# Chapter 8
# Non-metricity and the Nonlinear Mechanics of Distributed Point Defects

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**Abstract** We discuss the relevance of non-metricity in a metric-affine manifold (a manifold equipped with a connection and a metric) and the nonlinear mechanics of distributed point defects. We describe a geometric framework in which one can calculate analytically the residual stress field of nonlinear elastic solids with distributed point defects. In particular, we use Cartan's machinery of moving frames and construct the material manifold of a finite ball with a spherically-symmetric distribution of point defects. We then calculate the residual stress field when the ball is made of an arbitrary incompressible isotropic solid. We will show that an isotropic distribution of point defects cannot be represented by a distribution of purely dilatational eigenstrains. However, it can be represented by a distribution of radial eigenstrains. We also discuss an analogy between the residual stress field and the gravitational field of a spherical mass.

## 8.1 Introduction

The first mathematical study of line defects in solids goes back to the work of Volterra [Vol07] more than a century ago. The close connection between the mechanics of solids with distributed defects and non-Riemannian geometries was independently discovered in the 1950s by Kondo [Kon55a, Kon55b], Bilby et al. [BBS55], and Bilby and Smith [BS56]. Defects influence many of the mechanical properties of solids and have been the focus of intense research in the last few decades. Motivated by applications of metals in industry and the need to take into account plastic deformations, the micro mechanism of plasticity, i.e., dislocations have been studied by many researchers but mostly in the framework of linearized elasticity. Other

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line defects, e.g., disclinations have also been the subject of many investigations. However, point defects have not received much attention even in the linearized setting after the original works of Love [Lov27], and Eshelby [Esh54]. In particular, Love [Lov27] calculated the stress field of a single point defect in an infinite linear elastic solid and observed a $1/r^3$ singularity. Recently, we [YG12b] showed how one can calculate the residual stress field of a nonlinear elastic solid with distributed point defects. Non-metricity proved to be an essential geometric entity in describing the zero-stress configuration (material manifold) of solids with point defects. Here, material manifold $(\mathcal{B}, \mathbf{G}, \nabla)$ is a flat, torsion-free 3-manifold with non-metricity in which the body is stress free. We should mention that many researches have known the relevance of non-metricity to the mechanics of point defects [Fal81, deW81, Gra89, Kro90, MR02]. However, there has not been a concrete use of non-metricity in the literature for calculating residual stresses. The geometric framework discussed here has been recently used by the authors in the analysis of distributed dislocations and disclinations as well [YG12a, YG13b].

In this book chapter we review the results of [YG12b], extend the residual stress calculation to arbitrary incompressible isotropic solids, and make some new observations. In particular, we discuss an analogy with relativity and the gravitational field of a spherical ball of mass $m$ in an infinite empty space-time.

Another problem that can benefit from geometric ideas is the stress analysis of inclusions in nonlinear elastic solids [YG13a]. In [YG13a] we showed that collapsing a small spherical inclusion with pure dilatational eigenstrain while keeping the strength of the inclusion fixed one recovers the stress field of a single point defect in a linear elastic solid. Earlier in [YG12b] we had shown that using the nonlinear solution one can recover the classical linear solution for small strength point defect distributions supported in a small ball. Now one may be tempted to think that any isotropic distribution of point defects can be represented by pure dilatational eigenstrains. We will show that this is not the case and that material metric calculated in [YG12b] is equivalent to a distribution of radial eigenstrains with no circumferential eigenstrains.

## 8.2 Non-Riemannian Geometries and Anelasticity

To make this book chapter self contained, in the following we tersely review the necessary geometric background.

**Riemann-Cartan manifolds**. On a manifold $\mathcal{B}$ a linear (affine) connection is an operation $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ denotes the set of vector fields on $\mathcal{B}$. In a local coordinate chart $\{X^A\}$, $\nabla_{\partial_A}\partial_B = \Gamma^C{}_{AB}\partial_C$, where $\Gamma^C{}_{AB}$ are Christoffel symbols of the connection and $\partial_A = \frac{\partial}{\partial x^A}$ are the natural bases for the tangent space. $\nabla$ is compatible with a metric $\mathbf{G}$ of the manifold if $\nabla\mathbf{G} = \mathbf{0}$. The torsion of $\nabla$ is defined by $\boldsymbol{T}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$, where $[.,.]$ is the commutator of vector fields. $\nabla$ is symmetric if it is torsion-free, i.e., $\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$. In

$(\mathcal{B}, \nabla, \mathbf{G})$ the curvature is a map $\boldsymbol{\mathcal{R}} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by $\boldsymbol{\mathcal{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}$. A Riemann-Cartan manifold $(\mathcal{B}, \nabla, \mathbf{G})$ is a metric-affine manifold in which the metric and the connection are compatible.

**Cartan's moving frames**. Let us consider a frame field $\{\mathbf{e}_\alpha\}_{\alpha=1}^N$ that forms a basis for the tangent space of $\mathcal{B}$ everywhere. We assume that this frame is orthonormal, i.e., $\langle\!\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle\!\rangle_{\mathbf{G}} = \delta_{\alpha\beta}$. $\{\mathbf{e}_\alpha\}_{\alpha=1}^N$ is, in general, a non-coordinate basis for the tangent space, i.e., it is not necessarily induced from a coordinate chart. The frame field $\{\mathbf{e}_\alpha\}$ naturally defines the co-frame field $\{\vartheta^\alpha\}_{\alpha=1}^N$ such that $\vartheta^\alpha(\mathbf{e}_\beta) = \delta^\alpha_\beta$. The connection 1-forms are defined by $\nabla \mathbf{e}_\alpha = \mathbf{e}_\gamma \otimes \omega^\gamma{}_\alpha$. The corresponding connection coefficients are defined as $\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha = \langle \omega^\gamma{}_\alpha, \mathbf{e}_\beta \rangle \mathbf{e}_\gamma = \omega^\gamma{}_{\beta\alpha}\mathbf{e}_\gamma$, i.e., $\omega^\gamma{}_\alpha = \omega^\gamma{}_{\beta\alpha}\vartheta^\beta$. Similarly, $\nabla\vartheta^\alpha = -\omega^\alpha{}_\gamma\vartheta^\gamma$, and $\nabla_{\mathbf{e}_\beta}\vartheta^\alpha = -\omega^\alpha{}_{\beta\gamma}\vartheta^\gamma$. In an orthonormal frame, the metric has the simple representation $\mathbf{G} = \delta_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$.

**Non-metricity**. Given a metric-affine manifold $(\mathcal{B}, \nabla, \mathbf{G})$[1], the non-metricity is a map $\boldsymbol{\mathcal{Q}} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathbb{R}$ defined as $\boldsymbol{\mathcal{Q}}(\mathbf{U}, \mathbf{V}, \mathbf{W}) = \langle \nabla_{\mathbf{U}}\mathbf{V}, \mathbf{W} \rangle_{\mathbf{G}} + \langle \mathbf{V}, \nabla_{\mathbf{U}}\mathbf{W} \rangle_{\mathbf{G}} - \mathbf{U}[\langle \mathbf{V}, \mathbf{W} \rangle_{\mathbf{G}}]$. In other words, $\boldsymbol{\mathcal{Q}} = -\nabla\mathbf{G}$. In the frame $\{\mathbf{e}_\alpha\}$, $\mathcal{Q}_{\gamma\alpha\beta} = \boldsymbol{\mathcal{Q}}(\mathbf{e}_\gamma, \mathbf{e}_\alpha, \mathbf{e}_\beta)$. Non-metricity 1-forms are defined by $\mathcal{Q}_{\alpha\beta} = \mathcal{Q}_{\gamma\alpha\beta}\vartheta^\gamma$. One can show that $\mathcal{Q}_{\gamma\alpha\beta} = \omega_{\beta\gamma\alpha} + \omega_{\alpha\gamma\beta} - \langle dG_{\alpha\beta}, \mathbf{e}_\gamma \rangle$, where $d$ is the exterior derivative. Thus, $\mathcal{Q}_{\alpha\beta} = \omega_{\alpha\beta} + \omega_{\beta\alpha} - dG_{\alpha\beta} =: -DG_{\alpha\beta}$, where $D$ is the covariant exterior derivative. This is *Cartan's zeroth structural equation*. For an orthonormal frame $G_{\alpha\beta} = \delta_{\alpha\beta}$ and hence $\mathcal{Q}_{\alpha\beta} = \omega_{\alpha\beta} + \omega_{\beta\alpha}$. In a metric-affine manifold with non-metricity, the Weyl 1-form is defined as $Q = \frac{1}{n}\mathcal{Q}_{\alpha\beta}G^{\alpha\beta}$. Therefore, $\mathcal{Q}_{\alpha\beta} = \tilde{\mathcal{Q}}_{\alpha\beta} + QG_{\alpha\beta}$, where $\tilde{\boldsymbol{\mathcal{Q}}}$ is the traceless part of the non-metricity. If $\tilde{\boldsymbol{\mathcal{Q}}} = \mathbf{0}$ and if $\nabla$ is torsion-free, $(\mathcal{B}, \nabla, \mathbf{G})$ is called a Weyl manifold. The torsion and curvature 2-forms are defined by

$$\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta, \tag{8.2.1}$$
$$\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta. \tag{8.2.2}$$

These are, respectively, *Cartan's first and second structural equations*.

**The compatible volume element on a Weyl manifold**. A volume element on $\mathcal{B}$ is any non-vanishing $n$-form. In the orthonormal coframe field $\{\vartheta^\alpha\}$ the volume form is written as $\boldsymbol{\mu} = h\vartheta^1 \wedge ... \wedge \vartheta^n$, for some positive function $h$. In a coordinate chart $\{X^A\}$ we have $\boldsymbol{\mu} = h\sqrt{\det \mathbf{G}}\, dX^1 \wedge ... \wedge dX^n$. Divergence of an arbitrary vector field $\mathbf{W}$ on $\mathcal{B}$ is defined as $(\mathrm{Div}\,\mathbf{W})\boldsymbol{\mu} = \mathfrak{L}_{\mathbf{W}}\boldsymbol{\mu}$, where $\mathfrak{L}$ is the Lie derivative. Having a connection divergence can also be defined as the trace of the covariant derivative, i.e., $\mathrm{Div}_\nabla \mathbf{W} = W^A{}_{|A} = W^A{}_{,A} + \Gamma^A{}_{AB}W^B$. The volume element $\boldsymbol{\mu}$ is compatible with $\nabla$ if $\mathfrak{L}_{\mathbf{W}}\boldsymbol{\mu} = (W^A{}_{|A})\boldsymbol{\mu}$, which is equivalent to $D\left(h\sqrt{\det \mathbf{G}}\right) = 0$ [Saa95]. Thus, $\frac{dh}{h} = d\ln h = \frac{n}{2}Q$. Note that this implies that $dQ = 0$. Therefore, to be able to define a compatible volume element the Weyl one-form must be closed.

[1] In a metric-affine manifold the torsion, curvature, and non-metricity, are, in general, non-vanishing.

**Geometric anelasticity**. Let us briefly review geometric nonlinear elasticity. We identify a body $\mathcal{B}$ with a Riemannian manifold $\mathcal{B}$ and a configuration of $\mathcal{B}$ is a mapping $\varphi : \mathcal{B} \to \mathcal{S}$, where $\mathcal{S}$ is another Riemannian manifold $(\mathcal{B}, \mathbf{G})$. It is assumed that the body is stress free in the material manifold. The deformation gradient is the tangent map of $\varphi$ and is denoted by $\mathbf{F} = T\varphi$. At every point $\mathbf{X} \in \mathcal{B}$, $\mathbf{F}$ is a linear map $\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S}$. Choosing local coordinate charts $\{x^a\}$ and $\{X^A\}$ on $\mathcal{S}$ and $\mathcal{B}$, respectively, the components of $\mathbf{F}$ read

$$F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X}). \tag{8.2.3}$$

The transpose of $\mathbf{F}$ is defined by $\mathbf{F}^{\mathsf{T}} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{X}}\mathcal{B}$, $\langle\!\langle \mathbf{F}\mathbf{V}, \mathbf{v} \rangle\!\rangle_{\mathbf{g}} = \left\langle\!\left\langle \mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v} \right\rangle\!\right\rangle_{\mathbf{G}}$, for all $\mathbf{V} \in T_{\mathbf{X}}\mathcal{B}$, $\mathbf{v} \in T_{\mathbf{x}}\mathcal{S}$. In components $(F^{\mathsf{T}}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x}) F^b{}_B(\mathbf{X}) G^{AB}(\mathbf{X})$, where $\mathbf{g}$ and $\mathbf{G}$ are metric tensors on $\mathcal{S}$ and $\mathcal{B}$, respectively. The right Cauchy-Green deformation tensor $\mathbf{C}(X) : T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{X}}\mathcal{B}$ is defined by $\mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^{\mathsf{T}}\mathbf{F}(\mathbf{X}) = (g_{ab} \circ \varphi) F^a{}_A F^b{}_B$. The relation between the Riemannian volume element $dV$ at $\mathbf{X} \in \mathcal{B}$ and its corresponding deformed volume element at $\mathbf{x} = \varphi(\mathbf{X}) \in \mathcal{S}$ is $dv = J dV$, where $J = \sqrt{\det \mathbf{g} / \det \mathbf{G}} \det \mathbf{F}$ is the Jacobian.

The left Cauchy-Green deformation tensor is defined as $\mathbf{B}^\sharp = \varphi^*(\mathbf{g}^\sharp)$ with components $B^{AB} = (F^{-1})^A{}_a (F^{-1})^B{}_b \, g^{ab}$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are

$$\mathbf{c}^\flat = \varphi_*(\mathbf{G}), \quad c_{ab} = \left(F^{-1}\right)^A{}_a \left(F^{-1}\right)^B{}_b \, G_{AB}, \tag{8.2.4}$$

$$\mathbf{b}^\sharp = \varphi_*(\mathbf{G}^\sharp), \quad b^{ab} = F^a{}_A F^b{}_B G^{AB}, \tag{8.2.5}$$

where $\varphi^*$ and $\varphi_*$ are the pull-back and push-forward by $\varphi$, respectively. $\mathbf{b}^\sharp$ is called the Finger deformation tensor. Note that $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants denoted by $I_1$, $I_2$, and $I_3$ [Ogd84]. For an isotropic material the strain energy function $W$ depends only on the principal invariants of $\mathbf{b}$. One can show that for a compressible and isotropic material the Cauchy stress has the following representation [DE56, SM83]

$$\boldsymbol{\sigma} = 2\left(\frac{I_2}{I_3}\frac{\partial W}{\partial I_2} + \frac{\partial W}{\partial I_3}\right)\mathbf{g}^\sharp + 2\frac{\partial W}{\partial I_1}\mathbf{b}^\sharp - 2\frac{\partial W}{\partial I_2}\mathbf{b}^{-1}. \tag{8.2.6}$$

Similarly, for an incompressible and isotropic material the Cauchy stress has the following representation [DE56, SM83]

$$\boldsymbol{\sigma} = \left(-p + 2I_2\frac{\partial W}{\partial I_2}\right)\mathbf{g}^\sharp + 2\frac{\partial W}{\partial I_1}\mathbf{b}^\sharp - 2\frac{\partial W}{\partial I_2}\mathbf{b}^{-1}. \tag{8.2.7}$$

**Material manifold**. We assume that the defect-free body is stress-free in Euclidean space in the absence of external loads. This body may be made of a material with multiple stress-free configurations (corresponding to multiple wells of a strain-energy density) but we assume that all the material points are in the same energy well (that is we are not considering phase transformations). Next, we assume that some distribution of defects appears in this body and induces residual stresses (we are not considering nucleation or the work associated with the creation of defects). The stress-free configuration with defects (referred to as the material manifold) explicitly depends on the distribution of defects and their types but not on the constitutive equations. Of course, residual stresses explicitly depend on the choice of the constitutive equations.

**Point defects**. The classical continuum picture of a single vacancy is the following. Remove a small spherical ball from the body and identify all the points on its boundary sphere. A single interstitial or "extra matter" can be visualized by inserting a larger elastic ball inside the spherical cavity and letting the system relax. Consider a distribution of point defects or "extra matter" in a solid. If one imagines partitioning this body into a large number of small cubes of the same size and let them relax the resulting stress-free cubes will have different sizes (and hence volumes) depending on the distribution of point defects. The relaxed volumes are the local embedding of the underlying Riemannian material manifold into the ambient (Euclidean) space. In other words, in the stress-free configuration of the defective-solid volume elements vary depending on the distribution of point defects. It is known that in a metric-affine manifold $(\mathcal{B}, \nabla, \mathbf{G})$ with non-metricity the Riemannian volume element is not covariantly constant. More specifically, $D\sqrt{\det \mathbf{G}} = d\sqrt{\det \mathbf{G}} - \omega^{\alpha}{}_{\alpha}\sqrt{\det \mathbf{G}} = -\frac{n}{2} Q\sqrt{\det \mathbf{G}}$. This shows that the Weyl one-form $Q$ is somehow related to the volume density of point defects.

In the following example, we will use a semi-inverse method and start with a coframe field with some unknown function(s). This then implies that the material metric is known up to the unknown function(s). To relate this unknown function(s) to the volume density of point defects we will find a compatible volume element on the material manifold $\mathcal{B}$, i.e., a volume element that is covariantly constant.

## 8.3 Point Defects in an Incompressible Isotropic Ball

In [YG12b] we considered a ball of radius $R_o$ with a spherically-symmetric isotropic distribution of point defects. We constructed the material manifold and calculated the residual stress field for an incompressible neo-Hookean solid. Here we first construct the material manifold and then calculate the residual stress field when the defective ball is made of an arbitrary incompressible isotropic solid. We also consider a special class of compressible solids for a particular example of distributed point defects.

### *8.3.1 Construction of the Flat Weyl Material Manifold*

In a body with only point defects the material manifold is a flat Weyl manifold, i.e., the torsion and the curvature of the material connection both vanish. In order to find a solution, we start by an ansatz for the material coframe field. In the spherical coordinates $(R, \Theta, \Phi)$, $R \geq 0,\ 0 \leq \Theta \leq \pi,\ 0 \leq \Phi < 2\pi$, we assume the following coframe field

$$\vartheta^1 = f(R)dR, \quad \vartheta^2 = Rd\Theta, \quad \vartheta^3 = R\sin\Theta\, d\Phi, \tag{8.3.1}$$

for some unknown function $f(R) > 0$ to be determined. Assuming that the non-metricity is traceless and isotropic, i.e., $\mathcal{Q}_{\alpha\beta} = 2\delta_{\alpha\beta}\, q(R)\vartheta^1$, the matrix of connection 1-forms has the following form

$$\boldsymbol{\omega} = [\omega^\alpha{}_\beta] = \begin{pmatrix} \omega^1{}_1 & \omega^1{}_2 & -\omega^3{}_1 \\ -\omega^1{}_2 & \omega^2{}_2 & \omega^2{}_3 \\ \omega^3{}_1 & -\omega^2{}_3 & \omega^3{}_3 \end{pmatrix}, \tag{8.3.2}$$

where $\omega^1{}_1 = \omega^2{}_2 = \omega^3{}_3 = q(R)\vartheta^1$, for a function $q(R)$ to be determined. We now need to enforce $\mathcal{T}^\alpha = 0$. Note that

$$d\vartheta^1 = 0, \quad d\vartheta^2 = \frac{1}{Rf(R)}\vartheta^1 \wedge \vartheta^2, \quad d\vartheta^3 = -\frac{1}{Rf(R)}\vartheta^3 \wedge \vartheta^1 + \frac{\cot\Theta}{R}\vartheta^2 \wedge \vartheta^3. \tag{8.3.3}$$

Cartan's first structural equations read

$$\mathcal{T}^1 = \omega^1{}_2 \wedge \vartheta^2 - \omega^3{}_1 \wedge \vartheta^3 = 0, \tag{8.3.4}$$

$$\mathcal{T}^2 = \left[\frac{1}{Rf(R)} + q(R)\right]\vartheta^1 \wedge \vartheta^2 - \omega^1{}_2 \wedge \vartheta^1 + \omega^2{}_3 \wedge \vartheta^3 = 0, \tag{8.3.5}$$

$$\begin{aligned}\mathcal{T}^3 = {} & \frac{\cot\Theta}{R}\vartheta^2 \wedge \vartheta^3 - \left[\frac{1}{Rf(R)} - \frac{1}{R} + q(R)\right]\vartheta^3 \wedge \vartheta^1 \\ & + \omega^3{}_1 \wedge \vartheta^1 - \omega^2{}_3 \wedge \vartheta^2 = 0.\end{aligned} \tag{8.3.6}$$

This implies that

$$\omega^1{}_2 = -\left[\frac{1}{Rf(R)} + q(R)\right]\vartheta^2, \quad \omega^2{}_3 = -\frac{\cot\Theta}{R}\vartheta^3, \quad \omega^3{}_1 = \left[\frac{1}{Rf(R)} + q(R)\right]\vartheta^3. \tag{8.3.7}$$

It can be shown that $\mathcal{R}^1{}_1 = \mathcal{R}^2{}_2 = \mathcal{R}^3{}_3 = 0$ are trivially satisfied. The remaining Cartan's second structural equations read

$$\mathcal{R}^1{}_2 = -\mathcal{R}^2{}_1 = d\omega^1{}_2 + \omega^3{}_1 \wedge \omega^2{}_3 = 0, \tag{8.3.8}$$

$$\mathcal{R}^2{}_3 = -\mathcal{R}^3{}_2 = d\omega^2{}_3 + \omega^1{}_2 \wedge \omega^3{}_1 = 0, \tag{8.3.9}$$

$$\mathcal{R}^3{}_1 = -\mathcal{R}^1{}_3 = d\omega^3{}_1 + \omega^2{}_3 \wedge \omega^1{}_2 = 0. \tag{8.3.10}$$

The first equation gives us the following ODE

$$\frac{1}{f(R)}\frac{d}{dR}\left[\frac{1}{Rf(R)}+q(R)\right]+\frac{1}{Rf(R)}\left[\frac{1}{Rf(R)}+q(R)\right]=0. \tag{8.3.11}$$

The solution is

$$\frac{1}{Rf(R)}+q(R)=\frac{C}{R}. \tag{8.3.12}$$

Note that when $q(R)=0$, we have $f(R)=1$ and hence $C=1$. Therefore

$$q(R)=\frac{1}{R}\left[1-\frac{1}{f(R)}\right]. \tag{8.3.13}$$

In this example the Weyl 1-form is written as

$$Q=2q(R)\vartheta^1=\frac{2}{R}\left[1-\frac{1}{f(R)}\right]\vartheta^1=\frac{2(f(R)-1)}{R}dR. \tag{8.3.14}$$

The function $f(R)$ is determined using the volume density of point defects $\mathsf{n}(\mathbf{X})$ and using the equation $\boldsymbol{\mu}_0-\boldsymbol{\mu}=\mathsf{n}\boldsymbol{\mu}_0$ [YG12b]. In the particular example of a defective ball $\boldsymbol{\mu}_0=R^2\sin\Phi dR\wedge d\Theta\wedge d\Phi$ and $\boldsymbol{\mu}=f(R)h(R)\boldsymbol{\mu}_0$, and hence

$$f(R)=\frac{1-\mathsf{n}(R)}{1-\frac{1}{R^3}\int_0^R 3y^2\mathsf{n}(y)dy}. \tag{8.3.15}$$

### 8.3.2 Calculation of the Residual Stress Field

In this section we extend our previous calculation in [YG12b] to arbitrary incompressible isotropic solids and a certain class of compressible isotropic solids. We consider a ball of radius $R_o$ and assume that a point defect density $\mathsf{n}(R)$ is given.

#### Incompressible Isotropic Solids

The material metric has the following form

$$\mathbf{G}=\begin{pmatrix} f^2(R) & 0 & 0\\ 0 & R^2 & 0\\ 0 & 0 & R^2\sin^2\Theta\end{pmatrix},\qquad \mathbf{G}^\sharp=\begin{pmatrix} \frac{1}{f^2(R)} & 0 & 0\\ 0 & \frac{1}{R^2} & 0\\ 0 & 0 & \frac{1}{R^2\sin^2\Theta}\end{pmatrix}. \tag{8.3.16}$$

Having the underlying Riemannian material manifold, we obtain the residual stress field by embedding it into the Euclidean ambient space, which is the Euclidean

3-space. We look for solutions of the form $(r, \phi, z) = (r(R), \Phi, Z)$, and hence $\det \mathbf{F} = r'(R)$. Assuming an incompressible solid we have

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{r^2}{f(R)R^2} r'(R) = 1. \tag{8.3.17}$$

Assuming that $r(0) = 0$, we obtain

$$r(R) = \left( \int_0^R 3x^2 f(x) dx \right)^{\frac{1}{3}}. \tag{8.3.18}$$

The Finger tensor $\mathbf{b}^\sharp$ ($b^{ab} = F^a{}_A F^b{}_B G^{AB}$) is found to be

$$\mathbf{b}^\sharp = \begin{pmatrix} \frac{R^4}{r^4(R)} & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & \frac{1}{R^2 \sin^2 \Theta} \end{pmatrix}. \tag{8.3.19}$$

The principal invariants of $\mathbf{b}$ read

$$I_1 = 2\frac{r^2(R)}{R^2} + \frac{R^4}{r^4(R)}, \quad I_2 = 2\frac{R^2}{r^2(R)} + \frac{r^4(R)}{R^4}. \tag{8.3.20}$$

Now $(b^{-1})^{ab} = c^{ab} = g^{am} g^{bm} c_{mn}$ has the following representation

$$\mathbf{b}^{-1} = \begin{pmatrix} \frac{r^4(R)}{R^4} & 0 & 0 \\ 0 & \frac{R^2}{r^4(R)} & 0 \\ 0 & 0 & \frac{R^2}{r^4(R) \sin^2 \Theta} \end{pmatrix}. \tag{8.3.21}$$

Therefore, the Cauchy stress can be written as

$$\boldsymbol{\sigma} = \begin{pmatrix} -p + \alpha \frac{R^4}{r^4} + 2\beta \frac{R^2}{r^2} & 0 & 0 \\ 0 & \frac{-p}{r^2} + \frac{\alpha}{R^2} + \beta \left( \frac{R^2}{r^4} + \frac{r^2}{R^4} \right) & 0 \\ 0 & 0 & \frac{1}{\sin^2 \Theta} \left[ \frac{-p}{r^2} + \frac{\alpha}{R^2} + \beta \left( \frac{R^2}{r^4} + \frac{r^2}{R^4} \right) \right] \end{pmatrix}, \tag{8.3.22}$$

where $\alpha = 2\frac{\partial W}{\partial I_1}$ and $\beta = 2\frac{\partial W}{\partial I_2}$.

The only non-trivial equilibrium equation reads

$$\sigma^{rr}{}_{,r} + \frac{2}{r}\sigma^{rr} - r\sigma^{\theta\theta} - r \sin^2 \theta \, \sigma^{\phi\phi} = 0. \tag{8.3.23}$$

Or

$$\sigma^{rr}{}_{,R} + \frac{2fR^2}{r^2}\left(\frac{1}{r}\sigma^{rr} - r\sigma^{\theta\theta}\right) = 0. \tag{8.3.24}$$

This gives us the differential equation $p'(R) = k(R)$, where

$$k(R) = \alpha'(R)\frac{R^4}{r^4(R)} + 2\beta'(R)\frac{R^2}{r^2(R)} + 2\left(\beta(R) + \alpha(R)\frac{R^2}{r^2(R)}\right) \left[\frac{2R}{r^2(R)} - f(R)\frac{R^4}{r^5(R)} - f(R)\frac{r(R)}{R^2}\right]. \tag{8.3.25}$$

Suppose that at the boundary $\sigma^{rr}(R_o) = -p_\infty$. Thus

$$p(R) = p_\infty + \alpha(R_o)\frac{R_o^4}{r^4(R_o)} + 2\beta(R_o)\frac{R_o^2}{r^2(R_o)} - \int_R^{R_o} k(\xi)d\xi. \tag{8.3.26}$$

Once $p(R)$ is known all the stress components are easily calculated from (8.3.22).

*Example 8.3.1* In [YG12b] we considered the following point defect distribution

$$\mathsf{n}(R) = \begin{cases} \mathsf{n}_0 & 0 \le R \le R_i, \\ 0 & R_i < R \le R_0, \end{cases} \tag{8.3.27}$$

where $R_i < R_o$. Thus, one can see that

$$0 \le R \le R_i : \ f(R) = 1, \tag{8.3.28}$$

$$R > R_i : \ f(R) = \frac{1}{1 - \mathsf{n}_0\left(\frac{R_i}{R}\right)^3}. \tag{8.3.29}$$

This yields

$$0 \le R \le R_i : \ r(R) = R, \tag{8.3.30}$$

$$R > R_i : \ r(R) = \left[R^3 + \mathsf{n}_0 R_i^3 \ln\left(\frac{(R/R_i)^3 - \mathsf{n}_0}{1 - \mathsf{n}_0}\right)\right]^{\frac{1}{3}}. \tag{8.3.31}$$

Note that for $R < R_i$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and hence $I_1 = I_2 = 3$. Therefore, $\alpha = \alpha_0$ and $\beta = \beta_0$ are constants and consequently for $R < R_i$, $k(R) = 0$. Thus

$$0 \le R \le R_i : \ p(R) = p_\infty + \frac{\alpha(R_o)R_o^4}{r^4(R_o)} + 2\frac{\beta(R_o)R_o^2}{r^2(R_o)} - \int_{R_i}^{R_o} k(\xi)d\xi = p_i, \tag{8.3.32}$$

$$R_i \le R \le R_o : \ p(R) = p_\infty + \frac{\alpha(R_o)R_o^4}{r^4(R_o)} + 2\frac{\beta(R_o)R_o^2}{r^2(R_o)} - \int_R^{R_o} k(\xi)d\xi. \tag{8.3.33}$$

It is seen that pressure is uniform for $R < R_i$. Now the radial stress has the following distribution

$$0 \leq R \leq R_i : \ \sigma^{rr}(R) = \alpha_0 + 2\beta_0 - p_i = \sigma_i, \tag{8.3.34}$$

$$R_i \leq R \leq R_o : \ \sigma^{rr}(R) = \frac{\alpha(R)R^4}{r^4(R)} + 2\frac{\beta(R)R^2}{r^2(R)} - p_\infty - \frac{\alpha(R_o)R_o^4}{r^4(R_o)} - 2\frac{\beta(R_o)R_o^2}{r^2(R_o)} + \int_R^{R_o} k(\xi)d\xi. \tag{8.3.35}$$

It is seen that the radial stress is uniform and equal to $\sigma_i$ for $R < R_i$. Note that the other two physical components of stress are also equal to $\sigma_i$ for $R < R_i$.

*Remark 8.3.2* Note that the defective ball does not have to be homogenous. One can have different energy functions for $R < R_i$ and $R > R_i$. In this case $W = W(R, I_1, I_2)$ and hence $\alpha$ and $\beta$ will have jumps at $R = R_i$. This will not affect the pressure for $R < R_i$. However, for $R > R_i$, one should add the term $[\![\alpha + 2\beta]\!]_{R_i}$ to the pressure field.

**Compressible Isotropic Solids**

Next we consider a spherically-symmetric point defect distribution in a ball made of a compressible isotropic solid. For an isotropic solid instead of considering the strain energy density as a function of the principal invariants of $\mathbf{C}$ one can assume that $W$ explicitly depends on the principal invariants of $\mathbf{U}$, i.e. $W = \hat{W}(i_1, i_2, i_3)$, where

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{8.3.36}$$

Carroll [Car88] rewrote the representation of the Cauchy stress for isotropic elastic solids in terms of the left stretch tensor. In our geometric framework [Car88]'s Eq. (2.15) is rewritten as [YG13a]

$$\boldsymbol{\sigma} = \left( \frac{i_2}{i_3}\frac{\partial \hat{W}}{\partial i_2} + \frac{\partial \hat{W}}{\partial i_3} \right)\mathbf{g}^\sharp + \frac{1}{i_3}\frac{\partial \hat{W}}{\partial i_1}\mathbf{V}^\sharp - \frac{\partial \hat{W}}{\partial i_2}\mathbf{V}^{-1}. \tag{8.3.37}$$

In components this reads

$$\sigma^{ab} = \left( \frac{i_2}{i_3}\frac{\partial \hat{W}}{\partial i_2} + \frac{\partial \hat{W}}{\partial i_3} \right) g^{ab} + \frac{1}{i_3}\frac{\partial \hat{W}}{\partial i_1} V^{ab} - \frac{\partial \hat{W}}{\partial i_2} \left(V^{-1}\right)^{ab}, \tag{8.3.38}$$

where $b^{ab} = V^{am}V^{bn}g_{mn}$ and $c^{ab} = \left(V^{-1}\right)^{am}\left(V^{-1}\right)^{bn} g_{mn}$. Carroll [Car88] considered a special class of compressible materials for which $\hat{W}(i_1, i_2, i_3) =$

$u(i_1) + v(i_2) + w(i_3)$, where $u$, $v$, and $w$ are arbitrary $C^2$ functions. For this class of materials we have

$$\boldsymbol{\sigma} = \left(\frac{i_2}{i_3} v'(i_2) + w'(i_3)\right) \mathbf{g}^\sharp + \frac{u'(i_1)}{i_3} \mathbf{V}^\sharp - v'(i_2)\mathbf{V}^{-1}. \tag{8.3.39}$$

In the case of a ball with a spherically-symmetric point defect distribution, we have

$$\lambda_1 = \frac{r'(R)}{f(R)}, \quad \lambda_2 = \lambda_3 = \frac{r(R)}{R}. \tag{8.3.40}$$

Thus

$$i_1 = \frac{r'(R)}{f(R)} + \frac{2r(R)}{R}, \quad i_2 = \frac{2r(R)r'(R)}{Rf(R)} + \frac{r^2(R)}{R^2}, \quad i_3 = \frac{r'(R)r^2(R)}{R^2 f(R)}. \tag{8.3.41}$$

A simple calculation gives us

$$\mathbf{V}^\sharp = \begin{pmatrix} \frac{r'(R)}{f(R)} & 0 & 0 \\ 0 & \frac{1}{Rr(R)} & 0 \\ 0 & 0 & \frac{1}{Rr(R)\sin^2\Theta} \end{pmatrix}, \quad \mathbf{V}^{-1} = \begin{pmatrix} \frac{f(R)}{r'(R)} & 0 & 0 \\ 0 & \frac{R}{r^3(R)} & 0 \\ 0 & 0 & \frac{R}{r^3(R)\sin^2\Theta} \end{pmatrix}. \tag{8.3.42}$$

Hence, the non-zero stress components read

$$\sigma^{rr}(R) = u'(i_1)\frac{R^2}{r^2(R)} + v'(i_2)\frac{2R}{r(R)} + w'(i_3), \tag{8.3.43}$$

$$\sigma^{\theta\theta}(R) = u'(i_1)\frac{Rf(R)}{r^3(R)r'(R)} + v'(i_2)\left(\frac{R}{r^3(R)} + \frac{f(R)}{r^2(R)r'(R)}\right) + \frac{w'(i_3)}{r^2(R)}, \tag{8.3.44}$$

$$\sigma^{\phi\phi}(R) = \frac{1}{\sin^2\Theta}\sigma^{\theta\theta}(R). \tag{8.3.45}$$

The equilibrium equation (8.3.23) is simplified to read

$$\frac{R^2}{r^2}\frac{du'}{dr} + \frac{2R}{r}\frac{dv'}{dr} + \frac{dw'}{dr} + 2(1-f)\left(\frac{Ru'}{r^2 r'} + \frac{v'}{rr'}\right) = 0. \tag{8.3.46}$$

We first consider a harmonic material [Joh60] for which $v(i_2) = c_2(i_2 - 3)$ and $w(i_3) = c_3(i_3 - 1)$, where $c_2$ and $c_3$ are constants (Class I materials according to Carroll [Car88]). In this case the above ODE is reduced to

$$\frac{du'}{dr} + 2(1-f)\left(\frac{u'}{Rr'} + \frac{c_2 r}{R^2 r'}\right) = 0. \tag{8.3.47}$$

Let us now consider the point defect distribution (8.3.27). For $R < R_i$, $f(R) = 1$ and hence $\frac{du'}{dr} = 0$. This implies that $i_1$ must be a constant and therefore

$$r(R) = C_1 R + \frac{C_2}{R^2}. \tag{8.3.48}$$

For $r(R)$ to be bounded at the origin we must have $C_2 = 0$. Now for $R < R_i$, $i_1 = 3C_1$ and hence the physical components of Cauchy stress read

$$\bar{\sigma}^{rr}(R) = \sigma^{rr}(R) = \frac{u'(3C_1)}{C_1^2} + \frac{2c_2}{C_1} + c_3 = \sigma_1, \tag{8.3.49}$$

$$\bar{\sigma}^{\theta\theta}(R) = r^2(R)\sigma^{\theta\theta}(R) = \frac{u'(3C_1)}{C_1^2} + \frac{2c_2}{C_1} + c_3 = \sigma_1, \tag{8.3.50}$$

$$\bar{\sigma}^{\phi\phi}(R) = \bar{\sigma}^{\theta\theta}(R), \tag{8.3.51}$$

i.e., the Cauchy stress inside the point defect sphere is uniform and hydrostatic.

For Classes II and III materials according to Carroll [Car88], $u(i_1) = c_1(i_1 - 3)$, $w(i_3) = c_3(i_3 - 1)$ and $u(i_1) = c_1(i_1 - 3)$, $v(i_2) = c_2(i_2 - 3)$, respectively. For $R < R_i$, for Class II materials, we have $r^2(R) = C_1 R^2 + \frac{C_2}{R}$. Similarly, for Class III materials we have $r^3(R) = C_1 R^3 + C_2$. Assuming that $r(0) = 0$ in both cases $C_2 = 0$ and hence for $R < R_i$, we have $r(R) = \alpha R$, where $\alpha$ is a constant. This is identical to what we observed for harmonic materials. Therefore, the above result holds for materials of Types II and III as well. The unknown constant $C_1$ is determined after one solves a nonlinear second-order ODE for $r$ in the interval $R_i < R \leq R_o$ and imposes the continuity conditions $r(R_i^-) = r(R_i^+)$, $\sigma^{rr}(R_i^-) = \sigma^{rr}(R_i^-)$, and the boundary condition $\sigma^{rr}(R_o) = -p_\infty$.

### *8.3.3 An Analogy Between the Point Defect Metric and the Schwarzschild Metric*

Einstein's vacuum field equations can be solved exactly for a spherically-symmetric distribution of matter with gravitational mass $m$. The solution is called the Schwarzschild (exterior) solution. In the coordinates $(t, R, \Theta, \Phi)$ for space-time, the Schwarzschild metric reads [HE73]

$$dS^2 = -\left(1 - \frac{2m}{R}\right) dt^2 + \left(1 - \frac{2m}{R}\right)^{-1} dR^2 + R^2 d\Theta^2 + R^2 \sin^2\Theta d\Phi^2. \tag{8.3.52}$$

This metric represents the gravitational field outside of a ball of mass $m$. Note that this solution is valid only for $R > 2m$. The interior solution can be determined using the energy-momentum tensor of the matter inside the ball. When restricted to

$(R, \Theta, \Phi)$ this metric looks very similar to our point defect metric if one replaces $\left(1 - \frac{2m}{R}\right)^{-1}$ by $f(R)$. Next we consider a single point defect and observe another interesting similarity between our metric and that of Schwarzschild.

### 8.3.4 Singularity in the Material Metric in the Case of a Single Point Defect

We consider a single point defect with strength $\delta v$ at the center of the ball. In this case

$$\mathsf{n}(R) = \frac{\delta v}{4\pi R^2}\delta(R), \tag{8.3.53}$$

where $\delta(R)$ is the one-dimensional Dirac delta distribution. Therefore

$$h(R) = 1 - \frac{3\delta v}{4\pi R^3}. \tag{8.3.54}$$

Hence

$$f(R) = \frac{1 - \frac{\delta v}{4\pi R^2}\delta(R)}{1 - \frac{3\delta v}{4\pi R^3}} = \frac{R^3 - \frac{\delta v}{4\pi} R\delta(R)}{R^3 - \frac{3\delta v}{4\pi}} = \frac{1}{1 - \frac{3\delta v}{4\pi R^3}}. \tag{8.3.55}$$

Note that $f(R) > 0$ and hence this expression is meaningful only when

$$R > \left(\frac{3\delta v}{4\pi}\right)^{\frac{1}{3}}. \tag{8.3.56}$$

### 8.3.5 Exterior Residual Stress Field of a Ball of Point Defects

In Yavari and Goriely [YG12b] we considered a finite ball of radius $R_o$ with a uniform defect distribution $\mathsf{n}_0$ in a small ball of radius $R_i$ and showed that the stress inside the defective ball is uniform for $R < R_i$. Let us now assume that $\mathsf{n}(R) = 0$ for $R > R_i$ but is elsewhere arbitrary. The total volume of the point defects is

$$\delta v = \int_0^{R_i} 4\pi R^2 \mathsf{n}(R) dR. \tag{8.3.57}$$

Note that for $R > R_i$, we have

$$f(R) = \frac{1}{1 - \frac{3\delta v}{4\pi R^3}}. \tag{8.3.58}$$

It is seen that for $R > R_i$, $f(R)$ depends only on $\delta v$ and not on the specific distribution of $\mathsf{n}(R)$ for $R < R_i$. Therefore, $r(R)$ and consequently all the stress components for $R > R_i$ depend only on $\delta v$. Thus, we have proved the following proposition.

**Proposition 8.3.3** *Consider a ball of radius $R_o$ made of an isotropic elastic material or a material with anisotropy respecting the spherical symmetry. Assume that the ball is defect free outside a ball of radius $R_i < R_o$. Then, the residual stress field for $R > R_i$ depends only on the total volume of the point defects in $R < R_i$ and is independent of the specific form of $\mathsf{n}(R)$ for $R < R_i$.*

*Remark 8.3.4* Note that this result is similar to the effect of a spherical ball of matter with mass $m$ on the gravitational field. The gravitational field of the space-time outside the ball depends only on $m$ and not on the specific distribution of density inside the ball (as long as it is spherically symmetric).

### *8.3.6 Isotropic Distribution of Point Defects and Pure Dilatational Eigenstrains*

We know that in the limit of a vanishing inclusion with pure dilatational eigenstrain the linearized solution for a single point defect in an isotropic linear elastic solid is recovered as shown in [YG12b] by fixing $\delta v = 4\pi R_i^3 n_0/3$ and in the limit of small $R_i$. Note that our point defect metric is equivalent to that of a distributed radial eigenstrain. A natural question is: can we represent an isotropic distribution of point defects by a pure dilatational eigenstrain distribution? We will see in the following that the answer is negative.

Consider a coframe field of the following form

$$\vartheta^1 = K(R)dR, \quad \vartheta^2 = K(R)Rd\Theta, \quad \vartheta^3 = K(R)R\sin\Theta\, d\Phi, \tag{8.3.59}$$

for some unknown function $K(R)$ to be determined. Assuming that the non-metricity is traceless and isotropic $\mathcal{Q}_{\alpha\beta} = 2\delta_{\alpha\beta}\, q(R)\vartheta^1$, the matrix of connection 1-forms has the following form

$$\boldsymbol{\omega} = [\omega^\alpha{}_\beta] = \begin{pmatrix} \omega^1{}_1 & \omega^1{}_2 & -\omega^3{}_1 \\ -\omega^1{}_2 & \omega^2{}_2 & \omega^2{}_3 \\ \omega^3{}_1 & -\omega^2{}_3 & \omega^3{}_3 \end{pmatrix}, \tag{8.3.60}$$

where $\omega^1{}_1 = \omega^2{}_2 = \omega^3{}_3 = q(R)\vartheta^1$, for a function $q(R)$ to be calculated. Note that

$$\begin{aligned} d\vartheta^1 &= 0, \quad d\vartheta^2 = \frac{1}{K(R)}\left[\frac{1}{R} + \frac{K'(R)}{K(R)}\right]\vartheta^1 \wedge \vartheta^2, \\ d\vartheta^3 &= -\frac{1}{K(R)}\left[\frac{1}{R} + \frac{K'(R)}{K(R)}\right]\vartheta^3 \wedge \vartheta^1 + \frac{\cot\Theta}{RK(R)}\vartheta^2 \wedge \vartheta^3. \end{aligned} \tag{8.3.61}$$

Cartan's first structural equations read

$$\mathcal{T}^1 = \omega^1{}_2 \wedge \vartheta^2 - \omega^3{}_1 \wedge \vartheta^3 = 0, \tag{8.3.62}$$

$$\mathcal{T}^2 = \left\{ \frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] + q(R) \right\} \vartheta^1 \wedge \vartheta^2 - \omega^1{}_2 \wedge \vartheta^1 + \omega^2{}_3 \wedge \vartheta^3 = 0, \tag{8.3.63}$$

$$\mathcal{T}^3 = \frac{\cot \Theta}{RK(R)} \vartheta^2 \wedge \vartheta^3 - \left\{ \frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] + q(R) \right\} \vartheta^3 \wedge \vartheta^1 + \omega^3{}_1 \wedge \vartheta^1 - \omega^2{}_3 \wedge \vartheta^2 = 0. \tag{8.3.64}$$

This gives us

$$\omega^1{}_2 = -\left\{ \frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] + q(R) \right\} \vartheta^2, \quad \omega^2{}_3 = -\frac{\cot \Theta}{RK(R)} \vartheta^3,$$
$$\omega^3{}_1 = \left\{ \frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] + q(R) \right\} \vartheta^3. \tag{8.3.65}$$

It can be shown that $\mathcal{R}^1{}_1 = \mathcal{R}^2{}_2 = \mathcal{R}^3{}_3 = 0$ are trivially satisfied. The remaining Cartan's second structural equations read

$$\mathcal{R}^1{}_2 = -\mathcal{R}^2{}_1 = d\omega^1{}_2 + \omega^3{}_1 \wedge \omega^2{}_3 = 0, \tag{8.3.66}$$
$$\mathcal{R}^2{}_3 = -\mathcal{R}^3{}_2 = d\omega^2{}_3 + \omega^1{}_2 \wedge \omega^3{}_1 = 0, \tag{8.3.67}$$
$$\mathcal{R}^3{}_1 = -\mathcal{R}^1{}_3 = d\omega^3{}_1 + \omega^2{}_3 \wedge \omega^1{}_2 = 0. \tag{8.3.68}$$

The first equation gives us the following ODE

$$\frac{d}{dR} \left\{ \frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] + q(R) \right\} + \left( \frac{1}{R} + \frac{K'(R)}{K(R)} \right) \left\{ \frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] + q(R) \right\} = 0, \tag{8.3.69}$$

with solution

$$\frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] + q(R) = \frac{C}{RK(R)}. \tag{8.3.70}$$

Note that when $q(R) = 0$, we have $K(R) = 1$ and hence $C = 1$. Therefore

$$q(R) = \frac{1}{RK(R)} - \frac{1}{K(R)} \left[ \frac{1}{R} + \frac{K'(R)}{K(R)} \right] = -\frac{K'(R)}{K^2(R)} = \left( \frac{1}{K(R)} \right)'. \tag{8.3.71}$$

Interestingly, the other two curvature 2-forms trivially vanish. The Weyl 1-form is written as

$$Q = 2q(R)\vartheta^1 = \frac{-2K'(R)}{K(R)} dR = -2d \ln K(R). \tag{8.3.72}$$

The governing differential equation for the function $h(R)$ reads

$$d \ln h(R) = \frac{3}{2} Q = -3d \ln K(R). \tag{8.3.73}$$

Therefore, $h(R) = CK^{-3}(R)$. Because for $K(R) = 1$, $h(R) = 1$, $C = 1$, and hence $h(R) = K^{-3}(R)$. Note that $\boldsymbol{\mu} = h\vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 = h(R)K^3(R)\boldsymbol{\mu}_0 = \boldsymbol{\mu}_0$, and hence $\mathsf{n}(R) = 0$. This means that the metric (8.3.59) cannot represent a spherically-symmetric distribution of point defects.

## 8.4 Conclusions

We discussed the relevance of non-metricity in the nonlinear mechanics of distributed point defects. An anelasticity problem is transformed to a classical nonlinear elasticity problem if one can construct the material manifold, i.e., a 3-manifold in which the defective body is stress-free by construction. The material manifold of a solid with distributed point defects is a flat Weyl manifold, i.e., a manifold with a connection and metric such that the non-metricity is traceless and both the torsion and the curvature tensors vanish. We revisited the problem of a finite ball with a spherically-symmetric and isotropic distribution of point defects. We constructed the material manifold and calculated the residual stress field when the ball is made of an arbitrary incompressible isotropic solid. We observed an interesting analogy between the residual stress field and the gravitational field of space-time with a ball made of matter. We also showed that an isotropic distribution of point defects cannot be represented by a distribution of pure dilatational eigenstrains.

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