Apollonian Circle Packings

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Abstract Circle packings are a particularly elegant and simple way to construct quite complicated and elaborate sets in the plane. One systematically constructs a countable family of tangent circles whose radii tend to zero. Although there are many problems in understanding all of the individual values of their radii, there is a particularly simple asymptotic formula for the radii of the circles, originally due to Kontorovich and Oh. In this partly expository note we will discuss the history of this problem, explain the asymptotic result and present an alternative approach.

Keywords Apollonian circle packings • Poincaré series • Transfer operators • Asymptotic formulae

1 A Brief History of Apollonian Circles

Apollonius (c. 262–190 BC) was born in Perga (now in Turkey) and gave the names to various types of curves still used: ellipse, hyperbola and parabola. However, very little detail is known about his life and, although he wrote extensively on many topics, rather little of his work has survived (perhaps partly because it was considered too esoteric by his contemporaries). What has survived (partly in the form of translations into arabic) includes seven of his eight books on "conics". These include problems on tangencies of circles.

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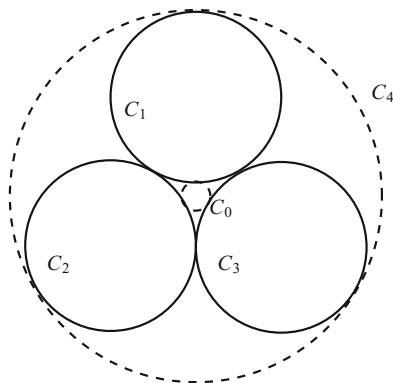e-mail: masdbl@warwick.ac.uk

Fig. 1 The three initial circles C_1, C_2, C_3, and the two mutually tangent circles C_0 and C_4 guaranteed by Apollonius' theorem

The result of Apollonius which is of particular interest to us is the following.

Theorem 1.1 *Given three mutually tangent circles C_1, C_2, C_3 with disjoint interiors there are precisely two circles C_0, C_4 which are tangent to each of the original three.*

This result is illustrated in Fig. 1b. The proof is so easy and short that we include it.

Proof We can apply a Möbius transformation which takes a point of tangency between two of the initial circles to infinity. These two circles are then mapped to two parallel lines, and the third initial circle to a circle between, and just touching, these parallel lines. We can then construct the two new circles by translating the middle circle between the parallel lines and then transforming back. Since a Möbius transformation preserves circles and lines we are done.   □

In 1643, **René Descartes** (1596–1650) wrote to Princess Elizabeth of Bohemia (1618–1680) stating a formula he had established on the radii a_1, a_2, a_3, a_4 of the tangent circles, and for which she independently provided a proof. The radii are related by the following formula.

Theorem 1.2 (Descartes-Princess Elizabeth) *Assume that the radii of the original circles are $a_1, a_2, a_3 > 0$ and the fourth mutually tangent circle has radius $a_4 > 0$ then*

$$2\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} + \frac{1}{a_4^2}\right) = \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right)^2.$$
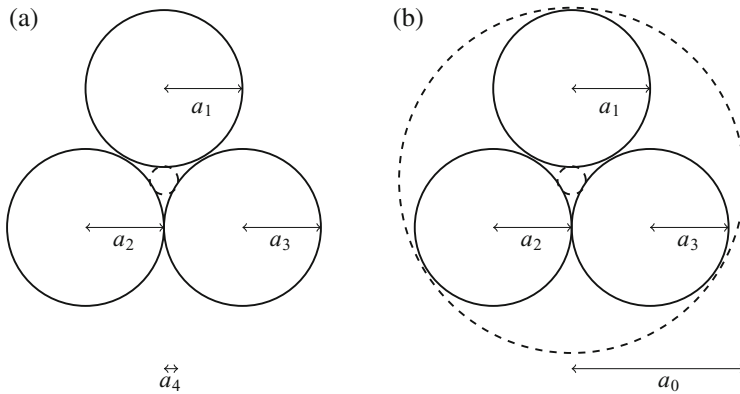
A simple proof appears in the notes of Sarnak [16].

Fig. 2 (**a**) The initial circles with radii a_1, a_2, a_3 and smaller choice of mutually tangent circle with radius a_4; (**b**) The initial circles with radii a_1, a_2, a_3 and the larger choice of mutually tangent circle with radius a_0

Notation 1.3 *The formula also applies where the radius a_4 of the inner circle is replaced by the radius of the outer circle a_0. However, in this case we adopt the convention that $a_0 < 0$, where $|a_0| > 0$ is the radius of the circle C_0 (Fig. 2).*

Princess Elizabeth was a genuine princess by virtue of being the daughter of Queen Elizabeth (1596–1662) and King Frederick V of Bohemia (whose reign lasted a brief 1 year and 4 days). Queen Elizabeth of Bohemia was in turn the daughter of King James I of England (Fig. 3).

In 1605, King James was the target of an unsuccessful assassination plan (the "gunpowder plot" of Guy Fawkes and co-conspirators, celebrated in England annually on 5th November) and Queen Elizabeth of Bohemia would have become Queen of England (aged 9) had the plot succeeded.

In 1646, Elizabeth's brother Philip stabbed to death Monsieur L'Espinay, for flirting with their mother and sister. In the ensuing family rift, Elizabeth wrote to Queen Christina of Sweden for an audience and help reinstating her Father's lands, but Christina invited Descartes to Stockholm instead, which proved unfortunate for him since he promptly died of pneumonia. Finally, Elizabeth entered a convent in Germany for the last few years of her life, where she worked her way up to the top job of abbess.

The formula of Descartes was subsequently rediscovered by Frederick Soddy (1877–1956), which is the reason that the circles are sometimes called "Soddy circles". Frederick Soddy is more famous (outside of Mathematics) for having won the Nobel Prize for Chemistry in 1921, and having introduced the terms "isotopes" and "chain reaction". However, most relevant to us, he rediscovered the formula of Descartes and published it in the distinguished scientific journal *Nature* in the form of a poem [18]:
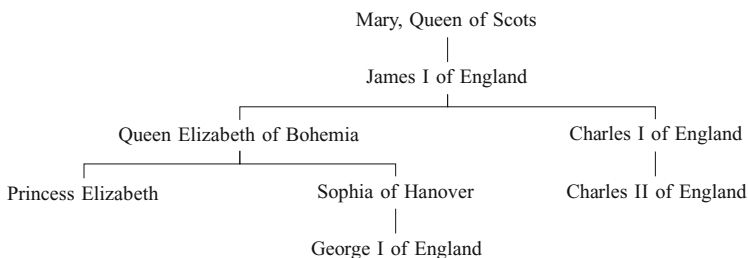
Fig. 3 The family tree of Princess Elizabeth. Her uncle, Charles I of England, was executed during the English revolution. Her nephew, George I, also became King of England and was the 6th Great-Grandfather of the present Queen

The kiss precise

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

To spy out spherical affairs
An ocular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends

Frederick Soddy (1877–1956)

2   Circle Counting

2.1   The Asymptotic Formulae

Starting from mutually tangent circles we can inscribe new circles inductively to arrive at what is known as an *Apollonian circle packing* consisting of infinitely many circles. We denote by $\mathcal{C}$ the set of such circles (Fig. 4).[1]

We can order these circles by (the reciprocal of) their radii, which we shall denote by a_n, for $n \geq 0$. It is easy to see that the sequence $(1/a_n)$ tends to infinity or, equivalently, the sequence of radii (a_n) tends to zero. This is because the total area of the disjoint disks enclosed by the circles $\sum_{n=1}^{\infty} \pi a_n^2$ which is in turn bounded by the area inside the outer circle. A natural question is then to ask: *How fast does the sequence $(1/a_n)$ grow, or, equivalently, how fast do the radii (a_n) tend to zero?* We begin with some notation.

Definition 2.1 Given, $T > 0$ we denote by $N(T)$ the finite number of circles with radii greater than $\frac{1}{T}$.

In particular, we see from our previous comments that $N(T) \to +\infty$ as $T \to 0$. A far stronger result is the following [5, 11].

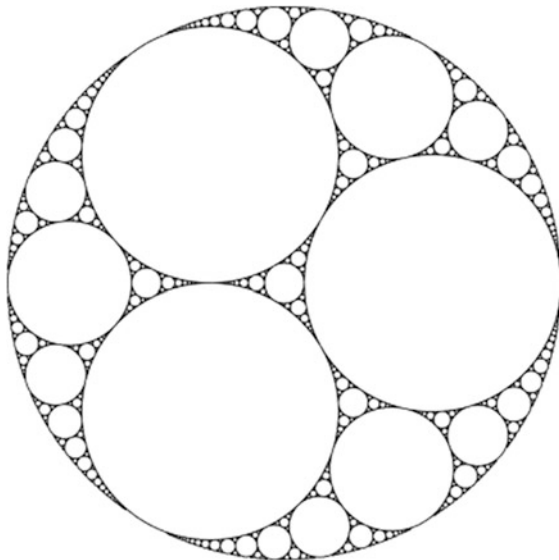**Fig. 4** An Apollonian circle packing consisting of infinitely many circles. The closure of their union is the Apollonian gasket denoted by Λ

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[1]For other aspects of the rich theory of circle packings, we refer the reader to [19].

Theorem 2.2 (Kontorovich-Oh, 2009) *There exists $C > 0$ and $\delta > 1$ such that the number $N(T)$ is asymptotic to CT^{δ} as T tends to infinity, i.e.,*

$$\lim_{T \to \infty} \frac{N(T)}{T^{\delta}} = C.$$

It is the convention to write $N(T) \sim CT^{\delta}$ as $T \to \infty$ (Fig. 5).

We can illustrate Theorem 2.2 with two examples.

Example 1 Assume that we begin with four mutually tangent circles the reciprocals of whose radii are $a_0 = -\frac{1}{3}$, $a_1 = \frac{1}{5}$, $a_2 = \frac{1}{8}$ and $a_3 = \frac{1}{8}$. Using Theorem 1.2 we can compute the following monotone increasing sequence of reciprocal radii:

$$\left(\frac{1}{a_n} \right)_{n=1}^{\infty} = 5, 8, 8, 12, 12, 20, 20, 21, 29, 29, 32, 32, \cdots$$

We will return to this in Example 4 in the Appendix.

Example 2 Assume that we begin with four mutually tangent circles the reciprocals of whose radii are $a_0 = -\frac{1}{2}$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{6}$ and $a_3 = \frac{1}{7}$. Using Theorem 1.2 we can compute the following monotone increasing sequence of reciprocal radii:

$$\left(\frac{1}{a_n} \right)_{n=1}^{\infty} = 3, 6, 7, 7, 10, 10, 15, 15, 19, 19, 22, 22, \cdots$$

We will return to this in Example 5 in the Appendix.

In the Appendix we also recall why the numbers in these sequences are all natural numbers.
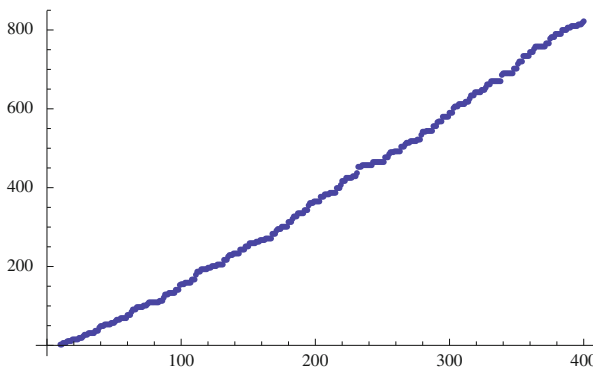

Fig. 5 A plot of $N(\epsilon)$ against $\frac{1}{\epsilon}$

2.2 The Exponent δ in Theorem 2.2

Of particular interest is the value of δ which controls the rate of growth of the radii. The next lemma provides an alternative characterisation of this number.

Lemma 2.3 *The value δ in Theorem 2.2 has the following alternative characterisation:*

$$\delta = \inf\left\{ t > 0 : \sum_{n=1}^{\infty} \frac{1}{a_n^t} < +\infty \right\}.$$

The expression for δ in Lemma 2.3 is usually called the *packing exponent*.

Notation 2.4 *We can denote by Λ the compact set given by the closure of the union of the circles in the Apollonian circle packing.*

This leads to a second useful alternative characterisation.

Lemma 2.5 *The value δ in Theorem 2.2 is equal to the Hausdorff Dimension $dim_H(\Lambda)$ of the limit set Λ.*

Remark 1 (The numerical value of δ) Unfortunately, there is no explicit expression for δ and it is rather difficult to estimate. The first rigorous estimates were due to Boyd [3] who, using the definition above, estimated $1 \cdot 300197 < \delta < 1 \cdot 314534$. A well known estimate is due to McMullen [10], who showed that $\delta = 1 \cdot 30568 \ldots$

Perhaps a little surprisingly, the value of δ is independent of the particular Apollonian circle packing being considered, as is shown by the next lemma.

Lemma 2.6 *For different Apollonian circle packings exactly the same value of δ arises (independently of the initial choices a_0, a_1, a_2, a_3, a_4).*

Again the idea of the proof is so simple that we recall the idea so as to dispel any mystery.

Proof Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two Apollonian circle packings and let Λ_1 and Λ_2 be the associated Apollonian gaskets. By Lemma 2.5 it suffices to show that $\dim_H(\Lambda_1) = \dim_H(\Lambda_2)$. We can then deduce the independence of the value δ using the following well known result: If there exists a smooth bijection $T : \mathcal{C}_1 \to \mathcal{C}_2$ then the sets share the same Hausdorff Dimension. Let us identify the plane with $\mathbb{C}$. Then it is a simple exercise to show that there is a *Möbius transformation $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$* of the form

$$g(z) = \frac{az + b}{\bar{b}z + \bar{a}} \text{ and } a, b \in \mathbb{C} \text{ with } |a|^2 - |b|^2 = 1,$$

such that $T(\mathcal{C}_1) = \mathcal{C}_2$. In particular, this follows easily since Möbius transformations necessarily take circles to circles. $\qquad\square$

3 Some Preliminaries for a Proof

We will describe a proof which differs from the original proof of Kontorovich-Oh and other subsequent proofs. This approach is more in the spirit of the classical proof of the Prime Number Theorem, except we use approximating Poincaré series in place of zeta functions.

3.1 An Analogy with the Prime Numbers

Purely for the purposes of motivation, we recall the classical Prime Number Theorem. Consider the prime numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots$$

Let $\pi(x)$ denote the number of primes numbers between 1 and x. Since there are infinitely many primes, we see that $\pi(x) \to \infty$ as x tends to infinity. This again poses the natural question: *How does $\pi(x)$ grow as $x \to +\infty$?* The solution is the classical prime number Theorem [4].

Theorem 3.1 (Prime Number Theorem: Hadamard, de la Vallée Poussin (1896)) *There is a simple asymptotic formula $\pi(x) \sim \frac{x}{\log x}$ as $x \to +\infty$, i.e.,*

$$\lim_{x \to +\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1.$$

The essence of the proof of the Prime Number Theorem is to analyse the associated complex function, the *Riemann zeta function*, defined formally by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

The Riemann zeta function has the following important basic properties [4].

Lemma 3.2 *The Riemann zeta function $\zeta(s)$ converges to a well defined function for $Re(s) > 1$. Moreover:*

1. *For $Re(s) > 1$ we have that $\zeta(s)$ is analytic and non-zero;*
2. *There exists a small neighbourhood of each $1 + it$ with $t \neq 0$ on which $\zeta(s)$ has a non-zero analytic extension [2]; and*
3. *$\zeta(s)$ has a simple pole at $s = 1$.*

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[2]The zeta function $\zeta(s)$ even has an analytic extension to $\mathbb{C} \setminus \{1\}$, but one does not need this to prove Theorem 3.1.

The result then follows by using a Tauberian theorem to convert this information on the domain of $\zeta(s)$ into information on prime numbers. For completeness, we recall the statement of the Ikehara-Wiener Tauberian Theorem [4].

Theorem 3.3 (Ikehara-Wiener Tauberian Theorem) *Assume that $\rho : \mathbb{R} \to \mathbb{R}$ is a monotone increasing function for which there exists $c > 0$, $\delta > 0$ such that the function*

$$F(s) := \int_0^\infty t^{-s} d\rho(t) - \frac{c}{s - \delta}$$

is analytic in a neighbourhood of $Re(s) \geq \delta$ then $\lim_{T \to +\infty} \frac{\rho(T)}{T^\delta} = c$.

Remark 2 The Prime Number Theorem easily follows from applying Theorem 3.3 to the auxiliary function $\rho(T) = \sum_{p^n \leq T} \log p$ and then relating the Stieltjes integral to $\zeta'(s)/\zeta(s)$. We refer the reader to [4] for further details of these now standard manipulations.

To adapt the proof of the Prime Number Theorem to the present setting, suggests considering a new complex function

$$\xi(s) = \sum_{n=1}^\infty a_n^s$$

where a_n are the radii of the circles in the Apollonian circle packing. In fact, it is more convenient to study a related function (a Poincaré series) and use an approximation argument to get the final result. However, to analyse such functions, we first introduce a dynamical ingredient.

3.2   An Iterated Function Scheme Viewpoint

Let us again identify the plane with the complex numbers $\mathbb{C}$, then we can introduce a transformation which preserves the circle packing $\mathcal{C}$. We want to define the "reflection" R in the circle $C = C(z_0, r)$ of radius r centered at z_0 (Fig. 6).

More precisely, let $z_0 \in \mathbb{C}$ and radius $r > 0$ then we associate a transformation

$$R : \mathbb{C} \setminus \{z_0\} \to \mathbb{C} \setminus \{z_0\}$$

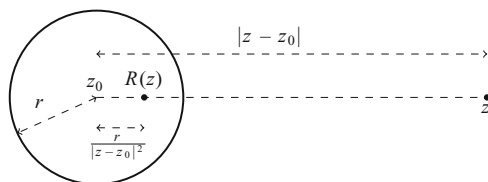$$R(z) = \frac{r^2(z - z_0)}{|z - z_0|^2} + z_0.$$

Fig. 6 Reflection in a circle

Rather than reflecting in the original Apollonian circles, we need to find four "dual circles" which we will reflect in. This point of view has a nice historical context. The original statement of the result was due to Philip Beecroft (1818–1862) who was a school teacher in Hyde, near Manchester, in England, and was the son of a miller and lived with his two elder sisters [1]. In his article he too had recovered Theorem 1.2.

Theorem 3.4 (Philip Beecroft, from "Lady's and Gentleman's diary" in 1842)
"If any four circles be described to touch each other mutually, another set of four circles of mutual contact may be described whose points of contact shall coincide with those of the first four."

As in [5], we associate to the four initial Apollonian circles a new family of "dual" tangent circles (the dotted circles in Fig. 7). We can then consider the four associated reflections $R_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in the four dual circles K_1, K_2, K_3, K_4 as shown in Fig. 7 ($i = 1, 2, 3, 4$).

The aim is to associate to the Apollonian circle packings complex functions, playing the rôle of the zeta function in number theory. These will be defined in terms of a family of contractions (i.e., an associated iterated function scheme) built out of the maps R_i on each of the four curvilinear triangles external to the initial four circles. For definiteness, let us fix the central curvilinear triangle Δ, whose sides are arcs from the circles C_1, C_2 and C_3 (with the other cases being similar) and let x_1, x_2, x_3 denote the vertices. We can consider the three natural linear fractional contractions $f_1, f_2, f_3 : \Delta \to \Delta$ defined by

$$f_i = R_4 \circ R_i, \quad i = 1, 2, 3,$$

each of which fixes the vertex x_i of Δ (Fig. 8). A simple calculation gives that:

- $|f_i'(z)| < 1$ for $z \in \Delta \setminus \{x_i\}$ for $i = 1, 2, 3$; and
- $|f_i'(x_i)| = 1$ for $i = 1, 2, 3$ (i.e., x_i is a parabolic point at the point of contact of K_4 with K_1, K_2 and K_3, respectively).
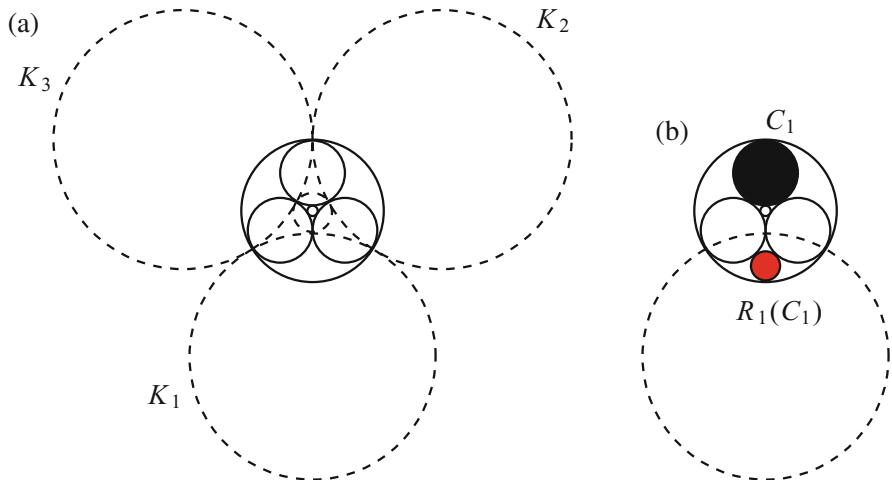
We recall the following explicit example from [7].

Fig. 7 (**a**) The four dual (*dotted*) circles K_1, K_2, K_3, K_4 associated to the original four Apollonian circles C_1, C_2, C_3, C_4; (**b**) The image of one of the original circles reflected in one of the dual circles begins the next generation of the circle packing
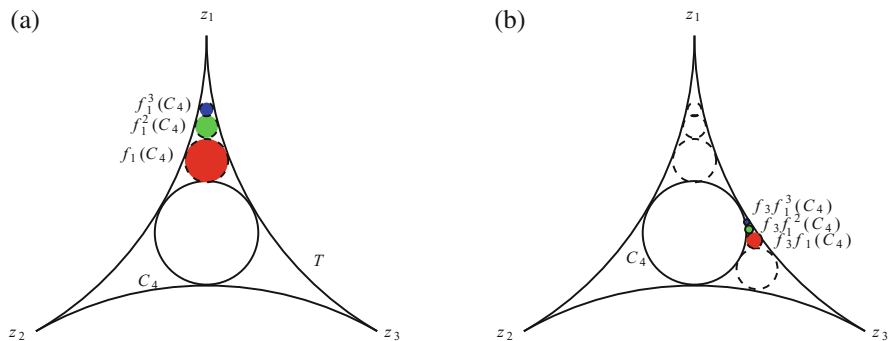
Fig. 8 (**a**) The central curvilinear triangle Δ and the images $f_1^n(C_4)$ of C_4 for $n = 1, 2, 3, \ldots$; (**b**) the images $f_3 f_1^n(C_4)$ of C_4 for $n = 1, 2, 3, \ldots$

Example 3 In the case of the Apollonian circle packing $\mathcal{C}$ with $a_0 = -1$ and $a_1 = a_2 = a_3$ we can explicitly write:

$$f_1(z) = \frac{az + b}{bz + a} \text{ where } a = -5\sqrt{\frac{4\sqrt{3} - 3}{78}} \text{ and } b = \sqrt{\frac{100\sqrt{3} - 153}{78}}$$

and $f_2(z) = e^{-2\pi i/3} f_1(e^{2\pi i/3} z)$ and $f_3(z) = e^{-2\pi i2/3} f_1(e^{2\pi i2/3} z)$.

Fig. 9 The radius of $g(C_4)$ is related to the derivative $|g'(0)|$ by the value of $g^{-1}(\infty)$

In particular, one can easily check that:

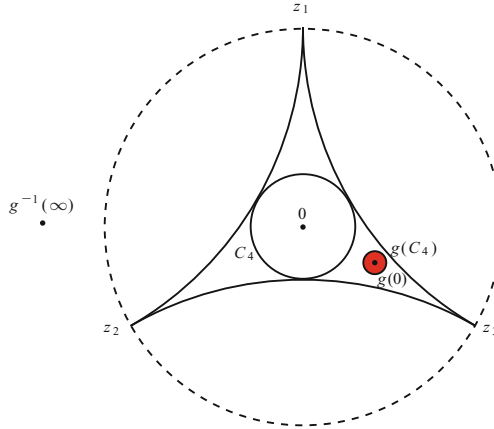1. For each $i = 1, 2, 3$ the iterates $f_i^k : \Delta \to \Delta$ ($k \geq 1$) have the effect of mapping the central circle C_4 on to a sequences of circles $\{f_i^k(C_4)\}_{k=1}^{\infty}$ occurring in $\mathcal{C}$ leading into the vertex x_i (cf. Fig. 9a); and
2. Any sequence of compositions of these three maps can be naturally written in the form $\bar{f} := f_{i_k}^{n_k} \cdots f_{i_1}^{n_1} : \Delta \to \Delta$, for $n_1, \cdots, n_k \geq 1$ and $i_1, \cdots, i_k \in \{1, 2, 3\}$ with $i_l \neq i_{l+1}$ for $1 \leq l \leq k - 1$.

The relevance of these maps to our present study is that we see that we can rewrite

$$\xi(s) = \sum_{\bar{f}} \operatorname{diam}(\bar{f}(C_0))^s,$$

at least for the contribution of circles in Δ, the other cases being similar, where the summation is over all such compositions $\bar{f} = f_{i_k}^{n_k} \cdots f_{i_1}^{n_1}$ in item 2 above.

3.3 Contracting Maps and Poincaré Series

The maps described above can be conveniently regrouped as follows:

$$\bar{f} := f_{i_k}^{n_k - 1} \circ (f_{i_k} \circ f_{i_{k-1}}^{n_{k-1}}) \circ \cdots \circ (f_{i_{k-2}} \circ f_{i_2}^{n_2}) \circ (f_{i_2} \circ f_{i_1}^{n_1}). \tag{3.1}$$

The advantage of this presentation is that at least part of this expression is contracting, in the following sense (cf. [7]).

Lemma 3.5 (After Mauldin-Urbanski) *For the Apollonion circle packings we have that the maps* $\phi_j = \phi_j^{(i_j,n_j)} := f_{i_{j-1}} \circ f_{i_j}^{n_j} : \Delta \to \Delta$ *are uniformly contracting (i.e.,* $\sup_j \sup_{z \in T} |\phi_j'(z)| < 1$).

This is illustrated in Fig. 9b with $f_3^n f_1$, $n \geq 1$.

Unfortunately, considering only compositions of the uniform contractions ϕ_j leads only to some of the circles in the circle packing $\mathcal{C}$. The rest of the circles require the final application of the maps $f_{i_k}^{n_k-1}$ in (3.1), which therefore also needs addressing. Moreover, the counting function we will actually use is a more localized version, which allows us to approximate the counting function for circles by a counting function for derivatives – for which the associated complex functions are easier to analyse. In particular, we want to analyse the following related complex functions.

Definition 3.6 Given $z_0 \in \Delta$ and an allowed word $\underline{j} = (j_1, \cdots, j_N)$, with $j_r \neq j_{r+1}$ for $r = 1, \cdots, N - 1$, we can associate a localised Poincaré function

$$\eta^{\underline{j}}(s) = \sum_{k=0}^{\infty} \sum_{\overline{\phi}} |(f_i^k \circ \overline{\phi} \circ \phi_{\underline{j}})'(z_0)|^s \tag{3.2}$$

where:

1. We first apply a fixed contraction $\phi_{\underline{j}} = \phi_{j_N} \circ \cdots \circ \phi_{j_1}$;
2. We next sum over all subsequent allowed hyperbolic compositions $\overline{\phi} := \phi_{i_n} \circ \cdots \circ \phi_{i_{N+1}} : \Delta \to \Delta$; and, finally,
3. We sum over the "parabolic tails" f_i^k (where i is associated to $\phi_{i_n} = f_i \circ f_l^n$, say).

The need to consider the contribution from different $\phi_{\underline{j}}$ is an artefact of our method of approximation in the proof.

Remark 3 Poincaré series are more familiar in the context of Kleinian groups Γ acting on three dimensional hyperbolic space and its boundary, the extended complex plane $\hat{\mathbb{C}}$. Our analysis applies to the Poincaré series of many such groups. In the particular case of classical Schottky groups the analysis is easier, since one can dispense with the parabolic tail (i.e., item 3 above).

As we will soon see, each such Poincaré series satisfies the hypotheses of Theorem 3.3, which allows us to estimate the corresponding counting function defined as follows.

Definition 3.7 We define an associated counting function

$$M^{\underline{j}}(T) = \#\{f_i^k \circ \overline{\phi} \circ \phi_{\underline{j}} : |(f_i^k \circ \overline{\phi} \circ \phi_{\underline{j}})'(z_0)| \leq T\} \text{ for } T > 0.$$

Let $\Sigma = \{(i_n)_{n=1}^{\infty} : i_n \neq i_{n+1} \text{ for } n \geq 0\}$ and consider the cylinder

$$[\underline{j}] = \{(i_n)_{n=1}^{\infty} \in \Sigma : i_r = j_r, \text{ for } 1 \leq r \leq N\}.$$

In particular, in the next section we will use the Poincaré series to deduce the following.

Proposition 3.8 *There exists $C > 0$ and a measure μ on Σ such that $M^{\underline{i}}(T) \sim C\mu([\underline{j}])T^{\delta}$ as $T \to +\infty$, i.e.,*

$$\lim_{T \to +\infty} \frac{M^{\underline{i}}(T)}{T^{\delta}} = 1.$$

There may be some circles whose radii we don't seem to capture with this coding, but their contribution doesn't effect the basic asymptotics.

4   The Proof of Theorem 2.2

To complete the proof of Theorem 2.2 we need to complete the proof of Proposition 3.8 (in Sect. 4.1 below) and then perform the approximation of the counting functions for circles by those for derivatives (in Sect. 4.2 below).

4.1   Extending the Poincaré Series

By the chain rule we can write

$$(f_i^k \circ \overline{\phi} \circ \phi_{\underline{j}})'(z_0) = (f_i^k)'(\overline{\phi} \circ \phi_{\underline{j}}z_0)\overline{\phi}'(\phi_{\underline{j}}z_0)\phi_{\underline{j}}'(z_0)$$

and, in particular, we can now rewrite the expression for $\eta^{\underline{i}}(s)$ in (3.2) as:

$$\begin{aligned}
\eta^{\underline{i}}(s) &= |\phi_{\underline{j}}'(z_0)|^s \sum_{n=0}^{\infty} \sum_{|\overline{\phi}|=n} \sum_{l=0}^{\infty} (f_i^k)'(\overline{\phi} \circ \phi_{\underline{j}}z_0)\overline{\phi}'(\phi_{\underline{j}}z_0) \\
&= |\phi_{\underline{j}}'(z_0)|^s \sum_{n=0}^{\infty} \sum_{|\overline{\phi}|=n} |\overline{\phi}'(z_0)|^s h_s(\overline{\phi}(z_0))
\end{aligned} \tag{4.1}$$

where the function $h_s : \Delta \to \mathbb{C}$ is defined by the summation

$$h_s(z) := \sum_{l=0}^{\infty} |(f_i^l)'(z)|^s \in C^1(\Delta)$$

is analytic in s. In particular, we see from the following lemma that $h_s(z)$ converges to a well defined function for $Re(s) > \frac{1}{2}$.

Lemma 4.1 *We can estimate* $||(f_i^l)'|_\Delta||_\infty = O(l^{-2})$.

We recall the simple proof (cf. [8]).

Proof By a linear fractional change of coordinates (mapping the vertex of Δ to infinity) the map f_i becomes a translation. Transforming this back to convenient coordinates we can write, say,

$$f_i^l(z) = \frac{(\sqrt{3} - l)z + l}{(-lz + l + \sqrt{3})}.$$

From this we see that

$$|(f_i^l)'(z)| = \frac{1}{|-lz + l + \sqrt{3}|^2}$$

and the required estimate follows.   □

The Poincaré series have the useful feature that they can be expressed simply in terms of linear operators on appropriate Banach spaces of functions.

Definition 4.2 Let $C^1(\Delta)$ be the Banach space of C^1 functions on Δ. We can consider the transfer operators $\mathcal{L}_s : C^1(\Delta) \to C^1(\Delta)$ ($s \in \mathbb{C}$) given by

$$\mathcal{L}_s w(x) = \sum_l |\phi_l'(x)|^s w(\phi_l x)$$

where $w \in C^1(\Delta)$. This converges provided $Re(s) > \frac{1}{2}$.

We are actually spoilt for choice of Banach spaces. Although the continuous functions $C^0(\Delta)$ is too large a space for our purposes, we could also work with Hölder continuous functions or suitable analytic functions (on some neighbourhood of the complexification of Δ thought of as a subset of $\mathbb{R}^2$). The choice of $C^1(\Delta)$ is perhaps the more familiar.

The approach in the rest of this subsection is now relatively well known (cf. [6, 8, 13], for example) and is a variant on the symbolic approach to Poincaré series and the hyperbolic circle problem [14, 15]. Recall that $\delta > 0$ is the exponent in Theorem 2.2.

Lemma 4.3 *The operators are well defined provided* $Re(s) > \frac{1}{2}$. *Moreover, for* $Re(s) > \delta$ *we have that the spectral radius satisfies*

$$\rho(\mathcal{L}_s) := \limsup_{n \to +\infty} \|\mathcal{L}_s^n\|^{\frac{1}{n}} < 1.$$

In particular, we see from the definition of $\mathcal{L}_s$ that we can write

$$\mathcal{L}_s^n w(z) = \sum_{\overline{\phi}} |\overline{\phi}'(z)|^s w(\overline{\phi}z), \text{ for } n \geq 2,$$

where the summation is over allowed compositions of contractions $\overline{\phi} = \phi_{i_n} \circ \cdots \circ \phi_{i_1}$. We can now rewrite the expression for the Poincaré series in (4.1) more concisely as

$$\eta^{\underline{j}}(s) = |\phi'_{\underline{j}}(z_0)|^s \sum_{n=0}^{\infty} \mathcal{L}_s^n h_s(\overline{\phi}_{\underline{j}} z_0).$$

In order to construct the required extension of $\eta^{\underline{j}}(s)$, we recall the following simple lemma improving on the result in Lemma 4.3.

Lemma 4.4 *Let $Re(s) = \delta$. Then*

1. *For $s = \delta + it$ with $t \neq 0$ we have that the spectral radius satisfies $\rho(\mathcal{L}_s) < 1$; and*
2. *For $s = \delta$ we can write $\mathcal{L}_\delta = Q + U$ where*

   (a) *Q is a (one dimensional) eigenprojection with $QU = UQ = 0$, $Q^2 = Q$, and*
   (b) *And $\limsup_{n \to +\infty} \|U^n\|^{1/n} < 1$.*

Remark 4 The spectral properties of $\mathcal{L}_s$ can be seen when the operator acts on C^1 functions. Alternatively, we could have looked at bounded analytic functions on a small enough neighbourhood $T \subset U \subset \mathbb{C}^2$ in the complexification (cf. [6]).

We can now deduce almost immediately from Lemmas 4.3 and 4.4 the following corollary for this Poincaré series.

Corollary 4.5 *The Poincaré series $\eta^{\underline{j}}(s)$ converges to a well defined function on $Re(s) > \delta$. Moreover,*

1. *For $Re(s) > \delta$ we have that $\eta^{\underline{j}}(s)$ is analytic;*
2. *There exists a small neighbourhood of each $\delta + it$ with $t \neq 0$ on which $\eta^{\underline{j}}(s)$ has an analytic extension; and*
3. *$\eta^{\underline{j}}(s)$ has a simple pole at $s = \delta$.*

Remark 5 In fact, we can deduce a little more which, if a little technical looking, is needed in the approximation argument below. In particular, we can also show that the simple pole for $\eta^{\underline{j}}(s)$ at $s = \delta$ has a residue of the form

$$C_{\underline{j}} := \frac{|(\phi_{\underline{j}})'(x_0)|^\delta \mu(h_s)}{\lambda'(\delta)}$$

where:

 (i) $\lambda(t)$ is an isolated eigenvalue equal to the spectral radius of $\mathcal{L}_t$ ($t \in \mathbb{R}$); and
(ii) $Q(h) = \mu(h)k$ where k is an associated eigenfunction, i.e., $\mathcal{L}_1 k = k$.

If we now write

$$\eta^{\underline{j}}(s) = \int_1^{\infty} t^{-s} dN^{\underline{j}}(t)$$

then comparing Corollary 4.5 with Theorem 3.3 gives the asymptotic formula for $N_j^i(T)$ in Proposition 3.8.

Let us now move on to the final step in the proof of Theorem 2.2.

4.2 The Approximation Argument

We can now approximate the radii $\text{rad}(g(C_4))$ of the circle $g(C_4)$ by suitably scaled values of $1/|g'(x_0)|$, where $g = f_i^l \circ \overline{\phi} \circ \phi_j$. Without loss of generality we can choose coordinates in $\mathbb{C}$ so that C_4 is the unit circle.

As a prelude to this we consider some simple geometric estimates on the sizes of the images of circles.

Lemma 4.6 *If $g(z) = (az + b)/(cz + d)$, with $ad - bc = 1$ and $a, b, c, d \in \mathbb{C}$, then the radius of the image circle $C = g(C_4)$ is equal to*

$$\frac{1}{||c|^2 - |d|^2|} = \frac{|g'(0)|}{||\frac{c}{d}|^2 - 1|}$$

The proof is a reassuringly elementary exercise:

Proof For the first part, we see that the image circle $g(C_0)$ has centre $z_c = (a\overline{c} - b\overline{d})/(|c|^2 - |d|^2)$ and radius $1/(||c|^2 - |d|^2|)$ since we can check that for $e^{i\theta} \in C_4 = \{z \in \mathbb{C} : |z| = 1\}$:

$$|g(e^{i\theta}) - z_c| = \left| \frac{ae^{i\theta} + b}{ce^{i\theta} + d} - \frac{a\overline{c} - b\overline{c}}{|c|^2 - |d|^2} \right| = \frac{1}{||c|^2 - |d|^2|}.$$

We then observe that $|g'(z)| = |cz + d|^{-2}$ and thus $|g'(0)| = |d|^{-2}$. Thus by the above we see that the radius of the image circle C is:

$$\text{rad}(C) = \frac{1}{||c|^2 - |d|^2|} = \frac{|g'(0)|}{\left| |\frac{c}{d}|^2 - 1 \right|}.$$

as claimed. □

We can write $g^{-1}(z) = (dz - b)/(-cz + a)$ and thus $g^{-1}(\infty) = d/c$.

Finally, we come to the crux of the approximation argument. The essential idea is to approximate the (technically more convenient) weighting of elements g by $|g'(z_0)|$, with a weighting by the more geometric weighting by reciprocals of the radii $\text{rad}(g(C_4))$. One simple approach is as follows. We are taking $z_0 = 0$, for definiteness, and then we want to use Proposition 3.8 to localise the counting to regions where

$$\frac{|g'(0)|}{\text{rad}(g(C_4))} = \left| \left| \frac{c}{d} \right|^2 - 1 \right|$$

is close to constant, using Lemma 4.6. Given an allowed string $(j_1, \cdots, j_N)$ we can write

$$
\begin{aligned}
g^{-1} &= \left((R_4 \circ R_{j_k})^{n_k} \circ \cdots \circ (R_4 \circ R_{j_{N+1}})^{n_{N+1}} \circ (R_4 \circ R_{j_N})^{n_N} \circ \cdots \circ (R_4 \circ R_{j_1})^{n_1} \right)^{-1} \\
&= (R_{j_1} \circ R_4)^{n_1} \circ \cdots \circ (R_{j_N} \circ R_4)^{n_N} \circ (R_{j_{N+1}} \circ R_4)^{n_{N+1}} \circ \cdots \circ (R_{j_k} \circ R_4)^{n_k} \\
&= \overline{f}_{j_k}^{n_k} \circ \cdots \circ \overline{f}_{j_1}^{n_1},
\end{aligned}
$$

where we denote $\overline{f}_j := R_j \circ R_4$ ($j = 1, 2, 3$) acting on the complement of the disk containing Δ (i.e., the dotted circle in Fig. 10). In particular, given $\eta > 0$, we can choose N sufficiently large such that for each $|\underline{j}| = N$ we can choose $K_{\underline{j}}$ such that for $g = f_i^l \circ \overline{\phi} \circ \phi_{\underline{j}}$:

$$
K_{\underline{j}} - \eta \le \frac{|g'(0)|}{\mathrm{rad}(g(C_4))} \le K_{\underline{j}} + \eta. \tag{4.2}
$$

We can define a local version of $N(T)$, which is useful to compare with $M^{\underline{j}}(T)$.

Definition 4.7 We define a restricted counting function

$$
N^{\underline{j}}(T) = \{ g : \mathrm{rad}(g(C_4)) \le T \},
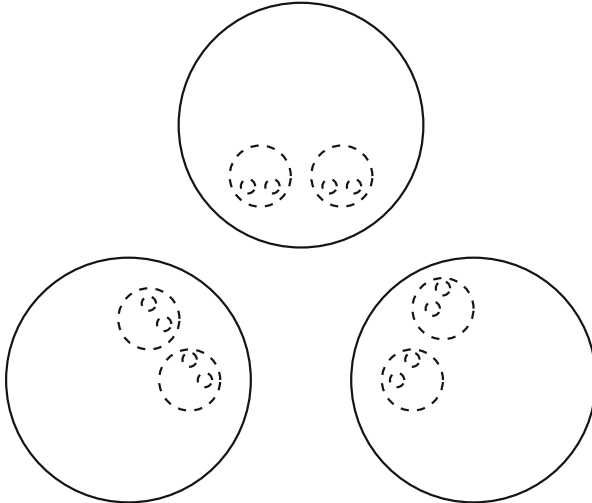$$

for $T > 0$.

Fig. 10 Sequences of circles generated by reflections in disjoint circles. The three initial circles are represented by *solid lines* and the first two generations of circles generated by reflections are represented by *dashed lines*

Using (4.2) we can write

$$M^{\underline{j}}\left(\frac{T}{K_{\underline{j}} + \eta}\right) \leq N^{\underline{j}}(T) \leq M^{\underline{j}}\left(\frac{T}{K_{\underline{j}} - \eta}\right).$$

and observe that $N(T) = \sum_{|\underline{j}|=N} N^{\underline{j}}(T)$. Using the asymptotic formula from Proposition 3.8 and summing over allowed strings $\underline{j}$ of length N, we have that

$$C\sum_{|\underline{j}|=N} \frac{\mu([\underline{j}])}{(K_{\underline{j}} + \eta)^{\delta}} \leq \liminf_{T\to\infty} \frac{N(T)}{T^{\delta}} \leq \limsup_{T\to\infty} \frac{N(T)}{T^{\delta}} \leq C\sum_{|\underline{j}|=N} \frac{\mu([\underline{j}])}{(K_{\underline{j}} - \eta)^{\delta}}$$

Letting $N \to +\infty$ (and thus $\epsilon \to 0$) gives the result in Theorem 2.2 with

$$K = \lim_{N\to+\infty} C\sum_{|\underline{j}|=N} \frac{\mu([\underline{j}])}{(K_{\underline{j}})^{\delta}}.$$

Remark 6 The existence of the limit, and its value K, can be understood in terms of an integral related to the natural measure μ on $\mathcal{C}$. A modified argument leads to an equidistribution result (expressed in terms of the measure μ, of course).

4.3 Generalizations

The approach to counting circles is more analytical than geometrical, and thus is somewhat oblivious to the specific setting of circle packings. In particular, the same method of proof works in a number of related settings where we ask for the radii of circles which are images under a suitable Kleinian group. For example:

1. Other circle packings for which the circles can be generated by the image of circles under reflections;
2. The radii of the images $g(C)$ of a circle C, where $\Gamma \subset SL(2, \mathbb{C})$ is a Schottky group (i.e., a convex cocompact Kleinian group generated by reflections in a finite number of circles with disjoint interiors);
3. The radii of the images $g(C)$ of a circle C, where $\Gamma \subset SL(2, \mathbb{C})$ is a quasi-Fuchsian group.

For more details of such examples, we refer the reader to [9].

The same basic method can also be used to prove other more subtle statistical properties of the radii of the circles.

Appendix: The Case of Reciprocal Integer Circles

The following is an interesting corollary to Descartes' Theorem.

Corollary 4.8 *If* $\frac{1}{a_0}, \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \in \mathbb{Z}$ *then* $\frac{1}{a_4} \in \mathbb{Z}$.

Proof In particular, this is a quadratic polynomial in $\frac{1}{a_4} > 0$, so given the radii of the initial circles a_1, a_2, a_3 we have two possible solutions

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \pm 2\sqrt{\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4}}.$$

and we denote these $\frac{1}{a_4} > 0$ (and $\frac{1}{a_0} < 0$). We use the convention that the smaller inner circle has radius $a_4 > 0$ and the larger outer circle has a negative "radius" a_4 (meaning its actually radius is $|a_4| > 0$ and the negative sign just tells us it is the outer circle). Adding these two solutions gives:

$$\frac{1}{a_0} + \frac{1}{a_4} = 2\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)$$

from which we easily deduce the result.   □

Proceeding inductively, for any subsequent configuration of four circles with radii $a_n, a_{n+1}, a_{n+2}, a_{n+3}$, for $n \geq 0$, we can similarly write

$$\frac{1}{a_{n+4}} = 2\left(\frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \frac{1}{a_{n+3}}\right) - \frac{1}{a_n}.$$

Proceeding inductively, then one gets infinitely many circles. Moreover, if the reciprocals of the initial four circles are integers then we easily see that this is true for all subsequence circles.

Corollary 4.9 *If the four initial Apollonian circles have that their radii a_0, a_1, a_2, a_3 are reciprocals of integers then all of the circles in $\mathcal{C}$ have that the reciprocals of their radii a_n, $n \geq 4$, are integers.*

Example 4 Let us consider the example starting with $a_0 = -\frac{1}{3}$, $a_1 = \frac{1}{5}$, $a_2 = \frac{1}{8}$, and $a_3 = \frac{1}{8}$. In Fig. 11 below we illustrate the iterative process of inscribing circles into each curved triangle formed by three previously constructed tangent circle and write $\frac{1}{a_n}$ inside the corresponding circle of radius a_n.

Example 5 Let us also consider the example with $a_0 = -\frac{1}{2}$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{6}$, $a_3 = \frac{1}{7}$. In Fig. 12 below we illustrate the iterative process of inscribing circles into each curved triangle formed by three previously constructed tangent circle and write $\frac{1}{a_n}$ inside the corresponding circle of radius a_n.
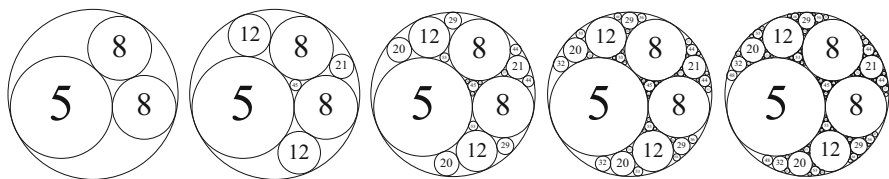
Fig. 11 We iteratively inscribe additional circles starting with circles of radii $a_0 = -\frac{1}{3}, a_1 = \frac{1}{5}$, $a_2 = \frac{1}{8}, a_3 = \frac{1}{8}$
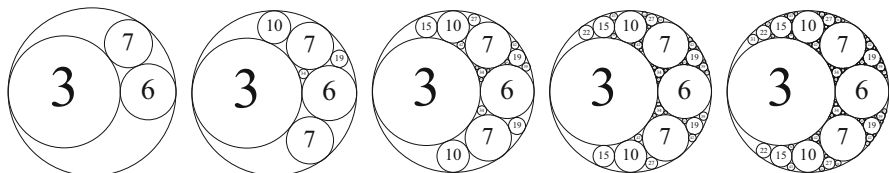

Fig. 12 We iteratively inscribe additional circles starting with circles with radii $a_0 = -\frac{1}{2}, a_1 = \frac{1}{3}$, $a_2 = \frac{1}{6}, a_3 = \frac{1}{7}$

Remark 7 An easy consequence of the fact $\delta > 1$ is that then $\frac{1}{a_n} \in \mathbb{N}$ some value must necessarily have high multiplicity (since we need to fit approximately $C\epsilon^{-\delta}$ inverse diameters into the first $[\epsilon^{-1}]$ natural numbers and the "pigeonhole principle" applies). In subsequent work, Oh-Shah showed that similar results are true for other sorts of circle packing [12]. Oh-Shah also gave an alternative approach to the original proof of Kontorovich-Oh using ideas of Roblin.

Remark 8 Another question we might ask is: *It we remove the repetitions in the sequence (a_n) then how many distinct diameters are greater than ϵ?* The following result was proved by Bourgain and Fuchs [2]: There exists $C > 0$ such that

$$\#\{\text{distinct diameters } a_n : a_n \geq \epsilon\} \geq \frac{C}{\epsilon}$$

for all sufficiently large ϵ.s Previously, Sarnak [17] had proved the slightly weaker result that there exists $C > 0$ such that

$$\#\{\text{distinct diameters } a_n : a_n \geq \epsilon\} \geq \frac{C}{\epsilon\sqrt{\log \epsilon}}.$$

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