Dimension of the Graphs of the Weierstrass-Type Functions

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Abstract We present a survey of results on dimension of the graphs of the Weierstrass-type functions on the real line.

Keywords Fractals • Hausdorff dimension • Weierstrass function

Mathematics Subject Classification (2000). Primary 28A78, 28A80; Secondary 37C45, 37C40, 37D25

1 Introduction

In this paper we consider continuous real functions of the form

$$f^{\phi}_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x) \tag{1.1}$$

for $x \in \mathbb{R}$, where $b > 1$, $1/b < \lambda < 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a non-constant, $\mathbb{Z}$-periodic, Lipschitz continuous, piecewise C^1 function. Probably the most famous function of that form is the *Weierstrass cosine function*

$$W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x),$$

introduced by Weierstrass in 1872 as one of the first examples of a continuous nowhere differentiable function on the real line. In fact, Weierstrass proved the non-differentiability for some values of the parameters λ, b, while the complete

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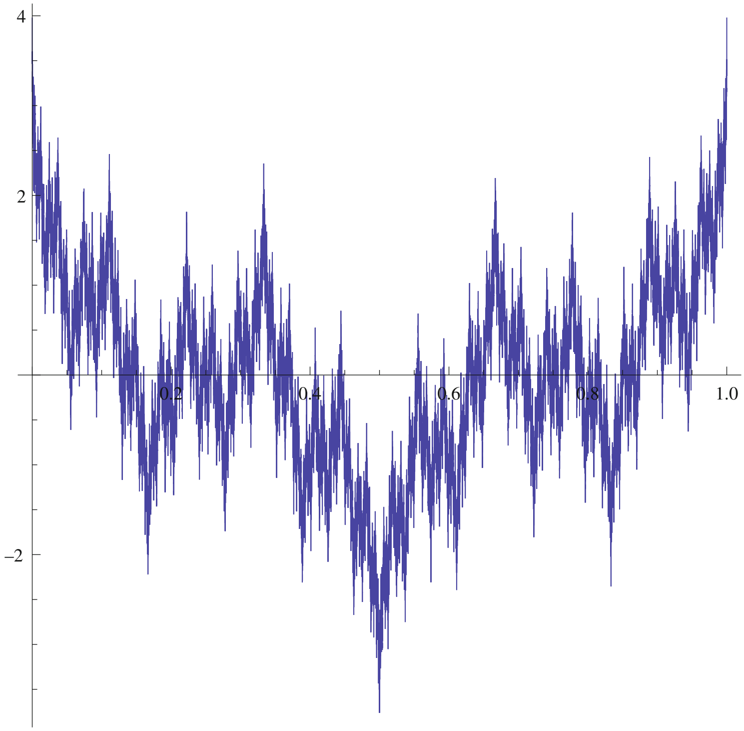

Fig. 1 The graph of the Weierstrass cosine nowhere differentiable function

proof for $b > 1$, $1/b < \lambda < 1$ was given by Hardy [16] in 1916. Later, starting from the work by Besicovitch and Ursell [8], the graphs of functions of the form (1.1) and related ones have been studied from a geometric point of view as fractal curves in the plane (Fig. 1).

In this paper we present a survey of recent results concerning various kinds of dimensions of the graphs of functions of the form (1.1).

Since

$$\lambda f_{\lambda,b}^{\phi}(bx) = f_{\lambda,b}^{\phi}(x) - \phi(x),$$

the graph of $f_{\lambda,b}^{\phi}$ exhibits a kind of approximate self-affinity with scales λ and $1/b$, which suggests a candidate for its dimension to be

$$D = 2 + \frac{\log \lambda}{\log b}.$$

We will see that this is indeed the case for the box dimension of the graph of $f_{\lambda,b}^{\phi}$ (unless it is a piecewise C^1 function with the graph of dimension 1), see Theorem 2.4. For the Hausdorff dimension, the situation is not so simple – we know some general lower estimates by constants smaller than D (see (3.1) and

Theorem 3.4), while the Hausdorff dimension of the graph is known to be equal to D only in some concrete cases (see Theorems 3.5 and 3.6), and for integer b and generic smooth function ϕ (see Theorem 4.2). On the other hand, we do not know any example of a function of the form (1.1), where the Hausdorff dimension of the graph is smaller than D.

Let us note that if b is an integer, then the graph of a function $f^{\phi}_{\lambda,b}$ of the form (1.1) is an invariant repeller for the expanding dynamical system

$$\Phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}, \qquad \Phi(x, y) = \left(bx \,(\mathrm{mod}\, 1),\ \frac{y - \phi(x)}{\lambda} \right) \tag{1.2}$$

with two different positive Lyapunov exponents $-\log \lambda < \log b$, which allows to use the methods of ergodic theory of smooth dynamical systems. In this case the graph of $f^{\phi}_{\lambda,b}$ is the common fractal boundary between the basins of attraction to (vertical) $+\infty$ and $-\infty$ on the cylinder $\mathbb{R}/\mathbb{Z}$ (see Fig. 2). Alternatively, the system

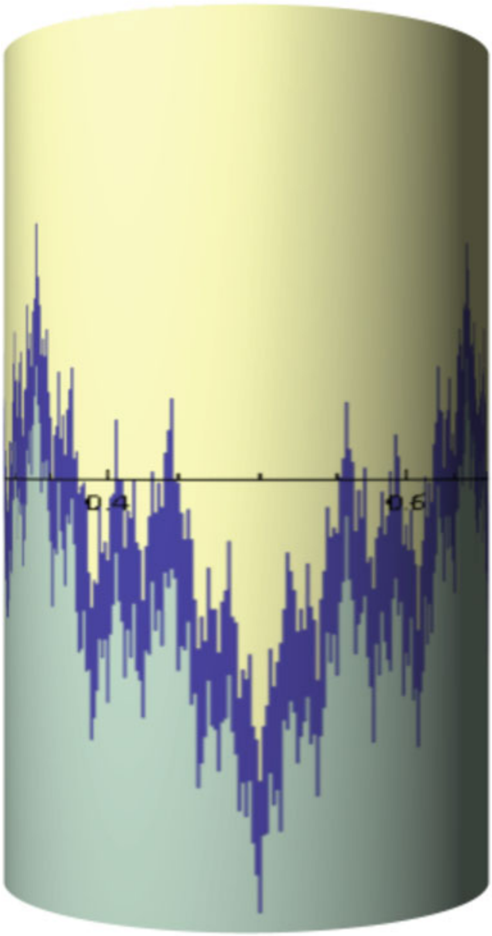

Fig. 2 The graph of the Weierstrass function as the boundary between two attracting basins on the cylinder

can be treated as a nonlinear iterated function system (IFS) on $[0, 1) \times \mathbb{R}$ composed of the maps

$$S_i(x, y) = \left(\frac{x}{b} + \frac{i}{b},\ \lambda y + \phi\left(\frac{x}{b} + \frac{i}{b}\right)\right), \qquad i = 0, \dots, b - 1.$$

Some results presented in this paper are valid also for a more general class of functions

$$f_{\lambda,b}^{\phi,\Theta}(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x + \theta_n), \tag{1.3}$$

where $\Theta = (\theta_1, \theta_2, \dots)$ for $\theta_n \in \mathbb{R}$ is a sequence of *phases* (with the previous assumptions on λ, b and ϕ).

We will consider the Hausdorff, packing and box dimension denoted, respectively, by $\dim_H$, $\dim_P$ and $\dim_B$. The upper and lower box dimension will be denoted, respectively, by $\overline{\dim}_B$ and $\underline{\dim}_B$. For an unbounded set, the (upper, lower) box dimension is defined as the supremum of (upper, lower) box dimensions of its bounded subsets.

For the definitions of the considered dimensions and their basic properties we refer to [14, 28]. We only note that for a set $X \subset \mathbb{R}^k$ we have

$$\dim_H(X) \le \underline{\dim}_B(X) \le \overline{\dim}_B(X)$$

and

$$\dim_H(X) \le \dim_P(X) \le \overline{\dim}_B(X).$$

The plan of the paper is as follows. In Sect. 2 we determine the box and packing dimension of the graphs of functions of the form (1.1). Results on the Hausdorff dimension are presented in Sects. 3–4. In Sect. 5 we deal with a randomization of functions of the form (1.3). Additional issues (complex extension of the Weierstrass cosine function, non-exponential sequences of scalings) are treated in Sects. 6–7.

Note that the quoted results are not necessarily presented in the chronological order and the formulation can be different from the original. Due to lack of space, the proofs are generally not included and the reader is referred to original articles.

There are a number of related issues which are not discussed in this paper (e.g. the case $\lambda = b$, wider classes of functions ϕ, dimension of the graphs of self-affine functions). The reader can find some information on these questions in the works included in the bibliography and the references therein.

2 Local Oscillations, Hölder Condition and Box Dimension

To determine the box dimension of the graphs of the considered functions, we examine their local oscillations in terms of the Hölder condition. By I we denote a non-trivial (not necessarily bounded) interval in $\mathbb{R}$.

Definition 2.1 We say that a function $f : I \to \mathbb{R}$ is *Hölder continuous* with exponent $\beta > 0$, if there exist $c, \delta > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^\beta$$

for every $x, y \in I$ such that $|x - y| < \delta$. Hölder continuous functions with exponent 1 are called *Lipschitz continuous* (with the *Lipschitz constant* c).

We say that f satisfies the *lower Hölder condition* with exponent $\beta > 0$, if there exist $\tilde{c}, \tilde{\delta} > 0$ such that the *oscillation*

$$\operatorname{osc}_J(f) = \sup_J f - \inf_J f$$

of f on every interval $J \subset I$ with $|J| < \tilde{\delta}$ satisfies

$$\operatorname{osc}_J(f) \geq \tilde{c}|J|^\beta$$

(where $|\cdot|$ denotes the length).

Note that the lower Hölder condition with exponent $\beta \in (0, 1)$ implies non-differentiability (i.e. non-existence of a finite derivative) of the function at every point.

The following proposition follows directly from the definitions of the upper and lower box dimension.

Proposition 2.2 *If $f : I \to \mathbb{R}$ is Hölder continuous with exponent $\beta \in (0, 1]$, then*

$$\overline{\dim}_B(\operatorname{graph} f) \leq 2 - \beta.$$

If a continuous function $f : I \to \mathbb{R}$ satisfies the lower Hölder condition with exponent $\beta \in (0, 1]$, then

$$\underline{\dim}_B(\operatorname{graph} f) \geq 2 - \beta.$$

Let

$$\alpha = -\frac{\log \lambda}{\log b} = 2 - D.$$

Note that by definition, $\alpha \in (0, 1)$ and $\lambda = b^{-\alpha}$. The following upper estimate of the box dimension of the graphs of the considered functions is a consequence of Proposition 2.2.

Proposition 2.3 *Every function* $f_{\lambda,b}^{\phi,\Theta}$ *of the form* (1.3) *is Hölder continuous with exponent* α*, and hence* $\overline{\dim}_B(\text{graph } f_{\lambda,b}^{\phi,\Theta}) \le D$.

Proof Let c be a Lipschitz constant of ϕ. Take $x, y \in I$ such that $0 < |x - y| \le 1$. Then, choosing $N \in \mathbb{N}$ with $1/b^N < |x - y| \le 1/b^{N-1}$, we have

$$|f_{\lambda,b}^{\phi,\Theta}(x) - f_{\lambda,b}^{\phi,\Theta}(y)| \le c|x-y| \sum_{n=0}^{N-1} (\lambda b)^n + 2 \max \phi \sum_{n=N}^{\infty} \lambda^n$$
$$< \left(\frac{cb}{\lambda b - 1} + \frac{2 \max \phi}{1 - \lambda} \right) \lambda^N < \left(\frac{cb}{\lambda b - 1} + \frac{2 \max \phi}{1 - \lambda} \right) |x - y|^\alpha.$$

□

One cannot expect a non-trivial lower estimate for the dimension of the graph, which holds for every function under consideration. Indeed, if

$$\phi(x) = g(x) - \lambda g(bx)$$

for an integer $b > 1$ and a $\mathbb{Z}$-periodic, Lipschitz continuous, piecewise C^1 function g, then $f_{\lambda,b}^{\phi}$ has the form (1.1) and $f_{\lambda,b}^{\phi} = g$, so its graph is a piecewise smooth curve of dimension 1. However, for functions of the form (1.1), the case of a piecewise C^1 curve is the only possible exception, when the box dimension of the graph is smaller than D. The following fact is a consequence of a result by Hu and Lau [18].

Theorem 2.4 *For every function* $f_{\lambda,b}^{\phi}$ *of the form* (1.1)*, exactly one of the two following possibilities holds.*

(a) $f_{\lambda,b}^{\phi}$ *is piecewise* C^1 (*and hence the dimension of its graph is* 1).
(b) $f_{\lambda,b}^{\phi}$ *satisfies the lower Hölder condition with exponent* α (*in particular it is nowhere differentiable*) *and* $\dim_B(\text{graph } f_{\lambda,b}^{\phi}) = D$.

Proof Adding a constant to $f_{\lambda,b}^{\phi}$, we can assume $\phi(0) = 0$. In [18, Theorem 4.1] it is proved that in this case, if the *Weierstrass–Mandelbrot function*

$$V(x) = \sum_{n=-\infty}^{\infty} \lambda^n \phi(b^n x) = f_{\lambda,b}^{\phi} + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} \phi\left(\frac{x}{b^n}\right)$$

is not identically zero, then $f_{\lambda,b}^{\phi}$ satisfies the lower Hölder condition with exponent α. Hence, if $V \not\equiv 0$, then we can use Propositions 2.2 and 2.3 to obtain the assertion (b). On the other hand, $\sum_{n=1}^{\infty}(1/\lambda^n)\phi(x/b^n)$ is a piecewise C^1 function, so the condition $V \equiv 0$ implies that $f_{\lambda,b}^{\phi}$ is piecewise C^1, which is the case (a). □

A consequence of Theorem 2.4 is that the graphs of functions of the form (1.1) have packing dimension equal to box dimension.

Proposition 2.5 *For every function $f^{\phi}_{\lambda,b}$ of the form* (1.1)*, we have*

$$\dim_P(\text{graph } f^{\phi}_{\lambda,b}) = \dim_B(\text{graph } f^{\phi}_{\lambda,b}).$$

Proof We can assume that we are in the case (b) of Theorem 2.4, i.e. $f^{\phi}_{\lambda,b}$ satisfies the lower Hölder condition with exponent α and $\dim_B(\text{graph } f^{\phi}_{\lambda,b}) = D$. It is a general fact (see [14, Corollary 3.9]), that for every compact set $X \subset \mathbb{R}^k$, if

$$\overline{\dim}_B(X \cap U) = \overline{\dim}_B(X)$$

for every open set U intersecting X, then $\dim_P(X) = \overline{\dim}_B(X)$. To prove the proposition, we check this condition for $X = \text{graph } f^{\phi}_{\lambda,b}|_I$, where I is an arbitrary non-trivial compact interval in R.

By the continuity of $f^{\phi}_{\lambda,b}$, for an open set U intersecting graph $f^{\phi}_{\lambda,b}|_I$, we can take a non-trivial interval $J \subset I$ such that graph $f^{\phi}_{\lambda,b}|_J \subset \text{graph } f^{\phi}_{\lambda,b}|_I \cap U$. Since $f^{\phi}_{\lambda,b}|_J$ satisfies the lower Hölder condition with exponent α, Proposition 2.2 implies

$$D \le \overline{\dim}_B(\text{graph } f^{\phi}_{\lambda,b}|_J) \le \overline{\dim}_B(\text{graph } f^{\phi}_{\lambda,b}|_I \cap U) \le \overline{\dim}_B(\text{graph } f^{\phi}_{\lambda,b}|_I) \le D,$$

which ends the proof. □

In particular, the Weierstrass cosine function $W_{\lambda,b}$ satisfies the lower Hölder condition with exponent α and

$$\dim_P(\text{graph } W_{\lambda,b}) = \dim_B(\text{graph } W_{\lambda,b}) = D$$

for $b > 1$, $1/b < \lambda < 1$. Similar results for various classes of functions ϕ were obtained, among others, by Kaplan, Mallet-Paret and Yorke [24], Rezakhanlou [34], Przytycki and Urbański [33] and Bousch and Heurteaux [9].

In [17], Heurteaux generalized the above results to the case of functions of the form (1.3) with transcendental b.

Theorem 2.6 *Every function $f^{\phi,\Theta}_{\lambda,b}$ of the form* (1.3)*, where b is a transcendental number, satisfies the lower Hölder condition with exponent α* (*in particular it is nowhere differentiable*). *Moreover,*

$$\dim_P(\text{graph } f^{\phi,\Theta}_{\lambda,b}) = \dim_B(\text{graph } f^{\phi,\Theta}_{\lambda,b}) = D.$$

3 Hausdorff Dimension

The question of determining the Hausdorff dimension of the graphs of the considered functions is much more delicate and far from being completely solved. Since the upper bound $\dim_H \le \overline{\dim}_B \le D$ is known, one looks for suitable lower estimates. A standard tool is to analyse local properties of a finite Borel measure on the graph.

Definition 3.1 Let μ be a finite Borel measure in a metric space X. The *upper* and *lower local dimension* of μ at a point $x \in X$ are defined, respectively, as

$$\overline{\dim}\ \mu(x) = \limsup_{r\to 0+} \frac{\log \mu(B_r(x))}{\log r}, \qquad \underline{\dim}\ \mu(x) = \liminf_{r\to 0+} \frac{\log \mu(B_r(x))}{\log r},$$

where $B_r(x)$ denotes the ball of radius r centered at x. If for some d the upper and lower local dimensions of μ at x coincide and are equal to d for μ-almost every x, then we say that μ has *local dimension* d and write $\dim \mu = d$. Such measures are also called *exact-dimensional*.

Estimating the Hausdorff dimension of a set, it is standard to use the following fact (see [14, 28]).

Lemma 3.2 *If for some* $d > 0$ *we have* $\underline{\dim}\ \mu(x) \geq d$ *for* μ*-almost every* x*, then every Borel set of positive measure* μ *has Hausdorff dimension at least* d*. In particular, this holds if* $\dim \mu \geq d$*.*

In [33], using Lemma 3.2 for the lift of the Lebesgue measure on $[0, 1]$ to the graph of the function, Przytycki and Urbański proved the following.

Theorem 3.3 *If* $f : I \to \mathbb{R}$ *is Hölder continuous with exponent* $\beta \in (0, 1)$ *and satisfies the lower Hölder condition with exponent* β*, then*

$$\dim_H(\text{graph } f) > C > 1,$$

where C *depends only on* β *and constants* $c, \tilde{c}$ *in Definition* 2.1.

This together with Proposition 2.3 and Theorem 2.4 implies that for every function $f^{\phi}_{\lambda,b}$ of the form (1.1), if $f^{\phi}_{\lambda,b}$ is not piecewise C^1, then

$$\dim_H(\text{graph } f^{\phi}_{\lambda,b}) > 1. \tag{3.1}$$

Better estimates can be obtained for large b, even in the presence of phases, as shown by Mauldin and Williams [29].

Theorem 3.4 *For every function* $f^{\phi,\Theta}_{\lambda,b}$ *of the form* (1.3)*, there exists a constant* $B > 0$ *depending only on* λ *and* ϕ*, such that*

$$\dim_H(\text{graph } f^{\phi,\Theta}_{\lambda,b}) > D - \frac{B}{\log b}$$

for every sufficiently large b*.*

The result was obtained by using Lemma 3.2 for the lift of a measure supported on a suitable Cantor set in $[0, 1]$ to the graph of the function.

The first example of a function of the form (1.1) with the graph of Hausdorff dimension equal to D was given by Ledrappier [25], who proved the following result, using the theory of invariant measures for non-uniformly hyperbolic smooth dynamical systems (Pesin theory) on manifolds [26].

Theorem 3.5 *For* $\phi(x) = \text{dist}(x, \mathbb{Z})$ (*the* sawtooth function) *and* $b = 2$,

$$\dim_H(\text{graph } f^{\phi}_{\lambda,2}) = D$$

for Lebesgue almost all $\lambda \in (1/2, 1)$.

In fact, the assertion holds provided the infinite Bernoulli convolution $\sum_{n=0}^{\infty} \pm 1/(2\lambda)^n$, with $\pm$ chosen independently with probabilities $(1/2, 1/2)$, has absolutely continuous distribution. As proved by Solomyak [39], the condition is fulfilled for almost all $\lambda \in (1/2, 1)$. By a recent result by Shmerkin [38], in fact it holds for all $\lambda \in (1/2, 1)$ except of a set of Hausdorff dimension 0.

In [40], Solomyak generalized the result from Theorem 3.5 to the case of some functions ϕ with discontinuous derivative (*nonlinear sawtooth functions*).

For the Weierstrass cosine function $W_{\lambda,b}$, the conjecture that the Hausdorff dimension of its graph is equal to D was formulated by Mandelbrot [27] in 1977 (see also [7]) and then repeated in a number of subsequent papers. Recently, Bárány, Romanowska and the author [5] proved the following result, showing that the conjecture is true for every nonzero integer b and a large set of parameters λ.

Theorem 3.6 *For every integer* $b > 1$ *there exist* $\lambda_b, \tilde{\lambda}_b \in (1/b, 1)$, *such that for every* $\lambda \in (\lambda_b, 1)$ *and Lebesgue almost every* $\lambda \in (\tilde{\lambda}_b, 1)$, *we have* $\dim \mu_{\lambda,b} = D$, *where* $\mu_{\lambda,b}$ *is the lift of the Lebesgue measure on* $[0, 1]$ *to* graph $W_{\lambda,b}$. *In particular,*

$$\dim_H(\text{graph } W_{\lambda,b}) = D$$

for every $\lambda \in (\lambda_b, 1)$ *and almost every* $\lambda \in (\tilde{\lambda}_b, 1)$. *We have*

$$\lambda_2 < 0.9352, \quad \lambda_3 < 0.7269, \quad \lambda_4 < 0.6083, \quad \lambda_b < 0.5448 \quad \text{for every } b \geq 5,$$
$$\tilde{\lambda}_2 < 0.81, \quad \tilde{\lambda}_3 < 0.55, \quad \tilde{\lambda}_4 < 0.44, \quad \tilde{\lambda}_b < 1.04/\sqrt{b} \quad \text{for every } b \geq 5$$

and

$$\lambda_b \to \frac{1}{\pi}, \quad \tilde{\lambda}_b\sqrt{b} \to \frac{1}{\sqrt{\pi}} \quad \text{as} \quad b \to \infty.$$

The proof uses the Ledrappier–Young theory from [26], Tsujii's results [41] on the Sinai–Bowen–Ruelle (SBR) measure for some smooth Anosov endomorphisms of the cylinder and the Peres–Solomyak transversality methods developed under the study of infinite Bernoulli convolutions (see e.g. [31, 32, 40]).

4 Dimension of Graphs of Generic Functions

In mathematics there are a number of definitions of a generic (typical) property. A *topologically generic* set in a space X is a set containing an open and dense set in X, or a *residual* set (containing a countable intersection of open and dense sets in X). A *measure-theoretic generic* set in $\mathbb{R}^k$ is a set of full Lebesgue measure. We use the following infinite-dimensional analogue of this property, which is called *prevalence* (see e.g. [30]).

Definition 4.1 A Borel set E in a real vector space V is *prevalent*, if there exists a finite set $\{v_1, \dots, v_k\} \subset V$ (called the *probe set*), such that for every $v \in V$, one has $v + \sum_{j=1}^{k} t_j v_j \in E$ for Lebesgue almost every $(t_1, \dots, t_k) \in \mathbb{R}^k$. A non-Borel subset of V is prevalent, if it contains a Borel prevalent set.

The topological and measure-theoretical genericity need not coincide. In fact, a topologically typical (residual) continuous function on $[0, 1]$ is nowhere differentiable (this follows from the Baire Theorem, see [1]) and has the graph of lower box dimension 1 (see [22]) and packing dimension 2 (see [19]), while a measure-theoretic typical (prevalent) continuous function on $[0, 1]$ is nowhere differentiable (see [20]) and has the graph of Hausdorff dimension 2 (see [15]). In [12] (see also [36]), using the wavelet technique, it was proved that functions with graphs of Hausdorff dimension $2 - \beta$ are prevalent within the space of Hölder continuous functions on $\mathbb{R}$ with given exponent $\beta \in (0, 1)$.

In [5], Bárány, Romanowska and the author proved that for functions $f^{\phi}_{\lambda,b}$ of the form (1.1) with integer b, the Hausdorff dimension of graph $f^{\phi}_{\lambda,b}$ is equal to D both for topologically and measure-theoretic typical C^3 function ϕ. To formulate the result precisely, consider the space $C^r(\mathbb{R}/\mathbb{Z})$, for $r = 3, 4, \dots, \infty$, of $\mathbb{Z}$-periodic C^r real functions on $\mathbb{R}$, treated as functions on $\mathbb{R}/\mathbb{Z}$. For $b > 1$ let

$$\mathcal{F}_b = \{(\lambda, \phi) \in (1/b, 1) \times C^3(\mathbb{R}/\mathbb{Z}) : \dim \mu^{\phi}_{\lambda,b} = D\},$$

where $\mu^{\phi}_{\lambda,b}$ is the lift of the Lebesgue measure on $[0, 1]$ to graph $f^{\phi}_{\lambda,b}$. Recall that

$$\dim_H(\text{graph } f^{\phi}_{\lambda,b}) = D \qquad \text{for every } (\lambda, \phi) \in \mathcal{F}_b.$$

For $\lambda \in (1/b, 1)$, let

$$\mathcal{E}_{\lambda,b} = \{\phi \in C^3(\mathbb{R}/\mathbb{Z}) : (\lambda, \phi) \in \operatorname{int} \mathcal{F}_b\},$$

where int denotes the interior with respect to the product of the Euclidean and C^3 topology in $(1/b, 1) \times C^3(\mathbb{R}/\mathbb{Z})$. In [5], the following result was proved.

Theorem 4.2 *For every integer $b > 1$ and $\lambda \in (1/b, 1)$, the set $\mathcal{E}_{\lambda,b}$ is prevalent as a subset of $C^3(\mathbb{R}/\mathbb{Z})$, with a probe set contained in $C^\infty(\mathbb{R}/\mathbb{Z})$. Consequently, for every $r = 3, 4, \dots, \infty$, the set $\mathcal{E}_{\lambda,b}$ is an open and dense subset of $C^r(\mathbb{R}/\mathbb{Z})$, and the set $\mathcal{F}_b$ contains an open and dense subset of $(1/b, 1) \times C^3(\mathbb{R}/\mathbb{Z})$.*

Similarly as for Theorem 3.6, the proof is based on the Ledrappier–Young theory from [26] and a result by Tsujii [41] on the generic absolute continuity of the SBR measure for some smooth Anosov endomorphisms of the cylinder.

5 Randomization

It is a well-known fact that introducing some additional parameters or stochastics to a system can sometimes help to answer questions which could not be solved in a standard setup. In studying dimension of the graphs of functions of the form (1.3), a number of results were obtained by randomizing suitable parameters.

Concerning the box dimension, Heurteaux [17] proved the following.

Theorem 5.1 *Let $f_{\lambda,b}^{\phi,\Theta}$ be a function of the form* (1.3)*. If one considers the phases θ_n as independent random variables with uniform distribution on* $[0, 1]$*, then almost surely, $f_{\lambda,b}^{\phi,\Theta}$ satisfies the lower Hölder condition with exponent $\alpha + \varepsilon$, for arbitrarily small $\varepsilon > 0$* (*in particular it is nowhere differentiable*)*. Moreover,*

$$\dim_P(\text{graph } f_{\lambda,b}^{\phi,\Theta}) = \dim_B(\text{graph } f_{\lambda,b}^{\phi,\Theta}) = D \quad \text{almost surely.}$$

An analogous result on the Hausdorff dimension can be obtained with stronger assumptions on the function ϕ, as proved by Hunt [21].

Theorem 5.2 *Let $f_{\lambda,b}^{\phi,\Theta}$ be a function of the form* (1.3)*, where ϕ is a C^∞ function with bounded set of orders of all its critical points. If one considers the phases θ_n as independent random variables with uniform distribution on* $[0, 1]$*, then*

$$\dim_H(\text{graph } f_{\lambda,b}^{\phi,\Theta}) = D \quad \text{almost surely.}$$

This includes the case, when ϕ is real-analytic, in particular (*for* $\phi(x) = \cos(2\pi x)$)*, when $f_{\lambda,b}^{\phi,\Theta}$ is the Weierstrass cosine function with phases θ_n.*

Similar results of that kind were obtained by Romanowska [35], using randomization of the parameter λ.

6 Complex Extension of the Weierstrass Function

It is interesting to notice that if b is an integer, then the Weierstrass cosine function $W_{\lambda,b}$ is the real part of the lacunary (Hadamard gaps) complex power series

$$w(z) = \sum_{n=0}^{\infty} \lambda^n z^{b^n}, \qquad z \in \mathbb{C}, \ |z| \leq 1$$

on the unit circle $\{|z| = 1\}$. In particular, $W_{\lambda,b}$ has a harmonic extension to the unit disc. This approach was already used by Hardy [16] to prove the non-differentiability of $W_{\lambda,b}$ in this case. The study of the boundary behaviour of the holomorphic map w is itself an interesting question. Salem and Zygmund [37] and Kahane, Weiss and Weiss [23] proved that for given b, if λ is sufficiently close to 1, then the image of the unit circle under w is a Peano curve, i.e. it covers an open subset of the plane. In [2], the author showed that in this case the box dimension of the graph of the function

$$x \mapsto \left(\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x), \ \sum_{n=0}^{\infty} \lambda^n \sin(2\pi b^n x) \right)$$

(which is a subset of $\mathbb{R}^3$), is equal to $3 - 2\alpha$. Moreover, the author [3] (see also Belov [6]) showed that for given b, if λ is sufficiently close to 1, then the map w does not preserve (forwardly) Borel sets on the unit circle. The boundary behaviour of w from a topological point of view was studied by Dong, Lau and Liu [13] (Fig. 3).

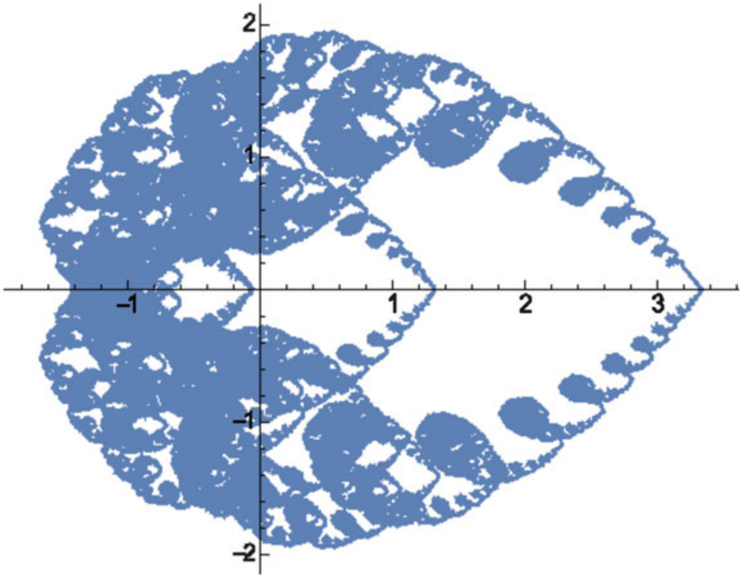

Fig. 3 The image of the unit circle under the map w

7 Other Sequences of Scalings

It is natural to study a generalization of functions of the forms (1.1) and (1.3), replacing λ^n, b^n by another sequences of scales λ_n, b_n, which are not exponential. More precisely, one can consider functions of the form

$$f(x) = \sum_{n=0}^{\infty} \lambda_n \phi(b_n x + \theta_n) \tag{7.1}$$

for $\lambda_n, b_n > 0$, $\sum_{n=0}^{\infty} \lambda_n < \infty$, $b_{n+1} > b_n$, $b_n \to \infty$ and a non-constant, $\mathbb{Z}$-periodic, Lipschitz continuous, piecewise C^1 function $\phi : \mathbb{R} \to \mathbb{R}$.

It turns out that the case of rapidly (faster than exponential) growing scales $1/\lambda_n$, b_n is easier to handle than the exponential one. In 1937, Besicovitch and Ursell [8] considered this case rather than the exponential one, and showed that for the sawtooth function $\phi(x) = \text{dist}(x, \mathbb{Z})$, if $\lambda_n = b_n^{-\alpha}$ for some $\alpha \in (0, 1)$ and b_{n+1}/b_n tends to ∞ sufficiently slowly, then $\dim_H(\text{graph } f) = 2 - \alpha$. The Hausdorff, upper and lower box dimension of the graphs of functions of the general form (7.1) for rapidly growing scales $1/\lambda_n$, b_n was computed by Carvalho [10] and the author [4]. More precisely, the following result was proved in [4].

Theorem 7.1 *For every function f of the form (7.1), if $\lambda_{n+1}/\lambda_n \to 0$, $b_{n+1}/b_n \to \infty$ as $n \to \infty$, then*

$$\dim_H(\text{graph } f) = \underline{\dim}_B(\text{graph } f) = 1 + \liminf_{n\to\infty} \frac{\log^+ d_n}{\log(b_{n+1} d_n / d_{n+1})},$$

$$\overline{\dim}_B(\text{graph } f) = 1 + \limsup_{n\to\infty} \frac{\log^+ d_n}{\log b_n},$$

where $\log^+ = \max(\log, 0)$ *and* $d_n = \lambda_1 b_1 + \cdots + \lambda_n b_n$.

If additionally, $\lambda_n = b_n^{-\alpha}$ *for some* $\alpha \in (0, 1)$ *and* $\log b_{n+1}/\log b_n \to 1$, *then*

$$\dim_H(\text{graph } f) = \dim_P(\text{graph } f) = \dim_B(\text{graph } f) = 2 - \alpha.$$

In particular, this shows that in the case of rapidly growing scales, the dimensions need not coincide. In fact, in [4] it is shown that for every $H, B \in [1, 2]$ with $H \le B$ one can find a function f satisfying the assumptions of Theorem 7.1, such that

$$\dim_H(\text{graph } f) = \underline{\dim}_B(\text{graph } f) = H, \qquad \overline{\dim}_B(\text{graph } f) = B.$$

The case when the scales $1/\lambda_n$, b_n grow slower than exponentially is much more difficult and almost nothing is known about the dimension of the graph of f. Some work was done for

$$R(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(2\pi n^2 x)$$

(called the *Riemann example*) and similar functions. In particular, Chamizo [11] determined the box dimension of the graph of R to be equal to $5/4$.

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