Chapter 22
Lacunarity Analysis of Fracture Intensity Maps: Are they Multifractals?

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Abstract Lacunarity (L) is a scale (r)-dependent parameter that was developed for quantifying clustering in fractals and has subsequently been employed to characterize various natural patterns. It has been further analytically proved that lacunarity analysis invoking the gliding-box algorithm can find the correlation dimension, D_2 of multifractals. The present research empirically tests this on a set of multifractal models generated with known D_2 values. The log-transformed lacunarityvalues of these models, log L were plotted as a function of the log-transformed box-size, log r. The slopes of these linear relations, estimated using regression analysis, were then used to calculate D_2 values that gave an approximately 1:1 relationship with the known values. Multifractal behavior can therefore, be checked without having to compute the whole spectrum of non-integer dimensions, D_q ($-\infty < q < +\infty$) that typically characterize a multifractal. The technique thus developed was applied to fracture intensity maps generated from a set of nested fracture networks from the Devonian Sandstone, Hornelen Basin, Norway in order to test if such intensity maps display multifractal behavior.

22.1 INTRODUCTION

Lacunarity is a parameter that characterizes the distribution of spaces or gaps in a pattern as a function of scale. Originally proposed for distinguishing between monofractal patterns having the same fractal dimension but different degrees of clustering [1], lacunarity has since been used for analyzing scale-dependent clustering in natural binary data sets, both fractal and non-fractal. The gliding-box

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algorithm can be used to quantify the lacunarity, L, at a given scale (or box-size, r). Since values vary with box-size the results are generally reported in terms of the function: $L(r)$. It has been theoretically established that when log-transformed values of $L(r)$ and r are plotted, both fractals and multifractals result in straight lines such that the slope plus the known embedding dimension equals the box or correlation dimension, respectively [2]. Although there have been a few lacunarity studies of both synthetic and natural multifractal patterns [3, 4], there is not enough evidence to unequivocally establish that lacunarity analysis can indicate multifractal behavior in non-binary data, and that the true correlation dimension, D_2 of a multifractal can be found by employing this technique. In the present paper, we test Allan and Cloitre's theory [2] using multifractal grayscale patterns with known correlation dimensions. As an application, we focus on four fracture intensity maps (grayscale / non-binary) that have been generated from a set of nested fracture network maps (binary) from the Devonian sandstone of Hornelen Basin, Norway [5]. Since this fracture network is fractal in nature [6] it can be hypothesized that intensity maps generated from them can possibly display multifractal properties.

22.2 GENERATING MULTIFRACTAL PATTERNS

A set of 2-dimensional multifractal grayscale patterns were constructed following the steps outlined in [7]. The process is akin to generating a Sierpinski carpet, only the zeros and ones in the pattern are replaced with fractional values. The algorithm involves normalizing mass-fractions calculated from the truncated binomial distribution for an average probability, p, of retaining a cell in the generator with a scale factor, b, in successive iterations. As described in [8] the locations of the normalized mass fractions can be spatially randomized so as to create a random geometrical multifractal grayscale pattern. Eight different patterns were created from a generator with a scale factor of $b = 3$ and probability values corresponding to $p = 1/9$, 2/9, 3/9, 4/9, 5/9, 6/9, 7/9, and 8/9 by iterating the system 5 times to produce grayscale fields of size $3^5 \times 3^5$ pixels. For each of the 8 models, 3 random realizations were constructed amounting to a total of 24 random multifractal grayscale patterns. Fig. 22.1 shows patterns corresponding to p = 8/9, 7/9, 6/9 and 5/9.

22.3 LACUNARITY AND MULTIFRACTAL BEHAVIOR

Lacunarity is a scale-dependent measure of textural heterogeneity that can be quantified by employing the gliding-box algorithm [2]. Essentially, this method involves sliding a window or an interrogator box of a given length, r, translated in
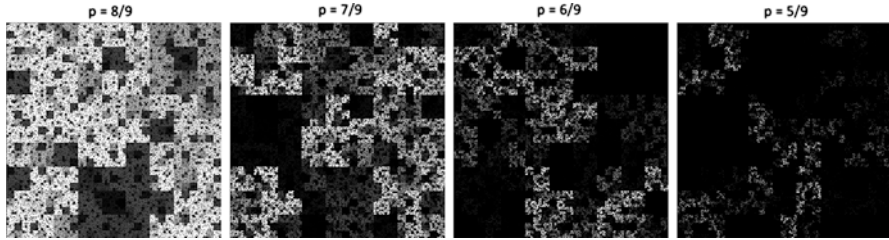
Fig. 22.1   Grayscale random multifractal patterns: $p = 8/9$ to 5/9. Lighter phases have higher mass fractions
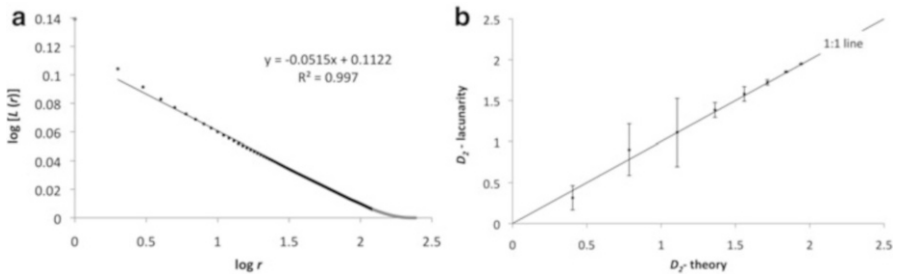

Fig. 22.2   (a) log $L(r)$ vs. log r plot for pattern p $= 8/9$; (b) D_2, estimated from lacunarity analysis compared to theoretical D_{2-}values for 8 random multifractal fields, each point is an average of 3 realizations with bars corresponding to the 95% confidence intervals

increments of a chosen unit length (usually that of a pixel) across the whole pattern such that the total number of steps is given by $(r_t - r + 1)^2$, r_t being the length of the entire pattern. The details of this technique employed in analyzing binary patterns can be found in [9]. In the case of grayscale patterns, the mass of all pixels, $s(r)$, contained within the interrogator box at each step is calculated and a distribution of this mass at the scale r is obtained by gliding the box through all the steps. Finally, the mean, $\overline{s(r)}$, and variance, $ss^2(r)$ of this distribution are used in calculating the lacunarity, L(r), at the scale r as: $L(r) = [ss^2(r) \ / \overline{s(r)}^2] + 1$. Typically, L(r) is calculated for a range of box-sizes, r and it has been theoretically demonstrated in [15] that in the case of multifractals, $L(r)$ is related to r by a power-law such that the power-term equals $D_2 - E$, where D_2 is the correlation dimension and E the Euclidean embedding dimension. Based on this result, it is expected that the log-transformed lacunarity functions for the 2-dimensional multifractal patterns described in the previous section can be fitted with a straight line, $y = mx + c$, such that the slope is, $m = D_2 - 2$. For each of the 24 multifractal patterns, the log $L(r)$ vs. log r values were calculated and a subset of the array of points thus obtained was fitted with a straight line. Fig. 22.2a shows one such plot for pattern $p = 8/9$ (random realization #3). In order to estimate the proper correlation dimension thus meeting a condition for multifractal behavior [2], only the relatively "straight" segment was considered for fitting a linear model to the points. For large r values,

the local slope starts to fluctuate and increases abruptly and so the points corresponding to box-sizes $r > r_t/2$ were not included in the fitting. For smaller r values, the coefficient of determination, R^2, of the fit improved when points were sequentially excluded from the left. However, it is the first point at $r = 1$ that mostly influences the change in slope such that there was a difference of ~4% in the estimated slope if this point was excluded from the fit. If two or more additional points were excluded there was no more than ~1% difference. This is mainly because the first few points on the left hand side of the log $L(r)$ vs. log r plot are sparsely distributed. Therefore, from the lower end, only the first point in the plot was excluded from the fitting. The points thus excluded from both ends are shown in grey. Correlation dimensions for all 24 patterns were calculated from their log $L(r)$ vs. log r plots by employing the above protocol. An average estimated D_2 value from all three realizations of each model corresponding to a particular p-value was computed and compared to the theoretical D_2 value [7]. The results are graphed in Fig. 22.2b. The 95% confidence intervals in the computed D_2 values arising from the three random realizations for each model are shown as vertical bars. As can be seen from the figure, the computed D_2 values overlap the 1:1 line (45° slope) and are thus are statistically equal to their theoretical counterparts. This analysis demonstrates empirically that an almost exact value of the correlation dimension for multifractals can be found from lacunarity analysis. It may therefore be concluded that lacunarity analysis can detect if a grayscale pattern exhibits possible multifractal behavior (on the basis of log-log linearity) and, if it does, provide an accurate estimate its correlation dimension.

22.4   FRACTURE INTENSITY MAPS

Fracture intensity may be defined as the summed fracture trace length per unit area [10]. For this study, we created four intensity maps from a subset of seven nested fracture networks that were mapped from the Devonian sandstones of Hornelen Basin, Norway [5]. These seven nested fracture maps are well studied and have been previously established as belonging to a fractal system [6]. Map 2, 4, 5 and 7 at length scales of 55m, 90m, 180m and 720m respectively, and represented by 1042 x 1042 pixels were chosen. These were converted into intensity maps by overlaying a grid with cell size 20 x 20 pixels and counting the fracture length in each cell. The bright areas indicate high fracture intensity as seen in Fig. 22.3.

22.5   RESULTS AND DISCUSSIONS

Figure 22.4 shows log L vs. log r plots for the intensity maps generated in the previous section. As seen here, plots of all four maps can be fitted by straight lines over a limited range thus confirming the hypothesis that they all display possible multifractal behavior. While the maps have somewhat similar correlation dimensions, the general trend is: larger the mapped area larger is the correlation dimension such that, map
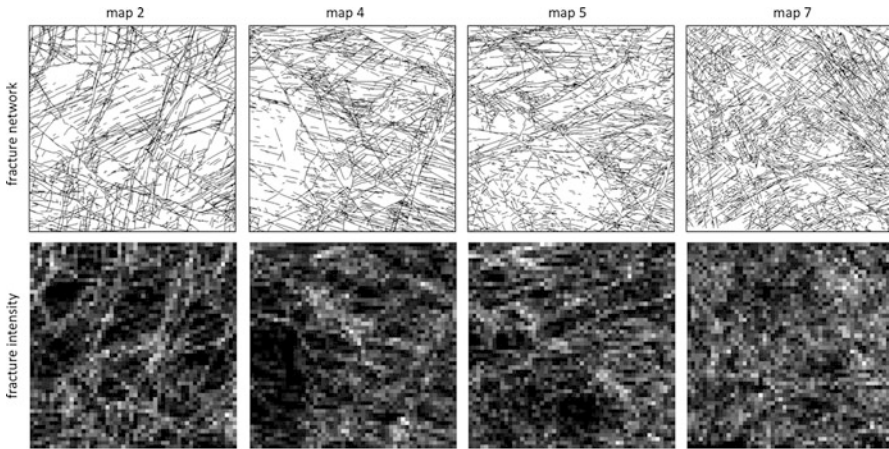
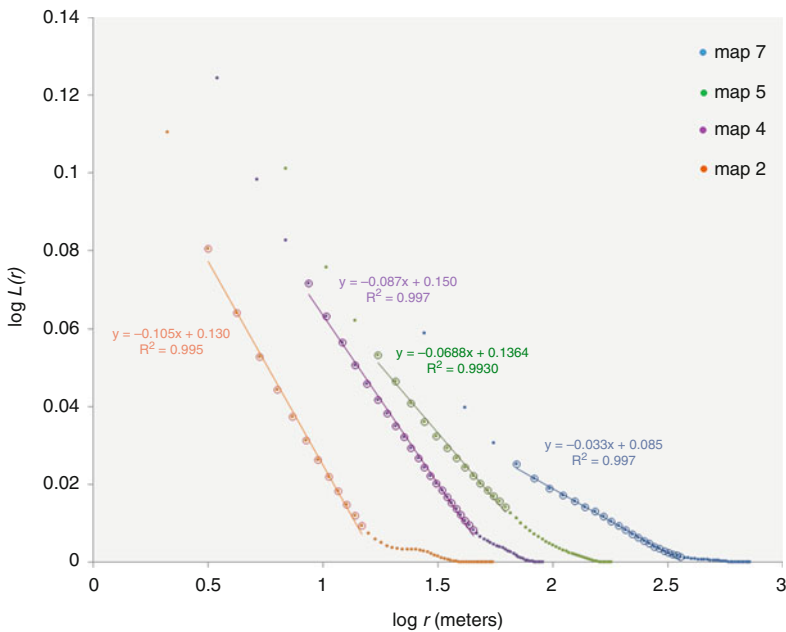**Fig. 22.3** Fracture intensity maps generated from fracture networks

Fig. 22.4 log L vs. log r plots for intensity maps 2, 4, 5 and 7

7 (720m) has the highest dimension (least steep slope). This general observation is in agreement with the trends in box-counting dimension values of the corresponding original fractal-fracture network maps as reported in [6]. The results therefore show that lacunarity analysis can delineate possible multifractal behavior in fracture intensity maps that are associated with underlying fractal-fracture geometries.

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