Martingale Marginals Do Not Always Determine Convergence

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Abstract Baéz-Duarte (J. Math. Anal. Appl. **36**, 149–150, 1971, http://dx.doi.org/10.1016/0022-247X(71)90025-4 [ISSN 0022-247x]) and Gilat (Ann. Math. Stat. **43**, 1374–1379, 1972, http://dx.doi.org/10.1214/aoms/1177692494 [ISSN 0003-4851]) gave examples of martingales that converge in probability (and hence in distribution) but not almost surely. Here such a martingale is constructed with uniformly bounded increments, and a construction is provided of two martingales with the same marginals, one of which converges almost surely, while the other does not converge in probability.

1 Introduction

Recent work of Marc Yor and coauthors [4] has drawn attention to how properties of a martingale are related to its family of marginal distributions. A fundamental result of this kind is Doob's martingale convergence theorem:

- if the marginal distributions $(\mu_n, n \geq 0)$ of a discrete time martingale $(M_n, n \geq 0)$ are such that $\int |x| \mu_n(dx)$ is bounded, then M_n converges almost surely.

Other well known results relating the behavior of a discrete time martingale M_n to its marginal laws μ_n are:

- for each $p > 1$, the sequence $\int |x|^p \mu_n(dx)$ is bounded if and only if M_n converges in L^p;
- $\lim_{y \to \infty} \sup_n \int_{|x|>y} |x| \mu_n(dx) = 0$, that is $(M_n)_{n \geq 0}$ is uniformly integrable, if and only if M_n converges in L^1.

We know also from Lévy that if μ_n is the distribution of a partial sum S_n of independent random variables, and μ_n converges in distribution as $n \to \infty$, then

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S_n converges almost surely. These results can be found in most modern graduate textbooks in probability. See for instance Durrett [2].

What if the marginals of a martingale converge in distribution? Does that imply the martingale converges a.s? Báez-Duarte [1] and Gilat [3] gave examples of martingales that converge in probability but not almost surely. So the answer to this question is no. But worse than that, there is a sequence of martingale marginals converging in distribution, such that some martingales with these marginals converge almost surely, while others diverge almost surely. So by mixing, the probability of convergence of a martingale with these marginals can be any number in [0, 1]. Moreover, the same phenomenon can be exhibited for convergence in probability: there is a sequence of martingale marginals converging in distribution, such that some martingales with these marginals converge in probability, but others do not.

The purpose of this brief note is to record these examples, and to draw attention to the following problems which they raise:

1. What is a necessary and sufficient condition on martingale marginals for every martingale with these marginals to converge almost surely?
2. What is a necessary and sufficient condition on martingale marginals for every martingale with these marginals to converge in probability?

Perhaps the condition for almost sure convergence is Doob's L^1-bounded condition. But this does not seem at all obvious. Moreover, L^1-bounded is not the right condition for convergence in probability: convergence in distribution to a point mass is obviously sufficient, and this condition can hold for marginals that are not bounded in L^1. See also Rao [5] for treatment of some other problems related to non-L^1-bounded martingales.

2 Examples

2.1 Almost Sure Convergence

This construction extends and simplifies the construction by Gilat [3, §2] of a martingale which converges in probability but not almost surely, with increments in the set $\{-1, 0, 1\}$ See also Báez-Duarte [1] for an earlier construction with unbounded increments, based on the double or nothing game instead of a random walk.

Let $(S_n, n = 0, 1, 2, \ldots)$ be a simple symmetric random walk started at $S_0 = 0$, with $(S_{n+1} - S_n, n = 0, 1, 2, \ldots)$ a sequence of independent $U(\pm 1)$ random variables, where $U(\pm 1)$ is the uniform distribution on the set $\{\pm 1\} := \{-1, +1\}$. Let $0 = T_0 < T_1 < T_2 < \cdots$ be the successive times n that $S_n = 0$. By recurrence of the simple random walk, $P(T_n < \infty) = 1$ for every n. For each $k = 1, 2, \ldots$ let

$M^{(k)}$ be the process which follows the walk S_n on the random interval $[T_{k-1}, T_k]$ of its kth excursion away from 0, and is otherwise identically 0:

$$M_n^{(k)} := S_n 1(T_{k-1} \leq n \leq T_k)$$

where $1(\cdots)$ is an indicator random variable with value 1 if $\cdots$ and 0 otherwise. Each of these processes $M^{(k)}$ is a martingale relative to the filtration $(\mathcal{F}_n)$ generated by the walk (S_n), by Doob's optional sampling theorem. Now let (A_k) be a sequence of events such that the σ-field $\mathcal{G}_0$ generated by these events is independent of the walk $(S_n, n \geq 0)$, and set

$$M_n := \sum_{k=1}^{\infty} M_n^{(k)} 1(A_k)$$

So M_n follows the path of S_n on its kth excursion away from 0 if A_k occurs, and otherwise M_n is identically 0. Let $\mathcal{G}_n$ for $n \geq 0$ be the σ-field generated by $\mathcal{G}_0$ and $\mathcal{F}_n$. Then it is clear that $(M_n, \mathcal{G}_n)$ is a martingale, no matter what choice of the sequence of events (A_k) independent of (S_n). The distribution of M_n is determined by the formula

$$P(M_n = x) = \sum_{k=1}^{\infty} P(S_n = x, T_{k-1} \leq n \leq T_k) P(A_k)$$

for all integers $x \neq 0$. A family of martingales with the same marginals is thus obtained by varying the structure of dependence between the events A_k for a given sequence of probabilities $P(A_k)$. The only way that a path of M_n can converge is if M_n is eventually absorbed in state 0. So if $N := \sum_k 1(A_k)$ denotes the number of events A_k that occur,

$$P(M_n \text{ converges}) = P(N < \infty).$$

Now take $P(A_k) = p_k$ for a decreasing sequence p_k with limit 0 but $\sum_k p_k = \infty$, for instance $p_k = 1/k$. Then (A_k) can be constructed so that the A_k are mutually independent, and $P(N = \infty) = 1$ by the Borel-Cantelli lemma. Or these events can be nested:

$$A_1 \supseteq A_2 \supseteq A_3 \cdots$$

in which case

$$P(N \geq k) = P(A_k) \downarrow 0 \text{ as } k \to \infty,$$

so $P(N = \infty) = 0$ in this case. Thus we obtain a sequence of marginal distributions for a martingale, such that some martingales with these marginals converge almost surely, while others diverge almost surely.

2.2 Convergence in Probability

Let us construct a martingale M_n which converges in distribution, but not in probability, following indications of such a construction by Gilat [3, §1].

This will be an inhomogeneous Markov chain with integer values, starting from $M_0 = 0$. Its first step will be to M_1 with $U(\pm 1)$ distribution. Thereafter, the idea is to force M_n to alternate between the values ± 1, with probability increasing to 1 as $n \to \infty$. This achieves $U(\pm 1)$ as its limit in distribution, while preventing convergence in probability by the alternation. The transition probabilities of M_n are as follows:

$$P(M_{n+1} = M_n \pm 1 \mid M_n \text{ with } M_n \notin \{\pm 1\}) = 1/2 \tag{1}$$

$$P(M_{n+1} = -1 \mid M_n = 1) = 1 - 2^{-n} \tag{2}$$

$$P(M_{n+1} = 2^{n+1} - 1 \mid M_n = 1) = 2^{-n} \tag{3}$$

$$P(M_{n+1} = +1 \mid M_n = -1) = 1 - 2^{-n} \tag{4}$$

$$P(M_{n+1} = -2^{n+1} + 1 \mid M_n = -1) = 2^{-n}. \tag{5}$$

The first line indicates that whenever M_n is away from the two point set $\{\pm 1\}$, it moves according to a simple symmetric random walk, until it eventually gets back to $\{\pm 1\}$ with probability one. Once it is back in $\{\pm 1\}$, it is forced to alternate between these values, with probability $1 - 2^{-n}$ for an alternation at step n, compensated by moving to $\pm(2^{n+1} - 1)$ with probability 2^{-n}. Since the probabilities 2^{-n} are summable, the Borel-Cantelli Lemma ensures that with probability one only finitely many exits from $\{\pm 1\}$ ever occur. After the last of these exits, the martingale eventually returns to $\{\pm 1\}$ with probability one. From that time onwards, the martingale flips back and forth deterministically between $\{\pm 1\}$.

A slight modification of these transition probabilities, gives another martingale with the same marginal distributions which converges almost surely and hence in probability. With $M_0 = 0$ as before, the modified scheme is as follows:

$$P(M_{n+1} = M_n \pm 1 \mid M_n \text{ with } M_n \notin \{\pm 1\}) = 1/2 \tag{6}$$

$$P(M_{n+1} = 1 \mid M_n = 1) = 1 - 2^{-n} \tag{7}$$

$$P(M_{n+1} = 2^{n+1} - 1 \mid M_n = 1) = 2^{-n} p_n \tag{8}$$

$$P(M_{n+1} = -2^{n+1} + 1 \mid M_n = 1) = 2^{-n} q_n \tag{9}$$

$$P(M_{n+1} = -1 \mid M_n = -1) = 1 - 2^{-n} \tag{10}$$

$$P(M_{n+1} = -2^{n+1} + 1 \mid M_n = -1) = 2^{-n} p_n \tag{11}$$

$$P(M_{n+1} = 2^{n+1} - 1 \mid M_n = -1) = 2^{-n} q_n \tag{12}$$

where

$$p_n := 1/(2 - 2^{-n}) \text{ and } q_n := 1 - p_n$$

are chosen so that the distribution with probability p_n at $2^{n+1}-1$ and q_n at $-2^{n+1}+1$ has mean

$$p_n(2^{n+1} - 1) + q_n(-2^{n+1} + 1) = 1.$$

In this modified process, the alternating transition out of states ±1 is replaced by holding in these states, while the previous compensating moves to $\pm(2^{n+1} - 1)$ are replaced by a nearly symmetric transitions from ±1 to these values. This preserves the martingale property, and also preserves the marginal laws. But the previous argument for eventual alternation now shows that the modified martingale is eventually absorbed almost surely in one of the states ±1. So the modified martingale converges almost surely to a limit which has $U(\pm1)$ distribution.

These martingales (M_n) have jumps that are unbounded. Gilat [3, §2] left open the question of whether there exist martingales with uniformly bounded increments which converge in distribution but not in probability. But such martingales can be created by a variation of the first construction of (M_n) above, as follows.

Run a simple symmetric random walk starting from 0. Each time the random walk makes an alternation between the two states ±1, make the walk delay for a random number of steps in its current state in ±1 before continuing, for some rapidly increasing sequence of random delays. Call the resulting martingale M_n. So by construction, M_1 has $U(\pm)$ distribution,

$$M_n = (-1)^{k-1} M_1 \text{ for } S_k \le n \le T_k$$

for some increasing sequence of randomized stopping times

$$1 = S_1 < T_1 < S_2 < T_2 < \cdots,$$

and during the kth crossing interval $[T_k, S_{k+1}]$ the process M_n follows a simple random walk path starting in state $(-1)^{k-1}M_1$ and stopping when it first reaches state $(-1)^k M_1$.

The claim is that a suitable construction of the delays $T_k - S_k$ will ensure that the distribution of M_n converges to $U(\pm1)$, while there is almost deterministic alternation for large k of the state M_{t_k} for some rapidly increasing deterministic sequence t_k. To achieve this end, let $t_1 = 1$ and suppose inductively for $k = 1, 2, \ldots$ that t_k has been chosen so that

$$P(M_{t_k} = (-1)^{k-1} M_1) > 1 - \epsilon_k \text{ for some } \epsilon_k \downarrow 0 \text{ as } k \to \infty. \tag{13}$$

Here $M_1 \in \{\pm 1\}$ is the first step of the simple random walk. The random number of steps required for random walk crossing between states ± 1 is a.s. finite. So having defined t_k, we can choose an even integer t_{k+1} so large, that $t_{k+1}/2 > t_k$ and all of the following events occur with probability at least $1 - \epsilon_{k+1}$:

- $M_{t_{k+1}/2} = (-1)^{k-1} M_1$, meaning that the $(k-1)$th crossing between ± 1 has been completed by time $S_k < t_{k+1}/2$;
- the kth crossing is started at time T_k that is uniform on $[t_{k+1}/2, t_{k+1})$ given $S_k < t_{k+1}/2$;
- the kth crossing is completed at time $S_{k+1} < t_{k+1}$, so $M_n = (-1)^k M_1$ for $S_{k+1} \leq n \leq t_{k+1}$.

Moreover, t_{k+1} can be chosen so large that the uniform random start time of the kth crossing given $S_k < t_{k+1}/2$ ensures that also

$$P(M_n \in \{\pm 1\}) \geq 1 - 2\epsilon_k \text{ for all } t_k \leq n \leq t_{k+1}$$

because with high probability the length $S_{k+1} - T_k$ of the kth crossing is negligible in comparison with the length $t_{k+1}/2$ of the interval $[t_{k+1}/2, t_{k+1}]$ in which this crossing is arranged to occur. It follows from this construction that M_n converges in distribution to $U(\pm 1)$, while the forced alternation (13) prevents M_n from having a limit in probability.

A feature of the previous example is that $\sup_n M_n = -\inf_n M_n = \infty$ almost surely, since in the end every step of the underlying simple symmetric random walk is made by the time-changed martingale M_n. A similar example can be created from a standard Brownian motion $(B_t, t \geq 0)$ using a predictable $\{0, 1\}$-valued process $(H_t, t \geq 0)$ to create successive switching between and holding in states ± 1 so that the martingale

$$M_t := \int_0^t H_t dB_t$$

converges in distribution to $U(\pm 1)$ while not converging in probability. In this example, $\int_0^\infty H_t dt = \sup_t M_t = -\inf_t M_t = \infty$ almost surely.

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References

1. L. Báez-Duarte, An a.e. divergent martingale that converges in probability. J. Math. Anal. Appl. **36**, 149–150 (1971). http://dx.doi.org/10.1016/0022-247X(71)90025-4 [ISSN 0022-247x]
2. R. Durrett, *Probability: Theory and Examples*, 4th edn. Cambridge Series in Statistical and Probabilistic Mathematics (Cambridge University Press, Cambridge, 2010). http://dx.doi.org/10.1017/CBO9780511779398 [ISBN 978-0-521-76539-8]
3. D. Gilat, Convergence in distribution, convergence in probability and almost sure convergence of discrete martingales. Ann. Math. Stat. **43**, 1374–1379 (1972). http://dx.doi.org/10.1214/aoms/1177692494 [ISSN 0003-4851]
4. F. Hirsch, C. Profeta, B. Roynette, M. Yor, *Peacocks and Associated Martingales, with Explicit Constructions*. Bocconi & Springer Series, vol. 3 (Springer, Milan, 2011). doi: 10.1007/978-88-470-1908-9. http://dx.doi.org/10.1007/978-88-470-1908-9 [ISBN 978-88-470-1907-2]
5. M.M. Rao, Non-L^1-bounded martingales, in *Stochastic Control Theory and Stochastic Differential Systems (Proc. Workshop, Deutsch. Forschungsgemeinsch., Univ. Bonn, Bad Honnef, 1979)*. Lecture Notes in Control and Information Science, vol. 16 (Springer, Berlin, 1979), pp. 527–538. http://link.springer.com/chapter/10.1007/BFb0009411