Chapter 13
Enric Trillas: One of the First Spaniards Interested in the Study of Aggregation Functions

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Everything a person can imagine,
another can make it happen.
Jules Verne (1828-1905)

Aggregation is the process of combining several input values into a single representative output value, and the functions that carried out this process are called aggregation functions. Perhaps the oldest example of aggregation function is the arithmetic mean, which has been used during the history of experimental sciences.

It is easy to understand that aggregation functions play an important role in many fields: pure and applied mathematics, computer and engineering sciences, fuzzy logic, knowledge-based systems, economics and finances, social sciences, etc. One of the main contributions of Prof. Trillas to the field of fuzzy logic has been the use of triangular norms and triangular conorms (associative and commutative aggregation functions on $[0, 1]$ with neutral 1 and 0 respectively) to the definition of connectives in fuzzy logic [5]: if an element u has partial degrees of membership $x_1, x_2 \in [0, 1]$ in two fuzzy sets A_1, A_2, then the overall membership value $x \in [0, 1]$ of u in the combined fuzzy set $A = A_1 * A_2$ (intersection, union, etc.) is calculated in this way: $x = F(x_1, x_2)$ where F is a t-norm, a t-conorm, etc.

In what follows I have wanted to do a little historical tour around the idea of Enric Trillas of using "flexible aggregation functions".

In the late sixties I was studying my degree in mathematics at the *University of Barcelona* and for this reason I met Prof. Enric Trillas who taught statistics and probability theory. From then I did not go back to meet him until the early eighties. It was around the year 1982 when Prof. Nadal Batle, which by then was elected rector of the *University of the Balearic Islands*, encouraged me to do a thesis on some topic that seemed interesting for both. By that time I was teaching Mathematics to future primary school teachers. In our first working meeting I heard talk about fuzzy sets I and liked what I heard about them. Thus, advised by Prof. Batle I contacted shortly thereafter with Prof. Trillas, pioneer and master of fuzzy logic in Spain, who

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opened to me the doors of this exciting theory and made it possible I meet Prof. Alsina and other relevant researchers working with him. I recall with some nostalgia my visits to the *Technical College of Architecture* in Barcelona where I learned a lot from them and spent unforgettable days from all points of view.

Fig. 13.1 Gala dinner of AGOP 2009 in Miramar (Mallorca, 9th of July 2009) – Enric Trillas between Antonia (my wife) and myself.

My early research interest was that of triangular norms [7, 15] and fuzzy intersection operations derived from them [5], but soon, advised by Prof. Trillas, I also began the study of operations that generalize t-norms and t-conorms that were called *aggregation functions*[20, 21]. The definition we gave in [20] is the following:

Definition 3 (Aggregation function, 1984)
$F : [0,1] \times [0,1] \to [0,1]$ *is an aggregation function if the following conditions hold for all* x, x', y, y'*:*

i) $F(x,0) = F(1,0)x$, $F(x,1) = (1 - F(1,0))x + F(1,0)$ *(boundary conditions)*
ii) $F(x,y) \leq F(x',y')$ *whenever* $x \leq x'$, $y \leq y'$ *(monotonicity)*
iii) $F(x,y) = F(y,x)$ *(symmetry)*

Note that the class of aggregation functions contains any convex linear combinations of a t-norm and a t-conorm. The difference between the class of t-norms and t-conorms and the class of aggregation functions is given by the associativity, as the following proposition states: *"An aggregation function is either a t-norm or a t-conorm if and only if it is associative"*.

For more general purposes, conditions i) and iii) in Definition 1 are somewhat too restrictive. The current definition of aggregation function [8, 14] is undemanding: it proposes a set of fundamental and minimal properties that group most of the precedent definitions related to "aggregation/merging processes" [1, 3, 4, 11, 13, 17, 20, 24, 25, 27], under weaker conditions.

Definition 4 (Aggregation function, today)
$F : [0,1] \times [0,1] \to [0,1]$ *is an aggregation function if the following conditions hold for all* x, x', y, y'*:*

i) $F(0,0) = 0$, $F(1,1) = 1$ *(boundary conditions)*
ii) $F(x,y) \leq F(x',y')$ *whenever* $x \leq x'$, $y \leq y'$ *(monotonicity)*

Obviously, an aggregation function (1984) is an aggregation function as it is accepted today.

The representation of continuous Archimedean t-norms by one-argument function and addition was discovered by C. H. Ling (1965) [16] as a variant of the Abel-Aczél representation of strict t-norms [2]. Since then, the representation of several classes of non-associative aggregation functions has also been established [6, 21]. These representations generalize those given for t-norms.

In 1985, Prof. Trillas, by then president of the *Spanish National Research Council* (CSIC), led a research project entitled "Flexible structures for the mathematical analysis of vagueness". One of the goals of that project was to introduce and study mixed connectives serving as a t-norm on a part $[0, k]$ (or $[k, 1]$) of the domain $[0, 1]$ and as a t-conorm on $[k, 1]$ (or $[0, k]$). I participated as a member of the research team and in 1986 we introduce a class of what we called "generalized logical operator" [9]. Using current terminology they are nothing but continuous symmetric aggregation functions.

Definition 5 (generalized logical operator, 1986)
$F : [0,1] \times [0,1] \to [0,1]$ *is a generalized logical operator if the following conditions hold for all* x, x', y, y'*:*

i) F *is continuous*
ii) $F(x,y) \leq F(x',y')$ *whenever* $x \leq x'$, $y \leq y'$
iii) $F(x,y) = F(y,x)$
iv) $F(0,0) = 0$, $F(1,1) = 1$.

In [9] we called t-operators those generalized logical operators which are associative. The structure of such aggregation functions is described in [9] as follows: "$F : [0,1] \times [0,1] \to [0,1]$ is a t-operator if and only if it is a continuous t-norm or a continuous t-conorm, or there exist a constant $k \in (0,1)$, a continuous t-norm T and a continuous t-conorm S such that:

$$F(x,y) = \begin{cases} kS\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } 0 \le x, y \le k \\ k + (1-k)T\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } k \le x, y \le 1 \\ k & \text{otherwise} \end{cases} \qquad (13.1)$$

Note that a t-operator F has k as absorbent element: $F(x,k) = F(k,x) = k$ for all x in $[0,1]$. A "non-continuous" version of this type of aggregation functions was introduced much later [18]:

Definition 6 (t-operator, 1999)
$F : [0,1] \times [0,1] \to [0,1]$ *is a t-operator if the following conditions hold for all* x, x', y, y', z*:*

i) $F(x,y) \le F(x',y')$ *whenever* $x \le x'$, $y \le y'$
ii) $F(x,y) = F(y,x)$
iii) $F(F(x,y),z) = F(x,F(y,z))$
iv) $F(0,0) = 0$, $F(1,1) = 1$
v) The horizontal sections $F(.,0)$ *and* $F(.,1)$ *are continuous.*

Few years later the concept of nullnorm was proposed [11]. In fact it is nothing more than a proper t-operator as it was defined in 1999.

Definition 7 (nullnorm, 2001)
$F : [0,1] \times [0,1] \to [0,1]$ *is a nullnorm if the following conditions hold for all* x, x', y, x', z*:*

i) $F(x,y) \le F(x',y')$ *whenever* $x \le x'$, $y \le y'$
ii) $F(x,y) = F(y,x)$
iii) $F(F(x,y),z) = F(x,F(y,z))$
iv) There exists k, $0 < k < 1$, *such that* $F(x,0) = x$ *for all* $x \in [0,k]$ *and* $F(x,1) = x$ *for all* $x \in [k,1]$.

Another variation of t-norms and t-conorms was introduced in 1996 [26] under the name of uninorm. This was a non-Spanish contribution to the idea of Prof. Trillas of creating flexible operational structures to be used as mathematical models of (flexible) fuzzy connectives.

Definition 8 (uninorm, 1996)
$F : [0,1] \times [0,1] \to [0,1]$ *is a uninorm if the following conditions hold for all* x, x', y, x', z*:*

i) $F(x,y) \leq F(x',y')$ *whenever* $x \leq x'$, $y \leq y'$
ii) $F(x,y) = F(y,x)$
iii) $F(F(x,y),z) = F(x,F(y,z))$
iv) There exists e, $0 \leq e \leq 1$, $F(x,e) = x$ *(neutral element).*

The structure of uninorms is similar to the t-operators. Using informal language:

"$F : [0,1] \times [0,1] \to [0,1]$ is a uninorm (with neutral element e) if and only if it is a t-norm on $[0,e]^2$, a t-conorm on $[e,1]^2$, and on the rest of the unit square F is bounded by the minimum and the maximum".

According to the fact that in most practical situations it is necessary to discretize the unit interval, we need to deal with logics where the set of truth values is modelled by a finite set $L = \{0,1,\ldots,n\}$. As it is expected, we use in the definition of discrete t-norms (and discrete t-conorms) the same set of axioms than for ordinary t-norms. In [23] a characterization of smooth discrete t-norms was obtained. Informally: "The smooth discrete t-norms are the ordinal sums of (discrete) Łukasiewicz t-norms".

Taking into account that smoothness is the proper equivalent of the continuity of ordinary t-norms, the given representation of smooth discrete t-norms is in full analogy to Ling's representation of ordinary continuous t-norms.

Having studied the discrete t-norms and discrete t-conorms, it was reasonable to consider discrete aggregation functions related to them with the same spirit that made for the continuous case. Thus, in 1999 [18] we introduced discrete t-operators and discrete uninorms and found characterizations for some classes of these operations. Now we can say that the idea that had Prof. Trillas in the early eighties to study flexible connectives has been satisfied.

So far we have treated the case of aggregation functions of two arguments but it is clear that in the process of aggregation the number of inputs can be any natural number. In 1997 [10] we first introduced the concept of extended aggregation function. Today [8, 14], an extended aggregation function is considered to be a sequence $F_1 = id, F_2, F_3, \ldots$, where F_n is an aggregation function of n arguments (an n-ary aggregation function).

Definition 9 (extended aggregation function, today)
$F : U_{n\geq 1}[0,1]^n \to [0,1]$*is an extended aggregation function if the following conditions hold for all* n *(*F_n *means the restriction of* F *to* $[0,1]^n$*):*

1. $F_1 = id$.
2. $F_n(0,\ldots,0) = 0$, $F_n(1,\ldots,1) = 1$
3. $F_n(x_1,\ldots,x_n) \leq F_n(y_1,\ldots,y_n)$ *whenever* $x_i \leq y_i$ *for all* $i = 1,\ldots,n$

Note that there is no relationship between the values taken by F on inputs $(x_1,\ldots,x_n)$ and $(x_1,\ldots,x_m)$ with $n \neq m$. It is clear that two members of the sequence F_n and F_m need to be related somehow in order to give consistency to the process of aggregation. That is what we did in our definition [10] by means of an extended/global monotonicity condition.

Definition 10 (extended aggregation function, 1997)
$F : U_{n\geq 1}[0,1]^n \rightarrow [0,1]$*is an extended aggregation function if the following conditions hold for all n (F_n means the restriction of F to $[0,1]^n$):*

1. $F_1 = id$.
2. $F_n(0,\ldots,0) = 0$, $F_n(1,\ldots,1) = 1$ *(n-ary monotonicity)*
3. $F_{n+1}(x_1,\ldots,x_n, min(x_1,\ldots,x_n)) \leq F_n(y_1,\ldots,y_n)$
   $\leq F_{n+1}(x_1,\ldots,x_n, max(x_1,\ldots,x_n))$ *(extended monotonicity)*

Fig. 13.2 Visit to the hermitage of Valldemossa (Mallorca, 23th of February 2012) – Enric Trillas and myself.

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