Inverse Problems in Multifractal Analysis

Julien Barral

Abstract We present recent results regarding the construction of positive measures with a prescribed multifractal nature, as well as their counterpart in multifractal analysis of Hölder continuous functions.

Keywords Multifractal formalism • Multifractal analysis • Hausdorff dimension • Packing dimension • Large deviations • Inverse problems

1 Foreword

This contribution deals with questions different from those considered by the author in his talk, which concerned joint work in progress with Stéphane Seuret on the multifractal nature of Choquet capacities obtained from Gibbs measures via percolation. The results presented here concern the construction of measures and functions with prescribed multifractal nature. Results for measures, the related comments and the sketch of proof given in Sect. 2 are extracted from [3]. Their application to multifractal analysis of functions constitutes the original part of the present paper, developed in Sect. 3.

J. Barral (✉)

Laboratoire d'Analyse, Géométrie et Applications (UMR 7539), Institut Galilée, Université Paris 13, Sorbonne-Paris-Cité, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France
e-mail: barral@math.univ-paris13.fr

2 Inverse Problems in Multifractal Analysis of Measures

2.1 Generalities About Multifractal Analysis

Let $\mathcal{M}_c^+(\mathbb{R}^d)$ stand for the set of compactly supported Borel positive and finite measures on $\mathbb{R}^d$ ($d \geq 1$). The upper box dimension of a bounded set $E \subset \mathbb{R}^d$ will be denoted $\overline{\dim}_B E$, and its Hausdorff and packing dimensions will be denoted by $\dim_H E$ and $\dim_P E$ respectively (see [20, 44, 52] for definitions).

Multifractal analysis is designed to finely describe geometrically the heterogeneity in the distribution at small scales of the elements of $\mathcal{M}_c^+(\mathbb{R}^d)$. If $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, this heterogeneity can be described via the lower and upper local dimensions of μ, namely

$$\underline{d}(\mu, x) = \liminf_{r \to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)} \quad \text{and} \quad \overline{d}(\mu, x) = \limsup_{r \to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)},$$

and the level sets

$$\underline{E}(\mu, \alpha) = \left\{ x \in \text{supp}(\mu) : \underline{d}(\mu, x) = \alpha \right\}, \quad \overline{E}(\mu, \alpha) = \left\{ x \in \text{supp}(\mu) : \overline{d}(\mu, x) = \alpha \right\},$$

and

$$E(\mu, \alpha) = \underline{E}(\mu, \alpha) \cap \overline{E}(\mu, \alpha) \quad (\alpha \in \mathbb{R} \cup \{\infty\}),$$

where $B(x, r)$ and $\text{supp}(\mu)$ stand for the closed ball of radius $r > 0$ centered at x and the topological support of μ respectively.

The *lower Hausdorff spectrum* of μ is the mapping defined as

$$f_{-\mu}^H : \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H \underline{E}(\mu, \alpha),$$

with the convention that $\dim_H \emptyset = -\infty$, so that $f_{-\mu}^H(\alpha) = -\infty$ if $\alpha < 0$. This spectrum provides a geometric hierarchy between the sets $\underline{E}(\mu, \alpha)$, which partition $\text{supp}(\mu)$. Here, the lower local dimension is emphasized as it provides at any point the best pointwise Hölder control one can have on the measure μ at small scales. However, the upper local dimension is of course of interest, and much attention is paid in general, especially in ergodic theory, to the sets $E(\mu, \alpha)$ of points at which one has an exact local dimension. The *Hausdorff spectrum* of μ is the mapping defined as

$$f_\mu^H : \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H E(\mu, \alpha).$$

After basic observations made by physicists [26, 27], mathematicians derived, and in many cases justified, the heuristic according to which, for a measure

possessing a self-conformal like property, f_μ^H should be the Legendre transform of a kind of free energy function, called the L^q-spectrum. This led to an abundant literature on what has become called multifractal formalism (see e.g. [12, 13, 39, 48, 49, 52]).

To be more specific we need some definitions. Given $I \in \{\mathbb{R}, \mathbb{R} \cup \{\infty\}\}$ and $f : I \to \mathbb{R} \cup \{-\infty\}$, the domain of f is defined as $\mathrm{dom}(f) = \{x \in I : f(x) > -\infty\}$. For $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, if $\mathrm{dom}(\tau) \neq \emptyset$, the concave Legendre-Fenchel transform of τ is the upper-semi continuous concave function defined as $\tau^* : \alpha \in \mathbb{R} \mapsto \inf\{\alpha q - \tau(q) : q \in \mathrm{dom}(\tau)\}$ (see [54]). If, moreover, $0 \in \mathrm{dom}(\tau)$, we define its (extended) concave Legendre-Fenchel transform as

$$\tau^* : \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \begin{cases} \inf\{\alpha q - \tau(q) : q \in \mathrm{dom}(\tau)\} & \text{if } \alpha \in \mathbb{R}, \\ \inf\{\alpha q - \tau(q) : q \in \mathrm{dom}(\tau) \cap \mathbb{R}_-\} & \text{if } \alpha = \infty, \end{cases}$$

with the conventions $\infty \times q = -\infty$ if $q < 0$ and $\infty \times 0 = 0$. Consequently, $\infty \in \mathrm{dom}(\tau^*)$ if and only if $0 = \min(\mathrm{dom}(\tau))$, and in this case $\tau^*(\infty) = -\tau(0) = \max(\tau^*)$. In any case, τ^* is upper semi-continuous over $\mathrm{dom}(\tau^*)$, and concave over the interval $\mathrm{dom}(\tau^*) \setminus \{\infty\}$ (here the notion of upper semi-continuous function is relative to $\mathbb{R} \cup \{\infty\}$ endowed with the topology generated by the open subsets of $\mathbb{R}$ and the sets $(\alpha, \infty) \cup \{\infty\}, \alpha \in \mathbb{R}$).

Now, define the L^q-spectrum of $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$ as

$$\tau_\mu : q \in \mathbb{R} \mapsto \liminf_{r \to 0^+} \frac{\log \sup \left\{ \sum_i \mu(B(x_i, r))^q \right\}}{\log(r)},$$

where the supremum is taken over all the centered packings of $\mathrm{supp}(\mu)$ by closed balls of radius r. The following properties are standard and proved for instance in [39].

Proposition 2.1 *Let $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$.*

1. *τ_μ is concave and non-decreasing; $\tau_\mu(1) = 0$, $-d \leq \tau_\mu(0) = -\overline{\dim}_B \mathrm{supp}(\mu) \leq 0$.*
2. *Either $\mathrm{dom}(\tau_\mu) = \mathbb{R}$, or $\mathrm{dom}(\tau_\mu) = \mathbb{R}_+$, according to whether the exponent $\limsup_{r \to 0^+} \dfrac{\log(\inf\{\mu(B(x, r)) : x \in \mathrm{supp}(\mu)\})}{\log(r)}$ is finite or not. Moreover τ_μ^* is non-negative on its domain, which is a closed subinterval of $\mathbb{R}_+ \cup \{\infty\}$.*

For $\alpha \in \mathbb{R}$ we always have (see [39, Section 3] or [49, Section 2.7])

$$f_\mu^H(\alpha) \leq \underline{f}_{-\mu}^H(\alpha) \leq \tau_\mu^*(\alpha) \leq \max(\alpha, -\tau_\mu(0)) \leq \max(\alpha, d); \qquad (2.1)$$

we also have

$$f_\mu^H(\infty) \leq \tau_\mu^*(\infty)$$

(see [3]), a dimension equal to $-\infty$ meaning that the set is empty. Notice that due to (2.1), if $f_\mu^H(\alpha) \geq \alpha$ at some α, then $0 \leq \alpha \leq d$ and $f_\mu^H(\alpha) = \tau_\mu^*(\alpha) = \alpha$, so that α is a fixed point of τ_μ^*. Moreover, since $\tau_\mu(1) = 0$ and τ_μ is concave, the set of fixed points of τ_μ^* is the interval $[\tau_\mu'(1^+), \tau_\mu'(1^-)]$.

We say that μ obeys the multifractal formalism at $\alpha \in \mathbb{R} \cup \{\infty\}$ if $\underline{f}_{-\mu}^H(\alpha) = \tau_\mu^*(\alpha)$, and that the multifractal formalism holds (globally) for μ if it holds at all $\alpha \in \mathbb{R} \cup \{\infty\}$. If $\underline{f}_{-\mu}^H(\alpha)$ can be replaced by $f_\mu^H(\alpha)$ in the previous definition, we say that the multifractal formalism holds strongly. In this case one has

$$\dim_H E(\mu, \alpha) = \dim_P E(\mu, \alpha) = \dim_H \underline{E}(\mu, \alpha) = \dim_H \overline{E}(\mu, \alpha) = \tau_\mu^*(\alpha).$$

The multifractal formalism turns out to hold globally, or on some non-trivial subinterval of $\mathrm{dom}(\tau_\mu^*)$, for some important classes of continuous measures, namely some classes of self-conformal measures (including certain Bernoulli convolutions), Gibbs and weak Gibbs measures on hyperbolic dynamical systems (see e.g. [13, 16, 21–25, 39, 42, 52, 53] and [3] for more references), and scale invariant limits of certain multiplicative chaos [1, 6, 17, 29, 46]; in these cases it also holds strongly. It also holds for some natural classes of discrete measures (see e.g. [2, 9, 34, 51] as well as references in [3]). Other examples are special self-affine or Gibbs measures on self-affine Sierpinski carpets [4, 7, 38, 50], or on almost all the attractors of IFS associated with certain families of $d \times d$ invertible matrices with small enough singular values [5, 18, 19], as well as generic probability measures on a compact subset of $\mathbb{R}^d$ [11, 14].

The measures mentioned above share the geometric property of being exact dimensional, i.e. for such a measure μ, there exists $D \in [0, d]$ such that $\underline{d}(\mu, x) = \overline{d}(\mu, x) = D$, μ-almost everywhere. This implies $D \in [\tau_\mu'(1^+), \tau_\mu'(1^-)]$ and μ strongly obeys the multifractal formalism at D. In fact, for any $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, for μ-almost every x one has $\tau_\mu'(1^+) \leq \underline{d}(\mu, x) \leq \overline{d}(\mu, x) \leq \tau_\mu'(1^-)$ ([48]), and for most of the continuous measures mentioned above, $\tau_\mu'(1)$ exists, hence equals D; also, τ_μ is piecewise C^1, and even analytic in certain cases, a typical example being Gibbs measures associated with Hölder potentials on repellers of $C^{1+\alpha}$ conformal mappings. Another property of these measures is that, when they obey the multifractal formalism globally, they are homogeneously multifractal (HM), in the sense that the lower Hausdorff spectrum of the restriction of μ to any closed ball whose interior intersects $\mathrm{supp}(\mu)$ is equal to the lower Hausdorff spectrum of μ.

2.2 Full Illustration of the Multifractal Formalism

Theorem 2.2 ([3]) *Let $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a concave function satisfying the necessary properties (see Proposition 2.1) to be the L^q-spectrum of some element of $\mathcal{M}_c^+(\mathbb{R}^d)$. Let $D \in [\tau'(1^+), \tau'(1^-)]$. There exists an (HM) measure $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$,*

exact dimensional with dimension D, and which strongly satisfies the multifractal formalism with $\tau_\mu = \tau$.

Remark 1 In [3] we develop much more general results by using a finer multifractal formalism to prescribe and distinguish Hausdorff and packing dimensions of the level sets $\{x \in \text{supp}(\mu) : \underline{d}(\mu, x) = \alpha, \ \overline{d}(\mu, x) = \beta\}$, $(\alpha \leq \beta \leq \infty)$. The connection with Olsen's multifractal formalism [49] is also studied.

It is interesting to complete this statement by describing the possible behaviors of (τ_μ, τ_μ^*) (see Figs. 1–6). For this we need to extend the notion of Legendre-Fentchel transform to functions $f : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{-\infty\}$: for such an f, if $\text{dom}(f) \cap \mathbb{R} \neq \emptyset$, we define the concave Legendre-Fenchel transform of f as

$$f^* : q \in \mathbb{R} \mapsto \inf\{q\alpha - f(\alpha) : \alpha \in \text{dom}(f)\},$$

with the conventions $q \times \infty = \text{sign}(q) \times \infty$ if $q \neq 0$ and $0 \times \infty = 0$. Consequently, if $\infty \in \text{dom}(f)$ and f is bounded from above, then $0 = \min(\text{dom}(f^*))$ and $f^*(0) = -\max(\sup(f)_{|\mathbb{R}}, f(\infty))$; moreover, f^* is concave over $\text{dom}(f^*)$, upper semi-continuous over $\text{dom}(f^*) \setminus \{0\}$, and upper semi-continuous at 0 only if and only if $f(\infty) = \max(f)$.

Proposition 2.3 ([3, 39]) *Suppose that $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfies the properties of the L^q-spectrum described in Proposition 2.1. One has $(\tau^*)^* = \tau$ on $\mathbb{R}$, and:*

1. *If $\text{dom}(\tau) = \mathbb{R}$, then $\text{dom}(\tau^*)$ is the compact interval $I = [\tau'(\infty), \tau'(-\infty)]$, τ^* is concave and continuous on its domain.*
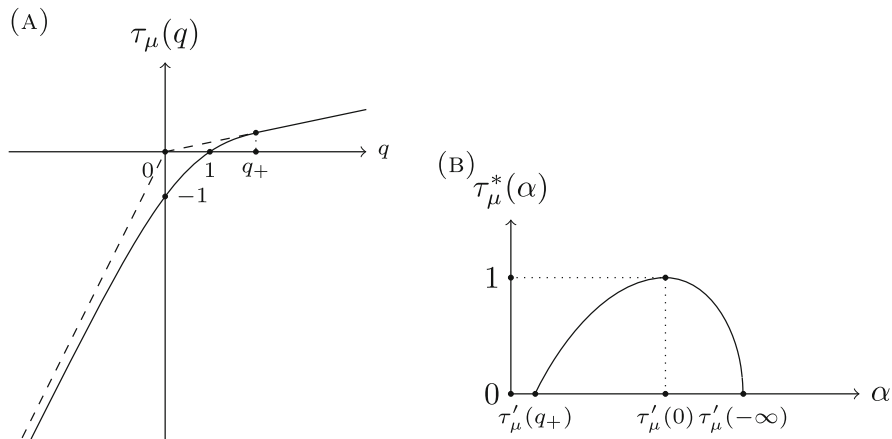

Fig. 1 Illustration of Proposition 2.3.1. when the domain of τ_μ is a non trivial interval and τ_μ is differentiable, with a second order phase transition at some $q_+ > 1$.
The case of a trivial interval $\{\alpha_0\}$ would correspond to a monofractal measure with $0 \leq \alpha_0 \leq d$, $\tau_\mu(q) = \alpha_0(q-1)$ for all $q \in \mathbb{R}$, $\tau_\mu^*(\alpha) = \alpha$ if $\alpha = \alpha_0$ and $\tau_\mu^*(\alpha) = -\infty$ otherwise. (**a**) The L^q spectrum of μ. (**b**) Its Legendre transform

(A)

$\tau_\mu(q)$

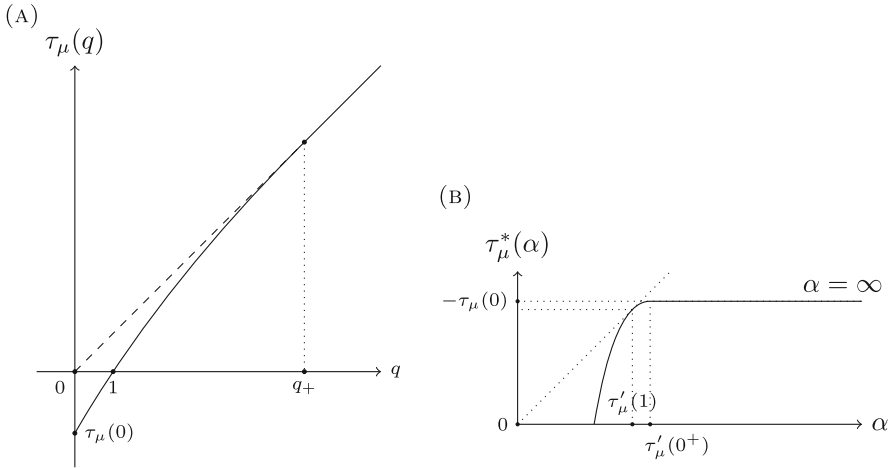(B)

$\tau_\mu^*(\alpha)$

Fig. 2 Illustration of Proposition 2.3.2(b) when $\tau_\mu'(0^+) < \infty$, τ_μ is differentiable, and it has a second order phase transition at some $q_+ > 1$. (**a**) The L^q spectrum of μ. (**b**) Its Legendre transform

(A)
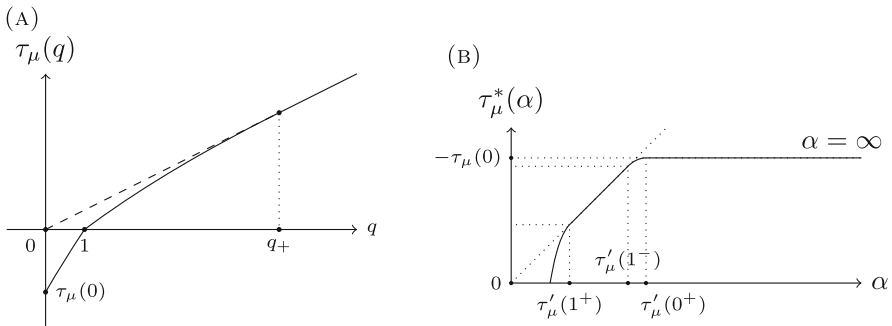
$\tau_\mu(q)$

(B)

$\tau_\mu^*(\alpha)$

Fig. 3 Illustration of Proposition 2.3.2(b) when $\tau_\mu'(0^+) < \infty$, τ_μ is not differentiable at 1, and it has a second order phase transition at some $q_+ > 1$. (**a**) The L^q spectrum of μ. (**b**) Its Legendre transform

2. If $dom(\tau) = \mathbb{R}_+$, then $\infty \in dom(\tau^*)$ with $\tau^*(\infty) = -\tau(0)$ and:

   (a) If $\tau(0) = 0$ then $\tau = 0$ over $\mathbb{R}_+$, $dom(\tau^*) = \mathbb{R}_+ \cup \{\infty\}$ and $\tau^* = 0$ over $\mathbb{R}_+ \cup \{\infty\}$.

   (b) If $\tau(0) < 0$ and τ is continuous at 0^+, then $dom(\tau^*) = [\tau'(\infty), \infty]$, τ^* is concave, continuous, and increasing over $[\tau'(\infty), \tau'(0^+))$, $\tau^*(\alpha) = -\tau(0) = \tau^*(\infty) = -\tau(0)$ for all $\alpha \in [\tau'(0^+), \infty)$ and τ^* is continuous at ∞; there are two distinct behaviors according to whether $\tau'(0^+) < \infty$ or not.

   (c) If $\tau(0) < 0$ and τ is discontinuous at 0^+, then $dom(\tau^*) = [\tau'(\infty), \infty]$. Moreover, for all $\alpha \in [\lim_{q\to 0+} \tau'(q^-), \infty)$ one has $\tau^*(\alpha) = -\tau(0^+) < \tau^*(\infty) = -\tau(0)$, so that τ^* is concave and continuous on $[\tau'(\infty), \infty)$ and
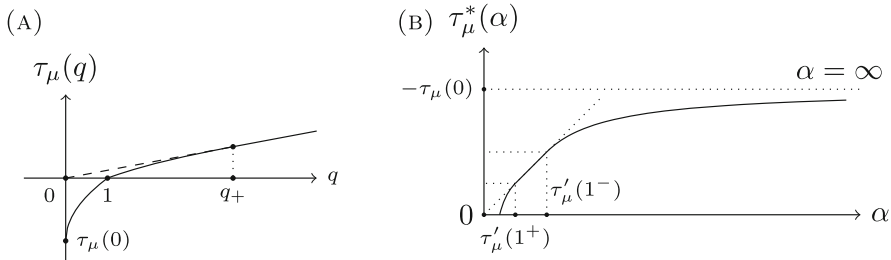
Fig. 4 Illustration of Proposition 2.3.2(b) when $\tau_\mu'(0^+) = \infty$, τ_μ is not differentiable at 1, and it has a second order phase transition at some $q_+ > 1$. (**a**) The L^q spectrum of μ. (**b**) Its Legendre transform
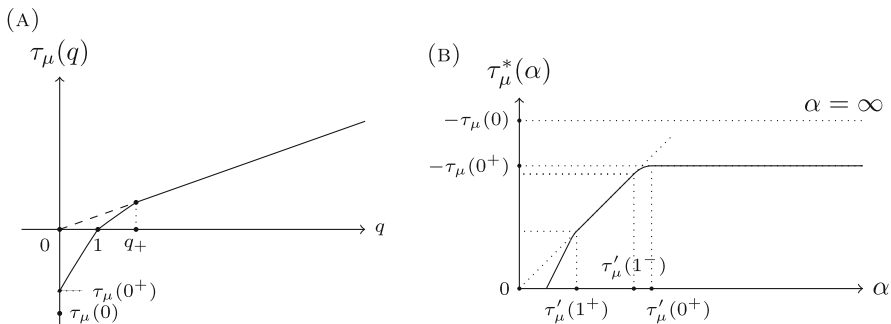

Fig. 5 Illustration of Proposition 2.3.2(c) when $\tau_\mu'(0^+) < \infty$, τ_μ is not differentiable at 1, and it has another first order phase transition at some $q_+ > 1$. (**a**) The L^q spectrum of μ. (**b**) Its Legendre transform
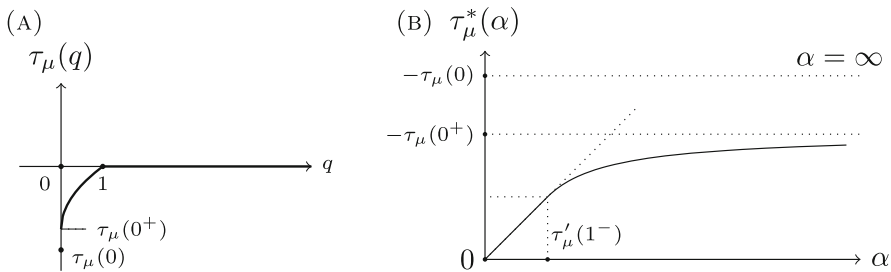

Fig. 6 Illustration of Proposition 2.3.2(c) when $\tau_\mu'(0^+) = \infty$, τ_μ is not differentiable at 1 and $\tau_\mu'(1^+)$ takes the minimal value 0. (**a**) The L^q spectrum of μ. (**b**) Its Legendre transform

discontinuous at ∞ (there are also two cases, according to $\lim_{q\to 0+} \tau'(q^-)$ equals ∞ or not).

Remark 2 The behavior described in Proposition 2.3.1 is illustrated, for instance, by (weak) Gibbs measures on conformal repellers [25, 49, 52]. The behaviors described by Proposition 2.3.2(b) are illustrated by some Gibbs measures on countable

Markov shifts and their geometric realizations [30, 42, 43], which also obey the multifractal formalism, though in [30, 43] the set $E(\mu, \infty)$ is not studied. The fact that the behaviors described in Proposition 2.3.2(a) and (c) be illustrated by measures obeying the mutifractal formalism seems to be new.

Remark 3 In [28], when $d = 1$, for each $D \in (0, 1)$ one finds an exact dimensional measure μ with dimension D and L^q-spectrum equal to $\min(q - 1, 0)$ over $\mathbb{R}_+$. It is also worth mentioning that in [10] one finds examples of inhomogeneous Bernoulli measures over $[0, 1]$ with an L^q-spectrum presenting countably many points of non-differentiability over $[1, +\infty)$.

2.3 Measures with prescribed lower Hausdorff spectrum

In general, $\mathrm{dom}(f_{-\mu}^H) = \{\alpha \in \mathbb{R} \cup \{\infty\} : \underline{E}(\mu, \alpha) \neq \emptyset\}$ is not a closed subinterval of $[0, \infty]$, and even when it is the case, the restriction of $f_{-\mu}^H$ to $\mathrm{dom}(f_{-\mu}^H) \cap \mathbb{R}_+$ is not necessarily concave. Consequently, it is also natural to study the inverse problem consisting of associating to a function $f : \mathbb{R} \cup \{\infty\} \to [0, d] \cup \{-\infty\}$ whose domain is a subset of $\mathbb{R}_+ \cup \{\infty\}$ and such that $f(\alpha) \leq \alpha$ for all $\alpha \geq 0$, an (HM) measure whose lower Hausdorff spectrum is equal to f. In [3] we construct such a measure μ, exact dimensional, when f shares important properties with τ_μ^*; specifically, f is taken in the family:

$$\mathcal{F}(d) = \left\{ f : \mathbb{R} \cup \{\infty\} \to [0, d] \cup \{-\infty\} : \begin{cases} \mathrm{dom}(f) \text{ is a closed subset of } [0, \infty] \\ f \text{ is u.s.c., } \mathrm{Fix}(f) \neq \emptyset \\ f(\alpha) \leq \alpha \text{ for all } \alpha \in \mathrm{dom}(f) \end{cases} \right\},$$

where $\mathrm{Fix}(f)$ ($\subset [0, d]$) stands for the set of fixed points of f.

Theorem 2.4 ([3]) *Let $f \in \mathcal{F}(d)$. For each $D \in \mathrm{Fix}(f)$, there exists an (HM) measure $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, exact dimensional with dimension D, such that $f_{-\mu}^H = f$.*

Remark 4

(1) The measures constructed in the proofs of Theorems 2.2 and 2.4 are continuous and supported on Cantor sets.
(2) Our approach does not make it possible to replace $f_{-\mu}^H = f$ by $f_\mu^H = f$ in the previous statement unless $\mathrm{dom}(f) = \mathrm{Fix}(f)$ or $\mathrm{dom}(f)$ is an interval and f is concave over $\mathrm{dom}(f) \cap \mathbb{R}_+$. It turns out that the proof given in [3] can be slightly improved so that μ is absolutely continuous with respect to Lebesgue measure when $D = d$.

Remark 5 (Related result by Z. Buczolich and S. Seuret) The prescription of the lower Hausdorff spectrum has also been studied in [15]. The authors work on $\mathbb{R}$ and construct (HM) continuous measures, not exact dimensional, but with upper Hausdorff dimension equal to 1, and whose support is equal to $[0, 1]$. Moreover, the lower Hausdorff spectrum is prescribed in the class $\mathcal{F}$ of functions $f : \mathbb{R}_+ \to [0, 1] \cup \{-\infty\}$ which satisfy: $f(1) = 1$, $\mathrm{dom}(f)$ is a closed subinterval of $[0, 1]$ of the form $[\alpha, 1]$ such that $\alpha > 0$, and $f_{|[\alpha,1)} = \max(g_{|[\alpha,1)}, 0)$, where the function g has the following properties: (i) g is the supremum of a sequence of functions $(g_n)_{n \geq 1}$, such that each g_n is constant over its domain supposed to be a closed subinterval of $[0, 1]$ and $g_n(\beta) \leq \beta$ for all $\beta \in [0, 1]$; (ii) $[\alpha, 1]$ is the smallest closed interval containing the support of g.

It is also shown that for an (HM) measure to be supported by the whole interval $[0, 1]$, it is necessary that the support of its lower Hausdorff spectrum contains an interval of the form $[\alpha, 1]$, $(0 \leq \alpha \leq 1)$.

The authors also study the case of non-(HM) measures. They construct measures that are non exact dimensional with upper Hausdorff dimension 1 whose support is equal to $[0, 1]$, with a prescribed lower Hausdorff spectrum in the broader class $\tilde{\mathcal{F}}$ of functions f which satisfy that $f(1) = 1$, $0 < \inf(\mathrm{dom}(f))$, and $f_{|\mathrm{dom}(f)\setminus\{1\}} = g_{|\mathrm{dom}(f)\setminus\{1\}}$, where g satisfies property (i). This includes all such functions f for which g is lower semi-continuous. Simultaneously, they also construct a non-(HM) measure with lower Hausdorff spectrum given by g.

Remark 6 The spectra previously defined make sense if measures are replaced by non-negative functions of subsets of $\mathbb{R}^d$ to which a notion of support is associated. This is the case for instance for Choquet capacities. In [40], the prescription of $\alpha \mapsto \dim_H E(C, \alpha)$ is studied, where C is a (HM) Choquet capacity on subsets of $[0, 1]$ but not a positive measure, which makes the situation easier to study; spectra are prescribed in a broader class than $\tilde{F}$, but defined in a similar spirit.

In [41], one finds non-(HM) non-negative functions C of subsets of $[0, 1]$, which are not measures, for which the spectrum $\alpha \mapsto \lim_{\epsilon \to 0+} \dim_H \bigcup_{s>0} \bigcap_{0<r<s}\{x \in \mathrm{supp}(C) : r^{\alpha+\epsilon} \leq C(B(x, r)) \leq r^{\alpha-\epsilon}\}$ is prescribed in the class of upper semi-continuous functions $f : \mathbb{R}_+ \mapsto [0, 1] \cup \{-\infty\}$ with non-empty compact domain.

2.4 Outline of the Proof of Theorem 2.4

Let us sketch the main ideas leading to the construction of the measure μ provided by Theorem 2.4. To establish Theorem 2.2 one must improve this approach in order to control both the finer level sets $E(\mu, \alpha)$ and the upper large deviations spectrum $\bar{f}_\mu^{\mathrm{LD}}$ of μ when $f = \tau_\mu^*$, and the relation $\tau_\mu = \bar{f}_\mu^{\mathrm{LD}*}$.

For simplicity, we assume that $\mathrm{dom}(f)$ is a non-trivial interval $[\alpha_{\min}, \alpha_{\max}] \subset \mathbb{R}_+$, f is continuous over $[\alpha_{\min}, \alpha_{\max}]$, $0 \leq f(\alpha) \leq \min(\alpha, d)$ over $[\alpha_{\min}, \alpha_{\max}]$, and $f(D) = D$ for a unique point D in $[\alpha_{\min}, \alpha_{\max}]$. The homogeneity of the construction of the measure μ automatically implies that the measure is (HM).

At first one shows (independently of f) that for each $\gamma \in [0, d]$ and $\alpha \geq \gamma$, one can find two Borel probability measures $\mu_{\alpha,\gamma}$ and $\nu_{\alpha,\gamma}$ supported on $[0, 1]^d$ such that $\mu_{\gamma,\gamma} = \nu_{\gamma,\gamma}$, $\nu_{\alpha,\gamma}$ is exact dimensional with dimension γ, and $\nu_{\alpha,\gamma}$ is concentrated on $E(\mu_{\alpha,\gamma}, \alpha)$, as well as on the set defined similarly but with $\alpha(\mu, x)$ replaced by $\lim_{n \to \infty} \frac{\log(\mu(I_n(x)))}{-n \log(2)}$, where $I_n(x)$ stands for the closure of dyadic cube semi-open to the right containing x.

Set $A_1 = \{\alpha_1 = D\}$, and for each integer $m \geq 1$, define $A_{m+1} = A_m \cup \{\alpha_{m+1}\}$, where $\alpha_{m+1} \in [\alpha_{\min}, \alpha_{\max}] \setminus A_m$, in such a way that the set $\{\alpha_m : m \geq 1\}$ is dense in $[\alpha_{\min}, \alpha_{\max}]$. By using the previous property with $\gamma = f(\alpha)$, for all $m \geq 1$ one gets an integer n_m such that for all $\alpha \in A_m$, for all $n \geq n_m$, there is a collection $G_{m,n}(\alpha)$ of about $2^{nf(\alpha)}$ dyadic subcubes of $[0, 1]^d$ such that for all $I \in G_{m,n}(\alpha)$ one has $\mu_{\alpha, f(\alpha)}(I) \approx 2^{-n\alpha}$, $\nu_{\alpha, f(\alpha)}(I) \approx 2^{-nf(\alpha)}$, and $\sum_{I \in G_{m,n}(\alpha)} \nu_{\alpha, f(\alpha)}(I) \in [1/2, 1]$.

For every integer $m \geq 2$, one considers m dyadic closed subcubes $L_{\alpha_1}, \ldots, L_{\alpha_m}$ of $[0, 1]^d$, of the same generation n'_m, so that the $2^{-n'_m/5}$ neighborhood of each L_{α_i} does not intersect any of the other L_{α_j}.

The measure μ is constructed on a Cantor set $K = \bigcap_{m \geq 1} \bigcup_{I \in \mathbf{G}_m}$, where the $\mathbf{G}_m$ are families of closed dyadic subcubes of $[0, 1]^d$ of generation g_m tending to ∞ as $m \to \infty$, constructed recursively according to a scheme roughly as follows:

One obtains $\mathbf{G}_1$ by considering the measure $\mu_{\alpha_1, f(\alpha_1)} = \mu_{D,D}$, an integer $N_1 \geq n_1$ much bigger than n'_2 and setting $\mathbf{G}_1 = G_{1,N_1}(\alpha_1) = G_{1,N_1}(D)$. This yields the probability measure μ_1 defined on $\mathbf{G}_1$ as

$$\mu_1(I) = \frac{\mu_{D,D}(I)}{\sum_{I' \in \mathbf{G}_1} \mu_{D,D}(I')}.$$

This measure satisfies $\mu_1(I) \approx 2^{-N_1 D}$. Suppose now that the set $\mathbf{G}_m$ has been constructed, as well as a probability measure μ_m on its elements. One takes $N_{m+1} \geq n_{m+1}$ an integer much bigger than $\max(g_m, n'_{m+2})$, and for each $1 \leq i \leq m + 1$, one considers the measure $\mu_{\alpha_i, f(\alpha_i)}$ and the associated set $G_{m+1}(\alpha_i) := G_{m+1, N_{m+1}}(\alpha_i)$. For each $1 \leq i \leq m + 1$ and $I_m \in \mathbf{G}_m$, one defines the set of the elements of $\mathbf{G}_{m+1}$ contained in I_m as $\bigcup_{i=1}^{m+1} \mathbf{G}_{m+1}(I_m, \alpha_i)$, where $\mathbf{G}_{m+1}(I_m, \alpha_i) = \{I_m \cdot L_{\alpha_i} \cdot I : I \in G_{m+1}(\alpha_i)\}$, and the concatenation $J \cdot J'$ of two closed subcubes of $[0, 1]^d$ is obtained as the cube $f_J(J')$, where f_J is the natural contracting similitude mapping $[0, 1]^d$ onto J (this operation is associative). One gets a probability measure μ_{m+1} on $\mathbf{G}_{m+1}$ by setting, for $I \in G_{m+1}(\alpha_i)$:

$$\mu_{m+1}(I_m \cdot L_{\alpha_i} \cdot I) = \mu_m(I_m) \frac{\mu_{\alpha_i, f(\alpha_i)}(I)}{\sum_{\alpha \in A_{m+1}} \sum_{I' \in G_{m+1}(\alpha)} \mu_{\alpha, f(\alpha)}(I')}. \tag{2.2}$$

This makes it possible to define a Borel probability measure carried on K and coinciding with μ_m over $\mathbf{G}_m$ for all $m \geq 1$.

Since $f(\alpha) < \alpha$ except for $\alpha = \alpha_1 = D$, if N_{m+1} is taken big enough, in (2.2) for each $i > 1$ the contribution of the elements of $G_{m+1}(\alpha)$ is roughly $2^{N_{m+1}(f(\alpha) - \alpha)}$ hence is negligible so that the denominator is equivalent to the single contribution

of $\sum_{I' \in G_{m+1}(D)} \mu_{D,D}(I') \in [1/2, 1]$. Consequently, for $I_{m+1} \in \mathbf{G}_{m+1}$ of the form $I_m \cdot L_{\alpha_i} \cdot I, I \in G_{m+1}(\alpha_i)$, we have the following estimate:

$$\mu(I_{m+1}) \approx \mu_m(I_m)\mu_{\alpha_i, f(\alpha_i)}(I) \approx \mu_m(I_m)2^{-\alpha_i N_{m+1}} \approx 2^{-\alpha_i g_{m+1}} \qquad (2.3)$$

because $g_m \ll N_{m+1}$. Also, we have that $\#G_{m+1}(\alpha_i) \approx 2^{f(\alpha_i)N_{m+1}}$, hence

$$\#\{I \in \mathbf{G}_{m+1} : I \in G_{m+1}(I_m, \alpha_i) \text{ with } I_m \in \mathbf{G}_m\}$$
$$= (\#\mathbf{G}_m)(\#G_{m+1}(\alpha_i)) \approx 2^{f(\alpha_i)g_{m+1}},$$

again because $g_m \ll N_{m+1}$. The previous estimate and the continuity of f essentially yield that f is an upper bound for f^H. Combined with (2.3), it shows that at generation $m + 1$, the mass of μ is essentially carried by the intervals $I_m \cdot L_D \cdot I, I \in G_{m+1}(D)$, since we have $1 = \|\mu\| \approx \sum_{i=1}^{m+1} 2^{f(\alpha_i)g_{m+1}} 2^{-\alpha_i g_{m+1}} = \sum_{i=1}^{m+1} 2^{(f(\alpha_i)-\alpha_i)g_{m+1}} \approx 2^{(f(\alpha_1)-\alpha_1)g_{m+1}} = 1$ (recall that $\alpha_1 = f(\alpha_1) = D$). This can be strengthened to show that μ is exact D-dimensional.

Another important fact is the natural existence of a family of auxiliary measures used to find a sharp lower bound for f^H: with each $\hat{\beta} = (\beta_m)_{m\geq 1} \in \prod_{m=1}^{\infty} A_m$ is associated the Cantor subset of K defined as $K_{\hat{\beta}} = \bigcap_{m\geq 1} \bigcup_{I \in \mathbf{G}_{\hat{\beta},m}} I$, where $\mathbf{G}_{\hat{\beta},m}$ is the subset of $\mathbf{G}_m$ obtained by selecting only the intervals of the construction for which one considers the exponent $\beta_i \in A_i$ at step i for all $1 \leq i \leq m$. Using (2.3) and finer properties of the measures $\mu_{\alpha,\gamma}$ one can show that $K_{\hat{\beta}} \subset \underline{E}(\mu, \beta)$, where $\beta = \liminf_{m\to\infty} \beta_m$. Moreover, the measures $\nu_{\beta_m, f(\beta_m)}$ can be used to construct a nice auxiliary probability measure $\nu_{\hat{\beta}}$ carried by $K_{\hat{\beta}}$. At first one defines recursively a sequence of measures $(\nu_{\hat{\beta},m})_{m\geq 1}$ on the atoms of the sets $\mathbf{G}_{\hat{\beta},m}$, $m \geq 1$, as follows: $\nu_{\hat{\beta},1}$ is the restriction of $\nu_{D,D}$ to $\mathbf{G}_{\hat{\beta},1}(= \mathbf{G}_1)$, and assuming that $\nu_{\hat{\beta},m}$ is constructed on $\mathbf{G}_{\hat{\beta},m}$, if $I_m \in \mathbf{G}_{\hat{\beta},m}$, for $I \in G_{m+1}(\beta_{m+1})$ one sets

$$\nu_{\hat{\beta},m+1}(I_m \cdot L_{\beta_{m+1}} \cdot I) = \nu_{\hat{\beta},m}(I_m) \frac{\nu_{\beta_{m+1}, f(\beta_{m+1})}(I)}{\sum_{I' \in G_{m+1}(\beta_{m+1})} \nu_{\beta_{m+1}, f(\beta_{m+1})}(I')}.$$

This yields a Borel probability measure $\nu_{\hat{\beta}}$ supported on $K_{\hat{\beta}}$ such that $\nu_{\hat{\beta}}(I_m \cdot L_{\beta_{m+1}} \cdot I) = \nu_{\hat{\beta},m+1}(I_m \cdot L_{\beta_{m+1}} \cdot I) \approx \nu_{\hat{\beta},m}(I_m)\nu_{\beta_{m+1}, f(\beta_{m+1})}(I)$, so that $\nu_{\hat{\beta}}(I_m \cdot L_{\beta_{m+1}} \cdot I) \approx \nu_{\beta_{m+1}, f(\beta_{m+1})}(I) \approx 2^{-f(\beta_{m+1})g_{m+1}}$ (again since $g_m \ll N_{m+1}$). This can be strengthened to $\dim_H(\nu_{\hat{\beta}}) = \liminf_{m\to\infty} f(\beta_m)$, which yields $\dim_H K_{\hat{\beta}} \geq \liminf_{m\to\infty} f(\beta_m)$ by the mass distribution principle (see [20]). Finally, if $\beta \in [\alpha_{\min}, \alpha_{\max}]$ and $\lim_{m\to\infty} \beta_m = \beta$, we get $\underline{f}^H(\beta) = \dim_H \underline{E}(\mu, \beta) \geq f(\beta)$ by continuity of f.

3 Application to Multifractal Analysis of Hölder Continuous Functions

Multifractal analysis of functions has developed in parallel to multifractal analysis of measures, mainly under the impulse of Frisch and Parisi's note about multifractality in fully developed turbulence [26], and with its own multifractal formalisms [33, 35–37, 47]. These are based on the link between pointwise Hölder regularity and the wavelet expansions of Hölder continuous functions [31].

Theorems 2.2 and 2.4 can be used to construct Hölder continuous wavelet series with prescribed upper semi-continuous lower Hausdorff spectra, and also to give a full illustration of the multifractal formalism for Hölder continuous functions based on the wavelet leaders [36], according to the bridge made in [8] between this formalism and the multifractal formalism for measures. We will restrict ourselves to the case $d = 1$.

To be more specific, recall first that if $F : \mathbb{R} \to \mathbb{R}$ is a bounded Hölder continuous function, for each $x_0 \in \mathbb{R}$, one defines the pointwise Hölder exponent of f at x_0 as

$$h_F(x_0) = \sup\{h \geq 0 : \text{for some polynomial } P,$$

$$|F(x) - P(x - x_0)| = O(|x - x_0|^h) \text{ as } |x - x_0| \to 0\},$$

where $|x - x_0|$ stands for the Euclidean norm of $x - x_0$. This exponent is the counterpart for functions of the lower local dimension for measures.

One usually calls the mapping

$$h \mapsto \dim_H\{x \in \mathbb{R} : h_F(x) = h\} \quad (h \in \mathbb{R} \cup \{\infty\})$$

the singularity spectrum of F (we keep the terminology lower Hausdorff spectrum for a slightly different spectrum defined below). Notice that if f is γ-Hölder, then $\{x \in \mathbb{R} : h_F(x) = h\} = \emptyset$ if $h < \gamma$.

We are going to restrict the study to $[0, 1]$. We fix a wavelet basis $\{\psi_I\}$ (I describing all the dyadic subintervals of $\mathbb{R}$), so that the mother wavelet is in the Schwartz class (see [45, Ch. 3]) and the ψ_I are normalized to have the same supremum norm.

Denoting $\{\lambda_I\}$ the collection of the wavelet coefficients of F in the basis $\{\psi_I\}$, let $L_I = \sup\{|\lambda_{I'}|\}$, the supremum being taken over all the dyadic intervals included either in I or in the two dyadic intervals of the same generation as I neighboring I. Then, let supp(F) be the closed set of those $x \in [0, 1]$ such that $|L(I_n(x))| > 0$ for all $n \geq 1$, where $I_n(x)$ stands for the closure of the unique semi-open to the right dyadic cube of generation n which contains x. According to [36], this set does not depend on ψ; moreover, for $x \in$ supp(F), one has

$$h_F(x) = \liminf_{n \to \infty} \frac{\log_2(|L(I_n(x))|)}{-n}.$$

For $h \in \mathbb{R} \cup \{\infty\}$, we set

$$\underline{E}(F, h) = \{x \in \operatorname{supp}(F) : h_F(x) = h\}.$$

The lower Hausdorff spectrum of F is the mapping

$$\underline{f}_F^H : h \mapsto \dim_H \underline{E}(F, h) \quad (h \in \mathbb{R} \cup \{\infty\}).$$

We say that F is homogeneously multifractal (HM) if for all $h \in \mathbb{R} \cup \{\infty\}$, the Hausdorff dimension of $\underline{E}(F, h) \cap B$ does not depend on the ball B whose interior intersects $\operatorname{supp}(F)$.

A basic idea [8] to relate multifractal analysis of functions to that of measures is to consider wavelet series of the form

$$F_{\mu, \gamma_1, \gamma_2} = \sum_{I \subset [0,1]} |I|^{\gamma_1} \mu(I)^{\gamma_2} \psi_I,$$

where $|I|$ stands for the diameter of I, $\gamma_1 \geq 0$, $\gamma_2 > 0$, $\mu \in \mathcal{M}_c^+(\mathbb{R})$ with $\operatorname{supp}(\mu) \subset [0, 1]$, and

$$\gamma = \gamma_1 + \gamma_2 \liminf_{n \to \infty} \frac{\log_2(\max\{\mu(I) : I \text{ dyadic } \subset [0, 1], |I| = 2^{-n}\})}{-n} > 0,$$

so that the function $F_{\mu, \gamma_1, \gamma_2}$ is β-Hölder continuous for all $0 < \beta < \gamma$. Then, the study achieved in [8] yields

$$\underline{E}(F_{\mu, \gamma_1, \gamma_2}, h) = \underline{E}\left(\mu, \frac{h - \gamma_1}{\gamma_2}\right) \tag{3.1}$$

for all $h \in \mathbb{R} \cup \{\infty\}$, so that any information about the multifractal structure of measures should transfer to a similar one for this class of wavelet series. In particular, it is clear from (3.1) that $\dim_H E(F_{\mu, \gamma_1, \gamma_2}, h) \leq \frac{h - \gamma_1}{\gamma_2}$.

3.1 Prescription of the Lower Hausdorff Spectrum

Theorem 3.1 *Let $f : \mathbb{R}_+ \cup \{\infty\} \to [0, 1] \cup \{-\infty\}$ be upper semi-continuous. Suppose that $\operatorname{dom}(f)$ is a closed subset $\mathcal{I}$ of $[0, \infty]$ such that $0 < \min(\mathcal{I}) < \infty$. There exists an (HM) Hölder continuous function F such that $\underline{f}_F^H = f$.*

Proof For $\lambda > 0$ set $\theta(\lambda) = \sup\{f(h)/\lambda h : h \in \mathcal{I}\}$, with the convention $x/\infty = 0$ for all $x \geq 0$. Since f is upper semi-continuous and bounded over its domain, $\theta(\lambda)$ is reached at some $h < \infty$. Moreover, the mapping θ is continuous, and we have $\theta(1/\min(\mathcal{I})) \leq 1$ by definition of f. Now we distinguish two cases.

If $f \not\equiv 0$ over $\mathcal{I}$, then $\theta(\lambda)$ tends to ∞ as λ tends to 0^+, so the continuity of θ yields $0 < \lambda_0 \le 1/\min(\mathcal{I})$ such that $\theta(\lambda_0) = 1$, hence $f(h) \le \lambda_0 h$ for all $h \in \mathcal{I}$, with equality at some h. Let $\tilde{f} = f(\lambda_0^{-1}\cdot)$. By construction we have $\tilde{f} \in \mathcal{F}(1)$. Put $\tilde{f}$ in Theorem 2.4 to get an (HM) measure in $\mathcal{M}_c^+(\mathbb{R})$ supported on $[0,1]$ whose lower Hausdorff spectrum is given by $\tilde{f}$. Then $F = F_{\mu,0,\lambda_0^{-1}}$ is Hölder continuous and has f as lower Hausdorff spectrum by (3.1).

If $f \equiv 0$ on $\mathcal{I}$, then $\tilde{f} = f(\cdot - \min(\mathcal{I}))$ belongs to $\mathcal{F}(1)$ (with 0 as unique fixed point). Put $\tilde{f}$ in Theorem 2.4 to get an (HM) measure in $\mathcal{M}_c^+(\mathbb{R})$ supported on $[0,1]$ whose lower Hausdorff spectrum is given by $\tilde{f}$. Then $F = F_{\mu,\min(\mathcal{I}),1}$ is Hölder continuous and has f as lower Hausdorff spectrum.

Remark 7 In [15], the measures described in Remark 5(1) are used to construct (HM) functions of the form $F = F_{\mu,\gamma_1,\gamma_2}$ with supp$(F) = [0,1]$. Previously in [32], S. Jaffard constructed non-(HM) wavelet series with prescribed spectrum in the class of functions $f : (0,\infty) \to [0,1]$ which are representable as the supremum of a countable collection of step functions.

3.2   Full Illustration of the Multifractal Formalism

Our results also yield a full illustration of the multifractal formalism for Hölder continuous functions whose support is a subset of $[0,1]$. This requires some preliminary definitions and facts.

If $F = \sum_I \lambda_I \psi_I$ is a non-trivial such function, i.e. $\emptyset \ne$ supp$(F) \subset [0,1]$, denote by $T(q)$ the L^q-spectrum associated with the wavelet leaders $(L_I)_{I \subset [0,1]}$, i.e. the concave non-decreasing function

$$T_F(q) = \liminf_{n \to \infty} \frac{-1}{n} \log_2 \sum_{I \in G_n^*} L_I^q \quad (q \in \mathbb{R}),$$

where G_n^* stands for the set of dyadic cubes I of generation n for which $L_I > 0$. Due to [36] again, this function does not depend on the choice of $\{\psi_I\}$ if the mother wavelet is in the Schwartz class. Moreover, if F takes the form $F_{\mu,\gamma_1,\gamma_2}$, one has almost immediately

$$\tau_F(q) = \tau_\mu(\gamma_2 q) - \gamma_1 q \quad (q \in \mathbb{R}). \tag{3.2}$$

From now on we discard the trivial case of $\lim_{n\to\infty} \frac{\log_2(\max\{L_I:I\in G_n^*\})}{-n} = \infty$, so that $T_F = -\overline{\dim}_B \text{supp}(F)\mathbf{1}_{\{0\}} + (-\infty)\mathbf{1}_{\mathbb{R}_-^*} + (\infty)\mathbf{1}_{\mathbb{R}_+^*}$ and supp$(F) = \underline{E}(F,\infty)$.

Now we have $\liminf_{n\to\infty} \frac{\log_2(\max\{L_I:I\in G_n^*\})}{-n} < \infty$, so there exists $\beta \ge 0$ such that $T_F(q) \le \beta q$ for all $q \ge 0$, which ensures that T_F takes values in $\mathbb{R} \cup \{-\infty\}$.

We say that F satisfies the multifractal formalism if $\dim_H \underline{E}(F,h) = T_F^*(h)$ for all $h \in \mathbb{R}_+ \cup \{\infty\}$. This is essentially the multifractal formalism considered in [36]. One

simple, but important, observation in [8] is that from (3.1) and (3.2) follows the fact that if $\mu \in \mathcal{M}_c^+(\mathbb{R})$ is supported on $[0, 1]$ and obeys the multifractal formalism for measures, then if $F_{\mu,\gamma_1,\gamma_2}$ is Hölder continuous, it obeys the multifractal formalism just defined above.

Let us now examine some features of the L^q-spectrum when the multifractal formalism holds. We distinguish three important properties denoted (*i*)–(*iii*): since $L_I = O(|I|^\alpha)$ for some $\alpha > 0$ by the Hölder continuity assumption, we have

(*i*) There exists $\alpha > 0$ and $c \in [0, 1]$ (here $c = \overline{\dim}_B \operatorname{supp}(F) = -T_F(0)$) such that $T_F(q) \geq \alpha q - c$ for all $q \geq 0$. Moreover,

(*ii*) T_F satisfies the same properties as τ in Proposition 2.1.2, in particular T_F^* is non-negative over its domain.

Due to (*i*), we can define $q_0 = \inf\{q \geq 0 : T_F(q) > 0\}$. If F satisfies the multifractal formalism, we must have the third property:

(*iii*) Either $q_0 > 0$, or $T_F'(0^+) > 0$ and $T_F(q) = T_F'(0^+)q$ for all $q > 0$.

Let us justify this fact. If $q_0 = 0$, there exists $c' \in \mathbb{R}_+$ such that $T_F(q) = T_F'(0^+)q + c'$ for all $q > 0$, for otherwise by concavity of T_F one has $-\infty < T_F^*(T_F'(q^-)) < 0$ for all $q > 0$ large enough so that $T_F'(q^-) < T_F'(0^+)$, while T_F^* must be non-negative over its domain. Also, since there exists $\beta > 0$ such that $0 < T_F(q) \leq \beta q$ for $q > 0$, we have $c' = 0$. If, moreover, F satisfies the multifractal formalism, we must have $T_F'(0^+) > 0$, otherwise $T_F = -\overline{\dim}_B \operatorname{supp}(F)$ over $\mathbb{R}_+^*$, and no Hölder continuous function F can fulfill the multifractal formalism with T_F as L^q-spectrum; indeed, this would imply $T_F^*(0) = \overline{\dim}_B \operatorname{supp}(F) \geq 0$ hence $\underline{E}(F_\mu, 0) \neq \emptyset$.

Theorem 3.2 *Suppose that a non-decreasing concave function T satisfies the above properties* (*i*)–(*iii*) *necessary to be the L^q-spectrum of a Hölder continuous function whose support is a non-empty subset of* $[0, 1]$. *Then there exists an (HM) Hölder continuous function F with* $\operatorname{supp}(F) \subset [0, 1]$, *which satisfies the multifractal formalism with $T_F = T$.*

Proof Let $q_0 = \inf\{q \geq 0 : T(q) > 0\}$. If $q_0 > 0$, then $\tau(q) = T(q_0 q)$ satisfies the properties of Proposition 2.1, so that it is the L^q-spectrum of an exact dimensional measure μ of dimension D, for any $D \in [q_0 T'(q_0^+), q_0 T'(q_0^-)] \subset [0, 1]$ by Theorem 2.2. Moreover, the inequality $\tau_\mu(q) \geq \alpha q_0 q - c$ implies that $\tau_\mu^*(\beta) = -\infty$ for all $\beta < \alpha q_0$. Consequently, the function $F = F_{\mu,0,1/q_0}$ is $(\alpha - \epsilon)$-Hölder continuous for all $\epsilon > 0$, and due to (3.1) and (3.2) it fulfills the multifractal formalism for wavelet leaders with $T_F : q \mapsto \tau_\mu(q/q_0) = T(q)$.

If $q_0 = 0$, the function defined as $\tau(q) = T(q) - T'(0^+)q$ satisfies the conditions required by Proposition 2.1. Take the (HM) measure μ associated with this function τ by Theorem 2.2. Then, the function $F_{\mu,T'(0^+),1}$ is $(T'(0^+) - \epsilon)$-Hölder continuous for all $\epsilon > 0$ and due to (3.1) and (3.2) it fulfills the multifractal formalism for wavelet leaders, with $T_F : q \mapsto \tau_\mu(q) + T'(0^+)q = T(q)$.

Remark 8 In [35], S. Jaffard uses a multifractal formalism associated with wavelet coefficients (not leaders). He introduces a class of concave functions such that to

each element τ of this class he can associate a Baire space V built from Besov spaces, so that generically an element of V has a non-decreasing Hausdorff spectrum obtained as the Legendre transform of τ computed by taking the infimum over a subdomain of $\mathbb{R}_+$.

References

1. N. Attia, J. Barral, Hausdorff and packing spectra, large deviations, and free energy for branching random walks in $\mathbb{R}^d$. Commun. Math. Phys. **331**, 139–187 (2014)
2. V. Aversa, C. Bandt, The multifractal spectrum of discrete measures. 18th winter school on abstract analysis. Acta Univ. Carolin. Math. Phys. **31**(2), 5–8 (1990)
3. J. Barral, Inverse problems in multifractal analysis of measures. Ann. Sci. Ec. Norm. Sup. (to appear). arXiv:1311.3895v3
4. J. Barral, D.-J. Feng, Weighted thermodynamic formalism and applications. Asian J. Math. **16**, 319–352 (2012)
5. J. Barral, D.-J. Feng, Multifractal formalism for almost all self-affine measures. Commun. Math. Phys. **318**, 473–504 (2013)
6. J. Barral, B. Mandelbrot, Random multiplicative multifractal measures, in *Fractal Geometry and Applications*. Proceedings of Symposia in Pure Mathematics, vol. 72, Part 2 (American Mathematical Society, Providence, 2004), pp. 3–90
7. J. Barral, M. Mensi, Gibbs measures on self-affine Sierpinski carpets and their singularity spectrum. Ergod. Theory Dyn. Syst. **27**, 1419–1443 (2007)
8. J. Barral, S. Seuret, From multifractal measures to multifractal wavelet series. J. Fourier Anal. Appl. **11**, 589–614 (2005)
9. J. Barral, S. Seuret, The singularity spectrum of Lévy processes in multifractal time. Adv. Math. **214**, 437–468 (2007)
10. A. Batakis, B. Testud, Multifractal analysis of inhomogeneous Bernoulli products. J. Stat. Phys. **142**, 1105–1120 (2011)
11. F. Bayard, Multifractal spectra of typical and prevalent measures. Nonlinearity **26**, 353–367 (2013)
12. F. Ben Nasr, Analyse multifractale de mesures, C. R. Acad. Sci. Paris, Série I **319**, 807–810 (1994)
13. G. Brown, G. Michon, J. Peyrière, On the multifractal analysis of measures. J. Stat. Phys. **66**, 775–790 (1992)
14. Z. Buczolich, S. Seuret, Typical measures on $[0, 1]^d$ satisfy a multifractal formalism. Nonlinearity **23**, 2905–2911 (2010)
15. Z. Buczolich, S. Seuret, Measures and functions with prescribed homogeneous spectrum. J. Fractal. Geom. **1**, 295–333 (2014)
16. G.A. Edgar, R.D. Mauldin, Multifractal decompositions of digraph recursive fractals. Proc. Lond. Math. Soc. **65**, 604–628 (1992)
17. K.J. Falconer, The multifractal spectrum of statistically self-similar measures. J. Theor. Probab. **7**, 681–702 (1994)
18. K.J. Falconer, Generalized dimensions of measures on self-affine sets. Nonlinearity **12**, 877–891 (1999).
19. K.J. Falconer, Representation of families of sets by measures, dimension spectra and Diophantine approximation. Math. Proc. Camb. Philos. Soc. **128**, 111–121 (2000)
20. K.J. Falconer, *Fractal Geometry – Mathematical Foundations and Applications*, 3rd ed. (Wiley, Hoboken, 2014)
21. A.H. Fan, Multifractal analysis of infinite products. J. Stat. Phys. **86**, 1313–1336 (1997)

22. D.-J. Feng, The limit Rademacher functions and Bernoulli convolutions associated with Pisot numbers. Adv. Math. **195**, 24–101 (2005)
23. D.-J. Feng, Gibbs properties of self-conformal measures and the multifractal formalism. Ergod. Theory Dyn. Syst. **27**, 787–812 (2007)
24. D.-J. Feng, K.S. Lau, Multifractal formalism for self-similar measures with weak separation condition. J. Math. Pures Appl. **92**, 407–428 (2009)
25. D.-J. Feng, E. Olivier, Multifractal analysis of the weak Gibbs measures and phase transition–Application to some Bernoulli convolutions. Ergod. Theory Dyn. Syst. **23**, 1751–1784 (2003)
26. U. Frisch, G. Parisi, On the singularity structure of fully developed turbulence, in *Turbulence, and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, ed. by M. Ghil et al. (North Holland, Amsterdam/New York, 1985), pp. 84–88
27. T. C. Halsey, M. H. Jensen, L.P. Kadanoff, I. Procaccia, B.I. Shraiman, Fractal measures and their singularities: the characterisation of strange sets. Phys. Rev. A **33**, 1141–1151 (1986)
28. Y. Heurteaux, Dimension of measures: the probabilistic approach. Publ. Mat. **51**, 243–290 (2007)
29. R. Holley, E.C. Waymire, Multifractal dimensions and scaling exponents for strongly bounded random fractals. Ann. Appl. Probab. **2**, 819–845 (1992)
30. G. Iommi, Multifractal Analysis for countable Markov shifts. Ergod. Theory Dyn. Syst. **25**, 1881–1907 (2005)
31. S. Jaffard, Exposants de Hölder en des points donnés et coefficients d'ondelettes. C. R. Acad. Sci. Paris Série I **308**, 79–81 (1989)
32. S. Jaffard, Construction de fonctions multifractales ayant un spectre de singularités prescrit. C. R. Acad. Sci. Paris, Sér. I **315**, 19–24 (1992)
33. S. Jaffard, Multifractal formalism for functions. I. Results valid for all functions. II Self-similar functions. SIAM J. Math. Anal. **28**, 944–970 & 971–998 (1997)
34. S. Jaffard, The multifractal nature of Lévy processes. Probab. Theory Relat. Fields **114**, 207–227 (1999)
35. S. Jaffard, On the Frisch-Parisi conjecture. J. Math. Pures Appl. **79**, 525–552 (2000)
36. S. Jaffard, Wavelet techniques in multifractal analysis, in *Fractal Geometry and Applications*. Proceedings of Symposia in Pure Mathematics, vol. 72, Part 2 (American Mathematical Society, Providence, 2004), pp. 91–152
37. S. Jaffard, Y. Meyer, On the pointwise regularity of functions in critical Besov spaces. J. Funct. Anal. **175**, 415–434 (2000)
38. J.F. King, The singularity spectrum for general Sierpinski carpets. Adv. Math. **116**, 1–8 (1995)
39. K.S. Lau, S.-M. Ngai, Multifractal measures and a weak separation condition. Adv. Math. **141**, 45–96 (1999)
40. J. Lévy-Véhel, R. Vojak, Multifractal analysis of Choquet capacities. Adv. Appl. Math. **20**, 1–43 (1998)
41. J. Lévy-Véhel, C. Tricot, On various multifractal spectra, in *Fractal Geometry and Stochastics III*. Progress in Probability, vol. 57 (Birkhäuser, Boston, 2004), pp. 23–42
42. B.B. Mandelbrot, C.J.G. Evertsz, Y. Hayakawa, Exactly self-similar left sided multifractal measures. Phys. Rev. A **42**, 4528–4536 (1990)
43. B.B. Mandelbrot, R. Riedi, Multifractal formalism for infinite multinomial measures. Adv. Appl. Math. **16**, 132–150 (1995)
44. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces* (Cambridge University Press, Cambridge, 1995)
45. Y. Meyer, *Ondelettes et Opérateurs I* (Hermann, Paris, 1990)
46. G.M. Molchan, Scaling exponents and multifractal dimensions for independent random cascades. Commun. Math. Phys. **179**, 681–702 (1996)
47. J.F. Muzy, A. Arneodo, E. Bacry, A multifractal formalism revisited with wavelets. Int. J. Bifur. Chaos Appl. Sci. Eng. **4**, 245–302 (1994)
48. S.-M. Ngai, A dimension result arising from the L^q-spectrum of a measure. Proc. Am. Math. Soc. **125**, 2943–2951 (1997)
49. L. Olsen, A multifractal formalism. Adv. Math. **116**, 82–196 (1995)

50. L. Olsen, Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$. Pac. J. Math. **183**, 143–199 (1998)
51. L. Olsen, N. Snigireva, Multifractal spectra of in-homogeneous self-similar measures. Indiana Univ. Math. J. **57**, 1789–1844 (2008)
52. Y. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications* (University of Chicago Press, Chicago, 1997)
53. D.A. Rand, The singularity spectrum $f(\alpha)$ for cookie-cutters. Ergod. Theory Dyn. Syst. **9**, 527–541 (1989)
54. R.T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, 1970)