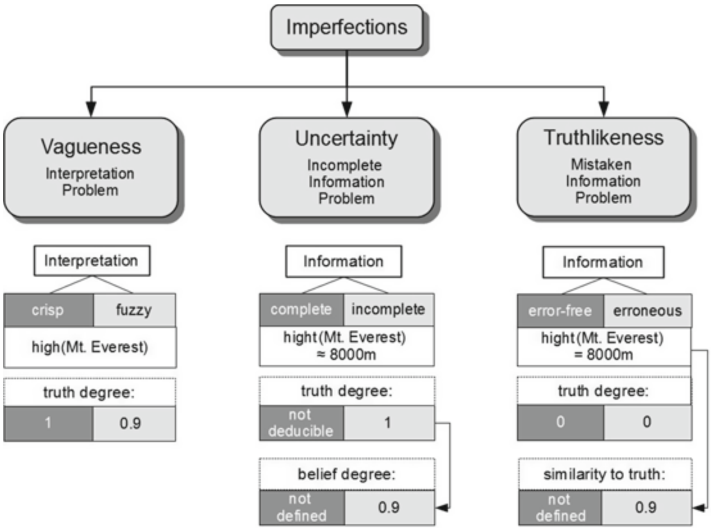


**Fig. 6.6** Ontology of Imperfections after Godo and Rodríguez [73]

**Contributing Imperfections: Possibility and Truthlikeness**

According to this ontology of imperfections, two types of imperfections contribute to geometric reasoning with position granules. These are (positional) *uncertainty* and *truthlikeness*. Positional uncertainty stems from the positional granulation of exact points and lines. Truthlikeness stems from the assumption that classical geometry is false but close to the truth, which we adopt from Lakoff's and Nunez' cognitive theory of mathematics [35]. For a given modality of *uncertainty*, the Granular Geometry Framework describes a generic way of defining and propagating *truthlikeness* in granular geometry. More specifically, it provides guidance of

1. how to define a truthlikeness measure for granular geometry based on a given modality of uncertainty (e.g., possibilistic or probabilistic),
2. how to axiomatize granular geometry in Fuzzy Logic with Evaluated Syntax (as a similarity logic) based on a given axiomatization of classical geometry.

The result is an axiomatic theory of granular geometry that treats reliability (truthlikeness) as an intrinsic part of the theory. Fuzzy Logic with Evaluated Syntax allows for propagating the integrated reliability degree, and geometric reasoning with position granules is sound by design.

## *6.3.2 The Granular Geometry Framework*

The Granular Geometry Framework provides a step-wise approach to axiomatize granular geometry based on an existing axiomatization of a classical geometry and for a given modality of uncertainty. It consists of four steps:

1. Choose a *classical geometry* and a corresponding *axiomatization* $\{\varphi_i\}_i$.
2. Define an *intended interpretation* of
   - the primitive geometric *objects* of the axiomatization by positional granules (such as points and lines with size) with a fixed modality of uncertainty (such as possibilistic, probabilistic, or veristic).
   - the *equality relation* between positional granules that is semantically consistent with the chosen modality of uncertainty (e.g., if points with size are interpreted as sets of possible exact points, a sensible interpretation of the *equality* predicate is the overlap relation, cf. Sect. 6.4.2).
   - the primitive geometric *relations* between these positional granules that is semantically consistent with the chosen modality of uncertainty.
3. Define a *truthlikeness measure* for all predicates in the intended interpretation.
4. *Fuzzify* the chosen *axiomatization* $\{\varphi_i\}_i$ in $FL_{ev}$(Ł∀):

(a) For every axiom $\varphi_i$ of the chosen axiomatization $\{\varphi_i\}_i$ use the truth-functional operators and quantifiers of Łukasiewicz fuzzy predicate logic ($FL_{ev}$(Ł∀))[8] to derive the truthlikeness degree $a_i$ of $\varphi_i$ from the truthlikeness measures of the involved predicates.
(b) For every axiom $\varphi_i$ assign the truthlikeness degree $a_i$ to $\varphi_i$ as a fuzzy degree of membership to the truth.
(c) Check if the truthlikeness degrees of all axioms $\varphi_i$ are greater than zero:

- If yes, $\{(\varphi_i; a_i)\}$ defines an axiomatization of a granular geometry.
- If no, extend the corresponding axioms (if possible), such that a positive truthlikeness degree results. The fuzzy extension must be consistent with the classical axiom. (I.e., for positional granules of size zero, fuzzy extension must coincide with the respective classical axiom).

Section 6.3.1 discussed the rationale of steps 1–3. In step 4, the classical axiomatization is fuzzified by attaching to every classical axiom its degree of truthlikeness. The result is a fuzzy set of axioms, $\{(\varphi_i; a_i)\}$, where the truthlikeness degrees (*signs*) $a_i$ attached to the classical axioms $\varphi_i$ indicate their fuzzy membership degrees to "real world geometry". The truthlikeness is interpreted as a measure of reliability of the respective axiom in the presence of positional granularity. It may happen that the truthlikeness of one or several of the axioms is zero,[9] which is equivalent to absolute falsity. In $FL_{ev}$(Ł∀)—as in classical (crisp) logical theories—it is not useful to list an absolutely false formula as an axiom of a fuzzy theory: from an absolutely false formula, the graded deduction apparatus of $FL_{ev}$(Ł∀) can only deduce absolutely false formulas, and absolutely false formulas are completely unreliable, i.e., useless. For this reason, it is necessary to ensure in step 4b that all fuzzified classical axioms have a positive truthlikeness degree. Only if this is the case, we call the resulting fuzzy theory a *granular geometry*.

*Remark 1* Notice that the steps 3 and 4a derive the truthlikeness measure *semantically* in the interpretation domain, while it is used in step 4b as part of the *syntax*. As a result, the intended interpretation is by design a model of the fuzzy theory $\{(\varphi_i; a_i)\}$. In $FL_{ev}$(Ł∀), a pair $(\varphi_i, a_i)$ is called a *signed formula*, and $a_i$ is called the *sign* or *syntactic evaluation* of $\varphi_i$.[10]

In the subsequent section, we apply the Granular Geometry Framework exemplarily to some axioms of projective plane geometry, based on a possibilistic interpretation of granular geometric objects.

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[8] cf, e.g., [27].

[9] Axioms are often universally quantified. In Łukasiewicz predicate logic, the universal quantifier ∀ is interpreted as the infimum operator. As a result, the truthlikeness degree of a universally quantified axiom $\forall x.[Statement(x)]$ is the infimum over all truthlikeness degrees of all instances $Statement(x)$, i.e., the worst case truthlikeness degree. Depending on the intended interpretation, this may result to zero.

[10] For details on signed formulas in $FL_{ev}$(Ł∀) see [41].

## 6.4 Towards an Approximate Tolerance Geometry

Wilke [73, 75] developed the Granular Geometry Framework for a special case of possibilistic uncertainty, where *possibilistic position granules* are (crisp) neighborhoods of possible positions of an exact point or line. The uncertain geometric relation between these possibilistic position granules as they occur in GIScience community is often referred to as *positional tolerance*, and Wilke calls the developed possibilistic granular geometry an *approximate tolerance geometry* (ATG). In step 1 of the Granular Geometry Framework, she chooses a standard axiomatization of projective plane geometry, and elaborates in steps 2–3 the definition of the intended interpretation of possibilistic position granules and their truthlikeness. In step 4 (the fuzzification step), she applies the fuzzification procedure only to a subset of the chosen axiomatization, and a complete fuzzy axiomatization of approximate tolerance geometry in $FL_{ev}$(Ł∀) is subject to further work.

Wilke shows that step 4(c) of the framework (validation of non-zero truthlikeness) is indeed necessary: In ATG, some of the axioms have a truthlikeness degree of zero. She proposes fuzzy extensions of these axioms that have positive truthlikeness degrees and are consistent with the corresponding classical axioms.

### *6.4.1 Step 1: Choose an Axiomatization*

Wilke [73, 74], chooses the following standard axiomatization of projective plane geometry[11] that uses points and lines as primitive objects, and equality and incidence as primitive relations:

(Pr1) For any two distinct points, at least one line is incident with them.
(Pr2) For any two distinct points, at most one line is incident with them.
(Pr3) For any two lines, at least one point is incident with both lines.
(Pr4) Every line is incident with at least three distinct points.
(Pr5) There are at least three points that are not incident with the same line.

The axioms (Pr1) and (Pr2) are usually called *Euclid's First Postulate*. They state that two distinct points can always be connected by a unique line. The projective axioms (Pr1)–(Pr5) can be formalized in classical predicate logic as follows:

(Pr1) $\forall \mathtt{p}, \mathtt{q}.\exists \mathtt{l}.\, [\neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \rightarrow \mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{l})]$,
(Pr2) $\forall \mathtt{p}, \mathtt{q}, \mathtt{l}, \mathtt{m}.\, [\neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{m}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{m}) \rightarrow \mathtt{E}(\mathtt{l}, \mathtt{m})]$,
(Pr3) $\forall \mathtt{l}, \mathtt{m}.\exists \mathtt{p}.\, [\mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{m})]$,
(Pr4) $\forall \mathtt{l}.\exists \mathtt{p}, \mathtt{q}, \mathtt{r}.\, [\neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \,\&\, \neg \mathtt{E}(\mathtt{q}, \mathtt{r}) \& \neg \mathtt{E}(\mathtt{r}, \mathtt{p}) \& \mathtt{I}(\mathtt{p}, \mathtt{l}) \& \mathtt{I}(\mathtt{q}, \mathtt{l}) \& \mathtt{I}(\mathtt{r}, \mathtt{l})]$,
(Pr5) $\exists \mathtt{p}, \mathtt{q}, \mathtt{r}.\forall \mathtt{l}.\neg\, [\mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{r}, \mathtt{l})]$.

[11] cf., e.g., [30].

Here, ∀ ("for all") denotes the the universal quantifier, and ∃ ("exists") denotes the existential quantifier. *Connectives* are used to formulate compound statements: stands & for a logical AND operator (conjunction),→ denotes implication, ¬ denotes negation. We use the symbol E to denote the equality predicate and the symbol I to denote the incidence predicate. Predicates can assume Boolean truth values, i.e. the value 1 (*true*) or the value 0 (*false*). E.g., $\mathtt{I(p, l)} = 1$ says that p is incident with l. The axioms employ two sorts of object variables, namely *points* and *line*s. Points are denoted by p, q, r, . . ., lines are denoted by l, m, n, . . . In logical theories, equality (equivalence) is usually treated as part of the background logic. Yet, in the context of uncertainty, equality can not be recognized with certainty in general. The Granular Geometry Framework accounts for this fact, and treats the axioms of geometric equality as part of the logical theory. They are axiomatized as follows:

(E1) $\forall \mathtt{x}.\,[\mathtt{E(x, x)}]$,
(E2) $\forall \mathtt{x, y}.\,[\mathtt{E(x, y)} \;\rightarrow\; \mathtt{E(y, x)}]$ ,
(E3) $\forall \mathtt{x, y, z}.\,[\mathtt{E(x, y)}\ \&\ \mathtt{E(y, z)} \;\rightarrow\; \mathtt{E(x, z)}]$ .

Here, x, y, z are either all points or all lines. (E1) is called reflexivity, (E2) is called symmetry, and (E3) is called transitivity.

### 6.4.2 Step 2: Define the Intended Interpretation

Step 2 of the framework requires the definition of the *intended interpretation* of the primitive geometric *objects* and *relations* of the chosen axiomatization in the context of granularity. For the axiom system (Pr1)–(Pr5), primitive objects are *points* and *lines*, and the primitive relations are *equality* and *incidence*.

#### The Intended Interpretation of Primitive Objects

According to Lakoff and Núñez, granules of positional information are the primitives of geometry as we perceive it in our interaction with the world around us, and we call them *position granules (PGs)*.

**Definition 1** Let $(X, d_X)$ be a metric space, and let $\tau_{d_X}$ be the induced metric topology on $X$. We call $P \subseteq X$ a position granule of type *approximate point in* $X$, if $P$ is either a $\tau_{d_X}$-topological neighborhood of a point $p \in X$ or a singleton, $P = \{p\}$. We denote the set of approximate points in $X$ by $\mathcal{P}_X$.

The definition of a PG $P$ as a neighborhood in a metric space (or a singleton) is motivated by examples from the GIS domain, cf., e.g., Fig. 6.1. Here, a PG is given as a *crisp* point set. Since Wilke address the *possibilistic* modality of uncertainty, she interprets a PG $P$ as the crisp set of *possible* positions of an exact point, i.e. as a *possibilistic position granule (PPG)*. I.e., a PPG of type approximate point can be specified by the simple possibility distribution

$$pos(x) = \begin{cases} 1 & \text{if } x \in P, \\ 0 & \text{if } x \notin P. \end{cases} \tag{6.1}$$

In many axiomatizations of geometry, both, points and lines are primitive objects. In the vector based representation model of GIS, geometric lines are not considered primitive objects, yet they are sometimes obtained as original measurements. It is therefore reasonable that the ATG framework adopts the approach of treating lines as primitives. A line feature in GIS consists of a number of connected line segments and is represented as a tuple $(p_1, \ldots, p_n)$ of coordinate points. Each pair $(p_i, p_{i+1})$, $i \in 1, \ldots, n-1$, of consecutive coordinate points of the tuple defines a line segment, together with its corresponding geometric line $l_i = p_i \vee p_{i+1}$, cf. Fig. 6.7a. If the points $p_i, p_{i+1}$ have positional tolerance, they can be represented by position granules $P_i, P_{i+1}$ of type approximate point. As a consequence, the corresponding line $l_i$ inherits positional tolerance from $p_i, p_{i+1}$, and it can be represented as a *set* $L = \{L_i\}_i$ of geometric lines, cf. Fig. 6.7b. It is consequently reasonable to interpret a geometric line with positional tolerance by a set $L_i$ of geometric lines that are *possible* candidates for an assumed ideal "true" line $l_i$.

Similar to the definition of PGs of the type approximate point, we call $L$ a PG of type approximate line. It is a set of geometric lines, which are "close" to the unknown "true" line and constrains its position:

**Definition 2** Let $L_X$ denote the set of geometric lines in a domain $X$. For a given geometric line $l \in L_X$, denote by $l'$ its dual point in a dual line parameter space $Y = X'$. Let $d_Y$ be a metric on $Y$, and let $\tau_{d_Y}$ be the induced metric topology on $Y$. $L \subseteq L_X$ is called a position granule of type *approximate line* in $X$, if $L' = \{l' | l \in L\}$ is a position granule of type approximate point in $Y$. We denote the set of approximate lines in $X$ by $\mathcal{L}_X$.

In analogy to the possibilistic semantic of approximate points, we attribute a possibilistic semantic to approximate lines, i.e., we consider PPGs of type approximate

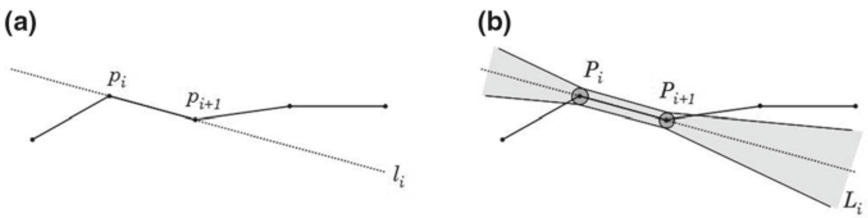


**Fig. 6.7** **a** A line feature and a geometric line. **b** Points with tolerance induce a geometric line with tolerance [75]

line. The definition of approximate points and lines extends the (corresponding) classical interpretation of geometric points and lines: An approximate point is a set $P$ of points that *possibly* coincide with the coordinates of an assumed "true" point $p$. If there is no uncertainty about the coordinates of $p$, $P = \{p\}$ holds. Similarly, if there is no uncertainty about the parameters of an exact line $l$, the set of possible coordinates of $l$ coincide with the one-element set containing $l$, i.e. $L = \{l\}$.

### The Intended Interpretation of Primitive Relations

Since PPGs represent the set of *possible* positions of an exact point or line, geometric relations between PPGs can not be recognized with certainty. In order to guarantee a correct representation of the available information, the primitive geometric relations between approximate points and lines are interpreted as *possible* relations of exact points and lines. More specifically, the geometric equality predicate is interpreted by *possible equality* (often also called *indistinguishability* [46, 47, 80, 85]) of the corresponding "true" points, cf. Fig. 6.8. It can be represented by overlapping neighborhoods of equal sort, i.e. the overlapping of an approximate point with an approximate point, or the overlapping of an approximate line with an approximate line. Notice that by "overlapping approximate lines", we mean that the approximate lines have a line (and not only a point) in common.

**Definition 3** The intended interpretation of the geometric equality predicate in ATG in a metric space $X$ is given by the Boolean overlap relations

$$e_{\mathbb{B}} : \mathcal{P}_X \times \mathcal{P}_X \to \{0, 1\},\ e_{\mathbb{B}}(P, Q) = (P \cap Q \neq \varnothing)\,, \tag{6.2}$$

$$e_{\mathbb{B}} : \mathcal{L}_X \times \mathcal{L}_X \to \{0, 1\},\ e_{\mathbb{B}}(L, M) = \left(L' \cap M' \neq \varnothing\right). \tag{6.3}$$

The relation $e_{\mathbb{B}}$ is called *equality with tolerance*.

Following the same semantic as for equality, the geometric incidence predicate is interpreted by *possible incidence* of an exact point with an exact line. In terms of

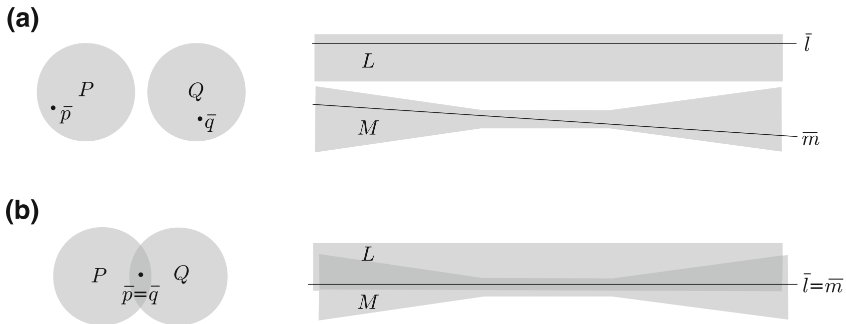


**Fig. 6.8** The exact points $\bar{p}$, $\bar{q}$ (the exact lines $\bar{l}$, $\bar{m}$) are **a** certainly distinct; **b** possibly equal [73]

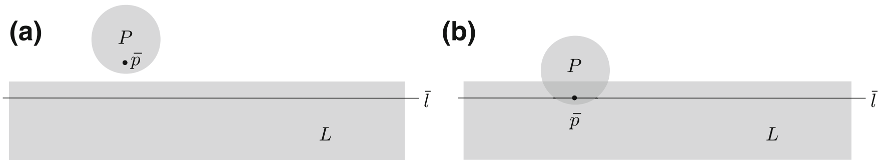


**Fig. 6.9** An exact point $\bar{p}$ and an exact line $\bar{l}$ are **a** certainly not incident; **b** possibly incident [73]

neighborhoods, possible incidence of exact objects translates into the overlap relation between constraints of different sort, i.e. into the overlapping of an approximate point with an approximate line, cf. Fig. 6.9.

**Definition 4** The intended interpretation of the incidence predicate in ATG in a metric space $X$ is given by the Boolean relation

$$i_{\mathbb{B}} : \mathcal{P}_X \times \mathcal{L}_X \to \{0, 1\}, \quad i_{\mathbb{B}}(P, L) = (P \cap L \neq \emptyset) . \tag{6.4}$$

The relation $i_{\mathbb{B}}$ is called *incidence with tolerance*.

### *6.4.3 Step 3: Define Truthlikeness*

Step 3 of the Granular Geometry Framework requires the definition of a *truthlikeness* measure of all predicates in the intended interpretation.

#### Truthlikeness of Primitive Relations

The relations *equality with tolerance,* $e_{\mathbb{B}}$, and *incidence with tolerance,* $i_{\mathbb{B}}$, are Boolean relations that specify the intended interpretation of the corresponding logical predicates *geometric equality* and *incidence* in an ATG. The interpretations specify what we intend to accept as truth. Similarity of $e_{\mathbb{B}}$ and $i_{\mathbb{B}}$ to the intended truth (truthlikeness) can be represented by fuzzy relations, that extend the corresponding Boolean relations. The fuzzy relations are chosen such that they assume the value 1 if the corresponding Boolean relations hold and their values decrease "the more wrong" it is to assume that the corresponding Boolean relations hold. The fuzzy membership degrees are understood as truthlikeness degrees. I.e. the fuzzy relations are graduated extensions of the intended truth.

We first quantify the similarity of the statement $e_{\mathbb{B}}(P, Q) = 1$ to the truth (i.e., the statement "$P$ and $Q$ are equal with tolerance"). The spatial setting of the problem statement suggests that a similarity measure that is dual to a spatial distance measure is appropriate. More specifically, Definition 3 of equality with tolerance, $e_{\mathbb{B}}(P, Q) := (P \cap Q \neq \emptyset)$, suggests using the following set distance measure, which is illustrated in Fig. 6.10:

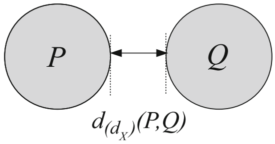


**Fig. 6.10** The set distance measure (6.5) [75]

$$d_{(d_X)}(P,Q) = \inf\{d_X(\bar{p},\bar{q})|\bar{p}\in P, \bar{q}\in Q\}, \quad P,Q\subseteq X. \tag{6.5}$$

Here, $d_X$ denotes the metric in the underlying metric space $X$. $d_{(d_X)}(P,Q)$ is an extensive distance (cf. Wilke [74], Gerla [20]), and measures the shortest distance between the subsets $P$ and $Q$: It quantifies the *semantic* distance of the statement $d_{(d_X)}(P,Q) = 0$ from the truth. Since

$$d_{(d_X)}(P,Q) = 0 \quad \Leftrightarrow \quad P\cap Q \neq \varnothing \quad \Leftrightarrow \quad e_{\mathbb{B}}(P,Q) = 1 \tag{6.6}$$

holds, $d_{(d_X)}(P,Q)$ is a dual measure of the of similarity of $e_{\mathbb{B}}(P,Q) = 1$ to the truth: The greater the distance $d_{(d_X)}(P,Q)$, the smaller the truthlikeness of $d_{(d_X)}(P,Q) = 0$. Similarly, the set distance

$$d_{(d_Y)}(L,M) = \inf\left\{d_Y(\bar{l}',\bar{m}')|\bar{l}'\in L', \bar{m}'\in M'\right\} \tag{6.7}$$

quantifies the semantic distance of the statement $d_{(d_Y)}(L,M) = 0$ from the truth, and is a dual measure of truthlikeness of $d_{(d_Y)}(L,M) = 0$. Here, $d_Y$ denotes the metric in the underlying line parameter space.

Similarity measures are often normalized to the interval [0, 1], and we adopt this. We assume that the distance measures (6.5) and (6.7) are normalized to the interval [0, 1] as well. As a result of this assumption, the degree of truthlikeness of a statement $e_{\mathbb{B}}(P,Q) = 1$, or $e_{\mathbb{B}}(L,M) = 1$, can be defined by $1 - d_{(d_X)}(P,Q)$ and $1 - d_{(d_Y)}(L,M)$, respectively. The assumption that $d_X$, and consequently $d_{(d_X)}$, can be normalized, i.e. that a maximal distance exists, is reasonable in the context of GIS, since all maps are bounded.

**Definition 5** The *fuzzy* interpretation of the *geometric equality predicate* in ATG is given by the fuzzy relations

$$e_{(d_X)} : \mathcal{P}_X \times \mathcal{P}_X \to [0,1], \quad e_{(d_X)}(P,Q) := 1 - d_{(d_X)}(P,Q), \quad \textit{and} \tag{6.8}$$
$$e_{(d_Y)} : \mathcal{L}_Y \times \mathcal{L}_Y \to [0,1], \quad e_{(d_Y)}(L,M) := 1 - d_{(d_Y)}(L,M). \tag{6.9}$$

$P, Q$ and $L, M$ are called *approximately equal with tolerance* to the degree $e_{(d_X)}(P,Q)$ and $e_{(d_Y)}(L,M)$, respectively. Here, $d_X$ and $d_Y$ are a normalized metrics in $X$ and $Y$, respectively.

The fuzzy relations $e_{(d_X)}$ and $e_{(d_Y)}$ extend the Boolean relations $e_{\mathbb{B}}$ and $e_{\mathbb{B}}$, respectively, because they coincide at the value 1:

$$e_{(d_X)}(P, Q) = 1 - d_{(d_X)}(P, Q) = 1 \quad \Leftrightarrow \quad e_{\mathbb{B}}(P, Q) = (P \cap Q = \varnothing) = 1, \tag{6.10}$$

$$e_{(d_Y)}(L, M) = 1 - d_{(d_Y)}(L, M) = 1 \quad \Leftrightarrow \quad e_{\mathbb{B}}(L, M) = \left(L' \cap M' = \varnothing\right) = 1. \tag{6.11}$$

As a second step, we quantify the similarity of the statement $i_{\mathbb{B}}(P, L) = 1$ to the truth (i.e., the statement "$P$ and $L$ are incident with tolerance"). A measure of truthlikeness for the primitive relation of *incidence with tolerance* is a measure that quantifies the distance of the statement $i_{\mathbb{B}}(P, L) = (P \subset L) = 1$ from the truth. The set distance measure used for specifying truthlikeness of equality with tolerance can not be used here: While the Boolean interpretation of equality with tolerance is the overlap relation between location constraints of the *same* sort, incidence with tolerance is interpreted by the overlap relation between location constraints of *different* sorts. The trivial solution for this problem is to keep the Boolean relation $(P \subset L)$ and interpret it as a discrete similarity measure. In order to simplify the modeling task at hand, [73] adopts this solution, and we keep with it. The integration of a more realistic graduated definition of truthlikeness of incidence with tolerance is a task for future work. In analogy to the definition of $e_{\mathbb{B}}$, we may understand $(P \subset L)$ as an inverse distance measure, $(P \subset L) = 1 - \Delta(P, L)$, where $\Delta$ denotes the discrete distance measure

$$\Delta(P, L) = 1 - (P \subset L) = \begin{cases} 1 & \text{if } (P \subset L) = 0, \\ 0 & \text{if } (P \subset L) = 1. \end{cases} \tag{6.12}$$

**Definition 6** The *fuzzy* interpretation of the incidence predicate in ATG coincides with Definition 4. It is a Boolean relation and it is given by

$$i_{\Delta} : \mathcal{P}_X \times \mathcal{L}_X \to \{0, 1\} \subset [0, 1], \quad i_{\Delta}(P, L) := 1 - \Delta(P, L) = P \subset L, \tag{6.13}$$

$P$ and $L$ are called *approximately incident with tolerance* to the degree $i_{\Delta}(P, L) \in \{0, 1\}$.

Since the fuzzy relation is chosen such that it coincides with the Boolean relation, it trivially extends the Boolean relation.

*Remark 2* A possible candidate for a graded definition of truthlikeness is the distance measure $d^{\perp}_{(d_X)}(P, L) = \inf \left\{ d^{\perp}_X(\bar{p}, \bar{l}) | \bar{p} \in P, \bar{l} \in L \right\}$, where $d^{\perp}_X(\bar{p}, \bar{l})$ denotes the orthogonal distance between Cartesian points and lines $\bar{p}, \bar{l}$. In order to integrate this measure in ATG, it is necessary to investigate if it works in concert with the definition of truthlikeness of the possible equality predicate in the sense that, together, they allow for a modification of classical geometric axioms such that a model of the modified axioms exists that complies with the properties of the intended interpretation.

### 6.4.4 Step 4: Fuzzification

In the fourth step of the Granular Geometry Framework, the truthlikeness measure derived in step 3 is used to fuzzify the classical axioms (E1)–(E3) and (Pr1)–(Pr5). As a first step towards an approximate tolerance geometry, Wilke [73] did this exemplarily for the equality axioms (E1)–(E3) and for Euclid's First Postulate, (Pr1) and (Pr2).

#### Step 4a: Truthlikeness of the Classical Axioms

Geometric axioms are compound geometric statements: they are composed of atomic formulas, connectives and quantifiers. In the present work, atomic formulas are statements that involve one of the predicates *geometric equality* $\mathtt{E}$, *incidence* $\mathtt{I}$, and in ATG, these are interpreted by the fuzzy relations *approximate equality with tolerance* $e_{(d_X)}$, *approximate incidence with tolerance* $i_{(d_X)}$, respectively. A truth-functional fuzzy logical system provides fuzzy interpretations of connectives and quantifiers, and since Łukasiewicz fuzzy predicate logic (L∀) is truthfunctional, it can be used to evaluate the truthlikeness degree of the classical axioms (E1)–(E3) and (Pr1)–(Pr2) from the truthlikeness degrees of the involved atoms. Attaching the derived truthlikeness degrees to the classical axioms results in a fuzzy set of axioms, and the truthlikeness degree of an axiom indicates its degree of membership to granular geometry. Wilke [73, 74] shows that in the intended interpretation of ATG and with the definition of truthlikeness given in the foregoing subsection, the axioms (E1), (E2) and (Pr1) have a truthlikeness degree of 1, while the transitivity axiom (E3) and the "uniqueness axiom" (Pr2) have a truthlikeness of zero, i.e., they are useless for geometric reasoning with position granules. The subsequent paragraph lists the resulting fuzzy set of axioms in $FL_{ev}$(Ł∀) that includes the "useless" fuzzy axioms, and the remainder of the subsection discusses fuzzy extensions of the two axioms that yield a positive truthlikeness degree.

#### Step 4b: Fuzzification of the Classical Axioms

Attaching the derived truthlikeness degrees to the axioms (E1), (E2), (E3), (Pr1) and (Pr2) yields the following fuzzy set of axioms in $FL_{ev}$(Ł∀):

(E1)$_{\text{ev}}$ $\left(\forall \mathtt{x}.\, [\mathtt{E}(\mathtt{x}, \mathtt{x})];\, 1\right)$,

(E2)$_{\text{ev}}$ $\left(\forall \mathtt{x}, \mathtt{y}.\, [\mathtt{E}(\mathtt{x}, \mathtt{y}) \rightarrow \mathtt{E}(\mathtt{y}, \mathtt{x})];\, 1\right)$,

(E3)$_{\text{ev}}$ $\left(\forall \mathtt{x}, \mathtt{y}, \mathtt{z}.\, [\mathtt{E}(\mathtt{x}, \mathtt{y}) \,\&\, \mathtt{E}(\mathtt{y}, \mathtt{z}) \rightarrow \mathtt{E}(\mathtt{x}, \mathtt{z})];\, 0\right)$,

and

$$(\mathsf{Pr1})_{\mathsf{ev}} \left( \forall \mathtt{p}, \mathtt{q}. \exists \mathtt{l}. [\neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \rightarrow \mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{l})]; 1 \right),$$

$$(\mathsf{Pr2})_{\mathsf{ev}} \left( \forall \mathtt{p}, \mathtt{q}, \mathtt{l}, \mathtt{m}. \Big[ \neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{m}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{m}) \rightarrow \mathtt{E}(\mathtt{l}, \mathtt{m}) \Big]; 0 \right),$$

respectively. Here, we write a fuzzy axiom in the form $(\varphi; a)$, where $\varphi$ is a classical axiom and $a$ is its truthlikeness degree [41]. Since the classical axioms (E3) and (Pr2) have a truthlikeness degree of zero, the corresponding evaluated axioms $(E3)_{ev}$ and $(Pr2)_{ev}$ are useless for geometric reasoning in granular geometry. The following subsection introduces fuzzy extensions of (E3) and (Pr2) that have a positive truthlikeness degree.

**Step 4c: A Consistent Extension of the Fuzzified Axioms**

Following Gerla [20], Wilke [73, 74] shows that a truthlikeness degree of 1 can be achieved for "weak transitivity". Weak transitivity (E3x) extends the classical transitivity axiom (E3) with a unary *exactness predicate* X that is an inverse measure of the *size* of position granules:

$$(\mathsf{E3x})_{\mathsf{ev}} \left( \forall \mathtt{x}, \mathtt{y}, \mathtt{z}. [\mathtt{E}(\mathtt{x}, \mathtt{y}) \,\&\, \mathtt{X}(\mathtt{y}) \,\&\, \mathtt{E}(\mathtt{y}, \mathtt{z}) \rightarrow \mathtt{E}(\mathtt{x}, \mathtt{z})]; 1 \right).$$

The weak transitivity axiom is a fuzzy extension of the classical transitivity axiom, i.e., in particular, it coincides with classical transitivity when only exact points and lines are involved. In this case, $\mathtt{X}(\mathtt{y}) = 1$ for arbitrary $\mathtt{y}$, and (E3x) $\equiv$ (E3).

Wilke [73] also shows that a truthlikeness degree of 1 can be achieved for a "weak uniqueness axiom": We mentioned in Sect. 6.2.1 that the connection of two approximate points $P$, $Q$ by an approximate line $L$ is not unique, and that, intuitively, the "degree of uniqueness" depends on the size and distance of the involved granular points $P$, $Q$, cf. Figs. 6.4 and 6.3. The weak uniqueness axiom (Pr2x) extends the classical uniqueness axiom (Pr2) with a binary *directionality predicate* $\mathtt{Dir}(\mathtt{p}, \mathtt{q})$ that captures the influence of the *size* and *extensive distance* on of the involved approximate points $\mathtt{p}$ and $\mathtt{q}$ on the uniqueness of $L$, and can be seen as a measure of the directionality of the approximate connection:

$$(\mathsf{Pr2x})_{\mathsf{ev}} \left( \forall \mathtt{p}, \mathtt{q}, \mathtt{l}, \mathtt{m}. \Big[ \neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \,\&\, \mathtt{Dir}(\mathtt{p}, \mathtt{q}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{m}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{m}) \rightarrow \mathtt{E}(\mathtt{l}, \mathtt{m}) \Big]; 1 \right),$$

The weak uniqueness axiom is a fuzzy extension of the classical uniqueness axiom, i.e., in particular, it coincides with the classical uniqueness axiom when only exact points and lines are involved. In this case, two distinct points always define an *exact* direction that is given by the unique connecting line. Here, $\mathtt{Dir}(\mathtt{p}, \mathtt{q}) = 1$ for arbitrary distinct points $\mathtt{p}$, $\mathtt{q}$, and (Pr2x) $\equiv$ (Pr2).

In the remaining paragraphs of this section, we give definitions of the exactness and directionality predicates, and briefly show how they can be derived from the intended interpretation introduced in Sect. 6.4.2.
The Exactness Predicate

We first define a Boolean version of the exactness predicate as part of the intended interpretation, and then discuss its fuzzy extension to define a corresponding truth-likeness measure. The Boolean version of the *exactness predicate* X is intended to single out the *exact* classical points and lines from the set of approximate points and lines.

**Definition 7** The intended interpretation of *exactness* of PGs of the type approximate point in a domain $X$ is the set

$$x_{\mathbb{B}} : \mathcal{P}_X \rightarrow \{0, 1\}, \quad x_{\mathbb{B}}(P) = (|P| = 1)\,. \tag{6.14}$$

Here, $|P|$ denotes the cardinality of the set $P$. If $x_{\mathbb{B}}(P) = 1$, $P$ is called *exact*. Similarly, the intended interpretation of *exactness* of PGs of the type approximate line in a domain $X$ is the set

$$x_{\mathbb{B}} : \mathcal{L}_X \rightarrow \{0, 1\}, \quad x_{\mathbb{B}}(L) = \left(\left|L'\right| = 1\right). \tag{6.15}$$

If $x_{\mathbb{B}}(L) = 1$, $L$ is called *exact*.

The exactness predicate $x_{\mathbb{B}}$ can be seen as an inverse bivalent size measure, $x_{\mathbb{B}} = 1 - s_{\mathbb{B}}$, where the size

$$s_{\mathbb{B}}(P) := \max\{\Delta(p, q) | p, q \in P\} \tag{6.16}$$

of an approximate point $P$ is measured based on the *discrete metric* $\Delta$,

$$\Delta(a, b) = \begin{cases} 1 & \text{if } a \neq b, \\ 0 & \text{if } a = b. \end{cases} \tag{6.17}$$

To measure the truthlikeness of the exactness of a granular point or line, we quantify the similarity of the statement "$X(P) = 1$" (i.e., the statement "$P$ is exact") to the truth. Points (or lines) that have non-zero positional tolerance are not exact. In this case $x_{\mathbb{B}}(P) = 0$ holds. Since we represent positional tolerance by position granules, the *size* (*diameter*) of a position granule $P$ can be used to quantify "how much" positional tolerance is involved: It measures the error made when assuming that all points $p \in P$ are equal. In terms of the exactness relation $x_{\mathbb{B}}$, the size of $P$ measures "how wrong" it is to assume that the statement $x_{\mathbb{B}}(P) = 1$ holds.

**Definition 8** For an approximate point $P \in \mathcal{P}_X$, and an approximate line $L \in \mathcal{L}_X$, define the *size* of $P$ and $L$, respectively, by

$$s_{(d_X)}(P) := \sup\{d_X(\bar{p}, \bar{q}) | \bar{p}, \bar{q} \in P\} \quad \text{and} \tag{6.18}$$
$$s_{(d_Y)}(L) := \sup\left\{d_Y(\bar{l}', \bar{m}') | \bar{l}', \bar{m}' \in L'\right\}. \tag{6.19}$$

The size measures $s_{(d_X)}(P)$ and $s_{(d_Y)}(L)$ quantify the semantic distance of the assumptions

$$[d_X(\bar{p}, \bar{q}) = 0 \quad \forall \bar{p}, \bar{q} \in P], \quad \text{and} \quad \left[d_Y(\bar{l}', \bar{m}') = 0 \quad \forall \bar{l}', \bar{m}' \in L'\right] \tag{6.20}$$

from the truth, respectively. Conversely, the truthlikeness degree of $x_{\mathbb{B}}(P)$ is intended to measure the similarity of the statement $x_{\mathbb{B}}(P) = 1$ to the truth. We define the truthlikeness degree $x_{(d_X)}(P)$ of $x_{\mathbb{B}}(P) = 1$ as an inverse size measure:

**Definition 9** The fuzzy interpretations of the *exactness predicate* in ATG are given by the fuzzy sets

$$x_{(d_X)} : \mathcal{P}_X \to [0, 1], \quad x_{(d_X)}(P) := 1 - s_{(d_X)}(P), \quad \text{and} \tag{6.21}$$
$$x_{(d_Y)} : \mathcal{L}_Y \to [0, 1], \quad x_{(d_Y)}(L) := 1 - s_{(d_Y)}(L), \tag{6.22}$$

respectively. Here, $d_X$ and $d_Y$ are a normalized metrics in $X$ and $Y$. $P$ and $L$ are called *approximately exact to the degree* $x_{(d_X)}(P)$ and $x_{(d_Y)}(L)$, respectively.

Approximate exactness $x_{(d_X)}$ is a fuzzy extension of exactness $x_{\mathbb{B}}$: The fuzzy set $x_{(d_X)} : \mathcal{P}_X \to [0, 1]$ is an extension of the classical set $x_{\mathbb{B}} : \mathcal{P}_X \to \{0, 1\}$, because both coincide at the value 1:

$$x_{(d_X)}(P) = 1 \;\Leftrightarrow\; s_{(d_X)}(P) = 0 \;\Leftrightarrow\; |P| = 1 \;\Leftrightarrow\; x_{\mathbb{B}}(P) = 1. \tag{6.23}$$

This holds analogously for the second object sort, approximate lines: The fuzzy set $x_{(d_Y)} : \mathcal{L}_X \to [0, 1]$ is an extension of the classical set $x_{\mathbb{B}} : \mathcal{L}_X \to \{0, 1\}$. because $x_{(d_Y)}(L) = 1 \;\Leftrightarrow\; x_{\mathbb{B}}(L) = 1$ holds for all $L \in \mathcal{L}_X$.
The Directionality Predicate

In classical projective geometry, the *join* of two distinct points is the unique line that is incident with them. Inspired by the classical terminology, we give the following definition of an approximate join:

**Definition 10** Let $P, Q \in \mathcal{P}_X$ be two approximate points in $(X, d_X)$. An approximate line $L \in \mathcal{L}_X$ that is approximately incident with both, $P$ and $Q$, is called an *approximate join* of $P$ and $Q$.

In the intended interpretation, we may define a *pencil* of approximate joins as follows:

**Definition 11** The *pencil of approximate lines through two approximate points* $P, Q \in \mathcal{P}_X$ is the set of approximate joins of $P$ and $Q$,

$$P \star Q := \{L \in \mathcal{L}_X | P, Q \subset L\}. \tag{6.24}$$

For short, we call $P \star Q$ the *pencil of approximate joins of* $P$ and $Q$. We call

$$S_{(d_{X'})}(P \star Q) := s_{(d_{(d_{X'})})}(P \star Q) = \sup\left\{d_{(d_{X'})}(L, M) | L, M \in P \star Q\right\} \in [0, 1] \tag{6.25}$$

the *extensive size* of $(P \star Q) \subseteq \mathcal{L}_X$, and we call

$$X_{(d_X)}(P \star Q) := 1 - S_{(d_X)}(P \star Q) \in [0, 1] \tag{6.26}$$

the *extensive approximate exactness degree* of $P \star Q$.

We mentioned in Sect. 6.2.1 that the connection of two approximate points $P, Q$ by an approximate line $L$ is not unique, and that, intuitively, the "degree of uniqueness" depends on the size and distance of the involved granular points $P, Q$, cf. Figs. 6.4 and 6.3. This intuition is formalized by the uniqueness axiom

(Pr2) $\forall \mathtt{p}, \mathtt{q}, \mathtt{l}, \mathtt{m}. [\neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{l}) \,\&\, \mathtt{I}(\mathtt{p}, \mathtt{m}) \,\&\, \mathtt{I}(\mathtt{q}, \mathtt{m}) \rightarrow \mathtt{E}(\mathtt{l}, \mathtt{m})]$.

This can be seen as follows: According to the Granular Geometry Framework, we determine the truthlikeness degree of (Pr2) by determining its value when interpreted in $FL_{ev}$(Ł∀). Following Gerla, Wilke [73] shows that the interpretation of (Pr2) in $FL_{ev}$(Ł∀) is equivalent with

$$\inf_{P, Q \in \mathcal{P}_X} \left[d_{(d_X)}(P, Q) \Rightarrow X_{(d_x)}(P \star Q)\right], \tag{6.27}$$

where

$$\left[d_{(d_X)}(P, Q) \Rightarrow X_{(d_x)}(P \star Q)\right] \tag{6.28}$$

is the extension of the formula

$$\forall \mathtt{l}, \mathtt{m}.\Big[\neg \mathtt{E}(\mathtt{p}, \mathtt{q}) \wedge \mathtt{I}(\mathtt{p}, \mathtt{l}) \wedge \mathtt{I}(\mathtt{q}, \mathtt{l}) \wedge \tag{6.29}$$

$$\mathtt{I}(\mathtt{p}, \mathtt{m}) \wedge \mathtt{I}(\mathtt{q}, \mathtt{m}) \rightarrow \mathtt{E}(\mathtt{l}, \mathtt{m})\Big]. \tag{6.30}$$

Equation (6.28) measures the truthlikeness of the assumption "Whenever the extensive distance of $P$ and $Q$ is large, the extensive exactness of $P \star Q$ is also large". Loosely formulated we may say that it measures the truthlikeness of the assumption "The larger the distance, the closer is the approximate join to being unique". We call (6.28) the *directionality degree* of $P$ and $Q$, and define the corresponding *directionality measure* and *directionality predicate* as follows:

**Definition 12** For $e_{(d_X)} : \mathcal{P}_X \times \mathcal{P}_X \rightarrow [0, 1]$ and $i_{:\triangle}\mathcal{P}_X \times \mathcal{L}_X \rightarrow \{0, 1\}$, the *directionality measure* induced by $e_{(d_X)}$ and $i_\triangle$ is the fuzzy relation $dir : \mathcal{P}_X \times \mathcal{P}_X \rightarrow [0, 1]$,

$$dir(P, Q) := \left[d_{(d_X)}(P, Q) \Rightarrow X_{(d_X)}(P \star Q)\right] \in [0, 1] \tag{6.31}$$

We call $dir(P, Q)$ the *directionality degree* of the pair $P, Q \in \mathcal{P}_X$ w.r.t. $e_{(d_X)}, i_\Delta$. The directionality measure $dir$ is the extension of the *directionality predicate*

$$\mathtt{Dir(p, q)} :\equiv \forall \mathtt{l, m}.\Big[\neg \mathtt{E(p, q)} \wedge \mathtt{I(p, l)} \wedge \mathtt{I(q, l)} \wedge \\ \mathtt{I(p, m)} \wedge \mathtt{I(q, m)} \rightarrow \mathtt{E(l, m)}\Big]. \tag{6.32}$$

The Extended Axioms

Wilke [73, 74] shows that the following extensions (E3x) and (Pr2x) of the transitivity and uniqueness axioms have a truthlikeness degree of 1. The resulting set of axioms in $F\L_{ev}$ is the following crisp axiom set:

$(\mathsf{E1})_{\mathsf{ev}}$ $\Big(\forall \mathtt{x}.\,[\mathtt{E(x, x)}]; 1\Big)$,
$(\mathsf{E2})_{\mathsf{ev}}$ $\Big(\forall \mathtt{x, y}.\,[\mathtt{E(x, y)} \rightarrow \mathtt{E(y, x)}]; 1\Big)$,
$(\mathsf{E3x})_{\mathsf{ev}}$ $\Big(\forall \mathtt{x, y, z}.\,[\mathtt{E(x, y)}\ \&\ \mathtt{X(y)}\ \&\ \mathtt{E(y, z)} \rightarrow \mathtt{E(x, z)}]; 1\Big)$,

and

$(\mathsf{Pr1})_{\mathsf{ev}}$ $\Big(\forall \mathtt{p, q}.\exists \mathtt{l}.\,[\neg\mathtt{E(p, q)} \rightarrow \mathtt{I(p, l)}\ \&\ \mathtt{I(q, l)}]; 1\Big)$,
$(\mathsf{Pr2x})_{\mathsf{ev}}$ $\Big(\forall \mathtt{p, q, l, m}.\Big[\neg\mathtt{E(p, q)}\ \&\ \mathtt{Dir(p, q)}\ \&\ \mathtt{I(p, l)}\ \&\ \mathtt{I(q, l)}\ \& \\ \mathtt{I(p, m)}\ \&\ \mathtt{I(q, m)} \rightarrow \mathtt{E(l, m)}\Big]; 1\Big)$,

Consequently, the above axiom set fulfills all requirements of the Granular Geometry Framework.

## 6.5 Granular Geometries and Zadeh's Restriction-Centered Theory of Truth and Meaning

In 2011, L. Zadeh [82] introduced the concept of a Z-number as a pair $(A, B)$, where "the first component, $A$, is a restriction (constraint) on the values which a real-valued uncertain variable, $\mathfrak{X}$, is allowed to take. The second component, $B$, is a measure of reliability (certainty) of the first component." [82, p. 2923]. A position granule in the sense of the Granular Geometry Framework can be seen as a precisiation of a positional restriction in the sense of Zadeh's restriction-centered theory of truth and meaning (RCT) [81, 83]: Here, Zadeh defines a *restriction* (or *generalized constraint*) in its canonical form by $R(X):\ X\ is_r\ A$, where X is the restricted variable and A is the restricting relation, both typically expressed in natural language. $r$ specifies the way in which $A$ restricts $X$ (the *modality* of $R$). Modalities are, for example,

$r = blank$ for possibilistic restriction, or $r = p$ for probabilistic restrictions. For $r = blank$, $R(X)$ can be written as $Poss(X = x) = \mu_A(x)$.

In Approximate Tolerance Geometry, an approximate point $P$ is a possibilistic restriction of the position of an exact point $p$. It can be written as $R(X) : \; X \; is \; P$, or, more specifically, $Poss(X = p) = \mu_P(p)$, where $\mu_P$ is given by the possibility distribution (6.1). To compute the degree to which $X$ satisfies $R$, it is necessary provide a precisiation. In ATG, the restriction is unprecisiated, if $P$ is given in natural language, e.g. $P =$ "Piccadilly Circus". It is precisiated, if it is given as a subset of a metric space.

The intended interpretations of many of the fuzzy geometric statements $(\varphi; a)$ in ATG are *Z-numbers* $(\phi, a)$, where $\phi$ is the intended interpretation of the classical formula $\varphi$. To see that, observe that a position granule $P \in \mathcal{P}_X$ is a restriction of a (possibilistic) uncertain variable $p \in X$. The intended interpretation $\phi$ of a statement $\varphi$ that includes $p$ is in many cases also a restriction of the corresponding domain. As an example consider the statement $\varphi \equiv (\mathrm{p} = \mathrm{q})$. Its intended interpretation $\phi$ in ATG is $e_{\mathbb{B}}(P, Q) = 1$ ("$P$ and $Q$ are possibly equal"). Since $P$ and $Q$ are restrictions of assumed "true" points $p, q \in X$, respectively, the relation $e_{\mathbb{B}}(P, Q) = 1$ can be seen as a restriction of the uncertain variable $(p, q) \in X \times X$ in the sense of RCT. Here, $p$ and $q$ are the assumed "true" exact points associated with the position granules $P$ and $Q$, respectively. Deviating from Zadeh's definition of a Z-number, $(p, q)$ is not real-valued, and $\phi$ is not a fuzzy *number*, but, more generally, a fuzzy *relation* in $X \times X$.[12] The second component, $a$, is the truthlikeness degree associated with $\varphi$. It is a fuzzy number that measures the reliability (certainty) of the information provided by $\phi$.

Zadeh [82] also proposes a generic schema for the computation with Z-numbers. Zadeh's schema applies on a *semantic* level, operating on the interpretations of statements and is based on the extension principle. In ATG, the use of fuzzy logic with evaluated syntax allows for the "trick" of evaluating the syntax: Calculations can be done on the *syntactic* level of logical propositions, while the evaluation component is derived from the "intended" possibilistic interpretation. It thereby allows for metamathematical considerations such as the possibility to guarantee *soundness* of a theory of granular geometry.

[12] More specifically, $\phi$ is a *crisp* relation. Yet, in principle, the Granular Geometry Framework can be extended to consider not only crisp position granules $P$, but also fuzzy position granules. In the GIS community, fuzzy position granules are discussed under the name "vague regions", cf., e.g. [10].

## 6.6 Conclusions and Outlook

### *6.6.1 Summary*

The introduction of the Granular Geometry Framework is motivated by the intention to provide a geometric calculus for granular geometry that is *sound*. I.e., geometric reasoning with position granules should be *reliable*, even if the introduced uncertainty is very big. For a given modality of uncertainty and a given classical axiomatization of geometry, the framework provides a guideline for fuzzifying the classical axioms so that the result is a sound logical theory of granular geometry. Soundness is achieved by augmenting every classical geometric axiom with a degree of reliability (truthlikeness), resulting in a fuzzy set of classical axioms. The reliability degree is derived from the intended interpretation (intended semantic) of granular geometry and incorporated in the logical theory as part of the syntax as a fuzzy membership degree. As a result, any classical geometric theory that is fuzzified based on the Granular Geometry Framework is sound by design. The underlying formal tool for representing and propagating reliability as an intrinsic part of the logical theory is Łukasiewicz Fuzzy Logic with Evaluated Syntax $FL_{ev}$(Ł∀), which is used as a similarity logic.

Approximate Tolerance Geometry is a *possibilistic* instantiation of the Granular Geometry Framework. We introduced its intended interpretation, derived corresponding truthlikeness (i.e., reliability) measures, and applied them to two fundamental geometric axioms groups, namely the equality axioms and Euclid's First Postulate. Research [73, 74] shows that some of the fuzzified axioms have a reliability degree of zero, and that a reliable fuzzy extension exists that is consistent with the classical axioms and that even has a reliability degree of 1.

### *6.6.2 Limitations*

In her thesis, Wilke [73] elaborates the interpretation of Approximate Tolerance Geometry introduced in Sect. 6.4.2 for the specific case of the *real projective plane* $X = \mathbb{P}^2$. Her research shows that, in the projective interpretation, the extended uniqueness axiom is always *trivially* fulfilled, which effectively renders it useless in practical application. She shows that the approximate join $P \star Q$ of two approximate points always has a directionality degree of zero, meaning that $P$ and $Q$, in the worst case, do not constrain the direction of a connecting approximate line at all, cf. Fig. 6.11. The reason for this is that approximate lines are allowed to be "arbitrarily broad".

With the benefit of the hindsight, this result is not surprising: A theory of granularity is concerned with relating and propagating size restrictionsand if no size restrictions are imposed on some of the objects, results can be arbitrary. In the proposed fuzzy extension of the equality axiom and Euclid's First Postulate, we introduced

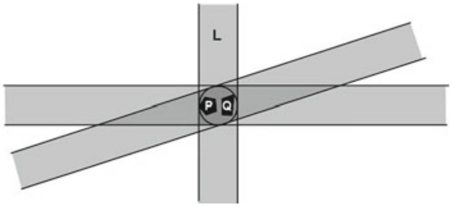


**Fig. 6.11** *P* and *Q* behave like one single point w.r.t. *L*. [76]

size restrictions for *some* of the involved objects, but not for all of them: In the case of the transitivity axiom $(\mathsf{E3x})_{\mathsf{ev}}$, we restricted (parametrized) the size of the middle element $y$ using the exactness predicate, while the sizes of the other two elements, $x$ and $z$, are arbitrary (cf. Sect. 6.4.4). Here, the sizes of $x$, $z$ do not impact the truthlikeness degree, which is why it is not necessary to restrict them. This is different in the case of the "uniqueness axiom" $(\mathsf{Pr2x})_{\mathsf{ev}}$: Here, we did not relate the sizes of the involved approximate points and lines to each other, resulting in an arbitrary statement. The following subsection addresses—amongst other topics—possible approaches to avoinding the problem by including further size restrictions.

### *6.6.3 Further Work*

#### Introducing Size Restrictions

To address the problem of missing size restrictions in ATG, the fuzzified "uniqueness axiom" need to be further extended by also adding exacness predicates for the lines. Wilke [73] also mentiones the presumably simpler solution of introducing *global* size restriction parameters, i.e., upper and lower bounds on the maximal allowable size of approximate points and lines that can be chosen according to the specific practical application scenario. Its incorporation in ATG, and in the Granular Geometry Framework in general, is subject to future work.

It is expected that the necessity of introducing size restrictions not only applies to ATG, but to granular geometries in general. A test should be added to the Granular Geometry Framework that checks, if an axiom is trivially fulfilled.

#### Elaborating ATG

In her PhD research, Wilke [73] applied the Granular Geometry Framework to the equality axioms and to Euclid's First Postulate. To define a full-fledged Approximate Tolerance Geometry of the projective plane, the remaining axioms (Pr3)–(Pr5) need to be fuzzified as well.

She also simplified the definition of the truthlikeness measure for approximate incidence, cf. Definition 6. The goal was to facilitate the formalization task: Instead of employing the overlap relation, she used the binary subset relation to define the truthlikeness degree of incidence of an approximate point with an approximate line.

The formalizations devised in her work should be extended to the overlap relation. It can be expected that this step will add considerable complexity to the formalism.

In order to be able to use approximate tolerance geometry for geometric constructions, e.g. in a geographic information system, it is a precondition that approximate versions of the geometric operators (constructors) *join* and *meet* are available. These are not yet defined in Approximate Tolerance Geometry. The reason for that is that the *approximate join* is not unique in general. As a consequence, an approximate version of the classical join operator is not definable. Instead, only the unique pencil of approximate joins can be assigned to a pair of approximate points. This seems not to be useful in practical application. Wilke suggests to instead use a choice function. Such a function must be defined in the interpretation domain.

Another interesting direction of future work is the elaboration of fuzzy projective notions in ATG, such as approximate direction, approximate angle, or approximate duality. Wilke [73] discusses first considerations in this directions in her thesis.

Finally, the intended interpretation of ATG may be further extended: While the proposed definition of approximate points and lines interprets them by *crisp* regions, *fuzzy* regions may be of interest as well. Here as well, it may be expected that considerable complexity will be added to the formalism with this step.

**Including Other Modalities of Uncertainty**

The Granular Geometry Framework allows for choosing the modality of uncertainty of position granules. In ATG, the intended interpretation of position granules is based on the assumption that location constraints describe possibilistic uncertainty. The Granular Geometry Framework may be applied to other kinds of imperfections in positional information. For example, *verity distributions* can also be modeled by location constraints. Their interpretation is not possibilistic, but veristic, cf. [81]: Instead of interpreting a location constraint as a set of possible positions of an exact point, it is interpreted as a set of points that are occupied at the same time. Examples are parcels or the footprints of buildings, cf. [72].

**Considering Other Gemometries**

The Granular Geometry Framework also allows for choosing the classical geometry and its axiomatization. In ATG, we referred to a standard axiomatization of the projective plane. This is sensible for applications in GIS, but for many other applications, e.g., Euclidean geometry is preferable, and the elaboration of a granular Euclidean geometry is desirable.

**Acknowledgments** Most of the work is part of the author's PhD thesis and was produced while having been employed as a research assistant at the Research Group Geoinformation at the Vienna University of Technology. It was supported by Scholarships of the Austrian Marshallplan Foundation and the Vienna University of Technology. The author would like to thank Prof. Andrew U. Frank, Dr. John Stell and Prof. Lotfi Zadeh for their continuous support and inspiration.

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# Chapter 7
# Inquiry About the Origin and Abundance of Vague Language: An Issue for the Future

**Alejandro Sobrino**

**Abstract** The origin of language has entered the current concern of the study of language as a topic of great interest and debate. Although vagueness is one of the most common features of common language, there are few references to about its roots. From a modern point of view, the main property characterizing human language is the ability to generate infinite sentences using recursion. Current linguistics emphasizes the role of the recursion in the consolidation of human language, underscoring center-embedded or coordinated sentences as the top of complexity in the generation process. But pragmatics, not only syntax, seems to play a relevant role in everyday language. Vague language, as previously said, is almost ubiquitous and present in many of the words we utter. We can hardly imagine a communication without using vague words. Thus, they are constitutive of human language as structural properties do. To inquire about how these words may have arisen in the language evolution and why they are so abundant seems to be an interesting challenge. Gossip is a kind of social communication that uses narratives to generally refer the rules that guide our behavior in the complexities of the social and cultural life. Gossip is related to vagueness as there is unthinkable gossiping using only precise meanings. Using expressions not completely defined, nonsense terms, generalities and vague language are common in gossiping. In this work we will show that many functions attributed to vague lexicon matches with many roles characteristics of gossiping. Thus, gossip seems to be a promising place to inquire the origins and abundance of vague language.

**Keywords** Vague language · Gossip · Linguistic vagueness · Abundance of vague language

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_7

## 7.1 Introduction: Vagueness in Syntax, Semantics and Pragmatics

From a philosophical point of view, the North American linguist Ch. Morris settled the study of language in three areas: syntax, semantic and pragmatics. In all of them can be traced the importance and interest to consider different aspects of vagueness, as a pervasive characteristic of natural language.

Vagueness has a double presence in syntax: as an attribute linked to undecidability and as a property of some grammatical categories that overlaps between them.

From modern linguistic (cf. [17]), recursion is a critical aspect of the language order and productivity. Recursion facilitates generating infinite sentences from a discrete set of symbols or alphabet, making language creative in essence. The discrete infinitude offers a demarcation criterion that distinguishes human language from animal communication systems. Natural languages are not recursive, but recursively enumerable, because exist a grammar to diagnose grammatical sentences, but not an effective procedure to prove ungrammatically ones. A sentence can fail to be grammatical due to its malformation or because it has not been fully formed. Because natural language is recursively enumerable, it leaves a loophole to vagueness. Adopting the computational theory of mind, Changizi [8], approached vague sentences as those for which human beings have no criterion to decide their truth or falsity; thus, they are neither true nor false. If an element is a borderline case of a vague property (as tall, round, truth, …), we have no way to ensure if the element holds or not under the extension of that property; there is no an algorithm to decide whether or not it has the property: it is an undecidable case.

Vagueness also holds in syntax due to the existence of grammatical categories that presents borderline cases. Grammar books recognize eight parts of speech, nouns and verbs regarded as primary. Bolinger [4], spoke about the gradience in grammar to highlight that linguistic does not pay attention, until then, to fringe cases. Chomsky [9], himself noted the convenience to attribute a degree of grammaticalness other than 0 or 1 to deviant sentences from the well-formed set of phrases generated by a generative grammar. The ideal hearer-speaker of a language is fully grammatical producing his/her sentences, but the acceptability of language requires gradedness. In this vein, Ross [30], argued that nouns, for example, falls into a hierarchy of nounniness, as the parsing shows that some nouns are 'nounier' that others. There are prototypical models of grammar categories, but also deviances from the dichotomy grammatical/ungrammatical that are better represented using degrees. In so far as grammar shows borderline cases, it is fuzzy.

Semantic deals with the meaning of the words or sentences. Traditionally, semantic relates linguistic symbols with the real or ideal entities they refer. Semantic places the language in the framework of communication, aiming to get a representation of the transmitted idea as close as of that the transmitter had in his mind. Meaning is frequently related with our perceptual capabilities and memory. We not only mean what we wont, but what we can. Vagueness emerges in this border. As our senses are limited, we need to use vague words in order to not commit too much with our assertions, to be respectful with our partners in the communication process. Vagueness is related to prudence in expression. Attending this property, semantics goes towards pragmatics.

Pragmatics is the use of context in order to make the message adequate and useful. In this task, vague language is an inescapable tool for communication due to, at least, two factors: (a) Vague language makes communication easier and fluid. Try to communicate in ordinary life using only precise language and see that it is impossible. (b) Vague language has proved useful in situations where, still having poor data, we need to make reasonable decisions using them. Vague language permits to manage incomplete sets of data with high efficacy. Decision-making theory offers abundant examples of rational choices based on poor data, endorsing the success of vague language in ordinary communication.

D. Everett [16], recently introduced a relevant factor characterizing human language from a pragmatic or cultural perspective. In his view, prior to communication is socialization or how we best interact with each other, back at the time of the child's interplay with the mother in the placenta. Socialization is also a key point in the Calvin and Bickerton's [5], approach to the origin of language. Even if in the emergence of the first language (protolanguage) had priority food and reproduction over socialization, the tribe is required if language is developed in its fullest. Education is the key element in the socialization. Protolanguage included perhaps verbs, denoting actions, and names, designating the subjects and objects of the actions. No trace of adjectives, quantifiers or other lexicon involved with vagueness. Nevertheless in order to clarify the situations related to survival scenarios, the use of vague vocabulary was perhaps paradoxically decisive. To defend against a predator maybe it was relevant to determine if there were many or few opponents in the environment, 'many' and 'few' vague quantifiers; to achieve food it was perhaps decisive to define if a berry is a little or almost nothing poisonous, 'a little' and 'almost nothing' semantic hedges, etc. Getting a map of the social relationships between human beings and between them and the environment was relevant to understand the evolution. Hearing about the adventures and misadventures of others qualify us to know the essential rules of our culture, including morals, traditions or laws, forwarding us to face them. Gossip is a linguistic practice that helps us in that task.

Pragmatics involved in socialization is relevant in our life, but behavior left as sediment structural patterns, such as reversibility in language, denoted for example through the word 'reciprocally'. 'Reciprocally' does not denote something specific; an object or a being. It denotes an action referring the own language. In 'Mary loves John and reciprocally', 'reciprocally' authorize us to reverse the order of subject and object in the alluded sentence. Calvin and Bickerton [5] emphasized syntax as a kind of scheme distilled from sentences denoting relevant practices in evolution. For example, reversible schemas, as symmetry, perhaps came from reversible practices, as delousing.

Dunbar [12], stressed the relationship between socialization and grooming and their relevance in the origin of language. Social grooming used by apes serves to bond and reinforce social structures, sharing that custom in their communities. In order to be effective and prevent diseases, a grooming activity as delousing should be practice in both directions: I will remove the louses to you and you, in correspondence, remove the louses to me. Cooperation as a means to acquire abstract patterns of behavior, finally reflected in the language by words as ‘reciprocally’ is, as aforementioned, a key factor in the development of the language syntax. But socialization involves behaviors that are not only immediately useful and readily visible. Human beings are defined by its ascription to a species, but also to a culture. Culture is often referred as an integrated pattern of human knowledge, belief, and behavior involving symbolic thought and social learning. Human beings need nature as much as nurture to live and nurture is mainly related to pragmatics, not to syntax.

A cultural event that has been given increasing importance in the development of the human species is that of gossip, the linguistic evolution of grooming. Gossip encourages talking extensively about personal or social topics, without few restrictions on the lexicon employed. To use gossip means to prioritize socialization or communication over representation, as Dan Everett pointed out regarding the origin of language. Emler [15], said that about 80 % of conversation time involved gossip. Our thesis is that in the origin of language, vague words played a relevant role favoring socialization through a possibly deficient but easy communication, encouraging the exchange of messages even if they did not have crisp meanings at hand. If precise words were required in the past to communicate, probably today we would have neither culture nor specie. To develop language gossip is needed and to fully develop gossip, vague language is demanded. Gossip is the oral correlate of the observational learning, allowing us to quickly gain information about our neighbors and contributing to cement our social links. As observational learning is always improvable if we have more and better observational instruments, vague language always tolerates precisification. But accuracy will always admits more precision, becoming futile any aimed ultimate precision.

Vague language is at the base of our community life, the engine of our communications and socialization. Vagueness shows its utility and function in the optimization of elections involving poor data, general contexts or ill-defined scenarios, common facts in our daily life. Vague words are widely used by humans in their conversational practices, helping to communicate our emotions, feelings or thoughts even if we have not an absolute or certain knowledge of what we mean; thus, vagueness prevents paralysis or silence. As the usefulness of gossip and vague language are closely related, we conjecture that gossip perhaps allows us to inquiry in the origin of the vague language.

In order to approach this conjecture, the paper will attend the following plan: In point 2 we will show that vagueness is present in both empirical language and formal language. In empirical language vagueness is related with observation or memory. Because memory and observation are common cognitive skills, perhaps vagueness is so abundant in natural language. In formal language, vagueness is linked with recursion. In point 3 we will describe some distinctive aspects of vagueness from related phenomena, as ambiguity or generality and the lexicon generally associated with vagueness are presented. In point 4 we will show gossip as a way to extensively use the language with the purpose of communication and socialization, including vagueness as a form of sociability. The use of a vague word carries an incomplete meaning that must be filled by another participant in the dialogue, fostering collaboration and negotiation rather than imposition. In point 5 the functions of vague language are highlighted and the vagueness as a linguistic device aiding to take good decisions in poor communicative scenarios is addressed. Finally, we conclude that gossip and vagueness show relations in so far as their functions match at a good extent. The conjecture that gossip can explain the origin and abundance of vague words in natural language is defended. Acknowledgements and references will complete the work.

Inquiry about the origin of vague language is an attractive task to deepening in the essentials of this feature of human language. Addressing the role and function of vagueness in human conversation perhaps help us to better understand this phenomenon.

## 7.2 The Pervasiveness of Vagueness: Vagueness in both an Empirical and a Formal Discourse

In an empirical frame, vagueness is related with observational predicates and with the limits of the human memory. In a formal setting, Changizi [8], related vagueness with undecidability, embracing the double hypothesis that human mind operates as a computer and that the meaning of the sentences produced by our mind can be delimited by an effective procedure that is not recursive, but recursive enumerable.

It is a common place to say that observational predicates are vague predicates (Cf. M. Dummet [13]). Color terms are a typical example of observational terms. Although it is possible to know with absolute precision the wavelength of a given color, we can still have serious doubts about naming it with a word or another (for example, with ‘red’ or ‘vermillion’ in English language). Berlin and Kay [2], put forwarded some conditional universals of color after studying 98 languages. The generalization they come is as follows: If a language has two terms of color, it discriminates between ‘black’ and ‘white’. If the same language has another term of color, then it names the ‘red’ color. If a language has four terms of color, them it has one term for ‘green’ or for ‘yellow’, but not for both, ..., and so on. Moral: although the color characterization in terms of the wavelength is unique, it can vary for different people or cultures and still for people sharing a culture. Colors are culture-dependent: and object -the sun- is named ‘yellow’ in English but ‘okora’ in Mbembe, a Nigeria language, denoting okora ‘red’, ‘orange’ and ‘yelow’ in English. Then, there is an overlap between different color denominations attending different cultures. Moreover, color transitions are fuzzy into the same culture. An English speaker distinguishes black and white color, but perhaps, in some situations, as poor lighting, he/she has no way to difference a dark gray from a black jersey. In sum and as Lakoff [19, p. 29] pointed out: ‘*colors categorization makes use of human biology, but color categories are more than merely a consequence of the nature plus human biology plus a cognitive mechanisms. They have some of the characteristics of fuzzy set theory plus a culture-specific choice of which basic color categories they are.*’

In his investigations about Piraha, D. Everett emphasized the cultural source of the observational predicates. He reported as Piraha language has only four words for naming colors. Those words matches the corresponding Berlin & Kay's conditional universals, although Everett stressed that using the word 'red', Piraha language denotes something that requires a kind of description, as 'It is like blood'. This fact leads Everett to conclude that Piraha has not exactly words for color, but sentences including the description of a prototype of the named color. Everett [16, p. 259] said that the moral is that it is possible to classify objects in the world without using words. But descriptions may be so lax: Having only four names for colors, Piraha people perhaps describes 'brown' as 'like a liana'. But 'like a liana' can refer to color but also to strength. In this sense, categorization becomes both an ambiguous and a fuzzy job.

Vagueness involved in observational predicates provokes sorites-like paradoxes. The popularization of the Sorites paradox often uses words as 'bald' or 'heap', both observational predicates; in fact, Sorites paradox is frequently named 'the bald paradox' or 'the heap paradox'. Although vague predicates are closely related with everyday language, Parikh [26], noted that they are also relevant in the scientific practice in so far as observation is a main part of the scientific method. In order to deal with observational predicates in science, Parikh analyzed the continuity of the observational predicates introducing the concept of '?-connected' and 'connected metrical space'. A metric space is $\alpha$-connected if it cannot be divided into two disjoint subsets $A$, $B$ ($A, B \neq \emptyset$), such that, for two arbitrary elements $x \in A$, $y \in B$, the distance $d(x, y) > \alpha$, $\alpha$ denoting a real number. A connected metric space satisfies the previous condition for any $\alpha$.

Observation can be practiced with our senses or helping us with a measuring instrument. If observation is made with our senses (vision), we can say for example, that the length of a pen is high, very high or perhaps, that it measures approximately 15 cm. Any effort to measure more precisely using our vision it is doomed to failure. Perhaps all we can adjust the previous measure is tuning it saying that the length of the pen is 15 cm more than 20 cm, because we feel that the top of the pen is closer to 15 than to 20 in an imaginary measuring tape. In the case that the observational predicates depend on our senses, it is pointless trying to refine the measure, but if we have a measure instrument, it does. Human senses may be perfected with measurements instruments. Telescope or microscope is an example of tool that allows us to see in detail absent realities hitherto. In the aforementioned case, we can employ a tape that measures in mm to better adjust the measure. A tape measure discriminating mm makes precise what was vague before showing that the new true measure is, for example, 155 mm.

It seems that with measurement instruments, vagueness has no place. But the '$\alpha$-connected' property changes that view. If the $\alpha$ is posed in cm. perhaps the tape measure is sufficient to specify the length, but if the $\alpha$ is posed now in mm, the old tape doesn't work because the measure can falls into a range that the cm. are incapable of discriminate. In order to avoid this, a new tape that discriminates mm. should be provided. But the same reasoning as we did before for centimeters, can be done now for millimeters. The moral we can draw is: in a connected metric space is not possible to define, one and for all and in a precise way, the exact denotation of the observational predicates. Although in a practical setting, the refinement of a measure usually ends at the moment in which is considered sufficiently accurate for a given purpose, in theory we can always increase its precision more and more by choosing a shrinking $\alpha$; small and small threshold. As Parikh said: … *this very exactness is a defect. And this restriction applies not only to ordinary people using their eyes and ears, but also to scientists using their instruments…What carpenters and physicists do is inherently inexact* [26, p. 246–248].

Observational predicates depending of our senses don't admit an incremental accuracy. Human senses if aided by measurement instruments can change a vague predicate denoting an observation by a crisp predicate. But the precision is only transitory, an illusion. If the matter demands more precision, marks that separates a predicate and its negation will show an area that is fuzzy, opaque to recognition. And this judge can be repeated again and again.

Vagueness is also related with memory. Human beings have limited senses and memory is an example or such constraints. Consider the following scenario: we go to a stadium and the speaker tell us the exact number of spectators. After a while, perhaps we can only remember vaguely that there were many or a lot of people in the stadium, 'many' or 'a lot of' vague quantifiers. Similarly, and in relation to perception, if someone gets a lost object on a beach, he/she may be able to remind the exact location of the find, but a few minutes later, perhaps he/she can only point vaguely the connected area. Human memory and perception are quite limited and seem to have an compelling tendency to the economy, the summarization and the approximation to what it is said or seen, probably to make room to other things and avoid a cognitive collapse.

In connection with memory and observation, vagueness has a relevant place in natural language. But also has a room in formal language. Changizi [8], made a pioneering study to accommodate vagueness in a formal language. He embraces the computational metaphor of mind advocating that human mind operates as a computer. From that hypothesis, he distinguishes decidable and semi-decidable computational problems, even for human beings. If the semantics of natural language were decidable, an effective procedure would determine if the meaning ascribed to a word is or is not the alleged one. But to appoint meanings is a semidecidable operation because still providing full information about the concerned sentence, in many cases we can have reasonably doubts whether to name (or not) with a word an event or action. For example, knowing the exact member of hairs one person has, don't afford us to name him/her as 'bold' or 'haired' with absolute certainty. Perhaps to build a robot that manages this essential uncertainty is a major challenge in Artificial Intelligence. Up till now AI aspires to design artifacts that solve imprecision with precise mathematic, be discrete, probabilistic or fuzzy. But the imprecision can be tackled only essentially if we manage it imprecisely; the opposite is to adjust (to regimentate) vagueness adapting it to the model used in the representation. However, today is really a chimera to know what a vague algorithm is.

Vagueness is a pervasive fact of natural language because to observe and to memorize are very common activities of our life. If our daily hypothesis are mainly about what we observe and those hypothesis are supported, contravened, influenced or backed by our memory, and observation and memorization are, as human activities, imperfect techniques, the use of vague language is motivated. In the next point, we will confirm the abundance of vague language.

## 7.3 Vagueness and Related Phenomena: The Vocabulary of Vagueness

Vagueness is a ubiquitous feature of natural language. Opposite to the language of science, ideally precise, natural language is mostly inaccurate, imperfect or vague. If you transcribe a conversation, you may notice that vague words and incomplete expressions are plentiful. If you read a novel, where the language is used in full, vague expressions and vague words are the norm, while words denoting accurately are the exception. It is a matter of fact that natural language includes extensively vague words. We will show that in this section.

But previously, some digressions about the specificity of vagueness are required. Usually vagueness is confused with ambiguity and generality. Although there are some borderline properties overlapping these phenomena, it is possible to make some key distinctions that separate one of the other. There are a lot of definitions of a vague word. The Sainbury's [32] one is brief, but substantial: the main feature of a vague word is that it denotes boundaryless concepts; i.e., concepts without fringes, without borders separating the objects that fall and not fall under the extension of the concept. 'Tall' is a typical example of vague word.

Vagueness is distinguished from ambiguity. Ambiguity emerges both at the level of words and at the level of sentences. At the level of words, ambiguity is related with polysemy. A polysemic word has a definition including alternative meanings, but one in each sense. Thus, the word 'bench' has two meanings: a kind of furniture used to sit (*I sat down on a park bench*) or a kind of table (*I'm sawing a table in the work bench*). In each sense, the reference is essentially crisp: we can distinguish in a park an object that is a bench from another that it is not.

At the level of the sentences, ambiguity is related with phrases that have more than one interpretation. The phrase "*I saw the man with the binoculars*" admits two interpretations: Binoculars are mine or I saw a man looking through his binoculars. Both sentences have different meanings, but in each interpretation, the meaning is essentially crisp. Thus, we can say that a word or a phrase is ambiguous if there are several contexts that can be unmistakably chosen to specify the meaning of the sentence or word. In this vein, if a word is vague, contexts can help to bring the margins near, but not to definitively demarcate them. This is a key difference between vagueness and ambiguity.

Vagueness is also different from generality. A term is general if it refers loosely. Thus, the word 'fish' is used with generality if we are demanding information of a kind of fish; for example a sardine. The word 'herd' is general if we need to know the exact number of 'heads' for a commercial transaction. A general term is a word not specified enough, but they could be completely defined if more information is requested and available. This does not happen with a vague word. 'Tall' remains a vague word even knowing the exact measure of the height of a person (170 cm); even approximating the measure as much as we want (1705 mm). That is the difference with generality.

Still a final differentiation should be quoted. Since the emergence of the fuzzy logic, it is usual to differentiate vagueness from fuzziness. E. Trillas [35, p. 12], advocates that fuzziness is vagueness accepting to be represented or modeled by fuzzy logic. Ungerer and Smidt [36], posited another distinction between vagueness and fuzziness, referring vagueness to entities or objects and fuzziness to properties. Vagueness is used to denote the boundaryless between one entity and another, both of which are parts of the same whole. For example at the Himalayas is vague where ends Mount Everest and begins Mount Lhotse. Mount Everest and Mount Lhotse are vague entities. Fuzziness however applies to the boundary between one category and another. The boundaries between the Mount Everest and the Silence Valley are ill-defined and consequently, fuzzy. Although conceptually flashy, I guess that the previous distinction has not main implications in the differentiation of vagueness from other related phenomena.

Vagueness is a property of natural language with its own peculiarities and it deserves to be pursued. Vagueness, involved in words or expressions, is present in most discourses, whether academic or informal. Opposed to precision, typical of numbers or words denoting numerical characteristics, common words are vague and are traditionally associated to a negative valuation. As Zadeh [33, p. 5], pointed out "there is a very deep-seated tradition according much more respect to numbers than to words…Numbers are respected, words are not ... (but) when true numbers are not known or are too costly to obtain ... words are good enough". Vague words facilitate communication and cooperation between human beings. The task to exchange a lot of messages everyday is not conceivable if vague words didn't have an extensive presence in the language. There are many words and expressions that count as vague language in English. We will show some of them below.

There are some types of highly vague words in natural language. In a pioneer investigation [11], advanced some general categories accommodating the vague lexicon: (a) placeholders, (*thingummy*, *thingy*); (b) summarizing lexical items (e.g., *and everything*, *and that*); (c) vague generic terms and collective nouns (e.g., *heaps*, *bold*), (d) approximate quantities (e.g. *around five*); (e) words with the suffix –ish (e.g. *boyish*, *fortyish*).

Although Crystal and Davy's approach was about language in general, the specific study of Channell [7], about the vague lexicon matches, in a good extent, the aforementioned classification. Semantic hedges, a term coined by Lakoff in [20], should be added to that classification. Hedges are expressions that permit the speaker to circumvent categorical or straightforward assertions, avoiding unwanted commitments in situations that are not clear enough or that we want to emphasize as very clear. It is not infrequently to approach hedges as a kind of approximator. Ruzaité [31], aimed to summarize all of these contributions in the following classification:

1. Placeholders
2. General extenders
3. Approximators
4. Vague quantifiers.

Vague words are highly context-dependent and subordinated to the background of the hearer. Categories showed above classify almost all prototypical vague words. Next we will approach some of their representative properties.

### *7.3.1 Placeholders (whatsisname, thingy)*

Channell [7, p. 164] defined placeholders as '*dummy nouns which stand for item names or for names of persons*'. They are words referring to objects or people whose names are irrelevant or unknown; words without meaning but serving some syntactic function. They are used for several purposes, playing a relevant role in oral parlance: using them, the speaker avoids be pretentious, ambitious or offensive with the hearer.

### *7.3.2 General Extenders (and son forth, something like that)*

General extenders refer vaguely to a set or category. Channell [7] supported that general extenders denote lexical gaps, as the lack of a label for an inexistent category (e.g., the label for alive and death in the Schrödinger's cat) or a superordinate category covering subordinate ones (e.g. *precipitation* as a category level for *rain*, *snow* or *hail*). General extenders as *and something like that, or whatever, and so on* are used to share a background between speaker and hearer. Simpson [34] approached general extenders in the frame of the academic discourse, concluding after an empirical study that *and so forth, and so on* favors teachers discourse and *something like that, things like that* are employed basically by pupils.

### *7.3.3 Approximators (approximately, more that, roughly)*

Approximators are words that refer vaguely to quantities. Typical approximators are *almost*, *or less*, *at least*, *more than*, defined by Jucker et al. [18], as either a lower or a higher level for quantities. Channell [7, p. 42], defined an approximator as a component of a sentence containing an exemplar number and usually a measure noun, matching the following pattern:

Approximator (*about*) + Exemplar number (*four*) + Measure noun (*trees*)

In some cases, approximators roughly can appear after an exemplar number and a measure noun, as occurs in *five weeks roughly*. Excluding this exception, we can say, following the above formula, that an approximator is a lexical category that precedes a cardinal number to make it less specific. If applied to degree adjectives or adverbs, approximators become Lakoff's semantic hedges.

### 7.3.4 Vague Quantifiers (few, many, most)

Quantifiers, as Ruzaité [31, p. 41] said, *are non-numerical expressions used for referring to quantities; they answer the question How-much?* Channell [6, p. 115], put the major utilities typical of vague quantities: (a) macrodiscourse constraints (structure, genre, readability); (b) giving the right amount of information; (c) persuading the reader; (d) lacking specific information (displacement); and (e) downgrading and highlighting. Moxey and Sanford [23] advocate for the use of quantifiers as they permits to easily remember situations difficult to recall if we use crisp numbers. They conclude that a vague approach of a contextualized sentence is better than a numerical representation of a decontextualized word.

Quantifiers can be negative or positive. Negative quantifiers,—*little*, *few* or *neither ... nor*—, serve to emphasize the poverty or even the absence of information in a field. They are referred also as 'downtoners', because they decrease the scalar intensity of verbs and adjectives they accompany. *Little* and *a bit* are called minimizers, *little* being a negative and *a bit* a nonassertive one. *Much* or *many* are intensifiers or amplifiers, playing a special role in topic generalizations.

Although not mentioned in the Ruzait'e's classification, comparatives deserve a special mention. Many philosophers and linguists consider comparatives as a clear index of vagueness. Comparatives provide a test for checking whether a predicate is inherently gradable or not. In effect, a vague predicate can be translated into a conditional sentence in where comparatives play a major role: If P is a vague predicate, $X$ is $P$ can be translated to $X$ is $P$-er than ($X$ is tall $\rightarrow$ $X$ is taller than ...). But comparatives can be used to precisificate vagueness. That happens if $X$ is $P$-er than is interpreted as:

(a) $X$ exceeds the *average* of $P$;
(b) $X$ beats the *majority* of the members of the set to which $X$ belongs to.

Some researchers use the previous interpretation for turning vagueness into precision, but the attempt is doomed to failure. In a connected metric space, it is not reasonable to argue that we can discriminate perfectly in terms of a predicate and its classical negation. If on the scale [1, 10] the average is 5, there is much more distance between 5.01 and 10 than between 4.99 and 5.01. Then, to say that 5.01 is true and 4.99 is false poses many problems as pass or fail a student with those grades. To show tolerance to degrees represent progress not only in human behavior, but also in technology. Setting a threshold as an absolute cut is a matter of convention rather than of reason.

Lexical vagueness concerns both with semantic and pragmatics. As a question of meaning, vague words denote loosely; as a question of use, vague words are context and knowledge dependent. To end this paragraph we will explore another possible place for linguistic vagueness: syntactic or structural vagueness.

Syntax and vagueness seem to be distant: syntax is governed by rules and vagueness, by uses in language games. It is possible to find a case of grammatical vagueness, even though linked to ambiguity, a structural feature of some sentences. Look at this example: two names together (*apple*, *pear*) don't involve vagueness. A name attached to a vague adjective leads to a vague expression—vagueness is infectious—(*big*, *apple*). But if we add a vague adjective to two nouns, then we reach both an ambiguous and a vague sentence (*big apple and pear*).

The inability to decide the membership of an element to a given property perhaps influences the bad reputation of vagueness. If vague language usually involves negative valuations, crisp language is normally associated to positive facets. In ordinary language sentences beginning with 'exactly' are highly valued and sentences including 'loosely speaking' are poor considered. But vague language is neither bad nor good; it should be used rightly. If used appropriately, the relevant is not precision, but clarity, as Popper [28], highlighted.

## 7.4 Gossip Language and Gossip Roles

In Chomsky's view, language is dependent on syntax and syntax requires a mathematical brain module, able to manage recursion. The emergence of language is an abrupt process culminating with a mutation. Evolutionary theories, that understand language as a gradual process, have no role in this formal explanation of language. But to imagine complex syntactic structures coming from other more simple perhaps surfacing from unstructured signals is not meaningless. Environment and social factors have a prominent role in this approach. R. Dunbar [12] championed the *Social Bonding Hypothesis* as the evolutionary thesis about the origin of the human language. In this approach, grooming and gossip play a key role. Although the hypotheses about the origin of language are pen and paper conjectures held by countless arguments, the Dumbar's one seems reasonable as put social intelligence, based on in the increasing interaction in groups, in the focus of the origin of language.

Grooming is a type of behavior by means of which apes belonging to a group clean or maintain one another's body or appearance. Grooming is characteristic of animals. Gossip can be defined as exchanging personal information about absent third parties providing a positive or negative evaluation. Gossip is typical of human beings.

One of the most important factors in the human socialization is grooming and gossip. First steps in mammal species communication seem to be always about relationships. Grooming favors that task, and gossip completes it. As Dunbar said: *... language evolved among humans to replace social grooming because grooming time required by our large groups made impossible demands on our time. Language, I argue, evolved to fill the gap because it allows us to use the time we have available* *for social interaction more efficiently* [12, p. 192]. In so far as primate groups become larger, more efficient instruments that grooming are required in order to maintain the social coherence. The hands should be free to use tools or bear arms. Other muscles should be employed in the task of social cohesion. The voice box, which includes some of the more subtle movements that human beings do, provided an efficient channel to exchange social information. Language favors 'grooming at a distance'.

When humans become bipedal in the African savannah, competitors threatening for surviving gain advantage. In that scenario, belonging to a group was essential to reach protection. Individual defense require allies to be effective. The Dunbar's thesis suggested that the glue maintaining alliances in higher primates is mutual grooming. But an increase in the size of groups requires a growth of alliances, in so far as the grooming time required to maintain bonds is proportional to the size of groups. Then, the substitution of grooming by vocal signals in humans provides the benefit to establish bonds without requiring physical presence. According to Dunbar, it was from this 'contact calls', characteristics of monkeys and apes, so that language emerged. Language appears in humans to replace grooming. But oral language not fully replaces the gestures involved in sign language.

Signals associated to language, as smile or laughing has the effect to introduce confidence in conversation that oral discourse don't have. Signs substitute grooming in socialization, but oral language doesn't substitute completely gestures. For example signals of proximity or distance are relevant marks to diagnose whether the neighbor is or is not an ally.

Isolated signs are necessary but they are not enough to form language. Language is, above all, sentences, i. e, structured words that serve a hierarchy showing a kind of geometry reflected in a syntactic tree. Novak et al. [24], attractively suggests that when the number of the signs required to socialization is too large, syntactic structures emerge: '*the crucial step that guided the transition from non-syntactic communication was an increase in the number of relevant events that could be referred to*' [24, p. 497]. While hominid communication uses gestures pointing to objects, syntactic communication characteristic of full language uses sentences representing the world.

Based on this thesis, language can be explained by a gradualist hypothesis, with fewer commitments than a nativist one, based on mutation and radical change. It is reasonable to think that natural selection favored individuals who employed words to liberate hands in social activities and that words were syntactically organized when the message become complex. M. Corbalis [10], a specialist in recursion at language, endorsed this thesis.

Language allows us to interchange messages each other even at distance, without the presence of the object. Contrarily, apes can only know what they see by direct observation. Gossip overtakes the observational learning: interchanging messages about the successes and misadventures of others using a loose language, people learn how to act in community. Undertaken this task, gossip is cheap, easy and effective.

Baumeister et al. [1, p. 112] suggested two different functions of gossiping as a tool cementing social relationships:

(a) gossiping, the teller and the hearer strength their links as they share information of mutual interest.
(b) gossiping, the teller and the hearer try to share a common context to favor communication and exchange of information.

In both tasks it is expected that the teller and the hearer use a vague lexicon. Unfortunately there are no yet studies about the lexicon of gossip. In some papers, gossip is related to the language of women, although today is tested that gossip is not only a female job. Further, the gossip is linked to the absence of specificity in discourse. In gossip what prevails is the act of communicating, not the accuracy of what is communicated. Gossip is to socialize, to share, no to prove, to make science. Briefly, we sum some linguistic characteristics of the gossiping activity:

(1) Gossip uses narratives to communicate.
Gossip uses narratives to communicate social norms or sanctions if the rules are broken in a community. It is a kind of low-cost control method that culture uses to standardize individuals' behavior. Gossip is not a kind of story for passive hearers, as it is a monologue, but a rather collaborative offer that encourages the hearer to complete the script. As that is a characteristic of vague words, gossiping ask for the use of vague language.
(2) Gossip facilitates information flow.
Gossip makes communications easier. It is considered an efficient tool for gathering and disseminating information. Wert and Salovey [38], argued that gossip avoids the risks of confrontation in communication: if the information requested is negative, gossip is the best election in order to mitigate a possible conflict. A typical feature of vague language is that it favors politeness and negotiation.
(3) Gossip provides stimulation for interaction at a low price. Rosnow and Fine [29], highlighted that gossip is a bulwark against life's monotony. Gossip favors human contact as money benefits trade.

Topics 1–3 need vague language. Next we will see that the uses of vague language are quite consistent with those required by the gossip ones.

## 7.5 The Usefulness of Vague Language

Vague lexicon is so abundant because they play a major role in human communication. Right after, we will approach some tasks associated to the use of vague language, convinced that, as Jucker [18] said, in some contexts, vague language can be more effective than precision and rarely leads to misunderstanding. Channell [7] highlights the following functions of vague language:

1. Give the right amount of information and withhold the needed information
2. Don't stop if precise information is not available. Look for approximate conclusions
3. Encourage negotiation and deal.
4. Favor politeness and self-protection
5. Contribute to informality

Next, we will briefly expand those points.

1. Vague language is frequently used in daily conversations because even if we dont't know all details involved in a scene, we want to speak about it. Humans spent much time talking about several subjects. Only a small part of these issues refer to 'serious' subjects, with a clear explanatory basis or with a vocabulary denoting with absolute precision. Most of the things we speak about are referred in an incomplete or inaccurate manner, without reaching a clear conclusion as a consequence from a delimited set of premises. In daily life we use language preferably not to represent reality, but to thrill, to empathize with our fellows. The success of a conversational thread is due more to the intention of the speaker and listener to respectively understand and be understood, than for the use of precise words. Try to have a conversation using only precise vocabulary and easily check yourself that it is crazy project. Even in scientific context, Popper said that: "*there can be no point in trying to be more precise than our problem demands.*" [28, p. 28].
2. In some cases vague lexicon serves to fix a kind of a maximum threshold beyond which our sentence would not be responsibly used. Channell [7] displayed as example of vague quantity the phrase: 'Some 200 million tonnes'. This sentence, used by an economist, perhaps becomes an argument both persuasive and honest, because even not having the exact figure to represent a crisp quantity, using a vague quantifier transmits the conviction that he/she is approximating the raw data. The use of vague language in daily argumentation serves to persuade the listener of our convictions. An example comes from approximate syllogism. In the Aristotelian frame, from the two following premises

   *Almost all young people are healthy*
   *Few salutary people use drugs*

   it is not strictly possible to derive any conclusion, because (1) the first and the second premise include uncovered vague quantifiers (almost, few) in the Aristotelian syllogistic and (2) there is no perfect matching between the words representing the middle term, even if '*healthy*' and '*salutary*' are synonyms. However, using common sense and vague quantifiers, most people would say that some conclusion reasonably follows from the premises: in particular, among others, that '*Few young people use drugs*'. This is a quite convincing conclusion, which may be shared with any listener, favoring a responsible communication in so far as the information included in the conclusion is proportional to the content of the premises.

3. According to Wardhaugh [37] power is best exercised subtly rather than overtly; because acting subtly you can offer others a chance. Accuracy leaves no room for cooperation. While crisp meanings have a definition once and for all, vague meanings require a pact, a deal, because they have open texture and change over the time and context. If either crisp language are employed by professionals as a tactic for mystification, as a strategy to produce a power-oriented discourse dividing professionals and people; vague language bring people closer, because it provides a space for consensus on matters that require interpretation, context and negotiation.
4. Vagueness has the role to reduce the conflict between the speaker and the hearer in a dialogue. Vague words made that sentences are not definitively finished, avoiding perhaps a strong breakup. Loosely speaking, vague words favor politeness. Leech [21, p. 132] suggested six maxims of Politeness related to the academic discourse: a) the Approbation maxim, summarized in: ($a_1$) Minimize the dispraise of other; ($a_2$) Maximize praise of other; ($b$) the Modesty Maxim, that says: ($b_1$) Minimize praise of self; ($b_2$) Maximize dispraise of self. The praise-dispraise scale can be applied in the context of the teacher-student communication. Students should practice the Modesty maxim whereas teachers should apply the Approbation Maxim. In order to communicate, we should minimize the use of impolite beliefs and maximize the utilization of polite thoughts. Vague language is for self-protecting and for protecting the hearer's face.

Vague words are typical of informal discourse. They are usually associated with oral language, not with writing language. Writing language is the language of science. There is no doubt that science is important to the people life, but people spend most time in prosaic themes than in those beyond the ordinary. And to communicate informal subjects uses mainly informal language. Our socialization facilitates a culture we belong to. In society we are individuals and, above all, members of a group. This important achievement depends largely on the extensive use of language, not on the specific or particular use of it. Vague language is extensively used in conversation.

Grice's Maxims (bellow, in cursive) collect the rational principles underlying conversation based on cooperation. In a suggestive work, Channell [7] posited a vague version of them (in bold) that matches, in a good extent, with the functions assigned to vagueness in discourse:

- Maxim of Quantity: (*Be as informative as required, but not too informative*)—**Giving the right amount of information**.
- Maxim of Quality: (*Say what you believe to be true on the basis of adequate evidence*)—**Persuading the reader even lacking specific information**.
- Maxim of Relation: (*Be relevant*)—**Be relevant**, *but if you can't, please try to communicate* (underlined, my added)
- Maxim of Manner: (*Avoid obscurity, ambiguity; be brief and orderly*)—**Graduating, highlighting**.

Vague language has an extensive presence in natural language, favoring communication and relationship between people, being more effective than precision in some tasks. Vagueness shows paradigmatically its usefulness in decision-making process. Next we will briefly exemplify this utility, largely explored since the paper of R. Parihk [27].

In a previous section we have seen the persistence of vagueness in ordinary language and even in formal language. Usually vagueness is linked to the limits of the human capacities to perceive and to recall data from memory. Our daily life is managed under these strong constraints that, however, don't prevent us to continually take decisions founded on scarce information and poor data. If the vagueness is so abundant in daily communication, it should have a role. A language feature that is frequent but useless would be an anomaly in language evolution. Next, we will see that vagueness improves solutions in environments where no accurate data are available.

Truly, sometimes the use of vague words is not appropriate. If a pilot does not receive specific instructions from the control tower on the runway location a foggy day, he/she could have serious trouble landing. This is the Lipman's thesis, advocated in an unpublished paper from 2006 [22]: if precise information is available, there is not a better pay-off function to rightly communicate. This thesis has motivated a lot of papers mainly from the game theory field applied to decision making (f. ex., García-Serra et al. [14, 25]). But if crisp data are not free to use, vagueness shows its usefulness.

Using an illustrative example R. Parikh showed that vagueness carry an advantage for communication when the words used by two people to refer to the same entity are vague. Parikh's game is a coordination game. Next, we briefly describe it.

Renato and Ada works in a school. Ada teaches Computer Science and Renato teaches Philosophy. One day Ada is in her office while Renato is working at home. Next day Renato has to explain the 'Philosophy of Wittgenstein' and ask Ada to bring the book entitled 'Notebook Brown', missed in his office. Ada knows nothing of Philosophy, but she recognizes colors. Suppose that the Notebook Brown has the cover of that color, an observational predicate shared by Ada and Renato, and that Renato has no other brown book on his shelf. Ada tells him what is the book he is asking for. Renato simply says: 'it is brown'. If both agree on what 'brown' refer, Ada will surely find the book and her mission will end with success, but if 'brown' refers differently for Ada and Renato, the book that is brown for Renato is not brown for Ada and conversely. If this mismatch occurs, to help Renato Ada would inspect all books in all shelves if the book is not among those she considers 'brown'. If however Renato and Ada think that each other have used 'brown' as a fuzzy term, Ada looks for books that in her opinion are clearly brown and, if she thinks the book has not founded, she reach among those that, not being clearly brown, are similar in color, overlapping with some of which Renato would classify as brown. The overlap causes to expand the chance to successfully satisfy the Renato request. Parikh demonstrated, using a specific numerical example that, the larger is the overlap between the meaning of 'brown' of Ada and Renato in proportion to the symmetric difference, the more is reduced the search time employed by Ada to satisfy the Renato's request. Moral: in the context of game theory, Parikh's game shows the utility of vague language providing solutions to decision-making problems involving overlapping meanings.

## 7.6 Concluding Remarks

Gossip is a remarkable approach to the study of the origin of language, as Dunbar advanced. The argument based on gossip puts the initial selection pressure based on social intelligence as the trigger of language. However, some primates show advanced interaction practices having a rudimentary theory of mind that does not result in the acquisition of language. Baboon monkeys, for example, are gathered together into large units, but they have not replaced grooming by gossip. Then, it is not explained why they don't develop language skills, even primitive, as those employed in gossiping. As Bickerton said, it is plausible to conjecture that gossip don't emerge until several hundred of words were available: *in its earliest studies its symbols were presumably enumerable in the single digits, and how many items of gossip could you convey with a single set of nine words?* [3, p. 514]. This remark from Bickerton leads us to conjecture that gossip, perhaps, is not related primarily with the origin of language, but with the immediate development of it. In the beginning were perhaps the names (designating subjects and objects) and verbs (designating actions). But once we have nouns and verbs, adjectives and adverbs emerged to refine and make our communication more sociable, less aggressive or abrupt. Gossiping we speak about a third part; thus, using vague language enables us to advance in the careful use of language, moving forward in the process of civilization, in the practices of cohabitation.

The functions of vague language match, in a good extent, with the utilities ascribed to gossip in the origin of language. Gossip contributes to solve a major adaptive problem in large groups, favoring friendship and mutual protection by exchanging socially relevant information. In that task vague language seems to play an inevitable role. Human communication is successful because is highly unspecific. Vague words help maintain communication even referring not accurately. Without vague words, communication would stop soon and silence abruptly emerges. Silence is useful for meditation, but useless in action. Vague language makes possible cooperative actions where consensus and deals, far from leading to optimal and permanent solutions, suggest contextual and evolutionary approaches.

To perform an empirically investigation about the role of vague language in gossip is still a challenge. Find a large sample of vague lexicon used in gossiping would yield some other evidences about the enigmatic question of the origin of vague language. If vague language is so abundant is because it had a prominent role in our evolution. Language is governed by rules of economy, as paradigmatically the Zipf's law shows. If vague language were superficial or useless, it would have disappeared. Vague language had a relevant role in our ascription as *homo sapiens*, because, as gossip suggests, it is in the birth of our linguistic capabilities and of our ability to socialize. To investigate the role of vague language in gossiping is an exciting task in itself, but also a relevant project to show why vagueness is consubstantial with our linguistic essence.

**Acknowledgments** This work was supported by the Spanish Ministry for Economy and Innovation and by the European Regional Development Fund (ERDF/FEDER) under grant TIN2011-29827-C02-02.

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# Chapter 8
# Fuzzy Natural Logic: Towards Mathematical Logic of Human Reasoning

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**Abstract** One of the often repeated proclaims appearing in the papers on fuzzy sets and fuzzy logic is their ability to model semantics of some linguistic expressions so that the inherent vagueness of the former is also captured. Recall that this direction of research was initiated by L.A. Zadeh already in his early papers and since then, most of the applications of fuzzy sets emphasize presence of natural language, at least in hidden form. In this paper we argue that the potential of fuzzy set theory and fuzzy logic is strong enough to enable developing not only a working model of linguistic semantics but even more—to develop a model of natural human reasoning that proceeds in natural language. We bring forward the concept of *fuzzy natural logic* (FNL) that is a mathematical theory whose roots lay in the concept of natural logic developed by linguists and logicians. Of course, this cannot be realized without cooperation with linguists. On the other hand, it seems reasonable not to try to solve all the problems raised by the linguistic research but rather to develop a simplified model that would capture the main features of the semantics of natural language and thus made it possible to realize sophisticated technical applications. In the paper, we will show that basic formalism of FNL has already been established and has potential for further development. We also outline how model of the meaning of basic constituents of natural language (nouns, adjectives, adverbs, verbs) can be developed and the human-like reasoning can proceed.

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_8

## 8.1 Fuzzy Set Theory, Natural Language and Human Reasoning

### 8.1.1 Motivation and History

Fuzzy set theory is the basis of methods that can be, in general, divided into two basic classes: (a) methods with linguistic motivation, and (b) methods with non-linguistic motivation. Typical example of (b) is fuzzy clustering (Cf. e.g., [16]), or the new and very powerful fuzzy transform (See, e.g., [53]).

This paper is focused on methods (a) with linguistic motivation. We suggest, as a possibility for future research, to focus on *fuzzy natural logic* (FNL)—the logic of natural human reasoning for which it is typical to use natural language. Our suggestion stems from the claim that fuzzy set theory has potential to serve as a good tool for modeling of linguistic semantics. This was argued by L.A. Zadeh in many of his papers since the very beginning (Cf. e.g., [63, 65, 66, 68]). It should also be noted, that the first necessary steps towards FNL have already been done.

The problem, however, is not so easy and it requires close cooperation with linguists. Zadeh suggested two simplified paradigms: *computing with words* (Cf. [69]) and *precisiated natural language* (See [70]). In the first case, it is assumed that we should confine to a small number of special linguistic expressions. In the literature, one can meet the term "linguistic label" (Cf. [23, 26, 64]). From the linguistic point of view, these are expressions consisting of degree or evaluative adjective together with (possibly) some hedge. This model, however, is oversimplified and one encounters quite often that the authors have in mind not the given linguistic expressions but linguistically named linearly ordered evaluative categories that are used in various kinds of questionaries. These are introduced to simplify the respondent's work. For example, instead of using the numbers 1–5, one is suggested to consider them as typical examples of very small" (1), "small" (2), "medium" (3), "big" (4) and "very big" (5). These categories are then taken as imprecise quantities whose meaning is modeled using triangular fuzzy sets. We cannot speak in this case, though, that we are using natural language.

The concept of a *precisiated natural language* is wider and it suggests to develop a "reasonable working formalization of the semantics of natural language without pretensions to capture it in detail and fineness." The goal is to provide an acceptable and applicable technical solution. The concept of PNL is based on two main premises:

(a) much of the world's knowledge is perception based,
(b) perception based information is intrinsically fuzzy.

It should be noted that the term *perception* is not considered here as a psychological term but rather as a result of intrinsically imprecise human measurement. The PNL methodology requires presence of *World Knowledge Database* and *Multiagent, Modular Deduction Database* where the former contains all the necessary information, including perception based propositions describing the knowledge acquired by direct human experience, which can be used in the deduction process. The latter contains various rules of deduction. Until recently, however, no exact formalization of PNL had been developed and so, it should be considered mainly as a reasonable methodology.

We are convinced that the potential of fuzzy set theory and fuzzy logic is strong enough to enable developing a working model of linguistic semantics and, on the basis of that, also a model of natural human reasoning. As has been convincingly argued by many authors,[1] vagueness is an unavoidable feature of natural language semantics. We argue that the idea of fuzzy sets and fuzzy logic provides a reasonable model of vagueness.[2]

Recall that in one of his early papers L.A. Zadeh [67]. suggested to model the *commonsense reasoning*. The idea to develop a logical model of the commonsense reasoning, however, is much older and has been proposed by J. McCarthy in 1959 [29] as a part of the program of logic-based artificial intelligence. Its paradigm is to develop formal commonsense theories and systems using mathematical logics that exhibit commonsense behavior. The reason is that commonsense reasoning is a central part of human thinking and we cannot imagine a real intelligence without it. The main drawback of the up-to-date formalizations of commonsense reasoning, in our opinion, is that it neglects the vagueness present in the meaning of natural language expressions (Cf. [5] and the citations therein). Therefore, a model of commonsense reasoning based on fuzzy sets and fuzzy logic can be more realistic.

The above concept was initiated in AI. A related concept came from linguists: in 1970, G. Lakoff published a paper [21] in which he introduced the concept of *natural logic* with the following goals:

- to express all concepts capable of being expressed in natural language,
- to characterize all the valid inferences that can be made in natural language,
- to mesh with adequate linguistic descriptions of all natural languages.

Natural logic is thus a collection of terms and rules that come with natural language and that allows us to reason and argue in it. According to G. Lakoff's hypothesis, natural language employs a relatively small finite number of atomic predicates that take sentential complements (sentential operators) and are related to each other by meaning-postulates that do not vary from language to language. The concept of natural logic has been further developed by several authors.[3]

In the following subsection and further we will try to convince the reader that it is reasonable to develop the concept of *fuzzy natural logic* (FNL) that continues the mentioned concept of natural logic. We will show that a good portion of work has already been done.

[1] Cf. e.g., [60].

[2] See, e.g., [39] where a well working mathematical model of the sorites paradox (a typical feature of vagueness) has been proposed. It is also proved in this paper that the developed model copes with the typical phenomenon of vagueness manifesting itself in the semantics of degree adjectives (adjectives such as "tall, small", etc.).

[3] See, e.g., [25, 58] and elsewhere.

### 8.1.2 The Paradigm of FNL

If we put all the ideas above together, we come to the concept of the fuzzy natural logic as a new theory that should be based on the results of linguists, logicians and AI specialists in natural logic, logical analysis of natural language (See, e.g. [7].), and commonsense reasoning. Our suggestion for the future is to develop FNL as an extension of *mathematical fuzzy logic*. The partly elaborated constituents of FNL till now can be summarized as follows:

(a) Formal theory of evaluative linguistic expressions.[4]
(b) Formal theory of fuzzy IF-THEN rules and approximate reasoning (derivation of a conclusion) [8, 10, 36, 46, 47].
(c) Formal theory of intermediate and generalized quantifiers [9, 15, 37, 40].

Let us remark that there are some other papers whose topics relate to the topic of FNL (Cf. [20, 62]). None of them, however, can be considered as a contribution to the consistent development of FNL as a formal logical theory.

The essential constituent of FNL is a model of linguistic semantics. Many logicians and linguists (Cf. [27, 28, 57]) have argued that the first order logic is not sufficient for this task. A suitable formal system has been chosen as the basis for further development of FNL is higher-order fuzzy logic called the *fuzzy type theory*.

### 8.1.3 Fuzzy Type Theory—The Mathematical Tool for FNL

The main mathematical tool for FNL is the *fuzzy type theory* (FTT), that is a *higher-order mathematical fuzzy logic*. There are more kinds of FTT that differ in the used algebra of truth values. For FNL, the most important is the Łukasiewicz fuzzy type theory (Ł-FTT) whose algebra of truth values is formed by an MV-algebra.

In this section we very briefly outline some of the main concepts of FTT. Details and full proofs of all theorems can be found in the literature [35, 42, 43]. Let us remark that FTT generalizes the classical type theory.

#### Syntax of Ł-FTT

The basic syntactical objects of Ł-FTT are classical (See [1].), namely the concepts of *type* and *formula*. The types are special subscripts (denoted by Greek letters) assigned to all formulas using which we distinguish kinds of objects represented by formulas. The atomic types are $\epsilon$ representing elements and $o$ representing truth values. The set of all types is denoted by *Types*.

The *language* $J$ of Ł-FTT consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ ($\alpha \in Types$), the symbol $\lambda$, and brackets. We will consider the following concrete

[4] [39]; see also [38].

special constants: $\mathbf{E}_{(o\alpha)\alpha}$ (fuzzy equality) for every $\alpha \in Types$, $\mathbf{C}_{(oo)o}$ (conjunction), $\mathbf{D}_{(oo)}$ (delta operation on truth values) and descriptions operator $\iota_{\epsilon(o\epsilon)}$.

Formulas are formed of variables, constants (each of specific type), and the symbol $\lambda$. As mentioned, each formula $A$ is assigned a type (we write $A_\alpha$). A set of formulas of type $\alpha$ is denoted by $Form_\alpha$ and a set of all formulas is $Form = \bigcup_{\alpha \in Types} Form_\alpha$.[5]

Recall that if $B \in Form_{\beta\alpha}$ and $A \in Form_\alpha$ then $(BA) \in Form_\beta$. Similarly, if $A \in Form_\beta$ and $x_\alpha \in J$, $\alpha \in Types$, is a variable then $(\lambda x_\alpha\, A) \in Form_{\beta\alpha}$.

The main connective is *equivalence* $\equiv$ defined by $\lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha$ for all types $\alpha \in Types$. As usual, we write $(A_\alpha \equiv B_\alpha)$ instead of $(\equiv A_\alpha) B_\alpha$. Note that this is a formula of type $o$.

Further connectives are *conjunction* ($\wedge := \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o}\, y_o) x_o$), *implication* ($\Rightarrow := \lambda x_o \lambda y_o\, (x_o \wedge y_o) \equiv x_o$), *negation* ($\neg := \lambda x_o (x_o \equiv \bot)$), *strong conjunction* ($\& := \lambda x_o (\lambda y_o (\neg(x_o \Rightarrow \neg y_o)))$), *disjunction* ($\vee := \lambda x_o (\lambda y_o (x_o \Rightarrow y_o) \Rightarrow y_o)$) and *delta* ($\boldsymbol{\Delta} := \lambda x_o \mathbf{D}_{oo} x_o$). The general ($\forall$) and existential ($\exists$) quantifiers are also defined as special formulas [35].

The fuzzy type theory has 17 logical axioms. Most of them are introduced to characterize properties of the considered algebra of truth values. For FNL, the considered algebra is MV-algebra. There are also inference rules:

(R) Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then infer $B'$ where $B'$ comes from $B$ by replacing one occurrence of $A_\alpha$, which is not preceded by $\lambda$, by $A'_\alpha$.

(N) Let $A_o \in Form_o$. Then, from $A_o$ infer $\boldsymbol{\Delta} A_o$.

The inference rules of modus ponens and generalization are derived rules in Ł-FTT. The concepts of *provability* and *proof* are defined in the same way as in classical logic. A *theory* $T$ over Ł-FTT is a set of formulas of type $o$ ($T \subset Form_o$). By $T \vdash A_o$ we mean that $A_o$ is provable in $T$. Many theorems characterizing syntactical properties of FTT were proved including deduction theorem and other ones.

### Semantics of Ł-FTT

The truth values form an MV-algebra (See [4, 49].) extended by the delta operation. It can be seen as the residuated lattice [13, 35] $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1}, \Delta \rangle$. An important special case is the standard Łukasiewicz $\mathrm{MV}_\Delta$-algebra

$$\mathcal{L} = \langle [0, 1], \vee, \wedge, \otimes, \to, 0, 1, \Delta \rangle \tag{8.1}$$

where

$$\begin{aligned} \wedge &= \text{minimum}, & \vee &= \text{maximum}, \\ a \otimes b &= \max(0, a + b - 1), & a \to b &= \min(1, 1 - a + b), \\ \neg a &= a \to 0 = 1 - a, & \Delta(a) &= \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

[5] In type theory, all syntactical objects including variables and connectives are taken as formulas.

We will also consider the operation $a \oplus b = \min(1, a + b)$. This algebra generates the variety of MV-algebras. Therefore, the MV-operations $\otimes, \oplus$ are often called *Łukasiewicz conjunction* and *Łukasiewicz disjunction*, respectively.

Let $J$ be a language of Ł-FTT and $(M_\alpha)_{\alpha \in Types}$ be a system of sets called *basic frame* such that $M_o$, $M_\epsilon$ are sets and for each $\alpha, \beta \in Types$, $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$, i.e. it is a set of functions from $M_\alpha$ to $M_\beta$.[6] The *general frame* is a tuple

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle \tag{8.2}$$

so that the following holds:

(i) The $\mathcal{L}_\Delta$ is a structure of truth values (i.e., an MV-algebra). We put $M_o = L$ and assume that the set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{L}_\Delta$.
(ii) $=_\alpha$ is a fuzzy equality on $M_\alpha$ and $=_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in Types$.

A *general model* is a general frame $\mathcal{M}$ such that for every $A_\alpha$, $\alpha \in Types$ interpretation $\mathcal{M}_p$ gives

$$\mathcal{M}_p(A_\alpha) \in M_\alpha$$

where $p$ is an assignment of elements from the sets $M_\alpha$ to variables (depending on the given type). This means that each set $M_\alpha$ from the frame $\mathcal{M}$ has enough elements so that the interpretation $\mathcal{M}_p(A_\alpha)$ is always defined. A general model $\mathcal{M}$ is a *model of a theory* $T$, $\mathcal{M} \models T$, if $\mathcal{M}(A_o) = \mathbf{1}$ holds for all axioms of $T$. If $A_o$ is true in the degree $\mathbf{1}$ in all general models of $T$ then we write $T \models A_o$.

Let $T$ be a theory. A formula $A_o$ is *true* in the degree $a \in L$ in $T$, if

$$a = \bigwedge \{\mathcal{M}_p(A_o) \mid \mathcal{M} \models T, p \in Asg(\mathcal{M})\}. \tag{8.3}$$

In this case, will write $T \models_a A_o$. If $a = 1$ then we omit the subscript.

The following completeness theorem was proved (See [35, 43]).

**Theorem 1** (completeness)

*(a) A theory $T$ is consistent iff it has a general model $\mathcal{M}$.*
*(b) For every theory $T$ and a formula $A_o$*

$$T \vdash A_o \iff T \models A_o.$$

The completeness theorem is an important result assuring us that FTT is a well founded mathematical tool that can be used for the development of FNL. We are thus able to formulate many results syntactically and, at the same time, to be sure that our results hold in all models. Hence, FNL is very powerful and encompasses most results in fuzzy set theory obtained semantically. Consequently, we should always try to formulate our problem syntactically and then use it in semantic interpretation.

[6] By currying, we may consider only unary functions.

### *8.1.4 Future Prospects of Fuzzy Set Theory in Linguistic Modeling*

As one possible direction in the future development of the fuzzy set theory and fuzzy logic we *suggest to focus on fuzzy natural logic*. This logic should be developed as an extension of the mathematical fuzzy logic in narrow sense.

The work on FNL consists of two tasks: (a) development of mathematical model of linguistic semantics and (b) characterization of fundamental reasoning schemes of human mind. Both tasks require close cooperate with linguists and logicians.

Natural language, however, is extremely complicated structure with many subtleties and exceptions. Even simple linguistic units, such as evaluative adjectives (e.g., "good, interesting", etc.) can be used in many contexts and ways, their meaning may depend on the position within topic-focus articulation (Cf. [14]) and so, at present stage we can hardly hope to be able to develop a mathematical model of the semantics of natural language that would capture all the details. It is therefore, questionable whether we should struggle for complete capturing semantics of natural language. It seems reasonable to relax our requirements and in line with the paradigm of Zadeh's precisiated natural language focus on smaller parts of natural language and try only to capture their essential properties, of course, with the perspective to improve and deepen continuously the model in parallel with the increase of linguistic knowledge. Our temporary goal should be to develop the model to such an extent that would make it possible to apply it in various technical and economical problems.

## 8.2 Linguistic Semantics and FNL

In this section, we will outline some aspects of linguistic semantics and relate them to the existing results in FNL. Let us remark that the model of semantics of FNL stems from the ideas presented in the book [33] on the *Alternative mathematical model of linguistic semantics* (AML).

We will turn our attention to a selection of specific linguistic units and phenomena, namely nouns, adjectives, adverbs, hedging and simple noun phrases and other ones. Our goal is to remind wealth and complexity of natural language and outline some of the problems and results of linguistic studies. In each case we at the same time outline how FNL, in its present state, copes with some of the discussed linguistic phenomena.

We will use the means of fuzzy type theory. Recall that FTT assigns a type to each formula. To make the notation more readable and transparent, we often introduce the type of a given formula only in the first occurrence and then write it without the type assuming that the reader still keeps it in mind.

### 8.2.1 Nouns and Objects

Nouns are names of objects. More specifically, they denote persons, places, things, events, substances, qualities, quantities, etc. Nouns are *original*, e.g., "house, horse, square", etc. and *derived*, namely from adjectives, e.g. "redness, beauty, simplicity", or verbs (we speak about *postverbal nouns*), e.g. "jumping, walk, writing", etc. There are also proper nouns denoting one specific object, e.g., "Earth, Saturn, Russia" and common ones that denote a class of objects, e.g., "planet, mammal, street", etc.

*Countable nouns* are common nouns that can take a plural, can be combined with numerals or counting quantifiers (e.g., one, two, several, every, most), and can take an indefinite article such as "a" or "an" (in languages which have such articles). Examples of count nouns are "chair, nose, occasion". *Mass nouns* or uncountable (or non-count) nouns differ from count nouns in precisely that respect: they cannot take plurals or combine with number words or the above type of quantifiers. For example, it is not possible to refer to a "furniture or three furnitures". Depending on the kind of objects, we can also distinguish *concrete* (e.g., "horse, table, house", etc.) and *abstract* nouns (e.g., "work, idea, feeling, happiness", etc.).

Objects are entities that can have very complicated properties. In general, an object is a phenomenon to which we concede its individuality that makes it distinct from its surrounding. In fact, we can construe arbitrary phenomenon as an object.

Till now, there is no more detailed model of nouns in FNL (or in general fuzzy set theory). The problem is in finding a satisfactory model of real objects because of too high complexity of them. It is possible to model some very special objects, such as 2-D or 3-D geometrical shapes, or so. But in full generality that includes also abstract objects such as those considered in postverbal nouns is this task so far too complicated.

In FNL, we suggest a simplification that can work in various AI applications and elsewhere. Namely, note that each object can be characterized by various kinds of features (characteristics), for example, "height, nationality, age, weight, strength, shape, intelligence", etc. Therefore, we can identify objects with sets of values of features. Semantically, an object can be represented by a tuple

$$\mathbf{o} = \langle v_1, v_2, \ldots, v_n \rangle \tag{8.4}$$

where $i = 1, \ldots, n$ are various features characterizing the object and $v_i$ are their values. The $n$ can in principle be even infinity. For example, if $i$ is a feature "length" then, for example, $v_i = 1.2\,\mathrm{m}$. We can see that the values can be real numbers, or some other kinds of characteristics. Since the set of real numbers is rich enough to represent all kinds of values, we will in practice usually take $v_i \in \mathbb{R}$.

A fuzzy set of objects $\mathbf{o}$ in (8.4) forms an *extension of a noun* if it is determined with respect to a certain context (possible world). Then an *intension* of a noun is represented by a function from the set of all contexts into a set of all objects.

The objects and nouns can be expressed in the syntax of FTT as follows. First, we must introduce special types:

(a) $\varphi$—features of objects. Any formula $A_\varphi$ represents some specific feature. Given a model, the interpretation $\mathcal{M}(A_\varphi) \in M_\varphi$ is a unique object representing one specific feature. In (8.4) it is a given $i \in \{1, \dots, n\}$.
(b) $\alpha$—*values* of concrete features. As mentioned above, in the model we will usually take $M_\alpha = \mathbb{R}$.
(c) Given a feature, it may attain various values dependently on its *local context*. Therefore, we will introduce a special type $\omega$ for local context of values of one specific feature. Recall that this type may even be itself more complex, for example, we may put $\omega := \alpha o$.
(d) $\omega\varphi$—global context which covers all features and can be taken as a context of the whole noun. We will use the variable $\mathbf{w}_{\omega\varphi}$ for global contexts. In a model $\mathcal{M}$, the interpretation $\mathcal{M}(\mathbf{w}_{\omega\varphi}) \in M_{\omega\varphi}$ is a function that assigns to each feature from $M_\varphi$ a context $w \in M_\omega$.
(e) $(\alpha\omega)\varphi$—type of objects that are elements of an extension of a noun. The objects are represented by sets of values of features in a global context. We will use the variable $\mathbf{h}_{(\alpha\omega)\varphi}$ for objects. In a model $\mathcal{M}$, the interpretation $\mathcal{M}(\mathbf{h}_{(\alpha\omega)\varphi}) \in M_{(\alpha\omega)\varphi}$ is the tuple of the form (8.4). For example, $\mathcal{M}(\mathbf{h}_{(\alpha\omega)\varphi})$ can be a Swede.
(f) $((o\alpha)\omega)\varphi$—type of a noun. Any formula $\mathbf{S}_{((o\alpha)\omega)\varphi}$ represents a formal way how a noun is construed. Its interpretation in a model $\mathcal{M}$ is a function which assigns to each feature from $M_\varphi$ its intension.

*Semantics of nouns*

In accordance with the results of analysis done in linguistics and logic, semantics of expressions of natural language is characterized by the concepts of possible world, intension and extension. Then, we can formalize semantics of nouns in FNL as follows.

Let $\mathbf{S}_{((o\alpha)\omega)\varphi}$ be a formula representing a **Noun**. For example, **Noun** can be **Swede**, **plate**, **house**, etc. Intension of **Noun** is defined by

$$\mathrm{Int}(\mathbf{Noun}) := \lambda \mathbf{w}_{\omega\varphi}\, \lambda \mathbf{h}_{(\alpha\omega)\varphi} \cdot (\forall c_\varphi)(\mathbf{S}c\,(\mathbf{w}c)(\mathbf{h}c(\mathbf{w}c))). \tag{8.5}$$

Thus, in a model $\mathcal{M}$, intension (8.5) of **Noun** is interpreted by a function assigning to each global context $\mathcal{M}(\mathbf{w})$ a fuzzy set of objects $\mathcal{M}(\mathbf{h})$. It can be seen from (8.5) that each feature $\mathcal{M}(c_\varphi)$ is in a local context $\mathcal{M}(\mathbf{w}c)$ assigned a value $\mathcal{M}(\mathbf{h}c(\mathbf{w}c))$ which also depends on the context $\mathcal{M}(\mathbf{w}c)$.

It follows from (8.5) that extension of **Noun** in a (global) context $\mathbf{w}$ is

$$\mathrm{Ext}_{\mathbf{w}}(\mathbf{Noun}) := \lambda \mathbf{h}_{(\alpha\omega)\varphi} \cdot (\forall c_\varphi)(\mathbf{S}c(\mathbf{w}c)(\mathbf{h}c(\mathbf{w}c))). \tag{8.6}$$

Clearly, in a model $\mathcal{M}$ it is a fuzzy set of elements $\mathcal{M}(\mathbf{h}_{(\alpha\omega)\varphi})$.

In this model, we easily obtain semantics of "a Noun" and "the Noun",[7] for example, "a Swede" and "the Swede". In the former case, for a given context, the

[7] We speak about *indefinite* and *specifying grammateme* of a noun, respectively, cf. [33, 57].

interpretation is an element from the kernel of the fuzzy set (8.6) and in the latter case it is one specific object taken from its support (see below).

### *8.2.2 Adjectives*

Adjectives are *names of properties* of objects. We can distinguish *proper* and *derived* adjectives. Example of the former are "red, tall, good", the latter are derived from other types of words, namely nouns (e.g., wood–wooden, grease–greasy) or verbs (*deverbal adjectives*), for example e.g., "smiling, washing.

We can distinguish also are other specific classes of adjectives. Important are (See [3].) *gradable adjectives* [2] (also called degree adjectives), for example "hot""small, tall", *evaluative adjectives* [59], for example "good, awful, fantastic, disasterous", and *absolute* (non-gradable) adjectives "green, freezing, dead, nuclear". The gradable adjectives can still be divided (See [19].) into absolute, for example "bent, straight" and relative gradable adjectives such as "expensive, tall, strong", etc.

The accepted hypothesis is that gradable adjectives denote functions that map objects onto representations of the degree to which they posses some gradable property. Hence, gradable adjectives may differ with respect to their scales. This corresponds with our model of objects in FNL outlined above.

Another specific feature of gradable but also of evaluative adjectives [56] are existence of pairs of antonyms, for example "short–long, clean–dirty, complete–incomplete". Antonymous pairs of gradable adjectives can be complementary and non-complementary. *Complementary adjectives* are pairs of antonymous adjectives that are furthermore each other's negation on their domain, e.g. complete–incomplete. *Non-complementary adjectives* may have an xtension gap which corresponds to the set of objects that the predicate is neither true nor false of in a particular context of utterance. This gives rise to what is called in FNL: the *fundamental evaluative trichotomy* that is a triple of expression consisting of the nominal adjective, its antonym and a middle member, for example "weak–medium (strong)–strong", etc. —see below. Crucially, the positive and negative extensions and the extension gap of a gradable predicate may vary across contexts of use, becoming more or less precise. In FNL, we propose a model covering both gradable as well as evaluative adjectives including also hedging discussed in the next subsection.

A special phenomenon is existence of the comparative and superlative of adjectives. The *comparative* is a name of a relation that does not necessarily corresponds with the given adjective. For example, bigger is a name of a relation $\geq$ that does not correspond with the adjective "big". Clearly, if "John is small" and "Charles is bigger", it does not imply that "Charles is big" (both John and Charles can be very small men). The comparative has degrees since we can say, for example, "much bigger, a little smaller", etc. The *superlative* is derived from comparative and, in fact, it corresponds to result of maximization (in mathematical sense). No satisfactory model of comparative and superlative phenomena is in FNL so far suggested.

### 8.2.3 Hedging

Hedging is a linguistic phenomenon that is used to specify more or less closely the topic of utterance. In the theory of fuzzy sets, we learned about hedges being special adverbs (such as "very, roughly"). However, hedging is a wider phenomenon that can be expressed by more complex expressions.

Examples of hedging are expressions such as "to some respect, in a sense, a sort of" but also "very roughly, approximately, more or less, about, almost". The most important in hedging is a class of adverbs called *intensifying ones*, among them we rank "very, extremely, typically, roughly", etc.

The concept of hedging was in linguistics in more detail analyzed first by G. Lakoff [22]. He also noticed that the general effect of hedging is either in inceras-ing fuzziness (widening effect) or decreasing fuzziness (narrowing effect). Thus, hedging is an important tool of natural language that enables us to specify more concretely what we have in mind. We may distinguish *narrowing hedges* (very, extremely, significantly, etc.), *widening hedges* (more or less, roughly, very roughly, etc.) and *specifying hedges* (approximately, about, rather, precisely, etc.).

### 8.2.4 Evaluative Linguistic Expressions in FNL

The analysis of nouns, adjectives and hedges gives rise to a more general concept. Namely, we can introduce a special class of linguistic expressions called *evaluative* ones. When putting together properties of gradable and evaluative adjectives discussed in the literature, we can specify evaluative linguistic expressions as special expressions of natural language using which people evaluate phenomena around them.

They include them the following classes of linguistic expressions:

(i) ⟨TE-adjective⟩, that belong to a class of special adjectives (TE stands for "trichotomic evaluative") that include gradable adjectives (big, cold, deep, fast, friendly, happy, high, hot, important, long, popular, rich, strong, tall, warm, weak, young), evaluative adjectives (good, bad, clever, stupid, ugly, etc.), but also adjectives such as *left, middle, medium*, etc. The TE-adjectives can usually be grouped to form a *fundamental evaluative trichotomy* that consists of two antonyms and a middle member, for example *low, medium, high*; *clever, average, stupid*; *good, normal, bad*, etc. The triple of adjectives *small, medium, big* will further be taken as canonical. An exception are complementary adjectives mentioned above that lack the middle member.

(ii) *Fuzzy numbers*. These include all linguistic expressions containing some number that is often completed by some hedge, for example "three hundred, roughly one hundred, about twenty five, approximately two million", etc.

(iii) *Simple evaluative linguistic expressions* (possibly with signs). They have a general form

$$\langle\text{linguistic hedge}\rangle\langle\text{TE-adjective}\rangle. \tag{8.7}$$

From the logical point of view, it is reasonable to introduce also *empty hedge*. Then we can consider as simple evaluative expressions also pure adjectives. Hence, examples of simple evaluative expressions are "small, rather medium, very big, more or less weak, medium strong, strong, quite silly, normal, extremely intelligent", etc. Note that from grammatical point of view simple evaluative expressions are adjective phrases. Note also, that we cannot apply the same hedge with all adjectives. For example "very medium" has no meaning and so, it is not an evaluative expression.

(iv) *Compound evaluative expressions* (roughly small or medium, small but not very (small), etc.). These expressions are formed from simple ones using connectives. However, these expressions never form a boolean structure since there are many combinations that have no sense. For example, the expression "very small or medium and extremely big" has no meaning.

(v) *Negative evaluative expressions* (not small, not very big, etc.). The use of negation is problematic and one encounters here a special linguistic phenomenon called *topic-focus articulation* (Cf. [14]). Namely, the particle "not" can act at least in two ways—either on the whole evaluative expression or only on the hedge. For example, *not very small* has (at least) two different meanings: either "(not very) small" where "very is negated" so that we deal with a new hedge "not very", or "not (very small)" where the whole expression "very small" is negated.

In the applications of fuzzy logic, evaluative linguistic expressions occur in the expressions of the form

$$X \text{ is } \langle\text{evaluative expression}\rangle \tag{8.8}$$

where $X$ is a variable whose values are the values of some measurable *feature of the noun* and $\mathcal{A}$ is an evaluative expression. They are a simplified from of a special class of verb phrases that are called *evaluative linguistic predications*. From linguistic point of view they are simple phrases of the form

$$\langle\text{noun}\rangle \text{ is } \langle\text{evaluative expression}\rangle \tag{8.9}$$

where "is" is a copula—the verb "to be". Examples are "temperature is low, very intelligent man, more or less weak force, medium tension, extremely long bridge, short distance and pleasant walk, roughly small or medium speed, etc.).

Evaluative predications semantically express a property of object(s) characterized by the given evaluative expression. If noun is concretely specified (e.g., John, my friend) then the meaning of (8.9) is a truth value. If it is general, e.g., "house is big" (i.e., without specification, which house) then the meaning of (8.9) (and also of (8.8)) is extension of all objects having the property denoted by ⟨evaluative expression⟩. In this case, the meaning of (8.9) is equal to the meaning of

$$\langle \text{evaluative expression} \rangle \langle \text{noun} \rangle,$$

for example "big house".

In more general way, evaluative linguistic expressions occur in the position of adjective phrases characterizing features of some objects and are used either in predicative or attributive role.

Example of the predicative use is "The man is very stupid". Example of the attributive one is "The very stupid man climbed a tree". The purpose of the first sentence is simply to communicate a particular quality of the sentence's subject. The purpose of the second sentence is primarily to tell us what the subject did i.e. climbed a tree; that the subject is very stupid is a secondary consideration.

*Semantics of evaluative expressions in FNL*

Formalization of the semantics of evaluative expressions in FNL is based on the standard assumptions of the theory of semantics developed in linguistics and logic (Cf. [14, 27, 28]). Namely, the fundamental concepts to be formalized are possible world, intension, and extension. This task is in FNL solved by introducing a special theory $T^{\text{Ev}}$ that is a special formal theory of Łukasiewicz fuzzy type theory (Ł-FTT). This theory[8] formalizes certain general characteristics of the semantics of evaluative expressions.

Let us remark that the model of semantics of evaluative expressions is very successful in applications. One of the reasons is that this semantics is based on the theory of ordered sets that is a well elaborated part of mathematics and it is relatively easy to construct the necessary models.

On e of essential concepts in the theory of semantics of natural language expressions is that of *possible world*. This concept can be traced back to Leibniz and in modern conception to Carnap as well as logicians such as Quine, Wittgenstein, Lewis, Kripke and many others.

In the theory of evaluative expressions, we will speak about *context* instead of a possible world. The latter is usually taken as a state of the world at a given point in time and space. It is very difficult to formalize such a definition. However, in [39], it is argued that extensions of evaluative expressions are classes of elements taken from some scale representing. Therefore, we can introduce a simplified concept of *context* that is a nonempty, linearly ordered and bounded set, in which three distinguished limit points can be determined: a *left bound* $v_L$, a *right bound* $v_R$, and a *central point* $v_S$. Hence, each context is identified with an ordered triple

$$w = \langle v_L, v_S, v_R \rangle$$

where $v_L, v_S, v_R \in U$. A straightforward example is the predication "$\mathcal{A}$ town", for example "small town", "very big town", etc. Then, the corresponding context for the Czech Republic can be ⟨3 000, 50 000, 1 000 000⟩, while for the USA it can be ⟨30 000, 200 000, 10 000 000⟩.

---

[8]The detailed presentation and an informal justification of $TEV$ can be found in [39].

We introduce a set $W$ of contexts. Each element $w \in W$ gives rise to an interval $w = [v_L, v_R] \subset U$.

*Intension* $\mathrm{Int}(\mathcal{A})$ of an evaluative expression $\mathcal{A}$ is a property that attains various truth values in various contexts but is invariant with respect to them. Therefore, it is modeled as a function

$$\mathrm{Int}(\mathcal{A}) : W \to \mathcal{F}(U) \tag{8.10}$$

where $\mathcal{F}(U)$ is a set of all fuzzy sets over $U$. Note that of an evaluative expression or predication $\mathcal{A}$ is obtained as interpretation of a formula $\lambda w\, \lambda x\, (Aw)x$ (in the language of Ł-FTT) in a special model $\mathcal{M}$.

*Extension* $\mathrm{Ext}_w(\mathcal{A})$ of an evaluative expression $\mathcal{A}$ in the context $w \in W$ is a fuzzy set of elements

$$\mathrm{Ext}_w(\mathcal{A}) = \mathrm{Int}(w)(\mathcal{A}) \subseteq_{\sim} w.$$

In our example, the truth value of a "small town having 30 000 inhabitants" could be, for example, 0.7 in the Czech Republic and 1 in the USA.

In the theory $T^{\mathrm{Ev}}$, the *extension* of an evaluative expression is obtained as a shifted horizon where the shift corresponds to a linguistic hedge, which is thus modeled by a function $L \to L$. A graphical scheme of such an interpretation in a specific context can be seen in Fig. 8.1.

Recall also our discussion above about gradable adjectives. We can distinguish absolute and relative ones. What is the difference between them? From our point of view, the solution is simple: absolute gradable adjectives have only one context and so, their intension coincides with their extension while the relative ones have (infinitely) many of contexts and so, their intension is the function (8.10).

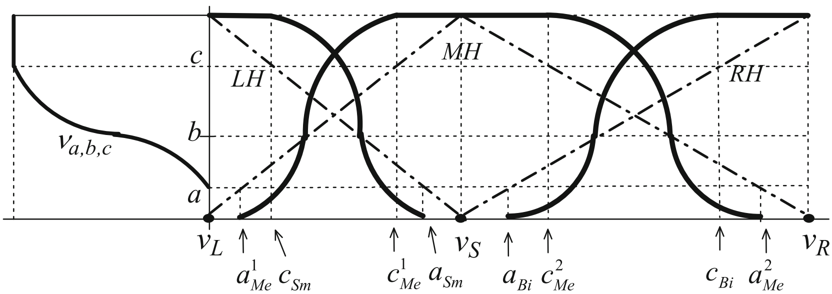


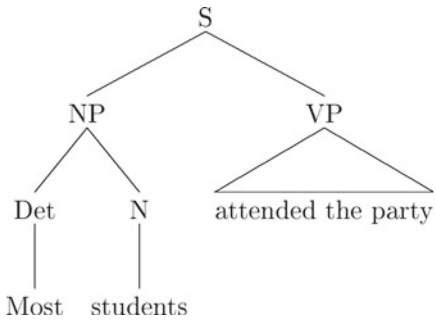

**Fig. 8.1** Graphical scheme of a construction of extensions of evaluative expressions in a given context. Each extension is obtained as a composition of a function representing a respective horizon *LH*, *MH*, *RH* (in the figure, it is linear because of the use of Ł-FTT) and the deformation function $\nu_{a,b,c}$ whose graph is for convenience depicted turned 90° anticlockwise

### 8.2.5 Linguistic Quantifiers and Determiners

A special phenomenon in natural language is a huge and apparently messy collection of expressions such as "not just every and some, but most, few, between five and ten, a lot of", and many others. These expressions occur in typical sentences, for instance:

(a) Few (both, enough, at lest ten, all but five) students attended the party.
(b) More male than female students attended the party
(c) John's mother arrived.
(d) Every student attended the party.

These sentences have the following standard syntactic structure:




The Det is a determiner that is a generalized (linguistic) quantifier. The NP is *noun phrase*. In our case it is a quantified noun phrase, also called determiner phrase.

In linguistic semantics, a generalized quantifier is an expression that denotes a property of a property (a higher-order property).

In linguistics, a determiner phrase (DP) is a type of phrase posited by some theories of syntax. The head of a DP is a determiner, as opposed to a noun. For example in the phrase "the car", "the" is a determiner and "car" is a noun; the two combine to form a phrase, and on the DP-analysis, the determiner "the" is head over the noun "car".

### 8.2.6 Intermediate (Fuzzy) Quantifiers in FNL

In logic, the linguistic quantifiers are modeled using the concept of *generalized quantifier* [18, 24, 31, 54, 61]. These were in fuzzy set theory generalized under the name *fuzzy (generalized) quantifiers*. The first paper in this topic was written by L.A. Zadeh [68]. His theory has been further elaborated by several authors (See, e.g., [9, 12, 15, 17]).

The theory, in the mentioned papers is focused on computation rather than on linguistics. The suggestion that took into account linguistic side and thus contributes to the theory of FNL was introduced by Novák [40]. This theory was inspired by the theory of intermediate quantifiers studied in detail by Peterson [55]. These are linguistic quantifiers whose meaning layes between classical and existential quantifiers. The basic idea consists in the assumption that these quantifiers are classical general or existential quantifiers for which the universe of quantification is modified and the modification can be imprecise.

We introduce a theory $T^{\mathrm{IQ}}$ which is a special theory of Ł-FTT extending the theory $T^{\mathrm{Ev}}$ of evaluative linguistic expressions introduced in the previous section by few more axioms, namely those for the measure function (see below).

The theory $T^{\mathrm{IQ}}$ is obtained from $T^{\mathrm{Ev}}$ by extending the latter by the concept of measure of fuzzy sets. In the frame of Ł-FTT, it can be introduced syntactically. Namely, the measure is represented by a special formula $\mu \in Form_{o(o\alpha)(o\alpha)}$ whose interpretation is a function $M_\alpha \to L$, i.e. values of the measure are taken from the set of truth values.[9] Moreover, the measure is noremed with respect to some reference fuzzy set (recall that a formula of type $o\alpha$ represents a function $M_\alpha \to L$, i.e. a fuzzy set). Namely,

$$\mu(z_{o\alpha})x_{o\alpha}$$

represents a *measure* of a fuzzy set $x_{o\alpha}$ normed with respect to the fuzzy set $z_{o\alpha}$ (i.e., $x_{o\alpha}$ is proportional to $z_{o\alpha}$). Its properties properties (and interpretation) can be found in the cited literature.

**Definition 1** Let $Ev \in Form_{oo}$ be intension of some evaluative expression, $A, B \in Form_{o\alpha}$ be formulas and $z \in Form_{o\alpha}$ and $x \in Form_\alpha$ variables where $\alpha \in \mathcal{S}$. Then a type $\langle 1, 1\rangle$ intermediate generalized quantifier interpreting the sentence

"⟨Quantifier⟩ $B$'s are $A$"

is one of the following formulas:

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\, \&\, (\forall x)(z\, x \Rightarrow Ax)) \wedge Ev((\mu B)z)). \tag{8.11}$$

$$(Q^{\exists}_{Ev}\, x)(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\, \&\, (\exists x)(z\, x \wedge Ax)) \wedge Ev((\mu B)z)). \tag{8.12}$$

For some syllogism figure, also presupposition requiring that only non-empty (fuzzy) subsets of $B$ are considered.

[9] This is quite natural because the set of truth values is assumed to be MV-algebra, namely the Łukasiewicz one whose support set is [0, 1].

Note that each formula above consists of three parts:

$$\underbrace{(\exists z)((\boldsymbol{\Delta}(z \subseteq B)}_{\text{“the greatest” part of } B\text{'s}} \;\&\; \underbrace{(\forall x)(z\,x \Rightarrow Ax))}_{\text{each of } B\text{'s has } A} \wedge \underbrace{Ev((\mu B)z))}_{\text{size of } z \text{ is evaluated by } Ev}$$

Thus, the concrete quantifiers are obtained when specifying the evaluative expression $Ev$.

Below are introduced several specific intermediate quantifiers based on the analysis provided by Peterson (Cf. [55]).

**A**: All $B$ are $A$ := $Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\forall x)(Bx \Rightarrow Ax)$,

**E**: No $B$ are $A$ := $Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax)$,

**P**: Almost all $B$ are $A$ := $Q^{\forall}_{Bi\,Ex}(B, A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\,Ex)((\mu B)z))$,
(*extremely big part of B has A*)

**B**: Few $B$ are $A$ := $Q^{\forall}_{Bi\,Ex}(B, \neg A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\,Ex)((\mu B)z))$,
(*extremely big part of B does not have A*)

**T**: Most $B$ are $A$ := $Q^{\forall}_{Bi\,Ve}(B, A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\,Ve)((\mu B)z))$,
(*very big part of B has A*)

**D**: Most $B$ are not $A$ := $Q^{\forall}_{Bi\,Ve}(B, \neg A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\,Ve)((\mu B)z))$,
(*very big part of B does not have A*)

**K**: Many $B$ are $A$ := $Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z))$,
(*not small part of B has A*)

**G**: Many $B$ are not $A$ := $Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, \neg A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z))$,
(*not small part of B does not have A*)

**I**: Some $B$ are $A$ := $Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\exists x)(Bx \wedge Ax)$,

**O**: Some $B$ are not $A$ := $Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax)$.

*Remark 3* (i) The evaluative expressions used in the definition of the quantifiers above are considered in the abstract context $w_{oo}$ and so, the variable $w_{oo}$ is omitted in the corresponding formulas.

(ii) The quantifier **B** is, in fact, defined as "Almost all $B$ are not $A$".

(iii) The quantifier "most" is considered in its shifted meaning as "relatively close to all" and *not* as "simple majority".

(iv) The quantifiers with the hedge $\boldsymbol{\Delta}$ are equivalent to the corresponding classical ones.

### *8.2.7 The Meaning of Noun Phrases and Simple Sentences*

Using the formal means of FNL, we can construct the meaning of simple noun phrases or simple sentences. Our construction in this section will be syntactical. However, after defining a model, it is straightforward to construct the concrete fuzzy relations representing the meaning of the given noun phrase or a sentence.

We will first demonstrate our construction on a simple example of the noun phrase *Very tall Swedes*.

*Special types*

Let us first introduce the following special types:

(a) $\beta$—features (characteristics) of objects. These can be, for example, "height, nationality, age, weight", and many other ones. In this section, we will consider only the first two ones.

(b) $\alpha$—*values* of concrete features. These are often real numbers, or some other kinds of characteristics.

(c) $\alpha\beta$—objects (people), i.e. people are in our model identified with sequences of values of features.

*Special constants and variables*

Recall that nouns are identified with functions $f_{\alpha\beta}$ that can be seen as sets of values of features. Each feature $c_\beta$ is a assigned some value $v_\alpha$ via the function $f_{\alpha\beta}$. Iterpretation of $f_{\alpha\beta}$ in any model $\mathcal{M}$ can be seen as the set

$$\mathcal{M}(f_{\alpha\beta}) = \{\langle c, v\rangle, \langle c', v'\rangle, \ldots\} \tag{8.13}$$

where $c, c', \ldots \in M_\beta, \ldots$ are various features and $v, v', \ldots \in M_\alpha \ldots$ are their values. The sets (8.13) represent people in this model. Furthermore:

(i) The constant $\mathbf{h}_\beta$ is the feature of *height*.

(ii) The constant $\mathbf{n}_\beta$ is the feature of *nationality*. Of course, this is a simplification. Instead, we could consider a set of several specific characteristics, such as *mother tongue, hair, skin*, etc. We do not need such a complicated model in this example.

(iii) The variable $\boldsymbol{\nu}_{oo}$ is a linguistic hedge. It is used as a general variable standing for specific hedges (very, roughly, etc.).
(iv) The variable $w_{\alpha o}$ is a context (possible world). Its interpretation is a function from the set of truth values into a set $M_\alpha$ of type $\alpha$. This trick enables to transfer ordering properties of the algebra of truth values into the set $M_\alpha$.

Every Swede is a human being and so, we must first introduce the property "to be human":

$$\mathrm{Int}(\mathbf{Human}) := \lambda \mathbf{w}_{\omega\varphi}\, \lambda \mathbf{h}_{(\alpha\omega)\varphi} \cdot (\forall c_\varphi)(\mathbf{H}c(\mathbf{w}c)(\mathbf{h}c(\mathbf{w}c))) \tag{8.14}$$

where $\mathbf{H}_{(o\alpha\omega)\varphi}$ is a formula representing people.

Among all features $c_\varphi$ we may distinguish also a feature of *nationality* $n_\varphi$ and *height* $v_\varphi$. Let the nationality of Swedes $n_\varphi$ be determined by the value (constant) $\mathbf{d}_\alpha$. Then intension of "to be Swede" is

$$\mathrm{Int}(\mathrm{Swedes}) := \lambda \mathbf{w}_{\omega\varphi}\, \lambda \mathbf{h}_{(\alpha\omega)\varphi} \cdot (\forall c_\varphi)(\mathbf{H}c(\mathbf{w}c)(\mathbf{h}c(\mathbf{w}c)))\, \boldsymbol{\&} (\mathbf{h}n(\mathbf{w}n) \equiv \mathbf{d}_\alpha). \tag{8.15}$$

Thus, intension of Swedes assigns to each context $\mathbf{w}_{\omega\varphi}$ a fuzzy set of people whose nationality $\mathbf{h}n(\mathbf{w}n)$ in the context $\mathbf{w}n$ corresponding to nationality has the value $\mathbf{d}_\alpha$, i.e. the nationality of Swedes.

To express intension of *Very tall Swedes*, we must consider the height $v_\varphi$ and characterize the truth of the proposition *the height is very big* in the context $\mathbf{w}v$:

$$\begin{aligned}\mathrm{Int}(\text{Very tall Swedes}) := \lambda \mathbf{w}_{\omega\varphi}\, \lambda \mathbf{h}_{(\alpha\omega)\varphi} \cdot (\forall c_\varphi)(\mathbf{H}c(\mathbf{w}c)(\mathbf{h}c(\mathbf{w}c)))\, \boldsymbol{\&} \\ (\mathbf{h}n(\mathbf{w}n) \equiv \mathbf{d}_\alpha)\, \boldsymbol{\&} (Bi\ Ve)(\mathbf{w}v)(\mathbf{h}v(\mathbf{w}v)).\end{aligned} \tag{8.16}$$

Thus, this intension is a function assigning to each (global) context a fuzzy set of people whose nationality is *to be Swede* and whose height $\mathbf{h}v(\mathbf{w}v)$ in the context $\mathbf{w}v$ is *very big* (i.e., they are very tall).

Now we can also construct the meaning of a simple sentence[10]:

⟨Quantifier⟩ *Swedes are tall.*

First we will introduce the formula Swede using $\lambda$-conversion: Swede $\equiv$ Int(Swedes) **wh**. Then intension of the proposition (sentence) "*All Swedes are very tall*" is

$$\lambda \mathbf{w}_{\omega\varphi} \cdot (\forall \mathbf{h}_{(\alpha\omega)\varphi})\mathrm{Swede}\, \mathbf{wh} \Rightarrow (Bi\ Ve)(\mathbf{w}v)(\mathbf{h}v(\mathbf{w}v)).$$

Its interpretation in a model $\mathcal{M}$ is a function which assigns to each context (possible world $w_{o\alpha}$) a truth value. We thus obtain intensions of various kinds of propositions, such as "*All Swedes are extremely tall*", "*All Swedes are more or less tall*", etc.

[10] This sentence is often discussed by L.A. Zadeh in his lectures.

Finally, we will analyze the proposition *Most Swedes are tall*. Using our theory of intermediate quantifiers, we obtain intension of this proposition as follows:

$$\lambda \mathbf{w}_{\omega\varphi} \cdot (\exists z_{o((\alpha\omega)\varphi)})(\boldsymbol{\Delta}(z \subseteq \mathbf{Swede}\ \mathbf{w}))\ \&$$
$$(\forall \mathbf{h}_{(\alpha\omega)\varphi})(z\mathbf{h} \Rightarrow Bi(\mathbf{w}v)(\mathbf{h}v(\mathbf{w}v)))\ \&\ Bi\ Ve((\mu(\mathbf{Swede}\ \mathbf{w})z)). \qquad (8.17)$$

Thus, interpretation of this formula in a model $\mathcal{M}$ is a function assigning to each (global) context $\mathbf{w}_{\omega\varphi}$ a truth value which is obtained as a minimum of the truth that the greatest fuzzy set $z$ of *tall Swedes* in the context $\mathbf{w}$ is *very big* (in the sense of the measure $\mu$).

### *8.2.8 Verbs and Other Linguistic Phenomena*

Verbs that are the most complicated units of natural language. They vary by type, and each type is determined by the kinds of words that follow it and the relationship those words have with the verb itself. There are six types of verbs: intransitive (to run how, to speak how), two kinds of transitive (to read what, to consider what), to-be verbs, linking (seem, become) and two-place transitive (to give whom what).

Verbs stay in the core of sentences and can have varying number of arguments (this is called *valency*) depending on the complexity of sentence. Furthermore, they are characterized by other features, namely by tense (present, future, past), modality (necessity, indicative, possibility), aspect (perfective, imperfective, continuous, etc.), direction of speech, gender and other ones. More about verbs can be found in an extensive literature (See, e.g., [6, 30, 57].)

The model of the meaning of verbs must cope with the problem of changing valency. This mathematically means that verbs behave as relations with changing arity. A possible model of the meaning of verbs can interpret them, for example, as a union of fuzzy sets of fuzzy relations of different arities that depends on time. Hence, we may construct intension of a verb as a function

$$\mathrm{Int}(\mathsf{verb}) : W \times T \to \bigcup_{n=1}^{K} \mathcal{F}(\mathcal{F}(U^n))$$

where $T$ is time (we can put $T = \mathbb{R}$), $W$ is a set of possible worlds (contexts), $U$ a universe and $K$ is a possible valency of the verb. The universe should consist of various kinds of elements that can be named by noun phrases. In syntax of FTT, we can express the meaning of verbs as a formula

$$\lambda w\, \lambda t \bigvee_{n=1}^{K} A_{o(o\alpha^n)}$$

where $\alpha^n = \underbrace{\alpha \cdots \alpha}_{n-\text{times}}$ and $\vee$ is a disjunction.

Till now, no model of the meaning of verbs using the means of fuzzy logic was suggested. A detailed and careful elaboration in cooperation with linguists is needed.

There are also other phenomena that need to be captured by the model of linguistic semantics. Among them let us recall the topic–focus articulation and de dicto/de re usage. The *topic–focus articulation* is a phenomenon that extremely extends expressive power of natural language. Roughly speaking, each more complex linguistic expression can be divided into two parts: the *topic* that is the known part and *focus*, the new information. Each expression is thus ambiguous and the meaning of it can be clear only after specifying both parts. For example, "John goes to the cinema" says either that JOHN goes to the cinema (and not somebody else), or that John goes to the CINEMA (and not to the theater), etc.[11]

The *de dicto/de re* distinction relates to the distinction about occupied office (e.g., that we speak about president—de dicto) or about Mr. Obama (de re). This problem seems to be well captured by the means of FTT since it enables us to distinguish between functions $A_{\epsilon\epsilon}$ and objects $B_\epsilon$ and manipulate with them (Cf. [7]).

## 8.3 Reasoning in FNL

Since FNL claims to be a *logic* of human reasoning, it must also suggest its model. Till now, two main inference models are available. The first is inference on the basis of linguistic description consisting of fuzzy/linguistic IF-THEN rules. The second are reasoning schemes with generalized quantifiers. The most elaborated part are intermediate syllogisms.

### *8.3.1 Fuzzy/Linguistic IF-THEN Rules*

The theory of fuzzy IF-THEN rules is the most widely discussed and most powerful area of fuzzy logic, which has a wide variety of applications. Recall the general form of a fuzzy IF-THEN rule:

$$\mathsf{IF}\ X \text{ is } \mathcal{A}\ \mathsf{THEN}\ Y \text{ is } \mathcal{B}, \tag{8.18}$$

[11]More about this phenomenon can be found, e.g., in [14].

where $X$ is $\mathcal{A}$, $Y$ is $\mathcal{B}$ are evaluative predications.[12] A typical example is

IF *temperature* is *small* THEN *the amount of gas* is *very big*.

From the linguistic point of view, this is a simple conditional clause, i.e. a conditional sentence with a clear structure consisting of the antecedent and consequent.

In FNL, we call a (finite) set of rules (8.18) a *linguistic description*. The rules (8.18) (and the linguistic descriptions) apparently characterize some kind of relation between values of $X$ and $Y$. In the fuzzy set theory, these rules are usually construed as special fuzzy relations. This is purely extensional approach and the rules, in fact, are not treated as sentences of natural language. Let us remark that this method of interpretation of fuzzy IF-THEN rules is very convenient when we need a well working tool for approximation of functions but it is less convenient as a model of human reasoning. Therefore, it does not fit the paradigm of FNL.

In FNL, the rules (8.18) are taken as genuine conditional clauses of natural language and the linguistic description is taken as a text characterizing some situation, strategy of behavior, control of some process, etc. The goal of the constructed FNL model is to mimic the way how people understand natural language. Then, a formal theory of Ł-FTT is considered so that intension of each rule (8.18) can be constructed:

$$\mathrm{Int}(\mathcal{R}) := \lambda w\, \lambda w' \cdot \lambda x\, \lambda y \cdot Ev^A\, wx \Rightarrow Ev^C\, w'y \tag{8.19}$$

where $w$, $w'$ are contexts of the antecedent and consequent of (8.18), respectively, $Ev^A$ is the intension of the predication in the antecedent and $Ev^C$ the intension of the predication in the consequent. The linguistic description is interpreted as a set of intensions (8.19).[13] When considering a suitable model, we obtain a formal interpretation of (8.19) as a function that assigns to each pair of contexts $w, w' \in W$ a fuzzy relation among objects.[14] It is important to realize that in this case, we introduce a consistent model of the context and provide a general rule for the construction of the extension in every context.

When a model of a linguistic description is given, we must be able to model human reasoning on the basis of it, given a perception in the form '$X$ is $\mathcal{A}_0$' where $\mathcal{A}_0$ may differ slightly from all the $\mathcal{A}_1, \dots, \mathcal{A}_n$. The well elaborated method is *perception-based logical deduction* (See [36, 47].) whose main idea is to consider the linguistic description as a specific text, which has a *topic* (what we are speaking about) and *focus* (what is the new information).[15] Each rule is understood as local but vague information about the relation between $X$ and $Y$. The given predication '$X$ is $\mathcal{A}_0$' is taken as a *perception* of some specific value of $X$. On this basis, the most proper rule from the linguistic description is applied (fires), and the best value of $Y$ with respect to this rule is taken as a result. Hence, despite the vagueness of the rules forming the linguistic description, the procedure can distinguish among them

[12] For simplicity, we consider only one variable in the antecedent.

[13] See [38, 46] for the details.

[14] Note that for specific elements assigned to $x$, $y$, the intension (8.19) provides a truth value.

[15] For the detailed linguistic analysis of these concepts, see, e.g., [14].

The rule of perception-based logical deduction can be formally expressed as

$$r_{PbLD}: \frac{LPerc^{LD}(x_0, w) = \mathrm{Int}(X \text{ is } \mathcal{A}_{i_0}), LD}{Eval(\hat{y}_{i_0}, w', \mathcal{B}_{i_0})},$$

where $LD$ is a linguistic description, and $\hat{y}_{i_0}$ is the resulting best value of $Y$, provided that the perception of $x_0$ in the context $w$ is the linguistic expression $\mathcal{A}_{i_0}$ and the dependence between $X$ and $Y$ is locally characterized by $LD$.

The perception-based logical deduction is a powerful reasoning method that well models the way of human reasoning and has a lot of various kinds of applications (See [44]).

### 8.3.2 Syllogistic Reasoning

In this section, we will discuss syllogistic reasoning on the basis of sentences containing intermediate quantifiers. We suppose to deal with the formal theory $T^{\mathrm{IQ}}$ mentioned above.

By a valid syllogism we understand a triple[16] of formulas $\langle P_1, P_2, C\rangle$ such that

$$T^{\mathrm{IQ}} \vdash P_1 \mathbin{\&} P_2 \Rightarrow C$$

(equivalently, if $T^{\mathrm{IQ}} \vdash P_1 \Rightarrow (P_2 \Rightarrow C)$). Note that, if a syllogism is valid then the inequality

$$\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) \leq \mathcal{M}(C) \tag{8.20}$$

holds in every model $\mathcal{M} \models T^{\mathrm{IQ}}$.

Let $Q_1, Q_2, Q_3$ be intermediate quantifiers and $X, Y, M \in Form_{o\alpha}$ be formulas representing properties. Analogously as in classical logic, we will consider four figures of syllogisms:

| **Figure I** | **Figure II** | **Figure III** | **Figure IV** |
|---|---|---|---|
| $Q_1$ $M$ is $Y$ | $Q_1$ $Y$ is $M$ | $Q_1$ $M$ is $Y$ | $Q_1$ $Y$ is $M$ |
| $Q_2$ $X$ is $M$ | $Q_2$ $X$ is $M$ | $Q_2$ $M$ is $X$ | $Q_2$ $M$ is $X$ |
| $Q_3$ $X$ is $Y$ | $Q_3$ $X$ is $Y$ | $Q_3$ $X$ is $Y$ | $Q_3$ $X$ is $Y$ |

Peterson in his book [55] demonstrated that there are 105 intermediate syllogisms that are valid. All these syllogisms contain the above introduced intermediate quantifiers. Validity of them is proved syntactically in FNL (See [32].) (in the formal theory $T^{\mathrm{IQ}}$) that is a very strong result assuring us that the inequality (8.20) holds in every model. For example, below is the list of valid intermediate syllogisms containing the intermediate quantifiers *almost all* (**P**), *few* (**B**), *most* (**T, D**), and *many* (**K, G**):

[16] In fact, we only need the number of formulas to be finite and reasonably small.

| *Figure I* | *Figure II* | *Figure III* | *Figure IV* |
|---|---|---|---|
| ***AAP*** | ***AEB*** | **(*P)AI** | **AEB** |
| ***APP*** | ***ABB*** | **E(*P)O** | **(*P)AI** |
| ***APT*** | ***ABD*** | **(*B)AO** | **E(*P)O** |
| ***APK*** | ***ABG*** | **A(*P)I** | |
| ***API*** | **A(*B)O** | **P(*P)I** | |
| ***EAB*** | ***EAB*** | **T(*P)I** | |
| ***EPB*** | ***EPB*** | **(*K)PI** | |
| ***EPD*** | ***EPD*** | **(*P)TI** | |
| ***EPG*** | ***EPG*** | **P(*K)I** | |
| **E(*P)O** | **E(*P)O** | **B(*P)O** | |
| | | **D(*P)O** | |
| | | **G(*P)O** | |
| | | **B(*T)O** | |
| | | **B(*K)O** | |

(the letters refer to the concrete quantifiers introduced introduced above and the asterisks denote quantifiers with presupposition of non-emptiness of the universe).

Examples of valid syllogisms are the following:

$$\textbf{ATT}\text{-I}: \frac{\text{All women are well dressed Most people in the party are women}}{\text{Most people in the party are well dressed}}$$

$$\textbf{ETO}\text{-II}: \frac{\text{No lazy people pass exam Most students pass exam}}{\text{Some students are not lazy people}}$$

$$\textbf{PPI}\text{-III}: \frac{\text{Almost all old people are ill Almost all old people have gray hair}}{\text{Some people with gray hair are ill}}$$

$$\textbf{TAI}\text{-IV}: \frac{\text{Most shares with downward trend are from energy industry All shares of energy industry are important}}{\text{Some important shares have downward trend}}$$

### *8.3.3 A Model of Commonsense Human Reasoning*

Finally, we will also demonstrate the power of FNL in a more complex model of human reasoning. This was shown on an example of reasoning of a detective Lt. Columbo based on one episode from the famous TV series.[17] Let us emphasize that the presented method can be taken as a more general methodology that has a variety of other specific applications (Cf. [11]).

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[17] For the details, see [45].

**The story:**

Mr. John Smith has been shot dead in his house. He was found by his friend, Mr. Robert Brown. Lt. Columbo suspects Mr. Brown to be the murderer.

**Mr. Brown's testimony:**

*I have started from my home at about 6:30, arrived at John's house at about 7, found John dead and went immediately to the phone booth to call police. They came immediately.*

**Evidence of Lt. Columbo:**

*Mr. Smith had a high quality suit and a broken wristwatch stopped at 5:45. There was no evidence of a hard blow to his body. Lt. Columbo touched the engine of Mr. Brown's car and found it to be more or less cold.*

Lt. Columbo concluded that Mr. Brown lied because of the following:

(i) Mr. Brown's car engine is *more or less cold*, so he must have been waiting *long* (more than about 30 min). Therefore, he could not have arrived and called the police (who came *immediately*).
(ii) A *high quality wristwatch* does not break after *not too hard blow*. A man having *high quality dress* and a *luxurious house* is supposed to also have a *high quality wristwatch*. The wristwatch of John Smith is of *low quality*, so it does not belong to him. It does not display the time of Mr. Smith's death.

The reasoning of Lt. Columbo based on FNL is modeled by means of a combination of logical rules, world knowledge and evidence with the help of *non-monotonic reasoning*.

The world knowledge includes common sense knowledge of the context and further knowledge, which can be characterized using linguistic descriptions applied in specific context for the included variable, e.g., drive duration to heat the engine, temperature of engine, etc. The used linguistic descriptions are, e.g., the following:

- *Logical rules* that are hereditary valid, for example

$$\mathsf{IF}\ X \text{ is } Sm_\nu\ \mathsf{THEN}\ X \text{ is } \neg Bi,$$
$$\mathsf{IF}\ X \text{ is } Bi_\nu\ \mathsf{THEN}\ X \text{ is } \neg Sm.$$

where $Sm$, $Bi$ are linguistic expressions "small, big" and $\nu$ is some linguistic hedge.
- *Common sense knowledge from physics:*

$$\mathsf{IF}\ \textit{drive duration} \text{ is } Bi\ \mathsf{THEN}\ \textit{engine temperature} \text{ is } Bi,$$
$$\mathsf{IF}\ \textit{drive duration} \text{ is } Sm\ \mathsf{THEN}\ \textit{engine temperature} \text{ is } ML\,Sm,$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

- *Common sense knowledge of customs of people:*

  IF quality of $x$'s suit is $Bi$ AND quality of $x$'s house is $Ve\,Bi$
  THEN wealth of $x$ is $Bi$,

  ..........................................................

- Some other kinds of common sense knowledge, for example, properties of products, etc.

On the basis of a formal analysis which includes the use of the perception-based logical deduction, Lt. Columbo concludes that the two special constructed theories are contradictory. Since his perceptions and the evidence cannot be doubted, Mr. Brown is lying, so he had an opportunity to kill Mr. Smith. It is important to emphasize that the contradictory theories were constructed as nodes of a graph representing the structure of *non-monotonic* reasoning.

## 8.4 Conclusion

In this paper, we suggest to focus more deeply on the proclaimed ability of fuzzy sets—to enable to model the semantics of natural language expressions. Our idea is to develop a special branch of mathematical fuzzy logic that we call Fuzzy Natural Logic as a generalization of an older classical concept of Natural Logic. Its paradigm is to model natural human reasoning that is based on the use of natural language. Therefore, it is necessary to have also a model of semantics of natural language at disposal. We argue that FNL should be developed in a close cooperation with linguists and logicians.

We gave a brief overview of some units and phenomena of natural language and outlined problems connected with their semantics. In parallel, we also outlined ways how their semantics can be modeled inside FNL. In the second part, we also outlined some of possible human reasoning schemes that can be modeled using FNL.

One can see that still many problems and open questions have to be solved and answered before we can say that FNL is a well developed theory that reached its goal—to model natural human reasoning. Even at this stage of research, though, there are various interesting and quite well working applications of FNL. Let us mention few of them:

- Identification of rock sequences on the basis of expert geologists' knowledge [34].
- Linguistic control of technological processes [41, 51].
- Multi-criteria decision-making (without need to define weights of criteria) [48].
- Forecasting of the trend-cycle of time series [50, 52] and linguistic evaluation of is trend (i.e., "steep increase/decrease, stagnating, rough increase/decrease", etc.).

**Acknowledgments** The research was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070).

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# Chapter 9
# From Lattice Valued Theories to Lattice Valued Analysis

**Irina Perfilieva and Alexandr Šostak**

**Abstract** We claim and justify that the future of a fuzzy logic is in the interconnection of various well-developed theories. We are focused on a lattice valued analysis that unifies the treatments of atomic elements, sets of atomic elements, functions between sets of atomic elements and their properties. We clarify the relationship between a fuzzy function and its ordinary core. We discuss the property of continuity of a fuzzy function in a lattice valued topology.

## 9.1 The Future of the Fuzzy Logic

Fuzzy logic and corresponding to it calculus gave birth to various lattice valued theories, such as fuzzy relation equations and inequalities, fuzzy measures and integrals, fuzzy operation research, fuzzy topology and many others. The development of each particular theory is more or less independent on others. The only common tool is a lattice ordered algebra. As a consequence of this, we have that similar results appear under different names.

In our opinion, the future of a fuzzy logic in a broader sense (by this we mean not only [0, 1]-valued structures, but lattice ordered ones) is in the interconnection of various well-developed theories. Let us explain this vision in more details.

Fuzzy logic as a theory provides other theories and applications with a generic language for various descriptions of objects and relations and formulation of results. The structure of this language is similar to the structure of a tree (or a forest): one or several basic notions (fuzzy sets, relations, partitions, etc.), relations and operations

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_9

over them. From this point of view, the future of the fuzzy logic is in preserving the fundamental structure and developing on its base other (specialized) theories.

We consider the fuzzy logic as a mathematical discipline, which means that once a new notion is formulated, it opens a door for a new direction (branch) where this notion becomes fundamental. On the other hand, in order to obtain a first-class theory, its local branches should intertwine and enrich each other.

Nowadays, there are many specialized theories (branches) that stemmed from the fuzzy logic, but do not influence each other. It is the right time to take them as basic tools and develop a higher level theory on their common framework. This contribution is an example of how it can be done.

## 9.2 Introduction

We are focused on what can be called as Lattice Valued Analysis—the name which we use instead of ordinary mathematical analysis. It develops foundations in the sense of a unified approach to the treatment of atomic elements, sets of atomic elements, functions between sets of atomic elements and their properties. The structure of the proposed lattice valued analysis mimics the way, how the modern mathematical analysis is presented.

We demonstrate that joint efforts of already established lattice valued theories, such as residuated algebraic structures, fuzzy relation equations and fuzzy topology lead to a calculus of fuzzy points and singletons (atomic units), fuzzy sets (collections of atomic units) and fuzzy functions (points-to-fuzzy sets mappings). In this contribution, we do not go beyond the notion of a continuous fuzzy function which we formulate in languages of the mentioned above lattice valued theories. Our presentation is given on rather elementary level in order to keep an interest of the reader to the proposed topic. We do not give a wide overview of achieved results, but trace the development of principal notions and ideas. The main attention is paid to the notion of fuzzy function, because it is crucial for many other areas and applications. We are focused on a relational representation of a fuzzy function (important for fuzzy IF-THEN rules models), fuzzy function properties, connection to an ordinary "prototype" (we call it a core) of a fuzzy function and its continuity. Below, is a short overview of various approaches to this notion.

Fuzzy function has at least two different meanings in fuzzy literature. On the one side (see e.g., [2–4, 6, 7, 14]), a fuzzy function is a special fuzzy relation with a generalized property of uniqueness. According to this approach, each element from the ordinary domain of thus defined fuzzy function is associated with a certain fuzzy set. Thus, a fuzzy function establishes a "point"-to-"fuzzy set" correspondence.

On the other hand (see [10–14]), a fuzzy function is a mapping between two universes of fuzzy sets, i.e. establishes a "fuzzy set"-to-"fuzzy set" correspondence. This approach is implicitly used in many papers devoted to fuzzy IF-THEN rule models where the latter are actually partially given fuzzy functions.

In this contribution, we show that both viewpoints can be connected by a natural generalization of the Extension Principle of L. Zadeh [15]. In details, a fuzzy function as a mapping is an extension of a fuzzy function as a relation to the domain of fuzzy sets. The similar approach is used in mathematical analysis where an image (pre-image) of a set under a function is commonly defined.

In order to establish the above mentioned extension, we introduce various spaces of fuzzy objects with fuzzy equivalence relations on them. We show that similar to the classical case, a (fuzzy) set is a collection of (fuzzy) points or (fuzzy) singletons.

We analyze a relationship between a surjective fuzzy function and its ordinary core function. The similar study has been attempted in [2] for a perfect fuzzy function and in [7] for one particular example of a fuzzy function. We propose a solution in the general case.

Last, but not least, we develop the notion of a continuous fuzzy function on the basis of a lattice valued topology, which is an extension of classical topology. Our goal is to extend the definition of a continuous (ordinary) function between two fuzzy topological spaces to the case where a function is fuzzy. Moreover, we analyze continuity of both surjective fuzzy function and its core with respect to the same fuzzy topological spaces.

The contribution is organized as follows. In Sect. 9.3, we give preliminary information about extension principle, residuated lattices, fuzzy sets and fuzzy spaces. Fuzzy functions and two approaches to this notion are discussed in Sect. 9.4. Section 9.5 contains main results about continuity of a surjective fuzzy function. We included proofs of some new statements to demonstrate how powerful are tools of involved lattice valued theories.

## 9.3 Preliminaries

### *9.3.1 Extension Principle and Its Relational Form*

An extension principle has been proposed by L. Zadeh [15] in 1975 and since then it is widely used in the fuzzy set theory and its applications. Let us recall the principle and propose its relational form which will be used later on in a correspondence with the notion of fuzzy function.

Assume that $X, Y$ are universal sets and $f : X \to Y$ is a function with the domain $X$. Let moreover, $\mathcal{F}(X), \mathcal{F}(Y)$ be respective universes of fuzzy sets on $X$ and $Y$ identified with their membership functions, i.e. $\mathcal{F}(X) = \{A : X \to [0, 1]\}$ and similarly, $\mathcal{F}(Y)$. By the extension principle, an image of a fuzzy set $A \in \mathcal{F}(X)$ under $f$ is fuzzy set $f^{\to}(A)$ on $\mathcal{F}(Y)$ such that

$$f^{\to}(A)(y) = \sup_{y=f(x)} A(x). \tag{9.1}$$

Let $R_f$ be a binary relation on $X \times Y$ which corresponds to the function $f$, i.e.

$$R_f(x, y) = 1 \Leftrightarrow y = f(x).$$

Then it is easy to see that (9.1) can be equivalently represented by

$$f^{\rightarrow}(A)(y) = \bigvee_{y \in Y} (A(x) \cdot R_f(x, y)), \tag{9.2}$$

where $\cdot$ is the multiplication on [0, 1].

Expression (9.2) is the *relational form of the extension principle*. The meaning of expression (9.2) will be more general, if we consider $A$ as an $L$-fuzzy set (see Definition 3), binary relation $R_f$ as a fuzzy relation, and multiplication $\cdot$ as a monoidal operation (see Sect. 9.3.2). In Sect. 9.4, we will discuss this generalization and its relationship to fuzzy functions.

### 9.3.2 Residuated Lattice

Let residuated lattice be a basic algebra of operations.

**Definition 1** A residuated lattice is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1 \rangle.$$

with a support $L$ and four binary operations and two constants such that

- $\langle L, \vee, \wedge, 0, 1 \rangle$ is a lattice where the ordering $\leq$ defined using operations $\vee, \wedge$ as usual, and 0, 1 are the least and the greatest elements, respectively;
- $\langle L, *, 1 \rangle$ is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $a * 1 = a$;
- the operation $\rightarrow$ is a residuation operation with respect to $*$, i.e.

$$a * b \leq c \iff a \leq b \rightarrow c.$$

A residuated lattice is complete if it is complete as a lattice.

The following is a binary operation of biresiduation on $\mathcal{L}$:

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

The well known examples of residuated lattices are: boolean algebra, Gödel, Łukasiewicz and product algebras. In the particular case $L = [0, 1]$, multiplication $*$ is a left continuous $t$-norm.

From now on we fix a complete residuated lattice $\mathcal{L}$.

### *9.3.3 L-fuzzy Sets, Fuzzy Relations and Fuzzy Spaces*

Below, we recall definitions of principal notions in the fuzzy set theory.

*Fuzzy sets with crisp equality*

Let $X$ be a non-empty universal set, $\mathcal{L}$ a complete residuated lattice. An $L$-fuzzy set $A$ of $X$ (fuzzy set, shortly) is a map $A : X \to L$ that establishes a relationship between elements of $X$ and degrees of *membership* to $A$.

Fuzzy set $A$ is *normal* if there exists $x_A \in X$ such that $A(x_A) = 1$. The (ordinary) set $\mathrm{Core}(A) = \{x \in X \mid A(x) = 1\}$ is a *core* of a normal fuzzy set $A$. The (ordinary) set $\mathrm{Supp}(A) = \{x \in X \mid A(x) > 0\}$ is a *support* set of fuzzy set $A$.

The class of $L$-fuzzy sets of $X$ will be denoted $L^X$. The couple $(L^X, =)$ is called an *ordinary fuzzy space* on $X$. The elements of $(L^X, =)$ are fuzzy sets equipped with the equality relation, i.e. for all $A, B \in L^X$,

$$A = B \textit{ if and only if } (\forall x \in X)\, A(x) = B(x).$$

In $(L^X, =)$, we strictly distinguish between fuzzy sets even if their membership functions differ in one point. On $(L^X, =)$, we can define the structure of residuated lattice using pointwise order relation $\leq$ and operations over fuzzy sets. Due to this convention, we have *inclusion*, *union* and *intersection* of two fuzzy sets $A, B \in L^X$ as follows:

$$A \subseteq B \textit{ if and only if } (\forall x \in X)\, A(x) \leq B(x),$$
$$(A \cup B)(x) = A(x) \vee B(x),$$
$$(A \cap B)(x) = A(x) \wedge B(x).$$

The lattice $\langle L^X, \cup, \cap, \mathbf{0}, \mathbf{1}\rangle$ is complete, where the bottom $\mathbf{0}$ and the top $\mathbf{1}$ are constant fuzzy sets, whose values are 0 and 1, respectively.

A class of normal $L$-fuzzy sets of $X$ will be denoted $\mathcal{N}(X)$. The space $(\mathcal{N}(X), =)$ is a subspace of $(L^X, =)$.

By identifying a point $u \in X$ with a fuzzy subset $I_u : X \to L$ such that $I_u(u) = 1$ and $I_u(x) = 0$ whenever $x \neq u$ we may view $X$ as a subspace of $(L^X, =)$ and as a subspace of $(\mathcal{N}(X), =)$.

*Space with fuzzy equivalence. Fuzzy points*

Let $X, Y$ be universal sets. Similarly to $L$-fuzzy sets, we define *(binary) (L)-fuzzy relations* as fuzzy sets of $X \times Y$. The class of $L$-fuzzy relations of $X \times Y$ will be denoted $L^{X \times Y}$. If $X = Y$, then a fuzzy set of $X \times X$ is called a ($L$)-fuzzy relation on $X$.

Fuzzy relation $E$ on $X$ is called *fuzzy equivalence* on $X$ (see [1, 5, 8])[1] if for all $x, y, z \in X$, the following holds:

1. $E(x, x) = 1$, *reflexivity*,
2. $E(x, y) = E(y, x)$, *symmetry*,
3. $E(x, y) * E(y, z) \leq E(x, z)$, *transitivity*.

If fuzzy equivalence $E$ fulfills

1. $E(x, y) = 1$ if and only if $x = y$,

then it is called *separated* or a *fuzzy equality* on $X$.

Fuzzy equivalence $E$ assigns a *degree of coincidence* to any two elements from $X$.

Let us remark that fuzzy equivalence $E$ creates fuzzy sets on $X$, we will call them $E$*-points*[2] of $X$. Every $E$-point is a class of fuzzy equivalence $E$ of just one point of $X$. In more details, if $t \in X$, then $E$-point $E_t$ is a fuzzy set $E_t : X \to L$ such that for all $x \in X$, $E_t(x) = E(t, x)$. It is easy to see that $E_t$ is a normal fuzzy set and $t \in \text{Core}(E_t)$.

The set of all $E$-points of $X$ will be denoted by

$$\mathcal{P}_X^E = \{E_t \mid t \in X\}.$$

Obviously, $(\mathcal{P}_X^E, =)$ is a subspace of $(L^X, =)$. If $E$ is a fuzzy equivalence on $X$, then it may happen that some element, say $E_t$ from $(\mathcal{P}_X^E, =)$ has more than one representation, i.e. there exists $u \in X$ such that $E_u = E_t$. It can be shown that this holds true if and only if $E(t, u) = 1$, or $u \in \text{Core}(E_t)$.

On the other side, if $E$ is a fuzzy equality on $X$, then the core of every $E$-point consists of just one element and thus, a representation of any $E$-point in the form $E_t$ is unique.

*Fuzzy singletons and sub-singletons*

Let us equip space $X$ with fuzzy equality $E$ and denote it by $(X, E)$. In this space, we are able to distinguish degrees of coincidence $E(t, u)$ of any two elements $t, u$ from $X$. As we discussed above, crisp and fuzzy equalities put into the correspondence with each element $t$ of $X$ its characteristic function $I_t$ and its $E$-point $E_t$. Both are normal fuzzy sets in $L^X$ with the same one-element core. Let us consider fuzzy sets $S_t \in L^X$, that are in between $I_t$ and $E_t$, i.e. for all $x \in X$,

$$I_t(x) \leq S_t(x) \leq E_t(x). \tag{9.3}$$

We will call them $E-$*singletons*. In [7], fuzzy singletons were introduced as normal fuzzy sets $S_t \in L^X$ with $\{t\}$ as a one-element core, i.e. $S_t(t) = 1$, and such that for all $x, y \in X$,

[1] Fuzzy equivalence appears in the literature under the names *similarity* or *indistinguishability* as well.

[2] This notion was introduced in [7].

$$S_t(x) * S_t(y) \leq E(x, y), \tag{9.4}$$

where $*$ is the monoidal operation from the chosen residuated lattice $L$. It is easy to show that this is equivalent to our characterization (9.3). Indeed, if $S_t$ fulfills (9.3), then it is normal, it has $\{t\}$ as a one-element core, and for all $x, y \in X$,

$$S_t(x) * S_t(y) \leq E(t, x) * E(t, y) \leq E(x, y).$$

On the other side, if $S_t$ has $\{t\}$ as a one-element core and fulfills (9.4), then for all $x \in X$, $I_t(x) \leq S_t(x)$ and

$$S_t(x) = S_t(x) * S_t(t) \leq E(t, x) = E_t(x).$$

From (9.4) and the discussion above it follows that $E$-point $E_t$ is the greatest $E$-singleton with the one-element core $\{t\}$. The space of all $E$-singletons will be denoted by $\mathcal{S}_X^E$. Obviously, $\mathcal{S}_X^E \subseteq L^X$ and $(\mathcal{S}_X^E, =)$ is a subspace of $(L^X, =)$.

Let us discard normality in the characterization (9.3) of $E$-singleton and define *$E$-sub-singleton* as a fuzzy set $U \in L^X$, that fulfills the following property:

$$(\exists t \in X)(0 < U(t))\&(0 < U(x) \leq E_t(x)). \tag{9.5}$$

In order to stress that an $E$-sub-singleton is connected with a certain point $t$, we will denote it as $U_t$. Similarly to the above, we can prove that any $E$-sub-singleton fulfills (9.4). The space of all $E$-sub-singletons will be denoted by $\mathcal{U}_X^E$. Obviously, $\mathcal{S}_X^E \subseteq \mathcal{U}_X^E \subseteq L^X$ and $(\mathcal{U}_X^E, =)$ is a subspace of $(L^X, =)$.

*Extensional hulls*

Let $(X, E)$ be a space with fuzzy equivalence. Similarly to the above, we can define notions of $E$-singleton and $E$-sub-singleton. However, different to the above, this notions refer to $t$ as to a certain, but not necessarily unique element of the core of $E_t$.

We remind [7] that fuzzy set $A$ is *extensional* (with respect to $E$) or $E$-extensional, if for all $x, y \in X$,

$$A(x) * E(x, y) \leq A(y).$$

The smallest extensional fuzzy set $A^E$ containing fuzzy set $A$ is called the *extensional hull* of $A$. It is not difficult to prove the following representation of $A^E$.

**Lemma 1** *The extensional hull $A^E$ of every fuzzy set $A \in L^X$ can be represented as follows:*

$$A^E(y) = \sup_{x \in X} \quad A(x) * E(x, y). \tag{9.6}$$

Representation (9.6) has been obtained in many papers (see e.g., [5]).

Lemma 1 has two important corollaries.

**Corollary 1** *Extensional hull of element $t \in X$ identified with $I_t$ is equal to $E$-point $E_t$.*

**Corollary 2** *Extensional hull of E-singleton* $S_t \in L^X$, $t \in X$, *is equal to the corresponding E-point* $E_t$.

*Decomposition of a fuzzy set into E-sub-singletons*

The purpose of this paragraph is to show that a fuzzy set can be decomposed into a collection of "atomic" singletons or sub-singletons and afterwards, can be represented as a union of these atomic elements. The proposed representation is similar to that in the theory of sets where a set is a collection of pairwise different elements.

In the below given text, we assume that atomic elements are defined in a space $(X, E)$, where $E$ is a fuzzy equivalence on $X$.

**Theorem 1** *Let* $A \in L^X$ *be a non-zero fuzzy set. Then* $A$ *can be represented by unions of E-sub-singletons* $U_t^A$ *or E-sub-singletons* $W_t^A$, *i.e.*

$$A = \bigcup_{t \in Supp(A)} U_t^A = \bigcup_{t \in Supp(A)} W_t^A, \tag{9.7}$$

*where*

$$U_t^A(x) = A(x) \wedge E_t(x), \; x \in X, \tag{9.8}$$

*and*

$$W_t^A(x) = A(x) * E_t(x), \; x \in X. \tag{9.9}$$

## 9.4 Fuzzy Functions

The notion of fuzzy function has many definitions in the literature, see e.g., [3, 4, 7, 11]. In [3, 4, 7], a fuzzy function is considered as a special fuzzy relation. Below, we remind the notion of fuzzy function as it appeared (independently) in [3, 6, 7]. We will use this definition in our analysis due to the following reason: it is based on the definition of a function as a special binary relation with the property "single value condition".

**Definition 2** Let $E$, $F$ be fuzzy equivalences on $X$ and $Y$, respectively. A fuzzy function is a binary fuzzy relation $\rho$ on $X \times Y$ such that for all $x, x' \in X$, $y, y' \in Y$ the following axioms hold true:

**FF.1** $E$- **extensionality** – $\rho(x, y) * E(x, x') \le \rho(x', y)$,
**FF.2** $F$- **extensionality** – $\rho(x, y) * F(y, y') \le \rho(x, y')$,
**FF.3** **single value condition** – $\rho(x, y) * \rho(x, y') \le F(y, y')$,

A fuzzy function is called perfect [2], if it additionally fulfills

**FF.4** for all $x \in X$, there exists $y \in Y$, such that $\rho(x, y) = 1$.

Semantically, $\rho(x, y)$ is a truth degree of the assertion: "$y$ is a functional value of $x$". Axiom **FF.4** characterize the property of being certainly defined on $X$.

A fuzzy function is called (strong) *surjective* [2], (see also [6, Sect. 4.2]) if

**FF.5** for all $y \in Y$, there exists $x \in X$, such that $\rho(x, y) = 1$.

Actually, fuzzy function $\rho$ is a point-to-fuzzy set mapping between $X$ and $L^Y$ such that for all $x \in X$, $\rho(x, \cdot)$ is a fuzzy set on $Y$. In other words, it is a *fuzzy-valued function*. If for all $x \in X$, $\rho(x, \cdot)$ is a normal fuzzy set then $\rho$ is perfect, and there is an ordinary function $g : X \to Y$ such that for all $y \in Y$, $\rho(x, y) = F(g(x), y)$ (see [3]). This means that every $F$-fuzzy point $F_{g(x)}$ of $Y$ determined by $g(x)$ is a fuzzy value of $\rho$ at $x \in X$.

In our study, we will consider the case where $\rho$ is surjective and defined on $X$, i.e.

$$(\forall x \in X)(\exists y \in Y) \quad \rho(x, y) > 0. \tag{9.10}$$

In this case, we will propose an analytic representation of $\rho$ and use $\rho$ in the generalized extension principle. Moreover, we will discover a relationship between a fuzzy function, its ordinary core function and its extension to a mapping over the domain of fuzzy sets.

### 9.4.1 Fuzzy Function and Its Core

In this section, we will show that each surjective fuzzy function $\rho$ on $X \times Y$ determines a corresponding ordinary core function $g : X' \to Y$, where $X' \subseteq X$ consists of all those points $x' \in X$ that guarantee surjectivity of $\rho$ as it is expressed in axiom **FF.5**. Let us give more details to the construction of $X'$.

We assume that $X$ is equipped with fuzzy equivalence $E$, $Y$ with fuzzy equality $F$ and $\rho$ is a surjective fuzzy function. For every $y \in Y$, let us choose the set $X_y = \{x_y \mid x_y \in \mathrm{Core}(\rho(\cdot, y))\}$ and show that for different $y_1, y_2 \in Y$, $X_{y_1} \cap X_{y_2} = \emptyset$. Indeed, if on contrary, there is $x \in X_{y_1} \cap X_{y_2}$, then $\rho(x, y_1) = 1$ and $\rho(x, y_2) = 1$, so that by axiom **FF.3**, we arrive to $F(y_1, y_2) = 1$ and thus, to $y_1 = y_2$.

To finish the construction of $X'$, we choose one (arbitrary) element $x_y \in X_y$ and put it into $X'$, so that $X' = \{x_y \mid x_y \in X_y, z \in Y\}$. We will refer to $X'$ as to the $\rho^{(-1)}(Y)$.

The proofs of the below given Theorems 2 and 3 are in [12].

**Theorem 2** *Let fuzzy relation $E$ be a fuzzy equivalence on $X$ and $F$ a fuzzy equality on $Y$. Let fuzzy relation $\rho$ on $X \times Y$ be a surjective fuzzy function and $X' = \rho^{(-1)}(Y)$. Then the following fuzzy relation on $X$*

$$E'(x, x') = \bigwedge_{y \in Y} (\rho(x, y) \leftrightarrow \rho(x', y)), \tag{9.11}$$

*is a fuzzy equivalence $E'$ such that*

1. *$E \le E'$ and $\rho$ is a fuzzy function with respect to fuzzy equivalences $E'$ and $F$,*
2. *for all $x \in X$, $y \in Y$,*

$$\rho(x, y) = E'(x, x_y), \tag{9.12}$$

3. *for all $y, y' \in Y$,*

$$E'(x_y, x_{y'}) = F(y, y'), \tag{9.13}$$

4. *the mapping $g : X' \to Y$ such that $g(x_y) = y$ is surjective and extensional with respect to $E'$ and $F$, i.e. for all $x, t \in X'$,*

$$E'(x, t) \le F(g(x), g(t)). \tag{9.14}$$

**Corollary 3** *Fuzzy equivalence $E'$, given by (9.11), is the greatest one (in the sense of $\le$) that makes $\rho$ to be a fuzzy function with respect to $E'$ and $F$.*

**Corollary 4** *Fuzzy equivalence $E'$, given by (9.11), covers $X$, i.e. for all $x \in X$ there exists $x_y \in X'$ such that $E'(x, x_y) > 0$.*

The meaning of the assertions below is that a surjective fuzzy function $\rho$ is indeed a fuzzified version of its core function $g : X' \to Y$, where $X' \subseteq X$. If $x \in X$, then the fuzzy value of $\rho(x, \cdot)$ is a "linear"-like combination of $F$-fuzzy points $F_{g(x')}(\cdot)$. In particular, if $x' \in X'$ - domain of $g$, then the fuzzy value of $\rho(x', \cdot)$ is equal to the corresponding $F$-fuzzy point $F_{g(x')}(\cdot)$.

**Theorem 3** *Let fuzzy relations $E$, $E'$, $F$, $\rho$ and function $g : X' \to Y$ where $X' = \rho^{(-1)}(Y)$ fulfil assumptions and conclusions of Theorem 2. Then*

1. *for all $x \in X$, $y \in Y$,*

$$\rho(x, y) = \bigvee_{x' \in X'} (E'_{x'}(x) * F_{g(x')}(y)), \tag{9.15}$$

2. *for all $t \in X'$, $y \in Y$,*

$$\rho(t, y) = F_{g(t)}(y). \tag{9.16}$$

On Fig. 9.1, we see an illustration of some surjective fuzzy function and its ordinary core function.

### 9.4.2 Generalized Extension Principle and Image of a Fuzzy Set

In this section, we show how an image of a fuzzy set under a fuzzy function can be computed. We propose the "Generalized extension principle", which defines an

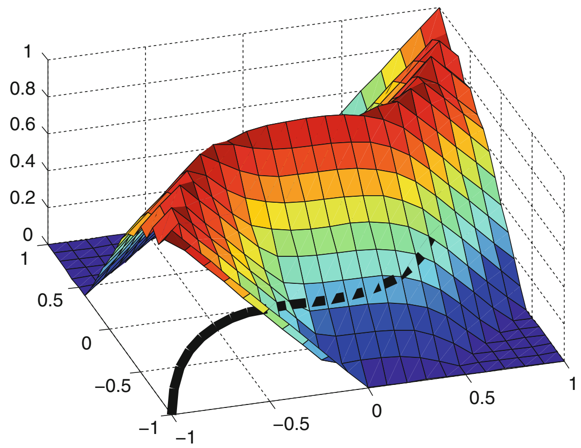


**Fig. 9.1** Example of a surjective fuzzy function and its core (*black line* on the coordinate plane)

image of a fuzzy set under a fuzzy function. We will be based on the "Extension principle" of L.A. Zadeh [15] in the relational form (9.2), where we replace the ordinary relation $R_f$ by fuzzy function $\rho$.

**Definition 3** (*Generalized extension principle*) Let $\mathcal{L}$ be a complete residuated lattice and $(L^X, =)$, $(L^Y, =)$ fuzzy spaces. Let $E$, $F$ be fuzzy equivalences on $X$ and $Y$, respectively, and fuzzy relation $\rho \in L^{X \times Y}$ be a fuzzy function. Then the $\rho$-image $f_\rho^{\rightarrow}(A)$ of a fuzzy set $A \in L^X$ is a fuzzy set in $(L^Y, =)$ such that

$$f_\rho^{\rightarrow}(A)(y) = \bigvee_{x \in X} (A(x) * \rho(x, y)). \quad (9.17)$$

Below, we will show that the image of a fuzzy set under a fuzzy function is a union of corresponding images of constituting fuzzy sub-singletons, or in particular, of $E'$-fuzzy points.

**Theorem 4** *Let $E$ be a fuzzy equivalence on $X$, $F$ a fuzzy equality on $Y$ and $\rho \in L^{X \times Y}$ a fuzzy function. Then for any $A \in L^X$,*

$$f_\rho^{\rightarrow}(A) = \bigvee_{t \in Supp(A)} f_\rho^{\rightarrow}(W_t^A), \quad (9.18)$$

*where $W_t^A$ is a fuzzy sub-singleton (9.9) in the space $(X, E')$.*

*In particular, if $\rho$ is surjective, $E'$ is a fuzzy equivalence given by (9.11) and fuzzy set $A$ is a union of fuzzy points $E'_t$, i.e. $A = \bigcup_{t \in Supp(A)} E'_t$, then*

$$f_\rho^{\rightarrow}(A)(y) = \bigvee_{t \in Supp(A)} f_\rho^{\rightarrow}(E'_t)(y) = \bigvee_{t \in Supp(A)} \rho(t, y). \quad (9.19)$$

*Proof* By Theorem 1, $A$ can be represented as a supremum of fuzzy sub-singletons $W_t^A$, $t \in \mathrm{Supp}(A)$, where $W_t^A(x) = A(x) * E'_t(x),\ x \in X$. Thence,

$$f_\rho^{\rightarrow}(A)(y) = \bigvee_{x\in X} (A(x) * \rho(x,y)) =$$

$$= \bigvee_{x\in X} \left( \bigvee_{t\in \mathrm{Supp}(A)} W_t^A(x) \right) * \rho(x,y) =$$

$$\bigvee_{t\in \mathrm{Supp}(A)} \bigvee_{x\in X} W_t^A(x) * \rho(x,y) = \bigvee_{t\in \mathrm{Supp}(A)} f_\rho^{\rightarrow}(W_t^A)(y).$$

To prove (9.19), we first decompose

$$f_\rho^{\rightarrow}(A)(y) = \bigvee_{t\in \mathrm{Supp}(A)} f_\rho^{\rightarrow}(E'_t)(y),$$

then choose $t \in \mathrm{Supp}(A)$ and continue with the following chain of equalities

$$f_\rho^{\rightarrow}(E'_t)(y) = \bigvee_{x\in X} (E'_t(x) * \rho(x,y)) =$$

$$= \bigvee_{x\in X} E'_t(x) * \bigvee_{x'\in X'} (E'_{x'}(x) * F_{g(x')}(y)) =$$

$$\bigvee_{x'\in X'} \left( \bigvee_{x\in X} E'_{x'}(x) * E'_t(x) \right) * F_{g(x')}(y) =$$

$$= \bigvee_{x'\in X'} E'_{x'}(t) * F_{g(x')}(y) = \rho(t,y),$$

where we made use of representation $\rho$ by (9.15). Finally,

$$f_\rho^{\rightarrow}(A)(y) = \bigvee_{t\in \mathrm{Supp}(A)} \rho(t,y).$$

The following properties of fuzzy set images are analogous to their classical prototypes. Let $A_i \in L^X$, $i \in I$, be any collection of fuzzy sets, then

- $f_\rho^{\rightarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f_\rho^{\rightarrow}(A_i)$,
- $f_\rho^{\rightarrow}(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f_\rho^{\rightarrow}(A_i)$.

### 9.4.3 Pre-image of a Fuzzy Set

The aim of this section is to define a pre-image of a fuzzy set under a fuzzy function. Our goal is to find a definition that allows to show that properties of fuzzy set pre-images are analogous to their classical prototypes.

**Definition 4** Let $\mathcal{L}$ be a complete residuated lattice and $(L^X, =)$, $(L^Y, =)$ fuzzy spaces. Let $E, F$ be fuzzy equivalences on $X$ and $Y$, respectively, and fuzzy relation $\rho \in L^{X\times Y}$ be a fuzzy function. Then the $\rho$-pre-image $f_\rho^{\leftarrow}(B)$ of a fuzzy set $B \in L^Y$ is a fuzzy set in $(L^X, =)$ such that

$$f_\rho^{\leftarrow}(B)(x) = \bigwedge_{y\in Y} (\rho(x, y) \rightarrow B(y)). \tag{9.20}$$

The following properties of fuzzy set pre-images are analogous to their classical prototypes. Let $A, A_i \in L^X$, $i \in I$, be any collection of fuzzy sets, then

- $f_\rho^{\leftarrow}(f_\rho^{\rightarrow}(A)) \supseteq A$;
- $f_\rho^{\leftarrow}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f_\rho^{\leftarrow}(A_i)$;
- $f_\rho^{\leftarrow}(\bigcup_{i\in I} A_i) \subseteq \bigcup_{i\in I} f_\rho^{\leftarrow}(A_i)$.

## 9.5 Continuous Fuzzy Functions

In Sect. 9.4, we have remarked that fuzzy function is a point-to-fuzzy set mapping or fuzzy valued function. The latter notion is widely used in the correspondence with fuzzy numbers and their applications in various numeric methods with fuzzy data. Continuous fuzzy valued functions are defined in the literature with the help of alpha-cuts and an adaptation of ordinary epsilon-delta definition.

In this contribution, we are focused on lattice valued fuzzy sets and therefore, the notion of a continuous fuzzy function should be defined with the help of a lattice valued topology ($L$-topology or fuzzy topology), which is an extension of classical topology. Let us remind that an $L$-topology [9] on a set $X$ is a subset $\tau$ of $L^X$ closed with respect to finite intersections and arbitrary unions. $(X, \tau)$ is called an $L$-topological space (fuzzy topological space). A (ordinary) function $f : (X, \tau) \rightarrow (Y, \nu)$ between two fuzzy topological spaces is called *continuous* if $f^{\leftarrow}(B) = Bf \in \tau$ for each $B \in \nu$.

Our goal is twofold: to extend the definition of a continuous function between two fuzzy topological spaces to the case where a function is fuzzy and to show that any surjective fuzzy function is continuous with respect to particular fuzzy topological spaces. To meet this goal, we extend the definition of a continuous function above by replacing the pre-image of an ordinary function by the pre-image of a fuzzy function.

**Definition 5** Let $(X, \tau)$ and $(Y, \nu)$ be two fuzzy topological spaces, and $\rho \in L^{X \times Y}$ be a fuzzy function. $\rho$ is called *continuous* with respect to $(X, \tau)$ and $(Y, \nu)$, if for each $B \in \nu$, $f_\rho^{\leftarrow}(B) \in \tau$, where $f_\rho^{\leftarrow}(B)$ is given by (9.20), i.e.

$$f_\rho^{\leftarrow}(B)(x) = \bigwedge_{y \in Y} (\rho(x, y) \to B(y)), \quad x \in X.$$

The example of a fuzzy topology in a space with fuzzy equivalence is given in Lemma 2. This particular fuzzy topology will be used in the below theorems where we show that a surjective fuzzy function and its core are continuous with respect to the same fuzzy topological spaces.

**Lemma 2** *Let $(X, E)$ be a space with fuzzy equivalence. The class $\Gamma(E)$ of all $E$-extensional $L$-fuzzy sets is a fuzzy topology.*

We say that $(X, \Gamma(E))$ is a fuzzy topological space of $E$-extensional $L$-fuzzy sets.

In Theorem 5, we prove that a core function of any surjective fuzzy function is continuous with respect to fuzzy topological spaces that are constituted of fuzzy sets that are extensional with respect corresponding fuzzy equivalences.

**Theorem 5** *Let $E$ be a fuzzy equivalence on $X$ and $F$ a fuzzy equality on $Y$. Let fuzzy relation $\rho$ on $X \times Y$ be a surjective fuzzy function with the core function $g : X' \to Y$, where $X' = \rho^{(-1)}(Y)$. Let moreover, $E'$ be the greatest fuzzy equivalence on $X$ (in the sense of $\le$) that makes $\rho$ a fuzzy function with respect to $E'$ and $F$. Then function $g$ between fuzzy topological spaces $(X', \Gamma(E'))$ and $(Y, \Gamma(F))$ is continuous.*

*Proof* The proof is based on Theorem 2 and especially on equality (9.13), that states $E'(x_y, x_{y'}) = F(y, y')$, where $x_y, x_{y'} \in X'$, $y, y' \in Y$.

Let $B \in \Gamma(F)$, i.e. $B$ is extensional with respect to $F$. We shall prove that $g^{\leftarrow}(B) \in \Gamma(E')$, where $g^{\leftarrow}(B) = Bg$. In other words, we shall prove that $Bg$ is extensional with respect to $E'$. This follows from

$$B(g(x_y)) * E'(x_y, x_z) = B(y) * F(y, z) \le B(z) = B(g(x_z)),$$

where $x_y, x_z \in X'$ and $z, y \in Y$ are corresponding elements. Thus, $g^{\leftarrow}(B) \in \Gamma(E')$ and $g$ is continuous. □

In Theorem 6, we prove that any surjective fuzzy function is continuous with respect to fuzzy topological spaces that are constituted of fuzzy sets that are extensional with respect corresponding fuzzy equivalences.

**Theorem 6** *Let the assumptions of the Theorem 5 be fulfilled. Then fuzzy function $\rho$ is continuous with respect to fuzzy topological spaces $(X', \Gamma(E'))$ and $(Y, \Gamma(F))$, where $X' = \rho^{(-1)}(Y)$.*

*Proof* The proof is based on Theorem 2 and especially on the two equalities: representation (9.12), i.e. $\rho(x, y) = E'(x, x_y)$, and equality (9.13), i.e. $E'(x_y, x_{y'}) = F(y, y')$, where $x_y, x_{y'} \in X'$, $y, y' \in Y$.

Let $B \in \Gamma(F)$, i.e. $B$ is extensional with respect to $F$. We shall prove that pre-image $f_\rho^{\leftarrow}(B)$ given by (9.20), is extensional with respect to $E'$, i.e. $f_\rho^{\leftarrow}(B) \in \Gamma(E')$. At first, we show that for all $x \in X'$, $f_\rho^{\leftarrow}(B)(x) = Bg(x)$. Let $x \in X'$, so that there exists $z \in Y$, such that $x = x_z$. Then

$$\bigwedge_{y \in Y} (\rho(x, y) \to B(y)) = \bigwedge_{y \in Y} (E'(x_z, x_y) \to B(y)) =$$
$$\bigwedge_{y \in Y} (F(z, y) \to B(y)) = B(z) = B(g(x_z)) = B(g(x)).$$

The proof that $Bg$ is extensional with respect to $E'$ is the same as at the end of the previous theorem. □

## 9.6 Conclusion

Our vision about the future of a fuzzy logic is in the interconnection of various well-developed theories. In this contribution, we were focused on the lattice valued analysis that unifies approaches to the treatment of atomic elements, sets of atomic elements, functions between sets of atomic elements and their properties. We demonstrate that joint efforts of already established lattice valued theories lead to a calculus of fuzzy points and fuzzy functions. We showed that similarly to the classical case, any fuzzy function can be extended to the domain of fuzzy subsets and this extension is similar to the Extension Principle of L. Zadeh. We clarified the relationship between a fuzzy function and its ordinary core function. We developed the notion of a continuous fuzzy function on the basis of a lattice valued topology and proved continuity of a surjective fuzzy function.

**Acknowledgments** The support of the two projects is kindly announced: ESF project 2013/0024/1DP/1.1.1.2.0/13/APIA/VIAA/045, European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070).

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# Chapter 10
# Applying Fuzzy Mathematics to Empirical Work in Political Science

**John N. Mordeson, Terry D. Clark and Mark J. Wierman**

**Abstract** This paper discusses the effort that scholars have been engaged in to develop the necessary theoretical basis for political scientists to apply fuzzy logic to empirical analysis in social choice. This paper discusses the many successes scholars have had in this endeavor. It also provides direction for future work in answering questions that have yet to be resolved adequately.

## 10.1 Past Work

Since 2006, Terry D. Clark (political science), John N. Mordeson (mathematics) and Mark J. Wierman (computer science) have been engaged in an effort to develop the necessary theoretical basis for political scientists to apply fuzzy logic to empirical analyses in social choice. We have been supported in that task by a goodly number of undergraduates and graduate students. Despite the fact that political scientists have made increasing resort to mathematics in the last two decades, relatively few articles making use of fuzzy approaches have appeared in the discipline's journals. Claudio Cioffi-Revilla [16] first argued that fuzzy set logic might offer some utility to research in international relations. In response, Taber [78] and Seitz [75] employed fuzzy expert systems to simulate decision-making in international relations and Koenig-Archibugi [43], Arfi [2], and Sanjian [68–73] used fuzzy mathematics to test hypotheses concerning decision-making in international relations.

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_10

Charles C. Ragin subsequently advanced a method for applying fuzzy mathematics in other subdisciplines of political science. In [65], he reformulated his earlier work [66] in which he argued that Boolean logic could be applied to test for multiple paths of causality across a modest number of nation-states. In his reformulation, Ragin abandons crisp sets required by Boolean logic (yes-no membership in a set) in favor of fuzzy set membership. He then demonstrates how fuzzy logic can be used to argue for necessary or sufficient conditions as well as multiple paths of causality between a set of conditions and an outcome. Few political scientists have taken Ragin up on the challenge to employ fuzzy mathematics. An exception is Penning [63]. The reticence of political scientists to employ Ragin's fuzzy approach owes largely to the *property ranking* problem, a problem related to the assignment of membership values, see [76, 79–82]. While Cioffi-Revilla [16] notes in his seminal article that statistical analyses constitute an approach frequently marked by measurement problems, data validity issues, and random error, the argument persists among political scientists that the primary methods for establishing correlations in fuzzy data (measures of subsethood) are particularly sensitive to whether the dimensions on which the data are coded are comparable [65, 66]. To correct for this, Ragin [65] suggests that researchers "adjust" measures to see if the test of necessity or sufficient is reached by doing so. While he cautions scholars to carefully consider these adjustments in light of theory, the fact that theory remains largely unspecified significantly reduces its ability to serve as a guide to such adjustments. In effect, Ragin's method is largely an inductive data mining approach that permits models to remain relatively poorly specified. Hence, scholars risk arriving at conclusions that are little more than consequences of how the data are measured.

Theoretical efforts attempted to deal with this problem are addressed by the authors in [53] with a follow-on empirical piece in [51]. These efforts were informed by the proposition that what are needed are fuzzy approaches and mathematical procedures that provide an estimate of the correlations among hypothesized variables derived from the models instead of measuring the degree to which one is a subsethood of another. It seemed that one promising avenue was offered by using the transitive closure [42], which permits the analyst to estimate the degree to which variables "belong" in the same set. The procedure works with the max-min product of matrices containing the values for dependent and independent variables. The resulting matrix provides an indirect estimate of the degree to which causal variables are inter-correlated as well as the degree to which each is correlated with the dependent variable. The latter permits re-analysis of correlations based on aggregating highly inter-correlated variables. The estimates are not dependent on measuring subsethood, an approach highly sensitive to the property ranking of data. Another approach considered was fuzzy clustering [9, 40], which categorizes variables on the basis of their degree of similarity. It too focuses on estimating the degree of correlation between variables.

Ultimately, however, much more fertile ground for fuzzy approaches was found in *social choice*, a highly formal and dynamic research agenda at the boundary of political science and economics that has drawn substantial scholarly attention for several decades. In spite of substantial criticism from Pena and Piggins [64] social choice theory has informed a goodly number of formal models. These models typically assume that political actors are rational and that they possess perfect information (they know their own political references as well as that of others). Thus, political actors' preference orderings possess the mathematical property of *comparability*: given any set of choices, she can state definitely whether she prefers one to the other or she is indifferent between them. Crisp approaches, however, effectively minimize the latter possibility or result in its being ignored. Thus, an alternative in relation to another is either strictly preferred or not preferred by a political actor. While this reduction in ambiguity makes it easier to apply Euclidean distance in spatial models to argue that all points within a circle (or ellipse) whose center is at the ideal point are preferred to those outside of the circle (only the set of options lying precisely on a circle or ellipse about the ideal point being equally preferred to one another), the conclusions derived from the models are significantly effected. The most important consequence is that in multi-dimensional models, there is no maximal set (a set of alternatives that is strictly preferred to all others). In formal terms, a maximal set is defined [4] as follows:

$$M(R, X) = \{x \in X \mid xRy \text{ for all } y \in X\}, \tag{10.1}$$

where $X$ is a set of alternatives and $R$ is a binary relation on $X$.

It seemed that a fuzzy approach might deal with this problem, among others, in social choice. The contours of a fuzzy approach to modeling thick indifference were laid out by the authors in [17]. The approach permits the calculation of a maximal set with relative ease under assumptions of non-separability. The approach eschews the conventional approach to fuzzy spatial modeling, which begins with fuzzy preference relations, see [9, 10, 12, 39, 40, 58, 60, 61] and follows instead the lead of Nurmi [59] by beginning with individual preferences. This raises a substantial number of issues related to how to derive fuzzy preference relations from fuzzy individual preferences, but the fact that social science data typically measures individual preferences rather than preference relations commends the approach. Moreover, in a subsequent published work it is demonstrated that one can determine the existence of a maximal set with relative ease no matter how irregular the geometric shape or number of dimensions individual preferences might take [52].

Since the intent of our efforts is to produce approaches that are usable for empirical analyses in political science, we demonstrated the empirical utility of this approach in [13, 18]. Despite the demonstrated utility of our fuzzy spatial models, it nonetheless remains that it merely reduces the likelihood of an empty majority maximal set. In what is our most important paper in this series, we demonstrate [50] that in all but a limited number of cases, the maximal set is empty in an $n$-dimensional spatial model if and only the Pareto set contains a union of cycles. Thus, while a fuzzy approach is far more likely to result in a maximal set than is a crisp approach, the problem persists. This is a decidedly negative result if one hopes to predict political ulotomies on the basis of the existence of a maximal set. That is, we expect that social choice is constrained by collective social preferences. The argument is vacuous in the absence of a maximal set.

Predictions concerning voting outcomes in crisp spatial models rely heavily on the existence of a core, in the absence of which political players choosing among a set of alternatives by majority rule will not be able to arrive at a stable choice. No matter which option they might initially choose, most voting rules will permit another option to defeat the previously chosen one. Such problems particularly plague majority rule spatial models at dimensionality greater than one. Mordeson, Bhutani and Clark [47] shown that a fuzzy core is more likely in two or more dimensions as the number of players increases. However, many of the results depend on fuzzy preferences which are bounded away from 0.

Let $X$ be a set and $\mathcal{P}^*(X)$ the set of all nonempty subsets of $X$. A choice function $C$ of $\mathcal{P}^*(X)$ into itself such that $C(S) \subseteq S$ for all $S \in \mathcal{P}^*(X)$. A choice function $C$ is called rationalizable if there exists a binary relation $R$ on $X$ such that for all $S \in \mathcal{P}^*(X)$,

$$C(S) = M(R, S); \tag{10.2}$$

in which case $C$ is said to be rationalized by $R$. If $C$ is rationalizable by $R$, then the choices determined by $C$ are consistent with the maximization relative to $R$. A set of consistency conditions for choice functions is studied in the crisp case. These conditions take the form of restrictions on how a choice function behaves as the subset of feasible alternatives is varied. Since human preferences are fuzzy in nature, Orlovsky [61] first considered them as fuzzy relations for drawing conclusions in decision making problems. Since then many researchers have studied them from different points of view. It is shown by Mordeson et al. [47] that if strict preferences are partial, then many of the crisp results carry over to the fuzzy case. In Barrett et al. [8], several rules for generating exact choices from fuzzy weak preference relations are introduced and the extent to which choice sets are generated by these rules satisfy weak rationality conditions is studied It is shown that max-min transitivity is crucial to obtaining a positive result. In Banerjee [6], conditions under which a fuzzy choice function is rationalizable by a fuzzy revealed preference relation satisfying certain regularity conditions is investigated. In this paper attention is confined to the case where the domain of the choice function consists of all crisp finite subsets of the universal set of alternatives. Two sets of weak and strong axioms of fuzzy revealed preference (WAFRP) and (SAFRP) are stated. In both cases it is shown that WAFRP is not equivalent to SAFRP and SAFRP does not characterize rationality. In the case where every set of available alternatives has at least one element which is unambiguously chosen, WAFRP is equivalent to SAFRP, but SAFRP still does not characterize rationality. A fuzzy congruence condition, stronger than SAFRP is proposed and is shown to be necessary and sufficient for rationality. Desai and Chaudhari [14], introduce weak fuzzy $T$-congruence axiom in order examine the full rationality of a fuzzy choice function. They characterize full rationality of fuzzy choice functions in terms of this axiom and the Chernoff axiom. They prove that $G$-rational choice functions with transitive rationalization satisfying the Chernoff axiom characterizes their full rationality. Desai [21] examines rationality of fuzzy choice functions with rationalization reflexive, complete, and quasi-transitive. Fuzzy path independence and the fuzzy Condorcet property are used. Chaudhari and Desai introduce two axioms called the fuzzy direct revelation axiom (FDRA) and fuzzy transitive-closure coherence axiom (FTCCA). They provide interrelations between full rationality, $G$-rationality, fuzzy congruence axiom, weak fuzzy congruence axiom, $F\alpha$, $F\beta$, FDRA and FTCCA. In Wu et al. [83], the authors consider rationality conditions of fuzzy choice functions, including Weak (Strong) Fuzzy Congruence Axiom, Weak (Strong) Axiom of Fuzzy revealed Preference, fuzzy versions of the crisp conditions $\alpha$, $\beta$, $\gamma$, $\delta$ and so on, in the framework of the Banerjee choice function. The relationships between these conditions under the assumption that every involved choice set is normal and verify some equivalence results for arbitrary $t$-norms. A fuzzy version of Arrow-Sen theorem is presented. Various types of transitivity play important roles in Arrowian type results. In Dasgupta and Deb [20], many types of transitivity are compared. These types include probabilistic-sum transitivity, min-max transitivity, weighted mean transitivity, max-min transitivity, weak max-min transitivity, max-product transitivity, max-$\Lambda$ transitivity, and IPreference sensitive transitivity. The notion of choice and transitivity have many different possible interpretations in the theory of choice with fuzzy preferences, very often, depend significantly on the definitions of choice set and of transitivity. In this paper, the authors analyze these two concepts to determine their appropriateness. In Georgescu [33], the degree of dominance of an alternative with respect to an available fuzzy set of alternatives is introduced. Interpreting an available fuzzy set of alternatives as a criterion in decision making the degree of dominance establishes a hierarchy of alternatives with respect to this criterion. With the degree of dominance new congruence axioms for fuzzy choice functions are formulated. Georgescu [35] has written an interesting book devoted entirely to fuzzy choice functions. The book has the goals (taken from the author's introduction):

1. to develop the main themes of revealed preference theory (rationality, revealed preference, congruence, consistency) for a large class of fuzzy choice functions;
2. to explore new topics (degree of dominance, similarity, indicators of rationality) specific to a fuzzy approach to choice functions;
3. to show the manner in which some problems of multicritical decision making problems can find natural solutions in fuzzy revealed preference theory.

The research on Arrow's Impossibility Theorem [48] expands upon the existence of a maximal set. In what many consider to be the single most important theorem in social choice, Arrow [3] identifies a conundrum associated with the aggregation of individual weak fuzzy preference relations: there is no aggregation rule that results in a collective social preference order that meets several well-known rationality and reasonableness conditions while simultaneously guaranteeing against dictatorship. The rationality conditions taken together (completeness and minimally acyclicity) assure the existence of a maximal set. The work by Mordeson and Clark [48, 50] seeks to identify fuzzy rationality and reasonableness conditions that might avoid this result.

In [55], an attempt to apply a fuzzy approach to Black's Median Voter Theorem [11] was made. It was found that a maximal set exists when strong restrictions are placed on the preference profile of a collective set of players. As is the case with Arrow's Impossibility Theorem, our work demonstrates that while a fuzzy approach reduces the problem of a lack of a maximal set, it does not guarantee against it [61]. Black's Median Voter Theorem is among the more useful mathematical tools available to political scientists for predicting choices of political actors based on their preferences over a finite set of alternatives within an institutional or constitutional setting. If the alternatives can be placed on a single-dimensional continuum such that the preferences of all players descend monotonically from their ideal point, then the outcome will be the alternative at the median position. In [19] it is demonstrated that the Median Voter Theorem holds for fuzzy preferences when strict preferences are of the type now sometimes called partial. The work in [37] extends the analysis of voting and simple rules in the fuzzy framework. They relax previous assumptions about strict preferences and by illustrating that Black's Median Voter Theorem does not hold under all conceptualizations of the fuzzy maximal set. Strict preferences here are regular.

Faced with these results, the Gibbard-Satterthwaite Theorem (G-S) was considered in [36, 74]. G-S states that a social choice function over three or more alternatives that does not incentivize individuals to misrepresent their sincere preferences must be dictatorial. It follows that voters in collective choice institutions will manipulate the voting procedure to obtain a more preferred social outcome by reporting insincere preferences. One major effort by social choice scholars to avoid this conclusion involves an application of Black's Median Voter Theorem, which restricts the domain of individual preferences to single-peaked profiles, under which there exists a strict ordering of all possible alternatives, individuals possess a single ideal alternative, and strict preference decreases monotonically in both directions from an ideal point. Under these assumptions, the augmented median rule emerges as a non-manipulable and non-dictatorial choice function [7, 15, 59, 77]. However, Penn et al. [62] extend the G-S results to a general case by demonstrating that even though individuals possess single-peaked preferences, there exist opportunities to manipulate the social choice when individuals report insincere preferences that violate the natural ordering of the alternatives. The crux of their argument rests on the empirical observation that no real-world voting rule actually has a ballot restriction that forces individuals to submit single-peaked preferences to the social choice function; hence, individuals will submit insincere, non-single-peaked preferences when it manipulates the social choice. Under these assumptions, a strategy-proof rule must be dictatorial. See also Moulin [56].

In many social decision-making contexts, a manipulator has incentives to change the social choice in his favor by strategically misrepresenting his preference. Gibbard [36] and Satterthwaite [74] have shown that any non-dictatorial voting choice procedure is vulnerable to strategic manipulation. Abdelaziz et al. [1] extend their result to the case of fuzzy weak preference relations. Three generalizations of the Gibbard-Satterthwaite theorem to the fuzzy context are provided. In Perota-Pena and Piggins [64], the structure of fuzzy aggregation rules which, for every permissible profile of fuzzy individual preferences, specify a fuzzy social preference is examined. It is shown that all fuzzy aggregation rules which are strategy-proof and satisfy a minimal range condition are dictatorial. That is, there is an individual whose fuzzy preferences determine the entire fuzzy social ranking at every profile in the domain of the aggregation rule. This result is proven by showing that all fuzzy aggregation rules which are strategy-proof and satisfy the minimal range condition must also satisfy counterparts of independence of irrelevant alternatives and the Pareto condition.

## 10.2 Where Do We Go from Here?

We lay out in this section some of the questions that have yet to be resolved adequately.

Abdelaziz et al. [1] introduce three new definitions of fuzzy manipulability were proposed. Under these definitions, it was proven that Gibbard and Satterthwaite's negative result remains true. It is stated that further research should be carried out to examine the validity of the G-S's theorem under other domains of fuzzy weak choice functions and to determine the minimum requirements for fuzzy strategy-proofness.

Spatial models dealing with infinite sets of alternatives provide an important research area for the future for those interested in applying fuzzy techniques. This area has received much less attention than that of spatial models dealing with finite sets of alternatives. The most interesting situation is that when the set of feasible alternatives is a subset of Euclidean space $\mathbb{R}^k$ for $k \geq 2$. For $k = 1$, the assumption of single peaked preferences generated nonempty cores for voting rules. However this is not the case for $k \geq 2$. Some results are known concerning core existence [49, 55]. However much work is needed on characterizing the core. One reason for the slow progress may be that an easily usable fuzzy calculus is not available. A possible approach might be use the techniques of Dubois and Prade [24–26], where ordinary calculus is used, but functions are evaluated at fuzzy points. Necessary restrictions on sets of gradient vectors at a point for the point to be in the core include the Plott conditions. Examining the Plott conditions provide a possible direction for those interested in applying fuzzy techniques.

A closely associated set of issues concerning instability and chaos (McKelvey [45, 46]) have hardly been touched as well by the fuzzy approach. In the crisp case, cores may not exist for relatively large sets of preferences. In the crisp case, questions such as how likely is it that the core of a simple rule is nonempty? For issue spaces of sufficiently large dimension, the likelihood of the core being nonempty is negligible. Hence for relatively large dimensions of the issue space, one cannot expect there to exist any one alternative which is at least as good as all others. Another question is, how badly behaved is the preference relation when the core fails to exist? It has been shown that social cycles fill the space in the crisp case [45, 46]. It would be an important research problem to determine to what extent fuzzy techniques could salvage the situation. It is also known that noncollegial simple rules can only be rational, in that core outcomes exist in high enough dimensions, only in very special circumstances. When the core outcomes fail to exist, collective preferences may appear almost anywhere in space, a well known chaos theorem. There are many ideas for the fuzzy researcher to consider.

The existence of a majority rule maximal set was determined in arbitrary $n$-dimensional spatial models by Mordeson and Clark [50]. It was assumed that each player was a fuzzy subset of the first quadrant of the plane and took on values from $T = \{0, .25, .50, .75, 1\}$. Some partial results on the top cycle set were obtained. The complete characterization of the top cycle set would be of interest. It would also be of interest to replace $T$ by an arbitrary finite set with $0, 1 \in T$.

A good deal of the results concerning fuzzy choice functions [14, 21, 35, 47] have made use of a very special strict fuzzy preference relation associated with a fuzzy preference relation $\rho$. This strict fuzzy preference relation $\pi$ is defined as follows: $\forall x, y \in X, \pi(x, y) = \rho(x, y)$ if $\rho(x, y) > \rho(y, x) = 0$ and $\pi(x, y) = 0$ otherwise. This strict fuzzy preference relation is associated with the drastic conorm in a factorization of $\rho$ involving $\pi$ as a component. A further research project would be to determine the extent to which the results involving this $\pi$ could be extended using other types of fuzzy strict preference relations. In the dissertation of Georgescu [34], eight open problems are presented and discussed.

A fuzzy version of Arrow's Theorem is proved in [48]. Under the definition in this paper and using a partial strict fuzzy preference relations Arrow's theorem remains intact even if levels of intensity of the players and levels of membership in the set of alternatives are considered. Fuzzy versions of Arrow's Theorem involving representation rules, oligarchies, and veto players is also considered. Richardson [67] has shown that with an alternative factorization of fuzzy weak preferences into symmetric and antisymmetric components, a fuzzy analogue of Arrow's Impossibility Theorem can be proved when the transitivity requirements on individual and social preferences are very weak. The notion of strong connectedness is needed here. However if strong connectedness is not assumed, another factorization of fuzzy weak preferences yields nondictatorial fuzzy aggregation rules satisfying weak transitivity. Additional results on factorization of fuzzy preference relations can be found in Llamazares [44] and Herr and Mordeson [38].

In [37, 55], an analysis of voting and simple rules in the fuzzy framework was undertaken. It was shown that Black's Median Voter Theorem does not hold under all conceptualizations of the fuzzy maximal set. Given that simple and voting rules are often used in spatial modes and given the more generalized fuzzy definitions presented here allow for strict preferences, further research is needed to investigate whether these models are as poorly behaved in two or more dimensional space, i.e. is the fuzzy maximal set under these rules generically empty in the spatial model?

In Dutta [28], it is shown that the impossibility result can be avoided by using fuzzy preferences having particularly weak versions of transitivity. Dutta states that further work needs to be done on the relationship between rationalizable choice functions and fuzzy preferences. Banerjee [5] shows that the choice of definitions for indifference and strict preference, given a fuzzy weak preference can also have Arrowian implications. Banerjee demonstrates that with fuzzy social preference relations, it is possible to distinguish between different degrees of power of the dictator. The power increases with the strength of the transitivity requirement. The conditions used to derive Dutta's version of strict preference imply a restriction on how fuzzy the original weak preference can be, namely that the fuzzy weak preference relation must be strongly connected. Richardson shows that without this restriction Dutta's conditions imply a third version of strict preference for which Dutta's possibility result under weak transitivity still holds. If strong connectedness is required for it to be valid, Richardson shows Dutta's dictatorship theorem can be strengthened to cover any version of transitivity for preferences no matter how weak and further that this dictatorship result holds for regular formulation of strict preference, including the one originally used by Dutta.

Duddy et al. [27] consider max-$*$ transitivity in their study of Arrow's impossibility Theorem, where $*$ is a $t$-norm [42]. They restrict their attention to fuzzy aggregation rules that satisfy counterparts on unanimity and independence of irrelevant alternatives. They characterize the set of triangular norms that permit preference aggregation to be nondictatorial. They show that this set contains exactly those norms that contain a zero divisor. In Fono and Andjiga citeFono:2005:FSP, some classical factorizations of a fuzzy relation into a symmetric component (Indifference) and an asymmetric and regular component (regular fuzzy strict preference) are generalized. Two properties of a fuzzy strict preference of a max-$*$ transitive relation, which are used to obtain new versions of Gibbard's oligarchy theorem and Arrow's impossibility theorem. In Fono et al. [31], appropriate properties of fuzzy preferences and fuzzy aggregation rules are introduced. They are used to provide fuzzy counterparts of Malawski and Zhou and Wilson's impossibility results concerning the aggregation of individual preferences which do not assume the Pareto principle. By weakening conditions on fuzzy social preferences a possibility result is obtained. In [54], several new definitions of independence of irrelevant alternatives were introduced and shown that they can be profitably used in the examination of Arrow's theorem. Some known nondictatorship results are generalized. Future research could be to determine how the various definitions of independence of irrelevent alternatives can be used to further study fuzzy Arrow's theorem.

Dutta states that social fuzzy preferences must ultimately be used to define exact social choice. Dutta also states that further work needs to be done on the relationship between rationalizable choice functions and fuzzy preferences and that a problem to consider is the derivation of restrictions imposed on choice functions by rationalizability in terms of general max-star transitive binary relations. Dutta further states that if these restrictions are mild, then it might be possible to define reasonably acceptable aggregation rules which map $n$-tuples of fuzzy individual preference orderings into exact social choices.

Another paper that is pertinent to the avoidance of the impossibility result is Garcia-Lapresta and Llamazares [32]. Aggregation rules which assign an aggregate fuzzy binary relation to each profile of reciprocal fuzzy binary relations are considered. The aggregate fuzzy binary relation does not necessarily have to be reciprocal, but it is desirable for it to be so. When this relation is reciprocal, collective decisions can be taken with different levels of qualification, depending upon the ordinary preference relation that is considered based on the aggregate fuzzy relation. Here neutral aggregation rules are characterized through functions from powers of [0, 1] in [0, 1]. In addition, a class of neutral aggregation rules based on arithmetic means associated with functions of [0, 1]. These rules are characterized through two properties:decomposability and anonymity. It is stated that when the set of alternatives has a group structure, then every group Chichilnisky $n$-rule (group homomorphism, anonymous and unanimous aggregation rule) is a convex mean. In order to bring the power of abstract algebra to bear, it would be of interest to examine Arrowian-type results when the set of alternatives has a group structure.

Finally, Fono et al. [29] establish by means of a large class of continuous $t$-representable intuitionistic fuzzy $t$-conorms, a factorization of an intuitionistic fuzzy relation (IFR) into a unique indifference component and a family of regular strict components. This result generalizes a previous factorization obtained in Dimitrov [22, 23] with the (max, min) intuitionistic fuzzy $t$-conorm. In this paper, a characterization of the $\mathcal{T}$-transitivity of an IFR for a continuous $t$-representable intuitionistic fuzzy $t$-norm. This allows for the determination of necessary and sufficient conditions on a $\mathcal{T}$-transitive IFR $\rho$ under which a strict component of $\rho$ satisfies pos-transitivity and negative transitivity. In Nana and Fono [57], the characterizations in Fono et al. [29] are used to obtain some intuitionistic fuzzy versions of Arrow's impossibility theorem. By weakening requirement to social preferences, they provide an example of a non-dictatorial intuitionistic fuzzy aggregation rule. They also establish an intuitionistic fuzzy version of Gibbard's oligarchy theorem. Not much has been done along these lines, i.e., the study of spatial models for fuzzy intuitionistic fuzzy preference relations. Consequently, these papers may be cornerstone papers for the development of intuitionistic fuzzy relations in spacial models. However, one must be wary of the following situation. Consider an intuitionistic fuzzy preference relation $\langle \rho_\mu, \rho_\nu \rangle$. Many the results concerning $\langle \rho_\mu, \rho_\nu \rangle$ involve two proofs, one for $\rho_\mu$ and one for $\rho_\nu$. The proofs are often dual in nature. That is, having one proof, the other proof can be mimicked from the other. It is possible to formalize this fact by use of an involutive complement of [0, 1].

## 10.3 A New Direction?

Our own efforts to extend a fuzzy approach to G-S have not reduced the problem significantly enough for political scientists to place sufficient confidence in social preference orders as a means to predict political outcomes.

Social choice theorists have turned to institutional design to provide the means to predict outcomes. Scholars have considered both formal voting rules as well as informal "rules of the game." The latter efforts have typically focused on game theory to capture the essence of the design of such institutions. We believe that social network analysis (SNA) offers a better vehicle for doing so. A newly emerging analytical approach based on graph theory, SNA permits scholars to consider an actor's choices within the full set of relations defining its environmental context.

## Appendix

In order to give a flavor for the concepts discussed above and given the importance of Arrow's Theorem, we offer the following formalizations for consideration in future research.

Let $N = \{1, 2, \ldots, n\}$ with $n \geq 2$. Let $X$ denote a set with three or more elements. The support of a fuzzy subset $\mu$ of $X$ is defined to be the set, $\text{Supp}(\mu) = \{x \in X \mid \mu(x) > 0\}$.

**Definition 1** Let $\rho$ be a fuzzy binary relation on $X$ and $*$ a $t$-norm. Then:

1. $\rho$ is called *reflexive* if $\forall x \in X$, $\rho(x, x) = 1$;
2. $\rho$ is called *complete* if $\forall x, y \in X$, $\rho(x, y) > 0$ or $\rho(y, x) > 0$;
3. $\rho$ is called *max-$*$ transitive* or *simply transitive* if $\forall x, y, x \in X$, $\rho(x, y) * \rho(y, z) \leq \rho(x, z)$;
4. $\rho$ is called *partially transitive* if $\forall x, y, z \in X$, $\rho(x, y) > 0$, $\rho(y, z) > 0$ implies $\rho(x, z) > 0$.

If $\rho$ satisfies (1), (2), and (4), then $\rho$ is called a *fuzzy weak order* on $X$.

**Definition 2** Let $\rho$ be a fuzzy binary relation on $X$. Define the fuzzy subsets $\pi$, $\iota$ of $X \times X$ as follows: $\forall x, y \in X$,

$$\rho(x, y) = \begin{cases} \rho(x, y) & \text{if } \rho(x, y) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10.3}$$

and

$$\iota(x, y) = \rho(x, y) \wedge \rho(y, x). \tag{10.4}$$

Let $\mathcal{FR}$ denote the set of all fuzzy weak orders on $X$. A *fuzzy preference profile* on $X$ is a $n$-tuple of fuzzy weak orders $\rho = (\rho_1, \ldots, \rho_n)$. Let $\mathcal{FR}^n$ denote the set of all fuzzy preference profiles. Let $\mathcal{FB}$ denote the set of all reflexive and complete fuzzy binary relations on $X$.

**Definition 3** A function $f : \mathcal{FR}^n \to \mathcal{FB}$ is called a *fuzzy preference aggregation rule*.

**Definition 4** Let $f$ be a fuzzy aggregation rule. Then

1. $f$ is said to be *non-dictatorial* if it is not the case that $\exists i \in N$ such that $\forall \rho \in \mathcal{FR}^n$, $\forall x, y \in X$, $\pi_i(x, y) > 0$ implies $\pi(x, y) > 0$;
2. $f$ is said to be *weakly Paretian* if $\forall \rho \in \mathcal{FR}^n$, $\forall x, y \in X$, $\forall i \in N$, $\pi_i(x, y) > 0$ implies $\pi(x, y) > 0$;
3. $f$ is said to be *independent of irrelevant alternatives* if $\forall \rho, \rho' \in \mathcal{FR}^n$, $\forall x, y \in X$,

$$\text{Supp}(\rho_i|_{\{x,y\}\times\{x,y\}}) = \text{Supp}(\rho_i'|_{\{x,y\}\times\{x,y\}}) \tag{10.5}$$

$\forall i \in N$ implies

$$\mathrm{Supp}(f(\rho)|_{\{x,y\}\times\{x,y\}}) = \mathrm{Supp}(f(\rho')|_{\{x,y\}\times\{x,y\}}) . \tag{10.6}$$

**Theorem 1** (A Fuzzy Arrow's Theorem) *Let $f$ be a fuzzy aggregation rule that is partially transitive, weakly Paretian, and independent of irrelevant alternatives. Then $f$ is dictatorial.*

**Definition 5** Let $\rho$ be a fuzzy binary relation on $X$. Then

1. $\rho$ is called *quasi-transitive* if $\forall x, y, z \in X, \pi(x, y) \wedge \pi(y, z) \le \pi(x, z)$;
2. $\rho$ is called *partially quasi-transitive* if $\forall x, y, z \in X, \pi(x, y) \wedge \pi(y, z) > 0$ implies $\pi(x, z) > 0$;
3. $\rho$ is called *acyclic* if $\forall x_1, x_2, \ldots, x_k \in X, \pi(x_1, x_2) \wedge \cdots \wedge \pi(x_{k-1}, x_k) \le \rho(x_1, x_k)$;
4. $\rho$ is called *partially acyclic* if $\forall x_1, x_2, \ldots, x_k \in X, \pi(x_1, x_2) \wedge \cdots \wedge \pi(x_{k-1}, x_k) > 0$ implies $\rho(x_1, x_k) > 0$.

**Definition 6** Let $f$ be a fuzzy aggregation rule. Then

1. $f$ is called *quasi-transitive* if $\forall \rho \in \mathcal{FR}^n$, $f(\rho)$ is quasi-transitive;
2. $f$ is called *partially quasi-transitive* if $\forall \rho \in \mathcal{FR}^n$, $f(\rho)$ is partially quasi-transitive;
3. $f$ is called *acyclic* if $\forall \rho \in \mathcal{FR}^n$, $f(\rho)$ is acyclic;
4. $f$ is called *partially acyclic* if $\forall \rho \in \mathcal{FR}^n$, $f(\rho)$ is partially acyclic.

**Theorem 2** *Let $f$ be a fuzzy aggregation rule. If $f$ is partially quasi-transitive, weakly Paretian, and independent of irrelevant alternatives, then it is oligarchic.*

**Theorem 3** *Suppose $|X| \ge n$. Let $f$ be a fuzzy aggregation rule. If $f$ is partially acyclic and weakly Paretian, then $f$ is collegial.*

**Definition 7** Let $f$ be a fuzzy aggregation rule. The $f$ is called neutral if $\forall \rho, \rho' \in \mathcal{FR}^n, \forall x, y, u, v \in X, P(x, y; \rho) = P(u, v; \rho')$ and $P(y, x; \rho) = P(v, u; \rho')$ imply $f(\rho)(x, y) > 0$ if and only $f(\rho')(u, v) > 0$.

**Theorem 4** *Suppose $|X| > n$. Let $f$ be a fuzzy aggregation rule. If $f$ is partially acyclic, weakly Paretian, and neutral, then there exists $i \in N$ such that $i$ has a veto.*

**Independence (I):** For all $(\rho_1, \ldots, \rho_n), (\rho'_1, \ldots, \rho'_n) \in H^n$ and for all $x, y \in X, \rho_j(x, y) = \rho'_j(x, y)$ for all $j \in N$ implies $\rho(x, y) = \rho'(x, y)$.

**Unanimity (U):** $(\rho_1, \ldots, \rho_n) \in H^n$ and for all $x, y \in X$ and for all $t \in [0, 1], \rho_j(x, y) = t$ for all $j \in N$ implies $\rho(x, y) = t$.

Let $H$ denote the set of all fuzzy binary relations that are reflexive, max-$*$ transitive, and satisfy the condition that for all $x, y \in X, \rho(x, y) = 0$ implies $\rho(y, x) = 1$, where $*$ is a $t$-norm.

**Theorem 5** *Let $f : H_n \to H$ be a fuzzy aggregation rule. If $*$ has no zero divisor, then any fuzzy aggregation rule satisfying $I$ and $U$ is dictatorial. Moreover, if $*$ has a zero divisor, then there exists a non-dictatorial fuzzy aggregation rule that satisfies $I$ and $U$.*

Let $\mathcal{FR}^*$ denote the set of all binary relations on $X$ with nonempty supports and $\mathcal{FP}^*$ the set of all fuzzy subsets of $X$ with nonempty supports.

**Definition 8** Define $M : \mathcal{FR}^* \times \mathcal{FP}^* \to \mathcal{FP}^*$ by $\forall(\rho, \mu) \in \mathcal{FR}^* \times \mathcal{FP}^*$, $M(\rho, \mu)(x) = \vee\{t \in [0, 1] \mid \mu(x) \geq t$ and $\rho(x, y) \geq t \forall y \in \text{Supp}(\mu)\}$ for all $x \in X$. Then $M(\rho, \mu)$ is called the *fuzzy maximal subset* associated with $(\rho, \mu)$.

**Theorem 6** *Let $\rho \in \mathcal{FR}^*$ be reflexive and complete. Then Supp$(M(\rho, \mu)) \neq \emptyset$ if and only if $\rho$ is partially acyclic.*

Let $C : \mathcal{FP}^* \to \mathcal{FP}^*$ be such that $C(\mu) \subseteq \mu \forall \mu \in \mathcal{FP}^*$. Then $C$ is called a *fuzzy choice function* on $X$.

**Definition 9** Let $C$ be a fuzzy choice function on $X$ Let $\rho \in \mathcal{FR}^*$. Then $C$ is called *rationalizable* with respect to $\rho$ if $C(\mu) = M(\rho, \mu)$ for all $\mu \in \mathcal{FP}^*$.

**Definition 10** Let $*$ be a continuous $t$-norm.

1. Define $M : \mathcal{FR}^* \times \mathcal{FP}^* \to \mathcal{FP}^*$ by $\forall(\rho, \mu) \in \mathcal{FR}^* \times \mathcal{FP}^*$, $M(\rho, \mu)(x) = \mu(x) * \wedge\{\mu(y) * \rho(y, x) \to \rho(x, u) | y \in X\}$ for all $x \in X$.
2. Define $G : \mathcal{FR}^* \times \mathcal{FP}^* \to \mathcal{FP}^*$ by $\forall(\rho, \mu) \in \mathcal{FR}^* \times \mathcal{FP}^*$, $G(\rho, \mu)(x) = \mu(x) * \wedge\{\mu(y) \to \rho(x, y) | y \in X\}$ for all $x \in X$.

Let $\mathcal{B}$ be a nonempty family of non zero fuzzy subsets of $X$ which contain characteristic functions of all singleton and two-element subsets of $X$. In the following let $C : \mathcal{B} \to \mathcal{FR}^*$.

**Definition 11** Suppose $C : \mathcal{B} \to \mathcal{FR}^*$. Define the fuzzy revealed preference relations $\rho$ and $\overline{\rho}$ on $X$ as follows: $\forall x, y \in X$,

$$\begin{aligned} \rho^*(x, y) &= \vee\{\{C(\mu)(x) \wedge \mu(y) | \mu \in \mathcal{B}\}, \\ \overline{\rho}(x, y) &= C([x, y])(x), \end{aligned} \tag{10.7}$$

where $[x, y]$ denotes the characteristic function of $\{x, y\}$ in $X$.

**Definition 12** A fuzzy choice function $C$ is said to be $G$-rational if there exists a fuzzy preference relation $\rho$ on $X$ such that $C(\mu) = G(\mu, \rho)$ for all $\mu \in \mathcal{B}$. The fuzzy preference relation $\rho$ for which the fuzzy choice function is $G$-rational is called a rationalization for $C$. The fuzzy choice function $C$ is full rational if $C$ is $G$-rational with reflexive, complete and transitive rationalization.

**Theorem 7** *If a choice function C defined on base domain is $G$-rational with rationalization $\rho$, then $\rho^* \subseteq \rho$.*

**Fuzzy T-congruence axiom:** For all $x, y, z \in X$ and for all $\mu \in \mathcal{B}$, $\rho^*(x, y) \wedge \rho^*(y, z) C(\mu)(z) \wedge \mu(x) \leq C(\mu)(x)$.

**Theorem 8** *A fuzzy choice function C defined on base domain is full rational if and only if C satisfies the fuzzy T-congruence* relation relation.

**Fuzzy Chernoff axiom:** For all $\mu_1, \mu_2 \in \mathcal{B}$ and for all $x \in X$, $I(\mu_1, \mu_2) \wedge \mu_1(x) \wedge C(\mu_2)(x) \leq I(\mu_1 \cap C(\mu_2), C(\mu_1))$.

**Theorem 9** *If a fuzzy choice function defined on base domain satisfies the Chernoff axiom, then $\rho^* = \overline{\rho}$.*

**Weak Fuzzy T-congruence axiom:** For all $x, y, z \in X$ and for all $\mu \in \mathcal{B}$,

$$\overline{\rho}(x, y) \wedge \overline{\rho}(y, z) \wedge C(\mu)(z) \wedge \mu(x) \leq C(\mu)(x). \tag{10.8}$$

**Theorem 10** *A fuzzy choice function $C$ defined on the base domain $\mathcal{B}$ is full rational if and only if it satisfies the weak fuzzy T-congruence relation and the Chernoff axiom.*

**Fuzzy Arrow Axiom (FAA):** For all $\mu_1, \mu_2 \in \mathcal{B}$ and for all $x \in X$,

$$I(\mu_1, \mu_2) \wedge \mu_1(x) \wedge C(\mu_2)(x) \leq E(\mu_1 \cap C(\mu_2), C(\mu_1)). \tag{10.9}$$

It can be shown that full rationality does not imply FAA on base domain.

**Definition 13** Let $\overline{A} = \{(x, y)|x, y \in X, x \neq y\}$. A fuzzy aggregation rule is said to satisfy the minimal range condition if and only if for all $(x, y) \in \overline{A}$, there exists $(\rho_1, \ldots, \rho_n), (\rho'_1, \ldots, \rho'_n) \in H^n$ such that $\rho(x, y) = 0$ and $\rho'(x, y) = 1$.

**Theorem 11** *Any strategy proof fuzzy aggregation rule which satisfies the minimal range condition is dictatorial.*

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# Chapter 11
# Crisis' Origin's Causes. Contributions from the Fuzzy Logic in the Sustainability on the Socio-Economic Systems

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**Abstract** The deep economic crisis where we are immersed, together with the recuperation perspectives, intensifies the efforts in the socioeconomic field to find new solutions to prevent the current involution process. One of the key factors to achieve this goal is based in rationalization and the outlay containment. But obviously, this purpose is not enough by itself, and will not make the real economy improve (the one which is perceived by all citizens, which becomes permeable in all social layers, and the one which allows a permanent welfare), and will never stay up longer than a short term. It is important to accompany these scarifying measures imposed to the citizenship with adequate politics, depending on the selected objectives. Just like that, we need to prioritize the expenses and inversions, which could generate wide multiplying effects in the economy.

## 11.1 From the Current Uncertainty to Tomorrow's Hope

As European citizens, we find ourselves immersed since some years in a crisis which is causing important disorders, affecting unequally every social layer, but with no exception. The reactions launched by the governments are not only late but improvised, unconnected and short-termed, as a result of politics and inappropriate strategies. The culmination of studies for the global adoption of measures capable to face the imbalances, which have caused a more or less intense depression in the European socio-economic systems, have been missed.

Even though, conscious about this situation, we may discern the future with great uncertainty, we strongly believe the solution will come. As Raymond Barre, ex Primer Minister of France and convinced pro-European, said: "*Europe will built up herself … even if sometimes it may have to go backwards … to take a breath and continue going on.*"

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_11

But to achieve this, it's necessary for absolutely all of us to recognize that this so called crisis is not only and economical and financial crisis … Its effects are much wider and deeper because they include typical aspects from the social, human and cultural dimension of our nation.

We are convinced we are going to sprout from this depressive situation, and we are sprouting by being stronger and solidly reinforced, as long as the first causes of the crisis' outbreak are approached with seriousness and firmness. In order to achieve this, a detailed itinerary through those rough paths of the thinking will be necessary; rough paths where the decision making is originated, in order to achieve the knowledge about why have we made so many mistakes.

We are going to draw in a brief way a plan of this itinerary, to make it possible to explain the rounds of the thinking "*through the human mind with all of its shortcuts, mistakes, defective wiring and abundant traps*" [7]. This can help at finding acceptable solutions, taking advantage of our formidable system of evolutionary adaptation, which allows us to talk about a summary of rationality and emotionality in every single moment.

## 11.2 Theoretical Bases of the Mistakes

It's already been more than three decades that the uncertainty school has been proclaiming the deficiency of the schemes traditionally used in the social sciences, based only on the Cartesian rationality. It has proposed a new principle able to enshrine the incidence of the Subjective in human decisions too at the same time.

From another point of view but still in the same direction, Kahneman together with the inestimable collaboration of Amos Tversky, has structured our decision-making architecture in two systems. Let's remember it very briefly[1]: Kahneman considers our decisions are a consequence of how system 1 and system 2 run, as he refers to as.

- System 1 is the one which "*thinks fast*": it is unconscious, intuitive and easy to use, effortless. It is the one which adds "black" when we say "white and …" without even needing to switch on the neurones. It is the one which decides "*in the blink of an eye*". System 1 recognises the patterns in series and answers questions in a tenth of second … although not always correct.
- System 2 is, on the other hand, the one which "*thinks slowly*": it is rational, needs effort and energy. It is deliberative, slow and suspicious. It considers, evaluates, reasons in hard stages and finally takes a decision which will be able to justify in all its extensions. It really knows why it takes it.

In the social sciences' field, it has been accepted in general terms that the human decisions are only product of system 2, that reasonable and wise friend, when we actually depend on both systems. According to what Kahneman said, system 2 and 1

[1]This abstract has been extracted from Daniel Kahneman's speech for his joining in the Royal Academy of Economic and Financial Sciences of Spain. Madrid, 14th of June 2012.

weight he same at the decision-making, even when the decision is taken by the most rationalist scientist.

With all respect, we dare dissenting from this last affirmation, as we are convinced that different moments and circumstances lead to different decisions taken by the same person, propelled by a major or minor incidence of the emotional component. The percentage of each component's weight, then changes.

If we accept this licence, we find evidence in the fact that the original hypothesis of the social sciences and its orthodoxy, dominant until recently and defending we are basically rational beings and therefore predictable, is usually imprecise and false in many occasions.

Indeed, "*the homo economicus, rationality's base in the social sciences, doesn't exist in the real world. The school where we belong has firmly defined since already a pretty lot of years that the decider subject doesn't only act rationally like a robot would, and that's why social sciences will never be exact sciences either. Social sciences will never be able to predict with accuracy the behaviour of the agents who act in their own bosom. Finances, for example, which are full of an econometric and statistic equipment, will always be conditioned by the human choices, again and again, in their decisions. And therefore, finances will always be fuzzy and hardly predictable, as we, people, are*" [6].[2]

Neither we are rational nor predictable, because the subjectivity component which coats the human decisions also penetrates in the models we elaborate. The result is, we have always committed, commit and will always commit mistakes.

Failing is effectively human, but does the human fail always in the same way? We can't give an accurate answer relating to the mistake's *cause*, but in the other hand, the behaviour science seems to admit their *effects* happen to meet, for now. We tend to make the same errors again and again even though the circumstances and their appearance may change. That's why the effects of our economic crisis are also recurrent. If this is it, the moment where establishing a relation of cyclic mistakes can be possible, must come. The mistakes will be then repeated in the origins of every crisis.

From this relation on, we would discover which the mechanisms that cause the decisive mistakes are. Coming to the current situation, we observe the same behaviours are detected both in the origin and development of the crisis, and the Dutch tulips crisis aged of four hundred years. Same happens with the 1929's crisis or 1892's one. The mistakes done, considered as effects, are the same. The only significant variables are their magnitude and length.

In all cases it's about the consequences of a different behaviour to the strictly rational one, owned by a mix of intuition and reason; prejudice and sanity; feeling and rationality. Nevertheless, we wish in all we just pointed out one thing to stay clear: we don't pretend a rotund sentence of the Cartesian in what it owns as pure rationalization, but to incorporate in it what we could call intuition, ambition, emotionalism and all those other singularities which cause in the human being a non-total rational behaviour. Their decisions are, finally, a subtle dance between reason and feelings.

[2]Gathered by J. Gil-Aluja's answer at Daniel Kahneman's speech, see footnote 1.

## 11.3 The Change in the Basis of the Crisis' Study

The acceptance of this approach would allow us to collaborate in the job of giving an "*end to the old dichotomy between emotion and reason, between intuition and reasoning. We are built up from both of them; we are complex creatures, resulting from a biologic evolution which chose the survival path between all the possible ways, which not always meant the straightest one*" [6].[3] This is the ultimate reason why we support the crisis' study must go on from that premise, and not from the false supposition which states we are robots and we take predictable decisions.

We believe the moment of formulating one of the fundamental questions has come; how do rationality and feelings integrate themselves in a unique formal system, which forms, in a certain way, a cause of subjectivity? The answer to this question cannot be as obvious as we would wish. The scientists who investigate in the social sciences' field exert themselves at understanding better, explaining more suitably and deal with harshness the most complex social phenomena, which mix with reason and emotion.

It seems like the more we get closer to the goal, the bigger the feeling of a necessary epistemological change is. A change related to the research lines which have been followed by the social sciences intellectuals, practically from their origins, when they used to focus on those physicians who used to observe the universe. They hoped to find enough elements to discover the future scenarios in which it was believed would occur the social activity. In this way, social laws followed nature laws. And also in the same way physicians used to wonder about the meaning of reality and the existence of time, the researchers of the social phenomena wondered about the existence of the happening in their environment, and how the forces which caused them, worked.

Reality and time in physics, on one side, and social phenomena with deep changes, on the other side, used to shape the worries of the scientists in this two plots of the knowledge.

The question which then imposes is: are concepts also parallel in both study spheres? In the social sciences field the current phenomena usually associate between themselves. The past has stopped "being", and the future "is not yet". It seems that our reasoning moves in a way where tomorrow's uncertainty

A lot of irreversible phenomena exists in the social sciences field. One could understand that an asymmetry of the objects in time does exist, although not an asymmetry in time. In this sense, therefore, asymmetry is a characteristic of the objects, not time's.

However, this essential perception directly crashes against rationality, with what physicians assume the concept of time. For them, a temporal landscape exists, where there were found all the events of the past, present and future. The time doesn't move, the objects do move in time. The time doesn't pass by, it merely is. Time's flow is unreal, what is real, is time. The eternity is present in all its infinite dimension.

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[3]See footnote 2.

The mailing sustained by the last years of Michele Besso's and Albert Einstein's life appears to be revealing. Facing the persistent question from Besso: What is time? What is irreversibility? Albert answers "*irreversibility is an illusion*". Einstein wrote a letter to Besso's sister and son as a matter of his passing, which contained the following words: "*Michelle has overtaken me at leaving this strange world. It lacks of importance. For us, convinced physicians, the difference between past, present and future is only an illusion, as persistent as it might be*" [2].

Facing the possibility of abandoning the almost secular mechanism, which has informed the social sciences, we wonder if it is possible to find a doctrine body able to sustain the theoretical and technique elements, both sensitive at representing the new social realities, increasingly being more complex and uncertain.

Facing this challenge [6] we have turned back to the past until stopping in the middle XIXth century, when a new adventure begins with the publication of Darwin's fundamental work in 1859, *The Origin of Species*. Two elements are combined in it: fluctuations and irreversibility. Fluctuations in species of the nature, thanks to the environment's selection, lead to an irreversible biologic evolution. This is the way how an auto organisation of systems with an increasing complexity takes place from the association between fluctuations (which comprehends the idea of chance, as we would call it "uncertainty") and irreversibility.

However, how does this structure creation take place, in other words, this auto organisation? Given the entropy of a system, if it is disturbed in a way a certain state stands still close enough to the balance, the system responds establishing the initial situation: we talk about a stable system. But if a certain state is brought far enough from the balance, it enters an unstable situation directly related with perturbation. In this context, the determinism doesn't allow to predict which way will be chosen between the possible ones. In many cases, an interruption of the symmetry is produced. We can say, then, that even though symmetry can exist in the equations which formalize the process, it doesn't usually appear in the solutions. The complexity of these processes makes unfeasible its comprehension and explication through single determinist laws.

In the social sciences field, new approaches which could possibly provide an answer to this big challenge have been sought. We point out the so called theory of the fuzzy subsets, whose epicentre is placed in a quarrel which is more than two thousand years old. Indeed, Aristotle (384–322 a. C.) used to tag: "A simple affirmation is the first kind of what we call simple preposition, and a simple negotiation is the second kind of them…Referring to the present or past things, the propositions, either positive or negative, are necessarily true or false. And the ones which remain contradictory, one will be true, and the other one, false" [1, pp. 258–260]. In this same line used to be placed the stoical thinking, particularly in one of their central characters, Crisipo de Soli (≈281–208 b. C.). The formulation of the so called *law of the excluded middle* (a preposition is true or false) was assigned to him. The epicurean answered vigorously this law, tagging that it is only acceptable if a third possibility is not given, *tertium non datur* (excluded middle). Even though its materialism, Epicure believed in willpower's freedom, even suggesting the atoms are free and move from time to time, with total spontaneity. This idea has evident connotations with Heisenberg's indeterminacy law.

Twenty two centuries must pass by for Łukasiewicz [9, pp. 372–373] to point out, returning to the epicurean idea that some prepositions, which are neither true nor false, but indeterminate, do exist. This allows him to formulate it *valence law* (each preposition has a truth value). He originally assigned three truth values: true (1), false (0) and indeterminate (0, 5). He then generalized to $n$ values, for $n$ same or major than 2. The development of the so called multivalent logics has begun.

At the SIGEF International Congress of Buenos Aires, professor Gil-Aluja [3] tried to establish the epicurean position of the new coordinates arisen from Zadeh's discovery [11] formulating the *gradual simultaneity law* (every preposition can be true and false at the same time, only if a truth and untruth degree is added to it). Before and after, a great number of scientists have been placing stone by stone the basis of what could be a new knowledge building.[4]

## 11.4 Restoration of the Forgotten Effects in the Crisis

In 1988, Professors Kaufmann and Gil-Aluja [8] elaborated the Forgotten Effects Theory, departing from the wide and deep studies about the incidence or causality relations which take place at all levels in the natural or social structures. The models built up from this theory allow us to obtain all the direct and indirect relations with no mistake possible through a series of processes based in the matrix operative. It allows us to get back all the elements which could have been partially or totally forgotten at the origin. In this approach, the facts, events and phenomena in our surroundings take part in a sort of system or subsystem; in other words, we can assure that practically every activity stays submitted to some kind of cause-effect incidence. Despite of the existence of a good control system, the possibility of not considering or forgetting any causality relation which not always happen to be explicit is always there, whether voluntary or involuntary, as evident they may be. These are usually not directly discerned.

It is usual for these incidence relations to stay hidden when they happen to be effects over effects, causing then an accumulation of causes which provoke them. The human intelligence needs to be supported by tools and models which are able to create a technique base. It must be possible to work on this base with all information, and also to contrast them with the ones obtained, in order to make all the direct and indirect causality relations surface.

The concept of incidence [5] could be associated to the idea of function, and it is present in all the actions of the living beings. Indeed, in all these sequential nature's processes, where the incidences communicate in a chained way, it is usual to forget a stage, whether voluntary or involuntary. Every oversight brings up secondary effects which affect the while net of incidence relations, in a sort of combinatory process.

[4]More details in this subject can be found in: [4, pp. 14–17].

The incidence in the social field owns an eminently subjective component, usually hard to measure, but its analysis allows to increase the reasoning at the decision taking. We will expose in rough lines, how the Forgotten Effects Theory works. In order to do so, we will briefly go into its methodological basis. So we depart from two sets of elements:

$$A = \{a_i | i = 1, 2, \ldots, n\}$$

$$B = \{b_j | j = 1, 2, \ldots, m\}$$

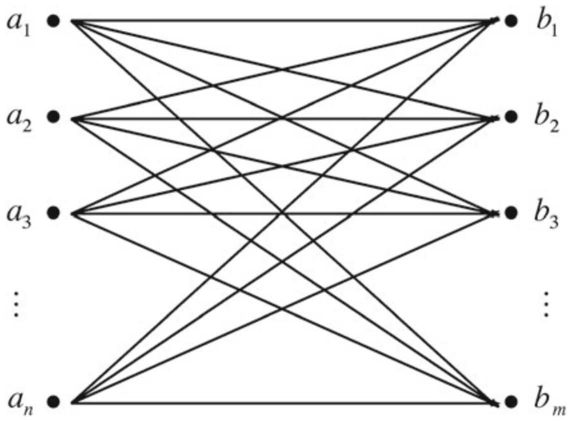
We will say there is an incidence of $a_i$ over $b_j$ if the characteristic function of property of the $(a_i, b_j)$ pair is valuated in [0, 1], that is:

$$\forall (a_i, b_j) \Rightarrow \mu (a_i, b_j) \in [0, 1]$$

The ensemble of the valuated element pairs will define what we call *direct incidence matrix*, which shows the cause-effects relations which produce with different degree between the elements from the $A$ (causes) set and the ones from the $B$ (effects) set:

$$\underset{\sim}{M} = \begin{array}{c|ccccc|c} & b_1 & b_2 & b_3 & b_4 & \cdots & b_j \\ \hline a_1 & \mu_{a_1 b_1} & \mu_{a_1 b_2} & \mu_{a_1 b_3} & \mu_{a_1 b_4} & \cdots & \mu_{a_1 b_j} \\ a_2 & \mu_{a_2 b_1} & \mu_{a_2 b_2} & \mu_{a_2 b_3} & \mu_{a_2 b_4} & \cdots & \mu_{a_2 b_j} \\ a_3 & \mu_{a_3 b_1} & \mu_{a_3 b_2} & \mu_{a_3 b_3} & \mu_{a_3 b_4} & \cdots & \mu_{a_3 b_j} \\ a_4 & \mu_{a_4 b_1} & \mu_{a_4 b_2} & \mu_{a_4 b_3} & \mu_{a_4 b_4} & \cdots & \mu_{a_4 b_j} \\ a_5 & \mu_{a_5 b_1} & \mu_{a_5 b_2} & \mu_{a_5 b_3} & \mu_{a_5 b_4} & \cdots & \mu_{a_5 b_j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_i & \mu_{a_i b_1} & \mu_{a_i b_2} & \mu_{a_i b_3} & \mu_{a_i b_4} & \cdots & \mu_{a_i b_j} \end{array}$$

This matrix can also be represented by an associated incidence graph. If the characteristic function of pertinence for a pair of elements was non-existing, the arch which joins the element from the $A$ set with the element from the $B$ set would be eliminated:



The group of incidences which is shown by the first of the two forms of presenting the cause-effect relations between the two groups of elements is known as direct incidences matrix (also called first order ones). It's about those who have been considered in the beginning of establishing the consequences that some elements have over some others. Indeed, it forms the first step of the built model in order to get back the incidence levels between the elements which haven't been detected, or simply forgotten at the beginning. But what if a third subset of elements appears?

$$C = \{c_k | k = 1, 2, \ldots, z\}$$

A subset which if formed by elements which are effects then the elements of the $B$ set act as causes, meaning:

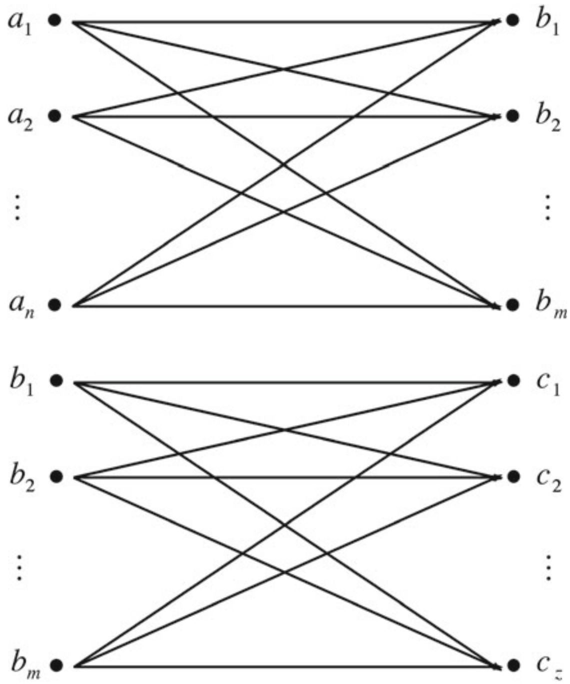
$$\underset{\sim}{N} = \begin{array}{c|c|c|c|c|}
\curvearrowright & c_1 & c_2 & \cdots & c_z \\
\hline
b_2 & \mu_{b_1c_1} & \mu_{b_1c_2} & \cdots & \mu_{b_1c_z} \\
b_2 & \mu_{b_2c_1} & \mu_{b_2c_2} & \cdots & \mu_{b_2c_z} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
b_m & \mu_{b_mc_1} & \mu_{b_mc_2} & \cdots & \mu_{b_mc_z} \\
\hline
\end{array}$$

We would obtain two matrixes of incidences, which would have the elements from the $B$ set in common:

$$\underset{\sim}{M} = \begin{array}{c|c|c|c|c|}
\curvearrowright & b_1 & b_2 & \cdots & b_m \\
\hline
a_1 & \mu_{a_1b_1} & \mu_{a_1b_2} & \cdots & \mu_{a_1b_m} \\
a_2 & \mu_{a_2b_1} & \mu_{a_2b_2} & \cdots & \mu_{a_2b_m} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
a_n & \mu_{a_nb_1} & \mu_{b_mc_2} & \cdots & \mu_{b_mc_z} \\
\hline
\end{array}$$

$$\underset{\sim}{N} = \begin{array}{c|c|c|c|c|}
\curvearrowright & c_1 & c_2 & \cdots & c_z \\
\hline
b_2 & \mu_{b_1c_1} & \mu_{b_1c_2} & \cdots & \mu_{b_1c_z} \\
b_2 & \mu_{b_2c_1} & \mu_{b_2c_2} & \cdots & \mu_{b_2c_z} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
b_m & \mu_{b_mc_1} & \mu_{b_mc_2} & \cdots & \mu_{b_mc_z} \\
\hline
\end{array}$$

The incidence graphs associated to each of the two matrixes would be the following:




There are two relations of incidence:

$$\mathsf{M} \subseteq \mathbf{A} \times \mathbf{B} \text{ and } \mathsf{N} \subseteq \mathbf{B} \times \mathbf{C}$$

The mathematical operator which, for example, we would use to establish the incidences of $A$ over $C$, is the max-min composition one. In fact, when three uncertain incidence relations are brought up:

$$\mathsf{M} \subseteq \mathbf{A} \times \mathbf{B}\ ,\ \mathsf{N} \subseteq \mathbf{B} \times \mathbf{C} \text{ and } \mathsf{P} \subseteq \mathbf{A} \times \mathbf{C}$$

The product of the composition is:

$$\mathsf{M} \circ \mathsf{N} = \mathsf{P}$$

where the $\circ$ symbol precisely represents the max-min composition. The composition of two uncertain relations is:

$$\mu\left(a_i, c_z\right)_{\mathsf{M} \circ \mathsf{N}} = \vee_{bj}\left(\mu_{\mathsf{M}}\left(a_i, b_j\right) \wedge \mu_{\mathsf{N}}\left(b_j, c_z\right)\right)$$

Therefore, we can state that the $P$ incidence matrix defines the causality relations between the elements of the first $A$ set and the elements of the third $C$ set, within the intensity or level which considering the elements from the $B$ set as intermediates entails.

After this brief analysis about the methodology used to know the incidence relations, considering three sets of elements, we decide to set out a powerful methodology, which, considering the indirect and indirect causality relations, will allow to acknowledge the cause-effect relations which stay hidden.

We start our approach [5] with the existence of a direct incidence relation; in other words, an uncertain cause-effect matrix defined by two sets of elements:

$$A = \left\{a_i \middle| i = 1, 2, \ldots, n\right\}, \text{ which act as causes,}$$

$$B = \left\{b_j \middle| j = 1, 2, \ldots, m\right\}, \text{ which act as effects,}$$

In addition, a $\mathsf{M}$ causality relation defined by the matrix:

$$\left[\mathsf{M}\right] = \left\{\mu_{a_i b_j} \in [0, 1] \middle| i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\right\}$$

where $\mu_{a_i b_j}$ are the belonging characteristic functions from each of the elements from the $\left[\mathsf{M}\right]$ matrix (formed by the corresponding rows of the elements from the $A$-*causes*-set and columns of the elements from the $B$-*effects*-set).

Therefore, it can be said that the $\left[\mathsf{M}\right]$ matrix is composed by the estimations carried out relating to all the effects which the $A$ elements apply over the $B$ elements. The more meaningful is this incidence relation, the higher will be the assigned valuation to each of the elements in the matrix. Supposing this situation, we understand that the higher is the incidence relation, the closer to 1 will result its valuation, as we have departed from the fact that the characteristic function of pertinence must belong to the [0, 1] interval. And on the contrary, the weaker a causality relation between the elements is considered, the closer its valuation will be to 0. We must underline the fact that this initial $\left[\mathsf{M}\right]$ matrix is elaborated since the direct cause-effect relations, that is, first generation relations.

Our aim is to obtain a new incidence matrix which reflects not only the direct causality relations, but also the indirect ones, called *second generation relations*.

To reach this goal it is necessary to establish the dispositive which will be able to make possible to represent the fact that the different causes can have an effect over themselves, as well as some effects can also have a certain incidence over themselves at the same time. For this reason it will be necessary to build up two additional causality relations, which will collect the possible derived effects from relating causes with causes, and effects with effects. These two auxiliary matrixes are defined as:

$$[\mathbf{A}] = \left\{\mu_{a_i a_j} \in [0, 1] \middle| i,\ j = 1, 2, \ldots, n\right\}$$

$$[\mathbf{B}] = \left\{\mu_{b_i b_j} \in [0, 1] \middle| i,\ j = 1, 2, \ldots, m\right\}$$

The [**A**] matrix collects the incidence relations which can be originated in one of the elements acting as causes, and the [**B**] matrix will respectively do so with the elements acting as effects. Both [**A**] and [**B**] matrixes are squared matrixes, and reflexive at the same time, so we have:

$$\mu_{a_i a_j} = 1 \big| i,\ j = 1, 2, \ldots, n$$

$$\mu_{b_i b_j} = 1 \big| i,\ j = 1, 2, \ldots, m$$

which expresses the fact that a certain element, whether cause or effect, influences the major presumption over itself.

In return, neither [**A**] nor [**B**] are symmetric matrixes:

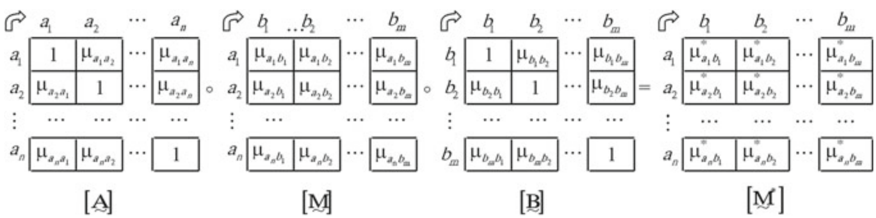
$$\mu_{a_i a_j} \neq \mu_{a_j a_i}, \big| i,\ j = 1, 2, \ldots, n$$

$$\mu_{b_i b_j} \neq \mu_{b_j b_i}, \big| i,\ j = 1, 2, \ldots, m$$

Once $[\mathsf{M}]$, [**A**] and [**B**] matrixes are built up, the establishment of direct or indirect incidences must be processed; in other words, the incidences where some interposed cause or effect takes part at the same time. In order to do so, a max-min composition of the three matrixes is realized:

$$[\mathbf{A}] \circ [\mathsf{M}] \circ [\mathbf{B}] = [\mathsf{M}^*]$$

The order in the composition must allow to always make the number of the elements in the row from the first matrix to meet with the number of elements in the column from the second matrix. The result will be a new $[\mathsf{M}*]$ matrix, which collects the first and second generation incidences between causes and effects, in other words, the initial causal relations, with the possible interposed incidence of a cause and/or effect, added. In this sense, we obtain:

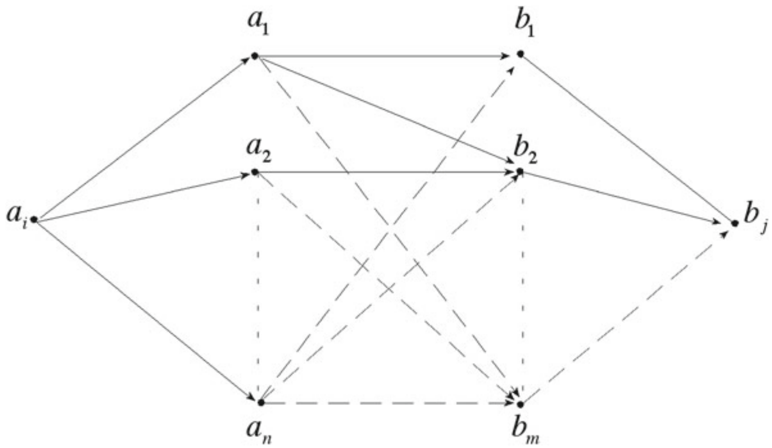



The difference between the first and second generation effects' matrix and the initial direct incidences matrix will allow to know the level in which some causality relations have been forgotten or omitted:

$$[\mathsf{O}] = [\mathsf{M}^*] \, (-) \, [\mathsf{M}]$$

$$[\mathrm{O}] = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \begin{array}{cccc} b_1 & b_2 & \cdots & b_m \\ \mu^*_{a_1b_1} - \mu_{a_1b_1} & \mu^*_{a_1b_2} - \mu_{a_1b_2} & \cdots & \mu^*_{a_1b_m} - \mu_{a_1b_m} \\ \mu^*_{a_2b_1} - \mu_{a_2b_1} & \mu^*_{a_2b_2} - \mu_{a_2b_2} & \cdots & \mu^*_{a_2b_m} - \mu_{a_2b_m} \\ \cdots & \cdots & \cdots & \cdots \\ \mu^*_{a_nb_1} - \mu_{a_nb_1} & \mu^*_{a_nb_2} - \mu_{a_nb_2} & \cdots & \mu^*_{a_nb_m} - \mu_{a_nb_m} \end{array}$$

It is also possible to know, since the forgotten level of any incidence, the element (whether cause or effect) which links them. We only have to follow the steps realized from the max-min composition of the last matrixes, in a graphic perspective, in order to obtain it:




It can finally be said that the higher is the value of the characteristic function of pertinence in the $[\mathrm{O}]$ matrix, the higher is the forgotten level in the initial incidence relation. This is translated as a wrong acting, caused by a lack of the right consideration of the incidences, or a wrong estimation of their intensity and their influence.

The Forgotten Effects Theory brings, as the causes which lead to a crisis and their effects are studied, a model of sequential nature which allows to introduce the incidence relations in the cyclic processes of the economic systems. The generated mix between the different elements which act in a direct or indirect way in the origin of the crisis, provokes causes which lead to effects with an accumulated intensity level, which is necessary to deal with in an appropriate way.

The last approaches exposed have been previously applied in various researches [6] where it is shown in which measure the economic cooperation between different countries can be used as a stimulus to revert the current economic process. We believe its use can be extended to the economic cycles' study.

These studies leaded to a series of conclusions which have been used as a base for the elaboration of other researches. Those researches were based in the incidence relations and aggregation operators, in order to establish in which measure, and under which criteria, the economic cooperation between countries through the developed actions of their enterprises, could be possible.

The approached tools show how the little moves in the whole working of the enterprises and institutions from the involved countries can generate, not only directly but basically indirectly, a multiplying effect over different economic fields in the society, and reach an economic value which can be a driving force to the growth and welfare.

These last years the efforts at looking for determinate actions addressed to finding new solutions for the current economic and social situation have been focussed. In the industrialized countries which take part of this situations, some of the most affected economies are gobbled in a spiral, where the increasing unemployment, the deficit and the damage of the public investment in strategic sectors, such as health and education, find themselves in a constant feedback which seriously affects their economy, society and welfare. It's important to make scientists' abilities and potentialities meet, as well as business men's and politicians', in order to create job places, values and wealth.

The goal of these approaches is to point out how punctual actions made by specific sectors can generate an added value, both directly in the related activities and indirectly, like the rest of socioeconomic activities from the so considered urgent sectors. It could be said that every recuperation process stays strengthened if a multiplying effect with clear beneficial effects exists. These effects echo on both economic sectors which receive inversions and economic sectors which generate inversions, because of the entailed feedback. So in this way, positive effects are generated and extended like a net, affecting all the socioeconomic agents.

## 11.5 The Underlying Reality: A Social Crisis of Humanism

The deep economic crisis where we are immersed, together with the recuperation perspectives, intensifies the efforts in the socioeconomic field to find new solutions to revert the current involution process. One of the key factors to achieve this goal is based in rationalization and the outlay containment. But obviously, this purpose is not enough for itself, and won't make the real economy improve (the one which is perceived by all citizens, the one which becomes permeable in all social layers, and the one which allows a permanent welfare), and will never stay up longer than a short term.

It's important to accompany these measures of sacrifice imposed to the citizenship with adequate politics depending on the objectives selected. In these austere budgets it is necessary to prioritize those expenses and inversions which could generate wide multiplying effects in the economy. In other words, make a part of the resources received by the first addressee circulate to a second addressee, and successively, causing the effects to maximize in wideness and quantity, to make them arrive to the major number possible of socioeconomic agents.

In this process the sustainable long-term growth germ is found, and it is only possible to generate a healthy economy with it, an economy with stable levels of social welfare. The ones which hold the reins of the biggest socioeconomic decisions usually get trapped in the short-term compromises' nets. With adequate studies, supported by adequate management techniques based on the uncertainty treatment, it is possible to make the short-term and long-term goals compatible.

But as long as our scientists are providing new research lines, able to supply useful instruments (which allow the decision taking in the new cohabitation map about to come) to the social sciences, the depressive and recessive process of an important part of the economic systems is contributing to a wide cry of intellectuals, politicians and business men, asking for urgent measures to the government's responsible people, destined to make a shortcut to our misfortunes.

The answers, almost all of them meeting the classical mechanism with its Boolean logic, have been placed in one of the two ends: sanitations or expansion, accepting for each of them the two shy positions and ineffective concessions to the other one. There hasn't been much improving since the most orthodox doctrine, typical from the XXth century origins, the golden era of the mechanism.

As we have pointed out, the new age, with is high evolutionism charge, is requiring the use of new concepts, models and algorithms, different from the used ones, held in too many occasions in the linearity and differential equations. Our offers are supported by the whole instrumental born in the Fuzzy Logic, which are reaching an enormous prominence. Looking for the maximization of the expansion with a minimum of sanitation, or minimizing the sanitations with the maximum expansion possible, are two options (between many others) which are placed in the approached schemes.

An equal order of ideas results with the paradox behaviour of the studious and analysts of the crisis when placing the causes of the current situation in the heavy acting of the economical deciders. They don't consider that these causes, serious indeed, are not the real *causes*, but the *effects* of a deep change operated in the mentality, habits and values of the new society, build up by all of us. Actually, the first *causes* of many of our problems is the existence of a *social crisis of humanism*.

In any case, and at each citizen's level, we observe that the looks are always leaded towards those *effects*, bringing up repeatedly two questions: when will the *crisis* end (meaning of course the *effects* of the crisis)? And, which *measures* to make the way shorter to this moment should be taken (meaning, how to appease the *effects* of the crisis)?

Relating to the first of these two questions, the end of the crisis isn't the one which will originate the recuperation, but the one where the recuperation will be overcome, and the economic system will be installed in a new and different welfare at the same time.

The answer to the second question requires certain considerations which will be briefly summarized right below.

In first place, special attention is earned by a repeated process and clearly and empirically validated: *an economic system can only be fully efficient as long as it proportionally rewards its agents with the benefits their activities provide*. This hasn't been the principle which has precisely ruled the previous period of the crisis. The intense activity of the economic agents and the lack of valid criteria requested to assign a value to their acting, have worsened and made more visible the disagreement between retribution and efficiency.

In second place, we repeatedly observe that in the origin of the crisis, the scorn towards the economic limitations to face the outlay and inversion are not acceptable. Occident Europe's big dilemma is prisoner in huge public debts, and limited by a lack of growth: can we adjust, cut, reduce the debts and grow up at the same time? Can we avoid the fitting from slowing down the growth? Can we increase the taxation system in order to find a better redistribution of the income without setting back the productive activity? Our emphatic answer is: yes. And we shelter from this clear and defined affirmation, where it's necessary and unavoidable to change our principles, methods and operators. This will require the transition of the geometric conception to the Darwinian conception in the social sciences. It is necessary to reject the typical mechanism of the economic science which has left the habit of thinking and acting following the Boolean logic: yes-no; clack-white; fitting-growth. It makes up the most genuine application of the already mentioned *law of the excluded middle*. Fortunately, we are able to adopt optimal decisions in intermediate positions. Basing on the gradual simultaneity law [3] it has already been possible to stabilize the social knowledge over the Darwinian thinking.

In third place, we validate that the idea of *spending money on anything* has dominated the citizen's habits. It is necessary to bring the negative consequences of the *dripping theory*,[5] to the general public opinion. This theory shows how even if the inversions are not well placed, they will always end by benefiting someone. The ultimate reason of our position is that the market is much less efficient when the money is introduced in an inadequate way in it, resulting more difficult to create welfare.

In fourth place, in the current depressive situation, the fact that all the countries involved by the crisis are being shaken by the same trembling, has been considered, so a common politic is enough.

In fifth place, we believe (as Stiglitz also does), that another world will be possible after this long recessive period. The immediate effects of the crisis happen to need an urgent correction if successfully starting a new stage in the economic and financial activity is desired. To achieve so, we need: a correct and adequate fiscal politic; a macro prudential regulation of the financial institutions; a major public investment to innovation, investigation and education; efficient encouragements towards earning for the middle citizen; an extension of the European Central Bank's mandate to stimulate the employment creation; the establishment of a New Deal [10] and a deep change in the citizen's soul, in order to demand a wise government, based in our human needs.

Finally, in sixth place, the hunt of the activities capable of generating a multiplying effect in local levels, results to be non-negotiable, avoiding in this way the stimulus of the great operations which externalize certain sources; sources which return to the circulation in a very dubitative way. Only in this way we will be able

[5]Gathered by Gil-Aluja, J.'s answer at Joseph E. Stiglitz's speech in his joining in the Royal Academy of the Economic and Financial Sciences: The price of inequality, RACEF 2012, p. 38.

to regenerate the economic tissue, refusing the speculative processes. The opposite means a degradation in the society at middle and long terms.

Those six reflexions are a sample of the wrong path followed by the leaders of our society since some decades. All of them take part in the necessary change to give sense to the human being, immersed in a pluriconnected and interdependent economic system.

We believe another world is possible, and we stand up to intensify the collaboration and cooperation efforts in the scientific research field. Absolutely all of us must be implicated with the social justice and the desired equality of opportunities. Only with this, the economic efficiency will be possible.

We are convinced that, helped by the public and private institutions, we will reach the highest success's overlooks, as we work on bringing our community towards a future of sustainable social progress.

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# Chapter 12
# Advanced Computing with Words: Status and Challenges

**Jerry M. Mendel and Mohammad Reza Rajati**

**Abstract** In this chapter, we focus on the status of Advanced Computing with Words (ACWW) and the challenges that it may encounter in the future. First, we elaborate on the notion of Computing with Words (CWW) and its various subareas. Then we present some non-engineering ACWW problems and connect them to more realistic engineering problems, after which we provide a roadmap for solving ACWW problems, and show how the Generalized Extension Principle (GEP) can be used to formulate their solutions. We also propose a syllogistic approach to solving ACWW problems that also uses the Extension Principle but in a different way. Finally, we discuss present and future challenges to ACWW, i.e. we explain what challenges ACWW encounters given the current trend of data abundance and widespread use of Internet for dealing with questions posed in natural languages.

## 12.1 Introduction: Computing with Words and Advanced Computing with Words

Computing with Words (CWW or CW) [11, 29, 34, 66, 69, 70, 75, 91, 104, 109, 142, 144, 147, 154] was probably conceived as the main area of application of fuzzy logic [132] when the field was originated. It is a methodology of computation whose objects are words rather than numbers, although those words are linked to numbers and classical calculations that can be carried out by computing machinery, via membership functions of fuzzy sets associated with words. To do that, a process called *precisiation of meaning* [152] is essential, which, in a nutshell, is determining membership functions of the fuzzy sets that model the words.

Zadeh distinguishes two subareas for CWW:

1. *Basic Computing with Words* (BCWW), which mainly deals with descriptions of complex systems in terms of fuzzy IF-THEN rules [134]. Soft constraints are generally possibilistic, i.e. they describe (physical) attributes like speed, temperature,

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_12

pressure, color, etc. The soft constraints are assigned explicitly by the IF-THEN rules. BCWW mainly deals with the function approximation properties of fuzzy systems [47, 107], and has been applied to engineering problems [63] such as modeling [38, 98], control systems design [78, 105], clustering [79] and data mining [106, 108].

2. *Advanced Computing with Words* (ACWW) [70, 152], which deals with the assignment of soft constraints through complicated natural language statements. Its related problems are closer to natural languages, in that the assignments of soft constraints are usually implicit, e.g. in the statement "Most Swedes are tall," the soft constraint "Most" is implicitly assigned to the *proportion* of Swedes who are tall. Besides possibilistic soft constraints, ACWW deals with veristic and probabilistic [148, 149] soft constraints. *World Knowledge*[1] or *Common Sense Knowledge* plays a pivotal role in solving ACWW problems.

The fuzzy logic community has mainly focused on the function approximation applications of fuzzy logic, especially after the seminal work of Mamdani and Assilian [58], which uses an interpretation of fuzzy systems that is particularly suitable for function approximation. Such an interpretation is different from the natural language aspect of fuzzy logic and from the idea of building computational machinery that is fed with words, computes with words, and communicates solutions of problems in terms of words.

Nevertheless, CWW has witnessed a resurrection through its application to various real-world problems including analysis of complex systems [102], control [59, 156], risk assessment [54], decision-making [21, 31, 35, 36, 40, 62, 65, 115], classification [1], website quality assessment [37], reinforcement learning [157], text processing [155], object oriented programming [5], information retrieval [4, 48], expert systems [45, 46, 77, 97], reputation systems [131], and natural language generation [41].

Many theoretical approaches have been proposed to formalize CWW, among which are: 2-tuple models [34], rough sets [80], concept algebra [111], fuzzy arithmetic [14], voting model semantics [49], ontologies [32, 90], Turing Machines [81, 103], fuzzy automata [130], formalization of the Generalized Constraint Language (GCL) [44] (GCL was proposed by Zadeh [143, 145, 148]), fuzzy Petri nets [10], meta-linguistic axioms [101], approximate reasoning [122], and Perceptual Computing (Per-C) [64, 65, 68, 74].

It is not surprising that different people have different viewpoints on CWW [70]. In many of the above contributions, the term CWW has been used inclusively, in the sense that any application of fuzzy logic for reasoning with words has been considered as an instance of CWW. While there is no problem with such a viewpoint (especially since the founding father even classifies rule-based function approximation applications of fuzzy logic as BCWW), we prefer to view CWW as a methodology of computation whose inputs, reasoning procedures, and outputs involve natural language words, phrases, and propositions rather than just numeric values. Nevertheless, the boundaries of CWW seem to be fuzzy themselves. In fact, there are CWW

[1] World Knowledge means information either given in the statement of a problem or needed to solve the problem.

applications for which the assignments of soft constraints are not implicit [112], and are even performed through fuzzy IF-THEN rules [114], but still the inputs and outputs are natural language words. Those applications also usually do not deal with linguistic truth, possibility, or probability, something that is expected from ACWW problems. On the other hand, there are applications of fuzzy logic for function approximation that deal with linguistic truth [28, 92] and linguistic probability [25, 50, 51, 99].

We adhere to two tests[2] for calling a computational method CWW [68] both of which we suggest must be passed or else the work should not be called CWW. A third test is optional but is strongly suggested. The tests are:

1. *A word must lead to a membership function rather than a membership function leading to a word.*
2. *The output from CWW must be at least a word and not just a number.*
3. *Because words mean different things to different people, they should be modeled using at least Interval Type-2 Fuzzy Sets (IT2 FSs).*

Test number 3 is "optional" so as not to exclude much research on CWW that uses Type-1 Fuzzy Sets (T1 FSs), even though we strongly believe that this test should also be a requirement for CWW, since a T1 FS cannot simultaneously capture intra-personal and inter-personal uncertainties about a word, whereas an IT2 FS can.[3]

Although Zadeh has partitioned CWW into only BCWW and ACWW, we feel that CWW should be partitioned into BCWW, Intermediate CWW (ICWW) and ACWW, as depicted in Fig. 12.1, which is our schema of the viewpoint of *Fuzzy Logic = Computing with Words*. To avoid confusion, in this chapter we call the function approximation applications of fuzzy logic, BCWW; they deal with fuzzy systems that have numerical outputs, like fuzzy rule-bases, fuzzy classifiers, and fuzzy clustering. We reserve the name ICWW for those CWW problems that involve simple assignments of attributes, including the usage of IF-THEN rules [116] or aggregation operators such as *Ordered Weighted Averages (OWAs)* [125], *Fuzzy Weighted Averages (FWAs)* [20, 30, 52] or *Linguistic Weighted Averages (LWAs)* [112], but whose outputs are linguistic. In Linguistic summarization, data lead to a model that may involve simple assignments of attributes through IF-THEN rules as well as implicit assignments of attributes through linguistic quantifiers (e.g., most people who aren't exposed to sunlight have serious vitamin D deficiency). Thus, linguistic summarization seems to have some features of ACWW, but those features appear in the answer that it yields, not necessarily in the world knowledge that it deals with, which is why we have located it in ICWW. Observe in Fig. 12.1 that fuzzy IF-THEN rules occur in both BCWW and ICWW problems. ACWW problems involve intricate assignments

[2]Detailed discussions about these tests are given in [68, pp. 312–313].

[3]Another test appears in [68]: *Numbers alone may not activate the CWW engine (e.g., IF-THEN rules). Numbers are modeled as singleton FSs and there is nothing fuzzy about them.* Jon Garibaldi (Univ. of Nottingham) has pointed out (in a conversation with the first author) that in e.g., a medical application, patient data may all be numbers, rules are used, and then outputs are not defuzzified, but instead are mapped into a linguistic output (with similarity). We agree with this, and have therefore dropped this test.

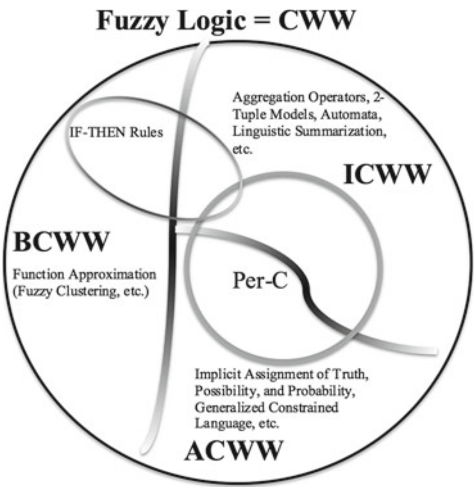


**Fig. 12.1** Fuzzy Logic = Computing with Words: BCWW, ICWW, and ACWW (The focus of this figure is the applied facet of fuzzy logic; it excludes other aspects of it, such as the logical and pure mathematical aspects)

of truth, probability, and possibility. Observe that the Perceptual Computer (Per-C) methodology can address both ICWW and ACWW problems, which is illustrated in Fig. 12.1 by showing an overlap between Per-C and both ICWW and ACWW.

In this chapter, we focus mainly on ACWW and provide a roadmap for solving ACWW problems. We briefly review how truth, probability, and possibility can be dealt with in the framework of ACWW, and sketch methodologies for solving some selected ACWW problems by different methodologies, and provide a perspective of the status, challenges and future of ACWW.

## 12.2 Some ACWW Problems

Zadeh has introduced some ACWW problems that involve everyday reasoning and decision making by linguistic probabilities [143, 145, 146, 150, 152]. Some of those problems are given in Table 12.1. To demonstrate how such problems occur for more realistic statements, we show, in Table 12.2, the Engineering versions of the problems for the first four rows of Table 12.1. We do this because we strongly believe that for ACWW to be taken more seriously, its problems must be shown to be of more practical relevance.

**Table 12.1** Some of Zadeh's ACWW problems involving linguistic probabilities and linguistic quantifiers

| World knowledge | Problem statement |
|---|---|
| • Most Swedes are tall | • What is the average height of Swedes? |
| • Probably John is tall | • What is the probability that John is short/very tall/not very tall? |
| • Most Swedes are much taller than most Italians | • What is the difference between the average height of Swedes and the average height of Italians? |
| • Usually Robert leaves the office at 5 p.m. everyday and usually it takes him about an hour to get home | • At what time does Robert get home? What is the probability that he is home before 6:15 p.m.? |
| • Vera has a son in mid-twenties and a daughter in mid-thirties. Usually mother's age at birth of a child is between approximately 20 and approximately 40 | • What is Vera's age? |
| • Most Swedes are tall, and most tall Swedes are blond | • What is the probability that Magnus (a Swede picked at random) is blond? |
| • Usually, most United flights from San Francisco leave on time. I am scheduled to take a United flight from San Francisco | • What is the probability that my flight will be delayed? |
| • Usually several cars are stolen every day in Berkeley | • What is the average number of cars stolen per month in Berkeley? |
| • $X$ is a real-valued random variable. Usually $X$ is much larger than approximately $a$ and usually $X$ is much smaller than approximately $b$ | • What is the probability that $X$ is approximately $c$, where $c$ is a number between $a$ and $b$? |
| • A and B are boxes, each containing 20 balls of various sizes. Most of the balls in A are large, a few are medium and a few are small; and most of the balls in B are small, a few are medium and a few are large. The balls in A and B are put into box C | • What is the number of balls in C which are neither large nor small? |
| • A box contains about 20 balls of various sizes and there are many more large balls than small balls | • What is the number of small balls? |

## 12.3 A Roadmap for Solving ACWW Problems

In this section, we provide a high-level roadmap for solving ACWW problems.

**Table 12.2** Engineering versions of some of Zadeh's ACWW problems

| World knowledge | Problem statement |
|---|---|
| • Most of the products of Company X have somewhat short life-times | • What is the average life-time that is expected from the products of Company X? |
| • Probably product X is highly reliable | • What is the probability that the reliability of X is low? |
| • Most of the products of Company X have much lower maintenance costs than most of the products of Company Y | • What is the difference between the average maintenace costs of the products of Company X and the average maintenace costs of the products of Company Y? |
| • Usually Product I lasts for about 3 years and is then replaced by a refurbished one, and usually, the refurbished Product I lasts for about 2 years | • What is the probability that a new Product I is not needed until the seventh year? |

### *12.3.1 Modeling of Words*

Because any ACWW problem involves words, the first step in the solution to an ACWW problem is to model those words. As was mentioned earlier, the process of assigning fuzzy sets to words is called *precisiation (of meaning)* [140, 146]. Mendel in [66] argues that a correct first order model for a word is an Interval Type-2 Fuzzy Set [9, 67, 135], since words mean different things to different people.[4] In other words, a fuzzy set model for a word should represent both the intra-uncertainty and the inter-uncertainty associated with that word. *Intra-uncertainty* is the uncertainty that an individual has about the meaning of the word, whereas *inter-uncertainty* is the uncertainty about the word that exists across a group of individuals. Both types of uncertainties cannot be modeled by T1 FSs, which is why the words need to be modeled using at least IT2 FSs.

Interestingly, Zadeh anticipates that fuzzy sets of higher type will play a central role in ACWW in the future [69].

The philosophy of precisiation in the Per-C is to build a vocabulary of IT2 FS models of words using data collected in the form of intervals from subjects by the Interval Approach (IA) [53] or the Enhanced Interval Approach (EIA) [117]. Those methods build IT2 FS models of words by processing data using various statistical tools; therefore, they can be seen as highly nonlinear transformations of the data that are collected in the form of intervals from subjects. This process of precisiation is called *encoding* in the Per-C language [3].

When data about words are collected from a group of subjects and are processed using probability and statistics, then the fuzzy set models are random, i.e. the randomness of the data does not disappear, but is propagated through the nonlinear transformations into the fuzzy set models.

[4]If one knows how to model a word using a general T2 FS [6, 7], that model can be called a second-order uncertainty model [3].

To implement solutions to ACWW problems, their veristic, probabilistic, and possibilistic constraints have to be modeled using data collected from subjects. In [85] linguistic probabilities and linguistic quantifiers (which are mathematically equivalent to linguistic probabilities [139]) were modeled by the EIA. A similar effort is needed to model linguistic truth values, usualities, and possibilities, so as to have a more complete collection of IT2 FS models of soft constraints for ACWW.

It was pointed out in [3] by Mendel, that for an ICWW problem, the linguistic answer depends on the size of the vocabulary that is selected for the output. This is also true for ACWW. If, e.g., height is described only by three terms (short, medium, and tall) then the output of an ACWW problem about height may be linguistically different from the output that is based on seven terms (very short, short, moderately short, medium, moderately tall, tall, and very tall). It may happen that, when the fuzzy set models in the two vocabularies are compared in terms of similarity, some of the terms from the three-word vocabulary will be quite similar to those in the seven-word vocabulary, so that they can be interchanged with one another. But, in general, the solution to an ACWW problem depends on the size of its vocabulary.

Although we are strong advocates of using IT2 FS models of words, to the best of our knowledge, the main tools pertinent to solving ACWW problems have only been developed for T1 FSs. Consequently, we explain the solutions to the ACWW problems using T1 FSs. Nevertheless, we still advocate obtaining those T1 FSs by using data collected from subjects because this reflects the intra- and inter- personal uncertainties about a word to some extent. One way to do this is to use the Upper Membership Functions (UMFs) of the IT2 FSs synthesized by the EIA (those UMFs will depend on the size of the vocabulary).

### *12.3.2 Computing Solutions to ACWW Problems*

The second step in the solution of an ACWW problem is to carry out the computations that are pertinent to the problem. In this step, the calculi of fuzzy sets are used to derive generalized (soft) constraints [148] on some variables of interest, given the generalized constraints on some other variables for which World Knowledge is available. The variables of interest can be described as functions of the variables on which World Knowledge is available, and consequently soft constraints provided by World Knowledge can be propagated to the variables of interest. The main tool of fuzzy logic for performing such calculations is the Extension Principle and its extended version, the Generalized Extension Principle (GEP) [142, 152]. This second step is called the *CWW engine* in the Per-C.

Fuzzy IF-THEN rules can play the role of a computational engine For ICWW, but they assign attributes directly, so they may not have a role in ACWW in their conventional form. For ICWW and ACWW, many other mathematical tools may be needed in the computations, such as antonyms and negations [13, 100], syllogisms [74, 84, 94, 141], and aggregation operators [35, 86, 125, 127].

Because, in ACWW problems, the assignment of the soft constraints are implicit, and the information is described in intricate phrases drawn from natural languages, the following framework of the GEP is of crucial importance [142]. Assume that $f_1(\cdot), \ldots, f_m(\cdot)$ and $g(\cdot)$ are real functions, i.e.

$$f_1, \ldots, f_m, g : U_1 \times U_2 \times \cdots \times U_n \longrightarrow V \tag{12.1}$$

The world knowledge or information pertinent to an ACWW problem gives some generalized constraints on the functions $f_1(\cdot), \ldots, f_m(\cdot)$. The goal is to derive the generalized constraint on $g(\cdot)$, given such information, i.e.:

$$\frac{f_j(X_1, X_2, \ldots, X_n) \text{ is } A_j, \;\; j = 1, \ldots, m}{g(X_1, X_2, \ldots, X_n) \text{ is } B}$$

where $A_i, \;(i = 1, 2, \ldots, m)$ and $B$ are T1 FSs. The $A_i$'s induce $B$ as follows:

$$\mu_B(v) = \begin{cases} \sup\limits_{\mathbf{u}|v=g(\mathbf{u})} \min\big(\mu_{A_1}(f_1(\mathbf{u})), \ldots, \mu_{A_m}(f_m(\mathbf{u}))\big) & \exists v = g(\mathbf{u}) \\ 0 & \nexists v = g(\mathbf{u}) \end{cases} \tag{12.2}$$

where $\mathbf{u}$ is shorthand for $(u_1, u_2, \ldots, u_n) \in U_1 \times U_2 \times \cdots \times U_n$. This is the GEP. Examples of $f_j$ and $g$ for some ACWW problems are given in Sect. 12.6.

When $f_j(X_1, X_2, \ldots, X_n) = X_j$, the GEP reduces to the Extension Principle [132, 135].

Equation (12.2) is a complicated (functional) optimization problem that is difficult to solve. To the best of our knowledge, [87] offers the only published effective methodology to carry out the calculations related to this optimization problem.

### *12.3.3 Translating the Solution Back into Words*

The third step in the solution of an ACWW problem is to map the results from the second step back into a word. This process is called *linguistic approximation* by Zadeh [136], *retranslation* by some other authors [61, 124], and *decoding* in Perceptual Computing. Subsethood and similarity measures [60, 68, 113] are often used to do this.

Some approaches to CWW, especially the 2-tuple approach [34, 35, 104, 110], circumvent the problem of information loss due to linguistic approximation by a symbolic approach. They process the input linguistic information using some aggregation operators that yield the index of a fuzzy set representing a word in an output vocabulary. The aggregation operators may not always yield an integer, in which case the output is rounded so that it represents the index of a word in the output

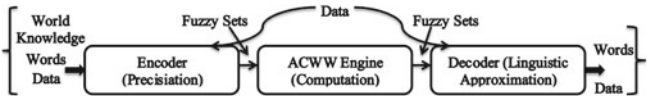


**Fig. 12.2** A Perceptual Computer for ACWW

vocabulary. The rounded output along with the difference between the index yielded by the aggregation operator and the rounded one are reported as a 2-tuple (a word associated with the rounded integer and a number in the interval $[-0.5, 0.5)$ that can represent the information loss).

### *12.3.4 Observation*

A block diagram of the Per-C is given in Fig. 12.2. Our discussions in Sects. 12.3.1–12.3.3 have demonstrated that solutions to ACWW problems fall exactly into the Per-C structure. What are different for ACWW from other applications of the Per-C is the use of World Knowledge and the ACWW engine, which uses the GEP. See [68] for other engines that are used in ICWW applications of the Per-C.

## 12.4 Dealing with Truth, Probability, and Possibility

In Zadeh's ACWW problems (see Table 12.1) words such as "probably", "most", and "usually" are used. Truth, possibility, and probability qualification principles are the main tools for handling linguistic truth, linguistic probability, and linguistic possibility in natural language statements [137].[5]

Assume that $A$ is a fuzzy set, and $\tau$ is a linguistic truth value for which $\mu_\tau$: $[0, 1] \to [0, 1]$. The *truth qualification principle* is [137, 153]:

$$\chi \text{ is } A \to \mu_A(x)$$
$$\chi \text{ is } A \text{ is } \tau \to \chi \text{ is } B,\ \ \mu_B(x) = \mu_\tau(\mu_A(x))$$

This principle gives a framework to deal with linguistic truth in ACWW problems through modification of the membership function of $A$ to have a new soft constraint $B$ on $\chi$.

Dealing with probability constraints is very important for solving ACWW problems. It relies on the definition of the probability of a fuzzy event [133]. Assume

[5]Truth and possibility qualification principles modify fuzzy sets. Truth qualification principle is a functional relationship that acts on the membership function of the fuzzy set that is associated with a truth value. Possibility qualification principle modifies the fuzzy set that is associated with it into an IT2 FS. This in turn calls for extension of ACWW methodologies to IT2 FSs.

that $C$ is a fuzzy event in $U$ and $p(\cdot)$ is a probability distribution function on $U$. The probability of the fuzzy event $C$ is then defined as:

$$\text{Prob}(C) = \int_U \mu_C(u)p(u)du \tag{12.3}$$

Assume that $\lambda$ is a fuzzy probability (e.g., Probable, Likely, Somewhat Improbable), $\mu_\lambda : [0,\ 1] \to [0,\ 1]$. The *probability qualification principle* starts from the following:

$$\begin{aligned} &\chi \text{ is } C \to \mu_C(x) \\ &\chi \text{ is } C \text{ is } \lambda \to \text{Prob}(C) \text{ is } \lambda \end{aligned}$$

Consequently, since $\text{Prob}(C) = \int_U \mu_C(u)p(u)du$, the probability qualification principle becomes

$$\int_U \mu_C(u)p(u)du \text{ is } \lambda \to \mu_\lambda\left(\int_U \mu_C(u)p(u)du\right)$$

The probability qualification principle is used very frequently for solving ACWW problems. When a soft constraint is assigned to the probability of an event, it is used by the GEP in (12.2). Therefore, saying that "the lifetime of a product is probably short" means we assign the soft constraint $A_j = Probable$ to the function $f_j(l(\cdot)) = \int_T \mu_{Short}(t)l(t)dt$ (which is a function of the pdf $l(\cdot)$), and $\mu_{Probable}(\int_T \mu_{Short}(t)l(t)dt)$ appears in the expression of the GEP, according to the probability qualification principle.

The *possibility qualification principle* is:

$$\begin{aligned} &\chi \text{ is } A \to \mu_A(x) \\ &\chi \text{ is } A \text{ is } \beta\text{−possible} \to x \text{ is } \widetilde{B},\ \ \widetilde{B} = A \cup \widetilde{\Pi}_\beta,\ \ \mu_{\widetilde{\Pi}_\beta}(x) = [0,\ \beta] \end{aligned}$$

The possibility qualification principle weakens the statement that is qualified by taking the union[6] of $A$ and an IT2 FS $\widetilde{\Pi}_\beta$, $\mu_{\widetilde{\Pi}_\beta}(u) = [0,\ \beta]$ which represents some level of indeterminacy (total indeterminacy occurs when $\beta = 1$), i.e. the membership value of each member of the universe of discourse $U$ may be any number in the interval $[0,\ \beta]$.

## 12.5 A General Guide for Formulation of ACWW Problems

Based on the above discussions the following framework is provided for solving an ACWW problem [87]:

[6] Zadeh uses the bounded sum t-conorm $T(a, b) = \min(1, a + b)$ as the union operator.

1. Determine the soft constraints and identify all of the words that are used for linguistic probability, quantifiers, usuality, truth, and possibility.
2. Model all of the words associated with the soft constraints using fuzzy sets (this is where the size of vocabulary is important).
3. Use truth qualification and possibility qualification principles to modify the soft constraints in the problem.
4. From the linguistic description of the problem, determine which quantities (e.g., numeric probabilities) are constrained by the words that were found in the previous step and how this occurs. This may be needed because such words may be implicitly assigned to the quantity (e.g., *Most* is assigned to the portion of Swedes who are tall, *Probable* is assigned to the probability that John is tall, etc.).
5. Formulate the numeric probabilities in the previous step:
   (a) Use definite integrals involving indicator functions of non-fuzzy events or membership functions of fuzzy events and the (generally unknown) probability distribution functions pertinent to those events (continuous case); or,
   (b) Use the fraction of the cardinality of the fuzzy event over the cardinality of the whole population (discrete case). Like the continuous case, for which the probability density functions might be unknown, some quantities related to the population (e.g., the average height of a population or the variance of the height of a population) may be unknown.
6. Formulate the quantity on which a fuzzy constraint must be calculated, in terms of the unknowns (e.g., probability distributions) of Steps 5a or 5b. This quantity may be an average, a numeric probability, or a function of some averages, etc. To do this, the calculi of fuzzy sets are needed.
7. Apply the GEP: knowing the soft constraints on the quantities formulated in Step 5, determine the soft constraint on the quantity that was formulated in Step 6. The sup of the GEP is taken over all possible unknowns of Steps 5a or 5b. For example, if a probability distribution is not known, or if the height of the individuals in a population is unknown, the sup is taken over all admissible probability distributions or admissible heights of individuals. The word "admissible" is crucial and implies that additional problem-specific World Knowledge is needed.
8. Carry out the calculations related to the GEP according to the algorithms that were presented in [87] to derive the membership function of the fuzzy constraint yielded by the GEP.
9. Decode the Step 8 solution back into a word (a numerical solution may also be provided). Similarity plays an important role in this step.

## 12.6 Two Examples

In this section,[7] we provide partial solutions[8] to two of Zadeh's ACWW problems, one of which involves linguistic probabilities, and the other linguistic quantifiers.

### *12.6.1 Tall Swedes Problem (AHS)*

The Tall Swedes problem[9] is about the average height of Swedes (AHS), and is (see Table 12.1): *Most Swedes are tall. What is the average height of Swedes?* This problem involves a linguistic quantifier (Most) and an implicit assignment of the linguistic quantifier to the portion of tall Swedes (Step 4). The portion of Swedes who are tall is equivalent to the probability of the fuzzy event Tall, and is calculated as (Step 5):

$$\text{Prob}(Tall) = \int_a^b \mu_{Tall}(h)p_H(h)dh \tag{12.4}$$

in which $a$ and $b$ are the minimum and maximum possible heights and $p_H$ is the probability distribution function of heights which clearly satisfies:

$$\int_a^b p_H(h)dh = 1 \tag{12.5}$$

The soft constraint "Most" is assigned to $\text{Prob}(Tall)$. On the other hand, the average height of Swedes is calculated as (Step 6):

$$AH = \int_a^b p_H(h)hdh \tag{12.6}$$

To derive the soft constraint imposed on $AH$ by the fact that $\text{Prob}(Tall)$ is constrained by "Most", one needs to use the framework of the GEP (Step 7):

In the Tall Swedes problem, $f(p_H) = \text{Prob}(Tall)$, where

$$f : \mathcal{X}_{[a,b]} \longrightarrow \mathbb{R} \tag{12.7}$$

in which $\mathcal{X}_{[a,b]}$ is the space of all possible height probability distribution functions on $[a, b]$; and, $g(p_H) = AH$, where

[7]Most of the material in this section is taken from [87].

[8]We are actually only setting up the ACWW engine (i.e., Steps 4 to 7, in our Sect. 12.5 procedure). Numerical solutions can be found in [87].

[9]We do not use the acronym "TSP" because it is already widely used for the famous Traveling Salesman Problem.

$$g : \mathcal{X}_{[a,b]} \longrightarrow \mathbb{R} \tag{12.8}$$

The GEP then implies that the soft constraint on the average height of Swedes is computed, as:

$$\mu_{AH}(v) = \sup_{\substack{p_H | v=\int_a^b p_H(h)hdh \\ \int_a^b p_H(h)dh=1}} \mu_{Most}\left(\int_a^b p_H(h)\mu_{Tall}(h)dh\right) \tag{12.9}$$

It is worth noting that other approaches to solving the Tall Swedes problem have been offered in [74, 89].

Assuming that the MFs for $Most$ and $Tall$ are determined by the precisiation process, to solve (12.9), the family of probability distributions over which the sup is taken has to be determined. The distribution of heights of men can be considered normal. The ranges of the parameters of those distributions are also part of the World Knowledge that has to be available. A comprehensive approach to solving problems involving the GEP is in [87].

### *12.6.2 Swedes and Italians Problem (SIP)*

The Swedes and Italian problem is (see Table 12.1): *Most Swedes are much taller than most Italians. What is the difference between the average height of Swedes and the average height of Italians?* Zadeh formulates the solution to SIP (see Table 12.1) using generalized constraints, as follows. Assume that the population of Swedes is represented by $\{S_1, S_2, \ldots, S_m\}$ and the population of Italians is represented by $\{I_1, I_2, \ldots, I_n\}$. The height of $S_i$ is denoted by $x_i$, $i = 1, \ldots, m$, and the height of $I_j$ is denoted by $y_j$, $j = 1, \ldots, n$. Let:

$$\begin{cases} \mathbf{x} \equiv (x_1, x_2, \ldots, x_m) \\ \mathbf{y} \equiv (y_1, y_2, \ldots, y_n) \end{cases} \tag{12.10}$$

$Much\ taller$ is defined as a fuzzy relation on $\mathcal{H}_S^m \times \mathcal{H}_I^n$, in which $\mathcal{H}_S^m$ and $\mathcal{H}_I^n$ are respectively the spaces of all possible heights of $m$ Swedes and $n$ Italians, and the degree to which $S_i$ is much taller than $I_j$ is $h_{ij}$, $(i = 1, \ldots, m;\ j = 1, \ldots, n)$, i.e.:

$$h_{ij} \equiv \mu_{Much\ taller}(x_i, y_j) \tag{12.11}$$

The cardinality $c_i$ of the set of Italians in relation to whom a Swede $S_i$ is much taller can be calculated using the following $\Sigma$-count [139]:

$$c_i = \sum_{j=1}^{n} \mu_{Much\ taller}(x_i, y_j) = \sum_{j=1}^{n} h_{ij} \tag{12.12}$$

The proportion of Italians in relation to whom $S_i$ is much taller, $\rho_i$, is then (Step 4):

$$\rho_i \equiv \frac{c_i}{n} \tag{12.13}$$

Using a T1 FS model for the linguistic quantifier $Most$, the degree $u_i$ to which a Swede $S_i$, is much taller than most Italians, is (Step 5):

$$u_i = \mu_{Most}(\rho_i) \tag{12.14}$$

The proportion of the $m$ Swedes who are much taller than most Italians can be derived via the division of the $\Sigma$-count of those Swedes by $m$:

$$v = \frac{1}{m}\sum_{i=1}^{m} u_i \tag{12.15}$$

Consequently, the degree to which $v$ belongs to the linguistic quantifier $Most$ is determined by:

$$Q(\mathbf{x}, \mathbf{y}) = \mu_{Most}(v) \tag{12.16}$$

in which the fact that $v$ is a function of $\mathbf{x}$ and $\mathbf{y}$ is emphasized in the argument of $Q$.

The difference in average height of Swedes and the average height of Italians, $d$, is calculated as (Step 6):

$$d = \frac{1}{m}\sum_{i} x_i - \frac{1}{n}\sum_{j} y_j \tag{12.17}$$

To derive the linguistic constraint imposed on $d$ by (12.16), one exploits the GEP (Step 7). Zadeh's approach states that there is a soft constraint "Most" on $v$, the $\Sigma$-count of Swedes who are much taller than most Italians, given by (12.15), and requires the calculation of the soft constraint on $d$, given by (12.17). Therefore, in (12.2), $f(\mathbf{x}, \mathbf{y}) = v$ and:

$$f : \mathcal{H}_S^m \times \mathcal{H}_I^n \longrightarrow \mathbb{R} \tag{12.18}$$

Also, $g(\mathbf{x}, \mathbf{y}) = d = 1/m \sum_i x_i - 1/n \sum_j y_j$, and:

$$g : \mathcal{H}_S^m \times \mathcal{H}_I^n \longrightarrow \mathbb{R} \tag{12.19}$$

The GEP implies that the soft constraint $D$ on the difference in average heights, $d$, is characterized by the following membership function:

$$\mu_D(d) = \sup_{\substack{(\mathbf{x},\mathbf{y})\in\mathcal{H}_S^m\times\mathcal{H}_I^n \\ d=\frac{1}{m}\sum_i x_i-\frac{1}{n}\sum_j y_j}} \mu_{Most}\left(\frac{1}{m}\sum_{i=1}^{m}\mu_{Most}\left(\frac{1}{n}\sum_{j=1}^{n}\mu_{Much\ taller}(x_i, y_j)\right)\right)$$
$$= \sup_{\substack{(\mathbf{x},\mathbf{y})\in\mathcal{H}_S^m\times\mathcal{H}_I^n \\ d=\frac{1}{m}\sum_i x_i-\frac{1}{n}\sum_j y_j}} Q(\mathbf{x},\mathbf{y}) \qquad (12.20)$$

in which $(\mathbf{x}, \mathbf{y})$ belongs to $\mathcal{H}_S^m \times \mathcal{H}_I^n$, the space of all possible heights that Swedes and Italians can have. The sup is taken over this space since we have no information on the height distributions among these two nationalities.

Assuming that the MFs for $Most$ and $Much\ taller$ are determined in the process of precisiation of meaning, to solve (12.20), $\mathcal{H}_S^m$ and $\mathcal{H}_I^n$ have to be determined as the World Knowledge, i.e. the set of all possible heights of Swedes and Italians, respectively.

## 12.7 A Syllogistic Approach to ACWW Problems

We believe that there can be more than one way to solve an ACWW problem, because World Knowledge can be used in different ways. We have already seen that to use the GEP requires World Knowledge that is not stated explicitly for a problem, e.g. pdf of height of Swedes, Italians, etc. In this section, we explain a syllogistic approach to solving ACWW problems [82–84, 88, 89]. Those solutions use the Extension Principle in a different way, namely by applying it to the calculation of Novel Weighted Averages (NWAs)[10] [68, 86]. To begin, we need to translate the ACWW problems into a form suitable for NWAs.The following syllogism [151] is used for handling the probability words and linguistic quantifiers in the ACWW problems:

$$\frac{\text{The probability that } \chi \text{ is } A \text{ is } P}{\text{The probability that } \chi \text{ is } A' \text{ is } \neg P}$$

in which $A'$ is the complement of the fuzzy set $A$, characterized by:

[10]NWAs refer to Interval Weighted Averages (IWAs), Fuzzy Weighted Averages (FWAs), and Linguistic Weighted Averages (LWAs) [68].

$$\mu_{A'}(u) = 1 - \mu_A(u) \tag{12.21}$$

and $\neg P$ is the antonym of the fuzzy set $P$, whose membership function is given by[11] [13]:

$$\mu_{\neg P}(u) = \mu_P(1-u), \ u \in [0, 1] \tag{12.22}$$

Using the above syllogism, we create the following *linguistic belief structure* for each linguistic probability:

$$\mathscr{B}_P = \{(A, P), (A', \neg P)\} \tag{12.23}$$

in which $A$ and $A'$ are focal elements of $\mathscr{B}$ and $P$ and $\neg P$ are fuzzy probability mass assignments for $\mathscr{B}$. The difference between these belief structures and those that are studied in e.g., [55, 118, 128, 129] is that the probability mass assignments are now words rather than numeric values.

Belief structures with fuzzy-valued probability mass assignments were first introduced by Zadeh [138]; however, they have not been in the mainstream of research in the evidential reasoning community. In the past two decades, there has been some research on belief structures with interval-valued probability mass assignments [16, 26, 27, 72, 123]. As a natural extension of this, some studies formulate fuzzy-valued probability mass assignments [12, 17].

The belief structure represented in (12.23) can be used to infer an average [120] or a probability [118, 138]. In the following, we explain how the probability of an event as well as an expected value can be inferred from a belief structure with fuzzy focal elements ($A_i$'s) and fuzzy mass assignments ($M_i$'s).

If the probability of an event must be calculated, it is known [15] that only lower and upper probabilities can be infered from non-fuzzy belief structures. Assume that

$$\mathscr{B} = \{(A_1, m_1), \ldots, (A_n, m_n)\} \tag{12.24}$$

is a non-fuzzy belief structure, for which $A_i \subseteq U$ are focal elements, and $0 < m_i \leq 1$, are probability mass assignments that satisfy $\sum_i m_i = 1$. Such a belief structure can be seen as the generalization of a pdf to the case when probability masses are assigned to sets rather than to points. The lower and upper probabilities $P^-$ and $P^+$ of an event $B \subseteq U$ (which is not necessarily one of $A_i$'s) can be calculated as [15]:

$$\begin{cases} P^- = \sum_{i|A_i \subseteq B} m_i \\ P^+ = \sum_{i|A_i \cap B \neq \emptyset} m_i \end{cases} \tag{12.25}$$

The concepts of lower and upper probabilities can be extended to belief structures with fuzzy focal elements and fuzzy mass assignments. Assume that the following belief structure represents some knowledge about a variable $\chi$:

[11] There are other ways to define the antonym of a fuzzy set, which are less common [100].

$$\mathscr{B} = \{(A_1, M_1), (A_2, M_2), \ldots, (A_n, M_n)\} \tag{12.26}$$

where $A_i$ are fuzzy sets over $U$ and $M_i$ are fuzzy sets over the unit interval of probabilities [0, 1] (e.g. $\{(Tall, Probable), (not\ Tall, Improbable)\}$). In order to infer $\mathrm{LProb}^-(B)$ and $\mathrm{LProb}^+(B)$ [the lower and upper probability that $\chi$ is $B$ (e.g., John is short)] from $\mathscr{B}$, the following *normalized sums* [86], which are extensions of (12.25), have to be calculated:

$$\begin{cases} \mu_{\mathrm{LProb}^-(B)}(z) = \sup_{\substack{z=\sum_{i=1}^n m_i x_i \\ \sum_{i=1}^n m_i = 1}} \min\left(\mu_{M_1}(m_1), \mu_{M_2}(m_2), \ldots, \mu_{M_n}(m_n)\right) \\ \mu_{\mathrm{LProb}^+(B)}(z) = \sup_{\substack{z=\sum_{i=1}^n m_i y_i \\ \sum_{i=1}^n m_i = 1}} \min\left(\mu_{M_1}(m_1), \mu_{M_2}(m_2), \ldots, \mu_{M_n}(m_n)\right) \end{cases} \tag{12.27}$$

In (12.27) $x_i = \mathcal{I}(A_i, B)$ is a *measure of inclusion* of $A_i$ in $B$, or more generally, a *pessimistic compatibility measure*, and $y_i = \mathcal{O}(A_i, B)$ is a measure of overlap between $A_i$ and $B$, and more generally, an *optimistic compatibility measure*.

Given the statement "The probability that $\chi$ is $A$ is $P$", one can infer a belief structure like the one in (12.23), and then apply (12.27) to calculate the probability that $\chi$ is $B$. It was shown in [86, pp. 138–139] that the optimization problems in (12.27) may have no solutions and that one can instead use the following FWAs to calculate $\mathrm{LProb}^-(B)$ and $\mathrm{LProb}^+(B)$:

$$\begin{cases} \mu_{\mathrm{LProb}^-(B)}(z) = \sup_{z=\frac{\sum_{i=1}^n m_i x_i}{\sum_{j=1}^n m_j}} \min\left(\mu_{M_1}(m_1), \mu_{M_2}(m_2), \cdots, \mu_{M_n}(m_n)\right) \\ \mu_{\mathrm{LProb}^+(B)}(z) = \sup_{z=\frac{\sum_{i=1}^n m_i y_i}{\sum_{j=1}^n m_j}} \min\left(\mu_{M_1}(m_1), \mu_{M_2}(m_2), \cdots, \mu_{M_n}(m_n)\right) \end{cases} \tag{12.28}$$

When the average of $\chi$ has to be inferred from the belief structure $\mathscr{B}$ that describes it, one can calculate its *expected value* $\mathbb{E}\{\mathscr{B}\}$ as:

$$\mu_{\mathbb{E}\{\mathscr{B}\}}(z) = \sup_{\substack{z=\sum_{i=1}^n m_i a_i \\ \sum_{i=1}^n m_i = 1}} \min\begin{pmatrix} \mu_{M_1}(m_1), \mu_{M_2}(m_2), \cdots, \mu_{M_n}(m_n), \\ \mu_{A_1}(a_1), \mu_{A_2}(a_2), \cdots, \mu_{A_n}(a_n) \end{pmatrix} \tag{12.29}$$

If the information about a variable $\chi$ is represented in more than one linguistic belief structure, combination rules for belief structures [18, 22–24, 95, 96, 121, 128] have to be extended to linguistic belief structures. It was shown in [84] that a combination rule for linguistic belief structures can be devised using an aggregation operator called *Doubly Normalized Linguistic Weighted Average* (DNLWA).

To solve an ACWW problem involving linguistic probabilities, using the syllogistic approach, the following steps have to be performed:

1. Determine the soft constraints and identify all of the words that are used for linguistic probability, quantifiers, usuality, truth, and possibility.

2. Model all of the words in the problem using fuzzy sets.
3. Determine the linguistic probabilities and linguistic quantifiers in the problem.
4. Use a syllogism for each linguistic probability to derive a linguistic belief structure like the one in (12.23).
5. Inference:
   (a) To calculate the average of the variable that the belief structure describes, use (12.29) that is computed by applying the $\alpha$-cut decomposition theorem and KM algorithms [52].
   (b) To calculate the lower and upper probabilities of an event, choose the measures of inclusion and overlap and then use (12.28).
   (c) To combine several belief structures or to calculate a belief structure describing a function of variables described by several linguistic belief structures use DNLWA [84]. For inference from the combined linguistic belief structures, use (12.28) that is computed by applying the $\alpha$-cut decomposition theorem and KM algorithms [52].
6. Decode the solution back into a word (a numerical solution may also be provided). Similarity plays an important role in this step.

## 12.8 Advanced Computing with Words: Status, Challenges, and Future

Zadeh [152] gives three rationales for CWW. He begins first with the following three premises: (a) Words are less precise than numbers; (b) Precision carries a cost; and (c) Numbers are respected, words are not. He then proceeds with the following rationales:

(a) *Use words when numbers are not known or are too costly to obtain.* Use of words is a necessity.
(b) *Words are good enough.* Numbers are known but there is a tolerance for imprecision which can be exploited by employing words in place of numbers, aiming at a reduction in cost and achieving simplicity. Use of words is advantageous.
(c) *Linguistic summarization.* Words are used to summarize numerical information. Use of words is expedient.

These rationales provide a guideline to identify the current status and challenges of ACWW, as well as its future. In this section, we provide a constructive critique of Zadeh's rationales.

The first rationale can be interpreted as being about the cost of obtaining data; it might still be valid in some cases, but is somehow losing its importance with the recent advances in information technology resulting in the abundance of data [8, 56], especially with the conversion of the *Internet* into the source of data [33] for almost everything.

We believe that the phrase "Computing With Words" carries the connotation of "the way in which a person (or people) solves a problem." This connotation raises the following very interesting question: How would a person solve an ACWW problem? To the best of our knowledge no answers to this question have been published. We feel that obtaining answers to this question is very important because they will shed light not only on how a person obtains a solution to an ACWW problem but also on the statement of the problem itself. To do this, we propose that research be conducted jointly with psychologists.

In many of the prototype problems that Zadeh provides (see Table 12.1), his bits of expert knowledge or World Knowledge (e.g., "Most Swedes are tall", "Usually Robert leaves the office at about 5 p.m.", and "Usually, several cars are stolen daily at Berkeley") are given. These bits of knowledge often replace the knowledge about probability distributions of some variables (e.g., height of Swedes, the time that Robert leaves his office, and the number of cars that are stolen daily in Berkeley.). However, in these examples, all of those probability distributions can easily be obtained by collecting data, from data on the Internet, or are already available on the Internet. When ACWW problems were initially posed the Internet did not exist and so easy access to much World Knowledge did not exist. This has changed dramatically, and we feel it cannot be ignored. Ask someone a question today and if they don't know the answer they go to the Internet. Of course, one can argue that some of the information on the Internet is wrong, but this is changing and arguably does not deter a person from using the Internet.

The implications of using the huge amount of data available through the Internet are very profound on ACWW. Consider the Tall Swedes Problem as a simple example. Today it is possible to find the answer to the question "What is the average height of Swedes" on the Internet (e.g., [19]). The fact that "Most Swedes are tall" is no longer relevant to this question. Interestingly, the same may or may not be true for the engineering version of this problem (Table 12.2) because a particular company may not make the relevant data available on the Internet; but, that will not deter a person from looking for the answer on the Internet because for a person to answer such a question they need some sort of data, numerical or linguistic, e.g., to answer the question: "What is the average lifetime of an ipad" see [57] (there are many more sites that can also be used.)

In some problems whose intents are to implement everyday decision making using human knowledge, the posed questions can be answered without referring to the provided World Knowledge, by collecting data or performing observations. Examples of such problems are about risk, reliability, or lifetime of products for which it is safer to rely on collecting data about those variables than to rely solely on the knowledge of some experts. This argument does not totally refute the plausibility of using ACWW for assessing the reliability or lifetime of a product, because the lifetime of a product may be so long, or the product may be so expensive and rare that collecting data is not practical, in which case the best one may be able to do is to rely on expert knowledge. In that case, the credibility of the results may then be a matter of question.

It is true that Zadeh's problems are only prototypes of what can be addressed by ACWW, but given the advancements in data collection and information retrieval methodologies, we question whether one can justify using expert knowledge about probabilities instead of obtaining the exact or approximate probability distributions. We have already shown that more World Knowledge is needed than is stated in order to solve an ACWW problem, e.g. the kind of pdf that is associated with a specific problem. Having such probability distributions, the answer to some of the prototype problems reduces to calculating the probability of a (fuzzy) event e.g., "What is the probability that Robert is home before 6:15" can be answered by collecting data about Robert's arrival times. Today, one can easily find what the travel time is between two addresses during a particular hour on the Internet (e.g., on Google Maps), without the need for expert knowledge. Focusing again on the Tall Swedes Problem, in order to solve the optimization problem in (12.9) one needs the density function $p_H(h)$. It is not reasonable to use arbitrary density functions, because $p_H(h)$ should be tied to the distribution of the heights of Swedes. But are these Swedish men, women or both? More World Knowledge is needed. In fact, height distributions may be bimodal (see [93]). In any event, by going to the Internet (there are many sites about the heights Swedes) it is possible to arrive at a family of pdfs that can be used for $p_H(h)$. The same is true for the engineering version of this problem where one needs to choose pdfs that are commensurate with a reliability lifetime problem. The site [71] is a good starting point. This argument may leave us with fewer instances for which ACWW is applicable, e.g., prediction of the probabilities of a future event or judgments about the probability of an event about which data collection is extremely difficult or impossible (a conceivable example is a social, or economic or political process taking place in a very closed country about which little information is available except expert knowledge/judgment).

Not using data (numerical or linguistic) to solve an ACWW problem is, to us, analogous to providing an a priori probability distribution about something, i.e. it provides a starting answer but one that gets replaced by answers that are based on facts (a posteriori distributions). Hopefully, as one acquires enough data, the answers converge to a fixed answer. Solutions to ACWW problems should experience a similar behavior.

We therefore feel that the time is right for experimental work to be done to establish how people solve ACWW problems and to modify or eliminate the ones whose solutions either can be found entirely on the Internet or be found by collecting data very easily.

This discussion leads us to two more questions: (1) "What does one do with the answer to an ACWW problem?" (2) How does one validate the answer to an ACWW problem?

The second question may be easier to answer than the first one, i.e. we believe that the answer to an ACWW problem can only be validated by acquiring more data, because without data the answer is speculative, even if it has been obtained by using the GEP or syllogistic reasoning. Speculative answers may be okay, but they lead to the first question, which may sound facetious, but it is not.

Mendel [66] claims that a Turing-type test can be used to validate a solution to a CWW problem. In a Turing-type test, a human is required to provide an answer to the CWW problem; however, ACWW problems are so complex that a human might be unable to provide an answer. It is clear from our discussions in Sects. 12.5–12.7 that there are many steps required to solve an ACWW problem, after which it seems natural to us to ask: Is the solution "correct"? We know of no simple way to answer this question. In [87] we have suggested a relatively simple way to check whether or not the numerical solution to an ACWW problem may be correct. Our suggestion is to formulate a special version of the ACWW problem for which a human can provide the answer, and see if the numerical solution to that problem agrees with the human's answer. For example, "Probably John is Tall. What is the probability that John is tall?" Common sense says that the answer is "Probable," i.e., "It is probable that John is tall." An ACWW methodology must yield the answer "Probable" for this problem. It was shown in [87] that sometimes the discrepancy between the ranges of parameters of the family of distributions (which is World Knowledge) and the membership functions of the probability words leads to the failure of such a validation process. Further research is needed to address this issue.

One answer to the first question may be: "I don't really care, because what I am really interested in is learning how a machine can provide a solution to this problem." To us, this is where the present state of ACWW is (e.g., [87]). There is nothing wrong with this because by learning how a machine can provide a solution to an ACWW problem one may discover new things, and that's what research should be about.

Another answer to this question may be: "I need the answer to the ACWW question in order to make a decision." For example, if I am designing a home in Sweden I will need to know how tall a doorway should be; or, if Robert's wife has prepared a surprise birthday dinner party for him she will need to know around what time to start cooking so that the food is not over-cooked. So, the intended use of the answer to the ACWW question informs the solution to the question (feedback is present). The builder of the home in Sweden cannot accept the answer: "The average height of Swedes is pretty tall;" he needs numbers. Similarly, Robert's wife may not be able to accept the answer: "Robert will be home somewhat close to 6:15;" she also needs numbers. Of course, such numbers can also be provided to them, but this requires that the World Knowledge involving "pretty tall" and "somewhat close" be accurate enough. Also, it is still unclear why one needs to use the given World Knowledge when the answer to some questions can be found on the Internet.

Zadeh's second rationale, *Words are good enough*, also loses its importance when data can be collected about a fact. Collecting data about many real-life events is becoming cheaper and more accessible, leaving us with less instances for which relying on intervals and words is necessary; however, words may be good enough or even unavoidable for problems that deal with subjective issues, e.g. assessing the quality of an article submitted for publication, or the desirability of an option, or the possibility of something to occur. On the other hand, when data can be collected about an event and the probability of that event is considered, we question whether it is practical to adhere to this rationale and rely only on subjective judgment about the probability of that event.

Zadeh's third rationale, (the need for) *Linguistic summarization*, is valid almost always when something has to be reported to a human. No matter how knowledgeable or expert someone is, one needs to communicate their ideas to other people in natural language, which introduces the issue of imprecision of the words. This rationale suggests that the subject of linguistic summarization [39, 73, 116, 119] is an important topic for ICWW. How to formulate linguistic summarizations so that they are also a part of ACWW is also an important direction for future research.

We have already mentioned that the GEP is an essential aggregation tool for ACWW, especially when dealing with probability constraints. Nevertheless, analytic solutions for the optimization problem in the GEP are in general presently hopeless when probability constraints are involved, and existing numerical algorithms (e.g., $\alpha$-cut decomposition) are not directly applicable. In [87], some of Zadeh's challenge problems that involve linguistic probabilities are solved using a novel algorithm for implementing the GEP. A limitation of this algorithm is that it includes "exhaustive search" on the space of the parameters involved in the problem. Implementing the GEP without exhaustive search is another challenge for future research.

As stated in Sect. 12.7, we also believe that there is more than one way to solve ACWW problems, e.g., GEP, and syllogistic reasoning. Syllogistic reasoning does not need World Knowledge about probability distributions, but it needs World Knowledge about the domains of variables involved in the problems (e.g., time and height). It also uses the Extension Principle, but in a different way from the GEP. Moreover, it involves the challenge of choosing appropriate compatibility measures [82], and may require multiple computational methodologies for fusion of inconsistent information [84]. Much more research is needed about this approach to solving ACWW problems.

Because the GEP is the main tool for manipulating Z-numbers [2, 42, 43, 76, 126, 151], the connections between ACWW and computing with Z-numbers is also an interesting topic for future research.

The reader may not agree with many aspects of the discussions that we have just presented above; but that's okay, because we have made them to provoke new thinking about ACWW. We believe that the time is right for shaking the tree of ACWW to see what golden apples fall from it.

**Acknowledgments** The authors would like to sincerely thank Prof. Lotfi A. Zadeh for his continuing support and enthusiasm, and for explanation of his solutions to ACWW problems to us. Mohammad Reza Rajati would like to acknowledge the generous support of the Annenberg Fellowship Program and the Summer Research Institute Program of the University of Southern California.

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# Chapter 13
# Informal Meditation on Empiricism and Approximation in Fuzzy Logic and Set Theory: Descriptive Normativity, Formal Informality and Objective Subjectivity

**Jordi Cat**

**Abstract** The paper defends the view that the application and construction of mathematics may prove to be empiricist, subjective, approximative, contextual and normative. These elements are both inseparable and central to the possibility and success of mathematical practice. The cases of fuzzy set theory and fuzzy logic illustrate and support this account. In turn, the framework the account offers also brings out the particular ways in which the noted elements distinctively characterize these related fuzzy projects. Their future will benefit from understanding them more critically and creatively.

## 13.1 Application and Interpretation of Mathematics: Scientific Empiricism

What makes fuzzy set theory and logic distinctively empirical, normative, contextual, subjective and approximate? I will sketch an answer to this question by considering how the application of mathematics more generally exhibits the five dimensions as inseparable, complicated and with diversity of significance. I will draw attention to standards of empiricism, objectivity and precision and emphasize the role of normative and pragmatic elements and the variety of interpretations and uses at work. Unlike other scientific projects, especially in applied mathematics, the project of application and combination of fuzzy logic and set theory does not measure its formal and empirical success and progress in terms the complete elimination of subjectivity and approximation.

A result of this factual empirical approach is what might seem a puzzling juxtaposition of two general methodological consequences: an enhanced power of empirical representation (see [39]) and the radical incompleteness of objective and precise representation. In the case of logic, I will note the issues that arise in the context of empiricism with a comparison to discussions of quantum logic. The fuzzy project is

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_13

a typical instance of empirical inquiry, with a distinguishing feature: it must face the challenge to acknowledge and represent informality, contextuality, imprecision and subjectivity, while retaining a sufficient degree of scientific objectivity and precision.

These dimensions are the properties that explain how the fuzzy project can take on the challenges of exceeding the bounds of classical semantic monism. The future of the fuzzy project passes through the critical appreciation and exploitation of the particular aspects I will note of these dimensions. The discussion will enrich a general picture of these issues in scientific practice more broadly, especially as a picture of the application of mathematics.

Classical semantic monism may be characterized as follows: By classical semantic monism I mean the commitment to axioms of bivalence and excluded middle and the existence of a natural, plausible, correct, universal, exact and unique truth value assignment for predications about any given objects (in the most general sense of subjects of prediction, whether individual objects, events or states).

The most basic challenges[1] the so-called problems of artificial precision or vagueness (boundary nature) and semantic indeterminacy (boundary location).

- Problem of artificial precision or vagueness: failure of mechanisms of fixation of interpretation—meaning or designation—to establish the complete satisfaction of predication or the exact truth of corresponding propositions. No fact or indicator suffices to establish any such assignation. The assignation of ANY particular, unique, precise numerical truth value, even a number between 0 and 1, is considered the artificial, that is, incorrect, unnatural, implausible or counterintuitive. This difficulty concerns the nature of the boundary between predicates or corresponding truth values.

The first problem may be solved by providing a vague interpretation, and this is precisely what fuzzy models or degrees of truth do. They represent vague predicates with a subset of the extension modeled by a function from the subset of members to the interval [0, 1]. Each case can thus be assigned a single value of degree of membership and of truth.[2] The objective semantics of the assigned degree of truth cannot be identified with a measurement value or degree of a quality on a scale (see [41, p. 215]). It is about the degrees, the precise measures are the members of the fuzzy set, the extension of the vague predicate that categorizes them, e.g., the function that assigns degrees of warmth to precise temperature measures. Below I will raise the issues of reduction and the joint objective and subjective dimensions of fuzzy semantics.

- Problem of semantic indeterminacy, intended interpretation or plurivaluation: The facts, data or indicators acting as mechanisms of fixation of interpretation for a predicate or proposition is not unique, it is underdetermined. The problem affects both classical and vague semantics and concerns the location of the boundary between predicates or truth values.

[1] Here I follow [41], esp. Chap. 6.

[2] See [16, 26, 45], etc.

The second problem may be solved by introducing constraints on the set of interpretations a community will deem acceptable. Both problems apply to language and categorization and, by implication, to reasoning. The fuzzy solution to both problems and their particular significance are matters of empiricism, subjectivity, contextuality, normativity and approximation. But rather that doing so against a standard of scientific practice defined by precision, objectivity, unity and universality, I suggest that the opposite is the norm, and only within that alternative picture we can identify and individuate the distinguishing features of fuzzy theory.

So I turn, first, to a brief and more general discussion of this alternative picture and its dimensions.

## 13.2 Scientific Empiricism: Subjectivity, Contextuality and Normativity

I consider a naturalized sense of the empirical and factual in terms of attention to scientific and more ordinary practices as they are the case; these include practices of reliable production and acceptance of factual information. Empiricism cannot be held to a standard that enforces a reduction or else exclusive attention to some form of perceptual information as a source of meaning and warrant. Expositions since Zadeh's earliest texts on the subject have noted that fuzzy set theory and logic are empirically discovered, justified and applied formalisms. It's an empiricist project. It formalizes facts about formalizing practices; one of them, categorization or predication, is itself an empirical practice, the application of linguistic terms in the conceptualization of facts. Also, at the empirical root of the cognitive practices, as fuzzy theory describes them, lie the roles of context and subjectivity.

Empiricism as naturalistic belief about matters of fact, about the world and epistemic practices is often opposed to logical and normative claims. The history of empiricism is based on distinctions endorsed as philosophical dichotomies (exhaustive binary classifications): analytic/synthetic and fact/value (or convention).[3] The a priori analytic kind has been long associated with classical logic and even mathematics (Frege). While both formal sciences and factual knowledge alike have been long touted to be universally valid, it is the analytic domain of logic that has been the locus of necessity, even objectivity, without reference to any kind of transcendental ontology. The necessity of the formal claims, especially analytic a priori claims of logic, is often cashed out in terms of the role of the rules of inference (what follows necessarily), and, connectedly, in terms of two cognitive conditions: (1) analyticity

[3]Synthetic judgments as statements of fact, ampliative and describing and testable only matters of experience (not pure subjective phenomenological states alone); opposed to analytical ones, relations of ideas, explicative and decidable only by rules or conventions (for Frege mathematics was analytic and analyticity was ampliative and grounded on derivation from basic logical principles, not explicative or revealed by self-contradicting negations as on the Kantian view; for logical empiricists mathematics was both analytic and explicative).

as the formal inconsistency or else the unintelligibility or inconceivability of the negation of truths or propositions; and (2) aprioricity as the internal, formal conditions of constructability and revisability of the different claims and the system they form. The only external source is the standard of cognitive intelligibility, in this case internal to the mind, its commitments and constraints, including adopted social conventions. Factual claims and their systems rely on the external sources of the revision linked to empirical reality. Necessity and objectivity have, since Kant, relativized to context and convention at the expense of universality (and uniqueness). Also revisability, even by empirical standards. The rejection of the dichotomies is suggested by judgments or sentences that aren't true by either fact or convention alone.

Here are general tenets of what we might call sophisticated empiricism and naturalism ([37], Chap. 7):

- Knowledge of (particular) facts presupposes knowledge of theories (connections and generalizations).
- Knowledge of theories presupposes knowledge of (particular) facts.
- Knowledge of facts presupposes knowledge of values: objective and indispensable in justification.
- Knowledge of values presupposes knowledge of facts. Additional source of objectivity and factual relevance and applicability.

Empirical science is about *empirical objectivity*, not *transcendental objectivity*. Empirical objectivity can be *interpretive objectivity*, objectivity of products, or *methodological objectivity*, objectivity of process. Empiricism has long been linked to considerations of subjectivity and objectivity about the formation and evaluation of hypotheses. Equally, subjectivity and objectivity have long been thought without any assumptions of metaphysical realism, without transcendent subjects and objects and truth about forever hidden or ideal facts and entities.

One step back from transcendent realism lies empirical objectivity, the objectivity of empirical objects as described within a relevant scientific theory, formalism or conceptual scheme. Objective truth will be the corresponding truth about them. This is the semantic objectivity of truth values and what they represent (not just what they are). This is objectivity of interpretation, or product. It is different, but not independent, from the objectivity of process, the methodological objectivity of formation and evaluation (and, a fortiori, of final endorsement or acceptance). In the case of mathematics, the structural objectivity of product relies on the methodological objectivity of rules and conditions involved in construction, calculation and proof. To each sense of objectivity corresponds a different but also related notion of subjectivity.

The variety of operative notions of objectivity suggests that objectivity is not just a feature of objects. It's about objects, objectives and objecting. Objectivity and other so-called epistemic considerations have become normative ideals (norms or prescriptions range from values to standards and conventions); and it is a condition of objective representations, without transcendent facts, that they be recognizable or (cognitively) accessible. Objectivity, even truth, are not always modes of description, more generally, they stand at least for forms of warranted assertability and deniability, and warranted appraisability or criticism (see [37], Chap. 6). To a weaker sense of objectivity corresponds an equally weaker sense of truth. Truth, in this sense, is an expression of commitment, both to the belief in fact that answers to true sentences, but to the system of language and categorization that make them possible and the evaluation criteria and procedures. Again, this sense of truth, and realism, extends to mathematical statements. For the case of formal conditions, abstract hypotheses and general principles, I want to add the distinction between the warrant of *construction* and the warrant of *application*. Two corresponding considerations of objectivity follow. For the weaker notion of truth to apply, standards of warrant and criticism had better be in place. This picture of empirical research endorses fallibilism and revisability, not skepticism: it seeks warrant for belief and doubt, certainty and uncertainty.

Judgments of fact and of value may be taken to be both relative and objective.[4] Schemes of categorization and classification demarcate facts through considerations of values (norms and standards) for instance prioritizing as more relevant and significant particular kinds of similarities over others. These systems are true and accurate to the extent that they are successfully calibrated against preferred known instances adopted as standards. It is not a requirement (as fuzzy-set schemes show) that the categories in a scheme have sharp boundaries and that their application be always applicable without indeterminacy. They must be also adequate, or right, to the extent that they can contribute to realize our objectives (ibid., p. 191). The informing values express or serve the interests that identify and motivate the particular kind of inquiry, including epistemic access to the instances (see also [5]). In the application of concepts and hypotheses, values and standards are inevitable (see [29]). Matters of fact are mainly theory-informed valuable phenomena; empiricism is not just about valuable (also constructed) data (see [4]).

Facts then are relative to value, because truth, conformation and objectivity of their representations are too. This is, I think, the sense in which answering questions and settling matters of application of conceptual schemes or language games require evaluation, not just non-cognitive judgments and their non-cognitive application of standards of correspondence and agreement, whether by model-theoretic correspondence, isomorphism or reliable causal connection; and this make truth—and objectivity—a normative notion (see [37, pp. 77–78]).

Objectivity is then secured by the criteria and techniques of evaluation adopted for the formulation and application by every system. And for empirical objectivity and objectivity of empirical application, this is objectivity enough. When one disregards the ontological sense of objectivity in terms of transcendental truth and inaccessible objects, the next best thing is the achievement of intersubjectivity—or, more precisely, cross-subjectivity: Generalizable independence from any personal or individual conditions concerning the constitution or choice of a concept or judgment.

[4]Elgin makes the case: "factual judgments are not objective unless value judgments are; and value judgments are not relative unless factual judgments are," in [17, p. 176].

Like subjectivity, objectivity without objects rests on a negative characterization; often relative to each other, as the denial of one or more associated more specific indicators.

The two related forms of the intersubjectivity, without reference to transcendental reality, are mechanical and formal modes of methodological and interpretive objectivity. Both involve the reliance on relations or operations. The mechanical criterion is based on the ethos of operations or manipulations, material or formal, that does not depend on exclusive individual contributions in their execution and testing. It is an independence criterion. Significantly, it applies to the outcomes of technological instruments and it includes pictorial representations (see [13]). The more purely formal one, so-called structural objectivity, depends on the application of a structure or set of formal relations established in terms of the conceptual manipulation of symbols.

Formal objectivity is not automatically empirical objectivity, the symbolic, formal or conceptual determination of empirical objects. Conceptual determinations—precise situations on conceptual maps—of empirical data that categorize, unify and distinguish empirical objects and, more generally, any form of objective empirical knowledge on which it is based rely on this notion of objectivity—or objective empirical reality. Since Kant, a long tradition of thought has grappled with the doctrine that the logical objectivity or constitution of concepts and the experience of particulars under those conceptual representations are understood as generally valid and unifying coherent configurations, structures of relations in or between judgments, according to "universal" functions, laws or rules.[5]

The constitutive rules may be internal conventions or definitions determining a formal concept, as in formal precision or in formal fuzziness as its negation; or in relation to the semantic rules of empirical application of the formal concept to particular cases. The problem of empirical application that grants empirical objectivity to the prior formal objectivity of autonomous mathematical structures is the *problem of coordination*. And this involves a problem of construction and selection of application conditions.

The question for the case of fuzziness is whether the objectivity of empirical representation requires precision and universality or else it can make do with contextuality and approximation. Formal characterizations of fuzziness suggest that representations of approximation may be objective, without their objectivity itself being an approximate matter of degree, unlike truth. This formal notion makes compatible formal precision with an objective epistemic interpretation (the normative rational basis of probability and decision theory).

From the perspective of a particular scheme or practice of inquiry, some such values may be deemed unwanted biases, brought out in a broader context of social diversity alongside alternative biases. This scenario is the main motivation behind

[5]Ernst Cassirer, Rudolf Carnap, Moritz Schlick and Hans Reichenbach were early 20th-century champions of accounts of knowledge and objectivity by means of formal conceptual determination by mathematical structures of relations and, especially Cassirer, invariants.

Longino's critical and social account of objectivity in terms of the recognition of the irreducible diversity of scientific and social values in a community.[6]

But in the account of empirical knowledge I am presenting not any concept or system is constructible (formal objectivity), applicable (empirical objectivity) or worth applying (empirical objective). While pluralism and relativism are valuable, they are so also to the extent that they are limited in range. The role of facts in motivating, challenging or bearing out a scheme, concept or value help establish the objectivity of the adopted values. Values may me assessed more generally in terms of objectives, that is, of their roles defining or implementing standards of warrant and reasonableness.[7] Objectivity itself, in its different forms, is a value in the same sense.

Other objectives constrain and inform scientific the production and assessment of representations giving way to the empirical practice of modeling, data and phenomena. Distinctive of modeling is its reliance on idealization, abstraction and simplification. In an general way, modeling is driven by simplification from the set of possible variables, parameters of values characterizing an assumption of different and more complex features, states and processes. Representation, in that sense, of phenomena has both a positive and negative dimensions and values, two faces of the same coin. The negative aspect of simplification is a commitment to a particular criterion of economy, a kind of compromise in form of qualitative approximation with, in mathematical cases, consequences in the form of quantitative approximation. The positive dimension on which the negative depends is guided by a criterion of relevance, interest and value: it expresses and is guided by a commitment to a given interest in selected variables or parameter or their values. The interest is relevant to goals of research and standards of intelligibility, explanation, prediction, promise, etc. The virtues, norms, conventions in place define a community that by upholding them also takes them for granted. They defend and define, for instance, objectivity.

Such a dependence of empirical modeling on specific conditions constitutes not just a form of normativity and approximation; it is also a form of relativity or contextuality. The contextual dimension is a double rejection of universality: it depends, in a way both principled and pragmatic, on limited resources and limited domain of applicability. It is expressive of a limited set of specific commitments to and application of specific standards of construction and evaluation. Moreover, the empirical application of models depend on the strings attached, on assumptions that specify in more or less detail the limited conditions of their relevance and validity. Finally, as a result of the precision of the embedded limits, many models involve theoretical inconsistencies in the assumptions involved in their construction or application.[8] Neither precision, universality or consistency constitute a reasonable standard for the construction, use, objectivity and credibility of models and generalizations in actual scientific practice. Objectivity, contextuality and imprecision—in theoretical and experimental approximation—are not incompatible. Rather than undermining

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[6] [29] An over-idealized alternative is negative talk of neutrality; see [15].

[7] See Putnam's pragmatist position, above.

[8] On the limits of modeling and their approximative and inconsistent application see [7].

modeling as empirical practice, normativity, limited objectivity, contextuality and approximation characterize and enable it.

Semantics has its place in this picture; some connecting remarks will help apply dimensions and issues I am raising with the picture to the case fuzzy theory. It's a widespread doctrine that logic and a set theory have in their core an algebra of functions, or operations, on members of ordered sets of truth values. The set-theoretic operations correspond to logical relations or connectives. A semantic interpretation of a language involves a model domain and a function that assigns values from a set of truth values to symbols for propositional constants and objects to symbols for individual constants. Interpretations will map sets of symbols onto sets of valuations and truth values, discrete and in degrees, and sets of objects and their properties satisfying them.

The classical semantic picture is based on an algebra with a two-element set of truth values and three connectives for operations/functions, union/disjunction, intersection/conjunction and complementation/negation. On this picture, vagueness and vague reasoning are epistemic matters of ignorance about the truth values about the assignation of predicates/properties to objects. Truth values get their objective interpretation from the properties they represent or "measure" in the objects satisfying the true or false, interpreted propositions.

Classical semantics, as a theory of how language relates to the world, also rests on metaphysical assumptions. One may speak of semantic realism, then. The relation between our terms and propositions and the world is one of reference and truth. Typically, one relation is considered primary, whether reference (reference-first semantics) or truth (truth-first semantics) (see [41, p. 46]). The ontology of classical models is based on the distinction between objects and properties, and the assumption that objects hold properties or relations precisely, univocally; either objects possess a property or they don't. We may call this view, correspondingly, classical metaphysics. The interpretation of language they provide is ideally unique and it supports the application of the classical logical principles of bivalence and excluded middle. One may speak in this case of classical semantic realism.

Truth in this combination of classical semantic and metaphysical pictures leaves little room for vagueness. But it may form a class of equivalent situations cognitively indistinguishable for us and, as a consequence, may determine vague meaning as a matter of use. This is a typical epistemic interpretation of vagueness.

The version I have outlined of the classical semantic picture might be weakened through two additional features: (1) partiality of scope of language to which it applies; (2) variability or contextuality of truth value or meaning for the same linguistic units and (3) plurality (in any context), so-called plurivaluation.

The radical alternative is the rejection of semantic realism and to deny the reality of either truth or reference (or both). As in the interpretation of theoretical elements in science, there is a corresponding semantic antirealist attitude that any notions beyond linguistic practice is instrumental or pragmatic, not intending any claim about the existence of anything else in the world and its properties. Different particular rejections lead to corresponding pictures often labeled as anti-representationalist. Realism emphasizes model theory, partly for methodological reasons, as a matter of clarity and precision; although the use of model theory doesn't necessarily indicate realism. I want to suggest that in general these realist-antirealist distinctions do not map neatly onto any distinction between objectivity and subjectivity, but rather often involve distinctions between different kinds of objectivity, empirical (realistic) and formal (structural, methodological or mechanical).

One such alternatives is a proof-theoretic approach without any reliance on semantic models. That is, on matters of reasoning, the approach adopts the assumption that vague predicates are undecidable and statements are accepted as true only by the practical success of getting away with asserting them. Proof-theoretic approaches are instances of a mechanical or formal form of objectivity.

My interest in the objectivity and subjectivity in empirical research concedes the role of mixed semantics. This is part of the picture I sketch here of fuzzy theory, in language and reasoning; and I consider non-semantic aspects too. But there is more than one picture of mixed semantics. The objectivist pictures come in different kinds, some more or less realists, some not at all. Moreover, there are mixed kinds where objective and subjective conditions entwine. The relevant kind of subjectivity is the subjectivity of interpretation.

Vagueness in language and reasoning can be modeled within the objectivity of semantic realism in terms of non-classical semantics, of indeterminacy. And this may rest on a picture of the world language relates to according to the semantic picture expressed by non-classical models or metaphysics. Fuzzy theory is a theory of vague but decidable predicates and reasoning, whose semantics rests on a fuzzy metaphysics: It is a collection of models of intrinsically vague properties—not vague objects—in the world, properties objects can possess in intermediate degrees [41, p. 122].

*The picture of empiricism will not be complete without taking seriously the place of subjectivity and approximation. Subjectivity and approximation are not just the absence of objectivity and precision as the sole active factors in empirical research. As I noted for the case of modeling practices, limitations in objectivity, universality and precision are enabling factors. Subjectivity too is part of the picture.*

In order to locate the place and role of subjectivity side by side objectivity, I distinguish three related views of subjectivity: *onto-psychological*, *interpretive* and *functional-methodological* and *volitional*. They make a residual appearance and play a residual role in scientific practice according to some of its characterizations, such as fuzzy set theory.

The first kind include such negative notions of the subjective are ontological and psychological.

- Ontological subjectivity without functional methodological or interpretive role: one may point to two capacities of the I. One is an active capacity, the other is a phenomenological one. The first has been identified as the site of will and mental spontaneity (intentional, emotional, intellectual, etc.). The second is perspectival, the center of perspectival self-consciousness, with self-attribution, self-knowledge. This is the domain of the mind as the domain of the irreducibly internal and private.[9]

- Psychological subjectivity with a methodological cost: subjectivity understood as condition of individual attachment (subjection) in the form of either impulse or rigidity, cognitive or emotional. The form of the negative value of subjectivity is associated with both aspects, the form of the attachment and the personal individuality. The attachment, whether as rigidity or arbitrariness, without consistency, is considered to undermine the possibility of objectivity in the empirical realist sense of truth to systems considered real precisely in terms of the sufficient independence of their representation and empirical reality from attachments (a residual level of relativity may be formal or material, as all our concepts or representations are ultimately forms of relativity, as are material connections, especially those whose empirical status is linked to cognitive access). At the same time, attachments are considered obstacles to objectivity in so far as they are indexed to a single and exclusive personal or particular individual source (as the extreme form of cognitive contextuality, regardless of additional considerations of privacy). Attachments may be found equally unacceptable insofar as their nature, individual or shared, are not amenable to revision or justification to any others. In this picture, subjectivity is the enemy, not the source of the normativity of reasons and reasoning that warrants information or representation.

For instance, the objectivity of expert judgment or analysis has been assessed in relation to the normative cognitive standard of rationality rather than a realist, factual norm (see [11]). For Cooke, expert opinion involves subjective assessment or analysis, e.g., of risk and uncertainty. The interpretation remains subjective, and the form of the subjectivity of interpretation is uncertainty in degree of belief. Instead, the methodological criterion of objectivity is formed by the conditions of intersubjective, rational consensus. This becomes, next, a problem of introducing conditions of aggregativity of expert opinion. Cooke's a set of minimal methodological guidelines: reproducibility (an independence condition), accountability (ability to track personal or institutional sources of expert subjective probabilities), empirical control (conditions of revisability), neutrality (true opinion), fairness (assumption of equal reliability of all experts). From this standpoint, Cooke claims that fuzzy methodology lacks a systematic criterion for revising uncertainty measures in the light of new information and a normative standard of evaluation of subjective opinion.

This notion of subjectivity connects with the broad issue of built-in or acquired individual biases and constraints, whether subpersonal—unconscious—or else allowed as internal representations and beliefs acting as means or guiding goals. In the formal and mechanical senses, one might distinguish between rule-following and rule-governing objectivity. One involves conscious self-represented symbolic rules followed to achieve, for instance, our cognitive states and performances; the other, which we might call *embedded objectivity*, involves functional forms of rigidity acquired or developed in interaction with our environment, such as automatic

[9] It is, in Roger Scruton's words, "the organizing principle of first-person awareness"; [42].

operations, habits or routines causally constraining or determining those processes and performances. These might be unstable, modifiable and contextual.

The embedded domain is a murky domain of the implicit or tacit. Objectivity and subjectivity can be found here, so do the factual and the normative: Objectivity of the factual, without normativity, and normative subjectivity. Do we consider embedded, subpersonal objectivity objective in the mechanical sense if we identify a stable and reliable, even if black-boxed, mechanisms? This is the functional objectivity of an instrument or technique relative to an end. It is a condition of the mechanical objectivity of procedures and representations. Computations carried out by machines may not involve explicit self-representation or self-monitoring. But such causal processes involve known, explicitly representable routines; rule-governing may embody in material and causal forms, is mappable onto, endorsed forms of rule-following, rules in reliable relation to certain commitments, norms, standards or desired goals. This is the context for addressing the ethical dimension of the behavior autonomous machines such as robots. *Embedded normativity* is implemented, vicarious normativity; at least some of it. In science, a naturalistic attitude will endorse habits and norms pragmatically in light of accepted empirical and formal results and methods.

Not everyone will agree that norms are explicit rules, rather than embedded and operative in implicit forms. Embedded objectivity might lack the normative force that justifies our judgments except in terms of functional reliability and independence from any individual cognitive subject. Collective subjectivity, supervening on individual dependence, is a spurious form of consensus or uniformity. Collectivity is compatible with subjectivity understood as form of individualism. Objectivity requires more than accidental or contextual consensus, like causality requires more than correlation. Embedded objectivity may be an individual mechanism without being individualistic or personal, subjective. It is the mystery of embedded normativity. Tacit knowledge depends on reliably acquired skills, habits and values that help perform tasks (play the piano, riding a bike, etc.) and, more abstractly and socially, train in and conduct scientific practice, as Michael Polanyi and Thomas Kuhn noted, without being reduced to the explicitly articulated content and rules.

Indeed, objectivity in the normative explicit sense may itself be relative and contextual, ex., relativity to conditions and constraints, even to those adopted only within a particular community in evolving sciences, such as sets of norms in methods and categorization schemes, as used in modeling, etc. Modeling is contextualization and condition-dependence. As forms of approximation to the more complex and the more general, relativity and contextuality alone do not entail subjectivity.

The overlapping functional notion of subjectivity, we may call it *methodological subjectivity*, emphasizes the explanatory and pragmatic functions of subjectivity within the negative limits just noted in the negative, normative, inter-subjective characterization. It appears as part of the characterization of the formal project. One positive role may be accommodated, like that of values, in terms of attributions of a function of tie breaking, but only after the relevance and value of alternatives has been otherwise established, that is, objectively; just like considerations of values are often considered as tie breakers in the face of empirically equivalent hypotheses, even if their role is far more diverse, fundamental and pervasive.

There is another functional sense in which one may speak of subjectivity, *volitional subjectivity*. It is a precondition for the application of taste, the execution of arbitrary preference, the adoption of convention and judgment, commitment, criteria and values that will motivate or warrant other judgments. Another role is more productive or constitutive. Beyond the theoretical and technological complexity and construction of descriptions, the mechanical ideal of objectivity has become supplemented with the practice of subjective expert judgment ([11, 13]). This is a mixed case.

What's the relation between subjectivity and expertise? Is tacit knowledge, rooted in training, practice and skill, a distinctive source of subjective judgment? Here lies the elusive difference between the cognitive role of *intuition* as a kind of embedded rigidity at the root of decision-making, and a *heuristic*, an objective representation and transferability of a constraint that is cognitively informed and with methodological value. Intuition is subjective insofar it lacks public representation and transferability. Yet it may be developed and adapted contextually through its exercise in repeated interaction with the environment and in the light of a record of feedback, uncertainty and external goals (in science, sports, crafts, policy, design, management, etc.). Intuition has embedded objective expression as an exercise of instrumental *contextual rationality* relative to cognitive and practical goals in situations of uncertainty. This is the performative and embodied nature of expertise or skill. The self-represented version with intersubjective expression is a heuristic normatively justified as an expression of objectivity derived from rationality, in this case known as *ecological rationality*.[10]

Subjectivity, without any mystifying ontological or cognitive meaning, may be considered as a negative methodological place, the eye of the hurricane of public practice. Its methodological value derives from its volitional expression and its boundary representation and control by means of external objective constraints, embedded or public. By negativity I mean both the uncertainty and the resistance to formal objectification in theory or method of the volitional necessary in method and practice for the purpose of adopting, selecting, deciding or accepting (besides elements of symbolic recognition or self-representation). It may have the effect of providing a rigid obstacle to the pursuit of goals but also a reliable source of action, cognitive and practical. This negativity, not just as activity, may be considered a form of valuable source of wiggle room or plasticity, whose valuable function may involve both creativity and reliable sensitivity to varying contexts, situations, goals.

Unsurprisingly, the most relevant and valuable applications and developments of fuzzy set theory have focused on process and projects featuring explicitly a role for the human factor, namely, cognitive and volitional exercises of subjectivity such as expert judgment, control and decision making, and, as a consequence, systems dependent on human elements of judgment, perception, etc., that Zadeh has called humanistic systems. As I have noted before, the role of judgment in scientific expertise, for

[10] See for instance [21].

instance, in risk evaluation, is perceived in need of constraining by the systematic application of objectifying criteria [11].

A third, related kind of subjectivity is what I have called *interpretive subjectivity*. It expresses individual psychological, cognitive or epistemic states providing the basis for a corresponding interpretation of formalism, ex. probabilities as measures of subjective or individual epistemic states of uncertainty, lack confidence or ignorance, especially as the theory first developed in the context of gambling and errors in astronomical observations. The formal articulation suggests a formalized subjectivity that may be reduced away to either an empirical psychological model or a normative model of rationality of belief (ex. rational betting strategies). The situation should recall measures of individual utility value and their role in rational decision-making. Uncertainty has come to be distinguished from (known) objective risk and associated, instead, with unknown factors. As a result this negative aspect suggests a mixed kind of semantics.

Fuzzy logic and fuzzy-set semantics, like decision theory, depend theoretically on the formal and quantitative modeling of subjective states. Formal subjectivity has received formal objective representation as a measure of uncertainty; and it has been given relabeled in terms of possibility, with formal properties different from the ones characterizing probabilities, a form of epistemic modality situated between logical and physical possibility, combining the psychological and the normative. From the normative methodological standpoint, interpretive subjectivity might pose a conflict with methodological and interpretive, semantic objectivity: it is the challenge of its apparent indistinguishability from error.

*The different roles of subjectivity in scientific practice may be considered effective, acceptable or irreducibly detrimental, depending on how each role is understood to serve prior commitments or values.*

This conclusion points to a valuable tension within the fuzzy set theory and logic projects: between (1) the scientific project qua scientific, guided by the methodological desideratum, among others, of objectivity (in terms of more specific criteria) and (2) the scientific project as empiricist or factual, aiming to extend the formal concepts in set theory and logic on the grounds aiming to represent actual modalities of human reasoning and categorization which include a role for subjectivity—the empiricist project yields the added methodological benefit of extending mathematical theory and, with it, empowering the formal methods in empirical sciences and technology.

Similar remarks can be made, on similar grounds, about approximation. The fuzzy project, one might argue, aims both to represent and to reduce, if not eliminate, approximation. However, the implicit methodological tension is dispelled by jettisoning the commitment to precision. There is no inconsistency in accepting a certain degree or kind of approximation in the representation of approximation.

*My claim is that the fuzzy project includes an implicit but valuable methodological tension between representing and eliminating subjectivity. We can ask, then, the following question: under what conditions can logic and set theory be empirical and, as a consequence, also subjective? Implicit in the project is the possibility to distinguish between the descriptive and the methodological contexts, and roles, of subjectivity. In the methodological context, as part of the application of fuzzy* *formalism as a formal instrument of representation and calculation, it adopts a more negative and residual expression, more constrained and also more precise, carved smaller on the procrustean altar to objectivity. In both contexts, the reference to a residual presence of subjectivity represents also a factual and methodological condition of application of these formalisms; that is, to the extent that they are claimed to be, at least partly, discovered, justified and applied on factual grounds. It's a valuable resource besides adequate factual information.*

*Empiricism, approximation and subjectivity are inseparable: the approximative character of fuzziness derives from the description and prescription of actual practices that are based on subjective conditions and actual epistemic interpretations. The project requires, by its own standards, the connected tasks describing the normative, formalizing the informal and objectifying the subjective, while eliminating neither. Above, I have noted the implicit methodological tension between representing and eliminating subjectivity. As in the case of vagueness, we have here a project not of elimination of the personal or subjective, but of its empirical and theoretical acknowledgment within the formalism and theory (and not the exclusion and elimination from it found in classical logic and set theory), capturing its objective situation within the objective constraints, and identifying its explanatory role in categorization, reasoning and intervention. I suggest that representation and understanding of subjectivity, contextuality and approximation may be not just factual and ineliminable features of description, but also a useful resource in explanation and application.*

## 13.3 From Empiricism to Approximation

*Empiricism is approximation. As I noted above, subjectivity and approximation are not just the absence of objectivity and precision as the sole active factors in empirical research. In modeling practices, limitations in objectivity, universality and precision are enabling factors. Approximation too is an enabling part of the empiricist picture.*

Can representation and explanation be satisfying, can objectivity be objectivity enough when they are all generally a matter of degree? Where does that leave references to the residual but ineliminable role of subjectivity? Can residual subjectivity, especially in its methodological role, be given any degree of epistemic value, for instance, degrees of fallibility and falsifiability that would establish a negative measure of approximation with testing significance? Can there be any significant difference between degree of membership, even understood subjectively as uncertainty, and degree of error? What issues, aims or means of research rest on considerations of approximation?

Part of fuzzy theory is the assumption of the formal idea of logical approximation and that it rests on semantic idea of set-theoretic approximation. To approximate meets the standard of success in empirical representation: fuzzy sets are told to "bear an approximate relation to the primary data" [46] and "fuzzy sets and fuzzy set operations are also employed, in general, as approximators of meanings of relevant linguistic terms in given contexts" [26, p. 281]. Therefore, determining which "operations on fuzzy sets best represents the intended operations" is equivalent to determining which "operations on fuzzy sets beat approximate the intended meaning" (ibid. pp. 280 and 281). On the same grounds, fuzzy logic provides "a model for approximate rather than precise reasoning" [3]. Zadeh elaborates: "Informally, by approximate or, equivalently, fuzzy reasoning we mean the process or processes by which a possibly imprecise conclusion is deduced from a collection of imprecise premises" [47].

*Consideration of approximation raises questions about the notion of fuzziness, its roles and significance, and similarly for (naturalistic) empiricism. In turn, considerations of fuzziness and empiricism raise questions about, and sheds light on, the notion of approximation, its complexity, roles and significance.* From the objective standpoint of realist semantics, one recent characterization of vagueness has the form familiar from fuzzy theory, in terms of degrees of truth. The core conception is a closeness condition: a predicate $F$ is vague just in case it for objects $x$ and $y$ it satisfies the condition that, if $x$ and $y$ are very close in $F$-relevant respects (not in respect of $F$!), the predications "$F_a$" and "$F_b$" are very close in respect to truth [41, p. 147].

There is still more to understanding fuzziness and fuzzy theory than this semantic picture. Judgments of closeness do not involve just an order relation, or a topology, but a criterion of distance, a metric [41, p. 149]. It's then a kind of approximation relation. The relation of being very close can be applied empirically only in relation to a fixed choice of standard, which should be prescriptive or at worst pragmatic, but it will be at best contextual or relative. The choice's objectivity might have different sources and forms, naturalized—say, the accuracy or power of discrimination of someone's cognitive capacity—constrained by the mechanics of fixed procedures, numerical form or motivating reasoning; or else subjective.

The truth-closeness part of the condition motivates introducing a role for numerical truth degrees. They can be found in fuzzy model theory, but then, for any two numerical values, establishing closeness will require also precise numerical standards. Their role is the semantic counterpart to that of standards of significance in the empirical application of statistics, another formal calculus, or an empirical standard in the application of a theory. The general dimensions I wish to introduce in the discussion of fuzzy theory start becoming relevant: Empiricism, contextuality, normativity, subjectivity and approximation.

It is in the context of empirical justification and application of formal structures that dimensions and issues concerning the significance of approximation arise. Then they can be compared to ones identified in the construction and application of non-fuzzy and fuzzy mathematical and logical formalisms.

Subjectivity as well as objectivity, in their transcendental metaphysical senses, suggest a qualitative virtual sense of approximation, towards a virtual focus, the real, something indeterminate existing outside the symbolic map, external to the formal framework and language of the theory and the representation of the data, phenomena or system, something off-metric.

The *metaphysical virtual approximation* must be contrasted with two other cases. The contrast helps bring out some of the more relevant dimensions of approximation.

Virtual approximation differs from the case of an *asymptotic approximation*, where the limit lies within the boundary of the formal metrical structure. It differs also from the *empirical approximation* that is part of the problem of coordination through measurement of the formal structure of "theory" and the structure of empirical data, the latter establishes and determines conceptually and mathematically—quantitatively or graphically—the empirical system as empirical; the coordination adds the theoretical determination (with predictive, unifying, explanatory or other value).

Empirical approximation involves two notions of approximation associated with the idea of *exactness*. The mathematical structure of the theoretical provides the basis for the internal notion of *precision*. The theoretical here extends to the representation of instruments of detection or measurement. The structure involved in the representation of data and its coordination with the model or theory is the basis for the empirical, 'external', notion of *accuracy*. The structure that enables the modeling of data is the context also of talk of *margin of error*.

Approximation is a measure of closeness and is, then, clearly a relation; but it is, like a degree of truth, a conceptual relation—i.e., degree—not even a semantic relation such as truth value *simpliciter*, between the conceptual and the world, index and indexed, sign and signified. The relation of approximation is in turn, relative to a relation of identity. The formal character of approximation gives it the sort of conceptual precision or determinateness that establishes judgments of objectivity also understood in formal, rule-based terms.

The issue becomes whether truth value can be in itself an extreme form of approximation only to facts under particular representations. Even introducing a formal structure to talk of degrees of truth might not render the semantic relation a matter of approximation. Even the notion of approximation to full truth doesn't make truth itself a matter of approximation. The notion of approximate truth, and degrees of truth, is intelligible only in terms of truth of formal approximation, within the mathematical structure in which the proposition is embedded (precision); otherwise, in relation to truth of empirical approximation, in terms of its corresponding structure.

Like approximation, also normativity and contextuality are forms of relativity, namely, to standards, conventions, goals, limits, etc. They act as formal conditions of construction and application—as connection or restriction. They apply to relations of approximation in this role. No notion of approximation relation is meaningful outside the framework fixed by the constraints from normative and constructive considerations, formal, epistemic and practical. They work as qualitative conditions of approximation insofar as they are effective conditions of any possible determination. The further moments of definition and application will depend on the so-called subjective act of fixation of standards, judgments of similarity, etc. reflecting the absence of an ideal absolute metric and its precise mechanical application.

*The normative dimension of approximation resides not only in assessing and defending degrees of approximation always within the negative perspective of failure to achieve precision. In fact, the values of approximation often express different forms of preferability to precision as a universal absolute value and goal or standard. In many contexts, relative to our effective goals, approximation is valuable precisely at the expense of the possibility of precision. Projects of cognition or intervention are* *determined by reference to more than one standard, aim or value; within the plural set, they often form trade-offs and demand compromises. From that perspective, precision presents different kinds of costs that different kinds of approximations can elude. Curve-fitting is a notorious example; the exactness of fit by a complex function often undermines not just auxiliary values or constraints such as tractability, but also predictive power. Description—or explanation—and prediction cannot be simultaneously maximized. Completeness of detail in mapping may also conflicts with applicability. We cannot handle the whole truth. In the cognitive world of optimization and compromise, the value of approximation trumps the ideal of (absolute) precision. The value of precision is a matter of context and degree.*

## 13.4 Approximation to Approximations: Kinds, Uses, Constraints and Significance

In this section I aim to sketch out and motivate a picture of approximation as concept, value and practice that is general, multidimensional and contextual. Approximation is a matter of construction, interpretation and application of formal structures. The concepts, values and practices of precision already have proven to have a rich history—with diverse interpretations such as truth, rationality, certainty or objectivity (see [44]). This case suggests a similarly complicated and contextual account of approximation.

Approximation appears in the context of approximative reasoning in calculation and construction. In earlier Greek literature, its contexts of significance had been philosophical and mathematical, and they concerned the abstract metaphysical relation between early concepts of finite and infinite quantities—especially in terms of analysis of wholes and parts—ex., in Zeno, Aristotle, Euclid, Archimedes, Eudoxus and Pappus. From a calculational point of view, the mathematical context of significance was the problem of calculating properties—ex., area—of one geometrical figure—ex., the circle, with an infinite number of sides—in terms of another—ex. a polygon, with a finite number. This problem of calculation is also a practical problem of empirical representation or measurement. This practical geometrical problem became central to cartography, geodesy and astronomy, even mechanics, and in modern times provided a more abstract foundational and methodological significance.

Here is my picture for an epistemology of approximation in a nutshell: The formal representation of approximation relations requires standards (of constitution, application and evaluation), an asymmetric structure (metric criterion), a target that establishes the direction of asymmetry (approximation to what), and significance (generalization, reduction, scope of validity, precision, accuracy, evidence, measure of degree of objective (ontological or empirical) relational properties of a system or of epistemic cognitive states, etc.), with a standard that connects to the context of significance and helps apply its norms and goals. My general picture of approximation and objectivity is thus not a theory of representation per se, or based on such a semantic theory of truth and its absolutist or essentialist companion talk of reality. But particular applications can simulate, so to speak, some such and weaker views of truth, knowledge and science. It is a matter of context, perspective and significance. And so, as a matter of a naturalistic theory of reasoning and linguistic behavior, fuzziness and degrees of truth are relatively easy to accommodate and distinguish.

To accommodate the complexity, I propose a set of mutually compatible properties. (1) The *construction* of approximation can be *qualitative* or, in addition, *quantitative*. (2) The *interpretation* of approximation can be *conceptual*, internal to the particular formal structure, or *applied*, external to it, in relation to particular empirical representations (data or phenomena, not without generalizations and theoretical assumptions attached). (3) Approximation can receive additional significance or consideration: in the form of evaluation, constraint or use, all within different kinds of *contexts of perspective* on research and application and according to different commitments, interests or purposes; the internal context is defined by empirically extended system of formal rules, tasks and relations; the *external* level can be either *epistemic*, *ontological* or *pragmatic*. The epistemic described cognitive states and their warrant; the ontological describes the contents of knowledge in a real world, and they overlap in the treatment of empirical systems (objects, phenomena); the pragmatic introduces external purposes of application. Each *context of perspective* includes corresponding kinds of regulative aims and standards, or conventions, and with these formulate and address different sorts of projects and problems. The specifics of each context sometimes differ by disciplines, sometimes by empirical situations or interests (in some cases connectedly, in others separately).

In order to bear the features distinguished above and to accommodate the various broader contexts, the practice of approximation rests on a minimal structure involving the following elements: *structure*, *metric target* and *standard*. The standards concern, often connectedly, constitution, calculation, evaluation and application; their role is informed by the context of significance or perspective.

As I suggested above, one basic interpretation is conceptual and internal to the mathematical formalism. Approximative reasoning in the application of mathematics depends on a level of approximative formal representation. For instance, mathematical functions receive infinite expansion representations, ex. algebraic representation in Taylor series and geometric representation in Fourier series. Series expansions, then, provide a formal structure that introduces specific meaning and measure in the idea of approximation. The structure provides the formal basis for a metric that helps determine a measure of approximation, that is, a countable degree of proximity or closeness.

Attention to proximity and closeness is more specific than generic considerations of basic distance. Reference to *partial ordering* relation is necessary but insufficient. The quantitative requirements of *additivity* and *transitivity* and the specific structure of a *measure* function will contribute towards a metric structure that imposes on the formal relations a criterion and measure of distance.

But outside the internal context of formal interpretation a concept of metric is necessary but still insufficient. In the application of formal structures, considerations of approximation, assume additional conceptual structure, although often it is not explicitly formalized. Proximity is not distance. Proximity, like distance, is a *symmetric* relation involving each term in the series. But the formulation and the interpretation of approximative claims typically assume at least two connected constraints: one of restricted *range*, the other of *centeredness*.

The implied restriction on the relevant range that fixes the bounds of proximity is typically left unspecified. The assumption is that, while the domain of proximity may in principle be indefinitely large, in practice it is assume to be a smaller interval of values or a subset of terms—e.g., in an ordered series. And it characterized relative to a fixed reference. The fixed reference may lie outside the proximity range; this is the case for asymptotic limits. Indeed, all partial approximations, embedded in a metric ordering or structure, share an effective reference point, which may be a value of a variable, a form of a function, etc. This reference imposes a preferred order and, relative to it, determines the intended measure of degree. This reference is what I call the target of approximation; it constitutes the standard of exactness and introduces an *asymmetry* in the relation of approximation. Note that there might be more than one target; for instance, bounds of an interval might provide separate significant references for description and evaluation. The potential structural or formal symmetry of those cases does not eliminate the descriptive or interpretive asymmetries of the approximation relations introduced.

At the target, the relation of approximation becomes a relation of identity. Formally, the inclusion of the target in the formal structure that establishes it as a fixed term in the approximation relation must be secured; this is done by definition or proof of a formal relation of order, limit or identity. For instance, the equality sign marks the warranted formal equivalence and bridge between an analytical function and its series expansion. Proofs of the relevant identities involve proving convergence for the approximation of algebraic polynomials to continuous functions (Weierstrass) or summability for the approximation of trigonometric polynomials (Fourier series) to certain periodical functions (Féjer and Hermite). In practice, explicit considerations of what constitutes relevant range and relevant target involve additional contexts of interpretation and significance, in relation to standards and perspectives.

Generally speaking, a value of variable or a function of a variable, etc. may be the target of more than one approximation relation. When the target is shared by a set of values or functions sharing the same measure of proximity to the target and standing in a symmetrical, inverse relation to each other relative to the target, we might call the target the center of that set, to which the members satisfy the asymmetry of approximation relation. The elements sharing a target stand in an equivalence relation of approximation and form an equivalence class. Examples of targets are exact measurement outcomes, or the true value, and analytical functions. Given the symmetry of equivalence relations, it is the target and its significance that establishes the asymmetry relation and its interpretation in broader contexts. The target plays the role of center of the approximation range. In the interpreted, applied context, it is sometimes a typically symmetric range around a designated center that is of relevance. In the positivist context of measurement, margins of measurement error, or degrees of accuracy, form equivalence classes relative to shared empirical target values. In the formal context of representation, analytical functions or curves provide the exact target of approximation by polynomials or series expansions; and continuous ordering or real numbers establish the representation of limiting or asymptotic behavior.

It's worth distinguishing further between two kinds of metric notions of approximation, if only because they signal different possible perspectives: (1) *positive*, whose semantic base may be either representational or constructional (computational), but it represents a commitment to meaningful, substantive, cumulative consideration of degrees of the relevant property, e.g., as an ordering of parts, subsets or quantitative degrees of the significant target such as exact or maximum truth or the maximum value of a variable, or as a measure of (in)completeness of performance of calculation or of conditional descriptions of the world as a unitary complex system only conceptually divisible or analyzable (this is the qualitative approach in idealization or modeling, often based on a picture of the world in terms of metaphysical holism or complexity); and (2) *negative*, based on the epistemic assumption that some relevant property is only available in absolute unitary form, e.g., degrees of negative distance as separation from the target, such as measures of error, uncertainty or probability, featured in Popper's notion of verisimilitude as degrees of falsehood without corresponding degrees truth.

The scope of the fuzzy project demands that reasoning or calculation be explicitly listed next to representation or categorization in discussions of approximation. Formally speaking, approximation isn't a relation that applies exclusively in the context of formal representation. It applies to the context of reasoning or calculation, but it does so in additional dimensions, beyond the semantic considerations of precision of premises and conclusions. To compute, it is said, is to approximate.[11] The number of terms operates, in turn, as measure of computation tasks or steps. Even representation often rests on calculation; meaning depends on construction The cognitive significance of this kind of approximation derives from its methodological computational role as a problem-solving strategy. Analytical functions provide the target of approximation as analytical solutions. Least-squares values provide statistical targets of approximation of error distributions. Discretization techniques in numerical approximations or simulations are approximations to analytical solutions of differential equations at discrete points on a grid. Calculations, rigor standards and solutions in theory provide targets for calculations, rigor standard and solutions in practical application. Negatively or positively, different perspectives will evaluate reasoning approximations beyond their representational outcomes. For instance, at the practical level, the negative perspective will look negatively at increased numbers of steps or terms of approximation insofar as they constitute a measure of computing time.

The application and interpretation of approximation relations implies a kind of higher-order approximation. I mean that it implies no unified, universal absolute framework. Each partial model of approximation may be considered as an approximation to a more abstract, general higher-order picture of approximation, like different models are relative to the complexity attributed to a phenomenon; and many of these

[11] See, for instance, [43, p. vi].

contexts may be even ordered relative to a higher-order target notion. Approximation relations receive different use, value and meaning through their relation to different external standards and associated contexts of significance, that is, in relation to varying kinds of situations and perspectives.

A standard, a baseline or threshold, is typically related to a context of significance as its condition of application of significant aims and constraints. The relevant perspective will incorporate additional standards or norms that provide the normative source of significance.

What are examples of the different contexts of significance in our understanding and use of approximations? Each kind illustrates a perspective, situation and associated values and purposes. Next I list a considerable variety of cases in order to emphasize the contextuality and diversity of perspectives (in the interest of length, I'll omit technical details and most references).

### *13.4.1 Formal Conceptual Contexts*

Within the context of the mathematical formalisms, one can distinguish and assess a variety of situations characterized by the formal behavior of the structures that characterize and measure approximations. For instance, when the structures of approximation take the form of limits, one may distinguish between singular and nonsingular behaviors; I have mentioned above the case of asymptotic behavior. The formal implications of the distinction depends on the situation and purpose of the assessment, e.g., conceptual classification or proof.

The entangled objectivity and rationality of definitions and theorems that characterize construction and justification in formal contexts rely on the application of formal constraints: rules and conditions applying formal values and standards adopted in the specific context at hand, e.g., uniqueness of choice, convergence, continuity, differentiability, etc. Often increasingly more abstract formal contexts are introduced in order to secure rational objectivity of particular structures. Geometry has become grounded on analysis and algebra; another source of abstract redescription has been category theory; and theories are more generally identified and unified by axiomatic systems.

### *13.4.2 Formal Calculational Contexts*

Whether in theoretical inquiry or in application, the formal character of approximation may be approached from the perspective of calculation. One may then distinguish a purely static formal view of structural relation from a dynamical view from the perspective of a process. From that perspective, approximation involves a methodological act and a strategy prior to any consideration of metric relations [40].

Approximations may be evaluated in terms of stability conditions: the evaluation of difference in functional performance relies on applying transformations and evaluating if the mapping of an expression onto another preserved the desired behavior within error bounds [28]. The desired behavior and the bounds are, of course, contextual choices. A strategy for approximating solutions to differential equations often rests on considerations of tractability and the choice to sacrifice continuity assumptions, for instance in the choice of numerical over algebraic approximations to analytical solutions. Numerical models of solution might, thus, involve a discretized approach based on calculating the solutions at different points on a lattice (this is the basis of many simulations of systems such as fluid behavior without intractable analytic solutions to the differential equations they ideally satisfy in the theoretical model).

### *13.4.3 Applied Conceptual Contexts*

When mathematical structures are applied to the expression of a theoretical or empirical concept—i.e., to the representation of its corresponding system or property—the context of application provides a different dimension of approximation. It is determined by comparisons between different possibilities and the corresponding range of the structures involved. This is the case, I have suggested already, of modeling: modeling as simplification by idealization (value selection and approximation) and abstraction (variable selection and approximation). The contextual understanding provides depends on the specific selection of particular variables and relations they obey for the construction of the model. Each choice can be considered a form of approximation in the sense of a counterfactual idealization. The comparison in the range of variables or their values determines assessments of their approximate character.

There is a more general sense in which every model presents an approximative character. Every model is considered a qualitative simplification based on the negative choices involved in its construction relative to the range of concepts established by a theory or from a broader one from the range of theories concerning a system, or a sense of qualitative complexity of conceptualization independent of any theory.

The selective and partial nature of the models responds to cognitive and practical choices based on perspectives, interests and standards considered relevant in the context. The difference between different map scales or between geological, political and thermal maps of the same geographical region illustrates the point [20]. The conceptual content identifying a phenomenon sufficiently distinguishes the success of a qualitative approximation to the kind of situation represented by a particular concept or variable in a given context and application from the quantitative approximation with merit in other contexts and applications [25].

By parity of reasoning, one may argue that as a selective targeted strategy of representation and understanding, modeling is a conceptual form of exactness insofar as it achieves conceptual isolation, it identifies an entity or an active factor in causal explanation. In that sense, idealization is not distorting simplification. The positive dimension of corrections (terms with empirical contextual meaning) compensate for the negative dimension of construction (idealization and abstraction). This context of isolation is typically delimited by so-called ceteris paribus conditions that identify the empirical or theoretical conditions of application of the model. They are a placeholder for the often unspecified (and unspecifiable) set of disturbing causal factors or simply requiring a blanket absence of other elements.

In that conceptual situation, approximation becomes a form of contextual exactness and accuracy. Exactness, conceptual and empirical, qualitative and quantitative is relative and limited to contexts at the expense of the universality of the alleged applicability of a so-called law, whether an empirical generalization or an abstract principle. It is the unqualified, decontextualized generalization that becomes, by contrast, approximate and cognitively deficient.

The approximative nature of theoretical or empirical concepts requires standards prior to their application to a given system; they are not necessarily part of the model. In that context and in the different situations over which it will vary, one may speak of benchmarks of categorization. The choice of benchmarks is as contextual as it is normative. This is part of the general phenomenon of categorization, subject to contextual constraints such as varying rules, generalizations, choices of prototypes, etc.

### *13.4.4 Experimental Contexts*

Experimental contexts concern the production, assessment and treatment of data. The application of measurement instruments–or instruments whose application involves measurement–incorporates two separate levels and meanings of approximation, precision and accuracy. Measurement precision, like the internal rule-based formal precision of mathematical relations, is embedded in the construction of the instrument, it's a type of built-in bias, constraint or rigidity, an instance of it is lack of calibration, with a deviation (often systematic) in mapping that applies the scale values onto the relevant property of the system (an indication of imprecision is the mapping of a null state onto a measurement value different from the null). Another is the instrument's sensitivity, the smallest different than receives numerical amplification and expression. The accuracy of the instrument, the measurement error away from the assumed true value is not reducible to the systematic degree of precision.

As a matter of contextuality of values, reference must be made to experimental concerns over the conditions of trade-offs involved in instrument design; for instance, controversy over the conflict between maximum accuracy and sensitivity of Kelvin's reflecting galvanometer for non-uniform fields.[12]

Measurement assessments rely on an additional statistical dimension of accuracy, that is, approximation in the context of populations of cases, relative to the population size: large numbers don't eliminate biases (imprecision) but increase accuracy

[12]For a discussions of the episode, see [22].

with the canceling out of average chance error so that average measurement value approaches the true value (relative to the description of error).

Part of the application of this empirical perspective in measurement is the normative standard as to what counts as an acceptable margin of error. Again, this is as normative as it is contextual, subject to a number of contingent conditions.

Within this context we encounter the epistemic, often described as subjective, interpretation of objectivity, not as a matter of error, or statistical distributions, but of ignorance. This is one interpretation of probability and its (misleading) associated talk of uncertainty. The subjective reading connects with the next kind of epistemic context.

### *13.4.5 Evidentiary Contexts*

In evidential contexts considerations of approximation respond to demands for warrant, for indications of truth, credibility, reliability or acceptability of hypotheses.

In relation with measurement statistics, mentioned above, a different kind of benchmark is required in addition to the semantic one for the purpose of categorization. In the new context and in relation to the new goals, one such constraint is the standard of statistical significance. It establishes when a hypotheses tested by the measured effect is least likely to be false, compared to the null hypothesis (chance event or correlation). In this context, the normative standard is doubly contextual, since the specific value required for its application to specific numbers needs additional determination and warrant, e.g., 0.05.

More generally, approximations themselves may receive justifications of different kinds, formal and empirical. In the empirical case, it is a matter of context and interpretation what is acceptable (see below).

An expression of this problem in general extends to the individual as well as statistical version of the problem of significance. It is the question of when an approximation ceases to support a hypothesis or instantiate a generalization, when the law is no longer stretched, but broken. Is the data negative evidence? or it can be dismissed as not even instantiating the relevant category necessary for evidentiary status? Here practice often has recourse to conditions of application or validity of the hypothesis or model in question.

The dynamical view of calculation mentioned above extends by application to the case of testing and the evaluation of evidence. The validity of precision of approximate calculation depends on reliability or justifiability of procedure [31, 40]. One proposal is testing and confirmation through stability (monotonic linearity) of approximate hypotheses: more accurate initial conditions lead to more accurate predictions [28]. Within any bounds adopted for the purpose at hand, approximative reasoning is a form of conditional reasoning. For instance, if, on one interpretation, the accuracy of hypotheses is considered a matter of probable truth, the probable truth of the calculated predictions and the generalization they support is dependent on the degree of precision of the initial or boundary conditions.

A more serious challenge is the language-dependence of accuracy of prediction. Approximation as closeness to truth associated with a target value of a relevant parameter is a relation with evidentiary use in comparative contexts for competing hypotheses. But for different hypotheses and parameters, the degree of predictive accuracy and the asymmetry of the relation may not be preserved under certain formal inter-translations [32].

### *13.4.6 Theoretical Contexts, Interpretive and Explanatory*

In this context, the defining issue is the objective interpretation, also referred to as the realist or ontological meaning or implication of approximations. That kind of status depends on the corresponding interpretation of the concepts, relations and hypotheses they involve, taken to be representing worldly entities and relations (whether in theory or experience). Examples abound.

First, one can distinguish between realist, fictionalist and positivist interpretations of approximation. Realism is often associated with the exact solution as its mathematical standard. Non-limiting physical approximations such as asymptotic or infinite or zero limits are often considered convenient fictions serving tractability and separate physical adequacy (idealizations may be no longer considered approximations). In the positivistic interpretation, the only objective significance is the empirical meaning attached to the structure or computational value; degrees of approximation with ranges or margins of error in measurement are grouped together in class of empirically equivalent values sharing the same interpretation.

In certain series approximating a function or dependent variable, dominant terms involving independent variables are interpreted as measures of dominant causal factors in the explanation of the dependent variable; other terms are declared correction terms and interpreted nonrealistically, as random noise.

More cases distinguish between computational and the realist physical interpretation. Zero or infinite limit are taken as formal idealizations contributing to computational simplicity or tractability but lacking literal realistic physical meaning, e.g. infinite potential, infinite limit of large numbers in populations sizes, frequencies, or infinite population size and volume in the thermodynamic limit. Yet, it's not that clear cut; in certain applications of the thermodynamic limit, the order of limits in the thermodynamic approximation alters physical interpretation; bulky macroscopic matter requires infinite volume limit [27]. Boundary problems are often denied physical interpretation and the approximations involved are granted only computational value, e.g., the discontinuity of expansions on opposite sides of a boundary layer and catastrophic limits at boundary layers [2, 24].

By contrast, linear approximations for the sake of computational simplicity obey superposition, which is nevertheless given material interpretation in terms of compositions of independent entities or causal factors. In small angle approximations, in which x can substitute for sin x, the substitution is used to eliminate interference terms in, for instance, quantum superpositions. In quantum field theory, Feynman diagrams assign to each term in the decomposition physical descriptions in terms of processes involving particles, their propagation and interactions, which receive either realistic or fictional status. Finally, the application of differential equations often motivates the introduction of continuous models of discontinuous structures: the stress-based dynamical explanation of incompressible fluids and elastic solids in continuum mechanics; or the use of continuous limits of stress levels to represent discontinuous transition from elastic cohesion to fracture.

An important theoretical use of approximation relations concerns the structure of science, its evolution and its interpretation. I am referring to the representation of relations of reduction between empirical theories. Approximation between concepts or quantities in different theories as model of reduction has explanatory and interpretive significance.

The asymmetry associated with reduction is projected onto an asymmetry in the understanding of approximation; and ultimately the asymmetry depends on assumptions of epistemological and ontological fundamentality, e.g., the reducing concepts and generalizations are more fundamental by virtue of being more explanatory and more "real", while the reduced representations appear as redescriptions with contextual value, higher scale in terms of size or organization, more empirical, familiar, tractable, relevant or useful.

In the more radical versions, reduced theories are meant to be revised and eventually eliminated and replaced, at least in the context of validity of the reduction. Local, partial revisions, however, are not as threatening as ones with universal scope. For this reason, talk of emergence–of entities, properties, etc.—signals a degree of conceptual autonomy beyond the merely instrumental value.[13]

Besides hierarchies of decomposition into smaller spatial and temporal parts, the form of reductive approximation is logical, but the logical derivation depends on mathematical derivation as the form of the relation between statements from different theories or models (the approximation relation), and bridge or translation principles, the identity conditions between properties that secure the semantic soundness of the derivation, with universal or contextual validity.

Examples: The limit of vanishing velocity is meant to show—and prove—the reduction of Newtonian mechanics to relativistic mechanics; the thermodynamic limit of infinite volume and infinite number of molecules to reduce thermodynamics to statistical mechanics; the infinite limit for energy levels (and number of particles) to reduce quantum to classical physics. The reduced and reducing theories must be formally (sufficiently) identical at the designated boundaries as contact points (i.e., at the limiting parameter values). Similar logical relations are formulated for theories such as genetics and molecular biology, psychology and neurobiology, etc. (see [9]).

[13]For instance: Local field operator expansions of effective, lower-energy field theory, are considered approximations to real, high-energy exact theories, whether known (top-down approximation) or unknown (bottom-up approximation); then, a higher interpretive autonomy and realism is assigned to bottom-up approximations.

A weaker relation is marked relations of generality and inclusion, that the scope of phenomena explained by one theory (i.e., the reducing one) includes the scope of the other (i.e., the reduced).

As I will briefly discuss below, also this context and this issue are relevant to the possible understanding of approximations in relation to fuzziness and fuzzy logic.

### 13.4.7 Disciplinary Contexts

This kind of context captures differences in interests and standards corresponding to different kinds of projects associated with different disciplines or communities.

The main distinction of this kind is probably between foundational and applied contexts: in foundational projects generalizability and rigor become valuable; in practical projects of computation and application, a measure of degree of approximation as a representation of functions corresponding to measures of practical cost of time, energy, money.

An example is due to Poincaré, as an illustration of the difference in approaches and standards between mathematicians and astronomers: [35, p. 1] the convergent or divergent status of a power series is relative to disciplinary interests and constraints. Mathematicians, interested in theoretical evaluation, will consider relevant as many terms as can be possibly identified. Astronomers, by contrast, interested in numerical calculation and its facilitation, will consider relevant a smaller initial subset especially if the convergence or divergence appears pronounced. This way, Poincaré argued, they might reach different conclusions, declaring the same series convergent or divergent.

Another examples concerns empirical disciplines. Since their introduction in the early 20th century, standards of statistical significance have not been consistent over time across disciplines. The 0.05 standard can be traced to publication requirements in experimental psychology. Physics has aimed at smaller values.

### 13.4.8 Practical Contexts

Despite the connections between all the kinds mentioned thus far, one may point to a broad practical meaning of conventions in the role of standards of significance or thresholds.

For different contexts of practical application, technological, medical, political, legal, etc., and a purpose $P$ in each, the general form of the approximative assessment is this: $X$ is sufficiently precise for $P$, or a degree of approximation sufficient for $P$, where sufficient condition is fixed by a convention $C$.

## 13.5 Fuzziness Is Empirical, Contextual, Approximative, Subjective and Objective in Construction and Application

In this section I sketch a brief discussion of the specific way each the themes presented above, empiricism, contextuality, approximation, subjectivity and objectivity, appear and play a role in the construction and application of fuzzy theory. I deal with set theory first and in the following section I will consider fuzzy logic. While each section and subsection will focus on a single theme for emphasis, it will become obvious, as I have suggested in the previous sections, that each theme overlaps with the others. Throughout the theme of normativity will appear as well. It is the connection between the general account I have presented and the specific fact and form fuzzy theory exemplifies it that I am contributing and can inform new reflection and motivate future conceptual and empirical research.

### *13.5.1 Fuzziness as Empirical*

As I emphasized at the beginning, the fuzzy project was born as an empirical project of formal scientific modeling with broader empirical and technological application. The fuzzy membership function is meant to be a good model of the way people actually perceive and apply categories, and reason with them [16, p. 255]. Another empirical dimension of fuzzy set theory (and confirmation of its empirical validity) is the more controversial experimental evidence for the success of fuzzy set operators—for experimentally set fuzzy membership values—modeling the structure of subjective human judgments as well as technological applications (for measurable input-output systems).[14]

As in the case of any empirical model, we can distinguish between the empirical grounds of construction, testing and the grounds of application. Application and construction, however, cannot be neatly separated. Often, the application of the formalism depends on the assignation of values understood in terms of construction of the model. The empirical grounds correspond to the human linguistic behavior conceptually identified as involving a distinctive element of vagueness. The grounds of empirical testing are, as is characteristic of empiricist methodology, the experimental cases that test the limits of empirical validity.[15]

The empirical grounds of application, independently of any additional evidence they provide, are the growing domain of phenomena to which the formalism is applied

[14] For a still adequate discussion of the human case, see also [16, pp. 261–263].

[15] As with any data-based testing of formal models, the evidentiary context might also challenge the empirical support. Thus claims to the effect that the empirical linguistic evidence doesn't support the fuzzy predication of truth by analogy with paradigmatic modified predicates and their opposites, [23, pp. 244–246].

and the technological developments that are guided by the formal possibilities and empirical opportunities (especially in human interfaces) the formal theory suggests.

The empirical grounds of construction share their enabling role with relatively a priori resources: the precise standards, rules and relations in the formal context that constrain possible constructions in mathematical formalism (regardless of any empirical roots of their own in turn). The non-factual sources establish the partially a priori status of fuzzy models. It's from this formal perspective that the characteristic set membership function for fuzzy subsets can be constructed as a generalization of the classical mapping, from the universal set to the pair {0, 1}, to the interval [0, 1]. For the same reason operations on the set members can be defined and the resulting theory can be embedded within the algebraic structure of lattices that supports logical systems (via isomorphisms between structures of set-theoretic operations—complement, union, intersection—and the structures of logical connectives—negation, and, or—).

The formal constraints on the construction of fuzzy structures can be said to provide formal justification, the context of internal normativity or necessity, even if by convention. In addition, they provide a valuable methodological service informing and guiding empirical application. The formalism underpins the fuzzy conceptual scheme within which vagueness and other properties, the empirical facts about linguistic and reasoning behavior, become intelligible, normative and empirically meaningful.

One of the constraints is to embed fuzzy set theory in the framework of algebraic and methodological resources that also help define classical set theory: Normative standards of derivation and organization, concepts schema from set theory such as concepts of classical set membership and operations, or related algebraic concepts from group theory such as commutativity and other properties of set-theoretic operations, or more general underlying concepts from function theory and analysis.

Standards of justification and resources of construction do not just include the rule-based standards of proof and the axiomatic derivational structure; they also include the commitment to justifying (deriving) with consistent and sufficient conditions the uniqueness of the choice of specific definitions for the basic set-theoretic operators union—as maximum membership value—or intersection—as minimum membership value.

Formal construction, then, actually involves the justifying application of prior rules and concepts. You might think of the process and the resulting structure to involve a sort of ordering in which each level of application is more formally revisable and more factually justifiable and revisable. The empirical roots of formal revision are neither simple nor direct; they are contextual, constrained by broader commitments, a situation often considered a pragmatic matter as much as formal and factual.[16] For the sake of formal, pragmatic or empiricist commitments, there is still an ordering of priorities as to what merits a firmer commitment. In the face of

[16]Carnap and Quine are standard references drawing conclusions about the challenge to formulate the conditions of factual meaning and justification.

an accepted factual hypothesis, not all formal statements or structures—logical or mathematical—relevant to modeling it are equally readily revisable.

This situation is the mirror image of the sort of formal holism that characterizes the meaning and justification of factual claims. Empirical testability involves not just a designated factual claim, but a variety of associated theoretical and formal assumptions, concepts, rules and commitments that make the testing both possible and significant. Background conditions concern both the system studied and the experimental or measurement conditions. In the light of a negative test result, all these assumptions appear in principle revisable, but, in a given context, not equally so, and some not at all. Their a priori character gets then established, but only relative to context. The limited factual status of fuzzy set theory is suggested also by the feature that the factual motivation behind formal construction and model application is stronger than their factual revision.

Logical and mathematical a priori propositions, as independent of factual justification and refutation, have been characterized also as being distinctively analytic. But the application of the concept can be traced to two criteria: (1) the exclusively logical or non-factual principles of justification, by virtue of the application of rules and definitions, and (2) the relation of contradiction that derives from the application of negation. With only the necessity of context and a priori choice, many statements are analytic merely by construction and convention.

If the distinction between analytic and synthetic has been hard to clarify and defend for classical cases, it is even more so in the fuzzy framework. Contradictory statements require special conditions supporting a disjunctive form in the absence of the principle of excluded middle. Then, negation, tautology and contradiction can be applied defined in terms of functions of degrees of truth. It seems then that analyticity, as a predicate in meta-language, turns out also fuzzy.

The picture of complex empiricism that fits scientific practice quite generally includes the introduction of claims, concepts, and rules that are contextually a priori or empirical. They provide the normative, constitutive and interpretive conditions of application formal model. One example is the a priori construction of causality criteria, which constraints, guides and adds interpretation to the empirical application of formal models (see [8]). Other examples are the concepts of information and entropy. Another, this kind of complex coordination in application is one of the contexts of application approximation, for calculation (constructive, normative constraint), interpretation and evaluation.

Mathematical claims in set theory and logic are justified by reference to both kinds of sources: factual information about human linguistic and reasoning behavior and formal general prior constraints—concepts, rules, standards and commitments, etc.

### *13.5.2 Fuzziness as Subjective and Objective*

Since Zadeh's early texts, presentations and discussions of fuzzy theory have been made repeated references to subjectivity like no other scientific literature (except, perhaps, discussions of the mystery of consciousness).[17] Although it has increasingly turned away from acknowledging this aspect in order to emphasize the formal objectivity of mathematical results and the equally mechanical objectivity of technological application. I have suggested above that the different roles of subjectivity in scientific practice may be considered effective, acceptable or irreducibly detrimental, depending on how each role is understood to serve prior commitments or values. To try to situate this case within my discussion of subjectivity and objectivity it will help here too to begin by keeping in place the distinction between the moments or contexts of construction, justification and application (and interpretation), also to note their inseparability.

What is subjective in fuzzy set theory? The following is a two-part list of elements recognized in different presentations: (1) subjective concept, subjective category, subjective observation, subjective event, subjective belief; (2) subjective method of construction, subjective guess, subjective judgment, subjective choice, subjective decision, subjective determination, subjective assignment, subjective assessment and subjective evaluation.

Each list concerns a type of dimension of subjectivity I have noted in my general picture. The list (1) classes together instances of subjective interpretation, personal states associated with a linguistic expression. List (2) classes together instances of subjective method or process; they involve construction, decision and evaluation.

Underlying both types is a no less important one, volitional subjectivity, subjective activity, spontaneity and will. The background subjectivity appears through a methodological supporting role of active voluntary subjectivity in episodes of construction, consideration and choice (selective decision). It helps contribute the practices and products that will be submitted to further use and evaluation relative to purposes, interests and norms at hand. It is both the background condition of consideration of ideas and commitment to ideals, where ideas and ideals get their hold on us. A modern tradition in psychology of mathematics going back to Poincaré and Haddamard (reluctant heir to the Kantian transcendental tradition) has stressed the active role of the mind in construction suggestion and justification—unconscious role in construction—often referred to as intuition.

In the technological context, talk of subjectivity ultimately resides in the human factor in construction and application. In the absence of the human factor, a metaphorical generalized notion could be introduced referring to technological models of embedded subjectivity. Might the notion of control an example? In more cognitive terms, an expert system aims to inform, constrained subjectivity into judgment as intelligent judgment and intelligent decision. The positive role of subjectivity has been incorporated, although not explored, to the extent that it has mechanically—formally and technologically—modeled, by modeling in effect subjective behavior, that is, its observable expressions or effects, e.g., particular assignments and

[17] I have counted 15 references in [16] and 19 in [26]. See also [33].

evaluations.[18] The picture I'm offering here is one of scientific practice that is centered on a methodologically normalized and naturalized subjectivity.[19]

What is what we may call the scandal of subjectivity? It's a noted deficiency and failure: a conflict with endorsed aims of knowledge and method, especially in science, oriented towards a real, empirical world, whose reality and its cognitive access are understood in terms of their public dimension, their independence from any single individual. Subjectivity is the failure to serve that aim, and value, by adopting a suspect self-oriented, self-dependent, internal and individual perspective—psychologically and even intellectually. The challenge of science, in that sense, is that humans are individual sources of cognitive and normative perspective and agency, ultimately it is they who are individually responsible for acting and understanding; yet the objects, processes and products of science are hoped to be de-individualized. That's the challenge of inter-subjectivity, as minimum objectivity condition, in the context of science understood today as human practice, by and for human individuals—as opposed to science without humans, by and for technological others.

A normative deficiency of the subjective character is indicted by the association with references to an element of arbitrariness of choice, bias and intuition. A trade-off is sensed between the volitional role and the epistemic role of warrant, in the sense that what valuable volitional role subjectivity plays also fails to provide objective warrant. But objective warrant is not warranting objectivity. The relation between rationality and objectivity is something like this: the objectivity of reasoned processes of construction and selection is coextensive with the reasoned form of objectivity.

More precisely, objectivity is normative as an aim and value in its own right, but that normative force does not depend directly on, much less is identical with, the normative character of justification processes, methods or rules per se, that is, as forms of warrant—e.g., think of proofs. It depends directly only on these conditions' inter-subjective character, which is contributed by the applicability and recognition of the justifications. It is not surprising, then, that objectivity conditions are often grounded on rationality conditions. It is a further matter that the source of that potentially intersubjective recognition and applicability is shared normative commitments.

At the very least, then, intersubjectivity is expected to explain and secure sufficient agreement among scientists (hence rationality as the normative mechanism that justifies and explain agreement); at most it is expected to warrant a reliable representation of an independent objective world.

Objectivity conditions or constraints serve typically the joint functions of construction and justification. Two kinds of such normative rules, objectively justified construction and objectively justified application. Raising the issue should draw more critical and creative attention to the standards, constraints and purposes of fuzzy theory.

[18] See, for instance, [33, pp. 319–320].

[19] A related discussion of Norbert Wiener's early cybernetic model of the enemy airplane pilot as subject can be found in [18].

Two different conceptual levels in fuzzy theory concern the notion of uncertainty. In fuzziness, uncertainty is conceptual or linguistic indeterminacy, as in the case of vagueness and its semantic expression in terms of membership functions and degrees of truth. The application of any rules for assigning membership grades or truth-values is considered an attempt to curb, but never quite eliminate the subjective as a matter of informed judgment.

Construction relies on the formal constraints from the larger scope of mathematical theory to which fuzzy theory is attached. Application and evaluation of formal structures—especially models—get their element of objectivity through methodological mechanical rules. Again, application and construction are not neatly separated. Often, the application of the formalism depends on the assignation of values understood in terms of construction of the model. At any rate, membership assignation, one can read, "is not totally arbitrary." The arbitrariness may be limited quantitatively (limits in the base concept such as percentage of something) or qualitatively by the motivated and informed choice of one or more standard assignations that constrain the full spectrum of application.

It's not a matter of artificial precision where numbers replace vague predicates. As a matter of semantics in terms of degrees of truth, the problem of assignation of membership degrees may concern precisely the application of vague predicates, e.g., "warm", to precise numbers, e.g., $70F$, in a base range. Fuzzy numbers provide a semantic tool to quantify—represent and apply—linguistic variables (including relations as $n$-tuples of variables). Another formal strategy applies objective interpretations through abstract redescription, for instance, within category theory.

Objectivity of application is linked to the warrant and rationality that informs the context of justification. The form of this approach is the idea of a method. We are speaking of strategies for warranted application with the aim of reducing the negative cognitive and methodological meaning of subjectivity as both irrational and inaccurate. That is, at work is the aim to reduce and replace a declared negative cognitive rigidity or bias, in contrast to positive rigidities, e.g., formal and methodological constraints in construction (see above), or the warranted and warranting heuristics and commitments in application.

So, one finds distinctions between methods, direct and indirect (see [16, 26]). One type involves a single cognitive subject (and agent), or expert; the other involves multiple experts. Indirect methods are less liable to "biases of subjective judgement" [26, p. 282].

First I give brief consideration to **direct methods** with one expert. A general strategy is based on a complete mathematical definition of a membership function for concepts based on two conditions: selective exemplification and similarity relations. Members are expected to present selective exemplification of the universal set or **ideal prototypes**; however, the determinating role of this criterion may be limited by the ideal prototype's location outside the category or the availability of many and incomparable prototypes. The compatibility relations to prototypes can be ordered and expressed mathematically by a **similarity function** representing a measure of degrees of similarity for compatible elements. The similarity functions must also incorporate, or be accompanied by, **error functions**. The standard of ideal prototype is also a benchmark for the standard of zero error (an instance of perfectly straight line).

A related criterion of membership construction is the representation of objective preferences. Unlike subjective preferences, **objective preferences** assign membership grades to input parameters according to degrees of objective property measurements in relation to a quantitative value or objective, for instance in the evaluation of properties of materials and designs in mechanical engineering. The objectivity, to any degree, of the approach resides in the objectivity of the mathematical representation of the information about objective material properties and the formal rules of evaluation all in relation to equally quantitative objectives (ex., minimum risk, cost or weight, etc.).

The following are a few additional examples of estimation methods: prototype, exemplification, deformable prototypes, implicit analytical definition, statistics, relative preference, subset comparison, filter function. In each method one may easily identify conditions of objectivity, approximativity and normativity, also the form and sources of residual subjectivity.

- Prototype theory: This influential model of categorization centers on a preferred member of the set selected often relative to the cognitive agent's own value of the category in question. Similarity judgments underdetermine the units and values of particular membership values. The preference is clearly contextual and normative.
- Exemplification: It emphasizes a broad range of exemplifications, as one would need to design a scale, including maximum and minimum matching cases, justifying true and false predication statements. The matching judgments re contextual and subjective in terms of the application of the contextual partial ordering, since the ordering underdetermines the values and the conventionality of units.
- Deformable prototypes: The relevant type of prototypes can be deformed by manipulating and modifying a set of parameters in order to achieve a maximum match with a given object. Dissimilarity judgments are quantified in terms of a distance metric and a so-called energy of deformation, or weighted distortion function. But assignation of values to each might involve a residual subjectivity for instance from the matching judgments.
- Statistics: Membership functions constructed from sample data, for instance, as measures of the statistical proportion of positive full answers to predication or membership questions in polls and normalized histograms
- Neural networks: It is another method of construction from sample data, from learning algorithms such as back-propagation. This method depends on preselection of functions and parameter values—activation functions, relative weights, etc.—that establish the properties of the neuron network that will help structure the membership function.
- Relative preferences: Categorization judgments are contextualized to different elements, without a prototype, with the semantic condition that the higher the relative membership value the higher the individual membership of an element compared to the other.

- Comparison of subsets: A set is induced from the fuzzy set if it's a subset of the power-set of the universe with elements from the fuzzy set. One induced set will match the fuzzy set better or worse to the degree that the average membership function of one subset's members is greater than the other's.
- Filter function: A S-shaped function models the distribution of membership values between 0 and 1 of the members of a fuzzy set. The function has a transition fuzzy zone centered around a neutral point at membership value 0.5 with an interval around it of size.

Multiple factors may be conjoined for complex concepts by a vector representation based on multiple scales and associated membership functions. The role of multiple experts requires the introduction of an interval and function for aggregating judgments. Indirect methods integrate calculations of membership functions for multiple dimensions of a concept performed by multiple experts: interpolation, least squares, neural networks,. . .

Uncertainty may be a cognitive, subjective form of indeterminacy beyond error and probability. It exemplifies at least two of the kinds of subjectivity I have listed, namely of application and its evaluation [16, pp. 126 and 144]. It is also an instance of subjective interpretation. The formal constructions are formally constrained, therefore with an element of objectivity; a more factual element of objectivity comes from its roots in empirical reality. But if we take the interpretation as a multidimensional vector, another element is irreducibly subjective in its dependence on personal cognitive states. It is a representation of human perceptions.

Degree of possibility is a truth-functional additive measure of a perception-based assessment. It's not about how likely it is that the room temperature has a specific value $T$ (subjective probability relative to missing information, or perception of objective random variables or chance for equally possible values for random sets, modeled by objective frequencies). Connectedly, it's also different from the epistemic uncertainty of measurement as error, whether room temperature is actually exactly $T$ (accuracy or margin or error in a statistical distribution of data). It's not about how true it is a given precise room temperature belongs is the fuzzy range corresponding to warm (fuzziness and fuzzy sets). Possibility measure is about how possible it is that an unknown temperature is perceived to belong in a fuzzy range corresponding to warm.

The interpretation of measures of uncertainty in terms of fuzziness and possibility is contextual. In the abstract, meaning gets its formal objectivity of quantification and axiomatization; in the context especially of technological modeling of human processes, it's perception-based. The subjective interpretation of possibility distributions identifies this dimension as the key general feature; it is Zadeh's original view, cited by several presentations: in Zadeh's own familiar words, "Intuitively, possibility relates to our perception of the degree of feasibility or ease of attainment whereas probability is associated with a degree of likelihood, belief, frequency or proportion."[20] Nguyen and Walker refer to the view as the primitive subjective concept of the interpretation [33, p. 258], Degrees of possibility may be

[20] In [16, p. 137].

operationalized in terms of degrees of similarity, closeness, to a prototype, either by objective measurement procedures or subjective expert judgment. But measures on possibility distribution measure "uncertainty in subjective evaluations" [33, p. 319].

### 13.5.3 Fuzziness as Approximative

Empiricism, I have suggested, is approximation, in form, method and target of representation. For fuzzy theory, one can read, "fuzzy sets and fuzzy set operations are also employed, in general, as approximators of meanings of relevant linguistic terms in given contexts" [26, p. 281]. Therefore, determining which "operations on fuzzy sets best represents the intended operations" is equivalent to determining which "operations on fuzzy sets beat approximate the intended meaning" (Ibid., pp. 280 and 281). In this section I try to give briefly a sense for the complexity and diversity of approximation I have suggested in the general picture, above. In that regard, a productive perspective should recognize fuzzy theory to be both typical and distinctive.

It is important to begin with a distinction between different levels of representation. Fuzziness as approximation isn't the same as fuzzy approximation, or a fuzzy model of a particular approximation relation. In the latter sense, relations of approximation may be generalized to a fuzzy relation. What about the construction, application or evaluation of fuzzy formalism?

Formal construction relies on the resources that help define the distinctive concepts of fuzzy set theory and develop its structure. In this context, formal conditions of fuzziness may be interpreted as formal models of approximation relations. Take the notion of a fuzzy subset as a set of membership mappings onto the interval [0, 1]. The metric structure of an approximation relation—ordering, additivity, measure, etc.—may be trivially imposed on the real interval. The classical values {0, 1} provide formal reference targets and associated partial asymmetric relations. Despite the dual-reference structure, distance from value 1, the classical standard of full membership and predication, may be considered setting a dominating asymmetry.

Also the so-called fuzzy numbers, more complex derivative formal constructs, are discussed in approximative terms: "should capture our intuitive conceptions of **approximate** numbers or intervals, such as 'numbers that are **close** to a given number' or 'numbers that are around a given interval of real numbers." Such concepts are essential for characterizing states of fuzzy variables and, consequently, play an important role in many applications, including fuzzy control, decision making, **approximate** reasoning, and statistics with **imprecise** probabilities."[21]

In addition, one may mention two discrete types of approximations to the fuzzy membership function: (1) alpha and strong alpha-cuts forming binary crisp sets, subsets of the support sets, linked to fixed points; and (2) discrete approximations of a membership function for a predicate in the form of discretized coarse-grained version of the range or argument of the fuzzy membership function.

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[21] [26, p. 97].

Alpha-cuts provide a bridge concept between fuzzy and crisp sets and their respective ranges of membership values. A fuzzy set can be represented by a decomposition in terms of crisp subsets set by alpha-cuts fixing the characteristic membership function$(a_1, a_2, a_3, \ldots)$. The set is represented in terms of the collection of membership assignations to its members $A = a_1/x_1 + a_2/x_2 + a_3/x_3 \ldots$ Cardinality measures provide a related metric, with probability distributions involving sets of cardinality 1. In addition, so-called extension principle for functions between sets provides an extending rule for mapping power sets and establishing fuzzified sharp functions between corresponding fuzzy sets defined on the original sets.

The context of application provides additional dimensions with occasions for applying approximative ways of thinking e.g., the semantic and the methodological. At the semantic level, we must recall the empirical aspect of fuzzy theory. It is this role that the words quoted above echo; as any scientific model, fuzzy formalism aims to model a target empirical phenomenon. The representation relation associated with modeling, in this sense, is what I have called an instance of conceptual approximation, qualitative and quantitative.

It involves features I have associated with modeling, abstraction and idealization: an asymmetric target, a standard of maximum representation–reality or truth-, a conceptual and empirical sense of conceptual and quantitative idealization, a concrete standard of relevant conceptualization in the context of necessary limitation that establishes the strength of the model, a sense of purpose that justifies the relevant selection, and a corresponding sense of accuracy—or faithfulness—of description and prediction. For the purposes of system modeling, for instance, in this spirit we can read that "the problem is to construct a mathematical description of the system, based upon available information, so that it will represent faithfully the 'true' system" [33, p. 235].

Another semantic dimension, in a narrower model-theoretic sense, is the objective interpretation of membership degrees in terms of degrees of truth. These are specified by the numerical models of membership or truth degrees. It is understood that the interpretation is objective in the sense that its intended target is some property of the system at hand.

Alternatively, as degrees of epistemic uncertainty they are generalizations of (classical and) probability measures. Probability (additive) measures are the intersection of plausibility (sub-additive) measures and belief (super-additive) measures, all types of fuzzy measures.

The methodological dimension of application illustrates what I have called the dynamical approach. It provides structures and techniques to inform the empirical application of the mathematical formalism as a process and a practice. Beyond the mere formal structure established by the range of membership degrees, one may pay attention to the objectifying techniques that constraint the application of the formal structure and inform the construction of the concrete model. I have listed a number of such methods. One, and most basic and intuitive, is the use of prototypes and similarity relations. It is based on a contextual and normative choice of standards, the prototypical member, that constitute the target reference for approximation relations that similarity relations help determine. A related method is deformation and similar notions of proximity to target prototypes. In other contexts, the representational strategy requires techniques and structures to re-connect with precise representations of the target system being modeled, i.e., so-called defuzzification procedures (more generally, strategies for the application of empirical models involve regulated deidealization procedures such as correction terms, etc.).

Approximation appears also in the methodological context of evaluation of applied formalism in two different dimensions: empirical testing and organization (unity and generality).

What is the basic empirical level of fuzzy description? Can a fuzzy model tested by an empirical support structure constituted by fuzzy data or does it require precise data? Basic conditions of empirical testability by precise data require coarse-graining and defuzzification procedures. This strategy requires contextual application of standards; but then it provides a structure of approximation that allows for the uniform—standardized—comparison between fuzzy models or between fuzzy and sharp models. Finally, one must introduce standards of significance that will establish for a given context whether the degree of proximity is good enough for the purpose at hand (analogous to the case of statistical benchmarks and standards of significance). The evaluation will be both normative and contextual, and the acceptable generality limited.

Alternatively, one must explain what fuzzy models of data look like and enable the testing of fuzzy models—or even sharp ones. Insofar as their semantic structure has the form of degrees of truth as a representation of degrees of an empirical property of the system, the challenge consists in distinguishing between degree of truth and error. This challenge is particularly challenging in cases of overlapping fuzzy models or fuzzy models with overlapping ranges—hot, warm, medium, cool, cold, etc.—informing their application to sharp data—temperatures values. As before, the coordination between fuzzy data models and the fuzzy or sharp theoretical models to be tested or constructed will require coarse-graining and defuzzification.

Also, in the context of evaluation, possibility theory models evidential claims. It involves a nesting of structures with an ordering centered around a focal element that plays the natural role of a target relative to which one can define a metric relation of length and approximation.

One methodological perspective on approximation will motivate the next. Assessment is often comparative also between alternative hypotheses or theories (or models or . . .); testing is a selective matter. In that situation, revision or rejection of a theory is involves assumptions about how different theories are related. Those assumptions effectively provide an ordering and organization of knowledge, assessing the theories' relative fundamentality and generality. This dimension of methodological assessment extends to formal theory as well as its application; and the strictest form it takes is a special unifying relation of reduction sought between these theories and between their concepts. From a formal standpoint, it effectively amounts to a relation of approximation, and it applies to the relevant context of application, local or global and more fundamental, like classical and quantum mechanics, or Newtonian and relativistic mechanics.

The approximation at stake is the relation of reduction between classical and fuzzy theories or concepts. In the empirical domain of application, fuzzy set theory revises and replaces classical set theory. It is said that one generalizes the other and in that sense approximates it. Does that mean that fuzzy set theory revises or replaces classical theory globally as the more fundamental theory? The justification for that claim might be its local empirical success; and the domain of validity of the claim might rest on a claim about fuzzy theory's more general domain of empirical application. In the shared domain of application, one can be said to reduce to the other to the point of identity; elsewhere it approximates it. But at the theoretical level this claim is formal, about an extended structure fixed by formal rules, definitions and justifications.

By analogy with empirical theories, one should be able to articulate, descriptively or normatively, the way in which fuzzy set theory reduces classical theory as more general and fundamental. Give my claims above, the fundamental and general status of fuzzy theory will not amount to a more marked analytic or a priori character of the claims of fuzzy set theory and its application. The relation between both theories does not seem to carry the ontological and explanatory significance that one can find in models of reduction of empirical theories. But determining the formal relation will contribute further to the task of clarifying the richness of the notion of approximation and its roles in fuzzy theory. In addition in interpretive contexts, the asymmetry of the relation proves significant beyond its purely formal features.

So, what is this reductive relation of approximation or, equivalently, the approximative relation of reduction? What characterizes and explains the difference in generality. Answering these questions require answering this one too, what are the relevant terms in the relation?

If we distinguish between the whole of a formal theory (relative to a given individuation criterion) from parts thereof, we can talk about inter-concept approximation upon which the relation between theories will depend, whether by mere inclusion or by formal integration. Parts may be concepts and axioms containing them.

Generalization models the relation of approximation. It supports not just the metric structure but also the additional asymmetry that characterizes its use in normative or interpretive contexts. The sense of generalization that applies to the notion of fuzzy set itself: "a fuzzy set is actually a generalized subset of a classical set" [16, p. 10]. Here formal generality of construction may be distinguished from empirical generality of application and evaluation. The latter depends on the former. That is, as a matter of empiricism, does fuzzy logic cover more cases and covers them better?

The universe from which fuzzy subsets get their members is represented by the classical concept of set. The formal generalization depends on generalizing the classical concept of membership or characteristic function: a subset is identified by a membership function as mapping from subset to valuation interval [0, 1]. Relations based on boundary conditions, e.g., at {0, 1}. The role of boundary conditions might be considered as a result of relation of inclusion of {0, 1} in [0, 1]. A metric measure for approximation may be defined over a partial ordering of membership grades, such as Hamming distance.

The boundary relation may be expressed as one of interchangeability at the boundaries. The criterion of approximation is the boundary condition, or set thereof, of value-identity for membership degrees 0 and 1. I specify value-identity because there is a sense in which the concepts remain different given their general theoretical definitions.

The generalization of the membership enables the generalization of operations. The standard elementary fuzzy operations of complement, union and intersection extend classical set operations for membership grade ranges generalized from {0, 1} to [0, 1], union $(A, B)$ as max $\{A(x), B(x)\}$ (smallest of the fuzzy unions based on membership grades) and intersection $(A, B)$ as min $\{A(x), B(x)\}$ (the largest of the fuzzy intersections based on membership grades). Boundary conditions for the crisp values 0 and 1 are multiple. Ex., fuzzy intersection $i(a, 1) = a$, which implies in turn (with monotonicity and commutativity) the following set of boundary conditions $i(1, b) = b$ and $i(a, 0) = i(0, b) = 0$, and $i(1, 0) = i(0, 1) = 0$, $i(1, 1) = 1$ and $i(0, 0, ) = 0$.

From the boundary constraints one could find a representation for the approximation relation in terms of concepts from fuzzy theory itself. One possibility is an algebraic limit relation; another is an inclusion relation. The difference consists in the fact that the first is shared by both fuzzy and classic theories, while the second must be proven first to be sufficiently shared, not just at the boundary. Which inclusion operation is sufficient for the job? If we consider the classic operation, we need to prove its applicability to elements of fuzzy theory, but this means that the first approximation is relative to a second, and, worse, this also pushes the conceptual problem back to the task of establishing the approximation relation between inclusion operations, on pain of a regress. How is the second relation to be established? the problem becomes that the second depends on itself, or alternatively we use the weaker criterion of the boundary condition.

A more elaborate approach is a logical relation based on the semantics of truth degrees for propositions containing the relevant set-theoretic predicates. The relevant propositions would be corresponding axioms introducing the relevant concepts in each theory. A concept of logical implication available from truth functions of logical connectives associated with fuzzy sets; classical logic should do. Now, of course, the condition of applicability of the implication connective becomes even more demanding because it involves a reduction or approximation relation that models the generalization relation between fuzzy and classical logics. Each logic involves a different implication connective. The connective in multivalent logics may be, for instance, Łukasiewicz's, with $v(p, q) = (1, 1 - v(p) + v(q))$. If we want to apply approximate reasoning theory, the relevant connective will be a t-norm.

The richer relation requires placing the related concepts in the context of the formal theory. The approximation must involve different set theories that include those concepts. For theoretical approximation the relevant units may be set theories in the form of lattices based on a set of fuzzy subsets of members of classical universe, with a partial ordering, operations of them (union, intersection, inclusion and complementation), satisfying a number of properties. The reductive approximation relation may connect poorer and richer embedded algebraic structures, Boolean lattices. We may say that fuzzy theory approximates the narrower target of classical theory if the basic concepts of subset and the operations do—as suggested.

The properties satisfied by the operations cannot be axioms in the theory or count as part of the algebraic structure or even of their definition; otherwise principles such as the principle of excluded middle will not naturally emerge by implication from classical theory or classical concepts.

The asymmetry of the approximation can be models different interpretations. On the semantics of truth degrees is, it models the approximation to sharp truth by degrees. The same classical target can model the epistemic interpretation of vagueness, in which the vagueness as indeterminacy of ignorance approximates the reality of features represented by sharp classical sets. In this context, the classical target of approximation clearly is granted normative value as the standard of truthful representation.

### *13.5.4 Fuzziness as Contextual*

Along the way I have been drawing attention to the role of context. The relevance of the role becomes amplified by the diversity of kinds. Each kind informs the significance of the concepts with a perspective, an issue, an interpretation, a norm or standard, etc.

In the context of application, for instance, one objectivity-driven mechanism renders the application of fuzzy concepts as local, specific or contextual: tallness predicated of people and of buildings involves different standards—different criteria & associated prototypes. The primary term assignment is specific to a set of propositions and may vary for another. Context dependence of application, or contextual meaning, is linked through standard(criteria)-dependence to standard(exemplar)-dependence.

In the same context, I have also mentioned cases of descriptive, interpretive conventions, such as causal criteria: they extend construction and constraint and enable the application of the formalism. Their relevance is particularly contextual; it is clearly a matter of limited, revisable descriptive and normative perspective, with its aim (explanation or control), standard (the criteria of causal relation), justification (formal and empirical grounds of the criterion) and interpretation (causal) of empirical reality. Approximation criteria share these kinds of features. They are the more contextual part of the theory's construction but their role is to provide contextual conditions of its application.

The picture I suggest of empirical practices with mathematical formalisms avoids the old view that applying a formalism is to given empirical interpretation to variables of a rigorous universal empty symbolic calculus. Part of the process of application is what I call preparation of the formalism; it involves formal and interpretive tailoring, and it is widely contextual in its relativity to descriptive and normative constraints from different situations, domains, problems, projects or, as the Poincaré example of approximations illustrates, disciplines. The formalism is often, as also the history of the calculus and fuzzy theory illustrates, made rigorous, purified, generalized after initial prototypical applications.

Empirical application is complicated at the level of application. The membership relation is not fixed just by convention, intentionally, beforehand. There is no fixed crisp "the concept" as conceptual gauge however operationalized, but by empirical judgment relative to contextualized standards embodying the predicate to some degree, max or min (relevant contextual sub-range). The sub-ranges characterizing each membership function lack cut-off points that are fixed, as in crisp variables, and are specified in overlapping, contextual, standard-dependent, but not in a completely arbitrary way.

The appeal to coarse-graining, defuzzifying, techniques and standards is a form of simplification; but based on a new kind of simplicity as conceptual, epistemic and pragmatic virtue. If thinking is a form of patterning by cutting or shaving, classical simplicity is shaving along the boundaries sharply with Occam's razor; fuzzy thinking is more like trimming, careful but not arbitrary.

In the context of assessment, then, we encounter the question of empirical evaluation and interpretation. Here a distinction between conditions of forced uncertainty and opted uncertainty differ in significance. Forced uncertainty includes margins of measurement error. Opted uncertainty introduces so-called task-relevance or decision-relevance summary conditions that eliminate unnecessary precision relative to pragmatic considerations of a cognitive or practical goal. It is misleading to draw the difference in terms the enhanced empirical significance at the expense of measures of fallibility: "Although the usual quantization of variables is capable of capturing limited resolutions of measuring instruments, it completely ignores the issue of measurement errors. While representing states of a variable by appropriate equivalence classes of its values is mathematically convenient, the ever-present measurement errors make this representation highly unrealistic. It can be made more realistic by expressing the states as fuzzy states" [26, p. 328].

Testability depends on defuzzification standards introduced externally, by means of a convention informed by contextual purposes and contextual exemplars. Testability depends on a dual translation or coordination. The constraints must translate fuzziness as conceptual approximation into empirical approximation, and degree of membership or truth into degree of measurement. The latter helps also assess the degree of error or accuracy. Only in this way, even contextually, the application of fuzzy formalism is empirical and fallible.

## 13.6 Fuzzy Logic Is Empirical, Subjective, Objective, Approximative and Contextual

In this final section, I sketch out an even briefer survey of how the same issues appear in relation to fuzzy logic. Needless to say, the kind and form of the issues is in many cases an extension or a replication derivative from their occurrence in set theory, warranted by the isomorphism between the lattice structure on fuzzy sets as sets of valuation mappings with operations on them and the lattice of propositions and connectives.

### *13.6.1 Fuzzy Logic as Empirical: The Case of Quantum Logic*

The issue of the empirical or factual status of logic has long been considered even more consequential (no pun intended) than the case of mathematics. From Frege and Poincaré to the logical empiricist philosophers it was recognized that, as Duhem has noted, the effective methodological—motivating and justifying—role of experience depended on rules, concepts and additional background knowledge that could not be immediately reduced to factual information linked to some form of perception (or equivalent factual mechanisms of interaction with the our environment). Above all are normative standards of reasoning that secure agreement in science and beyond and warrant the objectivity of their procedures and products. This emphasis connects with a long tradition that has privileged rationality, the so-called faculty of reason and its ways as cognitively—even morally—superior to any other. The kind of knowledge it yields was distinctively a priori and its claims analytic, independent of perception for the purposes of intelligibility and justification.

Needless to say, by way of a qualification, it is a related but separate issue whether empirical logic is not just empirical (and in what sense) but a logic (and in what sense).

A logic or logical system might be characterized as follows: a theory of consequence and validity in closed axiomatic systems of (deductive) inferences between propositions with a model-theoretic semantics of algebraic models or syntactic models of proof. Logic can be understood as a language or calculus characterized by a set of propositions (well formed formulas related by connectives) and a set of conditions and rules on them (axioms and rules of inference) [34].

For finite number of propositions or logical formula, classical logic is isomorphic to set theory and Boolean algebra; this is the source of the focus on the mathematical structure of Boolean lattices as common formal framework.

Classical logic is characterized by propositions that are semantically decidable (laws of excluded middle and non-contradiction), their meanings are sharp and unambiguous, and behave compositionally [10, p. 127].

Fuzzy logic (and Brouwer-Zadeh logic including fuzzy and intuitionistic axioms) and fuzzy set theory are an extension of quantum logic and set theory in which non-contradiction and excluded middle are violated by fuzzy negation (double negation satisfied). Quantum logic is a weaker extension of classical logic without distributive laws. In particular, quantum (computational) logic violates only weak distributivity, as well as laws of excluded middle and non-contradiction (and idempotence, so not a proper lattice); and it may be considered a fuzzy logic.

Logical principles are, then, at least quasi-tautologies, with truth value greater than 0, and in classical logic always 1. One may speak, accordingly, of quasi-analyticity, in the sense I have noted in my discussion of set theory. Quasi-analyticity opens the door to the relevance of empirical grounds to motivate or justify generalized logical principles.

As in set theory, also in fuzzy logic the project of fuzzy theory is a naturalistic one, and its normative force derives from its empirical relevance and pragmatic adequacy. It aims to represent factual information about behavior, human and technological (the theory has since been applied to an extended domain including useful descriptions of natural phenomena). To speak of naturalism and factual knowledge helps avoid the reductive misinterpretation of empiricism as a radical matter of pure experience. This is not the kind of naturalistic empiricism that I claim applies to the fuzzy project. I emphasize, instead, the realistic, complicated understanding of human and scientific practices.

As before here it helps to distinguish between the different, but related, contexts of empirical construction, application and evaluation. Testing involves all three but its significance lies in the third. Empirical testing has a dual character: a positive, support, and a negative, revisability. If logic, not just mathematics, is in any sense empirical, as a logic of the world, it's only partially, contextually, a priori and its analytic character provides no only partial, contextual immunity against revisability. It cannot be universally and exclusively true. As in the case of set theory, empirical grounds exist to motivate construction and provide a measure of support; but formal constraints in terms of pre-available concepts, rules, standards and commitments, are equally, if not more, determining, both constructively and normatively. From Poincaré to Carnap, Quine, Putnam and Haack, the matter of revisability of logical, mathematical and empirical claims is a relative, contextual and also a pragmatic matter.[22]

Again, one can identify two kinds of empirical grounds for fuzzy and approximate reasoning: So-called informal behavior and scientific or empirical natural results. Informal grounds, as in the case of fuzzy set theory, involve formalizing concepts expressed in natural language and rules involved in informal reasoning. By virtue of its relation to language and categorization, it rests on empirical and conceptual elements of fuzzy set theory. In Zadeh's words: "Informally, by approximate or, equivalently, fuzzy reasoning, we mean the process or processes by which a possibly imprecise conclusion is deduced from a collection of imprecise premises. Such reasoning is, for the most part, qualitative rather than quantitative in nature and almost all of it falls outside of the domain of applicability of classical logic."[23]

Additional empirical grounds mentioned are scientific, based on a historical connection of multi-valued logics to physics: "It is known that truth values of certain propositions in quantum mechanics are inherently indeterminate" [26, p. 217]. By the internal standards of our best representations of the factual world, classical and fuzzy logics are logic, and classic and fuzzy set theories are mathematical in the same way that Euclidean and non-Euclidean geometries are theories of geometry. This analogy has interpretive implications for the issue of empirical dimension of fuzzy logic

[22] See, for instance, [23, 38]. About Putnam, more below.

[23] Cited in [16, p. 173].

(and set theory). What's the privileged empirical status of quantum propositions that distinguishes them from the domain of application of fuzzy formalism? How can quantum theory add any degree of motivation or warrant to the general fuzzy project?

In the history and philosophy of quantum mechanics, the attention to its relevance to logic goes back to Hans Reichenbach, the physicist and empiricist philosopher proponent of three-valued logic. Two related commitments are in place here; that quantum physics is more general, that is, that in the same way as Zadeh notes about fuzzy logic, the domain of applicability and justification of quantum theory extends beyond the classical domain. Revisability is not a matter of local empirical success. Fuzzy theory is more general on the assumption that fuzzy logic, like fuzzy set theory, generalizes—and, thus, it also includes—classical theory. One is in that sense an approximation to the other. But generality is not the sole virtue. Quantum theory is also taken to be more fundamental; in part because the reduction is not just a matter of empirical generality, but of material and spatial decomposability.

The normative commitment is, besides empiricism, reductionism (Reichenbach and Quine leading the way); for some, such as Putnam (Reichenbach's student and Quine's colleague), it is realism about theories as well. Features of fundamental theories carry superior exemplary force; on that strength and interpretation, they are adopted as methodological standards. The question of revisability, its grounds and scope, of classical theory is linked, then, to its relation to its alleged generalization, quantum or fuzzy and to their ontological and methodological interpretation.

The meaning of set-theoretic operations and logical connectives alike and the methodological status of claims that rest on them become a matter of normative approximation, a relation between classical and non-classical theories. The two issues are inseparable.

The analogy to the case of quantum logic is instructive for another reason. The role of the normative methodological commitments–empiricism, naturalism pragmatism, conventionalism, reductionism, realism, etc.–is, by virtue of their contextual diversity and contingency, not trivially or generally determining. I have noted two opposite views: one, that logic is a priori and analytic and, therefore, unrevisable, empirically immune; and, the other, that alternative logical theories are in fact suggested and supported by empirical theory such as quantum mechanics (as general relativity theory does in geometry). On the second view, this is a matter of empiricism: Because empirical theory is not methodologically and structurally independent of formal claims, the application principles of classical logic to experimental propositions lead in the mathematical formalism of quantum mechanics to paradoxes.

The second position as a defense of alternative logics and the role of empirical theory has been challenged too. One extreme version of the challenge agrees, with Quine and others, that classical logic is not an unrevisable theory, but disagrees with Reichenbach and others that this is a matter of physical theory; instead, the claim is grounded on the diversity of formal alternatives—as it was once the reaction to non-Euclidean geometries. Quantum theory is unnecessary and insufficient for the job, and, in in fuzzy theory, to the extent that it is empirical, it is an empirically inadequate formal model of human behavior and technology that applies it but not a logic.[24] They provide local challenges, such as the alleged paradoxes, but ground no global reform in logic in the presence of less radical correcting adjustments.

The other, empiricist, version of the challenge is illustrated by Putnam's evolving positions on the issue. For Putnam and others on his footsteps, quantum logic prevents the formulation or derivation of paradoxes based on classical logic in the organization of quantum statements and reasoning, mathematical formalism of quantum mechanics (Hilbert vector spaces and probability calculus) [1, 14, 19, 36]. As a result, quantum logic could be considered an empirical calculus, much as probability theory can, namely, as a convenient abstraction from physical models—set of experimental propositions and statistical operators—of possible physical events defined on quantum phase space, suitably formalized; [12] or else as sets of "empirical" propositions of physical theory.

As an empirical matter of ontological interpretation, for Putnam it was neither a form of logical instrumentalism nor of realism. It was similar to the adoption of conventions, but not reducible to them. Logic and geometry are revisable but empirical, not purely conventional matters. For Putnam empirical claims may involve conventions, but these are neither empirical conventions (operationalism) nor pure conventions as definitions in disguise (radical conventionalism).

But theoretical and ontological commitments in interpretation may vary, and Putnam and others' later different views suggest that the argument about quantum logic depends on them. Different commitments lead to different positions. One such variant position, the global rejection of classical logic requires a rejection of any instrumental interpretations of quantum propositions, including Bohr's version of the Copenhagen interpretation. The justification of the empirical claim in favor of quantum logic depends, rather, on interpretations of quantum mechanics such as Everett's many-worlds interpretation [1].

In other interpretations classical logic will do, and hence arguably it may still be considered more fundamental: in de Broglie-Bohm, with the postulation of hidden variables, a conventional choice that preserves classical logic through physical hypotheses like Euclidean topology can be preserved in gravitation theory with additional physical hypotheses; or through a necessary complement (suited to the kinematics) for describing the non-standard dynamics of quantum states (spontaneous collapse) in terms of classical disjunctions. In those cases quantum logic cannot provide an alternative foundation for classical logic and its connectives. So the empirical matter is restricted to non-empirical grounds for the choice of interpretation (Ibid.).

For Putnam it is the commitment to realism that justifies a rejection of instrumentalist and Bohrian quantum interpretations and an endorsement of hidden-variable theories obeying classical physics with joint sharp, determinate values of those dynamical variables (despite any causal and other ontological anomalies concerning non-locality and non-separability) [38]. At the expense of (Reichenbach's) the commitment to standards of representation such as locality, empirically well-supported hidden-variable theories are nevertheless consistent with classical logic (with laws

[24]This is Haack's view in [23].

of excluded middle and distributivity organizing the description of the two-slit experiment).

It is normative matter of commitments and their ordered priorities in the contexts of methodology—naturalism, physicalism—interpretation—realism—and evaluation —sophisticated empiricism. If logic is the logic of the world—naturalism—our best theories about it and our preferred interpretations thereof, in the light of our commitments, lead Putnam to conclude that quantum theory supports classical logic. But, ironically, this also leaves quantum logic, as a matter of comparative evaluation, as an empirical hypothesis.

There is a lesson for semantics and set theory here as well. Putnam understands the empirical argument to be a matter of inference or proof, but not of semantics. The principles characterizing the properties of logical connectives, such as distributivity, may undergo modification without implying in turn a modification in the underlying semantics, other than, perhaps, indeterminacy—as in Reichenbach's third truth value in quantum logic and in intuitionism, as in Putnam's views on reasoning with vague predicates. Putnam denies that physicalism, even without his earlier reductionism, and quantum theory have anything to do with semantics. So he claims to avoid a commitment to a specific semantics based on his commitment to quantum realism. But one might challenge his assumption about the meaning of connectives; moreover, his two commitments become effectively coextensive as his quantum realism supports the sharp bivalent semantics of classical logic.[25]

The lesson for the empirical—or naturalist—status of fuzzy logic is this: once one adopts a naturalistic attitude towards logic and a sophisticated empiricist approach to methodology, the choice of fuzzy logic, like that of set theory, is a matter of commitments and as fallible, revisable and contextual as classical logic itself. From the same standpoint of a sophisticated methodological empiricism, if it is fallible, it can be a priori or analytic only in a contextual way and in relation to alternative empirical hypothesis.

### *13.6.2 Fuzzy Logic as Subjective and Objective*

In this section I can only add a point of emphasis concerning fuzzy logic. To the extent that it rests on set-theoretic semantic assumptions, the features presented in the earlier section carry over to logic case. The chief difference is that objectivity here relies more centrally on the mechanical formal rules that aim to constraint and represent what effectively is a descriptive and prescriptive model of human rationality.

[25]For an imprecise quantum set theory with references to Takeuti's non-extensive quantum sets and Da Costa's and Krause's quasi-sets see, for instance, [12]. Quantum logics will generally have semantic set-theoretic realizations or models that violate also the law of non-contradiction and even of excluded middle.

### 13.6.3 Fuzzy Logic as Approximative

Fuzzy operations enable approximate reasoning. What is approximate about approximate reasoning? It is approximate at least in the same way, but with a different function, that fuzzy set theory is approximate. The aim of fuzzy logic is inference, not just linguistic categorization or truth; a higher-order cognitive process.

Testability and empirical character rest on considerations of a comparative assessment, and hence the assumption of a reductive approximation to classical logic. The same consideration applies to set theory. Fuzzy logic, like quantum theory, assumes the character of a more general and more fundamental alternative to its classical counterpart. As in the case of set theory, there is a matter of what form the relation of reductive approximation is and what assumptions, contexts and norms may apply. One issue is the form of the relevant kind of relation between concepts or axioms and then, by inclusion or integration, the corresponding relation between logical systems.

As before, two connected relations require consideration: generalization and implication. How do fuzzy logical connectives generalize classical ones? Which implies the other? And, how? And what does it mean? Is there any asymmetry imposed by judgments of construction, application or evaluation?

As in set theory, formal generality of construction may be distinguished from empirical generality of application and evaluation. The latter depends on the former. That is, as a matter of empiricism, does fuzzy logic cover more cases and covers them better? The approximation in terms of formal generalization relies, as in set theory, on the relation of inclusion of a set of values, {0, 1} in another, [0, 1], which is larger: "All fuzzy implications are obtained by generalizing the implication operator of classical logic. That is, they collapse to the classical implication when truth values are restricted to 0 and 1" [26, p. 308]. The measure of approximation is parametrized by the ordered set of truth values. But boundary conditions must apply. In particular, as an a matter boundary conditions on which the inclusion relies, satisfaction for all values in the interval can be proven: the only necessary conditions, by Gaines' theorem, are set by the trivalent identity at {0, 1/2, 1}.

For implication connectives, the challenge arises within fuzzy theory. There is no unitary linear partial ordering for the family of connectives, which form a lattice with a partition between two equivalent classes.[26] Formal boundary conditions between classical and fuzzy rules such as modus ponens do not fix the form of the fuzzy implication connective uniquely. Moreover, validity itself acquires a continuous numerical representation missing in classical logic. Validity is replaced by a generalization by degrees.

At the level of implication, formally or truth-functionally, it is simpler to consider how fuzzy connectives will reduce, that is, approximate and imply, classical ones by taking the limit to the reduced bivalent set. The other direction poses new challenges: the possible consequences and mappings are not uniquely fixed, especially t-norms. The family itself yields no unitary ordering sequence of implications. Formally, then,

[26]See ibid., Fig. 11.2.

the relation of approximation is not symmetric. Also, as in set theory, one also requires to prove that classical implication will be satisfied in the fuzzy system, for each t-norm, for instance. So, as a difficulty, one finds that the relation between connectives already requires a larger structure, and is not independent from the relation between the corresponding logical systems.

In terms of axioms, to prove the both directions of implication is tricky too, since some principles, such as bivalence or excluded middle, do not have a counterpart in fuzzy logic, except in the form of their violation. So one would have to start from a reconstruction that already includes statements of the classical axioms and their violation. For the same reason, reduction of the classical theory by derivation ab initio is unrealistic, as it also is in the absence of auxiliary conditions coordinating or translating terms from different theories and specifying the conditions of the relevant relations of reduction. The theory or system with a higher generality of applicability does not, structurally speaking, contain the reduced or approximated one. In fact it's the reverse; the relation to the classical system is of sub-structurality, as the number of rules and axioms in the fuzzy one is reduced.

### *13.6.4 Fuzzy Logic as Contextual*

The relativity, locality and more broadly limited aspect of fuzzy theory and its uses constitute its contextual character. The elements that inform the different contexts, the diversity and interrelatedness of the contexts all provide the very conditions of possibility of the formal and empirical successes of fuzzy theory.

Perspectives for understanding the formalism of fuzzy logic, as well as set theory, include the contexts of construction, application, evaluation, interpretation, etc. Those roles are both distinctive and interrelated.

For instance, as in the case of set theory, the construction of the generalized formalism is constrained and justified by relations to classical resources and non-classical ones. But the justification of generalization by construction is different from the case of generalization in application. The former, with the extension to a broader set of truth values, is a condition of the latter, but the latter has its own additional particular empirical sources of motivation and testing, e.g., the contexts of human reasoning modeling, technological application and decision making, with specific and different aims and standards of representation and regulation. Here we encounter the pragmatic considerations of the external practical context.

The most direct source of contextuality derives directly from the same local, limited constraints imposed on the subjective source of fuzzy concepts and claims.

The same contextuality and normativity characterizes the question of appropriate choice of the fuzzy implication connective for reasoning in the different empirical situations. The form of fuzzy implication, like the value assignation of a membership function, isn't uniquely determined, or general across all relevant empirical contexts: "To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem. Although some theoretically supported guidelines are now available for some situations, we are still far from a general solution to this problem" [26, p. 312]. Meaningful criteria vary by context and purpose.

Another examples is the contextual character of the application of fuzzy validity. The fuzzy standard of sufficient validity rests conceptually on the contextual fuzzy notion of sufficient truth. Valid enough in a given context is relative to a standard selected for a purpose at hand, e.g., 0.5, but not arbitrarily [41, p. 221]. The application and normative roles depend on a particular choice; a logical equivalent to the evaluative role of benchmarks for statistical significance in empirical testing.

Contextuality is expressed in the formulation and implementation of normative criteria of admissibility such as Gaines': good enough means that no simpler or equally simple model is a better approximation to the data. Without such criteria one must simply opt for sacrificing one standard to a conflicting one, and in fuzzy dynamic systems, for instance, Zadeh warned with his principle of incompatibility that in the study of complex systems, significance or relevance is sacrificed to precision [16, p. 189].

The standard is also a form of precisification or defuzzification that connects with classical logic; in the same way that standards of significance contribute to the testability of empirical hypotheses against precise measurement results. This kind of complex coordination in application is also one of the contexts of application approximation, for calculation (constructive, normative constraint), interpretation and evaluation.

The contextuality of fuzzy logic, more than the contextuality of fuzzy set theory, stands in contrast with the case of classical logic as the normative ideal of precise and universal rationality and reasoning. It involves, then, a revision of normative ideals and standards.

Finally, taking seriously the contextuality of approximate as an enabling condition, rather than a negative limitation, of its conceptual and methodological force, will invite critical attention to the limits of generality as an aim of science. New lessons may still be learned, for instance, from recent criticisms of the generality or ecological validity of the causal and evidentiary conclusions of randomized control trials and of other statistical and probabilistic models [6, 30].

Fuzzy set theory has been applied to the classification of images as well as to the application of words associated with categorization. However, modeling approximate informal reasoning has neglected the diversity of roles images play in reasoning, ordinary and scientific, alongside recognition and representation. Fuzzy logic can extend the scope of its application to visual thinking, with images or about them.

## 13.7 Conclusion

I have examined fuzzy set theory and fuzzy logic broadly from a non-formal perspective and as a practice of application of mathematics. The present and future of the fuzzy project requires its critical and creative understanding. Sometimes the understanding exceeds the bound of formalism. This may be helpful because relevant intellectual considerations within and outside the technical practices lie also outside the formalism, both conceptually and technologically. It is another example of situational character or contextuality of real science.

The intellectual framework I have introduced to situate fuzzy theory focuses on several issues or dimensions: empiricism, subjectivity, approximation, contextuality and normativity. These issues are inseparable. My aim has been to integrate and differentiate fuzzy theory within this framework, to determine it in those terms, illustrating them, while pointing to its particularities. As a result, I have complicated both levels of exploration: a more complicated intellectual picture of the fuzzy project emerges, while the broad brushstrokes of my discussion should help inform and enrich the general picture of scientific practice in turn. From the more complicated particular picture of fuzzy theory emerges a model of formal practices that centers on the complex roles and value of subjectivity and approximation, radically contextual, relative to standards and goals.

A contextual and complex empiricism, as I endorse, requires the role of elements functioning contextually a priori and empirically; that is, a locally constructive and constraining source and function of concepts and rules, normative and interpretive. In application, this kind of complex coordination of formalism and factual information is one of the contexts of application approximation, for calculation (constructive, normative constraint), interpretation and evaluation. Approximation here is more complicated and general a concept that typically presented. This is part of my picture of the application of mathematics as involving contextual preparation of a constrained and often interpreted symbolic structure.

For the fuzzy project, this is critical and constructive contextualism: approximation, subjectivity and context should be perceived critically and creatively, as instruments and opportunities, not obstacles, in the pursuit of science and technology. They constitute sources of methodological plasticity that helps adjust in different contexts and to different and revisable goals and standards without fear or arbitrariness. Fuzzy logic and mathematics are not just true of technological systems; their truth and status are constructive also as formal technologies.

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# Chapter 14
# Formalizing the Informal, Precisiating the Imprecise: How Fuzzy Logic Can Help Mathematicians and Physicists by Formalizing Their Intuitive Ideas

**Olga Kosheleva, Renata Reiser and Vladik Kreinovich**

**Abstract** Fuzzy methodology transforms expert ideas—formulated in terms of words from natural language—into precise rules and formulas. In this paper, we show that by applying this methodology to intuitive physical and mathematical ideas, we can get known fundamental physical equations and known mathematical techniques for solving these equations. This fact makes us confident that in the future, fuzzy techniques will help physicists and mathematicians to transform their imprecise ideas into new physical equations and new techniques for solving these equations.

## 14.1 Fuzzy and Physics: Past and Present

**Fuzzy methodology: main objective**. Fuzzy methodology has been invented to transform expert ideas—formulated in terms of words from natural language—into precise rules and formulas, rules and formulas understandable by a computer (and implementable on a computer); see, e.g., [6, 10, 12].

**Fuzzy methodology: numerous successes**. Fuzzy methodology has led to many successful applications, especially in intelligent control [6, 10].

**What is fuzzy methodology good for? Traditional viewpoint**. When are soft fuzzy techniques mostly used now? Let us take, as an example, *control*, which is one of the major success stories of fuzzy methods.

In control, if we know the exact equations that describe the controlled system, and if we know the exact objective function of the control, then we can often apply

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_14

the optimal control techniques developed in traditional (crisp) control theory and compute the *optimal* control.

Even in these situations, we can, in principle, use soft computing methods instead: e.g., we can use simpler fuzzy control rules instead of (more complicated) traditional techniques. As a result, we may get a control that is much easier to compute but that it somewhat worse in quality.

However, the major application of fuzzy methodology in control is to the situations when we only have *partial* knowledge about the controlled system and about the objective functions and in which, therefore, traditional optimal control theory is not directly applicable. Here is where all known success stories come from: utilities like washing machines or camcoders, car parking automation, and other applications all share one thing: they all have to operate in a partially known environment.

From this viewpoint, as we gain more and more knowledge about a system, a moment comes when we do not need to use fuzzy techniques any longer: when we have accumulated enough knowledge, we will then be able to use traditional (crisp) techniques.

From this viewpoint, fuzzy techniques look like a (successful but still) *intermediate* step, "poor man's" data processing techniques, that need to be used only if we cannot apply "more optimal" traditional methods.

**Physics and applied mathematics as application areas**. When we study the physical world, our first objective is to find the *physical laws*, the *equations* that describe how the values of physical quantities change with time.

Once we have found these equations, the next task is *mathematical*: we need to solve these equations to predict the future values of physical quantities.

Both tasks are not easy. In both tasks, we start with informal ideas, and gradually move to exact equations and exact algorithms for solving these equations.

**Current use of fuzzy techniques in physics and applied mathematics**. The current use of fuzzy techniques in physics and applied mathematics following the similar lines as other applications to fuzzy methodology: fuzzy techniques are useful when we do not know the exact equations describing the corresponding system; see, e.g., [3, 6, 10, 11] and references therein.

Once we know these equations, more traditional (crisp) techniques can be used to solve these equations.

## 14.2 Fuzzy and Physics: Towards the Future

**A natural new idea of using fuzzy techniques**. As we have mentioned, physics research has two main objectives:

- to find the *physical laws*, the *equations* that describe how the values of physical quantities change with time, and

- to solve these equations, so that we will be able to predict the future values of physical quantities.

In both tasks, we start with *informal* ideas, and gradually move to *exact* equations and *exact* algorithms for solving these equations.

But transforming informal ideas into exact ones, their *precisiation* is exactly what fuzzy techniques have been invented for. It is therefore reasonable to use fuzzy techniques for this precisiation.

In this section, we will show that fuzzy logic techniques can indeed help in transforming informal ideas into exact equations and algorithms.

*Comment*. Preliminary versions of some of the results from this section first appeared in [5, 7, 8]; similar results are also presented in [9].

### *14.2.1 Case Study: Newton's Physics*

Let us start our analysis with the simplest example of physical equations: namely, with the equations of Newton's physics.

**Newton's physics: informal description**. Let us consider a simple case when we have a single body in a potential field $V(x)$. It is a commonsense knowledge that a body usually tries to go to the points $x$ where its potential energy $V(x)$ is the smallest. For example, a rock left at the top of the mountain, when it starts moving, it may sometimes move up (due to the original push), but mostly it tries to go down.

If we take friction into account, then a body also tries to stop. In the idealized case when there is no friction, there is a conservation of energy: the sum of the potential energy $V(x)$ and the kinetic energy $K = \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2$ is constant. Thus, when the body minimizes its potential energy, it thus tries to maximize its kinetic energy.

**What we plan to do**. The above text does not sound like a very accurate description of a physical system. However, we will show that when we apply the usual fuzzy methodology to this description, we get a very precise formulation—all the way to Newton's equations

$$m \cdot \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i}. \tag{14.1}$$

We will perform this derivation step-by-step.

**First step: selecting a physically meaningful membership function corresponding to "small $V(x)$"**. The body tries to get to the areas where the potential energy $V(x)$ is small. "Small" is an imprecise word from natural language. For such words, the fuzzy methodology recommends to select a membership function $\mu(V)$ describing, for each possible value $V$, to what extent this value $V$ is small.

**How membership functions are determined: one of the possible ways**. For each individual value $V$, the value $\mu(V)$ can be obtained, e.g., by polling several ($n$) experts and assigning, as $\mu(V)$, the ratio $\mu(V) = \dfrac{n(V)}{n}$, where $n(V)$ is the number of experts who answered that this value $V$ is small.

**Specifics of a physical system**. In assigning the appropriate membership function, we must take into account the specifics of the physical system. One of the features of this physical system (see, e.g., [4]) is that the potential energy has no absolute numerical value. All we know is the *relative* potential energy relative to some level. If we change that level by some value $V_0$, then all the numerical values of the potential energy get shifted, from $V$ to $V + V_0$.

Crudely speaking, this means that the numerical values $V$ and $V + V_0$ may represent the exact same value of the potential energy—but measured in comparison to different levels.

**How to describe these physical specifics: first try**. A seemingly natural formalization of this idea is to simply require that the degrees to which the values $V$ and $V + V_0$ are small should be the same: $\mu(V) = \mu(V + V_0)$.

**The first try does not work**. However, this formalization does not work: if we require that this equality holds for all $V$ and $V_0$, then for every two real numbers $V$ and $V'$, by taking $V_0 = V' - V$, we would be able to conclude that $\mu(V) = \mu(V + V_0) = \mu(V')$—and thus, that the resulting membership function is simply constant. This does not make intuitive sense, since we know that the smaller the value $V$, the larger should be our confidence that $V$ is small.

**A better idea**. We therefore cannot simply require that the functions $\mu(V)$ and $\mu(V + V_0)$ corresponding to two different levels are identical. However, we should require that these two membership functions be, to some extent, equivalent to each other.

**How to formalize this idea: re-analyzing the polling method**. How can we formalize this idea? To do that, let us go back to the polling method of determining a membership function.

Our objective is to find the value $\mu(V)$ as accurately as possible. It is known that in the poll, the more people we ask, the more accurate is the resulting opinion. Thus, a natural way to improve the accuracy of the poll is to ask more experts. However, there is a catch. When at first, we could only afford to poll $n$ people, we thus selected top experts in the field. Now that we add $m$ extra folks, these folks may be too intimidated by the original experts to voice their opinions—especially in case the original experts disagree. With the new experts mute, we still have the same number $n(V)$ of experts who believe that the value $V$ is small—but now we have to divide it not by the original number $n$, but by the new number $n + m$. As a result, instead of the original value $\mu(V) = \dfrac{n(N)}{n}$, we get a new value $\mu'(V) = \dfrac{n(N)}{n+m}$. It is easy to see that $\mu'(V) = c \cdot \mu(V)$, where $c = \dfrac{n}{n+m}$.

Thus, for the exact same opinion, by selecting two different numbers of experts $n$ and $n+m$, we get two numerically different membership functions: $\mu(V)$ and $c \cdot \mu(V)$. These two membership functions represent the same expert opinion and are, thus, equivalent in some reasonable sense.

**Resulting formalization of the physical intuition**. Now, we have a meaningful interpretation of the requirement that the membership functions $\mu(V)$ and $\mu(V+V_0)$—corresponding to two different starting levels for measuring potential energy—are equivalent: that for every $V_0$, there should be a value $c(V_0)$ for which $\mu(V+V_0) = c(V_0) \cdot \mu(V)$.

**Resulting selection of the membership function**. This functional equation is known (see, e.g., [1]). Its only monotonic solution is a function

$$\mu(V) = a \cdot \exp(-k \cdot V).$$

So we will use this exponential function to describe the fact that potential energy should be small.

*Comment*. Since, as we have mentioned, the membership function is determined modulo a factor $c$, we can, for simplicity, set $a$ to 1 and get an even simpler formula $\mu(V) = \exp(-k \cdot V)$.

**Second step: selecting a membership function corresponding to "large value of kinetic energy $K$"**. As we have mentioned, kinetic energy tends to increase, i.e., should be large.

Instead of starting a derivation from scratch, let us use the fact that we already have a physically meaningful membership function for "small". Intuitively, a value $K$ is large if $-K$ is small.

So, the statement "kinetic energy $K$ is large" is equivalent to saying "the value $-K$ is small". By using the above membership function for small, we thus conclude that the membership function describing our intuition about the kinetic energy is $\mu(K) = \exp(-k \cdot (-K)) = \exp(k \cdot K)$.

**Third step: selecting a physically meaningful t-norm ("and"-operation)**. We want to describe the intuition that the potential energy is small *and* that the kinetic energy is large *and* that the same is true at different moments of time. According to fuzzy methodology, we must therefore select an appropriate "and"-operation (t-norm) to combine our degrees of certainty in individual statements into a single degree describing the degree to which we believe in the composite statement.

Let us use physical intuition to select such a t-norm $f_\&(a, b)$.

**Specific of the physical system**. In principle, if we have two completely independent systems, we can consider them as a single system. Since these systems do not interact with each other, the total energy $E$ of the combined system is simply equal to the sum $E_1 + E_2$ of the energies of the components.

**Using the physical specifics**. Intuitively, if both component energies are small, then the resulting total energy should also be small. We can therefore estimate the smallness of the total energy in two different ways:

- first, we can simply apply the above membership function "small" to the total energy $E = E_1 + E_2$, and get the value $\mu(E_1 + E_2)$;
- second, we can first estimate the degrees $\mu(E_1)$ and $\mu(E_2)$ to which each of the components is small, and then use a t-norm $f_\&(a, b)$ to combine these degrees into a degree that $E_1$ is small *and* $E_2$ is small: $f_\&(\mu(E_1), \mu(E_2))$.

In view of the above motivation, it is reasonable to require that these two estimates should coincide, i.e., that we should have $\mu(E_1 + E_2) = f_\&((\mu(E_1), \mu(E_2))$. We know that $\mu(E) = \exp(-k \cdot E)$, thus, we conclude that the following equality should hold for all $E_1$ and $E_2$:

$$\exp(-k \cdot (E_1 + E_2)) = f_\&(\exp(-k \cdot E_1), \exp(-k \cdot E_2)).$$

This requirement enables us to uniquely determine the corresponding t-norm. Namely, to find the value $f_\&(a_1, a_2)$, we must first find the values $E_i$ for which $\exp(-k \cdot E_i) = a_i$. For these values, we then have

$$f_\&(a_1, a_2) = \exp(-k \cdot (E_1 + E_2)) = \exp(-k \cdot E_1) \cdot \exp(-k \cdot E_2) = a_1 \cdot a_2.$$

**Resulting selection of a t-norm**. Thus, the physically meaningful t-norm is the algebraic product $f_\&(a_1, a_2) = a_1 \cdot a_2$.

**Resulting model**. Now, we are ready to estimate to what extent a given trajectory $x(t)$ satisfies the intuitive ideas that the potential energy be small and the kinetic energy be large at all moments of time $t_1, \ldots, t_N$. We know the degrees to which each of these requirements is satisfied at each moment of time, so to get the overall degree, we can simply multiply all these degrees. As a result, we get the following product: $\prod_{i=1}^{N} \exp(-k \cdot V(t_i)) \cdot \prod_{i=1}^{N} \exp(k \cdot K(t_i))$. Since, as we have already mentioned,

$$\exp(-k \cdot a) \cdot \exp(-k \cdot b) = \exp(-k \cdot (a + b)),$$

this expression can be reformulated as $\exp(-k \cdot S)$, where $S \stackrel{\text{def}}{=} \sum_{i=1}^{N} (V(t_i) - K(t_i))$. It is reasonable to select, as the most reasonable, a trajectory for which our degree of confidence that this trajectory is reasonable is the highest. To find such a trajectory, we must maximize the value $\exp(-k \cdot S)$. Since the function $\exp(-k \cdot S)$ is strictly decreasing, this is equivalent to minimizing $S$.

So, we arrive at the requirement that we should minimize the sum $S$. In reality, the number of moments of time is infinite, so instead of a sum, we get an integral $S \sim \int L\,dt$, where we denoted

$$L = V(t) - K(t) = V(t) - \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2.$$

**This model leads to Newton's equations**. In modern physics, most physical laws are formulated in terms of the Principle of Least Action, according to which the trajectory is selected in such a way that the *action* $S = \int L\,dt$ is the smallest possible. In particular, for Newtonian physics, the exact same expression $S$—as we came up with based on fuzzy methodology—leads *exactly* to Newton's laws; see, Appendix 1.

*Comment*. With the fuzzy approach, we not only get the most reasonable Newton's trajectory, we also get the degree $\exp(-k \cdot S)$ with which all other trajectories are reasonable. In Newton's physics, only one trajectory is possible, but in quantum physics, non-Newtonian trajectories are also possible, and the "amplitude" of each trajectory is determined by exactly this formula $\exp(-k \cdot S)$, albeit with a complex value $k$. This fact makes the above derivation even more interesting.

### *14.2.2 Beyond the Simplest Netwon's Equations*

**Need to go beyond Newton's equations**. In our analysis of the Newton's equations, we assume that the expression for the potential energy $V(x)$ is given. However, in reality, this expression also needs to be determined. The potential energy represents a *field*—e.g., electrostatic, gravitational, etc.—so, in addition to mechanics, we must also find the equations that describe the corresponding field.

**Gravitational field: main idea**. Let us consider the simplest case of a gravitational field. We will consider it in the Newtonian approximation, where it is described by a scalar function $V(x)$.

The main physical property of the gravitational field is that it changes very slowly: gravitational pull of the Earth, for example, is caused by the Earth as a whole, so if we move a little bit, we still feel approximately the same gravitation. It is a known empirical fact that the differences in the gravitational field at different earth locations are very small (but, by the way, very important for geophysics, because they provide a good overall understanding of what is located below the Earth surface).

Thus, all the components $\partial V \partial x_i$ of the gradient of the gravitational field must be small. This situation is similar to kinetic energy and different from potential energy in the sense that we want these values to be close to 0. Similarly to the case of kinetic energy, this is equivalent to requiring that the squares of the derivatives be small.

**Derivation of the resulting model**. Thus, we arrive at the condition that for all locations $x$, all squares of partial derivatives must be small. For each location and for each $i$, the corresponding requirement that the square of the derivative is small can be described by the degree $\exp\left(-k\cdot\left(\dfrac{\partial V}{\partial x_i}\right)^2\right)$. By using the product t-norm to combine these values, we get the expression $\prod\limits_x \prod\limits_{i=1}^3 \exp\left(-k\cdot\left(\dfrac{\partial V}{\partial x_i}\right)^2\right)$. As in the Newton's case, this expression can be represented as $\exp(-k\cdot S)$, where $S = \sum\limits_x \sum\limits_{i=1}^3 \left(\dfrac{\partial V}{\partial x_i}\right)^2$. Taking into account that we have infinitely many spatial locations $x$, we get an integral instead of the sum: $S = \int L\,dx$, where $L(x) = \sum\limits_{i=1}^3 \left(\dfrac{\partial V}{\partial x_i}\right)^2$.

**This model leads to Newton's formulas for the gravitation force**. It is known that minimizing this expression leads to the equation $\sum\limits_{i=1}^3 \dfrac{\partial^2 V}{\partial x_i^2} = 0$, that leads to Newton's gravitational potential $V(x) \sim \dfrac{1}{r}$ that, in turns, leads to the known expression for the gravitational force $F \sim r^{-2}$.

*Comment*. Similar arguments can lead to other known action principles and thus, to other fundamental physical equations.

### *14.2.3 From Equations to Solutions: Fuzzy Techniques Help to Deal with Divergent Series*

**Small-parameter method**. Once we know the equations that describe the dynamics of the corresponding particles and/or fields, a natural next step is to solve these equations under the given information—and thus, predict the future values of the corresponding physical quantities. The equations are often complex, and in many situations, no analytical solution is known, so we have to consider approximate methods.

The complexity of solving a system of complex equations is often eased by the fact that our knowledge is usually incremental. At any given moment of time, we have a model which is a reasonably good approximation to reality, and then we find a new model which is even more accurate. The ideas behind the new model may be revolutionary—as they were for quantum physics or relativity theory—but in terms of predictions, the new theories usually provide a small adjustment to the previous known one. For example, General Relativity was confirmed when it turned out that it better describes the bending of light near the Sun—better by 1.75 arc-seconds.

Usually, by the time new complex equations appear, we already know how to solve previous equations. Thus, we can use the solution $x_0$ to the previous equations as a first approximation to the solution $x$ of the new equations.

The difference $x - x_0$ between these solutions can be characterized by some small parameter $q$. The original solution $x_0$ (to the previous theory) corresponds to taking into account only the 0th order term in the Taylor expansion of $x$ into a series in terms of $q$. To get a better approximation, we can take into account terms which are linear in $q$, terms which are quadratic in $q$, etc. Ideally, we thus get an expression for $x$ as an infinite power series; see, e.g., [4]:

$$x = \sum_{i=0}^{\infty} q^i \cdot x_i = x_0 + q \cdot x_1 + q^2 \cdot x_2 + \cdots \tag{14.2}$$

In practice, as an estimate for $x$, we compute the first few terms in this sum

$$s_k \stackrel{\text{def}}{=} \sum_{i=0}^{k} q^i \cdot x_i. \tag{14.3}$$

In general, the more terms we take—i.e., the larger cutoff value $k$ we use—the more accurate is the resulting estimate.

The Taylor series method also provides us with a reasonable estimate of the accuracy of the next approximation, i.e., of the approximation error

$$\Delta x_k \stackrel{\text{def}}{=} x - x_k = \sum_{i=k+1}^{\infty} q^i \cdot x_i;$$

namely, since we assume that the terms decrease with $k$, the first ignored term $q^{k+1} \cdot x_{k+1}$ provides a reasonably accurate description of the approximation error.

**This method often works well**. In many cases, this idea works very well [4]. It works, e.g., in celestial mechanics, when the two-body problem—which describes e.g., how the Earth goes around the Sun—has an explicit analytical solution, and we would like to analyze how the presence of the Moon affects this solution. In this problem, the Moon is much lighter than the Earth, so the ratio of these mass $m_{\text{Moon}}/m_{\text{Earth}}$ is the desired small dimensionless parameter $q$.

**Sometimes, this method leads to divergent series**. In some other cases, the small-parameter method only works for small $k$: we get a good approximation $s_0$, a more accurate approximation $s_1$, an even more accurate approximation $s_2$, etc.—until we reach a certain threshold $k_0$. Once this threshold is reached, the approximation accuracy decreases. In other words, the series (2) diverge. This is, e.g., of Taylor series corresponding to quantum electrodynamics (see, e.g., [2, 4]): the first few terms of the expansion in the weak interaction constant $\alpha \approx 1/137$ lead to very accurate predictions, but the whole series diverge.

**Divergence is one of the main problems of quantum field theory**. The above divergence is one of the main challenges preventing physicists from coming up with exact mathematical formulations of quantum field theory.

**Fuzzy techniques can potentially help in solving this problem**. Divergence is largely a theoretical problem; in practice, physicists use semi-heuristic methods to come up with meaningful predictions. Formalizing imprecise semi-heuristic ideas is one of the main reasons why fuzzy techniques were invented in the first place. Let us therefore try to use fuzzy techniques to formalize the physicists' reasoning.

**How physicists use divergent series**. Let us describe how physicists come up with answers when the series converge.

When the series representing the answer are divergent, physicists usually consider only the approximations until the remaining term $s_{k+1} - s_k$ starts increasing. In other words, the last approximation they consider is the one for which the difference $s_{k+1} - s_k$ is smaller than both the previous difference $s_k - s_{k-1}$ and the next difference $s_{k+2} - s_{k+1}$.

Usually, the difference is in the orders of magnitude—just like for Taylor series in general, so we have $s_{k+1} - s_k \ll s_k - s_{k-1}$ and $s_{k+1} - s_k \ll s_{k+2} - s_{k+1}$.

**Challenge**. The divergent character of the corresponding series presents a mathematical challenge: how do we formalize the idea that while the series diverge, its first terms serve as a good approximation?

Let us show that fuzzy logic allows us to come up with a mathematically rigorous formalization of this idea.

**Towards a fuzzy solution to the challenge**. Since the series (2) diverge, the corresponding sum makes no mathematical sense, so we cannot formulate equation (2) in the literal form. Instead, let us formalize exactly what the physicists are doing: they are claiming, in effect, that, for every $k$, $x \approx s_k$ with an accuracy proportional to the next ignored term, i.e., to the difference $s_{k+1} - s_k$. In other words, instead of the equation (2), we have a sequence of infinitely many imprecise rules of the type

$$x \approx x_0 + q \cdot x_1 + q^2 \cdot x_2 + \cdots + q^k \cdot x_k, \text{ with accuracy } q^{k+1} \cdot x_{k+1},$$

or, equivalently,

$$x \approx s_k \quad \text{with accuracy} \quad s_{k+1} - s_k. \tag{14.4}$$

**How to describe the relation "$x$ is approximately equal to $a$, with accuracy of order $\sigma$"**. We want to describe, for every three real numbers $x$, $a$, and $\sigma > 0$, the degree $\mu(x, a, \sigma)$ to which $x$ is approximately equal to $a$ with an accuracy of order $\sigma$.

**First natural condition**. Intuitively, when one of these values changes a little bit, this degree should also change only a little bit. Thus, this function should be *continuous*. To $a$ with an accuracy of order $\sigma$.

**Second natural condition**. This degree should be equal to 1 when $x = a$ and it should strictly decrease to 0 as $x$ increase up from $a$ or strictly decrease to 0 as $x$ decreases down from $a$.

**Third natural condition**. We want to apply this function to values of physical quantities. The numerical value of a physical quantity depends on the choice of a measuring unit and on the choice of a starting point. It is reasonable to require that the degree $\mu(x, a, \sigma)$ should not change if we simply re-scale the corresponding physical quantities by using a new measuring unit or a new starting point for measuring this quantity.

**Changing units**. If we replace a measuring unit by a new unit which is $\lambda$ times smaller, then the numerical value increases by a factor of $\lambda$: $x \to \lambda \cdot x$. For example, if we replace meters with centimeters which are $\lambda = 100$ times smaller, then a length of $x = 2$ m becomes $x' = 200$ cm.

Since accuracy is measured in the same units, in the new units, we have $\sigma' = \lambda \cdot \sigma$. To $a$ with an accuracy of order $\sigma$.

So, invariance relative to selection of a measuring unit means that for every $\lambda > 0$, we should have $\mu(\lambda \cdot x, \lambda \cdot a, \lambda \cdot \sigma) = \mu(x, a, \sigma)$.

**Changing sign**. Sometimes, the sign of a physical quantity is also arbitrary, so it can change $x \to -x$. For example, the direction of a spatial coordinate is a pure convention. It is also a pure convention that we consider electrons to be negatively changed and protons positively changed; all the formulas of electrodynamics remain the same if we simply change the signs of all the electric charges.

Accuracy $\sigma$ describes the *absolute value* $|x - a|$ of the difference $x - a$, so the value of $\sigma$ does not change if we simply change the sign of the corresponding physical quantity. To $a$ with an accuracy of order $\sigma$.

**Changing starting point**. Also, if we replace the original starting point with a new starting point which is $x_0$ units lower, then all numerical values are increased by $x_0$: $x \to x + x_0$. Since the accuracy $\sigma$ estimates the value of the difference $x - a$, the value of $\sigma$ does not change under this transformation. To $a$ with an accuracy of order $\sigma$.

**Fourth natural condition**. Finally, often, we have several estimates of this type; we should be able to combine them into a single estimate. In other words, for every finite set of values $a_i$ and $\sigma_i$, we should describe the "and"-combination of all the rules of these types by a single rule of a similar type. We have already argued that algebraic product is a good way to formalize "and". To formulate this requirement in precise terms, we need to take into account that we are usually interested in normalized membership functions, for which $\max\limits_x \mu(x) = 1$, but the product of two or more membership functions is not necessarily normalized. Thus, we need to normalize this product.

**Proposition 1** *Let $\mu(x, a, \sigma)$ be a* $[0, 1]$*-valued continuous function with the following properties:*

- *for any a and $\sigma$, $\mu(a, a, \sigma) = 1$;*
- *for any a and $\sigma$, the value $\mu(x, a, \sigma)$ strictly decreases for $x \geq a$ and strictly increases for $x \leq a$, and tends to 0 as $x \to \pm\infty$;*
- *for every x, a, $\sigma$, and $\lambda > 0$, we have $\mu(\lambda \cdot x, \lambda \cdot a, \lambda \cdot \sigma) = \mu(x, a, \sigma)$;*
- *for every x, a, $\sigma$, we have $\mu(-x, -a, \sigma) = \mu(x, a, \sigma)$;*
- *for every x, a, $\sigma$, and $x_0$, we have $\mu(x + x_0, a + x_0, \sigma) = \mu(x, a, \sigma)$;*
- *for every $a_1, \ldots, a_n$, $\sigma_1, \ldots, \sigma_n$, there exist values a, $\sigma$, and C for which, for all x, we have $\mu(x, a_1, \sigma_1) \cdot \ldots \cdot \mu(x, a_n, \sigma_n) = C \cdot \mu(x, a, \sigma)$.*

*Then, $\mu(x, a, \sigma) = \mu_0\left(\dfrac{x - a}{\sigma}\right)$, where $\mu_0(z) = \exp(-\beta \cdot z^2)$ for some $\beta > 0$.*

*Comment*. For readers' convenience, all the proofs are placed in the Appendix.

**Back to our problem**. Proposition 1 shows that a natural interpretation of the phrase "$x \approx a$ with accuracy $\sigma$" is provided by a membership function $\mu(x) = \exp\left(-\beta \cdot \dfrac{(x-a)^2}{\sigma^2}\right)$. So, the degree to which each rule is satisfied for a given $k$ is equal to $\exp\left(-\beta \cdot \dfrac{(x - s_k)^2}{(s_{k+1} - s_k)^2}\right)$. The degree to which the 1st rule is satisfied, *and* the 2nd rule is satisfied, ..., can be found by applying the corresponding "and"-operation (i.e., the product) of the above degrees. If we use the product as an "and"-operation, then the degree to which all the rules are satisfied is equal to

$$\prod_{k=0}^{\infty} \exp\left(-\beta \cdot \frac{(x - s_k)^2}{(s_{k+1} - s_k)^2}\right) = \exp\left(-\beta \cdot \sum_{k=0}^{\infty} \frac{(x - s_k)^2}{(s_{k+1} - s_k)^2}\right).$$

An infinite product can be zero, so we have to consider products corresponding to large $N$ and then tend $N$ to infinity in the resulting formulas. For a finite value $N$, we get

$$\mu_N(x) = \prod_{k=0}^{N} \exp\left(-\beta \cdot \frac{(x - s_k)^2}{(s_{k+1} - s_k)^2}\right) = \exp\left(-\beta \cdot \sum_{k=0}^{N} \frac{(x - s_k)^2}{(s_{k+1} - s_k)^2}\right). \quad (14.5)$$

If we need to select a single value $x$, it is reasonable to select a value for which the corresponding degree of belief is the largest possible. Since the function $\mu_N(x)$ is of the form $\exp(-z_N)$ for some $z_N$, and the function $\exp(-z)$ is strictly decreasing, its maximum is attained at a point $X_N$ at which $z_N$ attains its minimum: $\sum\limits_{k=0}^{N} \dfrac{(x - s_k)^2}{(s_{k+1} - s_k)^2} \to \min$. Differentiating this expression with respect to $x$, we conclude that $X_N = \dfrac{\sum_{k=0}^{N} s_k \cdot (s_{k+1} - s_k)^{-2}}{\sum_{k=0}^{N} (s_{k+1} - s_k)^{-2}}$. The actual solution corresponds to $N \to \infty$. Thus, we arrive at the following conclusion.

**The resulting solution: a formal definition**. In the general case of a series (convergent or divergent), we select

$$x = \lim_{N\to\infty} \frac{\sum_{k=0}^{N} s_k \cdot (s_{k+1} - s_k)^{-2}}{\sum_{k=0}^{N} (s_{k+1} - s_k)^{-2}}. \quad (14.6)$$

*Comment*. It is worth mentioning that if, instead of using fuzzy techniques, we use probabilistic techniques and assume that $x \approx s_k$ with Gaussian approximation error of 0 mean and standard deviation proportional to $|s_{k+1} - s_k|$, we get the exact same estimate.

**Analysis of the resulting formula**. Let us show that the above covers both the case of a convergent series—in which case it coincides with the limit $\lim s_k$—and of the divergent series.

**Case of divergent series**. For *divergent* series, the analysis is simple. If we have

$$|s_1 - s_0| \gg |s_2 - s_1| \gg \cdots \gg |s_k - s_{k-1}| \gg |s_{k+1} - s_k| \ll |s_{k+2} - s_{k+1}| \ll \cdots,$$

then

$$(s_1 - s_0)^{-2} \ll (s_2 - s_1)^{-2} \ll \cdots \ll (s_k - s_{k-1})^{-2} \gg$$

$$(s_{k+1} - s_k)^{-2} \gg (s_{k+2} - s_{k+1})^{-2} \gg \cdots$$

Thus, in the numerator $\mathcal{N}$ of the formula (6), the main term is the term $\mathcal{N} \approx s_k \cdot (s_{k+1} - s_k)^{-2}$ corresponding to the smallest possible difference $|s_{k+1} - s_k|$. Similarly, the denominator $\mathcal{D}$ is approximately equal to $\mathcal{D} \approx (s_{k+1} - s_k)^{-2}$. Thus, the ratio $\dfrac{\mathcal{N}}{\mathcal{D}}$ is approximately equal to $s_k$—which is exactly what physicists conclude now.

*Comment*. To be more precise, the physicists use the *next* value $s_{k+1}$, which, within the accuracy $s_{k+1} - s_k$, is the same as $s_k$.

**Case of convergent series**. For a *convergent* series, the result is the same as the limit:

**Proposition 2** *When $s_k \to x$, then the expression* (6) *coincides with the limit $x$.*

*Comment*. To provide a better understanding of the formula (6), in Appendix 4, we provide an example of applying this formula to the diverging geometric series $\sum z^i$ with $|z| \geq 1$.

**General comment**. The membership function corresponding to the approximation based on the first $N+1$ sums has the form

$$\mu_N(x) = \exp\left(-\frac{(x-x_N)^2}{\sum_{k=0}^{N}(s_{k+1}-s_k)^{-2}}\right).$$

Thus:

- For the case when $s_k \to x$, the sum $\sum_{k=0}^{N}(s_{k+1}-s_k)^{-2}$ tends to infinity. So, the membership value $\mu_N(x)$ tends to 1 for this $x$ and to 0 for all other $x$. Thus, in the limit $N \to \infty$, we have a crisp conclusion.
- In contract, in the divergence case, the sum is approximately equal to the value $(s_{k+1}-s_k)^{-2}$ corresponding to the smallest difference $|s_{k+1}-s_k|$.
  Thus, the limit values $\mu_N(x)$ remain non-zero within a neighborhood of size $\approx |s_k - s_{k+1}|$—which reflects the fact that in this case, we do not have a precise value of $x$, we can only determine this value with the accuracy $|s_{k+1}-s_k|$.

## 14.3 Fuzzy and Physics: Promising Future

**Fuzzy techniques will help to derive new physical equations**. In the previous section, we showed that fuzzy techniques can transform informal physical ideas into exact physical equations, and that in this way, we can derive many known physical equations. From the viewpoint of practical applications, from the viewpoint of being able to predict new physical phenomena, we have not yet achieved anything new: all we did was found a new justification for the already known equations.

However, the fact that the existing fuzzy methodology enables us to transform informal ("fuzzy") description of physical phenomena into well-known physical equations makes us confident that in the future, when new physical phenomena will be discovered, fuzzy methodology may help generate the equations describing these phenomena.

**Fuzzy techniques will help to solve physical equations**. Similarly, the fact that fuzzy techniques can lead to an explanation of the known heuristic methods for solving physical equations makes us confident that in the future, similarly fuzzy techniques will help to transform informal ideas into new successful mathematical techniques.

**Instead of conclusion: the future is fuzzy**. People often say "the future is fuzzy" meaning that it is difficult to predict the future exactly. But, based on what we observed, we can claim that "the future is fuzzy" in a completely different sense:

that the future will see more and more applications of fuzzy techniques, including applications to areas like theoretical physics and numerical mathematics, areas where, at present, there are not many applications of fuzzy. The future is fuzzy!

**Acknowledgments** This work was supported in part by the National Science Foundation grants HRD-0734825, HRD-124212, and DUE-0926721.

# Appendix

## Appendix 1: Variational Equations

**General derivation**. Let us recall how we can transform the Least Action Principle into a differential equation. Let us first do it on the example on Newton-type situation, where we need to find a function $x(t)$ that minimizes the following expression: $S = \int L(x, \dot{x})\,dt \to$ min. Minimizing means, in particular, that if we take any function $\Delta x(t)$ and consider a function $S(\alpha) = x + \alpha \cdot \Delta x$, then this function must attain its maximum for $\alpha = 0$. Thus, the derivative of $S(\alpha)$ at $\alpha = 0$ must be 0. Differentiating the expression

$$S(\alpha) = \int L(x + \alpha \cdot \Delta x, \dot{x} + \alpha \cdot \Delta \dot{x})\,dt$$

and equating the derivative to 0, we conclude that

$$\int \left( \frac{\partial L}{\partial x} \cdot \Delta x + \frac{\partial L}{\partial \dot{x}} \cdot \Delta \dot{x} \right) dt = \int \left( \frac{\partial L}{\partial x} \cdot \Delta x \right) dt + \int \left( \frac{\partial L}{\partial \dot{x}} \cdot \Delta \dot{x} \right) dt = 0.$$

Integrating the second term by parts, we conclude that

$$\int \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \right) \cdot \Delta x\,dt = 0.$$

This must be true for every function $\Delta x(t)$, in particular for a function that is equal to 0 everywhere except for a small vicinity of a moment $t$, For this function, the integral is proportional to the value of the expression $\frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right)$ at the point $t$. Since the integral is 0, this expression must also be equal to 0:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = 0.$$

The resulting equations are known as *Euler-Lagrange equations*.

**Case of Newton's laws.** In particular, for the Newton's case, when

$$L = V(x) - \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left( \frac{dx_i}{dt} \right)^2,$$

for each of the components $x_i(t)$, we have $\frac{\partial L}{\partial x_i} = \frac{\partial V}{\partial x_i}$ and $\frac{\partial L}{\partial \dot{x}_i} = -m \cdot \frac{dx_i}{dt}$. Thus, Euler-Lagrange's equations lead to $\frac{\partial V}{\partial x} + m \cdot \frac{d}{dt} \left( \frac{dx_i}{dt} \right) = 0$, i.e., to Newton's equations $m \cdot \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i}$.

**General case.** In the general case, Euler-Lagrange equations take the form $\frac{\partial L}{\partial \varphi} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial \varphi_{,i}} \right) = 0$, where $\varphi_{,i} \stackrel{\text{def}}{=} \frac{\partial \varphi}{\partial x_i}$.

## Appendix 2: Proof of Proposition 1

1°. Let us first apply the condition $\mu(x + x_0, a + x_0, \sigma) = \mu(x, a, \sigma)$ with $x_0 = -a$. Then, we get $\mu(x, a, \sigma) = \mu(x - a, 0, \sigma)$, or, equivalently,

$$\mu(x, a, \sigma) = \mu_1(x - a, \sigma),$$

where we denoted $\mu_1(z, \sigma) \stackrel{\text{def}}{=} \mu(z, 0, \sigma)$.

2°. In terms of the function $\mu_1$, the condition $\mu(\lambda \cdot x, \lambda \cdot a, \lambda \cdot \sigma) = \mu(x, a, \sigma)$ takes the form $\mu_1(\lambda \cdot (x - a), \lambda \cdot \sigma) = \mu_1(x - a, \sigma)$. Let us apply this condition for $\lambda = \sigma^{-1}$. Then, we conclude that $\mu_1(z, \sigma) = \mu_1 \left( \frac{z}{\sigma}, 1 \right)$, or, equivalently, $\mu_1(z, \sigma) = \mu_0 \left( \frac{z}{\sigma} \right)$, where we denoted $\mu_0(z) \stackrel{\text{def}}{=} \mu_1(z, 1)$.

Substituting this expression for $\mu_1(z, \sigma)$ in terms of $\mu_0$ in the expression for $\mu$ in terms of $\mu_1$, we conclude that $\mu(x, a, \sigma) = \mu_0 \left( \frac{x - a}{\sigma} \right)$.

3°. Substituting the expression for $\mu$ in terms of $\mu_0$ into the condition $\mu(-x, -a, \sigma) = \mu(x, a, \sigma)$, we conclude that $\mu_0(-z) = \mu_0(z)$. Thus, $\mu(x, a, \sigma) = \mu_0 \left( \frac{|x - a|}{\sigma} \right)$.

4°. For $a_1 = a_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, the fusion condition implies that

$$\mu_0(x) \cdot \mu_0(x) = C \cdot \mu_0 \left( \frac{x - a}{\sigma} \right)$$

for some $a$, $C$, and $\sigma$. The left-hand side attains its maximum (=1) at $x = 0$, the right-hand side attains its maximum (which is equal to $C$) for $x = a$. Since these two sides are one and the same function, we conclude that $a = 0$ and $C = 1$, i.e., that $\mu_0^2(x) = \mu(k_2 \cdot x)$ for some constant $k_2$ $(= 1/\sigma)$. For an auxiliary function $\ell(x) \stackrel{\text{def}}{=} \ln(\mu_0(x))$ we conclude that $2 \cdot \ell(x) = \ell(k_2 \cdot x)$.

Similarly, if we consider 3, 4, etc. terms, we conclude that $3 \cdot \ell(x) = \ell(k_3 \cdot x)$, $4 \cdot \ell(x) = \ell(k_4 \cdot x)$, etc.

4°. The function $\mu_0(x)$ for $x > 0$ is monotonously decreasing from 1 to 0. Therefore, $\ell(x)$ is monotonously decreasing from 0 to $-\infty$. Since $\mu$ (and thus, $\mu_0$) is continuous, the function $\ell(x)$ is also continuous, and hence, there exists an inverse function $i(x) = \ell^{-1}(x)$, i.e., such a function that $i(\ell(x)) = x$ for every $x$.

For this inverse function, the equality $n \cdot \ell(x) = \ell(k_n \cdot x)$ turns into $i(n \cdot \ell(x)) = i(\ell(k_n \cdot x)) = k_n \cdot x = k_n \cdot i(\ell(x))$. So, if we denote $\ell(x)$ by $X$, we conclude that for every $n$, there exists a $k_n$ such that $i(n \cdot X) = k_n \cdot i(X)$.

If we substitute $Y = n \cdot X$, we conclude that $i(Y) = k_n \cdot i\left(\dfrac{Y}{n}\right)$, and therefore, $i\left(\dfrac{Y}{n}\right) = \dfrac{1}{k_n} \cdot i(Y)$.

From these two equalities, we conclude that $i\left(\dfrac{m}{n} \cdot X\right) = \dfrac{1}{k_n} \cdot i(m \cdot X) = \dfrac{k_m}{k_n} \cdot i(X)$. So, for every rational number $r$, there exists a real number $k(r)$ such that $i(r \cdot X) = k(r) \cdot i(X)$. Therefore, the ratio $\dfrac{i(r \cdot X)}{i(X)}$ is constant for all rational $r$.

5°. Since $i(X)$ is a continuous function, and any real number can be represented as a limit of a sequence of rational numbers, we conclude that the ratio $\dfrac{i(r \cdot X)}{i(X)}$ is constant for real values of $r$ as well. Therefore, for every real number $r$, there exists a $k(r)$ such that $i(r \cdot X) = k(r) \cdot i(X)$.

We have thus arrived at a functional equation for which all monotonis solutions are known: they are $i(X) = A \cdot X^p$ for some $A$ and $p$; see, e.g., [1]. Therefore, the inverse function $\ell(x)$ $(x > 0)$ also takes the similar form $\ell(x) = B \cdot x^m$ for some $B$ and $m$. Taking into consideration that $\mu_0(x)$ and hence $\ell(x)$ are even functions, we conclude that $\ell(x) = B \cdot |x|^m$ for all $x$.

6°. Now, for every $a_1 > 0$, if we take $a_2 = -a_1$ and $\sigma_1 = \sigma_2 = 1$, then the fusion property implies that $\mu_0(x - a_1) \cdot \mu_0(x + a_1) = C \cdot \mu_0\left(\dfrac{x - a}{\sigma}\right)$ for some $a$ and $\sigma$. The left-hand side of this equation is an even function, so the right-hand side must also be even, and therefore $a = 0$. So, $\mu_0(x - a_1)\mu_0(x + a_1) = C \cdot \mu_0\left(\dfrac{x}{\sigma}\right)$. For $x = 0$ we get $\mu_0(a_1) \cdot \mu_0(a_1) = C$. Turning to logarithms, we conclude that for every $a_1$, there exists a $k(a_1)$ $(= 1/\sigma)$ such that $\ell(x - a_1) + \ell(x + a_1) = \ell(k(a_1) \cdot x) + 2 \cdot \ell(a_1)$. If we substitute here $\ell(x) = B \cdot |x|^m$, and divide both sides by $B$, we conclude that $|x - a_1|^m + |x + a_1|^m = (k(a_1))^m \cdot |x|^m + 2 \cdot a_1^m$.

Let us show that this equality is satisfied only when $m = 2$.

7°. When $x > 0$, and $a_1$ is sufficiently small, then $x + a_1$, $x$, and $x - a_1$ are all positive, and, therefore, $(x - a_1)^m + (x + a_1)^m = (k(a_1))^m \cdot x^m + 2 \cdot a_1^m$. If we move $2 \cdot a_1^m$ to the left-hand side, and divide both sides by $x^m$, we conclude that $\left(1 - \frac{a_1}{x}\right)^m + \left(1 + \frac{a_1}{x}\right)^m - 2 \cdot \left(\frac{a_1}{x}\right)^m = (k(a_1))^m$. The left-hand side of the resulting equality depends only on the ratio $z = \frac{a_1}{x}$, the right-hand side only on $a_1$. Therefore, if we choose any positive real number $\lambda$, and take $a_1' = \lambda \cdot a_1$ and $x' = \lambda \cdot x$ instead of $a_1$ and $x$, then we can conclude that the left-hand side will be still the same, and therefore, the right-hand side must be the same, i.e., $(k(a_1))^m = (k(\lambda \cdot a_1))^m$. Since $\lambda$ was an arbitrary number, we conclude that $k(a_1)$ does not depend on $a$ at all, i.e., that $(k(a_1))^m$ is a constant. Let us denote this constant by $k$.

So the equation takes the form $(1 - z)^m + (1 + z)^m = k + 2 \cdot z^m$. When $z \to 0$, then the left-hand side tends to 2 and right-hand side to $k$, so from their equality we conclude that $k = 2$, i.e., that $(1 - z)^m + (1 + z)^m = 2 + 2 \cdot z^m$.

The left-hand side is an analytical function of $z$ for $z$ close to 0. Therefore the right-hand side must also be a regular analytical function in the neighborhood of 0 (i.e., it must have a Taylor expansion for $z = 0$). Hence, $m$ must be an integer.

The values $m < 2$ are impossible, because for $m = 0$ our equality turns into a false equality $2 = 3$, and for $m = 1$ it turns into an equality $1 - z + 1 + z = 2 + z$, which is true only for $z = 0$. So $m \geq 2$.

Since both sides are analytical in $z$, the second derivatives of both sides at $z = 0$ must be equal to each other. The second derivative of the left-hand side at $z = 0$ is equal to $m \cdot (m - 1)$. The second derivative of the right-hand side is equal to $2m \cdot (m - 1) \cdot z^{m-2}$.

If $m > 2$, then this derivative equals 0 at $z = 0$ and therefore cannot be equal to $m \cdot (m - 1)$. So $m \geq 2$, and $m$ cannot be greater than 2. So, $m = 2$. Thus, $\ell(x) = B \cdot x^2$, and hence $\mu_0(x) = \exp(-\beta \cdot x^2)$ for some $\beta > 0$. The proposition is proven.

## Appendix 3: Proof of Proposition 2

Let us assume that $s_k \to x$, and let us prove that in this case, the ratios

$$X_N \stackrel{\text{def}}{=} \frac{\sum_{k=0}^{N} s_k \cdot (s_{k+1} - s_k)^{-2}}{\sum_{k=0}^{N} (s_{k+1} - s_k)^{-2}}$$

also tend to $x$, i.e., that for every $\varepsilon > 0$, there exists an $n$ for which, for all $N \geq n$, we have $|X_N - x| \leq \varepsilon$.

Since $s_k \to x$, there exists an integer $n_0$ such that for all $k \geq n_0$, we have $|s_k - x| \leq x + \frac{\varepsilon}{2}$. In particular, this means that for such $k$, we have $s_k \leq x + \frac{\varepsilon}{2}$. We can represent the numerator $\mathcal{N}$ of the ratio $X_N$ as

$$\mathcal{N} = \mathcal{N}_0 + \sum_{k=n_0+1}^{n_0} s_k \cdot (s_{k+1} - s_k)^{-2},$$

where $\mathcal{N}_0 \stackrel{\text{def}}{=} \sum_{k=0}^{n_0} s_k \cdot (s_{k+1} - s_k)^{-2}$. Since $s_k \leq x + \frac{\varepsilon}{2}$, we conclude that

$$\mathcal{N} \leq \mathcal{N}_0 + \left(x + \frac{\varepsilon}{2}\right) \cdot \Delta,$$

where we denoted $\Delta \stackrel{\text{defe}}{=} \sum_{k=n_0+1}^{N} (s_{k+1} - s_k)^{-2}$. Similarly, for the denominator $\mathcal{D}$ of the ratio $X_N$, we get an expression $\mathcal{D} = \mathcal{D}_0 + \Delta$, where

$$\mathcal{D}_0 \stackrel{\text{def}}{=} \sum_{k=0}^{n_0} (s_{k+1} - s_k)^{-2}.$$

Thus,

$$X_N = \frac{\mathcal{N}}{\mathcal{D}} \leq \frac{\mathcal{N}_0 + \left(x + \frac{\varepsilon}{2}\right) \cdot \Delta}{\mathcal{D}_0 + \Delta}.$$

The right-hand side of this inequality can be represented as

$$\frac{\mathcal{N}_0 + \left(x + \frac{\varepsilon}{2}\right) \cdot \Delta}{\mathcal{D}_0 + \Delta} = x + \frac{\varepsilon}{2} + \frac{\mathcal{N}_0 - \mathcal{D}_0 \cdot \left(x + \frac{\varepsilon}{2}\right)}{\mathcal{D}_0 + \Delta}.$$

Here, $|s_k - x| \leq \frac{\varepsilon}{2}$ and $|s_{k+1} - x| \leq \frac{\varepsilon}{2}$ implies that

$$|s_{k+1} - s_k| \leq |s_k - x| + |s_{k+1} - x| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(s_{k+1} - s_k)^{-2} \geq \varepsilon^{-2}$ and so, $\Delta \geq (N - n_0) \cdot \varepsilon^{-2}$. When $N \to \infty$, we have $\Delta \to \infty$ and thus,

$$\frac{\mathcal{N}_0 - \mathcal{D}_0 \cdot \left(x + \frac{\varepsilon}{2}\right)}{\mathcal{D}_0 + \Delta} \leq \frac{\varepsilon}{2}$$

for sufficiently large $N$. For such $N$, we get $X_n = \frac{\mathcal{N}_0}{\mathcal{D}_0} \le x + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = x + \varepsilon$. Similarly, for sufficiently large $N$, we get $X_N \ge x - \varepsilon$. The proposition is proven.

## Appendix 4: Example: Applying Formula (6) to the Divergent Geometric Series $\sum z^i$ for $|z| \ge 1$

When $|z| > 1$, the series $\sum z^i$ diverges. Here, $s_1 = 1$, $s_2 = 1+z$, ..., and, in general, $s_k = 1 + z + \cdots + z^k = \frac{z^{k+1} - 1}{z - 1}$. Thus, $s_{k+1} - s_k = \frac{z^{k+2} - z^{k+1}}{z - 1} = z^{k+1}$. So, the denominator $\mathcal{D}$ of the formula (6) has the form $\mathcal{D} = \sum_{k=0}^{N} z^{-2\cdot(k+1)}$. In the limit, when $N \to \infty$, we get $\mathcal{D} \to \frac{z^{-2}}{1 - z^{-2}}$.

For the numerator, we similarly have

$$\mathcal{N} = \sum_{k=0}^{N} \frac{z^{k+1} - 1}{z - 1} \cdot z^{-2\cdot(k+1)} = \frac{1}{z-1} \cdot \left( \sum_{k=0}^{N} z^{-k+1} - \sum_{k=0}^{N} z^{-2\cdot(k+1)} \right).$$

In the limit, when $N \to \infty$, we get $\mathcal{N} \to \frac{1}{z-1} \cdot \left( \frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-2}}{1 - z^{-2}} \right)$. Thus,

$$x = \lim_{N\to\infty} X_N = \lim_{N\to\infty} \frac{\mathcal{N}}{\mathcal{D}} = \frac{\frac{1}{z-1} \cdot \left( \frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-2}}{1 - z^{-2}} \right)}{\frac{z^{-2}}{1 - z^{-2}}} =$$

$$\frac{1}{z-1} \cdot \left( \frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-2}}{1 - z^{-2}} \right) \cdot \frac{1 - z^{-2}}{z^{-2}}.$$

Here,

$$\frac{1}{z-1} = \frac{1}{\frac{1}{z^{-1}} - 1} = \frac{z^{-1}}{1 - z^{-1}}.$$

Therefore,

$$x = \frac{z^{-1}}{1 - z^{-1}} \cdot \left( \frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-2}}{1 - z^{-2}} \right) \cdot \frac{1 - z^{-2}}{z^{-2}}.$$

Adding two fractions in parentheses, we get

$$\frac{z^{-1}}{1-z^{-1}} - \frac{z^{-2}}{1-z^{-2}} = \frac{z^{-1}\cdot(1+z^{-1}) - z^{-2}}{1-z^{-2}} = \frac{z^{-1}}{1-z^{-2}}.$$

Thus,

$$x = \frac{z^{-1}}{1-z^{-1}} \cdot \frac{z^{-1}}{1-z^{-2}} \cdot \frac{1-z^{-2}}{z^{-2}}.$$

The terms $z^{-1}$, $z^{-1}$, and $z^{-2}$ cancel each other, as well as the terms $1-z^{-2}$ in the numerator and in the denominator. Thus, we get $x = \dfrac{1}{1-z^{-1}}$.

For example, for $z = 2$, we get $x = 1 + 2 + 4 + \cdots = \dfrac{1}{1-1/2} = 2$.

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# Chapter 15
# Future Is Where Concepts, Theories and Applications Meet (also in Fuzzy Logic)

**Marco Elio Tabacchi and Settimo Termini**

**Abstract** No one knows where the future lies, and the idea of serendipity in science is now raised to something of a tropism. This does not impede our will to predict, if not the exact events, at least the short–term trends in the disciplines we live and breathe, and to point at the (subjective) glaring chances for a bright future. This volume is a clear example of the need that any living scientific discipline has for constant regrouping and redirection, in a never–ending process of consolidating results and finding new paths. In this contribution we will try and focus on a number of areas of fuzzy logic and, by extension, in the whole word of uncertainty, where (in our opinion) a number of interesting future developments can and will happen. While our comments and ideas about the technical aspects of the evolution hereby forecasted are proper to the realm of Fuzziness and much dependent on our previous work and experience in the field, the knowledge we have amassed and our personal preferences and quirks, the general remarks of a more epistemological nature interspersed and concluding this paper should and could be applied, in our view, to the development of any scientific endeavour.

## 15.1 Introduction

According to what the title of our present contribution explicitly states, we shall try to focus on a few loci in which—we think—interesting future developments can happen. Needless to say that for what regards the topics involved, our comments and indications are strictly connected and biased by our previous work, knowledge, and preferences. However, as it will be apparent soon, the majority of the pages which follows will deal with very general epistemological remarks; and these last ones, instead, should apply—in general—to the development of any field of investigation. This, at least, has been the conviction, which grow up while fighting against the

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_15

difficulty of understanding the dynamics and evolution of the concepts involved and their ability/inability to generate useful and adequate formal theories. Let us try to tell this better. While writing the paper, we had the impression that we should think about the problems and questions generated by the attempt to answer to the starting, initial question of looking at the future of fuzzy logic, by assuming a very general point of view, otherwise—it seemed to us—it would be impossible to proceed in a fruitful and satisfactory way.

So, we complemented and integrated what can be considered the natural bulk of a typical paper of the requested kind with many premises and digressions of general type, more epistemological in nature. We shall also present some specific comments, judgements—and also forecasts—in fields and territories of investigation which are not exactly those in which we have worked. All this, we now realise—at the end of the process of writing this paper—comes out from the fact that, before thinking about which specific developments are possible and desirable in fuzzy logic, in the back of our mind, there always was a more fundamental question: "**Which future is possible for fuzzy logic in itself**".

We can also anticipate another point: we shall present only very general and qualitative ideas. They will refer, and in a certain sense will also have a very strong reference to specific arguments of fuzzy logic, but we made the choice of not entering too much in the detail, for a very simple reason: we tried to interpret literally the title of the book. But if interpreted literally, "towards the future" is something very, very challenging and acknowledging this may cause many reasons for being perplexed.

### *15.1.1 A Perplexity*

Before proceeding further, let us confess: it is difficult to accept an invitation to speak about the future, although the future to be envisaged is limited to a tiny fragment of reality, as in our case—the development of a scientific theory. However the invitation was a serious one as well as the project. So we had to switch our mind with respect to the perplexity. Perhaps, the name of the project is a little bit risque from a scientific point of view. One could also ask why such an impressionistic name was given to it. A simple answer could be: it is brilliant—from the point of view of communication—and it is synthetic. We agree that it works from the point of view of communication. Let us, then, look at the second qualification. If it is synthetic that means that it is able to put together different things; but which are the elements that are synthetised? To ask what—presumably—will be the future of a certain field of investigation means to have a vision of the most innovative questions that the same field has posed and asked in relationship with (or with respect to) the answers that have been already given to the same questions in the same field as well as, maybe in different forms, in adjacent, related fields. Have the answers completely and fully "answered" the posed questions? If not, which is the quality of what remains to be answered? What is needed is a more general answer or we must only fill up specific cases which had remained outside the answers already obtained? Did the partial answers which were obtained modify the general conceptual scheme holding before these same partial answers were provided? Let us now consider the case in which the development of a certain field has allowed to completely and satisfactorily answer the questions asked. It seems that in this case there is nothing to say, unless to recognise a "cumulative" progress at least in the sense that both to the adjective and to the noun is usually assigned by the "canons" of a positivistic view of science.

### *15.1.2 We Will Go Along This Path, Anyway*

Let us try to look again at the problem just considered in this volume. The vantage point of taking into account the obtained results—the already obtained results—in the light of future developments. But developments of what? Now a complete and definitive answer to some open questions is very important since it allows to see the problem asked in a global and compact way and for what regards future developments it is useful to have some problems definitely solved and archived, since this allows to concentrate on new problems. But now—thinking to the theme of future—an interesting question arises. Did the solution of some open problems contribute to the birth of new interesting questions? In the case of an affirmative answer we can testify both the progress in the development of a certain area of investigation and its vitality: new questions to be answered related to, and generated from the old questions. In the contrary case we can only register the progress done but not the vitality of the field. Its vitality can be witnessed by new questions arisen in other regions of the same theory or discipline and not where the solution of some open problems and questions has, simply, been the purely technical solution of the problems without generating any new conceptual or technical question. The previous (very simple and trivial) remark, apply in general to every theory and discipline. But can we affirm something additional which specifically refer to new theories or disciplines having a strong innovative content at a conceptual level? In this case the situation is both plenty of interesting potentialities and more complex to analyze. The reason for that—in our view—resides in the fact that a mature discipline and a new field of investigation strongly differ in what could be called the level of "stabilization". Let us clarify what we intend, what we mean with this term. Let us limit ourselves now to providing the following observation. A theory can be considered stable if its basic notions are robust enough with respect to possible "perturbations" caused by the same development of the theory which produces an enrichment of the concepts and deep theorems involved without modifying its overall structure in a meaningful way. Now, it is clear that—in general—an old theory or discipline is by far more "stable" than a new emerging field. However, to clarify the origin of these kind of questions it is useful to remember Carnap's analysis of explication procedure illustrating it with examples taken from information sciences.

### 15.1.3 A Detour on Carnap's Explication Procedure

According to the well known analysis done by Rudolf Carnap on the notion of explication, in the process of the construction of a scientific theory we borrow notions and concepts from the natural language as it is used daily having in mind a more specific use. So, the informal notion, the explicandum, is transformed in a more or less formalised one, the explicatum. In doing this transformation from one side we obtain a more refined tool while on the other side we lost many other aspect of the informal notion. In general we welcome the loss of many aspects which are present in the too general informal use in everyday language of a certain notion, but must regret that we are not able—in some cases—to preserve in a certain formalization aspects of the informal notion which could play a role in the newborn theory but which are not easily handled in the proposed formalization. A consequence of this fact is that we can have different—non equivalent—formalizations of the same informal notion. In Carnap's terminology, we can have different explicata of the same explicandum; the fact that they are not equivalent correspond to the fact that the different formalizations, the various explicata, capture different. aspects, various facets of the informal notion. We could conclude: so many theories, so much better! And this is assured at a certain level. It is surely a positive result when we are able to discriminate aspects of the informal nature which are really very far apart and only a general linguistic legacy and tradition has preserved the use of one and the same word for different things. Just to clarify what I have in mind, let me consider the concept of nearness. The use (and the meaning looked for) in topology is very different from the one in colloquial language for expressing—for instance—the fact that one person is sharing the sorrow of another person. The informal concept is the same, seen at very different levels of abstraction and in different contexts; it is natural, however, that doing many additional specifications when regimenting the concept for the specific use and the needs of topology produces also the results of picking up specific features which are not relevant applicable (and may look out of place) when the concept of nearness—the adjective near of the colloquial language—is used to express the participation, our tuning with something unpleasant that has occurred to another person. A completely different situation is, however, the one occurred with the notion of information. In this case we have different formal theories each of which captures a different aspect of the informal notion. The situation seems similar to the one discussed above but it is not. We have different theories, hence we are unable to construct a unique theory which is comprehensive of the various different aspects. On the case of "nearness" we do not deem useful (before considering it feasible or affordable) the building of a "theory of nearness" so general that it would be applicable both in subtle questions of mathematics and in the daily use of the term. In the latter instance we feel that a general theory, in which the aspects of measurement, lossless transmission, interpretation and other related go hand in hand as different facets out of one and the same concept, would be useful and desirable. The different and non equivalent explicata of the informal notion of information are different theories which we have been unable to unify. The analogy with the previous case of nearness would be to consider the use of the term information for what is conveyed by the press and the media (in Italy the expression "mezzi di informazione" is used). It is obvious that we are not looking for a general theory of information, able to explain not only the flux of information but what happens in the world of media. In general, to an explicandum corresponds more than one explicatum not only when we compare—so to say—extreme uses of the involved notions but also when we try to formalise or at least to construct quantitative (mathematical) theories capturing very similar aspects of the informal notion. What was written above for the notion of information could be repeated verbatim for the notion of complexity. Also in this case we are able to pick up specific aspects of the informal motion which are suitably represented by formal/mathematical theories, but we have to see the different aspects as different specifications of one and the same notion, although leaving out the colloquial, everyday use of the word complexity. This happens to all the possible notions—with one well know and notable exception: the notion of computation. We shall not touch upon this question here, except for remembering that elsewhere it has been done the hypothesis that, perhaps, the reason why information sciences are correctly classified as hard sciences although for many aspects one could describe them in terms more similar to social sciences (due to their lack of solid and direct groundings in Nature) can be connected to the existence of one notion that is more stable than the other ones. But we have to recognise and admit that we have done a too long detour although in very nice areas rich of beautiful places to see. Let us assure the reader: we have not forgotten that the main theme of the volume (and of the paper) was fuzziness and its future. This long detour was only a preparation for having enough stuff to think about what can be seen as one crucial question in this discussion: How does the notion of fuzziness behave with respect to Carnap's procedure of explication? We shall take this question and the possible answers in the back of our mind. But now it is time to turn our attention back to the problem of the "stabilisation" of the theories. When a theory develops, enriching itself with many results, deep theorem, internal and external connections (with other deep results internal to the theory, as well as with strong results of other theories), it loses the necessity of referring back to its origin, to the original explicandum (or, explicanda, in the case of complex and sophisticated theories in which at the same time many different notions play a central role). The justification of the theory is provided by the corpus of all the results obtained and their connections. And in this sense it is stable enough to perturbations coming from different questions arising from the "outside world". When this happens one can say that the theory is stabilised. Let us however observe that if a theory is stabilised this does not necessarily mean that it is very productive, full of interesting questions, rich of future developments. Stable theories can be subsumed into more general theories, unified with others or simply remain there after having reached an important development. A theory is vital when it is able to ask new questions or ask again old questions in a different way or with more or less slight slippages of meaning of some of its defining terms. This statement is not as strange as it can appear since it is what has happened, for instance, for such a crucial concept as the one of function, as it is superbly illustrated by Imre Lakatos in his "Conjectures and Confutations". So the vitality of a theory is given by its capability of asking new questions (an act prior to the one of answering them) and this can happen in different situations. In some cases, also when it is stabilised enough, it can happen that one for same reasons one looks back to the basic notions, to their basic definitions and to a scrutiny of the origin of the explicatum used, looking for possible extensions starting from the informal notion. This is the reason why we maintain at the beginning of this contribution that particularly interesting situations are provided by all the new disciplines with a strong innovative content. In this case, in fact, the relationship with the informal notion is more strict than in other cases and this fact helps us in constructively thinking about our questions.

### *15.1.4 Let's Finally Focus on a Few Topics*

We have travelled half of our journey and have not provided yet any indication of technical developments or of interesting innovative applications. This at least could be a not too critical comment of the attentive reader who has till now patiently followed our considerations in a participatory way. Probably many other readers have stopped following our considerations long before. Well, let us say that we have not reached the end of our journey but only the end of the (fuzzy) limits of the length of the paper and of the (sharp) deadline for giving the manuscript (in a more or less final shape). Secondly and, perhaps, more importantly—let us affirm that only apparently we have not provided any technical indications. Firstly, all the epistemological and conceptual observations summed up in the previous pages grow out of our efforts and attempts of "imagining" what meaningful innovative developments could reasonably happen alongside paths and avenues without "smoothing" the innovative force, disruptive strength of the notion of fuzziness, by bringing back them inside the "channel" of existing streams. And at the same time without remaining an isolated republic. It is not productive (although it is obviously possible) either to construct "fuzziness in only one country" or to publicly hold the flag of fuzziness while in practice accepting a complete "normalisation". So, the general considerations of the present pages come out from real dialogues between the authors on specific questions as well as out of thought dialogues and imagined conversations with the editors of the volume and other fuzzy friends and scholars. In what follows, we shall briefly refer to a few topics in which the general considerations spread out in the paper as well as what was behind the written words, can apply. Let us add that, conversely and symmetrically, the reader can—perhaps—have the intuition of why and how the considerations we present emerged from our analysis. Let us, then, present some "concrete" themes. In Sect. 15.2 we shall list a few topics "inside" FST which present interesting unsolved questions, whilst in the subsequent Sect. 15.3 we shall present a few ideas of a possible very useful cross–fertilisation with Cognitive Sciences.

## 15.2 And What About the Future?

All the points in the following list refer to arguments in which a strong interaction among theoretical analysis, conceptual clarifications and applications not only can be useful but is a condition to be necessarily fulfilled for having a real advancement of our knowledge in the field along innovative directions. For each of these points we shall provide a few succinct comments making reference to other papers in which our ideas have been presented in a more detailed way. The treated topics are the following ones. The crucial theme of "Computing with words", in the first place. Secondly we refer to the idea of considering FST as an "experimental science". Measuring fuzziness could play a new important role in two directions. By analyzing in general, the notion of "sharpened order" and by looking at various families of measures introduced in a unitary way trying to develop the notion of infodynamics of fuzzy sets. Finally, by considering a crucial point for future developments the way in which fuzziness interact with other notions, we shall pinpoint differences of the interaction—with a standard theory like quantum logic (QL) or with new notions like "trust". As already observed, the subsequent Section will afford the problem of an interdisciplinary cross fertilization with cognitive sciences, outlining possible ways for solving past difficulties. But let us briefly consider the previous points with more detail.

### *15.2.1 Computing with Words*

For what has to do with CWW let us say something which can appear too radical. CWW is one of the most innovative ideas of the last decades, it is visionary in the good sense of the word: it is able to connect very distant concepts in a possible common framework. The concepts are really very distant. The simple act of computing has to do since its inception with numbers and, originally, with natural number. The "word", literally, comes from "another world", the one of religion, of myth and of the affirmation of unicity of man in Nature. Needless to remember that it is the crucial starting point of Saint John's Gospel, and similar considerations could be made with respect to the Veda. Also today to speak has to do with human sciences while computing is something typical of the scientific enterprise. A dialogue between a great linguist and a great physicist on the relationships between Science and Humanities not yet translated into English is entitled "Contare e raccontare", or Counting and telling (stories) [3]; it is interesting that in italian "contare" can also be a synonym of "raccontare". So, the simple idea of putting together these two very distant notions is of the utmost importance, a very brave operation. It really opens new horizons and the possibility of exploring new worlds. However, we must recognise that after twenty years no new crucial and meaningful ideas have been added to the visionary presentation of Zadeh [33]. What has been done, in the field, is certainly of very good quality, but the quality (as well the quantity) has to do with the variety of techniques and procedures that has been keenly envisaged. No really big steps have been done for really understanding what means to convert into a scientific theory, to find an explicatum of the explicandum "computing with words". We should dare to confront us, for instance, with the validity of Church-Turing Thesis (see [24]). Can we envisage a real model of computation which is based on *words*, in the sense of items of a natural language? And how can we connect these kinds of procedures to more traditional and usual ones? Here we have a situation that shows some contact points with the problem of computation over the reals. Also in this case we have the conundrum of confronting the results of the computation with the reading of the result which cannot but be a rational number (with the very limited exception provided by those reals which have a "name", like $\pi$ and $e$) but with specific additional very difficult (and unusual) conceptual questions.

### *15.2.2 FST as an Experimental Science*

We mainly refer to our recent joint paper for IPMU 2014 [20] which develops a personal analysis on the topic, limiting here to observe that an attitude like the one suggested by Trillas and Moraga [29] in which a particular attention is devoted to the specificities of the system modelled and, as a consequence, to the "design" of fuzzy sets representing meaningful features, in perfectly tuned with Zadeh's distinction between the two ways of considering the term "fuzzy logic" and—in its programmatic attitude against any "mechanical" application of whatever technique and result already obtained—allows to use all the richness and flexibility of the language of FST.

### *15.2.3 "Sharpened Order" and Measuring Fuzziness, Today.*

One of us wrote in [22] that "For what concerns the measures of fuzziness, in my view, the basic kernel of the theory may be considered fairly complete now, after the general classification of the various families of measures provided by Ebanks". Now one could rightly ask what is the reason for speaking of this theory again in a volume which speaks about future developments? The reason why it is reasonable to speak about measures of fuzziness again is related to two main questions. The first (more general) has to do with the fact that in the general vision we have tried to outline in the present paper, fuzziness should be considered in a dynamic way taking into account new nuances of the informal notion or an enlargement—under suitable conditions—of the ways in which the notion has been considered in the orthodox interpretation. Among these possible extensions, let us refer to the construction of an infodynamics, that is a theory that studies relationships existing among the different kinds of measuring fuzziness that have been proposed along the years (entropies, weighted cardinalities—energy, specificity), trying to find out (necessary) strong connections (equations) existing among them. Another interesting point is that we can consider the notion of sharpened order as a crucial tool for analysing fuzziness, independently from its use in the axiomatisation of the measures of fuzziness.

### *15.2.4 Relationships of Fuzziness with Other Emerging Notions (like "Trust")*

As we have repeated "ad nauseam", fuzziness is one of the (crucial) notions that have emerged along the scientific development of last Century. We repeat the statement here for the following two reasons. First, it would be better not using the verb "to emerge" since the same notion of emergence has taken in the last years a specific technical sense. So, we could say, that it has "popped up" from the turmoil of concepts and notions swimming in the primordial soup from which information sciences sprung out. However, if we take seriously, as we do, this image—not only as a metaphor but as a real indication of one of the ways in which science proceeds (at least in new fields) we have to take into account also the ways in which these notions arise and develop, thinking also to the fact that, in a darwinian vision, a few of them will play a central role, others will disappear. More importantly, from the point of view of the present paper, we must take into account their interactions (among the new ones and with the more stabilised notions). In [23], pp. 47–51, the attempt was done to study the similarity of the development of the notion of fuzziness and the one of "trust", which, recently and quickly developed into an interesting theory (see [5]). We refer to the previously mentioned paper for detail, limiting ourselves here to stress the importance for future developments of fuzzy sets theory to consider the "popping up" of new notions and interacting constructively with them.

### *15.2.5 Relationships of Fuzziness with Very Developed Different Theories (like QL)*

What is specifically very interesting in the case of the relationship between FST and QL is that one succeeds in completely reducing the specificities of QL to the language and formalism of FST as it is today. So we can conclude, for this specific case, that FST not only is a good and satisfactory theory and an adequate language for expressing situations in which degrees of membership play an important role but it is able to completely express also the specific properties another theory. As the author writes already in an early paper [14] the possibility of reinterpreting QL as a many valued logic "allows to explain the logical background of the wave-particle duality exhibited in the double-slit experiment in a better way than it can be done with the use of classical two-valued logic". But he goes, as we wrote, well beyond that. We refer to [15] where Pykacz summarizes the path which led him to completely reduce QL into FST. See some personal comments and remarks to this result from a conceptual point of view in [7]. Here we limit ourselves to stress that this result is not discussed with the intensity it would deserve (maybe it is not sufficiently known to a wide audience). In our view it is very important since it shows that FST in its present formulation can be enough for completely represent other theories. And among future possible developments one should take a particular care in looking for:

1. the (deep?) reasons which allowed this "complete" representation of QL into FST;
2. the possibility of doing the same for other theories;
3. the possibility of having such kind of representations but at the price of modifying (or enriching) the present formalism of FST.

The previous suggestions did not arose in a vacuum, but having in mind the development of Fuzzy topological spaces, which concludes our comments for this specific point. It is well known that Chang proposed a generalisation of the definition of topological space already in 1968 [6]. Why such an early attempt at cross fertilisation did not produce further results and additional interesting developments? Let's observe that conceptually, the basic ideas of topology seem more akin to the central notions of FST that the ones of QL. Why this topics did not rise the interest of other researchers? Let stress that we are asking this question not for its possible sociological interest, but from the point of view of possible interesting epistemological reasons (see Chang interview [17]).

## 15.3 In the Future Everyone and Everything Will Be a Fuzzy Set for Fifteen Minutes

The title of this section owes, apart from the obvious Warholian citation, to Eleanor Rosch contribution [16] on the "On Fuzziness" volume. While we remain skeptic about the conclusions and surprised by some details in the line of reasoning, the paper and its predecessor [8], along with the specific volume [1] in which the former is contained are a step forward toward some systematisation of categorisation in cognitive science with a little help from FST.

The possible directions towards which Fuzzy Theory will move and expand in the near and far future are limitless and probably unfathomable—whoever had tried to forecast the most recent developments we have discussed here and elsewhere [18–21, 25] would have been quite surprised by the expansion in the direction of domains usually destined to an exclusively qualitative analysis. Should we be pressed to pick a research domain ripe to benefit from the injection of FST, Cognitive Sciences would rank quite high in the shortlist.

The direction outlined by Zadeh with its Computing With Words paradigm [33] lends beautifully to a massive exercise in reviewing many of the discussions dealing with cognitive concepts through the light of a quasi–formal system imbued with the ability to use as its constitutive elements the same reasoning units we, as cognitive humans, employ, without the strict filtering of classical logic. This is a limit psychologists and cognitive scientists alike have always highlighted in their struggle to formalise (or, for some, their refusal to do so[1]), but in order to further progress models with a strong ground in cognition should be devised and implemented. This is not the place to discuss in its entirety the possible future relationships between Cognitive Science and FST, but a fruitful example (where some work has been already carried out) can be found in the link between Psychology of Concepts and FST. A recent and thoroughly researched book, the already cited [1], contains a good description of the state–of–the–art and clearly lists, especially through the chapters contributed by Hampton [10, 11] and Rosch [16], a number of research problems still worth of a deep and lengthy investigation. In the following we would like to outline a possible roadmap for research in this field (after all this is a volume on the future of FST!), with the only pretence to show the ampleness of the space for possible interactions between FST and a very specific branch of Cognitive Science, and the tools we have at our disposal.

1. **The ecological view of concept, coupled with a systemisation of the different kinds of prototypes, can lead to better modelling of concepts in cognition (and refutation by counterexample rarely works well in cognition, as everything there has the potential to be an exception)**. The presumed anomaly pointed by Osherson and Smith [13][2] as well as the inability of FST to represent concepts has been dissected and debunked at will (see eg. [2]), but still the goldfish example (as well as the striped apple) is used as the sign of an anomaly and an indestructible barrier between the way concepts are treated in "cognitive" logic and how they are treated by "formal" logic, a cauldron in which classical logic, different families of graded logics and FST are often lumped together. This is not necessary nor desirable. In an ecological view of concepts, a pet fish becomes a fish (treated as a/considered a) pet, with the implicit part of the concept designation coming from cultural, social and even memetic or genetic additional information, possibly culled and categorised from the list of prototype types. Classical logic would not be flexible enough to represent all the implicit knowledge and to connect the intensional meaning to the extensional meaning—an operation necessary to express the subtle nuances that are implicit in every human exchange of information. Once again, the "alien perplexity" caution is to be observed here: if you explain something to an alien and its degree of knowledge about the item after the explanation is much different than yours, then some implicit information is missing. Après Hampton [11], it is not a problem of choosing the right FST function, but more to supplement the system with more background information, up to a point where the right functions will emerge from the system itself. Is this information problem connected to the use of a specific formal model (any formal

[1] Again in [16], Rosch explicitly states "Mathematical models as such have a poor track record in psychology." We concur, but the explanation given—"[...]models do not seem to have the appropriate level of abstraction (not too much, not too little) or the connection to psychological reality that is generative of new knowledge in the field"— is a possible one amongst many, a set which includes other possibilities related more to the social aspects of doing science.

[2] The well known debacle, detailed in [2], strongly points to the necessity of a strong interdisciplinary link between hard and soft sciences [25], and an accurate assessment of each and any model lifted from hard sciences and used in cognitive theories.

model, for what matters) for representational purposes when the system we are trying to represent has no formal model to speak of—for what we know—as a base? We are not convinced: it is not clear what should take the place of any kind of formal model in the description, and why statistics should represent something different from mathematics or formal logic. A possibility would be to look at a meta–level, as ...

2. ... **intersection and other operators could be better modelled if fuzzy models are employed at the context level as well as at the term level**. This is an extension of the fuzzification of the concept of inclusion, and can be built on the examples by Verkuilen et al. [32] and especially Hampton on conceptual combinations [9]. The categories of nonintersective and intersective combinations alike, being usually defined as a specific relationship between fuzzy concepts, can also be affixed in a non–exclusive manner to a fuzzy graded membership functions that depends heavily on contextual implicit information. Such an approach would preserve the structure outlined by Hampton, adding more provisions for the treatment of ambiguity (zebra as the animal vs the prototypical striped object). This would require building a strong set of fuzzy ontologies (see eg. [4]), but this would really be necessary anyway (FST or not), as once again, most of the information we use to store and decode concepts come from data not directly present in the explicit formulation. A system based on this premises will be both compatible with an intensional approach to the theory of concepts (we are not even mentioning here the problem of vagueness and the aspects linked to semantics versus syntax), and could also ease some of the apparent contradictions between the use of logic connectives and quantifiers in classical inference such as ...
3. ... **Tversky and Kahneman conjunction fallacy** [31] (and a number of similar inconsistencies where the attempt to extend a perfectly logical structure—that gives nice and clean results when applied to theoretical dilemmas—to human behaviour, preferences and desires gets foiled by creativity, overextension, analogies and ambiguity). Many if not most of such fallacies are interesting discoveries by themselves, but hardly surprising: when we transmit concepts, every bit of the message is usually crafted in order to transmit very specific information. As such choosing to explicitly state that Linda was a radical is a supplement of information that inevitably reflects on the agreement on her being a feminist bank teller, and choosing apples instead of fruits reinforces opinion about a subset more than about a sub–subset. Jönsson and Hampton's [12] intensional version is no more or less intensional than the others—there seem to be no other ways of looking at it that are not both intensional AND extensional. Hampton's CPM model [9] can be a good basis toward building an FST system of concepts, but there is another way of looking at it:
4. **holism may be the way forward: we can evolve a system of concepts starting from scratch**. Turing's plan of how to build an AI system [30], a(nother) stroke of genius largely forgotten by the following AI research programs, can find a resource–hungry but powerful nonetheless application in the field of concept research; the whole machinery can be based on a series of CWW elements that include additional implicit information about the concepts itself, along with

a number of evolutionary rules, again based on FST, leading to a detailed system of concepts that includes all the different kind of prototypical types, and as much as possible of the nuances expressed by the noun–noun and the adjective–noun couples. Such evolutionary system may introduce new specifications for conjectual conjunctions that are not to be found in human cognition—after all an evolutionary system should be able to take different evolutive paths in response to different pressures—and research on cognitive treatment of concepts would be also carried out using a comparative paradigm. Most of prototype research, starting from Rosch's seminal paper [8] has been carried out starting by words and their relationships in canned sentences, and then working out by statistics and counterexamples, in a way resembling disciplines such as physics at its inception (experiment, measure, find exception, repeat). Models obtained as such are rarely all–encompassing, and the same shift made with post–galileian physics should also be afforded by cognition in general. Evolutionary systems, soft computing techniques and flexible, more human–like data structures (such FST in general and CWW in particular) could help in building better models, and also discover more efficient ways of dealing with concepts. It is obvious and necessary that such models be too complicated to be explained in linear terms and just be contained in a compact number of formulae, and we maintain that such kind of demand would be unjust. The idea that language, cognition or for that matter most of the human cognition epiphenomena could be expressed by simple rules or easily manageable systems of equations is dead and buried in the seventies, and there should be no reasons to go grave–digging for the sake of complexity savings: we have computers and big data for that, and what could come out from the machinery can usually be parcelled, tokenised and visualised for further analysis.

In conclusion of this section, we want to stress the idea that methodologies such FST and CWW are applicable (and will be applied!) to many different research fields in Cognitive Science, as their flexibility and ability to adapt to the human ways of reasoning and making inferences puts them one step above classical logic (see e.g. the recent work of Trillas [26–28]); what is more, FST as a discipline has demonstrated a level of adaptability which will render fruitful any exchange with soft sciences in all their spectrum. The means and the methods are already there, albeit in a sketched form. It is just a matter of implementation.

## 15.4 Conclusions

Before concluding the paper, let us point out that in our view, the real reason for looking for important and crucial future developments enhancing the growth of the theory, is not only the one of picking up reasonable specific points. This is something which is not only feasible (as we have just shown), but can also provide useful suggestions. It is patently clear that any suggestion cannot but be biased by what one knows better or prefer, independently from a general vision of the development of the field as a whole, as a living whole. One could think that a comprehensive vision can be provided by summing up the contributions of many people, giving a specific weight to the most outstanding contributors in the last decades. This is only partially and relatively true for at least two main reasons. First we must remember what Max Planck wrote about the acceptance of new ideas: “A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.” The second reason is that a general vision, a Weltanschaung cannot be obtained by summing up specific proposals and ideas, however interesting and innovative they can be. This must be envisaged and shaped as such—also using and starting from specifically argued topics and ideas, of course, but not seeing them as a simple cluster, but as part of an organised structure. This is the point we shall briefly discuss in the following paragraphs. Let us try to drive a few (very preliminary) conclusions after this hard (we must confess) and tiring journey. We strongly believe that fuzziness has been one of the most innovative concept of the last Century (on the same foot of complexity, emergency, or Bohr’s complementarity) and that in the few decades from its inception, it has not had the opportunity to fully exploit its potentialities. In the previous pages we have also, implicitly, indicated some of the reasons why this happened. Let us observe that also for the other notions mentioned above something similar hold. Similar, but not identical. Each of these notions (and other few) has particular features and a specific history which deserves to be specifically analysed. In our view, among the things which is worthwhile to take into account—surely for what regard fuzziness, but probably also for the other notions—there are also two apparently minor aspects which played a crucial role in establishing modes and times of (uncritical) rejection and refusal, of criticism, of appreciation and acceptance. These are—on one side—the way in which the community reacted to criticism and—on the other side—the fact that it did not look for a constructive dialogue and a confrontation with other similar proposals and the corresponding minoritarian communities. Fuzziness, in the very intuitive formulation provided by Zadeh is a very good explicatum of the informal, linguistic and philosophical notion of vagueness. It captures many interesting aspects and nuances of the informal intuitive notion and it is not by chance that this “locally” coincides (modulo the different languages used and the background motivations) with the parallel proposal done, independently, a few months later by Dieter Klaua and his school while searching to build a “set–theoretic” counterpart of Łukasiewicz’s many valued logics. However, this is the beginning and not the end of the story. The logical paradigm, constructed starting from crucial questions arising with the foundations of mathematics (seen, moreover, in the conceptual background of classical two–valued logic) is not the only context (and perhaps, the most suitable one) in which the new conceptual problems and questions can be posed, framed, and settled. More than to see how and in which measure such crucial questions for classical logic such as completeness, coherence, soundness can be studied and afforded in a world in which vagueness (and fuzziness) are pervasive and can (and should) not be eliminated, the crucial and most challenging questions—at least from the point of view of fuzziness and vagueness—are how to find out new notions that could play in the new universe the role that completeness, coherence and soundness played classical logic. Not that such logical studies has no contact at all with the construction of our brave new world. They are crucial for a profound understanding of many valued logic, for completing and extending the towering enterprise started from Jan Lukasiewicz and other giants of his level. However the relevance of these logical aspects will presumably affect only locally and tangentially the new Weltanschaung, the new paradigm—if this will be really able to largely develop and nurture new innovative concepts. In a sense we could dare to affirm that fuzzy mathematical logic will remain a very important province, extension and enrichment of mathematical logic but it will play no role at all (at least not for what regards the conceptually innovative aspects) in the outlining and building of a framework in which vagueness and fuzziness (and, possibly, other different explicata of the informal notion of vagueness) will be the crucial pillars. This in a sense is what Lotfi has tried to suggest in a very polite and diplomatic way when distinguishing two senses of fuzzy logic. But we should be brave enough to think that new ideas and avenues can probably, respectively, appear, in the future, outside the view to which we are accustomed today. In this sense Trillas' suggestion of looking at the theory of fuzzy sets as an experimental science is crucial. We have really to construct an entirely new framework of which the work done in these past 50 years is only an anticipation, a tiny fragment of the whole enterprise. More, in these past years one had to fight a conceptual battle to see how to modify the received view on many questions. Now we can begin to look for new questions to pose, before trying to obtain the answers. In this way new really innovative developments can take place. And among the new things it is probable that the scientific paradigm will be enriched by new notions, put until now, outside the door. We intend what we consider basic in the scientific vision of the world. For instance, some of the ideas envisaged by Husserl should necessarily be reconsidered and taken into account in some way, if the suggestions of Zadeh to start from perceptions (and compute with words) will be taken seriously, as scientific indications to follow and not only as interesting metaphors. If all these considerations looks verbose and abstract, we take the blame: this is due to our inability to convey them in a palatable form and not to the content of the suggestions by themselves. We are, in fact deeply convinced that the informal idea of fuzziness is only at the beginning of its development. But to really blossom it has to fight against many things, among which there are two different—but equally terrible—enemies. One is the tendency to acknowledge every small "translation" of whatsoever traditional results into the language of FL and FST as something meaningful for them; the other is the subordination to the traditional paradigms. A corollary of both is the fear to innovate, also with respect to what is considered acceptable in the community, WRT to the orthodox paradigms of FL. We must dare to profoundly innovate at all levels, and one of the ways to check that we are moving in the right direction is to see if in the newly explored lands concepts, theories and applications meet.

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# Chapter 16
# Graduated Conjectures

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**Abstract** The study of the relationships between conjectures, hypotheses, refutations and speculations have been studied by Professor Enric Trillas and coworkers in the classical case to the point of having well clarified its main properties in rather general structures as the orthocomplemented lattices. In the framework of a possibilistic interpretation of fuzzy logic, these models have been studied from the point of view of a crisp reasoning. In this work these models defined by graduated consequences relations are studied under a fuzzy algebraic structure general enough that it can accommodate various common phenomena in natural language reasoning.

## 16.1 Introduction

Fuzzy logic, in the broad sense [18], tries to account for approximate reasoning processes which involved graduates concepts, vague, or, generally, uncertain. These concepts are used in common reasoning processes in natural language and, under this vision, fuzzy logic attempts to explain some of the reasoning processes of common sense used by human reasoning. During the short but intense history of Artificial Intelligence, the goals of this branch of human knowledge has changed from a very general research about "intelligence" with the objective to build intelligence machines to a more narrow perspective, trying to solve concrete, more specific and domain limited problems. Recently, many recognized researchers ([3–5], among others) have expressed the wish that the Artificial Intelligence returns to its origins (see also for example the series of conferences on Artificial General Intelligence) and aim to build machines with the level and type of human intelligence. To walk in that path, to give account for the common human reasoning processes and model them is imperative if we want to continue with the goals of Artificial Intelligence in its broadest sense.

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_16

Deductive reasoning has been the most formalized reasoning in the history of human thought. From Aristotle to Bertrand Russell, many great authors have been devoted to describe, justify and clarify the mechanisms of deductive reasoning of human thought. However these mechanisms can not be justified in an absolute way, as we know from a tortoise friend of Lewis Carroll [1], but must simply be accepted by humans as basic reasoning processes, free from excessive doubts. This is essentially the process of justification of common sense reasoning, when a large majority of people consider it an appropriate reasoning. Probably in the common sense reasoning there are jumps, no monotonic changes due to changing information, situations that may be difficult to justify step by step, but they must be nevertheless common to the community, keeping some minimum level of consistency, without contradiction inside them.

In spite of these efforts to justify deductive reasoning, the scientific method is based largely on inductive reasoning to develop hypotheses and, from the work of Popper [6] we know that a theory must be considered true until it is refuted. In fact, in Mathematics, the paradigmatic deductive science, in the process to find out new results, guessing and making conjectures play a crucial role. From this, not only deductive processes represent human intelligence; establishing hypotheses, construction of refutations and overall ability to conjecture new propositions is essential in human reasoning.

In several works ([2, 12–14]), Professor E. Trillas and colleagues have built a formal model as general as possible to account for the sets of consequences, hypotheses, refutations, conjectures or speculations in a framework near to commonsense reasoning. As it is said in [12], starting with general algebraic structures in a crisp context, this model in ortholattices and De Morgan algebras is almost complete [2, 10]. In the work [16] a graduated case of these models is analyzed but the model is restricted to a multivalued logic setting based on complete residuated lattices.

However, to reach the goal of building a commonsense reasoning model, work with fuzzy sets can be imperative. According to Zadeh's view of Computing with Words [17] natural languages are essentially a mechanism to describe perceptions, and for reasoning in natural languages, it will be necessary to be able to express complex sentences in fuzzy logic. To reach these goals, first the fuzzy sets must be included, but it is likely that many cases not considered in the theories of fuzzy sets in its narrow sense should be taken into account. For instance, many reasoning schemes show that the conjunction is not commutative, specially if time is involved, and that's something that is not usually considered. To cover more aspects of ordinary reasoning, more general structures are required.

In [12], a general framework for a crisp form of reasoning with fuzzy sets using models of "conjecturing + refuting" was given. In the setting of a basic and quite general fuzzy algebra these models are built, with the only restrictions of commutative and associative conjunctions and a strong negation. However the model is a degree-free model, it is crisp reasoning with fuzzy sets, without a degree between the steps of the deductive paths, without a concept of "graduated consequences". In this work that task is addressed.

## 16.2 A Basic Fuzzy Algebra

To formalize a frame to analyze the sets of ordinary reasoning we use a Basic Fuzzy Algebra. The concept of Basic Fuzzy Algebra (BFA) was introduced in [9] to define a very general algebra upon the class of fuzzy sets $\mathcal{F}(X) = \{\mu : X \longrightarrow [0,1]\}$ on a given universe $X$. The standard framework for almost all fuzzy logics, the Standard Fuzzy Algebra $(\mathcal{F}(X), T, S, N)$, based on a continuous t-norm $T$, a continuous t-conorm $S$ and a strong negation $N$ has many strong properties to give account of several basic phenomena in natural language [12]. In particular, formal connectives in a BFA are not presumed to be functionally expressible, associative, commutative, distributive or dual. They try to be more general that any Standard Fuzzy Algebra but they maintain a basic characteristic of fuzzy sets: having crisp sets as a particular case. Hence Basic Fuzzy Algebras have as a basic requirement that a boolean subalgebra isomorphic to $(\mathbb{P}(X), \cap, \cup, ^c)$ must be exists inside them.

In their basic formulation, a BFA uses the pointwise ordering between fuzzy sets, which of course is a crisp ordering between fuzzy sets. In this work to obtain a graded relation between conjectures a more general ordering is defined. From now $\preccurlyeq: \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow [0,1]$ will be a fuzzy relation between fuzzy sets. We usually use an infix notation for the fuzzy relation $\preccurlyeq$ . The symbols $=, \leq, \geq$ will be reserved for the equality and ordering in grades and the symbol $\equiv$ for the equality between fuzzy sets.[1]

The relation $\mu \preccurlyeq \eta$ represents a conditional relation between propositions $\mu, \eta$ from the habitual interpretation in mathematical logic

$$\mu \preccurlyeq \eta = r \Longleftrightarrow \mu \to \eta \text{ to grade } r.$$

Natural requirements for relations $\preccurlyeq$ are the reflexive property and the $T$-transitive property. Specially last one gives the quite basic requirement of knowing some minimun level to which a reasoning with several propositions can be chained.

Several examples of graded $T$-preorders can be shown. Taking

$$\overrightarrow{T}(x, y) = \arg\sup_{z \in X}\{T(x, z) \leq y\},$$

known as the residuum implication of the t-norm $T$, which always exists for left-continuous t-norms. All operators defined by

$$\mu \preccurlyeq \eta = \inf_x \overrightarrow{T}(\mu(x), \eta(x)), \tag{16.1}$$

are $T$-preorders. In (16.1), $\overrightarrow{T}$ can be substituted for any $T$-transitive implication $I$, because the relation $\preccurlyeq$ inherites the transitive property.

[1] It will be the pointwise equally between fuzzy sets, but it is not strictly necessary.

Another method to obtain graded $T$-preorders is to apply the representation theorem for $T$-preorders [15]. From a family $F$ of functions $f_i : \mathcal{F}(X) \longrightarrow [0, 1]$, the operator defined by

$$\mu \preccurlyeq \eta = \inf_{i \in F} \overrightarrow{T}(f_i(\mu), f_i(\eta)) \tag{16.2}$$

are graded $T$-preorders.

From the relation $\preccurlyeq$, classical relations between fuzzy sets can be defined fixing a threshold $r$:

$$\mu \preccurlyeq_r \sigma \text{ iff } \mu \preccurlyeq \sigma \geq r.$$

In particular, the classical relation $\preccurlyeq_1$, called the kernel of $\preccurlyeq$, is a classical preorder.

**Definition 1** (*Basic Fuzzy Algebra*) A Basic Fuzzy Algebra is a five tuple $\Omega = (\mathcal{F}(X), \preccurlyeq, \cdot, +, ')$, where, for $\mu, \sigma, \rho \in \mathcal{F}(X)$, it holds:

1. $\preccurlyeq: \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow [0, 1]$ is a $T$-preorder, with $T$ a continuous t-norm, so
   (a) for all $\mu \in \mathcal{F}(X)$, $\mu \preccurlyeq_1 \mu$;
   (b) for all $\mu, \sigma, \rho \in \mathcal{F}(X)$, $T(\mu \preccurlyeq \sigma, \sigma \preccurlyeq \rho) \leq \mu \preccurlyeq \rho$.[2]
2. With the functions $\hat{0}(x) = 0$, and $\hat{1}(x) = 1$, for all $x$ in $X$, the binary operations $\cdot$ and $+$, verifying
   (a) $\mu \cdot \hat{0} \equiv \hat{0} \cdot \mu \equiv \hat{0}$; $\hat{1} \cdot \mu \equiv \mu \cdot \hat{1} \equiv \mu$
   (b) $\mu + \hat{0} \equiv \hat{0} + \mu \equiv \mu$; $\hat{1} + \mu \equiv \mu + \hat{1} \equiv \hat{1}$.
3. The unary $'$, and the preordering $\preccurlyeq$, jointly verifying,
   (a) $\hat{0}' \equiv \hat{1}$; $\hat{1}' \equiv \hat{0}$
   (b) $\mu' \preccurlyeq \sigma' \geq \sigma \preccurlyeq \mu$
   (c) $\hat{0} \preccurlyeq_1 \mu$ and $\mu \preccurlyeq_1 \hat{1}$
4. for all $\mu, \sigma, \rho \in \mathcal{F}(X)$, is:
   (a) $\mu \cdot \rho \preccurlyeq \sigma \cdot \rho \geq \mu \preccurlyeq \sigma$ and $\rho \cdot \mu \preccurlyeq \rho \cdot \sigma \geq \mu \preccurlyeq \sigma$;
   (b) $\mu + \rho \preccurlyeq \sigma + \rho \geq \mu \preccurlyeq \sigma$ and $\rho + \mu \preccurlyeq \rho + \sigma \geq \mu \preccurlyeq \sigma$;
5. The subset $\{0, 1\}^X$, endowed with the restriction of the order and the three operations defined in $\Omega$, $\Omega_0 = (\{0, 1\}^X, \min, \max, 1 - id)$, is isomorphic to the boolean algebra $\mathbb{P}(X)$, the power set of $X$.

Observe that the properties of a BFA just establish $\hat{0}$ and $\hat{1}$ as the neutral element and absorbent element to $+$ and reciprocally as the absorbent and neutral element to $\cdot$; the complementary relation between $\hat{0}$ and $\hat{1}$ with respect to the negation $'$, the reverse ordering relation between negate fuzzy sets with respect to the ordering relation $\preccurlyeq$ and $\hat{0}$ and $\hat{1}$ as the minimun and maximun elements respectively. The operations $+$ and $\cdot$ are monotonic with respect the ordering relation $\preccurlyeq$ and finally the restrictions of the operations and negation to $\mathbb{P}(X)$ is a boolean algebra.

[2]With $T = \min$, the classical relations $\preccurlyeq_r$ are reflexive and transitive relations.

The relation $\preccurlyeq$ has as a particular case the pointwise ordering (pointwise ordering is a crisp reflexive and min-transitive ordering relation) and in [9] was proben that, only with $\cdot = \min$ and $+ = \max$ BFAs are lattices, namely DeMorgan-Kleene algebras, provided the negation is strong $((\mu)')' = \mu'' = \mu$, for all $\mu \in \mathcal{F}(X)$. So, no BFA on $\mathcal{F}(X)$ is a boolean algebra, nor even an ortholattice [9]. It is clear that standard algebras of fuzzy sets $(\mathcal{F}(X), \leq, T, S, ')$, with $\leq$ the pointwise ordering, $T$ a functional expressible conjunction by a t-norm, $S$ a functional expressible disjunction by a t-conorm, and $'$ with a functional expressible negation by a unary operation $n : [0, 1] \longrightarrow [0, 1]$ which is a strong negation function [11] are particular cases of BFA. In fact, the boolean algebra of the crisp sets $\mathbb{P}(X)$, isomorphic to $(\{0, 1\}^X, \min, \max, 1 - id)$ is also a BFA.

Thanks to the monotony of operations $+, \cdot$ with respect to the ordering relation $\preccurlyeq$ and neutral and absorbent properties of $\hat{0}$ and $\hat{1}$ is always $\mu \cdot \eta \preccurlyeq_1 \mu$ and $\mu \cdot \eta \preccurlyeq_1 \eta$, together with $\mu \preccurlyeq_1 \mu + \eta$ and $\mu \preccurlyeq_1 \mu + \eta$.

## 16.3 Conjectures and Refutations in a BFA with a Graded Ordering

The goal of BFAs is to create a mathematical framework as general as possible which be capable to model ordinary reasoning connectives as conjunction $\cdot$, disjunction $+$ and negation $'$ with a minimal set of properties. This framework is a possibilistic view of Fuzzy Logic and the fuzzy sets represent sentences. The relation $\preccurlyeq$ represent conditional information and $\mu \preccurlyeq \rho$ means that $\rho$ is derived from $\mu$ to the grade $r = \mu \preccurlyeq \rho$.

As it is well known, the classical sense of the non-contradiction principle in boolean logic, i.e. that the grade of truth of $p \wedge p^c$ is always 0, is not valid in general in standard fuzzy logic algebras [8]. However in any human reasoning the non-contradiction principle appears as a important principle, the relation of being contradictory plays a fundamental role. Our reasoning goes well if, at least, there is not a contradiction in it. In a BFA, a fuzzy set $\mu$ is said contradictory to degree $r$ with another fuzzy set $\rho$ if $\mu \preccurlyeq \rho' > r > 0$.[3] That means the negation of $\rho$ is derived from $\mu$ in some degree. When $\mu \preccurlyeq \rho' = 0$, $\mu$ and $\rho$ are said non-contradictories fuzzy sets. A fuzzy set $\mu$ will be named self-contradictory if it is the case that $\mu \preccurlyeq \mu' > 0$. The fuzzy set $\mu_0$ is self-contradictory because $\mu_0 \preccurlyeq_1 \mu_1 \equiv \mu_0'$. In classical logic, empty set is the unique self-contradictory set, but in Fuzzy Logic with a different negation function can exist many self-contradictory fuzzy sets [11]. In fact, in any BFA an expression of the principle of Non-contradiction is always valid because $\mu \cdot \mu'$ is self-contradictory, i.e.

$$\mu \cdot \mu' \preccurlyeq_1 (\mu \cdot \mu')' \tag{16.3}$$

[3] If the negation is strong, if $\mu$ is contradictory with $\rho$ is equivalent to $\rho$ is contradictory with $\mu$ because $0 < r < \mu \preccurlyeq \rho' \leq \rho'' \preccurlyeq \mu' = \rho \preccurlyeq \mu'$.

because from $\mu \cdot \mu' \preccurlyeq_1 \mu$ it is valid that $\mu' \preccurlyeq_1 (\mu \cdot \mu')'$ and with $\mu \cdot \mu' \preccurlyeq_1 \mu'$ results (16.3).

### *16.3.1 Sets of Admissible Premises*

**Definition 2** (*Admissible premises*) In a BFA $\Omega$, let's consider those sets $P = \{\rho_1, \ldots, \rho_n\} \subseteq \mathcal{F}(X)$, such that:

1. for all $i, j \in 1, \ldots, n$ $\rho_i \preccurlyeq \rho_j = 0$;
2. its résumé, $\rho_P \equiv \rho_1 \cdot (\rho_2 \cdot (\cdots (\rho_{n-1} \cdot \rho_n) \cdots)$, verifies $\rho \preccurlyeq \rho' = 0$, and $\rho' \preccurlyeq \rho = 0$.[4]

Last definition tries to fix a kind of minimal precautions on the available information from of which some conclusions should be extracted based on forbidding self-contradictory information.[5] Also note that in lack of associativity property, premises must be numerated to properly define $\rho$.

In next sense, the résumé implies any premise.

**Theorem 1** *For all $\rho_i \in P$, it is $\rho \preccurlyeq_1 \rho_i$.*

*Proof* By induction. For $n = 2$, from $\rho_1 \preccurlyeq_1 \hat{1}$, and property 3 of definition 1, $\rho_1 \cdot \rho_2 \preccurlyeq \hat{1} \cdot \rho_2 \geq \rho_1 \preccurlyeq \hat{1}$, and $\hat{1} \cdot \rho_2 \equiv \rho_2$ so it results that $\rho \equiv \rho_1 \cdot \rho_2 \preccurlyeq_1 \rho_2$. The same can be done for $\rho_1$.

For the general case, let $\rho \equiv \rho_1 \cdot (\rho_2 \cdot (\cdots (\rho_{n-1} \cdot \rho_n) \cdots) \equiv \rho_1 \cdot \hat{\rho}$. By induction, it is valid that $\hat{\rho} \preccurlyeq_1 \rho_i$ for all $i \in 2, \ldots, n$. Following the case $n = 2$ can be proved that $\rho_1 \cdot \hat{\rho} \preccurlyeq_1 \rho_1$ and $\rho_1 \cdot \hat{\rho} \preccurlyeq_1 \hat{\rho}$. Thanks to the T-transitivity, it is valid that $\rho_1 \cdot \hat{\rho} \preccurlyeq_1 \rho_i$ for $i \in 2, \ldots, n$. □

**Corollary 1** *In a set of admissible premises, no pair of contradictory premises can exist.*

*Proof* If $\rho_i \preccurlyeq \rho'_j > 0$ for any pair of premises of a set $P$ of admissible premises, as $\rho \preccurlyeq_1 \rho_i$ it follows $\rho \preccurlyeq {\rho'}_j > 0$, and from $\rho \preccurlyeq_1 \rho_j$ it is valid that ${\rho'}_j \preccurlyeq_1 \rho'$ and it would result that $\rho \preccurlyeq \rho' > 0$, something impossible in an admissible set of premises. □

[4]If $P$ is clear from the context we use $\rho$ instead of $\rho_P$ to reference the résumé of $P$.

[5]Note that in a framework of a grade ordering relation, first condition is a strong one but necessary to avoid self-contradiction.

### 16.3.2 Sets of Consequences

Given a set of admissible premises $P$, consider next definition:

**Definition 3** (*Weak Consequences of grade r*)

$$\mathrm{C}^{\,r}_{\cdot}(P) = \{\mu \in \mathcal{F}(X) \ : \ \rho \preccurlyeq \mu \geq r\}$$

The set of sentences $\mathrm{C}^{\,r}_{\cdot}(P)$ represents the notion of consequences of level $r$. These sets are anti-monotonic with respect to the grade $r$, i.e. if $s > r$ then $\mathrm{C}^{\,s}_{\cdot}(P) \subseteq \mathrm{C}^{\,r}_{\cdot}(P)$.

Usually the concept of consequence is given by means a Tarski' consequence operator $C$ [7] which satisfies three properties

1. extensibility property $P \subseteq C(P)$,
2. monotony property $P \subseteq Q \Rightarrow C(P) \subseteq C(Q)$, and
3. clousure property $C(P) = C(C(P))$.

Observe the sets of weak consequences are crisp sets, not fuzzy, and since $\mathrm{C}^{\,r}_{\cdot}(P)$ can be infinite, it results the impossibility to define $C(C(P))$. Following Corollary 1, it can be shown that $\mathrm{C}^{\,r}_{\cdot}(P)$ are consistent and it results they satisfie properties 1 and 2 of Tarsky Consequence Operators.

**Theorem 2** (Extensitivity of operators $C^r_{\cdot}$) *It is always valid that* $P \subseteq C^{\,r}_{\cdot}(P)$, *for any set* $P$ *of admissible premises and any* $r \in [0, 1]$.

*Proof* $\rho \preccurlyeq \rho_i = 1 \geq r$ for any $r \in [0, 1]$. □

**Theorem 3** *If* $P$ *and* $Q$ *are sets of admissible premises, and it is* $P \subseteq Q$, *then* $C^{\,r}_{\cdot}(P) \subseteq C^{\,r}_{\cdot}(Q)$.

*Proof* Let $P = \{\rho_1, \ldots, \rho_n\}$ and $Q = P \cup \{\rho_{n+1}, \ldots, \rho_m\}$ be two addmisible sets with $P \subseteq Q$. It is evident that $\rho_n \cdot (\rho_{n+1} \cdot \cdots (\rho_{m-1} \cdot \rho_m) \cdots) \preccurlyeq_1 \rho_n$. Adding fuzzy sets by the left, it is valid that $\rho_Q \equiv \rho_1 \cdot (\cdots (\rho_n \cdot (\cdots (\rho_{m-1} \cdot \rho_m) \cdots) \preccurlyeq_1 \rho_1 \cdot (\cdots (\rho_{n-1} \cdot \rho_n) \cdots) \equiv \rho_P$, so $\rho_Q \preccurlyeq_1 \rho_P$.

Then, if for any $\mu \in \mathcal{F}(X)$, is $\rho_P \preccurlyeq \mu \geq r$ by T-transitivity, $T(\rho_P \preccurlyeq \mu, \rho_Q \preccurlyeq_1 \rho_P) = \rho_P \preccurlyeq \mu \leq \rho_Q \preccurlyeq \mu$, so $\mu \in \mathrm{C}^{\,r}_{\cdot}(Q)$. □

The $T$-transitivity together with the two conditions $\rho \preccurlyeq \rho' = 0$ and $\rho' \preccurlyeq \rho = 0$ constitue, in the case of non-nilpotent t-norms, a strong condition. Using the first one, $T$-transitivity and the basic property of the operator of negation, the relation

$$T(\rho \preccurlyeq \mu', \rho \preccurlyeq \mu) \leq T(\rho \preccurlyeq \mu', \mu' \preccurlyeq \rho') \leq \rho \preccurlyeq \rho' = 0, \tag{16.4}$$

is always true for any fuzzy set $\mu$, so either $\rho \preccurlyeq \mu = 0$, i.e. $\mu$ is not a consequence of $\rho$ to any level, or $\rho \preccurlyeq \mu' = 0$, i.e. $\rho$ is not contradictory with $\mu'$. Of course, also both cases are possible simultaneously.

Analogously, using the condition $\rho' \preccurlyeq \rho = 0$, for any $\mu$, either $\rho$ is not a consequence of $\mu$ or it is not a consequence of $\mu'$. Observe that it is not necessary true for nilpotent t-norms. But in some cases a threshold could be establish between the value of $\mu \preccurlyeq \rho$, $\mu' \preccurlyeq \rho$, $\rho \preccurlyeq \mu$, and $\rho \preccurlyeq \rho'$.

### *16.3.3 Sets of Refutations*

Operators $C^r_{\cdot}$ are, in the sense explain before, good candidates to represent consequences. We consider now how to define good candidates to define set of graded conjectures and refutations. In the crisp case, if $\rho$ represents the résumé, a conjecture is a propositions $p$ whose negation $p'$ is not deduced from the résumé, i.e. $\rho. \not\leq p'$. That sense means that a conjecture is any proposition which is not contradictory with the knowledge generate by the résumé, or which is compatible with that knowledge. Or put another way, a conjecture is a candidate to increase the knowledge consistently in some way. Refutations, on the contrary, are propositions which are in contradiction to the current knowledge.

The set of refutations is defined as follows:

**Definition 4** (*C.-refutations of grade r*)

$$\mathrm{Ref}^r_{\cdot}(P) = \{\mu \in \mathcal{F}(X) \ : \ \rho \preccurlyeq \mu' \geq r\}$$

The next theorem shows how with this definition, refutations introduce some level of contradiction in the information given by the résumé.

**Theorem 4** *If* $\mu \in Ref^r_{\cdot}(P)$, *then* $\mu \cdot \rho \leq (\mu \cdot \rho)' \geq r$.

*Proof* Because $\mu \in \mathrm{Ref}^r_{\cdot}(P)$, it is valid that

$$\begin{aligned} r &\leq \rho \preccurlyeq \mu' \\ &= T(\rho \preccurlyeq \mu', 1) = T(\rho \preccurlyeq \mu', \mu \cdot \mu' \preccurlyeq \mu') \\ &\leq T(\mu \cdot \rho \preccurlyeq \mu \cdot \mu', \mu \cdot \mu' \preccurlyeq \mu') \\ &\leq \mu \cdot \rho \preccurlyeq \mu' \\ &= T(\mu \cdot \rho \preccurlyeq \mu', \mu \cdot \rho \preccurlyeq \mu) \\ &\leq T(\mu \cdot \rho \preccurlyeq \mu', \mu' \preccurlyeq (\mu \cdot \rho)') \\ &\leq \mu \cdot \rho \preccurlyeq (\mu \cdot \rho)'. \end{aligned}$$

□

So from $0 < r \leq \rho \preccurlyeq \mu'$ we obtain a level $r$ of contradiction with $0 < \mu \cdot \rho \preccurlyeq (\mu \cdot \rho)'$. It shows that a refutation added by the left to the set of premises causes a level of contradiction in our knowledge.

Note that in case of a non-nilpotent t-norm $T$, a refutation never is a consequence to any grade because

$$T(\rho \preccurlyeq \mu', \rho \preccurlyeq \mu) \leq T(\rho \preccurlyeq \mu', \mu' \preccurlyeq \rho') \leq \rho \preccurlyeq \rho',$$

and it is not possible to have $\rho \preccurlyeq \mu' > 0$ and $\rho \preccurlyeq \mu > 0$, simultaneously. However with nilpotent t-norms it can be the case that a refutation to some grade can be a consequence to a non-zero grade. In particular we have

**Theorem 5** *For a non-nilpotent t-norm $T$, if $\mu \in C^r_\cdot(P)$, then $\mu' \notin C^r_\cdot(P)$.*

As the weak-consequences, the sets of refutations are a family of anti-monotonic sets with respect to the grades $r$.

### *16.3.4 Sets of Conjectures*

If the set of refutations contains contradictory fuzzy sets with the current knowledge to some level, it is seems natural define the set of fuzzy sets which allows to increase our knowledge, named conjectures, to some degree $r$ as the complementary set

$$\text{Conj}^r_\cdot(P) = \{\mu \in \mathcal{F}(X) \ : \ \rho \preccurlyeq \mu' < r\}. \tag{16.5}$$

$\mu$ is a conjecture to level $r$ if it is the case than $\rho$ is not contradictory with $\mu$ to level greater or equal than $r$.

Obviously, $\mathcal{F}(X) = \text{Conj}^r_\cdot(P) \cup \text{Ref}^r_\cdot(P)$ and $\emptyset = \text{Conj}^r_\cdot(P) \cap \text{Ref}^r_\cdot(P)$. The sets of conjectures are monotonic with respect to the grades, i.e. if $r < s$, then $\text{Conj}^r_\cdot(P) \subseteq \text{Conj}^s_\cdot(P)$.

The behavior of conjectures when the set of premises increase is the contrary of the consequences.

**Theorem 6** *If $P$ and $Q$ are sets of admissible premises, and it is $P \subseteq Q$, then $Conj^r_\cdot(Q) \subseteq Conj^r_\cdot(P)$ for any $r \in [0, 1]$. It is said that the operators $Conj^r_\cdot$ are anti-monotonic with respect to the set of premises.*

*Proof* If any $\mu \in \mathcal{F}(X)$ satisfies $\mu \in \text{Conj}^r_\cdot(Q)$, then $\rho_Q \preccurlyeq \mu' < r$. If it were that $\rho_P \preccurlyeq \mu' \geq r$ by T-transitivity

$$\rho_P \preccurlyeq \mu' = T(\rho_P \preccurlyeq \mu', \rho_Q \preccurlyeq_1 \rho_P) \leq \rho_Q \preccurlyeq \mu',$$

giving a contradiction. □

**Theorem 7** *For any non-nilpotent t-norm $T$ and any strong negation, any set of admissible premises $P$ and any $r \in (0, 1]$ is $C^r_\cdot(P) \subseteq Conj^r_\cdot(P)$.*

*Proof* As

$$T(\rho_P \preccurlyeq \mu, \rho_P \preccurlyeq \mu') \leq T(\rho_P \preccurlyeq \mu, \mu'' \preccurlyeq \rho_P) \leq \rho_P \preccurlyeq \rho'_P = 0,$$

if $T$ is non-nilpotent and $\rho_P \preccurlyeq \mu \geq r > 0$ then $\rho_P \preccurlyeq \mu' = 0$ and $\mu$ is a conjecture to any level. □

**Theorem 8** *Operators* $Ref^r_{\cdot}$ *are not extensive for any* $r \in (0, 1]$*, and they are monotonic.*

*Proof* if it were $P \subseteq \mathrm{Ref}^r_{\cdot}(P)$, from $\rho \preccurlyeq_r \rho'_i$ and $\rho \preccurlyeq_1 \rho_i$, as $\rho \preccurlyeq \rho_i \leq \rho'_i \preccurlyeq \rho'$, thanks to T-transitivity of preorder $\preccurlyeq$ it would be $\rho \preccurlyeq \rho'_i = T(\rho \preccurlyeq \rho'_i, \rho'_i \preccurlyeq_1 \rho') \leq \rho \preccurlyeq \rho'$ something impossible.

If $P \subseteq Q$, from $\rho_P \preccurlyeq_r \mu$ and $\rho_Q \preccurlyeq_1 \rho_P$, it is valid $\rho_Q \preccurlyeq_r \mu$, and then $\mathrm{Ref}^r_{\cdot}(P) \subseteq \mathrm{Ref}^r_{\cdot}(Q)$. □

Another interesting study would be what happens when the conjunction change. For that, it is necessary to establish a relationship between the fuzzy preorder and the conjunction given by min.

**Definition 5** The $T$-preorder $\preccurlyeq$ preserves the min pointwise if for any $\mu, \eta, \sigma \in \mathcal{F}(X)$ when $\mu \preccurlyeq_1 \eta$ and $\mu \preccurlyeq_1 \sigma$ then $\mu \preccurlyeq_1 \min(\eta, \sigma)$ defined pointwise.

The last definition can be described by the ordering principle: $\mu \preccurlyeq_1 \eta \Leftrightarrow \mu(x) \leq \eta(x) \quad \forall x$. Many $T$-preorders contains the pointwise ordering, in particular those defined by residuation.

Another strong form to establish the last property could be

$$\min(\mu \preccurlyeq \eta, \mu \preccurlyeq \sigma) \leq \mu \preccurlyeq \min(\mu, \sigma). \tag{16.6}$$

Implications defined by residuation satisfy

$$\min(\overrightarrow{T}(x, y), \overrightarrow{T}(x, z)) = \overrightarrow{T}(x, \min(y, z))$$

so they give examples of $T$-preorder which satisfy (16.6).

**Theorem 9** *For all conjunction* $\cdot$*, and a* $T$*-preorder which preserves the* min *pointwise is*

1. $Conj^r_{\cdot}(P) \subseteq Conj^r_{\min}(P)$,
2. $Ref^r_{\min}(P) \subseteq Ref^r_{\cdot}(P)$,
3. $C^r_{\min}(P) \subseteq C^r_{\cdot}(P)$.

*Proof* Point 3 follows of $\rho \preccurlyeq_1 \rho_{\min}$, where $\rho_{\min}$ is the résumé taking min as the conjunction because $\preccurlyeq$ preserves the min pointwise. Point 2 follows from $\mu \in \mathrm{Ref}^r_{\cdot}(P) \Leftrightarrow \mu' \in \mathrm{C}^r_{\cdot}(P)$ for any conjunction, and finally point 1 follows by complementation of point 2. □

### *16.3.5 Sets of Hypothesis and Speculations*

Take into consideration the difference-set:

$$\mathrm{Conj}^r_{\cdot}(P) - \mathrm{C}^r_{\cdot}(P) = \{\mu \in \mathcal{F}(X) : \rho \preccurlyeq \mu < r \;\&\; \rho \preccurlyeq \mu' < r\} \tag{16.7}$$

Remember that the set $C^r_{\cdot}(P)$ is not necessarily a subset of $\mathrm{Conj}^r_{\cdot}(P)$, as it is shown in Theorem 6 and, in fact, $\mathrm{Conj}^r_{\cdot}(P)$ can be empty. The set (16.7) represents the conjectures, if they exist, that are not consequences.

Consider next two sets:

$$\mathrm{Hyp}^r_{\cdot}(P) = \{\mu \in \mathcal{F}(X) : \rho \preccurlyeq \mu < r \ \& \ \rho \preccurlyeq \mu' < r \ \& \ \mu \preccurlyeq \rho \geq r\},$$
$$\mathrm{Sp}^r_{\cdot}(P) = \{\mu \in \mathcal{F}(X) : \rho \preccurlyeq \mu < r \ \& \ \rho \preccurlyeq \mu' < r \ \& \ \mu \preccurlyeq \rho < r\},$$

**Theorem 10** *If* $\mathit{Conj}^r_{\cdot}(P) \neq \emptyset$*, then* $\mathit{Conj}^r_{\cdot}(P) = C^r_{\cdot}(P) \cup \mathit{Hyp}^r_{\cdot}(P) \cup \mathit{Sp}^r_{\cdot}(P)$.

The conjectures, if they exist, are classifies in consequences, hypothesis and speculations.

With respect to the grade $r$, if $r \leq r'$, $\mathrm{Hyp}^r_{\cdot}(P)$ is incomparable with $\mathrm{Hyp}^{r'}_{\cdot}(P)$, it can be that the hypotheses grow or not. The set of conjectures increase with $r$ but the set of consequences decrease and the set of speculations increase as it is easy to probe. For the set of hypothesis nothing can be said.

**Theorem 11** *It is satisfied that*

$$\mu \in \mathit{Hyp}^r_{\cdot}(P) \Rightarrow \rho \in C^r_{\cdot}(\{\mu\}).$$

*Proof* If $\mu \in \mathrm{Hyp}^r_{\cdot}(P)$, then $\mu \preccurlyeq \rho \geq r$ and $\rho \preccurlyeq \mu < r$ then $\rho \neq \mu$ and $\rho \in C^r_{\cdot}(\{\mu\})$. □

A natural requirement for a hypothesis is that explain the consequences, in particular, any consequence of a premise must be a consequence of a hypothesis. In our case that depends of the t-norm because if $\sigma \in \mathrm{Hyp}^r_{\cdot}(P)$ and $\mu \in C^r_{\cdot}(P)$, then

$$T(r, r) \leq T(\sigma \preccurlyeq \rho, \rho \preccurlyeq \mu) \leq \sigma \preccurlyeq \mu.$$

So $\mu$ is a consequence of $\sigma$ to level $T(r, r)$.

**Theorem 12** *For a fixed r, operators* $\mathit{Hyp}^r_{\cdot}$ *are anti-monotonic.*

*Proof* From $P \subseteq Q$ is $\rho_Q \preccurlyeq_1 \rho_P$, if $\mu \in \mathrm{Hyp}^r_{\cdot}(Q)$, from $T(\mu \preccurlyeq \rho_Q, \rho_Q \preccurlyeq_1 \rho_P) = \mu \preccurlyeq \rho_Q \leq \mu \preccurlyeq \rho_P$, follows $\mu \preccurlyeq \rho_P \geq r$. Now if it were $\rho_P \preccurlyeq \mu \geq r$, from $T(\rho_P \preccurlyeq \mu, \rho_Q \preccurlyeq_1 \rho_P) = \rho_P \preccurlyeq \mu \leq \rho_Q \preccurlyeq \mu$, reaching a contradiction. The same can be done for proving that $\rho_P \preccurlyeq \mu' < r$, so $\mu \in \mathrm{Hyp}^r_{\cdot}(P)$. □

Operator $\mathrm{Sp}^r_{\cdot}(P)$ are neither monotonic nor antimonotonic.

## 16.4 Examples

A couple of examples show the relations between these sets in a graded context.

### *16.4.1 Example 1*

First consider a T-preorder defined by residuation:

$$\rho \preccurlyeq^1 \mu = \inf_{x \in X} \overrightarrow{\min}(\rho(x), \mu(x)) = \begin{cases} 1 & \text{if } \forall x \ \ \rho(x) \leq \mu(x) \\ \inf_{x:\rho(x)>\mu(x)} \mu(x) & \text{otherwise} \end{cases} \tag{16.8}$$

That fuzzy ordering contains the pointwise ordering because $\rho \preccurlyeq^1 \mu = 1 \Leftrightarrow \rho(x) \leq \mu(x) \ \ \forall x$. From now we consider that the set of premises contains only a fuzzy set $\rho$ and the fuzzy complement is given by the classical pointwise strong negation $1 - id$. We try to give an idea about the sets of consequences, refutations, conjectures, hypotheses and speculations in that case.

To evaluate how is the set a consequences, a necessary condition for a fuzzy set $\mu$ to belong to the set $\mathrm{C}^r_\cdot(\rho)$ is that $\mathrm{Supp}(\rho) \subseteq \mathrm{Supp}(\mu)$, where $\mathrm{Supp}(\mu) = \{x \in X : \mu(x) > 0\}$. When $r = 0$, the set of consequences will be the universal set of fuzzy sets, and when $r = 1$, the set of consequences is just the fuzzy sets which are pointwise greater than $\rho$.

With respect to the set of refutations, again a necessary condition for a fuzzy set $\mu$ to belong to $\mathrm{Ref}^r_\cdot(\rho)$ is that $\mathrm{Supp}(\rho) \subseteq \mathrm{Supp}(1 - \mu)$. The set $\mathrm{Ref}^0_\cdot(\rho) = \mathcal{F}(X)$ and $\mathrm{Ref}^1_\cdot(\rho) = \{\mu \in \mathcal{F}(X) \ : \ \rho(x) \leq \mu(x) \ \forall x \in X\}$.

The set of conjectures for $r \in (0, 1]$ can be expresed as follows:

$$\begin{aligned} \mathrm{Conj}^r_\cdot(\rho) &= \{\mu \in \mathcal{F}(X) \ : \ \rho \preccurlyeq^1 \mu' < r\} \\ &= \{\mu \in \mathcal{F}(X) \ : \ \mathrm{Supp}(\rho) \cap \mathrm{Supp}(\mu')^c \neq \emptyset\} \cup \\ &= \{\mu \in \mathcal{F}(X) \ : \ \mathrm{Supp}(\rho) \subseteq \mathrm{Supp}(\mu') \wedge \inf_{x:\rho(x)>\mu'(x)} (1 - \mu(x)) < r\}. \end{aligned}$$

When $r = 0$, $\mathrm{Conj}^0_\cdot(\rho) = \emptyset$, and $\mathrm{Conj}^1_\cdot(\rho) = \{\mu \in \mathcal{F}(X) \ : \ \rho \not\leq \mu' \text{ pointwise}\}$.

A sufficient condition for a fuzzy set $\mu$ to belong to the set of hypotheses $\mathrm{Hyp}^r_\cdot(\rho)$ is that the support of $\rho$ is neither included in the support of $\mu$ nor included in the support of $\mu'$ together that the support of $\mu$ is included in the support of $\rho$. Speculations can be calculated in a analogous way and Fig. 16.1 shows an example of elements of these sets for a triangular fuzzy set.

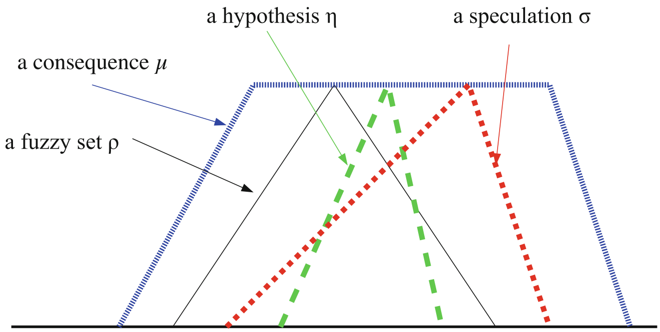


**Fig. 16.1** Examples of a consequence $\mu$ (*blue ultrathin dashed line*), a hypothesis $\eta$ (*green dashed line*), and a speculation $\sigma$ (*dotted line*) with the ordering $\preccurlyeq^1$ for a triangular fuzzy set $\rho$ (*black solid line*)

### *16.4.2 Example 2*

Consider now a probabilized boolean algebra $L$ with a probability $p$. This is an example of a BFA and the ordering on $L$ given the expression

$$a \preccurlyeq^2 b = p(a' + b) = 1 - p(a) + p(a \cdot b), \tag{16.9}$$

is an example of a $W$-preorder, with $W$ the Lukasiewicz t-norm. Considering that the set of premises is just one element $a$ of $L$. It is possible to calculate the sets in this case.

For the consequences:

$$\begin{aligned} \mathrm{C}^r_\cdot(a) &= \{b \in L \;:\; a \preccurlyeq^2 b \geq r\} \\ &= \{b \in L \;:\; p(a \cdot b) \geq p(a) - (1 - r)\}. \end{aligned}$$

For the refutations:

$$\begin{aligned} \mathrm{Ref}^r_\cdot(a) &= \{b \in L \;:\; a \preccurlyeq^2 b' \geq r\} \\ &= \{b \in L \;:\; p(a \cdot b') \geq p(a) - (1 - r)\} \\ &= \{b \in L \;:\; p(a \cdot b) \leq 1 - r\}, \end{aligned}$$

where the equality $p(a) = p(a \cdot b) + p(a \cdot b')$ has been applied.

The set of conjectures is $\mathrm{Conj}^r_\cdot(a) = \mathrm{Ref}^r_\cdot(a)^c = \{b \in L \;:\; p(a \cdot b) > 1 - r\}$.

The hypothesis are characterized by

$$\mathrm{Hyp}^r_\cdot(a) = \{b \in L : 1 - r < p(a \cdot b) < p(a) - (1 - r) \wedge p(b) - (1 - r) \leq p(a \cdot b)\}.$$

Finally, the set of speculations is given by

$$\mathrm{Sp}^{r}_{\cdot}(a) = \{b \in L \,:\, 1 - r < p(a \cdot b) < \min(p(a) - (1 - r), p(b) - (1 - r))\}.$$

## 16.5 Conclusions

In this work we have studied the models of conjectures, consequences, hypothesis and speculation initiated by Professor Enric Trillas in various works for the case in which we have a graded consequence relation given by a T-preorder, with T be a t-norm, on an algebra of fuzzy sets. For this we have used a model of algebra of fuzzy sets as general as possible named Basic Fuzzy Algebra in which the properties required to the operations have been reduced as much as possible so that they may account for the largest number of phenomena that occur in natural language reasoning. We prescinded from the commutativity of conjunction or its associativity, or it has been considered weak negations, and yet interesting results that relate the sets of propositions studied from a minimally coherent set of assumptions are obtained.

The results basically differ depending on whether the selected t-norm T is nilpotent or not. When the t-norm is not nilpotent results resemble those obtained considering a consequence relation given by the pointwise ordering between fuzzy propositions. However with nilpotent t-norms weaker results are obtained because the thresholds are not maintained and they can collapse to zero.

**Acknowledgments** I would like to thank Professor Enric Trillas for their support in carrying out this work and for an enriching collaboration during my research career.

Also I acknowledge the support of the Spanish Ministry for Economy and Innovation and the European Regional Development Fund (ERDF/FEDER) under grant TIN2011-29827-C02-02.

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# Chapter 17
# Fuzzy Concepts and Fuzzy Logic in Historical and Genetic Epistemology

**Rudolf Seising**

**Abstract** This paper discusses epistemology in two variations: Genetic and Historical Epistemology. Historical Epistemology combines research in history and philosophy of science to study developments in scientific research whereas Genetic Epistemology is the study of the cognitive development in childhood. We consider Kuhn's theory of scientific paradigms and paradigm changes on the one hand and Piagest's Genetic Epistemology of cognitive development on the other. We present parallels of these two structuralist approaches and we introduce "unsharp concepts" into these views. Then the paper throws a glance at their fuzzy extensions. Because of these "fuzzifications" we make an argument for using fuzzy instead of crisp concepts in Genetic and Historical Epistemology.

## 17.1 Introduction

Knowledge, its emergence and its developments are inherently interesting subjects in science and research. In the 20th century slogans or buzz words were "knowledge society" and "knowledge explosion" and the last turn of the century was called the beginning of the "knowledge age".

Knowledge, its emergence and its developments were under examination from various perspectives and in this chapter I will near to these subjects from a historical and philosophical point of view. The branch of philosophy that is concerned with knowledge is named Epistemology.

The *Stanford Encyclopedia of Philosophy* defines "narrowly, epistemology is the study of knowledge and justified belief. As the study of knowledge, epistemology is concerned with the following questions: What are the necessary and sufficient conditions of knowledge? What are its sources? What is its structure, and what are its limits?" [46].

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R. Seising et al. (eds.), *Towards the Future of Fuzzy Logic*,
Studies in Fuzziness and Soft Computing 325,
DOI 10.1007/978-3-319-18750-1_17

To answer these questions Immanuel Kant (1724–1804) tried to unify the two main views in former philosophy of science, empiricism and rationalism. The rationalist approach came to fundamental, logical and theoretical investigations using logics and mathematics to formulate axioms and laws, however from the empiricist point of view the source of our knowledge is sense experience and we have to use experiments to find or prove or refute natural laws. In both directions—from experimental results to theoretical laws or from theoretical laws to experimental proves or refutations—scientists have to bridge the gap that separates theory and practice in science. In his influential book *Critique of Pure Reason (Kritik der reinen Vernunft)* Kant came from the view that knowledge increases through, and that cognition starts from experience. The "a priori forms of cognition" he named "concepts" and he proposed that these concepts exist within the subject of cognition. On the other hand, he said, that the object of cognition is established when the sensory content coming from the object is put in order by the subject's concepts.

The *Stanford Encyclopedia of Philosophy* differentiates at least between three prevailing ways to understand what a concept is in contemporary philosophy [25]:

- Concepts as mental representations, i.e. entities that exist in the brain;
- Concepts as abilities, peculiar to cognitive agents;
- Concepts as abstract objects, where these objects are the constituents of propositions that mediate between thought, language, and referents.

In their book *Concepts and Fuzzy Logic* Belohlavek and Klir [1] use the concept of concepts from Cognitive science and they refer to Edouard Machery: "In cognitive science, concepts are the bodies of knowledge that are stored in long-term memory and are used by default in the higher cognitive processes (categorization, inductive and deductive reasoning, analogy making, language understanding, etc.)" [23]. In this chapter we will argue that those concepts are unsharp or fuzzy and we will show that those fuzzy concept are also fruitful in epistemology.

In Kant's epistemology the human mind provides a structure that shapes all sensory experience and thought. Perceptions and thoughts must conform into this structure in order to be representations. Kant says that humans have an active mind that produces our conception of reality by acting as a filter, and also it is organizing and enhancing. He continues that objective reality is made possible by the form of its representation. Our minds structure includes space, time, and causation, they are preconditions of our perceptions, and they are fundamental conditions for human experience.

The construction of the system of science with a progression from the formal to the most empirical phases Kant named an "Architectonic of Science": "by the term architectonic I mean the art of constructing a system. Without systematic unity, our knowledge cannot become science; it will be an aggregate, and not a system. Thus architectonic is the doctrine of the scientific in cognition, and therefore necessarily forms part of our methodology" [16]. He also developed a system of physical nature starting with "the most formal act of human cognition, called by him the transcendental unity of apperception, and its various aspects, called the logical functions of judgment. He then proceeds to the pure categories of the understanding, and then to the schematized categories, and finally to the transcendental principles of nature in general" [7]. It is this concept of a "schema" that Kant used to both the structure of human knowledge itself and the procedure by which the human mind produces and uses such structures.

In the 20th century two different research programs in different academic disciplines became well-known under the name "epistemology"—*Genetic* Epistemology was founded by the philosopher and psychologist Jean Piaget (1896–1980) to study the cognitive development of children and *Historical* Epistemology was established as a field of research in history and philosophy of science by the molecular biologist and historian of science Hans-Jörg Rheinberger (born 1946).

Education science and Psychology as well as History and Philosophy of science are concerned with preconditions, foundations, methods, implications and goals of science; philosophers of science reflect scientific theories and their changes, educationists and psychologists want to know how human beings—especially young human beings—learn. They develop and apply theories of human cognitive developments. In science of education pedagogues and psychologists are concerned with the development of conceptual knowledge as individual mental processes of children and students. In the 20th century parts of these two academic fields, history and philosophy of science on the one hand and education science and psychology on the other hand emerged approaches that adopted Kant's concept of a "schema"—these approaches have been called Structuralism.

Structuralism in education science and psychology was intended by the Swiss developmental psychologist and philosopher Piaget who used the structuralism of the Swiss linguist and semiotician Ferdinand de Saussure (1857–1913) as a methodology in psychology. However, in Piaget's view humans have adaptive mental structures. These mental structures assimilate external events; humans convert these external events to fit their mental structures. The most simple level of these mental structures is that of the so-called schema. Piaget introduced schemata as categories of knowledge to describe mental or physical actions.

Philosophy of science concerns scientific explanations of real systems and phenomena. Scientists connect these real systems and phenomena with theoretical structures and in modern science these structures are characterized in terms of logics and mathematics. Therefore, these structures show a "mapping" from the real world to the world of logics and mathematics. It is supposed that there is a connection between the real world and the logical-mathematical world—otherwise it doesn't make sense to speak about empirical science. The German-US-American philosopher Rudolf Carnap (1891–1970) proposed an approach to axiomatize scientific theories in a formal language. In contrast to that the American mathematician and philosopher Patrick Suppes (1922–2014) suggested to determine the mathematical structure of a theory by use of informal set theory [47–49]. Thus, one is able to axiomatize theories in a precise way without recourse to formal languages. This approach can be traced back to the meta-mathematical programme of the French group Bourbaki [5] to include the axiomatization of empirical theories in the metamathematical programme of the French group Bourbaki [5].

In the last third of the 20th century this approach in philosophy of science has been elaborated and continued by Joseph D. Sneed (born 1938), Wolfgang Stegmüller (1923–1991), C. Ulises Moulines (born 1946) and Wolfgang Balzer (born 1947) and others. Their new developments act as a bridge between Philosophy of science and History of science because they could reconstruct theory dynamic in the sense of Thomas S. Kuhn's (1922–1996) concept of paradigm and paradigm change, i.e. scientific revolutions.

In the following we give brief surveys of the structuralist approaches in history of science (Sect. 17.2) and in education science and psychology (Sect. 17.6). We outline the different ways of looking at concepts—"sharp" and "unsharp"—and we show that "unsharp" concepts play an important role in both of the two lines of epistemology (Sect. 17.4 and Sect. 17.6.3). We discuss parallels between the Kuhnian and the Piagetian theories of knowledge development (Sect. 17.6.2) and because of these parallels, after a brief introduction of our proposal for a fuzzy structuralism in philosophy of science (Sect. 17.5), as future prospects we argue for a fuzzy structuralist approach to Genetic and Historical Epistemology (Sect. 17.8).

## 17.2 An Architectonic for Science

Drawing our attention back to the philosophical topic of the gap between theoretical and real entities in science we mentioned already Kant named his construction of the system of science with a progression from the formal to the most empirical phases an "Architectonic of Science".

*An Architectonic for Science* was published as *A structuralist program* in 1987 by Balzer, Moulines and Sneed [3]. Their program was based on the structuralist approach that was established by Suppes to axiomatize real physical theories in a precise way without recourse to formal languages. In the 1970s, one of Suppes' Ph D.-students, the US-American physicist Joseph D. Sneed (born 1938), developed informal semantics meant to include not only mathematical aspects, but also application subjects of scientific theories in the framework, based on this method. In [44] he presented the view that all empirical claims of phys- ical theories have the form "$x$ is an $S$", where "is an $S$" is a set-theoretical predicate (e.g., "$x$ is a classical particle mechanics"). Every physical system that fulfills this predicate is called a model of the theory. For example, the class $M$ of a theory's models is characterized by empirical laws that consist of conditions governing the connection of the components of physical systems. Therefore, we have models of a scientific theory, and by removing their empirical laws, we get the class $M_p$ of so-called potential models of the theory. Potential models of an empirical theory consist of theoretical terms, i.e. observables with values that can be measured in accordance with the theory. This connection between theory and empiricism is the basis of the philosophical "problem of theoretical terms". If we remove the theoretical terms of a theory in its potential models, we get structures that are to be treated on a purely empirical layer; we call the class $M_{pp}$ of these structures of a scientific theory its "partial potential models". Finally, every

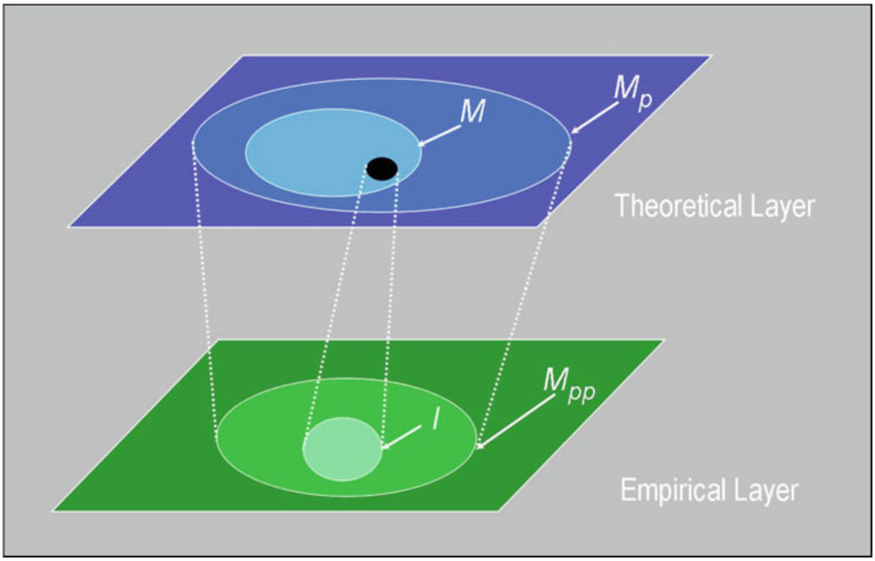


**Fig. 17.1** Empirical and theoretical structural layers

physical theory has a class $I$ of intended systems (or applications) and, of course, different intended systems of a theory may partially overlap. This means that there is a class $C$ of constraints that produces cross connections between the overlapping intended systems. In brief, this structuralist view of scientific theories regards the core $K$ of a theory as a quadruple $K = < M_p, M_{pp}, M, C >$. This core can be supplemented by the class $I$ of intended applications of the theory $T = < K, I >$. To make it clear that this concept reflects both sides of scientific theories, these classes of $K$ and $I$ are shown in Fig. 17.1. Thus we notice that $M_{pp}$ and $I$ are entities of an empirical layer, whereas $M_p$ and $M_{pp}$ are structures in a theoretical layer of the schema.

Stegmüller, Sneed, Moulines and Balzer also started to reconstruct the change of theories, i.e. "scientific revolutions" or "paradigm shifts", e.g. the change from Ptolemy's geocentric universe to Copernicus' heliocentric world picture or from Newtonian Mechanics to Einstein's Special Relativity Theory by using the structuralist program in philosophy of science. For this purpose they defined an intertheorical relation that is called "Approximate Reduction". However, this "Approximate Reduction" is—as its name says, an approximation of the "pure" intertheoretical relation that is called "Reduction" (see Fig. 17.2) and it is defined as follows:

There are two theories, say $T_{old}$ and $T_{new}$. $T_{old}$ reduces $T_{new}$ by the reduction relation $\rho$ if the following two conditions are fulfilled:

$$\rho \subseteq M_p(T_{old}) \times M_p(T_{new}). \tag{17.1}$$

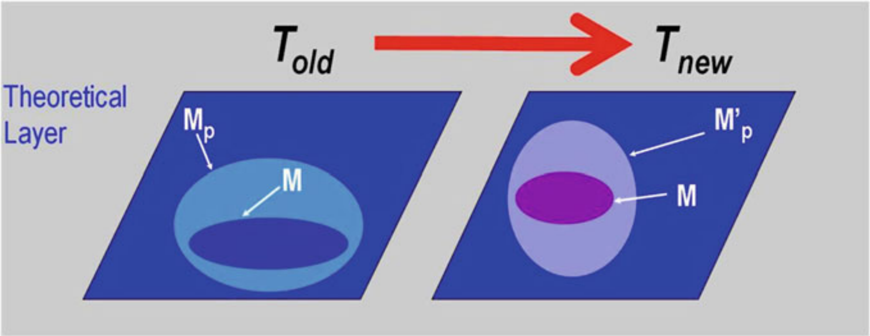


**Fig. 17.2** Intertheoretical relation "Reduction"

$$\forall x, x' : \langle x, x' \rangle \in \rho \text{ and } x' \in M(T_{new}), \text{ then } x \in M(T_{old}) \tag{17.2}$$

To sum up: the structuralist approach in philosophy of science is a way to describe the logical structures and the dynamics of scientific theories.

However, from our point of view the version of the "approximate reduction" that the "classical" structuralists in philosophy of science proposed is not the best choice to represent theory changes due to scientific revolutions because paradigm shifts in the sense of Kuhn are not pure rational changes and there is no one-to-one-relation between the concepts of the old and the new theory. Moreover, Kuhn claimed that scientists do not employ *rules* in reaching their decisions—however, his philosophical reviewers said that he claimed "that science is irrational" [4]. What he stressed was that a paradigm shift is not a logically determinate procedure but a mixture of processes that result from other reasons, e.g. sociological and psychological. An example is his already five years earlier published analysis of the Copernican Revolution [19]: he pointed out that in the beginning of this paradigm shift the new view could not offer more accurate predictions of the positions of planets or stars in the future than the old system of Ptolemy. But what made it more successful was that the new system dangled better and simpler mathematical solutions.

We don't say that these "external factors" are irrational but to respect these kinds of "unsharpness" of concepts when modeling (or reconstructing) the "approximate reduction" we need "a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions."[1]

However, whereas the "classical" structuralists proposed to reconstruct an "approximative reduction" of scientific theories by methods within classical mathematics, e.g. converging series of models of a theory or topological entities in spaces of such models, we think that it is more suitable to define a different version of the "approximative reduction" by using fuzzy sets and fuzzy relations. This "Fuzzy

[1]This is a quotation of Lotfi Zadeh's paper [56] that appeared in 1962—three years before he founded the theory of fuzzy sets!

Reduction" is a fuzzy relation and in our opinion it fits as a fuzzy model for paradigm changes in history of science!

## 17.3 Structures of the Scientific and the Cognitive Development

Concerning the dynamics of scientific theories the US-American historian and philosopher of science Thomas S. Kuhn (1922–1996) argued in *The Structure of Scientific Revolutions* [18] that there is no linear accumulation of new knowledge in the development of science. Moreover, he wrote that science undergoes periodic revolutions and that there are conceptual changes—initially he called them "paradigm shifts"—in history of science, in which the nature of scientific inquiry within a particular field is transformed. Kuhn claimed that there are three different stages of science: *Prescience* (also called the "pre-paradigm-phase") lacks a central paradigm. Later, when scientists attempt to enlarge the central paradigm by "puzzle-solving" prescience is followed by *normal science*. Normal science reaches a crisis when anomalous results build up. At this point a new paradigm can emerge, which subsumes the old results along with the anomalous results into one framework. This new paradigm is termed *revolutionary science* [18].

Among other elements (research techniques, sets of accepted data, theory, etc.) a paradigm includes conceptual schemes. During a paradigm shifts therefore concept are replaced by different concepts or—as the historian of science David Kaiser recently explained: "Sometimes we do change our conceptual filters". He continued: "These radical ruptures, though rare, Kuhn thought, could really reshuffle the basic facts of science" [6].

Kuhn also gave in his seminal book a hint to the work of Piaget: "A footnote encountered by chance led me to the experiments by which Jean Piaget has illuminated both the various worlds of the growing child and the process of transition from one to the next" [18] and he referred to Piaget works [28, 29] in his own footnote. Many years later he clarified that he "discovered Piaget" and especially the last mentioned book [28] when he read the important thesis on history of science, written by American sociologist Robert Merton (1910–2003) [26]. He then thought that "these children develop ideas just the way scientists do, except—and this was something I felt Piaget did not himself sufficiently understand, and I'm not sure that I realized it early—they are being taught, they are being socialized, in this not spontaneous learning, but learning what it is that is already in place. And that was important" [17, p. 279].

With reference to [20, p. 172f] also Nigerian philosopher Douglas I. O Anele says that "Kuhn applied some of the insights revealed by Jean Piaget's classic laboratory experiments on how children interpreted motion in his analysis of how a child learns to differentiate various kinds of birds in perceptual space through a means of processing data into similarity sets or categories which does not depend on a prior answer to the question: similar with respect to what" [2].

Nevertheless, the methods that the structuralists proposed to reconstruct the "Approximative Reduction" are within classical mathematics, e.g. converging series of models of a theory or topological entities in spaces of such models. In our view it will be more suitable to define another "approximative" version of the intertheoretic relation of "reduction" by using fuzzy sets and fuzzy relations. The "Fuzzy Reduction" that we propose to use is a fuzzy relation and in our opinion this fuzzy reduction relation fits as a fuzzy model for paradigm changes in history of science!

## 17.4 Sharp and Unsharp Concepts in Epistemology

The German philosopher and mathematician Gottlob Frege (1848–1925) published in 1879 his revolutionary book *Concept Script* (*Begriffsschrift*). When formalizing the mathematical principle of complete induction he confronted the problem of vagueness: he saw that some predicates are not inductive, viz. they have been defined for all natural numbers, but they result in false conclusions, e.g. the predicate "heap" cannot be evaluated for all natural numbers [9]. Ten years later, when he revised the basics of this book for a lecture to the Society of Medicine and Science in Jena at the beginning year 1891, he reinterpreted concept functions and subsequently he introduced these functions of concepts everywhere. He stated: If "$x + 1$" is meaningless for all arguments $x$, then the function $x + 1 = 10$ has no value and no truth value either. Thus, the concept "that which when increased by 1 yields 10" would have no sharp boundaries. Accordingly, for functions the demand on sharp boundaries entails that they must have a value for every argument [10]. This is a mathematical verbalization of what is called the classical sorites paradox that can be traced back to the old Greek word $\sigma o \rho o \varsigma$ (for 'heap') used by Eubulid of Alexandria (4th century BC).

In his book *Foundations of Arithmetic* (Grundgesetze der Arithmetik) that appeared in the years 1893–1903, Frege called for concepts with sharp boundaries, because otherwise we could break logical rules and, moreover, the conclusions we draw could be false [11] and later he wrote: "A definition of a concept (of a possible predicate) must be complete; it must unambiguously determine, as regards any object, whether or not it falls under the concept (whether or not the predicate is truly ascribable to it). [...] We may express this metaphorically as follows: the concept must have a sharp boundary" [11, Sect. 56].

Frege called for concepts with sharp boundaries because otherwise we could break logical rules and, moreover, the conclusions we draw could be false [11] but about half a century later Wittgenstein argued the converse in his *Philosophical Investigations*:

> One might say that the concept 'game' is a concept with blurred edges. 'But is a blurred concept a concept at all?' Is an indistinct photograph a picture of a person at all? Is it even always an advantage to replace an indistinct picture by a sharp one? Isn't the indistinct one often exactly what we need? Frege compares a concept to an area and says that an area with

> vague boundaries cannot be called an area at all. This presumably means that we cannot do anything with it. But is it senseless to say: 'Stand roughly there'? [53, Sect. 71]

Thus, Wittgenstein had turned away from the epistemological system of the *Tractatus* with its ideal mapping between the objects of reality and a logically precise language. If we are not able to find such an exact logical language, then we have to accept the fact that there is unsharp or vague linguistic usage in all languages. Then the images, models, and theories that we build with the words and propositions of our languages to communicate with them are and will also be unsharp or vague. In other words, our conceptions, images, and symbols of external things or objects are entities without sharp borders. They are fuzzy entities and it is time to establish a "fuzzy epistemological system" to master these complex circumstances with an appropriate theory of science!

## 17.5 Fuzzy Structuralism

Referring to *The Structure and Dynamics of Theories* [45] by the Austrian philosopher of science Stegmüller, the Iranian-German philosopher and physician Kazem Sadegh-Zadeh (born 1942) stated in his article "The Fuzzy Revolution: Goodbye to the Aristotelian Weltanschauung" that concepts as Kuhn's and his combatants "are still too vague and inadequate to be useful". [35, p. 3] He continued that

> we may, nevertheless, learn from these studies that in contrast to our accustomed views on the development of science and scientific knowledge, this very development is not a cumulative process. Science does not progress continuously and by accumulating knowledge. It does not add to an antecedent knowledge or theory $T_i$ a subsequent knowledge or theory $T_{i+1}$ of the same type such that one could reasonably consider science as the open ordered series of related theories $T_1, T_2, \ldots, T_{i+1}$. Scientific ideas, theories, and worldviews evolve discontinuously in that a body of knowledge or theory $T_i$, which is held over a particular period of time, is dislodged by another body of knowledge or theory $T_j$, because the disciplinary matrix[2] within which the former theory $T_i$ had grown, changes to another disciplinary matrix which gives rise to the new theory, $T_j$, that is incompatible and incommensurable with its predecessor $T_i$.

12 years later, in his *Handbook on Analytical philosophy of Medicine*, he demands an "overhaul" to adapt the structuralist approach in philosophy of science to fuzzy set theory [34, p. 439f]. He requires "to render the metatheory applicable to real world scientific theories, it needs to be fuzzified because like everything else in science, scientific theories are vague entities and implicitly or explicitly fuzzy." Then he lists two ways of scientific theories' explicit fuzzifications:

1. Introduction of the theory's set-theoretical predicate as a fuzzy predicate ("$x$ is a fuzzy $S$" instead of "$x$ is an $S$").
2. In addition to 1. also any other component of the theory appearing in the structure that defines the predicate may be fuzzified.

[2] In later times Kuhn used the term "disciplinary matrix" instead of "paradigm".

Unfortunately, Sadegh-Zadeh did not go into details at this point but he concludes this section with an outlook: "Fuzzifications of both types will impact the application and applicability of theories as well as the nature of the knowledge produced by using them. This is true because fuzzification will change the conception of models; potential models; partial, potential models; and the core and intended applications of a theory, on the one hand; and the epistemological relationships between empirical claims of the theory and the 'real world', on the other, e.g., support, confirmation, falsification, etc." At the end of this chapter he wrote: "The above considerations suggest that the entities a theory is concerned with, be construed as vague entities. For similar analyses and assessments he referred to [38, 39, 42]" [34, p. 441].

I name the gap between the "real" (or "empirical") and the "theoretical layer" (see Fig. 17.3) the "Fuzzy Space" of unsharp concepts or perceptions and I introduced the variables *Theoretization* $T$ and *Empirization* $E$ of concepts to be more or less in the class of theoretical entities or of real phenomena. A concept (or perception) $c$ with $T(c) = 1$ is completely theoretical and if $T(c) = 0$ then $c$ is completely empirical. Variable empirization $E$ is the complement of variable theoretization $T$. If $E(c) = 1$ then $c$ is completely empirical and if $E(c) = 0$ then $c$ is completely theoretical. However, in most cases $c$'s $T$- and $E$-values lay between 0 and 1 because our concepts are unsharp or fuzzy.

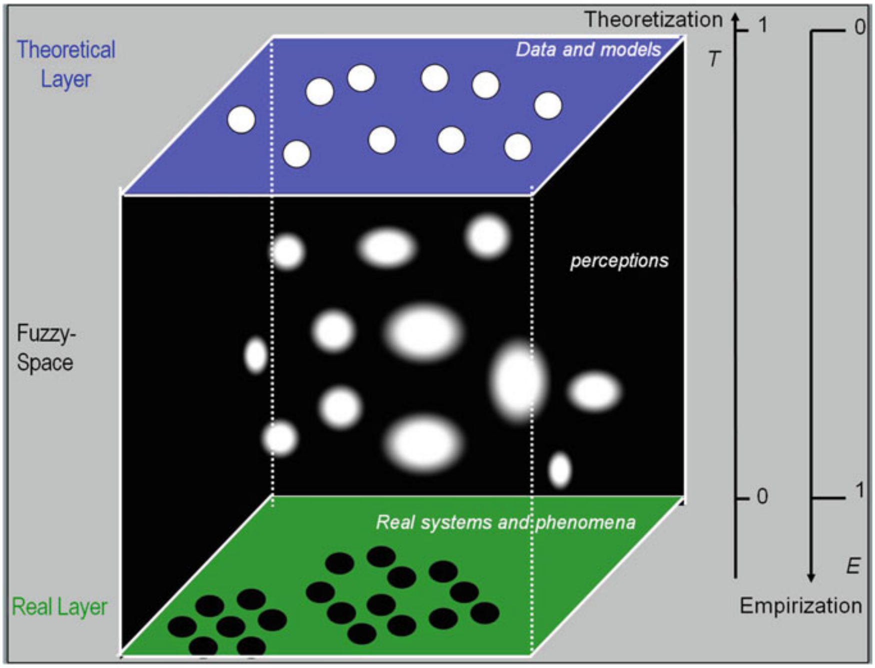


**Fig. 17.3** The Fuzzy space between the empirical and the theoretical layer

## 17.6 Genetic Epistemology

In the early 20th century history some psychologists became interested in the field of child development, and in the subsequent decades in Developmental Psychology mainly two doctrines are significant to date:

- In his Analytic Psychology (*Psychoanalysis*) Sigmund Freud (1856–1939) stressed the importance of childhood events and experiences. In his *Three Essays on the Theory of Sexuality* he described child development as a series of 'psychosexual stages' or phases: oral (ages 0–2), anal (2–4), phallic (3–6), latency (6-puberty), and mature genital (puberty-onward). Each phase involves the satisfaction of a libidinal desire and can later play a role in adult personality [13].
- When he was a young scientist, the Swiss developmental psychologist and philosopher Jean Piaget (1896–1980) was engaged in Psychoanalysis, but later he established a very different stage theory of children's cognitive development: *Genetic Epistemology*. In his theory it is assumed that children think differently than adults and they play an active role in gaining knowledge of the world. They actively construct their knowledge!

### *17.6.1 Structuralism in Education Science and Psychology*

The structuralist approach in education science and psychology was intended by Piaget who used the structuralism of the Swiss linguist and semiotician Ferdinand de Saussure as a methodology in psychology. Sassure used structuralism as a methodology in psychology when he focused not on the use of language (parole), but rather on the underlying system (structure) of language (langue). He called this theory *semiotics*. He was interested in how the elements of language are related to each other. Piaget abstracted Saussure's theory of structural organization. In his view, language depends upon operational intelligence (logical structures) and, therefore, it is not the grounding of cognition. However, in Piaget's view humans have adaptive mental structures. These mental structures assimilate external events; humans convert these external events to fit their mental structures. The most simple level of these mental structures is that of the so-called schema. Piaget introduced schemata as categories of knowledge to describe mental or physical actions.

Piaget was the first psychologist to make a systematic study of cognitive development. His contributions include a theory of cognitive child development, detailed observational studies of cognition in children, and a series of simple but ingenious tests to reveal different cognitive abilities. Before Piaget's work, the common assumption in psychology was that children are merely less competent thinkers than adults. Piaget showed that young children think in strikingly different ways compared to adults.

### 17.6.2 Parallels Between Piaget's and Kuhn's Theories

As a biologist by training Piaget was skilled to observe that organism adapted to their environment. He then applied this model to cognitive development and in his theory the mind organizes internalized regularities or operations into dynamic cognitive structures. Piaged named these structures "schema". Schemata are structured clusters of concepts, that can be used to represent objects, scenarios or sequences of events or relations between concepts. As we mentioned already (see Sect. 17.2), the original idea was proposed by philosopher Kant as innate structures used to help us perceive the world. Piaget's concept of a schema covers category of knowledge and a process to receive that knowledge. When we obtain new knowledge then our schema may be modified or changed. In his view, the cognitive development (the adaption to the environment) is a process of four aspects: *schema*, *assimilation*, *accommodation*, and *equilibrium*.

- *Schema*: As we said already, humans develop "cognitive structures" or "mental categories" that Piaget named schemata in order to name and organize, and to make sense of life and reality [50, p. 10].
- *Assimilation*: An individual uses its existing schemata to make sense of a new event. This process involves trying to understand something new by fitting it into what we already know.
- *Accommodation*: Also existing schemata can change to respond to a new situation. If new information cannot be made to fit into existing schemata, a new, more appropriate structure must be developed. There are also instances when an individual encounters new information that is too unfamiliar that neither assimilation nor accommodation will occur because the individual may choose to ignore it.
- *Equilibration*: This is the complex act of searching for the balance in organizing, assimilating, and accommodating. On the other hand, it is the state of Disequilibrium that motivates us to search for a solution through assimilation or accommodation.

In *Psychogenesis and the History of Science*—a co-authored and posthumous published book of Piaget with the Argentinian physicist Rolando Garcia (1919–2012), Piaget refers to Kuhn's view of theory change in science as physicist Daniel L. MacIsaac [24] quoted: "Our notion of epistemic framework ...", which describes "...an explanatory schema for the interpretation of the evolution of knowledge, both at the level of the individual and that of social evolution". Piaget further recognizes the role of social construction in schemata, stating "... when language becomes the dominating means of communication ... what we might call direct experience of objects becomes subordinated ... to the system of interpretations attributed to it by the social environment". He claimed that there is a connection of the psychogenesis of logico-mathematical thinking and historical development, he distinguished between "prescientific" (other authors name it "immature") (see Fig. 17.4) and "scientific" periods in each field, he mentions "analogies" between ontogenesis and historical development in both periods [27, p. 63].

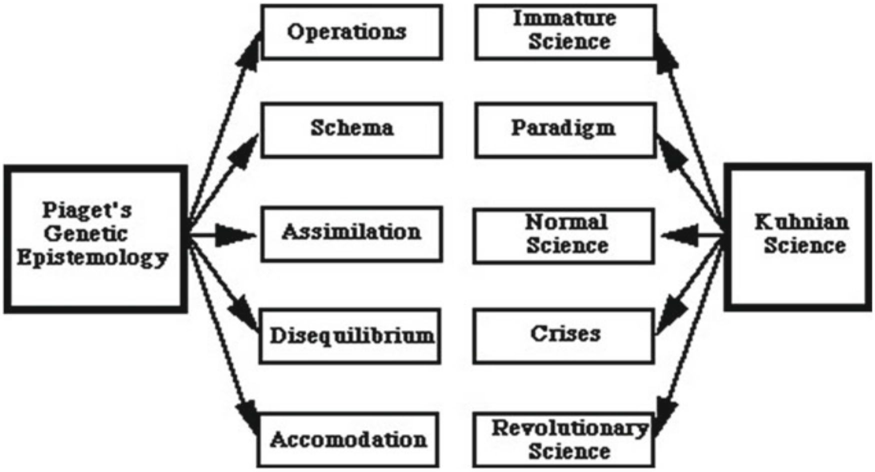


**Fig. 17.4** Parallels between Piaget's and Kuhn's theories, see [24]

Following MacIsaac, Piaget believed that his notion of epistemic framework "... includes that of paradigm ... but is simply different" (ibid). Paradigm belongs to the sociology of knowledge rather than with epistemology (concerned with the acquisition of knowledge)" [24].

MacIsaac quotes the seminal *Introduction for Students of Psychology and Education* into *Piaget's Theory of Cognitive Development* by education scientist Barry J. Wadsworth: "... the child's active assimilation of objects and events results in the development of structures (schemata) that reflect the childs' concepts of the world or reality. As the child develops these structures, reality or his knowledge of the world changes" [51]. He concludes: "However, schema[ta] are not directly derived from raw objects and events, but from operations. While schema[ta] are the highest-order mental organizations, schema[ta] never appear alone; structures are always related to other structures and many structures are substructures of larger structures [24, 51].

### *17.6.3 Unsharp Concepts in Genetic Epistemology*

Piaget defined four stages of cognitive development:

1. *Sensorimotor stage*: Birth through ages 18–24 months.
   "In this stage, infants construct an understanding of the world by coordinating experiences (such as seeing and hearing) with physical, motoric actions. Infants gain knowledge of the world from the physical actions they perform on it. An infant progresses from reflexive, instinctual action at birth to the beginning of symbolic thought toward the end of the stage" [37].

2. *Preoperational stage*: 18–24 months—7 years.
   The children do not yet understand concrete logic and cannot mentally manipulate information. They increase in playing but this is mainly categorized by symbolic play and manipulating symbols. They still have trouble seeing things from different points of view. The children lack basic logic. An example of transitive inference would be when a child is presented with the information “$a$ is greater than $b$ and $b$ is greater than $c$”. This child may have difficulty here understanding that $a$ is also greater than $c$ [22, Chap. 3] [36, Chap. 8].
3. *Concrete operational stage*: Ages 7–12 years.
   In this stage children use logic appropriate. They start solving problems in a more logical fashion but abstract, hypothetical thinking has not yet developed. The children can only solve problems that apply to concrete events or objects. They can draw inferences from observations in order to make a generalization (inductive reasoning) but they struggle with deductive reasoning, which involves using a generalized principle in order to try to predict the outcome of an event. “The Children can not figure out logic, e.g., a child will understand $A > B$ and $B > C$, however when asked is $A > C$, the child might not be able to logically figure the question out in their heads”.
4. *Formal operational stage*: Adolescence—adulthood.
   Now, the person has the ability to distinguish between their own thoughts and the thoughts of others. He or she recognizes that their thoughts and perceptions may be different from those around them. The children are now able to classify objects by their number, mass, and weight, they can think logically about objects and events and they have the ability to fluently perform mathematical problems in both addition and subtraction.

However, it was quite plain to Piaget that not all children may pass through the stages at the exactly the same age. Also he acknowledged that children could show at a given time the characteristics of more than one of the stages above. Nevertheless he emphasized that cognitive development always follows the sequence of the enumeration above. No stage can be skipped, and every stage prepares the child with new intellectual abilities and a more complex understanding of the world. Therefore, in our view, Piaget’s four stages of cognitive development are unharp concepts. Also MacIsaac’s analysis of the parallels between Piaget’s and Kuhn’s model leads to the assumption that Piaget’s model comprises non-crisp concepts, e.g.: “Comparisons between Kuhnian normal science and Piagetian assimilation reveal that assimilation is a much more complex and fluid activity. There is a great deal of flexibility displayed within assimilation, while normal science is rigidly constrained” [24].

## 17.7 Historical Epistemology and Fuzzy Set Theory

The Swiss historian of science and biologist Hans-Jörg Rheinberger proposed in the last decade of the 20th century the program of “historical epistemology” as a turnaround in philosophy and history of science. He made allowance for the situation that historians and philosophers of science in the 1980s and 1990s concentrated their attention to experiments in science. Historical epistemology deals with the concept of so-called "experimental systems". What is an experimental system?—Rheinberger illustrated: "This notion is firmly entrenched in the everyday practice and vernacular of the twentieth-century life scientists, especially of biochemists and molecular biologists. Scientists use the term to characterize the scope, as well as the limits and the constraints, of their research activities. Ask a laboratory scientist what he is doing, and he will speak to you about his 'system'. Experimental systems constitute integral, locally manageable, functional units of scientific research" [31, p. 246].

Historical epistemology deals with the concept of so-called "epistemic things"—"fluctuating objects", "imprecise concepts"—as he also called them in his historical work on "Gene Concepts". He argued: "If there are concepts endowed with organizing power in a research field, they are embedded in experimental operations. The practices in which the sciences are grounded engender epistemic objects, epistemic things as I call them, as targets of research. Despite their vagueness, these entities move the world of science. As a rule, disciplines become organized around one or a few of these "boundary objects" that underlie the conceptual translation between different domains" [32, p. 220].

In Rheinberger's epistemology we find again two parts of scientific research, that he named epistemic and technical, respectively, but he emphasizes that there is no sharp boundary between, moreover this boundary is vague or fuzzy. We follow Rheinberger's arguments to show that "epistemic things" are vague or fuzzy and that they "move the world of science" [32, p. 220]. He considered these "fluctuating objects" and "imprecise concepts"—as he also called them—in detail in his historical work.

Rheinberger followed the gene as an "epistemic thing" in the history of biology but there are many more of objects in other scientific disciplines too, that fluctuate through their history: "For a long time in physics, such an object has been the atom; in chemistry, the molecule; in classical genetics, it became the gene. It is the historically changing set of epistemic practice that gives contours to these objects. According to received accounts, that I need not question here in depth, the boundary object of classical genetics has worked as a formal unit: That which, in an ever more sophisticated context of breeding experiments, accounts for the appearance or disappearance of certain characters that can be traced through subsequent generations" [32, p. 220f]. Finally, Rheinberger summarized: "There has never been a generally accepted definition of the 'gene' in genetics. There exist several, different accounts of the historical development and diversification of the gene concept as well. Today, along with the completion of the human genome sequence and the beginning of what has been called the era of postgenomics, genetics is again experiencing a time of conceptual change, voices even being raised to abandon the concept of the gene altogether" [30].

### 17.7.1 Fuzzy Sets and Fuzzy Systems

In the early 1960s Lotfi A. Zadeh, a professor of Electrical Engineering at the University of California at Berkeley "began to feel that complex systems cannot be dealt with effectively by the use of conventional approaches largely because the description languages based on classical mathematics are not sufficiently expressive to serve as a means of characterization of input-output relations in an environment of imprecision, uncertainty and incompleteness of information" [54]. In the year 1964 he discovered how he could describe real systems as they appeared to people. "I'm always sort of gravitated toward something that would be closer to the real world" [41]. "In order to provide a mathematically exact expression of experimental research with real systems, it was necessary to employ meticulous case differentiations, differentiated terminology and definitions that were extremely specific to the actual circumstances, a feat for which the language normally used in mathematics could not provide well. The circumstances observed in reality could no longer simply be described using the available mathematical means. These thoughts indicate the beginning of the genesis of Fuzzy Set Theory" [58, p. 7]. In his first article "Fuzzy Sets" he launched new mathematical entities as classes or sets that "are not classes or sets in the usual sense of these terms, since they do not dichotomize all objects into those that belong to the class and those that do not". He introduced "the concept of a fuzzy set, that is a class in which there may be a continuous infinity of grades of membership, with the grade of membership of an object $x$ in a fuzzy set $A$ represented by a number $f_A(x)$ in the interval [0, 1]" [43, 57].

Some years later Zadeh compared the strategies of problem solving by computers on the one hand and by humans on the other hand. He called it a paradox that the human brain is always solving problems by manipulating "fuzzy concepts" and "multidimensional fuzzy sensory inputs" whereas "the computing power of the most powerful, the most sophisticated digital computer in existence" is not able to do this. Therefore, he stated that "in many instances, the solution to a problem need not be exact", so that a considerable measure of fuzziness in its formulation and results may be tolerable. "The human brain is designed to take advantage of this tolerance for imprecision whereas a digital computer, with its need for precise data and instructions, is not" [55, p. 132].

### 17.7.2 Fuzzy Logic and Fuzzy Concepts

Zadeh had served as first reviewer and his Berkeley-colleague and mathematician Hans-Joachim Bremermann (1926–1996), as second for the Ph.D. thesis of mathematician Joseph A. Goguen's (1941–2006) entitled "Categories of Fuzzy Sets". Reference [14] Here, Goguen generalized the fuzzy sets to so-called "$L$-sets". An $L$-set is a function that maps the fuzzy set carrier $X$ into a partially ordered set $L$. The partially ordered set $L$ Goguen called the "truth set" of the fuzzy set.

The elements of $L$ can thus be interpreted as "truth values"; in this respect, Goguen then also referred to a "Logic of Inexact Concepts" [15].

Zadeh's efforts to use his fuzzy sets in linguistics led to an interdisciplinary scientific exchange between him and Goguen on the one hand and between Berkeley-psychologist Eleanor Rosch (Heider) (born 1938) and the Berkeley-linguist George Lakoff (born 1941) on the other. In her psychological experiments Rosch could show that concept categories are graded. Consequently she argued that concepts are not adequately represented by classical sets. Rosch developed her prototype theory on the basis of these empirical studies. This theory assumes that people perceive objects in the real world by comparing them to prototypes and then ordering them accordingly. In this way, according to Rosch, word meanings are formed from prototypical details and scenes and then incorporated into lexical contexts depending on the context or situation. It could therefore be assumed that different societies process perceptions differently depending on how they go about solving problems [33].

Lakoff referred to the fact that also statements in natural language are graded, "that sentences of natural languages (at least declarative sentences) are either true or false or, at worst, lack a truth value, or have a third value". He argued "that natural language concepts have vague boundaries and fuzzy edges and that, consequently, natural language sentences will very often be neither true, nor false, nor nonsensical, but rather true to a certain extent and false to a certain extent, true in certain respects and false in other respects" [21, p. 458]. In this paper Lakoff wrote that Zadeh's fuzzy set theory is an appropriate tool of dealing with degrees of membership, and with (concept) categories that have unsharp boundaries. Because he used the term "fuzzy logic" he deserves the credit for first introducing this expression in the scientific literature but based on his later research, however, he came to find that fuzzy logic is not an appropriate logic for linguistics [40]. But for all that we think that fuzzy logic is an appropriate logic for psychology and epistemology—in it's genetic and in its historical variation!

## 17.8 Outlook: Towards Fuzzy Epistemology

After Zadeh's approach to generalize systems to fuzzy systems many concepts in science—epistemic things—have been "fuzzified" already in engineering sciences and in mathematics there exist a theory of fuzzy algebra, theories of fuzzy topology and fuzzy probability. What is strongly needed is an epistemological foundation of fuzzy concepts and a fuzzy set in its core. Rheinberger stressed "that the fruitfulness of boundary objects in research does not depend on whether they can be given a precise and codified meaning from the outset. Stated otherwise, it is not necessary, indeed it can be rather counterproductive, to try to sharpen the conceptual boundaries of vaguely bounded research objects while in operation. As long as objects are in flux, too, the corresponding concepts must remain in flux, too". In other words, he wrote: "Boundary objects require boundary concepts" [32, p. 221] which we will interpret as fuzzy objects require fuzzy concepts.

Rheinberger accentuates the value of imprecision, vagueness or fuzziness in science when he wrote: “Lofti Zadeh claims that ‘there is a rapidly growing interest in inexact reasoning and processing of knowledge that is imprecise, incomplete, or totally reliable. And it is in this connection that it will become more and more widely recognized that classical logical systems are inadequate for dealing with uncertainty and that something like fuzzy logic is needed for that purpose’”. Rheinberger went back to the question “whether we need, in order to understand conceptual tinkering in research, more rigid metaconcepts than those first-order concepts that we, as epistemologists, analyze. I am inclined to deny this. Why should historians and epistemologists be less imprecise, less operational, and less opportunistic after all, than scientists?” [32, p. 236].

In analogy to our “fuzzification” of the structuralist approach in philosophy of science that we presented in Sect. 17.5 and because of the parallelism between the Kuhnian model of theory change and the Piagetian model of conceptual development that we discussed in Sect. 17.6.2 we propose now to establish a “fuzzy” view on Piaget’s Genetic Epistemology. In our view the concepts that Piaget named “schema” are fuzzy concepts, Piaget’s processes of assimilation and accomodation are fuzzy relations, and the stages in Piaget’s constitution of the series of the cognitive development of children have no sharp borders but are fuzzy stages. This fuzzy approach to Piaget’s structuralist theory, i.e. a “Fuzzy Genetic Epistemology” could be established as follows:

Fuzzy structuralism uses fuzzy sets and fuzzy relations instead of usual sets and relations. Fuzzy structures are models for concepts or perceptions that scientist use to think on and to create new scientific theories. Fuzzy relations are appropriate to model intertheoretical relations between fuzzy structures to model developments, e.g. scientific revolutions or paradigm changes. A “Fuzzy Genetic Epistemology” could be useful to fuzzy Piaget’s theory of conceptual development. Fuzzy structures are suitable to build fuzzy models for unsharp concepts and perceptions (“fuzzy schemata”) that children use to represent objects, scenarios or sequences of events or relations between them. Fuzzy relations will model changes of fuzzy schemata like Piaget’s “assimilation” and “accomodation”.

**Acknowledgments** Work leading to this paper was partially supported by the *Foundation for the Advancement of Soft Computing* Mieres, Asturias (Spain).

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