Some Recent Developments in Quantization of Fractal Measures

Marc Kesseböhmer and Sanguo Zhu

Abstract We give an overview on the quantization problem for fractal measures, including some related results and methods which have been developed in the last decades. Based on the work of Graf and Luschgy, we propose a three-step procedure to estimate the quantization errors. We survey some recent progress, which makes use of this procedure, including the quantization for self-affine measures, Markov-type measures on graph-directed fractals, and product measures on multiscale Moran sets. Several open problems are mentioned.

Keywords Quantization dimension • Quantization coefficient • Bedford-McMullen carpets • Self-affine measures • Markov measures • Moran measures

Mathematics Subject Classification (2000). Primary 28A75, Secondary 28A80, 94A15

1 Introduction

The quantization problem for probability measures originated in information theory and certain areas of engineering technology such as image compression and data processing. In the past decades, this problem has been rigorously studied by mathematicians and the field of quantization theory emerged. Recently, this theory

---

S. Zhu was supported by China Scholarship Council No. 201308320049

M. Kesseböhmer (✉)
Fachbereich 3 – Mathematik und Informatik, Universität Bremen, Bibliothekstr. 1, Bremen 28359, Germany
e-mail: mhk@math.uni-bremen.de

S. Zhu
School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, China
e-mail: sgzhu@jsut.edu.cn

C. Bandt et al. (eds.), *Fractal Geometry and Stochastics V*, Progress in Probability 70, DOI 10.1007/978-3-319-18660-3_7

has also been found to have promising applications in numerical integrations and mathematical finance (see e.g. [22–24]). Mathematically we are concerned with the asymptotics of the errors in the approximation of a given probability measure with finitely supported probability measures in the sense of L_r-metrics. More precisely, for every $n \in \mathbb{N}$, we set $\mathcal{D}_n := \{\alpha \subset \mathbb{R}^q : 1 \leq \mathrm{card}(\alpha) \leq n\}$. Let μ be a Borel probability measure on $\mathbb{R}^q$, $q \in \mathbb{N}$, and let $r \in [0, \infty)$. The n-th quantization error for μ of order r is given by [6]

$$e_{n,r}(\mu) := \begin{cases} \inf_{\alpha \in \mathcal{D}_n} \left(\int d(x,\alpha)^r \, \mathrm{d}\mu(x) \right)^{1/r}, & r > 0, \\ \inf_{\alpha \in \mathcal{D}_n} \exp \int \log d(x,\alpha) \, \mathrm{d}\mu(x), & r = 0. \end{cases} \tag{1.1}$$

According to [6], $e_{n,r}(\mu)$ equals the error with respect to the L_r-metrics in the approximation of μ with discrete probability measures supported on at most n points. See [6, 13] for various equivalent definitions for the quantization error. In the following we will focus on the L_r-quantization problem with $r > 0$. For the quantization with respect to the geometric mean error, we refer to [8] for rigorous foundations and [37, 41, 42, 44] for more related results.

The upper and lower quantization dimension for μ of order r, as defined below, characterize the asymptotic quantization error in a natural manner:

$$\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)}, \quad \underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)}.$$

If $\overline{D}_r(\mu) = \underline{D}_r(\mu)$, we call the common value the quantization dimension of μ of order r and denote it by $D_r(\mu)$. To obtain more accurate information about the asymptotic quantization error, we define the s-dimensional upper and lower quantization coefficient (cf. [6, 26]):

$$\overline{Q}_r^s(\mu) := \limsup_{n\to\infty} n^{1/s} e_{n,r}(\mu), \quad \underline{Q}_r^s(\mu) := \liminf_{n\to\infty} n^{1/s} e_{n,r}(\mu), \quad s > 0.$$

By [6, 26], the upper (lower) quantization dimension is exactly the critical point at which the upper (lower) quantization coefficient jumps from zero to infinity.

The following theorem by Zador is a classical result on quantization of absolutely continuous measures. It was first proposed by Zador [32] and then generalized by Bucklew and Wise [2]; we refer to [6, Theorem 6.2] for a rigorous proof.

Theorem 1.1 ([6]) *Let μ be absolutely continuous Borel probability measure on $\mathbb{R}^q$ with density h with respect to the q-dimensional Lebesgue measure λ^q. Assume that for some $\delta > 0$, we have $\int |x|^{r+\delta} \, \mathrm{d}\mu\,(x) < \infty$. Then for all $r > 0$ we have*

$$\underline{Q}_r^q(\mu) = \overline{Q}_r^q(\mu) = C(r,q) \left(\int h^{\frac{q}{q+r}}(x) \, \mathrm{d}\lambda^q\,(x) \right)^{\frac{q+r}{q}},$$

where $C(r,q)$ is a constant independent of μ.

While engineers are mainly dealing with absolutely continuous distributions, the quantization problem is significant for all Borel probability measures satisfying the moment condition $\int |x|^r \mathrm{d}\mu(x) < \infty$. For later use we define the subset of Borel probabilities $\mathcal{M}_r := \{\mu\colon \mu(\mathbb{R}) = 1, \int |x|^r \mathrm{d}\mu(x) < \infty\}$ and let $\mathcal{M}_\infty$ denote the set of Borel probability measures with compact support. This condition ensures that the set of n-optimal sets of order r denoted by $C_{n,r}(\mu)$ is non-empty. Also note that $\mathcal{M}_\infty \subset \mathcal{M}_r$ for all $r > 0$. The most prominent aspects in quantization of probability measures are the following:

Find the exact value of the upper/lower quantization dimension for μ of order r: In the case where the quantization dimension does not exist, it is usually difficult to obtain the exact value of the upper or lower one (cf. [30]). Up to now, in such a situation, the upper and lower quantization dimension could only be explicitly determined for very special cases.

Determine the s-dimensional upper and lower quantization coefficient: We are mainly concerned about the finiteness and positivity of these quantities. This question is analogous to the question of whether a fractal is an s-set. Typically, this question is much harder to answer than finding the quantization dimension. So far, the quantization coefficient has been studied for absolutely continuous probability measures ([6]) and several classes of singular measures, including self-similar and self-conformal [19, 29, 33, 39] measures, Markov-type measures [16, 29, 44] and self-affine measures on Bedford-McMullen carpets [15, 38].

Properties of the point density measure $\boldsymbol{\mu}_{\mathbf{r}}$: Fix a sequence of n-optimal sets $\alpha_n \in C_{n,r}(\mu)$ of order r, $n \in \mathbb{N}$, and consider the weak limit of the empirical measures, whenever it exists,

$$\mu_r := \lim_{n\to\infty} \frac{1}{n} \sum_{a\in\alpha_n} \delta_a.$$

The point density measure characterizes the frequency at which optimal points fall into a given open set. Up to now, the point density measure is determined only for absolutely continuous measures [6, Theorem 7.5] and certain self-similar measures [9, Theorem 5.5].

Local properties and Voronoi partitions: Fix a finite subset α of $\mathbb{R}^q$. A Voronoi partition with respect to α refers to a partition $(P_a(\alpha))_{a\in\alpha}$ of $\mathbb{R}^q$ such that

$$P_a(\alpha) \subset \{x \in \mathbb{R}^q : d(x,\alpha) = d(x,a)\}, \ a \in \alpha.$$

It is natural to ask, if there exists constants $0 < C_1 \leq C_2 < \infty$ such that for all $\alpha_n \in C_{n,r}(\mu)$ and $n \in \mathbb{N}$ we have

$$\frac{C_1 e_{n,r}^r}{n} \leq \min_{a\in\alpha_n} \int_{P_a(\alpha_n)} \mathrm{d}(x,\alpha_n)^r \, \mathrm{d}\mu(x) \leq \max_{a\in\alpha_n} \int_{P_a(\alpha_n)} \mathrm{d}(x,\alpha_n)^r \, \mathrm{d}\mu(x) \leq \frac{C_2 e_{n,r}^r}{n}.$$

This question is essentially a weaker version of Gersho's conjecture [5]. Graf, Luschgy and Pagès proved in [10] that this is in fact true for a large class of absolutely continuous probability measures. An affirmative answer is also given for self-similar measures under the assumption of the strong separation condition (SSC) for the corresponding iterated function system [39, 43].

In the final analysis, the study of the quantization problem addresses the optimal sets. Where do the points of an optimal set lie? Unfortunately, it is almost impossible to determine the optimal sets for a general probability measure. It is therefore desirable to seek for an "approximately explicit" description of such sets. In other words, even though we do not know exactly where the points of an optimal set lie, we want to know how many points are lying in a given open set. This would in return enable us to obtain precise estimates for the quantization error.

Connection to fractal geometry: To this end, some typical techniques in fractal geometry are often very helpful. In fact, the quantization problem is closely connected with important notions in fractal geometry. One may compare the upper (lower) quantization dimension for measures to the packing (Hausdorff) dimension for sets; accordingly, the upper (lower) quantization coefficient may be compared to the packing (Hausdorff) measure for sets. Although they are substantially different, they do have some close connections, as all these quantities can be defined in terms of coverings, partitions and packings. In fact, we have

(1) $\dim_H^* \mu \leq \underline{D}_r(\mu) \leq \underline{\dim}_B^* \mu$ and $\dim_P^* \mu \leq \overline{D}_r(\mu) \leq \overline{\dim}_B^* \mu$, for $r = 2$ these inequalities were presented in [26], and for measures with compact support and all $r \in (0, \infty]$ they were independently proved in [6].

(2) In [14] we have studied the *stability* of the upper and lower quantization dimension in some detail. In [14], for $r \in [1, \infty]$, we proved the following statements:

  (i) For all $\mu \in \mathcal{M}_r$ we have $\overline{D}_r(\mu) = \max_{1\leq i\leq n} \overline{D}_r(\mu_i)$ with $\mu_i \in \mathcal{M}_r$, $s_i > 0$, $i = 1, \ldots, n$, $n \in \mathbb{N}$ and $\mu = \sum_{i=1}^n s_i \mu_i$.

  (ii) $\dim_P^*(\mu) = \inf\left\{\sup_{i\in\mathbb{N}} \overline{D}_r(\mu_i) : \mu = \sum_{i\in\mathbb{N}} s_i\mu_i, \mu_i \in \mathcal{M}_\infty, s_i > 0\right\}$ for all $\mu \in \mathcal{M}_\infty$.

  (iii) There exists $\mu \in \mathcal{M}_\infty$ such that $\overline{D}_r(\mu) \neq \dim_P^*(\mu)$.

(iv) There exists $\mu \in \mathcal{M}_\infty$ such that $\overline{D}_r(\mu) > \underline{D}_r(\mu)$.
(v) There exists $\mu \in \mathcal{M}_\infty$ such that $\underline{D}_r(\mu) \neq \max_{1\le i\le n} \underline{D}_r(\mu_i)$ for some $\mu_i \in \mathcal{M}_\infty, s_i > 0, i = 1, \ldots, n, n \in \mathbb{N}$ with $\mu = \sum_{i=1}^n s_i\mu_i$.

(3) For certain measures arising from dynamical systems, the quantization dimension can be expressed within the thermodynamic formalism in terms of appropriate *temperature functions* (see [15, 19, 27, 28]).
(4) The upper and lower quantization dimension of order zero are closely connected with the *upper and lower local dimension*. As it is shown in [43], if ν-almost everywhere the upper and lower local dimension are both equal to s, then $D_0(\nu)$ exists and equals s.

We end this section with Graf and Luschgy's results on self-similar measures. These results and the methods involved in their proofs have a significant influence on subsequent work on the quantization for non-self-similar measures.

Let $(S_i)_{i=1}^N$ be a family of contractive similitudes on $\mathbb{R}^q$ with contraction ratios $(s_i)_{i=1}^N$. According to [12], there exists a unique non-empty compact subset E of $\mathbb{R}^q$ such that $E = \bigcup_{i=1}^N S_i(E)$. The set E is called the self-similar set associated with $(S_i)_{i=1}^N$. Also, there exists a unique Borel probability measure on $\mathbb{R}^q$, such that $\mu = \sum_{i=1}^N p_i\mu \circ S_i^{-1}$, called the self-similar measure associated with $(S_i)_{i=1}^N$ and the probability vector $(p_i)_{i=1}^N$. We say that $(S_i)_{i=1}^N$ satisfies the *strong separation condition* (SSC) if the sets $S_i(E), i = 1, \cdots, N$, are pairwise disjoint. We say that it satisfies the *open set condition* (OSC) if there exists a non-empty open set U such that $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$ and $S_i(U) \subset U$ for all $i = 1, \cdots, N$. For $r \in [0, \infty)$, let k_r be the positive real number given by

$$k_0 := \frac{\sum_{i=1}^N p_i \log p_i}{\sum_{i=1}^N p_i \log s_i}, \quad \sum_{i=1}^N (p_i c_i^r)^{\frac{k_r}{k_r+r}} = 1. \tag{1.2}$$

Theorem 1.2 ([7, 8]) *Assume that $(S_i)_{i=1}^N$ satisfies the open set condition. Then for all $r \in [0, \infty)$, we have*

$$0 < \underline{Q}_r^{k_r}(\mu) \le \overline{Q}_r^{k_r}(\mu) < \infty.$$

In particular, we have $D_r(\mu) = k_r$.

This is the first complete result on the quantization for (typically) singular measures. In its proof, Hölder's inequality with an exponent less than one plays a crucial role, from which the exponent $k_r/(k_r + r)$ comes out in a natural manner.

2 The Three-Step Procedure

Following the ideas of Graf-Luschgy we propose a three-step procedure for the estimation of the quantization errors by means of partitions, coverings and packings. This procedure is applicable to a large class of fractal measures, including Moran measures, self-affine measures and Markov-type measures, provided that some suitable separation condition is satisfied; it even allows us to obtain useful information on the quantization for general Borel probability measures on $\mathbb{R}^q$ with compact support.

Step 1 (Partitioning). For each n, we partition the (compact) support of μ into φ_n small parts $(F_{nk})_{k=1}^{\varphi_n}$, such that $\mu(F_{nk})|F_{nk}|^r$ are uniformly comparable, namely, for some constant $C > 1$ independent of $k, j \in \{1, \ldots, \varphi_n\}$ and $n \in \mathbb{N}$, we have

$$C^{-1}\mu(F_{nk})|F_{nk}|^r \le \mu(F_{nj})|F_{nj}|^r \le C\mu(F_{nk})|F_{nk}|^r,$$

where $|A|$ denotes the diameter of a set $A \subset \mathbb{R}^d$. This idea was first used by Graf and Luschgy to treat the quantization problem for self-similar measures, we refer to [6] for a construction of this type. The underlying idea is to seek for some uniformity while μ generally is not uniform.

Step 2 (Covering). With a suitable separation condition, we may also assume that for some $\delta > 0$, we have that

$$d(F_{nk}, F_{nj}) \ge \delta \max\{|F_{nk}|, |F_{nj}|\},\ k \neq j,\ n \ge 1.$$

In this step, uniformity and separation allow us to verify that any φ_n-optimal set distributes its points equally among suitable neighborhoods of F_{nk}, $1 \le k \le \varphi_n$, in other words, each F_{nk} "owns" a bounded number of points of the φ_n-optimal set. More precisely, we prove that there exists some constant L_1, which is independent of n, such that for every $\alpha \in C_{\varphi_n,r}(\mu)$, we have

$$\max_{1\le k\le\varphi_n} \operatorname{card}\left(\alpha \cap (F_{nk})_{4^{-1}\delta|F_{nk}|}\right) \le L_1,$$

where A_s denotes the *s-parallel set* of A. This can often be done inductively by means of contradiction.

Step 3 (Packing). In the last step we have to find a constant L_2 and subsets β_{nk} of F_{nk} with cardinality at most L_2 such that for all $\alpha \in C_{\varphi_n,r}(\mu)$ and $x \in F_{nk}$ we have

$$d(x, \alpha) \ge d(x, (\alpha \cap (F_{nk})_{4^{-1}\delta|F_{nk}|}) \cup \beta_{nk}).$$

This reduces the global situation to a local one and enables us to restrict our attention to an arbitrary small set F_{nk}. We have

$$e^r_{\varphi_n,r}(\mu) \geq \sum_{k=1}^{\varphi_n} \int_{F_{nk}} d(x, (\alpha \cap (F_{nk})_{4^{-1}\delta|F_{nk}|}) \cup \beta_{nk})^r \,\mathrm{d}\mu(x).$$

Note that card $\big((\alpha \cap (F_{nk})_{4^{-1}\delta|F_{nk}|}) \cup \beta_{nk}\big) \leq L_1 + L_2$. For measures with explicit mass distributions, we often have

$$\int_{F_{nk}} d(x, \gamma \cup \beta_k)^r \,\mathrm{d}\mu(x) \geq D\mu(F_{nj})|F_{nj}|^r$$

for any subset γ of $\mathbb{R}^q$ with cardinality not greater than $L_1 + L_2$ and an appropriate constant D. Thus, we get a lower estimate for the quantization error:

$$e^r_{\varphi_n,r}(\mu) \geq D \sum_{k=1}^{\varphi_n} \mu(F_{nk})|F_{nk}|^r.$$

On the other hand, by choosing some arbitrary points $b_k \in F_{nk}$, $k \in \{1, \ldots, \varphi_n\}$, one can easily see

$$e^r_{\varphi_n,r}(\mu) \leq \sum_{k=1}^{\varphi_n} \int_{F_{nk}} d(x, b_k)^r d\mu(x) \leq \sum_{k=1}^{\varphi_n} \mu(F_{nk})|F_{nk}|^r.$$

After these three steps, for sufficiently "nice" measures, we may additionally assume that $\varphi_n \leq \varphi_{n+1} \leq C\varphi_n$ for some constant $C > 1$ (cf. [34–36]). To determine the dimension it is then enough to estimate the growth rate of φ_n. Here, ideas from Thermodynamic Formalism – such as critical exponents or zeros of some pressure function – often come into play: E.g., for $r > 0$ we often have

$$\frac{D_r(\mu)}{D_r(\mu) + r} = \inf\left\{t \in \mathbb{R} : \sum_{n\in\mathbb{N}} \sum_{k=1}^{\varphi_n} (\mu(F_{nk})|F_{nk}|^r)^t < \infty\right\}$$

allowing us to find explicit formulae for the quantization dimension for a given problem (see [15] for an instance of this). Typically, for a non-self-similar measure such as a self-affine measures on Bedford-McMullen carpets, this requires a detailed analysis of the asymptotic quantization errors. In order to formulate a rigorous proof, we usually need to make more effort according to the particular properties of the measures under consideration. As general measures do not enjoy strict self-similarity, it seems unrealistic to expect to establish simple quantities for the quantization errors as Graf and Luschgy did for self-similar measures [6, Lemma 14.10]. However, the above-mentioned three-step procedure often provides us with estimates of the quantization errors which is usually a promising starting point.

Moreover, in order to examine the finiteness or positivity of the upper and lower s-dimensional quantization coefficient of order r, it suffices to check that (cf. [40])

$$0 < \liminf_{n\to\infty} \sum_{k=1}^{\varphi_n} (\mu(F_{nj})|F_{nj}|^r)^{\frac{s}{s+r}} \leq \limsup_{n\to\infty} \sum_{k=1}^{\varphi_n} (\mu(F_{nj})|F_{nj}|^r)^{\frac{s}{s+r}} < \infty.$$

An effective way to do this is to construct some auxiliary probability measures. Such a measure should closely reflect the information carried by $(\mu(F_{nj})|F_{nj}|^r)^{\frac{s}{s+r}}$. For a self-similar measure, as Graf-Luschgy's work shows, an auxiliary probability measure is the self-similar measure associated with $(S_i)_{i=1}^N$ and the probability vector $((p_i c_i^r)^{\frac{k_r}{k_r+r}})_{i=1}^N$. It is interesting to note that this measure coincides with the point density measure provided that the k_r-dimensional quantization coefficient exists. For a self-similar measure, as Graf and Luschgy showed, we can use the above auxiliary probability measure and obtain the finiteness or positivity of the upper and lower k_r-dimensional quantization coefficient, which also implies that the quantization dimension exists and equals k_r. In the non-self-similar situation, due to the complexity of the topological support, it is often not easy to construct a suitable auxiliary probability measure to estimate the quantization coefficients.

3 Recent Work on the Quantization for Fractal Measures

3.1 Self-Affine Measures on Bedford-McMullen Carpets

Fix two positive integers m, n with $2 \leq m \leq n$ and fix a set

$$G \subset \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$$

with $N := \text{card}(G) \geq 2$. We define a family of affine mappings on $\mathbb{R}^2$ by

$$f_{ij} : (x, y) \mapsto \left(n^{-1}x + n^{-1}i, m^{-1}y + m^{-1}j\right), \quad (i,j) \in G. \tag{3.1}$$

By [12], there exists a unique non-empty compact set E satisfying $E = \bigcup_{(i,j)\in G} f_{ij}(E)$, which is called the Bedford-McMullen carpet determined by $(f_{ij})_{(i,j)\in G}$. Given a probability vector $(p_{ij})_{(i,j)\in G}$ with $p_{ij} > 0$, for all $(i,j) \in G$, the self-affine measure associated with $(p_{ij})_{(i,j)\in G}$ and $(f_{ij})_{(i,j)\in G}$ refers to the unique Borel probability measure μ on $\mathbb{R}^2$ satisfying

$$\mu = \sum_{(i,j)\in G} p_{ij} \mu \circ f_{ij}^{-1}. \tag{3.2}$$

Sets and measures of this form have been intensively studied in the past decades, see e.g. [1, 4, 11, 17, 18, 21, 25] for many interesting results. We write

$$G_x := \{i : (i,j) \in G \text{ for some } j\}\,,\ G_y := \{j : (i,j) \in G \text{ for some } i\}\,,$$
$$G_{x,j} := \{i : (i,j) \in G\}\,,\ \ q_j := \sum_{i \in G_{x,j}} p_{ij}.$$

We carry out the three-step procedure and obtain an estimate for the quantization errors. This allows us to conjecture that the quantization dimension exists and equals s_r, where

$$\Big(\sum_{(i,j)\in G} (p_{ij} m^{-r})^{\frac{s_r}{s_r+r}}\Big)^{\theta} \Big(\sum_{j \in G_y} (q_j m^{-r})^{\frac{s_r}{s_r+r}}\Big)^{1-\theta} = 1,\ \theta := \frac{\log m}{\log n}. \tag{3.3}$$

However, it seems rather difficult to find a suitable auxiliary measure for a proof of this conjecture. A cornerstone is the crucial observation that the number s_r coincide with a Poincare-like exponent [15]. Using the property of sup-additive sequences, we are able to prove that $D_r(\mu)$ exists and also coincides with κ_r. Finally, we consider the self-affine measure associated with $((p_{ij}m^{-r})^{\frac{s_r}{s_r+r}}/C_r)_{(i,j)\in G}$ as an auxiliary measure, where $C_r := \sum_{(i,j)\in G}(p_{ij}m^{-r})^{\frac{s_r}{s_r+r}}$. This measure and the above-mentioned estimate enable us to obtain sufficient conditions for the upper and lower quantization coefficient to be both positive and finite. We have

Theorem 3.1 ([15]) *Let μ be as defined in (3.2). Then for each $r \in (0,\infty)$ we have that $D_r(\mu)$ exists and equals s_r, Moreover, $0 < \underline{Q}_r^{s_r}(\mu) \leq \overline{Q}_r^{s_r}(\mu) < \infty$ if one of the following conditions is fulfilled:*

(A) $\sum_{i\in G_{x,j}}(p_{ij}q_j^{-1})^{\frac{s_r}{s_r+r}}$ *are identical for all $j \in G_y$,*
(B) q_j *are identical for all $j \in G_y$.*

Open problem: Is it true that $0 < \underline{Q}_r^{s_r}(\mu) \leq \overline{Q}_r^{s_r}(\mu) < \infty$ if and only if condition (A) or (B) holds?

3.2 Quantization for Markov-Type Measures

3.2.1 Mauldin-Williams Fractals

Let J_i, non-empty compact subsets of $\mathbb{R}^d$ with $J_i = \text{cl}(\text{int}(J_i))$, $1 \leq i \leq N$, where $\text{cl}(A)$ and $\text{int}(A)$ denote the closure and interior in $\mathbb{R}^d$ of a set $A \subset \mathbb{R}^d$. For the integer $N \geq 2$ let $P = (p_{ij})_{1\leq i,j\leq N}$ be a row-stochastic matrix, i.e., $p_{ij} \geq 0, 1 \leq i,j \leq N,$

and $\sum_{j=1}^{N} p_{ij} = 1, 1 \leq i \leq N$. Let θ denote the empty word and set

$$\Omega_0 := \{\theta\},\ \Omega_1 := \{1, \ldots N\},$$
$$\Omega_k := \{\sigma \in \Omega_1^k : p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k} > 0\},\ k \geq 2,$$
$$\Omega^* := \bigcup_{k\geq 0} \Omega_k,\ \Omega_\infty := \{\sigma \in \Omega_1^{\mathbb{N}} : p_{\sigma_h\sigma_{h+1}} > 0 \ \text{ for all } \ h \geq 1\}.$$

We call J_i, $1 \leq i \leq N$, cylinder sets of order one. For each $1 \leq i \leq N$, let $J_{ij},\ (i,j) \in \Omega_2$, be non-overlapping subsets of J_i such that J_{ij} is geometrically similar to J_j and $\operatorname{diam}(J_{ij})/\operatorname{diam}(J_j) = c_{ij}$. We call these sets cylinder sets of order two. Assume that cylinder sets of order k are determined, namely, for each $\sigma \in \Omega_k$, we have a cylinder set J_σ. Let $J_{\sigma * i_{k+1}}, \sigma * i_{k+1} \in \Omega_{k+1}$, be non-overlapping subsets of J_σ such that $J_{\sigma * i_{k+1}}$ is geometrically similar to $J_{i_{k+1}}$. Inductively, cylinder sets of order k are determined for all $k \geq 1$. The *(ratio specified) Mauldin-Williams fractal* is given by

$$E := \bigcap_{k\geq 1} \bigcup_{\sigma\in\Omega_k} J_\sigma.$$

3.2.2 Markov-Type Measures

Let $(\chi_i)_{i=1}^N$ be an arbitrary probability vector with $\min_{1\leq i\leq N} \chi_i > 0$. By Kolmogorov consistency theorem, there exists a unique probability measure $\tilde{\mu}$ on Ω_∞ such that $\tilde{\mu}([\sigma]) := \chi_{\sigma_1} p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k}$ for every $k \geq 1$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in \Omega_k$, where $[\sigma] := \{\omega \in \Omega_\infty : \omega|_{|\sigma|} = \sigma\}$. Let π denote the projection from Ω_∞ to E given by $\pi(\sigma) := x$, where

$$\{x\} := \bigcap_{k\geq 1} J_{\sigma|_k}, \quad \text{for } \ \sigma \in \Omega_\infty.$$

Let us assume the following:

(A1) $\operatorname{card}(\{j : p_{ij} > 0\}) \geq 2$ for all $1 \leq i \leq N$.
(A2) There exists a constant $t \in (0,1)$ such that for every $\sigma \in \Omega^*$ and distinct $i_1, i_2 \in \Omega_1$ with $\sigma * i_l \in \Omega_{|\sigma|+1}$, $l = 1, 2$,

$$d(J_{\sigma * i_1}, J_{\sigma * i_2}) \geq t \max\{|J_{\sigma * i_1}|, |J_{\sigma * i_2}|\}.$$

Under this assumption, π is a bijection. We consider the image measure of $\tilde{\mu}$ under the projection π given by $\mu := \tilde{\mu} \circ \pi^{-1}$. We call μ a Markov-type measure which satisfies

$$\mu(J_\sigma) = \chi_{\sigma_1} p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k} \ \text{ for } \ \sigma = (\sigma_1 \ldots \sigma_k) \in \Omega_k. \tag{3.4}$$

For $1 \leq i, j \leq N$, we define $a_{ij}(s) := (p_{ij}c_{ij}^r)^s$. Then we get an $N \times N$ matrix $A(s) = (a_{ij}(s))_{N\times N}$. Let $\psi(s)$ denote the spectral radius of $A(s)$. By [20, Theorem 2], $\psi(s)$ is continuous and strictly decreasing. Note that, by the assumption (A1), the Perron-Frobenius theorem and intermediate-value theorem, there exists a unique number $\xi \in (0, 1)$ such that $\psi(\xi) = 1$. Thus, for every $r > 0$, there exists a unique positive number s_r such that $\psi(\frac{s_r}{s_r+r}) = 1$.

We consider the directed graph G associated with the transition matrix $(p_{ij})_{N\times N}$. Namely, G has vertices $1, 2, \ldots, N$. There is an edge from i to j if and only if $p_{ij} > 0$. In the following, we will simply denote by $G = \{1, \ldots, N\}$ both the directed graph and its vertex sets. We also write

$$b_{ij}(s) := (p_{ij}c_{ij}^r)^{\frac{s}{s+r}}, \quad A_{G,s} := (b_{ij}(s))_{N\times N}, \quad \Psi_G(s) := \psi\left(\frac{s}{s+r}\right).$$

Let $\mathrm{SC}(G)$ denote the set of all strongly connected components of G. For $H_1, H_2 \in \mathrm{SC}(G)$, we write $H_1 \prec H_2$, if there is a path initiating at some $i_1 \in H_1$ and terminating at some $i_k \in H_2$. If we have neither $H_1 \prec H_2$ nor $H_2 \prec H_1$, then we say H_1, H_2 are incomparable. For every $H \in \mathrm{SC}(G)$, we denote by $A_{H,s}$ the sub-matrix $(b_{ij}(s))_{i,j\in H}$ of $A_G(s)$. Let $\Psi_H(s)$ be the spectral radius of $A_{H,s}$ and $s_r(H)$ be the unique positive number satisfying $\Psi_H(s_r(H)) = 1$.

Again, we apply the three-step procedure in Sect. 2 and obtain upper and lower estimates for the quantization error. Using these estimates and auxiliary measures of Mauldin-Williams type, we are able to prove that, when the transition matrix is irreducible, the upper and lower quantization coefficient are both positive and finite. This fact also leads to the positivity of the lower quantization coefficient in the general case. Then, based on a detailed analysis of the corresponding directed graph (not strongly connected) and some techniques in matrix theory, we are able to prove the formula for the quantization dimension. Finally, by using auxiliary measures of Mauldin-Williams type once more, we establish a necessary and sufficient condition for the upper quantization coefficient to be finite as stated next.

Theorem 3.2 ([16]) *Assume that (A1) and (A2) are satisfied. Let μ be the Markov-type measure as defined in (3.4) and s_r the unique positive number satisfying $\Psi_G(s_r) = 1$. Then, $D_r(\mu) = s_r$ and $\underline{Q}_r^{s_r}(\mu) > 0$. Furthermore, $\overline{Q}_r^{s_r}(\mu) < \infty$ if and only if $\mathcal{M} := \{H \in \mathrm{SC}(G) : s_r(H) = s_r\}$ consists of incomparable elements, otherwise, we have $\underline{Q}_r^{s_r}(\mu) = \infty$.*

3.3 *Quantization for Moran Measures*

3.3.1 Moran Sets

Let J be a non-empty compact subset of $\mathbb{R}^d$ with $J = \mathrm{cl}(\mathrm{int}(J))$. Let $|A|$ denote the diameter of a set $A \subset \mathbb{R}^d$. Let $(n_k)_{k=1}^{\infty}$ be a sequence of integers with $\min_{k\geq 1} n_k \geq 2$

and θ denote the empty word. Set

$$\Omega_0 := \{\theta\},\ \Omega_k := \prod_{j=1}^{k}\{1, 2, \cdots, n_j\},\ \Omega^* := \bigcup_{k=0}^{\infty}\Omega_k.$$

For $\sigma = \sigma_1 \cdots \sigma_k \in \Omega_k$ and $j \in \{1, \cdots, n_{k+1}\}$, we write $\sigma * j = \sigma_1 \cdots \sigma_k j$.

Set $J_\theta := J$ and let J_σ for $\sigma \in \Omega_1$ be non-overlapping subsets of J_θ such that each of them is geometrically similar to J_θ. Assume that J_σ is determined for every $\sigma \in \Omega_k$. Let $J_{\sigma*j}, 1 \leq j \leq n_{k+1}$ be non-overlapping subsets of J_σ which are geometrically similar to J_σ. Inductively, all sets $J_\sigma, \sigma \in \Omega^*$ are determined in this way. The Moran set is then defined by

$$E := \bigcap_{k=1}^{\infty}\bigcup_{\sigma\in\Omega_k} J_\sigma. \tag{3.5}$$

We call $J_\sigma, \sigma \in \Omega_k$, cylinders of order k. It is well known that the Moran sets E are generally not self-similar (cf. [3, 31]). For $k \geq 0$ and $\sigma \in \Omega_k$, we set

$$|\sigma| := k,\ c_{\sigma,j} := \frac{|J_{\sigma*j}|}{|J_\sigma|},\ 1 \leq j \leq n_{k+1}.$$

We assume that there exist some constants $c, \beta \in (0, 1)$ such that

(B1) $\inf_{\sigma\in\Omega^*} \min_{1\leq j\leq n_{|\sigma|+1}} c_{\sigma,j} = c > 0,$

(B2) $\mathrm{dist}(J_{\sigma*i}, J_{\sigma*j}) \geq \beta \max\{|J_{\sigma*i}|, |J_{\sigma*j}|\}$ for $1 \leq i \neq j \leq n_{|\sigma|+1}$ and $\sigma \in \Omega^*$.

3.3.2 Moran Measures

For each $k \geq 1$, let $(p_{kj})_{j=1}^{n_k}$ be a probability vector. By the Kolmogorov consistency theorem, there exists a probability measure ν on $\Omega_\infty := \prod_{k=1}^{\infty}\{1, 2, \cdots, n_k\}$ such that

$$\nu([\sigma_1, \cdots, \sigma_k]) = p_{1\sigma_1} \cdots p_{k\sigma_k},\ \sigma_1 \cdots \sigma_k \in \Omega_k,$$

where $[\sigma_1, \cdots, \sigma_k] = \{\tau \in \Omega_\infty : \tau_j = \sigma_j, 1 \leq j \leq k\}$. Let $\Pi : \Omega_\infty \to E$ be defined by $\Pi(\sigma) = \bigcap_{k\geq 1} J_{\sigma|_k}$ with $\sigma|_k = \sigma_1 \cdots \sigma_k$. Then, with the assumption (B2), Π is a continuous bijection. We define $\mu := \nu \circ \Pi^{-1}$. Then, we have

$$\mu(J) = 1,\ \mu(J_\sigma) := p_{1\sigma_1} \cdots p_{k\sigma_k},\ \ \sigma = \sigma_1 \cdots \sigma_k \in \Omega_k,\ k \geq 1.$$

We call the measure μ the Moran measure on E. It is known that the quantization dimension for μ of order r does not necessarily exist. Let $d_{k,r}, \overline{d}_r, \underline{d}_r$ be given by

$$\sum_{\sigma\in\Omega_k}(p_\sigma c_\sigma^r)^{\frac{d_{k,r}}{d_{k,r}+r}} = 1,\ \ \overline{d}_r := \limsup_{k\to\infty} d_{k,r},\ \ d_{k,r}\underline{d}_r := \liminf_{k\to\infty} d_{k,r}.$$

Open problem Is it true that $\overline{D}_r(\mu) = \overline{d}_r, \underline{D}_r(\mu) = \underline{d}_r$?

3.3.3 Multiscale Moran Sets

A multiscale Moran set is Moran set with some additional structure encoded in the infinite sequence $\omega = (\omega_l)_{l=1}^{\infty} \in \Upsilon := \{1,\ldots,m\}^{\mathbb{N}}$ for some $m \geq 2$. For this fix some positive integers $N_i \geq 2, 1 \leq i \leq m$ and for every $1 \leq i \leq m$, let $(g_{ij})_{j=1}^{N_i}$ be the contraction vector with $g_{ij} \in (0,1)$ and $(p_{ij})_{j=1}^{N_i}$ a probability vector with $p_{ij} > 0$ for all $1 \leq j \leq N_i$. Now using the notation in the definition of Moran sets, we set

$$n_{l+1} := N_{\omega_{l+1}},\ \ \left(c_{\sigma,j}\right)_{j=1}^{N_{\omega_{l+1}}} := (g_{\omega_{l+1}j})_{j=1}^{N_{\omega_{l+1}}},\ \sigma\in\Omega_l,\ l\geq 0. \tag{3.6}$$

If, for some $l \geq 0$, we have $\omega_{l+1} = i$, then for every $\sigma \in \Omega_l$, we have a continuum of choices of $\{J_{\sigma*j}\}_{j=1}^{N_i}$ fulfilling (B1),(B2) and (3.6), because we only fix the contraction ratios of the similitudes. Hence, to every $\omega \in \Upsilon$, there corresponds a class $\mathcal{M}_\omega$ of Moran sets according to (3.5). We call these Moran sets *multiscale Moran sets*.

For each $\omega \in \Upsilon$, we write

$$N_{k,i}(\omega) := \mathrm{card}\{1 \leq l \leq k : \omega_l = i\},\ \ 1 \leq i \leq m.$$

Fix a probability vector $\chi = (\chi_i)_{i=1}^m$ with $\chi_i > 0$ for all $1 \leq i \leq m$ and define

$$G(\chi) := \{\omega \in \Upsilon : \lim_{k\to\infty} k^{-1}N_{k,i}(\omega) = \chi_i,\ 1 \leq i \leq m\},$$

$$G_0(\chi) := \{\omega \in \Upsilon : \limsup_{k\to\infty} \left|N_{k,i}(\omega) - k\chi_i\right| < \infty, 1 \leq i \leq m\}.$$

3.3.4 Multiscale Moran Measures

Fix an $\omega \in G(\chi)$. According to Kolmogorov consistency theorem, there exists a probability measure ν_ω on the product space $\Omega_\infty := \prod_{k=1}^{\infty}\{1,2,\cdots,N_{\omega_k}\}$ such that

$$\nu_\omega([\sigma_1,\cdots,\sigma_k]) = p_{\omega_1\sigma_1}\cdots p_{\omega_k\sigma_k},\ \sigma_1\cdots\sigma_k \in \Omega_k,$$

where $[\sigma_1, \cdots, \sigma_k] = \{\tau \in \Omega_\infty : \tau_j = \sigma_j, 1 \leq j \leq k\}$. We define $\mu_\omega := \nu_\omega \circ \Pi^{-1}$. Then, we have

$$\mu(J) = 1,\ \mu_\omega(J_\sigma) := p_{\omega_1\sigma_1} \cdots p_{\omega_k\sigma_k},\ \ \sigma = \sigma_1 \cdots \sigma_k \in \Omega_k,\ k \geq 1.$$

We call the measure μ_ω the infinite product measure on $E(\omega)$ associated with ω and $(p_{ij})_{j=1}^{N_i}, 1 \leq i \leq m$.

For every $\omega \in G(\chi)$ and $k \in \mathbb{N}$, let $s_{k,r}(\omega), s_r$ and $\underline{H}_r(\omega), \overline{H}_r(\omega)$ be defined by

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{N_i}(p_{ij}g_{ij}^r)^{\frac{s_{k,r}(\omega)}{s_{k,r}(\omega)+r}}\right)^{N_{k,i}(\omega)} = 1,\ \ \prod_{i=1}^{m}\left(\sum_{j=1}^{N_i}(p_{ij}g_{ij}^r)^{\frac{s_r}{s_r+r}}\right)^{\chi_i} = 1, \quad (3.7)$$

$$\underline{H}_r(\omega) := \liminf_{k\to\infty} k|s_{k,r}(\omega) - s_r|,\ \ \overline{H}_r(\omega) := \limsup_{k\to\infty} k|s_{k,r}(\omega) - s_r|.$$

Compared with Mauldin-Williams fractals, the disadvantage is that we have more patterns in the construction of multiscale Moran sets. However, the pattern we use at the $(k+1)$-th step is independent of words of length k, which is an advantage. After we carry out the three-step procedure in Sect. 2, we conveniently obtain the exact value of the quantization dimension by considering some measure-like auxiliary functions. This also enables us to transfer the question of the upper and lower quantization coefficient to the convergence order of $(s_{k,r}(\omega))_{k=1}^{\infty}$. For the latter, we need a detailed analysis of some auxiliary functions related to (3.7). One may see [40] for more details. Our main result is summarized in the following theorem.

Theorem 3.3 ([40]) *For every $\omega \in G(\chi)$, we have*

(i) *$D_r(\mu_\omega)$ exists and equals s_r, it is independent of $\omega \in G(\chi)$,*
(ii) *If $s_{k,r}(\omega) \geq s_r$ for all large k, then $\underline{Q}_r^{s_r}(\mu_\omega) > 0$. If in addition $\underline{H}_r(\omega) = \infty$, then we have $\overline{Q}_r^{s_r}(\mu_\omega) = \infty$,*
(iii) *If $s_{k,r}(\omega) \leq s_r$ for all large k, then $\overline{Q}_r^{s_r}(\mu_\omega) < \infty$; if, in addition, $\underline{H}_r(\omega) = \infty$, then we have $\underline{Q}_r^{s_r}(\mu_\omega) = 0$,*
(iv) *If $\overline{H}_r(\omega) < \infty$, then $\underline{Q}_r^{s_r}(\mu_\omega)$ and $\overline{Q}_r^{s_r}(\mu_\omega)$ are both positive and finite,*
(v) *If $\omega \in G_0(\chi)$, then the assertion in* (iv) *holds.*

Open problem: What can we say about necessary conditions for $\underline{Q}_r^{s_r}(\mu_\omega)$ and $\overline{Q}_r^{s_r}(\mu_\omega)$ to be both positive and finite?

References

1. T. Bedford, Crinkly curves, Markov partitions and box dimensions in self-similar sets, Ph.D. thesis, University of Warwick (1984)
2. J.A. Bucklew, G.L. Wise, Multidimensional asymptotic quantization with rth power distortion measures. IEEE Trans. Inform. Theory **28**, 239–247 (1982)

3. R. Cawley, R.D. Mauldin, Multifractal decompositions of Moran fractals. Adv. Math. **92**, 196–236 (1992)
4. K.J. Falconer, Generalized dimensions of measures on almost self-affine sets. Nonlinearity **23**, 1047–1069 (2010)
5. A. Gersho, Asymptotically optimal block quantization. IEEE Trans. Inform. Theory **25**, 373–380 (1979)
6. S. Graf, H. Luschgy, *Foundations of Quantization for Probability Distributions*. Lecture Notes in Mathematics, vol. 1730. (Springer, Berlin/New York, 2000)
7. S. Graf, H. Luschgy, Asymptotics of the quantization error for self-similar probabilities. Real. Anal. Exch. **26**, 795–810 (2001)
8. S. Graf, H. Luschgy, Quantization for probabilitiy measures with respect to the geometric mean error. Math. Proc. Camb. Phil. Soc. **136**, 687–717 (2004)
9. S. Graf, H. Luschgy, The point density measure in the quantization of self-similar probabilities. Math. Proc. Camb. Phil. Soc. **138**, 513–531 (2005)
10. S. Graf, H. Luschgy, G. Pagès, The local quantization behavior of absolutely continuous probabilities. Ann. Probab. **40**(4), 1795–1828 (2012)
11. Y. Gui, W.X. Li, Multiscale self-affine Sierpinski carpets. Nonlinearity **23**, 495–512 (2010)
12. J.E. Hutchinson, Fractals and self-similarity. Indiana Univ. Math. J. **30**, 713–747; 7–156 (1981)
13. M. Kesseböhmer, S. Zhu, Quantization dimension via quantization numbers. Real Anal. Exch. **29**(2), 857–866 (2003/2004)
14. M. Kesseböhmer, S. Zhu, Stability of quantization dimension and quantization for homogeneous Cantor measures. Math. Nachr. **280**(8), 866–881 (2007)
15. M. Kesseböhmer, S. Zhu, On the quantization for self-affine measures on Bedford-McMullen carpets. arXiv:1312.3289 (2013)
16. M. Kesseböhmer, S. Zhu, The quantization for Markov-type measures on a class of ratio-specified graph directed fractals. preprint arXiv:1406.3257 (2014)
17. J.F. King, The singularity spectrum for general Sierpiński carpets. Adv. Math. **116**, 1–11 (1995)
18. S.P. Lalley, D. Gatzouras, Hausdorff and box dimensions of certain self-affine fractals. Indiana Univ. Math. J. **41**, 533–568 (1992)
19. L.J. Lindsay, R.D. Mauldin Quantization dimension for conformal iterated function systems. Nonlinearity **15**, 189–199 (2002)
20. R.D. Mauldin, S.C. Williams, Hausdorff dimension Graph-directed constructions. Trans. AMS. Math. **309**, 811–829 (1988)
21. C. McMullen, The Hausdorff dimension of general Sierpiński carpetes. Nagoya Math. J. **96**, 1–9 (1984)
22. G. Pagès, A space quantization method for numerical integration. J. Comput. Appl. Math. **89**(1), 1–38 (1998)
23. G. Pagès, B. Wilbertz, Dual quantization for random walks with application to credit derivatives. J. Comput. Finance **16**, 33–60 (2012)
24. G. Pagès, B. Wilbertz, Optimal Delaunay and Voronoi quantization schemes for pricing American style options, in *Numerical Methods in Finance*. Springer Proc. Math., Vol. 12. (Springer, Heidelberg, 2012), pp. 171–213
25. Y. Peres, The self-affine carpetes of McMullen and Bedford have infinite Hausdorff measure. Math. Proc. Camb. Phil. Soc. **116**, 513–26 (1994)
26. K. Pözelberger, The quantization dimension of distributions. Math. Proc. Camb. Phil. Soc. **131**, 507–519 (2001)
27. M.K. Roychowdhury, Quantization dimension for some Moran measures. Proc. Am. Soc. **138**(11), 4045–4057 (2010)
28. M.K. Roychowdhury, Quantization dimension function and Gibbs measure associated with Moran set. J. Math. Anal. Appl. **373**(1), 73–82 (2011)
29. M.K. Roychowdhury, Quantization dimension and temperature function for recurrent self-similar measures. Chaos Solitons Fractals **44**(11), 947–953 (2011)
30. M.K. Roychowdhury, Quantization dimension estimate of probability measures on hyperbolic recurrent sets. Dyn. Syst. **29**(2), 225–238 (2014)

31. Z.Y. Wen, Moran sets and Moran classes. Chinese Sci. Bull. **46**, 1849–1856 (2001)
32. P.L. Zador, Development and evaluation of procedures for quantizing multivariate distributions. Ph.D. thesis, Stanford University (1964)
33. S. Zhu, The lower quantization coefficient of the F-conformal measure is positive. Nonlinear Anal. **69**(2), 448–455 (2008)
34. S. Zhu, Quantization dimension of probability measures supported on Cantor-like sets. J. Math. Anal. Appl. **338**(1), 742–750 (2008)
35. S. Zhu, Quantization dimension for condensation systems. Math. Z. **259**(1), 33–43 (2008)
36. S. Zhu, Quantization dimension for condensation systems. II. The geometric mean error. J. Math. Anal. Appl. **344**(1), 583–591 (2008)
37. S. Zhu, The quantization for self-conformal measures with respect to the geometric mean error. Nonlinearity **23**(11), 2849–2866 (2010)
38. S. Zhu, The quantization dimension for self-affine measures on general Sierpiński carpets. Monatsh. Math. **162**, 355–374 (2011)
39. S. Zhu, Asymptotic uniformity of the quantization error of self-similar measures. Math. Z. **267**(3–4), 915–929 (2011)
40. S. Zhu, On the upper and lower quantization coefficient for probability measures on multiscale Moran sets. Chaos, Solitons Fractals **45**, 1437–1443 (2012)
41. S. Zhu, A note on the quantization for probability measures with respect to the geometric mean error. Monatsh. Math. **167**(2), 295–304 (2012)
42. S. Zhu, Asymptotics of the geometric mean error in the quantization for product measures on Moran sets. J. Math. Anal. Appl. **403**(1), 252–261 (2013)
43. S. Zhu, A characterization of the optimal sets for self-similar measures with respect to the geometric mean error. Acta Math. Hungar. **138**(3), 201–225 (2013)
44. S. Zhu, Convergence order of the geometric mean errors for Markov-type measures. Chaos, Solitons Fractals **71**, 14–21 (2015)