Chapter 4
Lectures on the Isometric Embedding Problem $(M^n, g) \to \mathbb{R}^m$, $m = \frac{n}{2}(n+1)$

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Abstract This work derives the basic balance laws of Codazzi, Ricci, and Gauss for the isometric embedding of an n-dimensional Riemannian manifold into the $m = \frac{n}{2}(n+1)$-dimensional Euclidean space. It is shown how the balance laws can be expressed in quasi-linear symmetric form and how weak solutions for the linearized problem can be established from the Lax-Milgram theorem.

4.1 Introduction

Riemann introduced the notion of an abstract manifold with metric structure, his motivation being the problem of defining a surface in Euclidean space independently of the underlying Euclidean space. The isometric embedding problem seeks to establish conditions for the Riemannian manifold to be a submanifold of a Euclidean space having the same metric. For example, consider the smooth n-dimensional Riemannian manifold M^n with metric tensor g. In terms of local coordinates x_i, $i = 1, 2, \ldots, n$ the distance on M^n between neighbouring points is

$$ds^2 = g_{ij} dx_i dx_j, \qquad i, j = 1, 2, \ldots n, \tag{4.1.1}$$

where here and throughout the standard summation convention is adopted. Now let $\mathbb{R}^m$ be m-dimensional Euclidean space, and let $y : M^n \to \mathbb{R}^m$ be a smooth map so that the distance between neighbouring points is given by

$$d\bar{s}^2 = dy.dy = y^i_{,j} y^i_{,k} dx_j dx_k, \tag{4.1.2}$$

where the subscript comma denotes partial differentiation with respect to the local coordinates x_i. *Global embedding* of M^n in $\mathbb{R}^m$ is equivalent to proving the existence

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of the smooth map y for each $x \in M^n$. *Isometric embedding* requires the existence of maps y for which the distances (4.1.1) and (4.1.2) are equal. That is,

$$g_{ij}dx_i dx_j = y^i_{,j} y^i_{,k} dx_j dx_k, \tag{4.1.3}$$

or

$$y^i_{,j} y^i_{,k} = g_{jk}, \tag{4.1.4}$$

which may be compactly rewritten as

$$\partial_i y \cdot \partial_j y = g_{ij}, \tag{4.1.5}$$

where

$$\partial_i = \frac{\partial}{\partial x_i}, \tag{4.1.6}$$

and the inner product in $\mathbb{R}^m$ is denoted by the symbol "·".

The classical isometric embedding of a 2-dimensional Riemannian manifold into a 3-dimensional Euclidean space is comparatively well studied and comprehensively discussed in the book by Han and Hong [HH06]. By contrast, the embedding of n-dimensional Riemannian manifolds into $n(n+1)/2$ Euclidean space has only a comparatively small literature. When $n = 3$, the main results are due to Bryant et al. [BGY83], Nakamura and Maeda [NM86, NM89], Goodman and Yang [GY88], and most recently to Poole [Poo10]. The general, but related, case when $n \geq 3$ is considered by Han and Khuri [HK12]. These studies all rely on a linearization of the full nonlinear system (4.1.4) to establish the embedding y for given metric g_{ij} of the Riemannian manifold.

Applied analysts familiar with continuum mechanics and quasi-linear balance laws might find a presentation of the embedding problem within the context of symmetric quasi-linear forms appealing since there is an accompanying extensive literature originating with Friedrichs [Fri56]. For this and related references, the reader may consult Han and Hong [HH06]. It appears, however, that when the critical Janet dimension is $m = n(n+1)/2$ the isometric embedding problem $(M^n, g) \to \mathbb{R}^m$ has not yet been expressed in symmetric quasi-linear form. The purpose of these self-contained notes is to demonstrate how this may be achieved using the Gauss, Codazzi, and Ricci relations. The existence and uniqueness of a weak solution to these equations is then proved by means of the Lax-Milgram theorem.

4.2 Basic Isometric Embedding Equations

Let (X, g) denote an n-dimensional Riemannian manifold with ascribed metric tensor g. Suppose the manifold (X, g) can be embedded globally into $\mathbb{R}^m$. (The term *immersion* is used when the embedding is local.) As stated in Sect. 4.1, this assumption implies that there exist a system of local coordinates x_i, $i = 1, 2, \ldots n$ on X and embeddings $y_j(x_i)$, $j = 1, 2, \ldots m$ such that (4.1.5) holds.

As an example, consider the 2-dimensional Riemannian manifold viewed as a surface in $\mathbb{R}^3$ and given by $y^1 = x_1$, $y^2 = x_2$, $y^3 = f(x_1, x_2)$, for a smooth function f. See Fig. 4.1.

In introductory courses, Pythagoras' theorem is used to write the distance along the surface as

$$
\begin{aligned}
(ds)^2 &= (dx_1)^2 + (dx_2)^2 + (df)^2 \\
&= (dx_1)^2 + (dx_2)^2 + \left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2\right)^2 \\
&= \left\{1 + \left(\frac{\partial f}{\partial x_1}\right)^2\right\} (dx_1)^2 + 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} dx_1 dx_2 \\
&\quad + \left\{1 + \left(\frac{\partial f}{\partial x_2}\right)^2\right\} (dx_2)^2,
\end{aligned}
$$

and consequently the corresponding metric is

$$
\begin{aligned}
1 + \left(\frac{\partial f}{\partial x_1}\right)^2 &= g_{11}, \\
2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} &= 2g_{12}, \qquad (g_{12} = g_{21}), \\
1 + \left(\frac{\partial f}{\partial x_2}\right)^2 &= g_{22}.
\end{aligned}
\tag{4.2.1}
$$

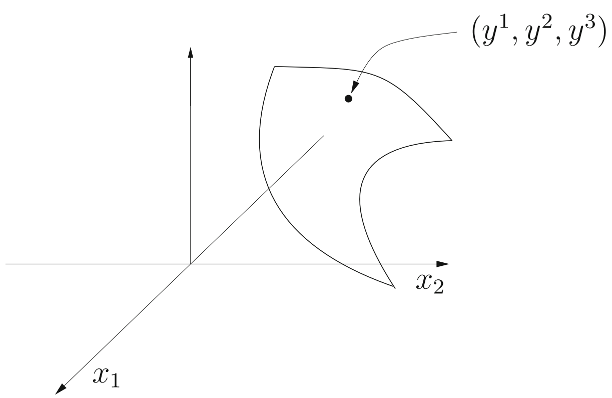

Fig. 4.1 Embedding

Now consider the inverse problem: given the metric as a positive-definite covariant symmetric tensor, to find components y^1, y^2, y^3 that determine the surface. The components $y^1 = x_1$, $y^2 = x_2$ are known, so the question is, can the nonlinear system of partial differential equations (4.2.1) be solved for f given g? (The general system is provided by (4.1.5).) For the example of the embedding of (M^2, g) into $\mathbb{R}^3$, the metric tensor may be displayed in the matrix form

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}$$

which shows that for the system (4.2.1), there is an equation for each component of g. More generally, the symmetry of g_{jk} reduces (4.1.5) to *three* equations for *three* unknowns y^1, y^2, y^2, leading to a *determined* system. On other hand, the embedding of (M^2, g) in $\mathbb{R}^2$ still has three equations but only two components y^1, y^2 of the unknown vector y, (the **overdetermined** case), while the embedding of (M^2, g) into $\mathbb{R}^4$ has three equations to determine *four* unknown components (y^1, y^2, y^3, y^4) (the **underdetermined** case).

For an n-dimensional Riemannian manifold the components of the corresponding metric tensor may be represented by the $n \times n$ symmetric matrix

$$g = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ & \ddots & \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}. \tag{4.2.2}$$

There are $n(n + 1)/2$ entries on and above the diagonal, and we conclude in general that the isometric embedding problem (recovering the "surface" from the metric) is

$$\begin{aligned} &\textbf{underdetermined} \quad \text{when } m > \frac{n}{2}(n+1), \\ &\textbf{determined} \quad \text{when } m = \frac{n}{2}(n+1), \\ &\textbf{overdetermined} \quad \text{when } m < \frac{n}{2}(n+1), \end{aligned}$$

where m is the number of unknowns $(y^1, y^2, \dots, y^m)$, and $n(n+1)/2$ are the number of equations. The crucial number

$$\frac{n}{2}(n+1)$$

is called the *Janet dimension*.

Not too many solutions can be expected in the overdetermined case, and the question of uniqueness has been pursued by several mathematicians. The underdetermined case provides the flexibility of more unknowns than equations rendering superfluous

Riemann's concept of an abstract surface. Specifically, for m sufficiently large, the manifold (M^n, g) embeds globally and smoothly into $\mathbb{R}^m$, and (M^n, g) looks exactly like a surface. The following theorem is the precise statement.

Theorem 4.2.1 (Nash [Nas56]) *Let* $3 \leq k \leq \infty$. *A* C^k*-Riemannian manifold* (M^n, g) *has a* C^k*-embedding into* $\mathbb{R}^m$ *(globally) if*

$$m = n(3n + 11)/2, \quad \textit{compact case},$$
$$m = n(n + 1)(3n + 11)/2, \quad \textit{non-compact case}.$$

Nash's theorem has been improved but the main point to note is that results for global embedding always refer to the underdetermined system. Global embedding (smoothly) is in general not possible for determined systems, where the number of equations equals the number of unknowns, and which conceptually is more familiar in applied mathematics.

It is appropriate to quote from the following relevant section in the paper by S-T Yau [Yau06]:

> **Section 3.13**. **Isometric embedding**. Given a metric tensor on a manifold, the problem of isometric embedding is equivalent to finding enough functions $f_1, \ldots, f_N$ so that the metric can be written as $\Sigma(df_i)^2$. Much work was accomplished for two-dimensional surfaces (as mentioned in Sect. 2.1.2). Isometric embedding for general dimensions was solved in the famous work of J. Nash. Nash used his implicit function theorem which depends on various smoothing operations to gain derivatives. In a remarkable work, Gunther was able to avoid the Nash procedure. He used only standard Hölder regularity estimates for the Laplacian to reproduce the Nash isometric embedding with the same regularity result. In his book, Gromov was able to lower the codimension of the work of Nash. He called his method the h-principle.
>
> When the dimension of the manifold is n, the expected dimension of the Euclidean space for the manifold to be isometrically embedded is $n(n + 1)/2$. It is important to understand manifolds isometrically embedded into Euclidean space with this optimal dimension. Only in such a dimension does it make sense to talk about rigidity questions. *It remains a major open problem whether one can find a nontrivial family of isometric embeddings of a closed manifold into Euclidean space with an optimal dimension.....*
>
> Chern told me that he and Levy studied local isometric embedding of a three manifold into six dimensional Euclidean space, but they did not write any manuscript on it. The major work in this subject is due to E. Berger, Bryant, Griffiths, and Yang. They show that a generic three dimensional embedding system is strictly hyperbolic, and the generic four dimensional system is of principle type. Local existence is true for a generic metric using a hyperbolic operator and the Nash-Moser implicit function theorem...

Remark 4.2.1 The theory of isometric embedding is a classical subject, but our knowledge is still rather limited, especially in dimensions greater than three. Many difficult problems are related to nonlinear mixed type equations or hyperbolic differential equations over closed manifolds.

4.2.1 Preliminary Lemmas

In this section, we state and prove some lemmas of subsequent interest.

Lemma 4.2.1 *Let $X = X' \times I \subset \mathbb{R}^n$, where $X' \subset \mathbb{R}^{n-1}$ is an open domain and I is a connected open interval. Given smooth functions $f : X \times \mathbb{R}^m \to \mathbb{R}^m$ and $A_0 : X' \to \mathbb{R}^m$, where $t \in I$, there exists a unique solution $A : X \to \mathbb{R}^m$ to the system of ordinary differential equations*

$$\partial_n A = f(x', x_n, A),$$
$$A|_{x_n=t} = A_0(x') \qquad \text{for } x' \in X',$$

where $\partial_n = \partial_{x_n}$.

Proof The proof is just that of the standard existence-uniqueness theorem for ordinary differential equations. Here, the independent variable x_n is "time", t is the initial time where the data $A_0(x')$ is specified, x' are parameters on which the data $A_0(x')$ and prescribed $f(x', x_n, A)$ may depend, and A is the unknown function (dependent variable) that is required to be determined.

Lemma 4.2.2 *Let $X \subset \mathbb{R}^n$ be an open contractible domain and let $f_i : X \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy*

$$\frac{\partial f_i^a}{\partial x_j} + \frac{\partial f_i^a}{\partial A^b} f_j^b = \frac{\partial f_j^a}{\partial x_i} + \frac{\partial f_j^a}{\partial A^b} f_i^b \tag{4.2.3}$$

for each $(x, A) \in X \times \mathbb{R}^m$, where the Einstein summation convention is used here and throughout unless otherwise stated. Then given $x_0 \in X$ and $A_0 \in \mathbb{R}^m$, there exists a unique solution $A : X \to \mathbb{R}^m$ to the system

$$\partial_i A = f_i(x, A), \qquad A(x_0) = A_0, \tag{4.2.4}$$

where $\partial_i = \partial_{x_i}$, and $x = (x_1, \ldots, x_n)$.

Proof Lemma 4.2.1 establishes existence and uniqueness provided the system of ordinary differential equations is consistent. But differentiation gives

$$\partial_i \partial_j A = \partial_i f_j(x, A),$$
$$\partial_j \partial_i A = \partial_j f_i(x, A),$$

and the required condition is given by

$$\partial_i f_j(x, A) = \partial_j f_i(x, A).$$

On expanding the partial derivatives, we obtain

$$\frac{\partial f_j}{\partial x_i} + \frac{\partial f_j}{\partial A^b}\frac{\partial A^b}{\partial x_i} = \frac{\partial f_i}{\partial x_j} + \frac{\partial f_i}{\partial A^b}\frac{\partial A^b}{\partial x_j},$$

which by (4.2.4) reduces to

$$\frac{\partial f_j}{\partial x_i} + \frac{\partial f_j}{\partial A^b} f_i^b = \frac{\partial f_i}{\partial x_j} + \frac{\partial f_i}{\partial A^b} f_j^b,$$

which is hypothesis (4.2.3) stipulated in the Lemma. □

Remark 4.2.2 Lemma 4.2.2 is a nonlinear version of the Poincaré lemma, which rather than the fundamental theorem of the calculus uses instead the existence and uniqueness theorem of ordinary differential equations. In the standard Poincaré lemma, the functions f_i do not depend upon A and the statement

$$\frac{\partial f_i^a}{\partial x_j} = \frac{\partial f_j^a}{\partial x_i}$$

implies the existence of a "potential" A with

$$f_i^a = \frac{\partial A^a}{\partial x_i},$$

where

$$\frac{\partial^2 A^a}{\partial x_j \partial x_i} = \frac{\partial^2 A^a}{\partial x_i \partial x_j}.$$

4.2.2 Riemannian Structure in Local Coordinates

We recall some standard results whose derivation and further discussion may be found in most textbooks on differential geometry or tensor analysis.

Let (X, g) be an n-dimensional Riemannian manifold with metric g, and denote the kth covariant derivative by ∇_k. This derivative permits differentiation along the manifold, and for scalars ϕ, vectors ϕ_i and second order tensors ϕ_{ij} is given respectively by

$$\nabla_k \phi = \partial_k \phi, \tag{4.2.5}$$

$$\nabla_k \phi_j = \partial_k \phi_j - \Gamma^l_{jk} \phi_l, \tag{4.2.6}$$

$$\nabla_k \phi_{ij} = \partial_k \phi_{ij} - \Gamma^l_{ik} \phi_{lj} - \Gamma^l_{jk} \phi_{il} \tag{4.2.7}$$

where the *Christoffel symbols* are calculated from the metric g by the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{ij} - \partial_j g_{il} - \partial_l g_{ij} \right). \tag{4.2.8}$$

The metric tensor with components g^{kl} (upper indices) is the inverse of that with components g_{ij} (lower indices) so that

$$g^{kl} g_{pl} = \delta^k_l, \tag{4.2.9}$$

where δ^k_l is the usual Kronecker delta of mixed order defined by

$$\delta^k_l = 1, \qquad \text{when } k = l, \tag{4.2.10}$$
$$= 0, \qquad \text{when } k \neq l. \tag{4.2.11}$$

Kronecker deltas of upper and lower order are defined similarly.

It is well-known that the following identities hold between the above quantities:

$$\nabla_k g_{ij} = 0, \tag{4.2.12}$$
$$\Gamma^k_{ij} = \Gamma^k_{ji}, \tag{4.2.13}$$
$$\partial_k g_{ij} = g_{ip}\Gamma^p_{kj} + g_{jp}\Gamma^p_{ik}, \tag{4.2.14}$$
$$\nabla_i \partial_j = \Gamma^l_{ij} \nabla_l. \tag{4.2.15}$$

The *Riemann curvature tensor*, R^l_{ijk}, defined in terms of Christoffel symbols by

$$R^l_{ijk} = \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \Gamma^l_{jp}\Gamma^p_{ki} - \Gamma^l_{kp}\Gamma^p_{ji}, \tag{4.2.16}$$

is known to satisfy the operator identity

$$R^l_{ijk}\partial_l = -\nabla_j \nabla_k \partial_i + \nabla_k \nabla_j \partial_i. \tag{4.2.17}$$

By lowering indices, we have the *covariant Riemann curvature tensor*

$$R_{ijkl} = g_{iq} R^q_{jkl}, \tag{4.2.18}$$

or

$$R_{ijkl} = g_{iq}\left(\partial_k \Gamma^q_{lj} - \partial_l \Gamma^q_{kj} + \Gamma^q_{kp}\Gamma^p_{lj} - \Gamma^q_{lp}\Gamma^p_{kj}\right), \tag{4.2.19}$$

which possesses the minor *skew-symmetries*

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}, \tag{4.2.20}$$

and the *interchange* (or major) symmetry

$$R_{ijkl} = R_{klij}. \tag{4.2.21}$$

Cyclic interchange of indices leads to the *first Bianchi identity*;

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \tag{4.2.22}$$

and also to the *second Bianchi identity*;

$$\nabla_s R_{ijkl} + \nabla_k R_{ijls} + \nabla_l R_{ijsk} = 0. \tag{4.2.23}$$

Remark 4.2.3 (Special case n = 2) When $n = 2$, the covariant Riemann curvature tensor reduces to

$$R_{ijkl} = K\left(g_{ik}g_{lj} - g_{il}g_{jk}\right), \tag{4.2.24}$$

where K is the *Gauss* curvature given by

$$K = \frac{R_{ijkl}\xi^i\xi^k\eta^j\eta^l}{\left(g_{pq}g_{rs} - g_{pr}g_{qs}\right)\xi^p\xi^q\eta^r\eta^s}, \tag{4.2.25}$$

for any vectors $\xi,\ \eta$.

Remark 4.2.4 The mixed and covariant Riemann curvature tensors involve the first derivatives of Christoffel tensors and therefore second derivatives of the metric g. Consequently, the Gauss curvature is expressed in terms of first and second derivatives of the metric. This is *Gauss' Theorema Egregium*.

4.2.3 Non-commutativity of Covariant Derivatives of Vectors

We establish the operator identity (4.2.17) when applied to a vector. That is, we prove the formula

$$\nabla_k\nabla_j\phi_i - \nabla_j\nabla_k\phi_i = R^l_{ijk}\phi_l, \tag{4.2.26}$$

demonstrating that the second covariant derivative of a vector does not commute.

It follows from (4.2.6) to (4.2.7) that

$$\begin{aligned}\nabla_k\nabla_j\phi_i &= \partial^2_{kj}\phi_i - \partial_k\Gamma^p_{ji}\phi_l - \Gamma^p_{ji}\partial_k\phi_p - \Gamma^q_{ik}\partial_j\phi_q + \Gamma^q_{ik}\Gamma^p_{qj}\phi_p - \Gamma^q_{jk}\partial_q\phi_i + \Gamma^q_{jk}\Gamma^p_{iq}\phi_p\\ &= \partial^2_{kj}\phi_i - \nabla_k\Gamma^p_{ji}\phi_p - \Gamma^p_{ji}\partial_k\phi_p - \Gamma^q_{ik}\partial_j\phi_q - \Gamma^q_{jk}\partial_q\phi_i,\end{aligned} \tag{4.2.27}$$

since relation (4.2.7) yields

$$\nabla_k\Gamma^l_{ij} = \partial_k\Gamma^l_{ij} - \Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{jk}\Gamma^l_{ip}.$$

Similarly, it may be shown that

$$\nabla_j\nabla_k\phi_i = \partial^2_{jk}\phi_i - \nabla_j\Gamma^p_{ki} - \Gamma^p_{ki}\partial_j\phi_p - \Gamma^q_{ij}\partial_k\phi_q - \Gamma^q_{kj}\partial_q\phi_i, \tag{4.2.28}$$

on using the relation

$$\nabla_j \Gamma^l_{ik} = \partial_j \Gamma^l_{ik} - \Gamma^p_{ij} \Gamma^l_{pk} - \Gamma^p_{kj} \Gamma^l_{ip}.$$

Subtraction of (4.2.28) from (4.2.27) gives

$$\begin{aligned}\nabla_k \nabla_j \phi_i - \nabla_j \nabla_k \phi_i &= \nabla_k \Gamma^p_{ij} \phi_p - \nabla_j \Gamma^p_{ik} \phi_p \\ &= R^p_{ijk} \phi_p,\end{aligned}$$

because by definition (4.2.16) we have

$$\begin{aligned}R^l_{ijk} &= \partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^p_{ij} \Gamma^l_{pk} - \Gamma^p_{ik} \Gamma^l_{pj} \\ &= \nabla_k \Gamma^l_{ij} - \nabla_j \Gamma^l_{ik}\end{aligned}$$

4.3 Isometric Immersion

As before, we let (X, g) be an n-dimensional Riemannian manifold with metric g. An *isometric immersion* is a $\mathbb{R}^m$-valued function $y : (X, g) \to (\mathbb{R}^m, .)$ when the induced metric is the same as the original. That is, in terms of local coordinates $(x^1, x^2, \ldots, x^n)$ there holds

$$\partial_i y \cdot \partial_j y = g_{ij}, \qquad \text{for each } 1 \leq i, j \leq n, \tag{4.3.1}$$

where the dot "·" denotes the canonical Euclidean metric in the coordinate patch $(y^1, \ldots, y^m)$ in $\mathbb{R}^n$.

On letting ds be the distance between neighbouring points in $\mathbb{R}^m$, when y is known, we have from the Pythagoras theorem that

$$ds^2 = \partial_i y \cdot \partial_j y \, dx_i dx_j.$$

On the other hand, the general distance formula for the abstract Riemannian manifold (X, g) due to Riemann is given by

$$ds^2 = g_{ij} dx_i dx_j.$$

It is then natural to ask under what conditions can the two expressions for the distance be equated to determine a *realization* of the manifold.

We investigate this question by again first considering the case $n = 2$, $m = 3$. The tangents to the surface (manifold) are given by $\partial_1 y$ and $\partial_2 y$ and span the tangent space at the point $y(x) = \left(y^1(x^1, x^2), y^2(x^1, x^2), y^3(x^1, x^2)\right)$. The unit normal vector at this point is defined (up to a sign) by the usual vector cross product

$$N = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|}.$$

In higher dimensions, although there is no cross-product, similar ideas may be used. Indeed, on the manifold (X, g) the coordinate patch $y = (y^1, \ldots, y^m)$ generates the collection of tangents

$$\{\partial_1 y(x), \ldots, \partial_n y(x)\}$$

that span the tangent space to the manifold. Define this tangent space to be $T_x X$ and note that it is n-dimensional. Let $N_x X$ denote the $(m - n)$-dimensional subspace orthogonal and complementary to $T_x X$, and for each x choose a fixed orthogonal basis of $N_x X$ given by

$$\{N_{n+1}(x), \ldots, N_m(x)\},$$

where each N_r, $r = n + 1, \ldots, m$, is assumed to depend smoothly on x.

4.3.1 The Second Derivative of an Immersion

Now, for each x, the vectors $\{\partial_1 y(x), \ldots, \partial_n y(x), N_{n+1}(x), \ldots, N_m(x)\}$ comprise a basis of $\mathbb{R}^m$, and as such are linearly independent. Therefore, for each pair of indices $1 \leq i, j \leq n$, the vector $\partial^2_{ij} y(x)$ can be written as a linear combination of these base vectors. In other words, there exist unique coefficients $\tilde{\Gamma}^k_{ij}$, $1 \leq k \leq n$ and H^μ_{ij}, $n + 1 \leq \mu \leq m$ such that

$$\partial^2_{ij} y(x) = \tilde{\Gamma}^k_{ij}(x)\partial_k y(x) + H^\mu_{ij}(x)N_\mu(x), \tag{4.3.2}$$

or in components,

$$\partial^2_{ij} y^p(x) = \tilde{\Gamma}^k_{ij}(x)\partial_k y^p(x) + H^\mu_{ij}(x)N^p_\mu, \qquad p = 1, \ldots, m. \tag{4.3.3}$$

Since partial derivatives commute, the decomposition (4.3.2) implies

$$\left(\tilde{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ji}\right)\partial_k y(x) + \left(H^\mu_{ij} - H^\mu_{ji}\right)N_\mu = 0.$$

As just mentioned, the set $\{\partial_1 y(x), \ldots, N_m\}$ is a basis in $\mathbb{R}^m$, and therefore we have the symmetries

$$\tilde{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ji}, \tag{4.3.4}$$

$$H^\mu_{ij} = H^\mu_{ji}. \tag{4.3.5}$$

The notation $\tilde{\Gamma}^k_{ij}$ is intentional since it will be proved in Sect. "The Coefficients $\tilde{\Gamma}^k_{ij}$" that the coefficients are precisely the Christoffel symbols Γ^k_{ij} defined in (4.2.8). It will then follow from (4.2.7) that in terms of the covariant derivative, the relation (4.3.2) can be expressed as

$$\nabla_i \partial_j y(x) = H^{\mu}_{ij}(x) N_{\mu}(x). \tag{4.3.6}$$

The Coefficients $\tilde{\Gamma}^k_{ij}$

We prove that in expressions (4.3.2) and (4.3.3) for the tangent direction of the second derivatives $\partial^2_{ij} y(x)$, the coefficients $\tilde{\Gamma}^k_{ij}$ are precisely the Christoffel symbols Γ^k_{ij}. On taking the scalar product of both sides of (4.3.2) with the tangent vector $\partial_q y(x)$, and after noting that $\partial_q y(x) \cdot N_{\mu}(x) = 0$, we obtain

$$\begin{aligned} \partial^2_{ij} y(x) \cdot \partial_q y(x) &= \tilde{\Gamma}^k_{ij} \partial_k y(x) \cdot \partial_q y(x) \\ &= \Gamma^k_{ij} g_{kq}. \end{aligned} \tag{4.3.7}$$

The last equation follows since $y(x)$ is an immersion and therefore

$$\partial_k y(x) \cdot \partial_q y(x) = g_{kq}. \tag{4.3.8}$$

Differentiation with respect to x_i of relation (4.3.8) yields the identity

$$\partial_i g_{jq} = \partial^2_{ij} y \cdot \partial_q y + \partial_j y \cdot \partial^2_{iq} y,$$

which by (4.3.7) reduces to

$$\partial_i g_{jq}(x) = \tilde{\Gamma}^k_{ij} g_{kq}(x) + \tilde{\Gamma}^k_{iq} g_{kj}(x). \tag{4.3.9}$$

This expression, together with the symmetry (4.3.4) of $\tilde{\Gamma}^k_{ij}$, is now used in definition (4.2.8) to give

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \right) \\ &= \frac{1}{2} g^{kl} \left(\tilde{\Gamma}^p_{il} g_{pj} + \tilde{\Gamma}^p_{ij} g_{pl} + \tilde{\Gamma}^p_{ji} g_{pl} + \tilde{\Gamma}^p_{jl} g_{pi} - \tilde{\Gamma}^p_{il} g_{pj} - \tilde{\Gamma}^p_{lj} g_{pi} \right) \\ &= \frac{1}{2} \left(\tilde{\Gamma}^p_{il} g^{kl} g_{pj} + 2\tilde{\Gamma}^k_{ij} + \tilde{\Gamma}^p_{jl} g^{kl} g_{pi} - \tilde{\Gamma}^p_{il} g^{kl} g_{pj} - \tilde{\Gamma}^p_{lj} g^{kl} g_{pi} \right) \\ &= \tilde{\Gamma}^k_{ij}, \end{aligned}$$

which establishes the assertion. Note that these derivations have employed the formula (4.2.9).

Henceforth, the superposed tilde is removed from the coefficient $\tilde{\Gamma}^k_{ij}$ in the decomposition (4.3.2).

The Coefficients H^{μ}_{ij}

Let us further consider the decomposition (4.3.2). The assumed orthogonality of the set $\{\partial_1 y(x), \ldots, N_m(x)\}$, and in particular that of the set $\{N_{n+1}(x), \ldots, N_m(x)\}$, so that

$$N_{\mu}(x) \cdot N_{\nu}(x) = \delta_{\mu\nu}, \tag{4.3.10}$$

enables us to write

$$\begin{aligned} \partial^2_{ij} y(x) \cdot N_{\nu}(x) &= H^{\mu}_{ij}(x) N_{\mu}(x) \cdot N_{\nu}(x) \\ &= H^{\mu}_{ij}(x). \end{aligned} \tag{4.3.11}$$

The tensors $H^{\mu}_{ij}(x),\ \mu = n+1, \ldots, m$, as already shown in (4.3.5), are symmetric with respect to i, j and form the *second fundamental form*. The *first fundamental form* is given by the tensor g.

4.3.2 Decomposition of First Derivative of $N_{\mu}(x)$

In this section, the first derivative of the normals $N_{\mu}(x)$ is treated analogously to that of the decomposition of the first derivative of the tangent vectors expressed by (4.3.2). We prove

Lemma 4.3.1 *There exist functions (the induced connection on the normal bundle over the embedding)*

$$A^{\nu}_{\mu i} = -A^{\mu}_{\nu i} \tag{4.3.12}$$

such that

$$\partial_i N_{\mu} = -g^{jk} H^{\mu}_{ik} \partial_j y + A^{\nu}_{\mu i} N_{\nu}, \tag{4.3.13}$$

whose component version is given by

$$\partial_i N^p_{\mu} = -g^{jk} H^{\mu}_{ik} \partial_j y^p + A^{\nu}_{\mu i} N^p_{\nu}, \qquad p = 1, \ldots, m. \tag{4.3.14}$$

Proof The normals N_{μ} are postulated to form an orthonormal set in $\mathbb{R}^m$ so that differentiation of (4.3.10) gives

$$0 = \partial_i \left(N_{\mu} \cdot N_{\nu} \right) = N_{\nu} \cdot \partial_i N_{\mu} + N_{\mu} \cdot \partial_i N_{\nu}. \tag{4.3.15}$$

Moreover, because the tangents and normals form a full set of orthonormal vectors that span $\mathbb{R}^m$, we have the decomposition

$$\partial_i N_\mu = B^j_{i\mu} \partial_j y + A^\nu_{\mu i} N_\nu, \tag{4.3.16}$$

which on scalar multiplication by N_ν and use of (4.3.10) leads to

$$\begin{aligned} N_\nu \cdot \partial_i N_\mu &= A^\alpha_{\mu i} N_\alpha \cdot N_\nu = A^\alpha_{\mu i} \delta_{\alpha\nu} = A^\nu_{\mu i}, \\ N_\mu \cdot \partial_i N_\nu &= A^\alpha_{\nu i} N_\alpha \cdot N_\mu = A^\alpha_{\nu i} \delta_{\alpha\mu} = A^\mu_{\nu i}. \end{aligned}$$

Upon substitution in (4.3.15), we conclude that

$$A^\nu_{\mu i} + A^\mu_{\nu i} = 0, \tag{4.3.17}$$

as stated in the Lemma.

On the other hand, we also have

$$N_\mu \cdot \partial_k y = 0, \qquad \forall \mu, k, \tag{4.3.18}$$

and on recalling (4.3.10) and (4.3.11), we deduce that

$$\begin{aligned} 0 &= g^{jk} \partial_i \left(N_\mu \cdot \partial_k y \right) \\ &= g^{jk} \left(\partial_i N_\mu \cdot \partial_k y + N_\mu \cdot \partial^2_{ik} y \right) \\ &= g^{jk} \left(\partial_i N_\mu \cdot \partial_k y + H^\mu_{ik} \right) \\ &= g^{jk} \left(\partial_k y \cdot \partial_p y B^p_{i\mu} + H^\mu_{ik} \right) \\ &= g^{jk} \left(g_{kp} B^p_{i\mu} + H^\mu_{ik} \right) \\ &= B^j_{i\mu} + g^{jk} H^\mu_{ik}, \end{aligned}$$

where we again recall the relation $g^{jk} g_{kp} = \delta^j_p$. We conclude that

$$B^j_{i\mu} = -g^{jk} H^\mu_{ik}, \tag{4.3.19}$$

which after substitution in (4.3.16) and in conjunction with (4.3.17) proves the Lemma. □.

4.3.3 The Second Partial Derivatives of Normal Vectors

In this section we establish the well-known *Codazzi* and *Ricci equations* as a consequence of the property that second partial derivatives of the normal vectors commute. The *Gauss* equations are derived in the next section after further discussion of the Codazzi equations.

Now, differentiation of (4.3.13) gives

$$\partial_j \left(\partial_i N_\mu\right) = -\partial_j \left(g^{qp} H^\mu_{ip} \partial_q y\right) + \partial_j \left(A^\nu_{\mu i} N_\nu\right)$$
$$= -\partial_j \left(g^{qp} H^\mu_{ip}\right) \partial_q y - g^{qp} H^\mu_{ip} \partial^2_{jq} y + \left(\partial_j A^\nu_{\mu i}\right) N_\nu + A^\nu_{\mu i} \partial_j N_\nu,$$

which after substitution from (4.3.2) to (4.3.13) leads to

$$\partial_j \left(\partial_i N_\mu\right) = -\partial_j \left(g^{qp} H^\mu_{ip}\right) \partial_q y - g^{qp} H^\mu_{ip} \left(\Gamma^k_{jq} \partial_k y + H^\nu_{jq} N_\nu\right)$$
$$+ \left(\partial_j A^\nu_{\mu i}\right) N_\nu + A^\nu_{\mu i} \left(-g^{pq} H^\nu_{pj} \partial_q y + A^\eta_{\nu j} N_\eta\right).$$

On collecting terms in the tangential and normal directions, we rewrite the last equation as

$$\partial_j \left(\partial_i N_\mu\right) = -\left(\partial_j \left(g^{pq} H^\mu_{ip}\right) + g^{pk} \Gamma^q_{jk} H^\mu_{ip} + g^{pq} A^\nu_{\mu i} H^\nu_{pj}\right) \partial_q y$$
$$+ \left(\partial_j A^\nu_{\mu i} - g^{pq} H^\mu_{ip} H^\nu_{jq} + A^\eta_{\mu i} A^\nu_{\eta j}\right) N_\nu,$$

But the second derivatives of the normal commute, so that

$$\partial_j \left(\partial_i N_\mu\right) = \partial_i \left(\partial_j N_\mu\right), \tag{4.3.20}$$

and from the terms in the tangent direction, we can read off the *Codazzi equations*

$$\partial_j \left(g^{pq} H^\mu_{ip}\right) + g^{pk} \Gamma^q_{jk} H^\mu_{ip} + g^{pq} A^\nu_{\mu i} H^\nu_{pj} = \partial_i \left(g^{pq} H^\mu_{jp}\right)$$
$$+ g^{pk} \Gamma^q_{ik} H^\mu_{jp} + g^{pq} A^\nu_{\mu j} H^\nu_{pi}. \tag{4.3.21}$$

Similarly, terms in the normal direction lead to the *Ricci equations*

$$\partial_j A^\nu_{\mu i} - g^{pq} H^\mu_{ip} H^\nu_{jq} + A^\eta_{\mu i} A^\nu_{\eta j} = \partial_i A^\nu_{\mu j} - g^{pq} H^\mu_{jp} H^\nu_{iq} + A^\eta_{\mu j} A^\nu_{\eta i}. \tag{4.3.22}$$

The more traditional form of the Codazzi equations is recovered by the following simple computation. On differentiation of (4.2.9), we obtain

$$0 = \partial_j \left(g^{pq} g_{pr}\right)$$
$$= \partial_j \left(g^{pq}\right) g_{pr} + g^{pq} \partial_j \left(g_{pr}\right),$$

which after multiplying by g^{rs} and appealing to (4.2.9) leads to

$$\begin{aligned} 0 &= \partial_j\left(g^{pq}\right)g_{pr}g^{rs} + g^{rs}g^{pq}\partial_j\left(g_{pr}\right) \\ &= \partial_j\left(g^{pq}\right)\delta^s_p + g^{rs}g^{pq}\partial_j\left(g_{pr}\right) \qquad (4.3.23)\\ &= \partial_j\left(g^{sq}\right) + g^{rs}g^{pq}\partial_j\left(\partial_p y\cdot\partial_r y\right) \qquad (4.3.24)\\ &= \partial_j\left(g^{sq}\right) + g^{rs}g^{pq}\left(\partial^2_{jp}y\cdot\partial_r y + \partial_p y\cdot\partial^2_{jr}y\right), \end{aligned}$$

where (4.3.1) is used. We now conclude from (4.3.2) in conjunction with the orthogonality relations (4.3.18) and (4.2.9) that

$$\begin{aligned} \partial_j\left(g^{sq}\right) &= -g^{rs}g^{pq}\left(\Gamma^k_{jp}\partial_k y\cdot\partial_r y + \partial_p y\cdot\Gamma^k_{jr}\partial_k y\right) \qquad (4.3.25)\\ &= -g^{rs}g^{pq}\left(\Gamma^k_{jp}g_{rk} + \Gamma^k_{jr}g_{kp}\right)\\ &= -g^{pq}\left(\Gamma^k_{jp}\delta^s_k\right) - g^{rs}\left(\Gamma^k_{jr}\delta^q_k\right)\\ &= -g^{pq}\Gamma^s_{jp} - g^{rs}\Gamma^q_{jr}. \qquad (4.3.26) \end{aligned}$$

We perform the differentiation of the first term on the left and right of the Codazzi equations (4.3.21), and then substitute from (4.3.26) after suitably changing indices to obtain

$$\begin{aligned} &g^{pq}\partial_j H^\mu_{ip} + \left(-g^{sq}\Gamma^p_{js} - g^{rp}\Gamma^q_{jr}\right)H^\mu_{ip} + g^{pk}\Gamma^q_{jk}H^\mu_{ip} + g^{pq}A^\nu_{\mu i}H^\nu_{pj} = \\ &g^{pq}\partial_i H^\mu_{jp} + \left(-g^{sq}\Gamma^p_{is} - g^{rp}\Gamma^q_{ir}\right)H^\mu_{jp} + g^{pk}\Gamma^q_{ik}H^\mu_{jp} + g^{pq}A^\nu_{\mu j}H^\nu_{pi}. \end{aligned}$$

Multiplication of both sides of the last equation by $g_{q\alpha}$ together with (4.2.9) yields

$$\begin{aligned} &\partial_j H^\mu_{i\alpha} - \Gamma^p_{j\alpha}H^\mu_{ip} - g_{q\alpha}g^{rp}\Gamma^q_{jr}H^\mu_{ip} + g_{q\alpha}g^{pk}\Gamma^q_{jk}H^\mu_{ip} + A^\nu_{\mu i}H^\nu_{\alpha j} = \\ &\partial_i H^\mu_{j\alpha} - \Gamma^p_{i\alpha}H^\mu_{jp} - g_{q\alpha}g^{rp}\Gamma^q_{ir}H^\mu_{jp} + g_{q\alpha}g^{pk}\Gamma^q_{ik}H^\mu_{jp} + A^\nu_{\mu j}H^\nu_{\alpha i}. \end{aligned}$$

By virtue of the symmetry $g^{rp} = g^{pr}$, and by changing dummy superscripts, the third and fourth terms on either side cancel to give

$$\partial_j H^\mu_{i\alpha} - \Gamma^p_{j\alpha}H^\mu_{ip} + A^\nu_{\mu i}H^\nu_{\alpha j} = \partial_i H^\mu_{j\alpha} - \Gamma^p_{i\alpha}H^\mu_{jp} + A^\nu_{\mu j}H^\nu_{\alpha i}.$$

The usual form of the Codazzi equations is now obtained by the subtraction of $\Gamma^p_{ij}H^\mu_{\alpha p}$ from both sides of the last equation. This gives

$$\partial_j H^\mu_{i\alpha} - \Gamma^p_{j\alpha}H^\mu_{ip} - \Gamma^p_{ij}H^\mu_{\alpha p} + A^\nu_{\mu i}H^\nu_{\alpha j} = \partial_i H^\mu_{j\alpha} - \Gamma^p_{i\alpha}H^\mu_{jp} - \Gamma^p_{ij}H^\mu_{\alpha p} + A^\nu_{\mu j}H^\nu_{\alpha i}. \qquad (4.3.27)$$

We apply the formula (4.2.7) for the covariant derivative of a second order tensor to write

$$\nabla_j H^\mu_{i\alpha} = \partial_j H^\mu_{i\alpha} - \Gamma^p_{ij} H^\mu_{p\alpha} - \Gamma^p_{\alpha j} H^\mu_{ip},$$
$$\nabla_i H^\mu_{j\alpha} = \partial_i H^\mu_{j\alpha} - \Gamma^p_{ji} H^\mu_{p\alpha} - \Gamma^p_{\alpha i} H^\mu_{jp},$$

and use the symmetry of the Christoffel symbols and of the coefficients H^μ_{ij} to derive the *Codazzi equations* in the form

$$\nabla_j H^\mu_{i\alpha} - \nabla_i H^\mu_{j\alpha} + A^\nu_{\mu i} H^\nu_{\alpha j} - A^\nu_{\mu j} H^\nu_{\alpha i} = 0. \tag{4.3.28}$$

Remark 4.3.1 (The hypersurface) When the manifold is a hypersurface, we have $m = n + 1$ and there is only one normal N_{n+1}, since $n + 1 \le \nu \le m = n + 1$. But N_{n+1} is a unit vector so that

$$N_{n+1} \cdot N_{n+1} = 1,$$

and consequently

$$\partial_i N_{n+1} \cdot N_{n+1} = 0. \tag{4.3.29}$$

The appropriate member of the system (4.3.13) is

$$\partial_i N_{n+1} = -g^{pq} H^{n+1}_{ip} \partial_q y + A^{n+1}_{(n+1)i} N_{n+1},$$

which after using (4.3.29) and the orthogonal set $\partial_1 y, \ldots, N_{n+1}$ leads us to

$$0 = N_{n+1} \cdot \partial_i N_{n+1} = A^{n+1}_{(n+1)i},$$

and therefore $A^{n+1}_{(n+1)i} = 0$. The conclusion, which can be alternatively derived by applying the skew-symmetry $A^\nu_{\mu i} = -A^\mu_{\nu i}$, implies that for a hypersurface the Codazzi equations simplify to

$$\nabla_j H^\mu_{i\alpha} - \nabla_i H^\mu_{j\alpha} = 0. \tag{4.3.30}$$

Remark 4.3.2 (Determined case for hypersurfaces) When dealing with hypersurfaces in the determined case, we have $m = n(n + 1)/2 = (n + 1)$ so that $n = 2$ and $m = 3$. This is the classical case of (M^2, g) embedded into $\mathbb{R}^3$.

4.4 The Gauss and Codazzi Equations

This section further discusses the derivation of equations obtained in the previous section.

We commute partial derivatives and then use (4.3.2) to obtain

$$
\begin{aligned}
0 &= \partial_k \left(\partial^2_{ij} y\right) - \partial_j \left(\partial^2_{ik} y\right) \\
&= \partial_k \left(\Gamma^p_{ij} \partial_p y + H^\mu_{ij} N_\mu\right) - \partial_j \left(\Gamma^p_{ik} \partial_p y + H^\mu_{ik} N_\mu\right) \\
&= \left(\partial_k \Gamma^p_{ij} - \partial_j \Gamma^p_{ik}\right) \partial_p y + \Gamma^p_{ij} \partial^2_{kp} y - \Gamma^p_{ik} \partial^2_{jp} y \\
&\quad + \left(\partial_k H^\mu_{ij} - \partial_j H^\mu_{ik}\right) N_\mu + H^\mu_{ij} \partial_k N_\mu - H^\mu_{ik} \partial_j N_\mu. \qquad (4.4.1)
\end{aligned}
$$

On appealing again to (4.3.2) and also to (4.3.13), we can reduce (4.4.1) to

$$
\begin{aligned}
0 &= \left(\partial_k \Gamma^p_{ij} - \partial_j \Gamma^p_{ik}\right) \partial_p y + \Gamma^p_{ij} \left(\Gamma^q_{kp} \partial_q y + H^\mu_{kp} N_\mu\right) - \Gamma^p_{ik} \left(\Gamma^q_{jp} \partial_q y + H^\mu_{jp} N_\mu\right) \\
&\quad + \left(\partial_k H^\mu_{ij} - \partial_j H^\mu_{ik}\right) N_\mu \\
&\quad + H^\mu_{ij} \left(-g^{pq} H^\mu_{kq} \partial_p y + A^\nu_{\mu k} N_\nu\right) - H^\mu_{ik} \left(-g^{pq} H^\mu_{jq} \partial_p y + A^\nu_{\mu j} N_\nu\right) \\
&= \left[\partial_k \Gamma^p_{ij} - \partial_j \Gamma^p_{ik} + \Gamma^q_{ij} \Gamma^p_{kq} - \Gamma^q_{ik} \Gamma^p_{jq} - g^{pq} \left(H^\mu_{ij} \cdot H^\mu_{kq} - H^\mu_{ik} \cdot H^\mu_{jq}\right)\right] \partial_p y \\
&\quad + \left[\Gamma^p_{ij} H^\mu_{kp} - \Gamma^p_{ik} H^\mu_{jp} + \partial_k H^\mu_{ij} - \partial_j H^\mu_{ik} + H^\nu_{ij} A^\mu_{\nu k} - H^\nu_{ik} A^\mu_{\nu j}\right] N_\mu, \qquad (4.4.2)
\end{aligned}
$$

where the last expression has been separated into tangential and normal components. In consequence, the orthogonality relation (4.3.18) implies that each component must vanish. We have

$$
\begin{aligned}
0 &= \partial_k H^\mu_{ij} - \partial_j H^\mu_{ik} + \Gamma^p_{ij} H^\mu_{kp} - \Gamma^p_{ik} H^\mu_{jp} + H^\nu_{ij} A^\mu_{\mu k} - H^\nu_{ik} A^\mu_{\nu j} \\
&= \partial_k H^\mu_{ij} - \partial_j H^\mu_{ik} + \Gamma^p_{ij} H^\mu_{kp} - \Gamma^p_{ik} H^\mu_{jp} + H^\nu_{ik} A^\nu_{\mu j} - H^\nu_{ij} A^\nu_{\mu k}, \qquad (4.4.3)
\end{aligned}
$$

where the antisymmetry relation (4.3.17) for the vectors $A^\nu_{\mu k}$ is employed. The system (4.4.3) is the previously derived *Codazzi equations*.

From the tangential component in (4.4.2), we have

$$
0 = \left[\partial_k \Gamma^p_{ij} - \partial_j \Gamma^p_{ik} + \Gamma^q_{ij} \Gamma^p_{kq} - \Gamma^q_{ik} \Gamma^p_{jq} - g^{pq} \left(H^\mu_{ij} \cdot H^\mu_{kq} - H^\mu_{ik} \cdot H^\mu_{jq}\right)\right],
$$

which upon noting the expression (4.2.16) for the Riemann curvature tensor becomes

$$
g^{pq} \left(-R_{qijk} - H^\mu_{ij} \cdot H^\mu_{kq} + H^\mu_{ik} \cdot H^\mu_{jq}\right) = 0,
$$

from which follows the *Gauss relation*

$$
H^\mu_{ij} \cdot H^\mu_{qk} - H^\mu_{ik} \cdot H^\mu_{jq} = R_{iqjk}, \qquad (4.4.4)
$$

on recalling the antisymmetry $R_{qijk} = -R_{iqjk}$, and that summation over repeated superscripts is implied.

4.5 Summary for $(M^n, g) \to (\mathbb{R}^m, \cdot)$

We summarise the conclusions obtained so far. Notice that $A^{\nu}_{\mu j}$ are components of *vectors* for $j = 1, 2, 3, \ldots, n$ with the indices ν, μ accounting only for the dimensions $n + 1 \le \mu, \nu \le m$.

A *necessary* condition for the existence of an isometric embedding is that there exist functions

$$H^{\mu}_{ij} = H^{\mu}_{ji}, \qquad A^{\nu}_{\mu i} = -A^{\mu}_{\nu i}, \qquad 1 \le i, j \le n, n = 1 \le \mu, \nu \le m,$$

such that the *Gauss equations* hold

$$\sum_{\mu=n+1}^{m} \left(H^{\mu}_{ik} H^{\mu}_{jl} - H^{\mu}_{il} H^{\mu}_{jk} \right) = R_{ijkl}, \tag{4.5.1}$$

along with the *Codazzi equations*

$$\partial_k H^{\mu}_{ij} + A^{\mu}_{\nu k} H^{\nu}_{ij} - \Gamma^{p}_{ki} H^{\mu}_{pj} - \Gamma^{p}_{kj} H^{\mu}_{ip} = \partial_j H^{\mu}_{ik} + A^{\mu}_{\nu j} - \Gamma^{p}_{ji} H^{\mu}_{pk} - \Gamma^{p}_{jk} H^{\mu}_{ip}, \tag{4.5.2}$$

and the *Ricci equations*

$$\partial_i A^{\nu}_{\mu j} - \partial_j A^{\nu}_{\mu i} + A^{\nu}_{\eta i} A^{\eta}_{\mu j} - A^{\nu}_{\eta j} A^{\eta}_{\mu i} = g^{pq} \left(H^{\mu}_{ip} H^{\nu}_{jq} - H^{\mu}_{jp} H^{\nu}_{iq} \right). \tag{4.5.3}$$

The Ricci system (4.5.3) can be expressed in covariant form by the addition and subtraction of the term

$$\Gamma^{q}_{ij} A^{\nu}_{\mu q}$$

to obtain

$$\nabla_i A^{\nu}_{\mu j} - \nabla_j A^{\nu}_{\mu i} + A^{\nu}_{\eta i} A^{\eta}_{\mu j} - A^{\nu}_{\eta j} A^{\eta}_{\mu i} = g^{pq} \left(H^{\mu}_{ip} H^{\nu}_{jq} - H^{\mu}_{jp} H^{\nu}_{iq} \right). \tag{4.5.4}$$

4.6 Reconstruction of an Isometric Embedding

In this section we state and sketch of the proof of a theorem giving necessary and sufficient conditions for the existence of an isometric embedding. We have

Theorem 4.6.1 *Consider a simply connected n-dimensional Riemannian manifold X with coordinates* $(x^1, \ldots, x^n)$ *and Riemannian metric* g $(=g_{ij})$. *Let* $1 \le i, j \le n$, *and suppose there exist symmetric functions* $H^\mu_{ij} = H^\mu_{ji}$ *and anti-symmetric functions*

$$A^\nu_{\mu i} = -A^\mu_{\nu i}, \qquad n+1 \le \mu, \nu \le m,$$

such that Eqs. (4.5.1)–(4.5.3) are satisfied.

Then there exist functions $N_{n+1}, \ldots, N_m : X \to \mathbb{R}^n$ *and a function* $y : X \to \mathbb{R}^m$ *for which the following formulae hold*

$$N_\mu \cdot N_\nu = \delta_{\mu\nu}, \tag{4.6.1}$$

$$N_\mu \cdot \partial_i y = 0, \tag{4.6.2}$$

$$\partial_i y \cdot \partial_j y = g_{ij}, \tag{4.6.3}$$

and

$$\partial^2_{ij} y = \Gamma^k_{ij} \partial_k y + H^\mu_{ij} N_\mu, \tag{4.6.4}$$

$$\partial_i N_\mu = -g^{jk} H^\mu_{ik} \partial_j y + A^\nu_{\mu i} N_\nu. \tag{4.6.5}$$

Remark 4.6.1 The theorem states that the conditions on H^μ_{ij}, $A^\nu_{\mu i}$ together with (4.6.1)–(4.6.3) are both necessary and sufficient for the embedding $(M^n, g) \to (\mathbb{R}^m, \cdot)$, $X = M^n$; that is, the conditions are necessary and sufficient for the existence of vector functions $y(x)$.

Sketch of Proof

Let $\{e_1, \ldots, e_m\}$ be the standard orthonormal basis of $\mathbb{R}^m$. For a fixed point $x_0 \in X$, define $\{\partial_1 y(x_0), \ldots, \partial_n y(x_0), N_{n+1}(x_0), \ldots, N_m(x_0)\}$ to satisfy (4.5.3)–(4.6.2). As a possible choice, we set $N_\mu(x_0) = e_\mu$ and $y(x_0) = 0$, and select $\{\partial_1 y(x_0), \ldots, \partial_n y(x_0)\}$ to be a linear combination of $\{e_1, \ldots, e_n\}$ such that (4.6.2) holds at x_0.

Remark 4.6.2 When $g_{ij}(x_0) = \delta_{ij}$, we may choose

$$N_\mu(x_0) = e_\mu, \qquad n+1 \le \mu \le m,$$
$$\partial_p y(x_0) = e_p, \qquad 1 \le p \le n.$$

Let $\phi_p = \partial_p y(x_0)$, and observe that (4.6.4)–(4.6.5) form a total differential system for the unknown $\mathbb{R}^m$-valued function $\{\phi_1, \ldots, \phi_n, N_{n+1}, \ldots, N_m\}$. This conclusion may be checked by first differentiating equations (4.6.4) and (4.6.5) to show that the compatibility conditions obtained by constructing partial derivatives are consequences of the Gauss equations (4.5.1), Codazzi equations (4.5.2), Ricci equations (4.5.3), and the original equations (4.6.4) and (4.6.5). In consequence, and by Lemma 4.2.2, we conclude that there exists a unique solution (the "potential" ϕ_p) that extends the initial data specified at x_0.

Moreover, the differentials of Eqs. (4.6.1)–(4.6.3) are consequences of (4.6.4) and (4.6.5). Therefore, they hold not only at x_0 but also on all of X.

Finally, the symmetry of the right side of (4.6.4) implies $\partial_j \phi_i = \partial_i \phi_j$, and consequently by Lemma 4.2.2, there exists a unique $\mathbb{R}^m$-valued function y on X such that

$$y(x_0) = 0, \quad \text{and} \qquad \partial_i y = \phi_i, \qquad 1 \leq i \leq n.$$

The proof of Theorem 4.6.1 is complete. □

4.6.1 Examples

It is important that the number of independent equations matches the number of independent unknowns. The following examples illustrate this aspect, and also serve as introduction to a counting process developed by Blum.

Example 1. $(M^2, g) \to (\mathbb{R}^3, \cdot)$

In this example, we have $n = 2$ and $m = 3$ so that $1 \leq i, j, k \leq 2$ and $\mu = \nu = 3$. The second fundamental form therefore can be represented as the matrix

$$H = \begin{bmatrix} H^3_{11} & H^3_{12} \\ H^3_{21} & H^3_{22} \end{bmatrix}. \tag{4.6.6}$$

Furthermore, since $n = 2$, we may use (4.2.24) to write

$$\begin{aligned} R_{1212} &= K \left(g_{11} g_{22} - g^2_{12} \right) \\ &= K \det g, \qquad \det g > 0. \end{aligned}$$

where K is the Gauss curvature. Consequently, the Gauss equations (4.4.4) reduce to the single equation

$$H^3_{ik} H^3_{jl} - H^3_{il} H^3_{jk} = K \det g. \tag{4.6.7}$$

Upon slight rearrangement, the Codazzi equations (4.5.2) become

$$\partial_k H^\mu_{ij} - \partial_j H^\mu_{ik} = \Gamma^p_{ki} H^\mu_{pj} + \Gamma^p_{kj} H^\mu_{ip} - \Gamma^p_{ji} H^\mu_{pk} - \Gamma^p_{jk} H^\mu_{ip}, \tag{4.6.8}$$

which on specialisation to the example under consideration reduce to

$$\partial_2 H^3_{11} - \partial_1 H^3_{12} = \dots, \tag{4.6.9}$$
$$\partial_2 H^3_{12} - \partial_1 H^3_{22} = \dots. \tag{4.6.10}$$

Consequently, there are three equations (4.6.7), (4.6.9) and (4.6.10) in the three unknowns H^3_{11}, H^3_{12}, H^3_{22}.

On employing the Gauss equations (4.6.7) to eliminate one of the unknowns, we obtain a *quasi-linear system*. Accordingly, *the Gauss relation becomes a "constitutive relation"*.

Example 2. $(M^3, g) \to (\mathbb{R}^6, \cdot)$

In this example, we have $1 \leq i, j \leq 3$ and $4 \leq \mu, \nu \leq 6$, and the Gauss equations (4.4.4) reduce to

$$\sum_{\mu=4}^{6} \left(H^\mu_{ik} H^\mu_{jl} - H^\mu_{il} H^\mu_{jk} \right) = R_{ijkl}, \tag{4.6.11}$$

where the six non-zero components of the Riemann curvature tensor are

$$R_{1212},\ R_{1313},\ R_{2323},\ R_{1223},\ R_{1332},\ R_{1231}. \tag{4.6.12}$$

We are left, therefore, with *six* non-trivial Gauss equations, the remainder being identically satisfied.

The second fundamental form may be expressed as the matrix array of 6 independent entries for each μ:

$$\begin{bmatrix} H^\mu_{11} & H^\mu_{12} & H^\mu_{13} \\ H^\mu_{21} & H^\mu_{22} & H^\mu_{23} \\ H^\mu_{31} & H^\mu_{32} & H^\mu_{33} \end{bmatrix}, \tag{4.6.13}$$

from which it can be seen that the Codazzi equations (4.5.2) are just a statement about cross derivatives along rows (or columns since H^μ_{ij} is symmetric). Apparently, there are 3 equations across each row, but the couplings

$$\partial_1 H^\mu_{23} - \partial_3 H^\mu_{21} = \dots,$$
$$\partial_1 H^\mu_{32} - \partial_2 H^\mu_{31} = \dots,$$

after subtraction yield

$$\partial_2 H^\mu_{31} - \partial_3 H^\mu_{21} = \dots$$

Thus instead of 9 couplings for each μ, there are only 8. In consequence, as $\mu = 4, 5, 6$ there are 24 Codazzi equations. In summary, we have

1. Equations
   (a) 6 Gauss equations.
   (b) 24 Codazzi equations.

(c) 9 Ricci equations.
(d) Thus, there are a total of 39 equations.

2. Unknowns

(a) $6 \times 3 = 18$ independent components H^{μ}_{ij} of the second fundamental form.
(b) $3 \times 3 = 9$ coefficients $A^{\nu}_{\mu k} = -A^{\mu}_{\nu k}$.
(c) Thus, there are a total of 27 unknowns.

We conclude that there are more equations than unknowns despite the embedding problem $(M^3, g) \to (\mathbb{R}^6, \cdot)$ being *determined* ($m = n(n+1)/2$; $n = 3, m = 6$), which implies that *not all equations are independent in the Gauss, Codazzi, Ricci system.*

4.6.2 Blum's Counting Process

The rather painful counting process illustrated in the previous examples is examined in a series of papers published in the 1940s and 1950s by R. Blum [Blu55, Blu46, Blu47] and further described in the excellent survey by Goenner [Goe77].

The description in [Goe77, p. 143] of Blum's counting result for the embedding $(M^n, g) \to \mathbb{R}^m, \cdot)$ may be paraphrased as follows.

Theorem 4.6.2 *When the Gauss equations (4.4.4) are satisfied, and Goenner's matrices M and N, defined below, are of maximal rank, then (i) for $0 \le p = m - n \le n(n-2)/8$ all Codazzi and Ricci equations are consequences of the Gauss equations; (ii) for $n(n-2)/8 < p = m - n \le n(n-1)/2$ a system of*

$$\frac{1}{3}n(n^2-1)\left[p - \frac{1}{8}n(n-2)\right]$$

Codazzi equations are independent. The remainder of the Codazzi equations and all the Ricci equations are a consequence of the independent Codazzi system and of the Gauss equations.

Goenner's matrices M and N are given by

$$M^{\mu kij}_{abcde} = \left\{\frac{1}{2}(\delta^i_c\delta^j_d - \delta^j_c\delta^i_d)H^{\mu}_{eb} + \frac{1}{2}(\delta^i_e\delta^j_c - \delta^j_e\delta^i_c)H^{\mu}_{db} + \frac{1}{2}(\delta^i_d\delta^j_e - \delta^j_d\delta^i_c)H^{\mu}_{cb}\right\}\delta^k_a,$$

$$N^{\mu ij}_{abcd} = \frac{1}{2}(\delta^i_c\delta^j_d - \delta^j_c\delta^i_d)H^{\mu}_{ab} + \frac{1}{2}(\delta^i_b\delta^j_c - \delta^j_b\delta^i_c)H^{\mu}_{ad} + \frac{1}{2}(\delta^i_d\delta^j_b - \delta^j_d\delta^i_b)H^{\mu}_{ac}.$$

Of course even these definitions are not particularly enlightening, and Goenner has given results that are easier to state but which we will not repeat here. Also since the above notation may be confusing, we note that M, N are the coefficient matrices in systems (4.2.5) and (4.2.6) of Goenner, i.e.,

$$\sum_{\mu=n+1}^{m} M^{\mu kij}_{abcde}\mathcal{C}^{\mu}_{kij} = 0, \quad \sum_{\mu=n+1}^{m} N^{\mu ij}_{abcd}\mathcal{K}^{\mu}_{\nu ij} = 0.$$

The matrix M has $\frac{1}{2}\binom{n+1}{2}\binom{n}{3}$ rows and $\frac{1}{3}pn(n^2-1)$ columns, the matrix N has $\frac{p}{2}\binom{n+1}{2}\binom{n-1}{2}$ rows and $\binom{p}{2}\binom{n}{2}$ columns. Notice that

$$M^{\mu kij}_{abcde} = N^{\mu ij}_{bcde}\delta^k_a,$$

and

$$N^{\mu ij}_{bcde} = N^{\mu ij}_{bdec} = N^{\mu ij}_{becd}, \quad N^{\mu ij}_{bcde} = -N^{\mu ij}_{bced}, \quad N^{\mu ij}_{bcde} = -N^{\mu ij}_{bdce}.$$

A useful example is given by the case $n = 3, m = 6, p = 3$. In this case, the symmetries in $N^{\mu ij}_{abcd}$ yield that only non-zero terms are of the form $N^{\mu ij}_{a123}$ and the equations

$$\sum_{\mu=n+1}^{m} N^{\mu ij}_{abcd}\mathcal{K}^{\mu}_{\nu ij} = 0$$

become

$$\begin{bmatrix} 0 & H^5 & H^6 \\ H^4 & 0 & H^6 \\ H^4 & H^5 & 0 \end{bmatrix} \mathcal{K} = 0,$$

where

$$\mathcal{K} = (\mathcal{K}^4_{523}, \mathcal{K}^4_{513}, \mathcal{K}^4_{5i12}, \mathcal{K}^5_{623}, \mathcal{K}^5_{613}, \mathcal{K}^5_{612}, \mathcal{K}^6_{423}, \mathcal{K}^6_{413}, \mathcal{K}^6_{412})^T, \text{ i.e., } N\mathcal{K} = 0.$$

But row operations reduce the coefficient matrix N to obtain

$$\begin{bmatrix} H^4 & 0 & 0 \\ 0 & H^5 & 0 \\ 0 & 0 & H^6 \end{bmatrix},$$

and the condition on N of Blum is just that H^4, H^5, H^6 each be of full rank 3. The matrix N is 9×9 as predicted by Blum's theorem and the matrix M is 3×24. We can write the system

$$\sum_{\mu=n+1}^{m} M^{\mu kij}_{abcde}\mathcal{C}^{\mu}_{kij} = 0$$

in the form

$$\left[H^4\ H^5\ H^6\ H^4\ H^5\ H^6\ H^4\ H^5\ H^6\right] C = 0.$$

In this representation the three repetitions for $\mathcal{C}^{\mu}_{123}$ are not accounted for and hence the vector $\mathcal{C}$ has 27 entries instead of the 24 predicted by Blum's theorem. If any one of the H^{μ} has full rank 3 then M will have full rank 3.

The example "$(M^3, g) \to (\mathbb{R}^6, \cdot)$" in Sect. 4.6.1, for which $m = 6, n = 3$ and $p = 3$, satisfies the condition in category (ii) of the above theorem which gives.

$$\frac{1}{8} \times 3 \times 1 \leq 3 \leq 3,$$

and there are

$$\frac{1}{8} \times 3 \times 8 \times \left[\frac{24}{8} - \frac{3}{8}\right] = 21$$

independent Codazzi equations. All the Ricci equations are implied by these independent Codazzi equations and the Gauss equations. Thus, Blum's count gives 21 independent Codazzi equations, whereas the elementary count conducted in the example produced 24 Codazzi equations.

The discrepancy is explained by observing that the elementary counting *omitted to include the three equations in Bianchi's second identity.* Substitution of the Gauss equations in these three equations gives three more relations between derivatives of the second fundamental forms and consequently there are only 21 and not 24 independent Codazzi equations.

Combined with the 6 Gauss equations there are 27 equations for the 27 unknowns consisting, as already shown, of 18 entries of the second fundamental forms and 9 coefficients $A^{\nu}_{\mu k}$. Nevertheless, it is unclear how even local existence can be proved for this system.

In the determined system, we have $m = n(n+1)/2$, and category (ii) of Blum's theorem again applies with $p = n(n-1)/2$ so that there are $n^2(n^2-1)(3n-2)/24$ independent Codazzi equations. Under the maximal rank condition, the Codazzi and Gauss equations imply the Ricci equations.

Sketch of the Proof of Blum's Theorem When $n = 3, m = 6$

Throughout this section, unless otherwise stated, the summation convention is suspended for repeated indices μ.

Step 1

Particular forms of the covariant Codazzi equations (4.3.28) are

$$\nabla_1 H^{\mu}_{23} - \nabla_3 H^{\mu}_{21} + A^{\nu}_{\mu 3} H^{\nu}_{21} - A^{\nu}_{\mu 1} H^{\nu}_{23} = 0, \tag{4.6.14}$$

$$\nabla_1 H^{\mu}_{32} - \nabla_2 H^{\mu}_{31} + A^{\nu}_{\mu 2} H^{\nu}_{31} - A^{\nu}_{\mu 1} H^{\nu}_{32} = 0, \tag{4.6.15}$$

which by subtraction yield the equation

$$\nabla_2 H^{\mu}_{31} - \nabla_3 H^{\mu}_{21} + A^{\nu}_{\mu 3} H^{\nu}_{21} - A^{\nu}_{\mu 2} H^{\nu}_{31} = 0. \tag{4.6.16}$$

We conclude that the Codazzi equations (4.6.16) are implied by the pair (4.6.14) and (4.6.15) so that for $n = 3,\ m = 6$ the number of independent Codazzi equations is reduced by 3.

Step 2

Next, we rewrite the Codazzi equations (4.3.28) as

$$\epsilon_{lji} \nabla_j H^{\mu}_{ik} + \epsilon_{lji} A^{\nu}_{\mu i} H^{\nu}_{kj} = 0, \tag{4.6.17}$$

where ϵ_{ijk} is the standard Einstein alternating tensor given by

$$\begin{aligned} \epsilon_{ijk} &= +1, \quad \text{when } i, j, k, \text{ is an even permutation of } 1, 2, 3, \\ &= -1, \quad \text{when } i, j, k, \text{ is a odd permutation of } 1, 2, 3, \\ &= 0, \quad \text{otherwise} . \end{aligned}$$

Let cof A be the cofactor of the entry A in the matrix $[A]$. Then we have

$$\operatorname{cof} H^{\mu}_{il} = \frac{1}{2} \epsilon_{ijk} \epsilon_{lmn} H^{\mu}_{kn} H^{\mu}_{jm}, \tag{4.6.18}$$

and consequently

$$\begin{aligned} \nabla_l \left(\operatorname{cof} H^{\mu}_{il} \right) &= \frac{1}{2} \epsilon_{ijk} \epsilon_{lmn} \left(\nabla_l H^{\mu}_{kn} \right) H^{\mu}_{jm} \\ &\quad + \frac{1}{2} \epsilon_{ijk} \epsilon_{lmn} H^{\mu}_{kn} \left(\nabla_l H^{\mu}_{jm} \right) \qquad (4.6.19) \\ &= \epsilon_{ijk} \epsilon_{lmn} H^{\mu}_{jm} \left(\nabla_l H^{\mu}_{kn} \right), \qquad \text{no sum on } \mu, \qquad (4.6.20) \end{aligned}$$

where the last expression is obtained by interchange of suffixes $j \leftrightarrow k,\ m \leftrightarrow n$.

After a further interchange of suffixes, the Codazzi equations (4.6.17) may be written as

$$\epsilon_{lmn} \nabla_l H^{\mu}_{kn} + \epsilon_{lmn} A^{\nu}_{\mu n} H^{\nu}_{kl} = 0, \tag{4.6.21}$$

$$\epsilon_{lmn} \nabla_l H^{\mu}_{jm} + \epsilon_{lmn} A^{\nu}_{\mu m} H^{\nu}_{jl} = 0, \tag{4.6.22}$$

and substituting these relations in (4.6.19) yields

$$\nabla_l \left(\operatorname{cof} H^{\mu}_{il} \right) + \frac{1}{2} \epsilon_{ijk} \epsilon_{lmn} A^{\nu}_{\mu n} H^{\nu}_{kl} H^{\mu}_{jm} + \frac{1}{2} \epsilon_{ijk} \epsilon_{lmn} A^{\nu}_{\mu m} H^{\nu}_{jl} H^{\mu}_{km} = 0,$$

where there is no sum on μ. The interchange $m \leftrightarrow n,\ j \leftrightarrow k$ in the last expression then gives

$$\nabla_l \left(\text{cof } H^{\mu}_{il}\right) + \epsilon_{ijk}\epsilon_{lmn} A^{\nu}_{\mu m} H^{\nu}_{jl} H^{\mu}_{kn} = 0. \tag{4.6.23}$$

Now sum over μ to obtain

$$\sum_{\mu=4}^{6} \nabla_l \left(\text{cof } H^{\mu}_{il}\right) + \epsilon_{ijk}\epsilon_{lmn} \sum_{\mu=4}^{6} A^{\nu}_{\mu m} H^{\nu}_{jl} H^{\mu}_{kn} = 0. \tag{4.6.24}$$

Next, define the second order Ricci tensor R to be

$$R_{ps} = \frac{1}{4}\epsilon_{pjk}\epsilon_{siq} R_{iqjk}, \tag{4.6.25}$$

which can be concisely written in matrix form as

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} R_{2323} & R_{2331} & R_{2312} \\ R_{2331} & R_{3131} & R_{3112} \\ R_{2312} & R_{3112} & R_{1212} \end{bmatrix}. \tag{4.6.26}$$

It then follows from the Gauss equations (4.6.11) that

$$R_{ps} = \frac{1}{4}\epsilon_{pjk}\epsilon_{siq} \sum_{\mu=4}^{6} \left(H^{\mu}_{kq} H^{\mu}_{ji} - H^{\mu}_{ki} H^{\mu}_{jq}\right) \tag{4.6.27}$$

$$= \frac{1}{2}\epsilon_{pjk}\epsilon_{sqi} \sum_{\mu=4}^{6} H^{\mu}_{ki} H^{\mu}_{jq} \tag{4.6.28}$$

$$= \sum_{\mu=4}^{6} \text{cof } H^{\mu}_{ps}, \tag{4.6.29}$$

and on substituting in (4.6.24) to eliminate the cofactor term, we obtain

$$\nabla_l R_{il} + \epsilon_{ijk}\epsilon_{mnl} \sum_{\mu=4}^{6} A^{\nu}_{\mu m} H^{\nu}_{jl} H^{\mu}_{kn} = 0, \qquad i = 1, 2, 3. \tag{4.6.30}$$

It is easy to infer from the second Bianchi identity (4.2.23) that the first term on left in the last equation vanishes, i.e.,

$$\nabla_l \left(R_{1l}\right) = \nabla_l \left(R_{2l}\right) = \nabla_l \left(R_{3l}\right) = 0.$$

The second term on the left of (4.6.30) is zero as $A^{\mu}_{\nu k}$ is skew-symmetric in μ, ν (see (4.3.12)). Consequently, the left side of (4.6.30) is identically zero. The combination, therefore, of the Codazzi and Gauss equations leads to three trivial relations which reduce the number of independent Codazzi equations by an additional 3.
Step 3.

It is convenient to introduce extra notation with respect to the covariant Codazzi (4.6.17) and Ricci (4.5.4) equations as follows;

$$C^{\mu}_{k\alpha} \equiv \epsilon_{ijk}\nabla_j H^{\mu}_{i\alpha} + \epsilon_{ijk} A^{\nu}_{\mu i} H^{\nu}_{\alpha j} = 0, \tag{4.6.31}$$

$$K^{\nu}_{k\mu} \equiv \epsilon_{ijk}\nabla_l A^{\nu}_{\mu j} + \epsilon_{ijk} A^{\nu}_{\eta i} A^{\eta}_{\mu j} - g^{pq}\epsilon_{ijk} H^{\mu}_{ip} H^{\nu}_{jq}. \tag{4.6.32}$$

Covariant differentiation of (4.6.31) yields

$$\epsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha} + \epsilon_{ijk}\left(\nabla_k A^{\nu}_{\mu i}\right) H^{\nu}_{\alpha j} + \epsilon_{ijk} A^{\nu}_{\mu i}\left(\nabla_k H^{\nu}_{\alpha j}\right) = 0. \tag{4.6.33}$$

The Codazzi equations (4.6.31) enable the last term to be expressed as

$$\epsilon_{ijk}\nabla_k H^{\nu}_{\alpha j} = -\epsilon_{ijk} A^{\eta}_{\nu j} H^{\eta}_{\alpha k}$$

and (4.6.33) then is reduced to

$$\epsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha} + \epsilon_{ijk}\left(\nabla_k A^{\nu}_{\mu i}\right) H^{\nu}_{\alpha j} - \epsilon_{ijk} A^{\eta}_{\nu i} H^{\eta}_{\alpha k} A^{\nu}_{\mu i} = 0.$$

The interchange of indices $i \to j \to k \to i$ in the last term leads to the further reduction

$$\epsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha} + \epsilon_{ijk} H^{\nu}_{\alpha j}\left(\nabla_k A^{\nu}_{\mu i} - A^{\eta}_{\nu k} A^{\eta}_{\mu i}\right) = 0. \tag{4.6.34}$$

But from (4.2.26), we may derive the commutation relation

$$\nabla_k\nabla_j H^{\mu}_{i\alpha} - \nabla_j\nabla_k H^{\mu}_{\alpha i} = R^{l}_{ijk} H^{\mu}_{l\alpha},$$

which may be expressed as

$$\begin{aligned}\epsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha} &= R^{l}_{ijk} H^{\mu}_{l\alpha}\\ &= g^{pq} R_{ijkq} H^{\mu}_{p\alpha}\\ &= g^{pq}\epsilon_{ijk} H^{\nu}_{ik} H^{\nu}_{jq} H^{\mu}_{p\alpha},\end{aligned}$$

where Gauss' equations (4.5.1) are employed in the derivation of the last line of the previous equation.

Finally, on appealing to the formula for the commutativity of $\epsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha}$, and the Gauss equations, we find from (4.6.34) that

$$H^{\nu}_{lj} K^{\nu}_{i\mu} = 0. \tag{4.6.35}$$

The maximal rank condition on H^{ν}_{lj} implies (4.6.35) has a unique solution

$$K^{\nu}_{i\mu} = 0, \tag{4.6.36}$$

and the 9 Ricci equations are satisfied.

4.7 Symmetrization of the Codazzi Equations

The required symmetrization is achieved by using the Codazzi equations to derive a certain matrix equation.

On noting the skew-symmetric relation (4.3.12), we may rewrite the Codazzi equations (4.6.17) in the slightly different form

$$\epsilon_{lji}\nabla_j H^{\mu}_{ik} + \epsilon_{lji} A^{\mu}_{\nu i} H^{\nu}_{jk} = 0, \tag{4.7.1}$$

a subset of which is

$$\nabla_1 H^{\mu}_{ij} - \nabla_j H^{\mu}_{i1} + A^{\mu}_{\nu 1} H^{\nu}_{ij} - A^{\mu}_{\nu j} H^{\nu}_{i1} = 0. \tag{4.7.2}$$

The Codazzi equations (4.7.1) may now be used to eliminate the covariant derivative on the right of the identity (4.6.19) to obtain

$$\begin{aligned}\nabla_l \mathrm{cof}\ H^{\mu}_{il} &= -\frac{1}{2}\epsilon_{ijk}\epsilon_{lmn} A^{\mu}_{\nu n} H^{\mu}_{jm} H^{\nu}_{lk} - \frac{1}{2}\epsilon_{ijk}\epsilon_{lmn} A^{\mu}_{\nu m} H^{\mu}_{kn} H^{\nu}_{lj} \\ &= -\epsilon_{ijk}\epsilon_{lmn} A^{\mu}_{\nu n} H^{\mu}_{jm} H^{\nu}_{lk}. \end{aligned} \tag{4.7.3}$$

Now let

$$W^{\mu} = \det H^{\mu}_{ij}, \tag{4.7.4}$$

so that by standard algebra of determinants, we have

$$\frac{\partial^2 W^{\mu}}{\partial H^{\mu}_{jk} \partial H^{\mu}_{il}} = \epsilon_{jim}\epsilon_{kln} H^{\mu}_{mn}, \tag{4.7.5}$$

which together with the expression (4.6.18) enables the identity (4.6.23) to be written as

$$\epsilon_{jim}\epsilon_{kln}\nabla_l \left(H^{\mu}_{mn}\right) H^{\mu}_{kj} + \epsilon_{jim}\epsilon_{kln} H^{\mu}_{mn}\nabla_l \left(H^{\mu}_{jk}\right) = -2\frac{\partial^2 W^{\mu}}{\partial H^{\mu}_{il}\partial H^{\mu}_{kn}} A^{\mu}_{\nu n} H^{\nu}_{lk}.$$

Terms on the left may be simplified on further appeal to (4.7.5) to give

$$\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{mn}}\nabla_l H^\mu_{mn} + \frac{\partial^2 W^\mu}{\partial H^\mu_{jk}\partial H^\mu_{il}}\nabla_l H^\mu_{jk} = -2\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{kn}} A^\mu_{\nu n} H^\nu_{lk},$$

and consequently,

$$\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\nabla_l H^\mu_{jk} = -\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jm}} A^\mu_{\nu n} H^\nu_{lk}, \qquad \text{no sum on } \mu. \tag{4.7.6}$$

The next part of the construction of a matrix equation involves the multiplication of (4.7.2) by

$$\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}$$

to obtain

$$\begin{aligned} -\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\nabla_1 H^\mu_{il} &+\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\nabla_l H^\mu_{i1} \\ &-\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}} A^\mu_{\nu 1} H^\nu_{il} + \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}} A^\mu_{\nu l} H^\nu_{i1} = 0. \end{aligned} \tag{4.7.7}$$

We combine the systems (4.7.6) and (4.7.7) into the matrix array of equations given by

$$\begin{aligned} &\begin{bmatrix} 0 & 0 \\ 0 & -\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}} \end{bmatrix} \nabla_1 \begin{bmatrix} H^\mu_{i1} \\ H^\mu_{il} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}} \\ \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{kn}} & 0 \end{bmatrix} \nabla_l \begin{bmatrix} H^\mu_{i1} \\ H^\mu_{jk} \end{bmatrix} \\ &\qquad + \begin{bmatrix} \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{kn}} A^\mu_{\nu n} H^\mu_{lk} \\ \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}} \left(-A^\mu_{\nu 1} H^\mu_{il} + A^\mu_{\nu l} H^\mu_{i1}\right) \end{bmatrix} \\ &\qquad = 0. \end{aligned} \tag{4.7.8}$$

We examine in detail the terms in this matrix equation and for this purpose introduce further notation. For example, the block matrices in the matrix coefficient of ∇_2 are given by

$$L^\mu_2 = \begin{bmatrix} \frac{\partial^2 W^\mu}{\partial H^\mu_{11}\partial H^\mu_{12}} & \frac{\partial^2 W^\mu}{\partial H^\mu_{12}\partial H^\mu_{12}} & \cdots & \frac{\partial^2 W^\mu}{\partial H^\mu_{33}\partial H^\mu_{12}} \\ \frac{\partial^2 W^\mu}{\partial H^\mu_{11}\partial H^\mu_{22}} & \frac{\partial^2 W^\mu}{\partial H^\mu_{12}\partial H^\mu_{22}} & \cdots & \frac{\partial^2 W^\mu}{\partial H^\mu_{33}\partial H^\mu_{22}} \\ \frac{\partial^2 W^\mu}{\partial H^\mu_{11}\partial H^\mu_{32}} & \cdots & \cdots & \frac{\partial^2 W^\mu}{\partial H^\mu_{33}\partial H^\mu_{32}} \end{bmatrix}_{3\times 9}, \tag{4.7.9}$$

$$\left(L_2^\mu\right)^T = \begin{bmatrix} \frac{\partial^2 W^\mu}{\partial H_{12}^\mu \partial H_{11}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{22}^\mu \partial H_{11}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{32}^\mu \partial H_{11}^\mu} \\ \frac{\partial^2 W^\mu}{\partial H_{12}^\mu \partial H_{12}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{22}^\mu \partial H_{12}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{32}^\mu \partial H_{12}^\mu} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 W^\mu}{\partial H_{12}^\mu \partial H_{33}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{22}^\mu \partial H_{33}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{32}^\mu \partial H_{33}^\mu} \end{bmatrix}_{9\times 3}, \tag{4.7.10}$$

so that the 12×12 composite matrix B_2^μ defined as

$$B_2^\mu = \begin{bmatrix} 0 & L_2^\mu \\ \left(L_2^\mu\right)^T & 0 \end{bmatrix}, \tag{4.7.11}$$

is symmetric, and the second term on the left in (4.7.8) involving ∇_2 becomes

$$B_2^\mu \nabla_2 U^\mu,$$

where

$$\left(U^\mu\right)^T = \left(H_{11}^\mu,\ H_{21}^\mu,\ H_{31}^\mu,\ H_{11}^\mu,\ H_{12}^\mu,\ H_{13}^\mu,\ \ldots H_{33}^\mu\right). \tag{4.7.12}$$

The matrices B_l^μ appearing in the coefficient of ∇_l are defined in a manner similar to (4.7.11).

Every coefficient matrix in (4.7.8) is symmetric including that for $l = 1$, but a separate argument is used to check the first coefficient matrix in the first term on the left of (4.7.8). This matrix, denoted by B_0^μ, is written as

$$B_0^\mu = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\partial^2 W^\mu}{\partial H_{il}^\mu \partial H_{jk}^\mu} \end{bmatrix} = \begin{bmatrix} 0^{3\times 3} & 0^{9\times 3} \\ 0^{3\times 9} & -\left(L_0^\mu\right)_{3\times 9} \end{bmatrix}, \tag{4.7.13}$$

where

$$L_0^\mu = \begin{bmatrix} \frac{\partial^2 W^\mu}{\partial H_{11}^\mu \partial H_{11}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{12}^\mu \partial H_{11}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{13}^\mu \partial H_{11}^\mu} \\ \frac{\partial^2 W^\mu}{\partial H_{11}^\mu \partial H_{12}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{12}^\mu \partial H_{12}^\mu} & \frac{\partial^2 W^\mu}{\partial H_{33}^\mu \partial H_{12}^\mu} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 W^\mu}{\partial H_{11}^\mu \partial H_{33}^\mu} & \cdots & \frac{\partial^2 W^\mu}{\partial H_{33}^\mu \partial H_{33}^\mu} \end{bmatrix}. \tag{4.7.14}$$

Consequently, in terms of the vector U^μ given by (4.7.12), the Codazzi system (4.7.8) may be expressed as

$$B_0^\mu \nabla_1 U^\mu + B_l^\mu \nabla_l U^\mu + Q^\mu = 0, \tag{4.7.15}$$

where Q^μ, the third term on the left of (4.7.8), is given explicitly by the 12-dimensional vector

$$Q^\mu = \begin{bmatrix} \frac{\partial^2 W^\mu}{\partial H^\mu_{il} \partial H^\mu_{kn}} A^\mu_{\nu n} H^\mu_{lk} \\ \frac{\partial^2 W^\mu}{\partial H^\mu_{il} \partial H^\mu_{jk}} \left(-A^\mu_{\nu 1} H^\mu_{il} + A^\mu_{\nu l} H^\mu_{i1}\right) \end{bmatrix}. \tag{4.7.16}$$

4.8 Symmetrization of the Linearized Codazzi Equations

4.8.1 Remarks on linearization

Let $\epsilon > 0$ be a small positive parameter, and suppose that a small perturbation in the variable y_i is given by

$$y_i = \bar{y}_i + \epsilon \dot{y}_i, \tag{4.8.1}$$

with corresponding small perturbations in other quantities given , for example, by

$$H^\mu_{ij} = \bar{H}^\mu_{ij} + \epsilon \dot{H}^\mu_{ij}, \tag{4.8.2}$$

$$A^\mu_{\nu i} = \bar{A}^\mu_{\nu i} + \epsilon \dot{A}^\mu_{\nu i}, \tag{4.8.3}$$

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + \epsilon \dot{\Gamma}^i_{jk}. \tag{4.8.4}$$

In these expansions, the superposed dot is intended to suggest differentiation with respect to ϵ.

4.8.2 Linearization of the Codazzi Equations

We now linearize (4.7.2) and (4.7.1) in the sense that after substitution from (4.8.2)–(4.8.4) all terms of order higher than the first in ϵ are neglected. Moreover, in the linearization it is convenient to remove the overbar without risk of confusion. Then, in view of the definition of the covariant derivative (see (4.2.5)–(4.2.7)), linearization of (4.7.2) and (4.7.1) respectively yields

$$\begin{aligned} \nabla_1 \dot{H}^\mu_{il} - \nabla_l \dot{H}^\mu_{i1} + \dot{A}^\mu_{\nu 1} H^\nu_{il} + A^\mu_{\nu 1} \dot{H}^\nu_{il} - \dot{A}^\mu_{\nu l} H^\nu_{i1} \\ - A^\mu_{\nu l} \dot{H}^\nu_{i1} - \dot{\Gamma}^q_{i1} H^\mu_{lq} - \dot{\Gamma}^q_{l1} H^\mu_{iq} + \dot{\Gamma}^q_{il} H^\mu_{1q} + \dot{\Gamma}^q_{1l} H^\mu_{iq} = 0; \end{aligned} \tag{4.8.5}$$

and

$$\epsilon_{lji} \left(\nabla_j \dot{H}^\mu_{ip} - \dot{\Gamma}^q_{jp} H^\mu_{iq} - \dot{\Gamma}^q_{ji} H^\mu_{pq}\right) + \epsilon_{lji} \left(\dot{A}^\mu_{\nu i} H^\nu_{jp} + A^\mu_{\nu i} \dot{H}^\nu_{jp}\right) = 0, \tag{4.8.6}$$

which by interchange of indices becomes

$$\epsilon_{kln}\left(\nabla_l \dot{H}^\mu_{mn} - \dot{\Gamma}^q_{lm} H^\mu_{nq} - \dot{\Gamma}^q_{ln} H^\mu_{mq}\right) + \epsilon_{kln}\left(\dot{A}^\mu_{\nu n} H^\nu_{lm} + A^\mu_{\nu n}\dot{H}^\nu_{lm}\right) = 0. \tag{4.8.7}$$

On multiplying (4.8.7) by $\epsilon_{jim} H^\mu_{kj}$ and suspending summation over the repeated index μ, we obtain

$$\epsilon_{jim}\epsilon_{kln} H^\mu_{kj}\left(\nabla_l \dot{H}^\mu_{mn} - \dot{\Gamma}^q_{lm} H^\mu_{nq} - \dot{\Gamma}^q_{ln} H^\mu_{mq}\right) + \epsilon_{kln} H^\mu_{jk}\epsilon_{lji}\left(\dot{A}^\mu_{\nu n} H^\nu_{lm} + A^\mu_{\nu n}\dot{H}^\nu_{lm}\right) = 0, \tag{4.8.8}$$

which on recalling (4.7.5), we rewrite as

$$\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{mn}}\left(\nabla_l \dot{H}^\mu_{mn} - \dot{\Gamma}^q_{lm} H^\mu_{nq} - \dot{\Gamma}^q_{ln} H^\mu_{mq}\right) + \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{mn}}\left(\dot{A}^\mu_{\nu n} H^\nu_{lm} + A^\mu_{\nu n}\dot{H}^\nu_{lm}\right) = 0. \tag{4.8.9}$$

Next, consider the particular equation (4.8.5), which after multiplication by

$$-\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}, \qquad \text{no sum on } \mu$$

becomes

$$\begin{aligned}
&-\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\nabla_1 \dot{H}^\mu_{il} + \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\nabla_l \dot{H}^\mu_{i1}\\
&-\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\Big(\dot{A}^\mu_{\nu 1} H^\nu_{il} + A^\mu_{\nu 1}\dot{H}^\nu_{il} - \dot{A}^\mu_{\nu l} H^\nu_{i1}\\
&-A^\mu_{\nu l}\dot{H}^\nu_{i1} - \dot{\Gamma}^q_{i1} H^\mu_{lq} - \dot{\Gamma}^q_{l1} H^\mu_{iq} + \dot{\Gamma}^q_{il} H^\mu_{1q} + \dot{\Gamma}^q_{1l} H^\mu_{iq}\Big) = 0.
\end{aligned} \tag{4.8.10}$$

The linearized Codazzi system (4.8.9) and (4.8.10) may be concisely expressed by introducing the definitions

$$\dot{Q}^\mu = \begin{bmatrix} \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{kn}}\left(\dot{A}^\mu_{\nu n} H^\nu_{lk} + A^\mu_{\nu n}\dot{H}^\nu_{lk}\right) \\ \frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\left(-\dot{A}^\mu_{\nu 1} H^\nu_{il} - A^\mu_{\nu 1}\dot{H}^\nu_{il} + \dot{A}^\mu_{\nu l} H^\nu_{i1} + A^\mu_{\nu l}\dot{H}^\nu_{i1}\right) \end{bmatrix}, \tag{4.8.11}$$

$$\dot{S}^\mu = \begin{bmatrix} -\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{mn}}\left(\dot{\Gamma}^q_{lm} H^\mu_{nq} + \dot{\Gamma}^q_{ln} H^\mu_{mq}\right) \\ -\frac{\partial^2 W^\mu}{\partial H^\mu_{il}\partial H^\mu_{jk}}\left(\dot{\Gamma}^q_{i1} H^\mu_{lq} + \dot{\Gamma}^q_{l1} H^\mu_{iq} + \dot{\Gamma}^q_{il} H^\mu_{1q} + \dot{\Gamma}^q_{1l} H^\mu_{iq}\right) \end{bmatrix}, \tag{4.8.12}$$

when the system may be written as the single matrix equation

$$B_0^{\mu}\nabla_1\dot{U}^{\mu} + B_l^{\mu}\nabla_l\dot{U}^{\mu} + \dot{Q}^{\mu} + \dot{S}^{\mu} = 0, \tag{4.8.13}$$

where B_l^{μ} is defined analogously to (4.7.11), and $\dot{U}^{\mu}$ is the linearization of the vector (4.7.12).

4.9 The Ricci Equations

We next discuss the Ricci equations (4.5.3), and without loss of generality[1] set

$$A_{\mu 1}^{\nu} = 0, \tag{4.9.1}$$

and (4.5.2) simplify to

$$\partial_1 A_{\mu 2}^{\nu} = g^{pq}\left(H_{1p}^{\mu}H_{2q}^{\nu} - H_{2p}^{\mu}H_{1q}^{\nu}\right), \tag{4.9.2}$$

$$\partial_1 A_{\mu 3}^{\nu} = g^{pq}\left(H_{1p}^{\mu}H_{3q}^{\nu} - H_{3p}^{\mu}H_{1q}^{\nu}\right). \tag{4.9.3}$$

We note that $A_{\mu 2}^{\nu}$, $A_{\mu 3}^{\nu}$ are therefore completely determined by their data on a plane x_1 = constant $= -L$ and on the set H_{jk}^{ν}. Accordingly, we may introduce the substitutions

$$A_{\mu 2}^{\nu}(x_1, x_2, x_3) = A_{\mu 2}^{\nu}(-L, x_2, x_3) + \int_{-L}^{x_1} g^{pq}\left(H_{1p}^{\mu}H_{2q}^{\nu} - H_{2p}^{\mu}H_{1q}^{\nu}\right) dx_1', \tag{4.9.4}$$

$$A_{\mu 3}^{\nu}(x_1, x_2, x_3) = A_{\mu 3}^{\nu}(-L, x_2, x_3) + \int_{-L}^{x_1} g^{pq}\left(H_{1p}^{\mu}H_{3q}^{\nu} - H_{3p}^{\mu}H_{1q}^{\nu}\right) dx_1', \tag{4.9.5}$$

in the expression (4.7.16) for the matrix Q to eliminate explicit dependence on $A_{\nu l}^{\mu}$. Observe that dependence on $A_{\nu l}^{\mu}$ is reduced to dependence on data provided on $x_1 = -L$. This data, of course, must be consistent with the additional Ricci equations.

[1] Deane Yang pointed out this equality to me and called it a "gauge condition". An analogy with continuum mechanics might be setting the pressure equal to zero on the surface of a water wave.

4.10 The Full Nonlinear System

We emphasize analogies with *continuum mechanics* by restating the full nonlinear system in terms employed by that theory.

The *balance laws* are given by the quasi-linear Codazzi equations (4.7.15):

$$B_0^{\mu}\nabla_1 U^{\mu} + B_l^{\mu}\nabla_l U^{\mu} + Q^{\mu} = 0,$$

where from (4.7.12) we have $U^{\mu} \in \mathbb{R}^{12}$ for each $\mu = 4, 5, 6$, and Q^{μ} is given by (4.7.16).

The *constitutive relations* are provided by the Gauss equations (4.5.1)

$$\sum_{\mu}\left(H_{ik}^{\mu}H_{jl}^{\mu} - H_{il}^{\mu}H_{jk}^{\mu}\right) = R_{ijkl}, \tag{4.10.1}$$

together with constitutive relations for $A_{\nu l}^{\mu}$ given by (4.9.1), (4.9.4), and (4.9.5).

According to Blum's theorem [Blu55], when the elements H_{jk}^{μ} form a full rank matrix, there are 27 independent equations in 27 unknowns H_{jk}^{μ} and $A_{\nu l}^{\mu}$ since the Ricci equations follow from the Gauss and Codazzi equations. Observe, however, that relations (4.9.4) and (4.9.5) do not completely eliminate the terms $A_{\nu l}^{\mu}$ in favour of the terms H_{ij}^{μ}, because initial data on $x_1 = -L$ still enter into the values of $A_{\nu l}^{\mu}$.

4.11 The Linearized Ricci Equations

In the notation of Sect. 4.8.1, the linearized Ricci equations (4.9.1), (4.9.4), and (4.9.5) are given by

$$\dot{A}_{\mu 1}^{\nu} = 0, \tag{4.11.1}$$

$$\begin{aligned}\dot{A}_{\mu 2}^{\nu}(x_1, x_2, x_3) = \dot{A}_{\mu 2}^{\nu}(-L, x_2, x_3) + \int_{-L}^{x_1} \Big\{ &\dot{g}^{pq}\left(H_{1p}^{\mu}H_{2q}^{\nu} - H_{2p}^{\mu}H_{1q}^{\nu}\right) \\ &+ g^{pq}\left(\dot{H}_{1p}^{\mu}H_{2q}^{\nu} + H_{1p}^{\mu}\dot{H}_{2q}^{\nu} - \dot{H}_{1p}^{\mu}H_{2q}^{\nu} - H_{1p}^{\mu}\dot{H}_{2q}^{\nu}\right)\Big\}\, dx_1',\end{aligned} \tag{4.11.2}$$

$$\begin{aligned}\dot{A}_{\mu 3}^{\nu}(x_1, x_2, x_3) = \dot{A}_{\mu 3}^{\nu}(-L, x_2, x_3) + \int_{-L}^{x_1} \Big\{ &\dot{g}^{pq}\left(H_{1p}^{\mu}H_{2q}^{\nu} - H_{2p}^{\mu}H_{1q}^{\nu}\right) \\ &+ g^{pq}\left(\dot{H}_{1p}^{\mu}H_{2q}^{\nu} + H_{1p}^{\mu}\dot{H}_{3q}^{\nu} - \dot{H}_{1p}^{\mu}H_{3q}^{\nu} - H_{1p}^{\mu}\dot{H}_{3q}^{\nu}\right)\Big\}\, dx_1'.\end{aligned} \tag{4.11.3}$$

When $\dot{A}_{\mu 2}^{\nu}(-L, x_2, x_3)$ and $\dot{A}_{\mu 3}^{\nu}(-L, x_2, x_3)$ vanish on the boundary of the domain, their contribution to (4.11.2) and (4.11.3) is zero. Furthermore, the integral terms in (4.11.1) and (4.11.3) are bounded by $K\,\mathrm{vol}\,(\Omega)$, where

$$K = \|\dot{g}^{pq}\|_{L^2(\Omega)} \sup_{\Omega,\mu,j,k} |H^{\mu}_{jk}|^2 + \sup_{\Omega} \|g^{pq}\| \sup_{\Omega,\mu,j,k} |H^{\mu}_{jk}| \|\dot{H}^{\mu}_{jk}\|_{L^2(\Omega,\mathbb{R}^{27})},$$

and in consequence, we obtain

Proposition 4.11.1 *The quantities $\dot{A}^{\nu}_{\mu 2}$ and $\dot{A}^{\nu}_{\mu 3}$ satisfy the bounds*

$$|\dot{A}^{\nu}_{\mu 2}| \le K\,vol(\Omega)^{1/3}, \tag{4.11.4}$$

$$|\dot{A}^{\nu}_{\mu 3}| \le K\,vol(\Omega)^{1/3}. \tag{4.11.5}$$

Proof of (4.11.4)
Typical terms in the relation (4.11.2) may be expressed as

$$a(x_1, x_2, x_3) = \int_{-L}^{x_1} \dot{g}^{pq} \left(H^{\mu}_{1p} H^{\nu}_{2q} \right) dx_1',$$

$$b(x_1, x_2, x_3) = \int_{-L}^{x_1} g^{pq} \left(\dot{H}^{\mu}_{1p} H^{\nu}_{2q} \right) dx_1',$$

where there is no sum on p, q.

The Cauchy-Schwarz inequality applied to the first expression leads to the bounds

$$\begin{aligned} |a(x_1, x_2, x_3)| &\le \sup_{\Omega} |H^{\mu}_{1p} H^{\nu}_{2q}| \left(\int_{-L}^{x_1} dx_1' \right)^{1/2} \left(\int_{-L}^{x_1} |\dot{g}^{pq}|^2 \, dx_1' \right)^{1/2} \\ &\le \sup_{\Omega} |H^{\mu}_{1p} H^{\nu}_{2q}| \, (2L)^{1/2} \left(\int_{-L}^{L} |\dot{g}^{pq}|^2 \, dx_1' \right)^{1/2}, \end{aligned}$$

and consequently, on noting that the term on the right is independent of x_1, we have

$$\begin{aligned} \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} |a(x_1, x_2, x_3)|^2 \, dx_1 dx_2 dx_3 \le 4L^2 &\left(\sup_{\Omega} |H^{\mu}_{1p} H^{\nu}_{2q}| \right)^2 \\ &\times \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} |\dot{g}^{pq}|^2 \, dx_1' dx_2 dx_3, \end{aligned}$$

or

$$\|a\|_{L^2(\Omega)} \le 2L \sup_{\Omega} |H^{\mu}_{1p} H^{\nu}_{2q}| \|\dot{g}^{pq}\|_{L^2(\Omega)}.$$

A similar argument gives

$$|b(x_1, x_2, x_3)| \le \sup_{\Omega} |g^{pq} H^{\nu}_{2q}| \int_{-L}^{L} |\dot{H}^{\mu}_{1p}(x_1', x_2, x_3)| \, dx_1',$$

where the expression on the right is again independent of x_1.

Thus we conclude that

$$|b(x_1, x_2, x_3)|^2 \leq 2L \left(\sup_{\Omega} |g^{pq} H^{\nu}_{2q} \right)^2 \int_{-L}^{L} |\dot{H}^{\mu}_{1p}|^2 (x_1', x_2, x_3)\, dx_1',$$

from which follows

$$\int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} |b(x_1, x_2, x_3)|^2\, dx_1 dx_2 dx_3 \leq 4L^2 \sup_{\Omega} |g^{pq} H^{\nu}_{2q}|^2 \times \int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} |\dot{H}^{\mu}_{1p}(x_1', x_2, x_3)|^2\, dx_1' dx_2 dx_3,$$

which leads to the final bound

$$\|b\|_{L^2(\Omega)} \leq 2L \sup_{\Omega} |g^{pq} H^{\nu}_{2q}| \|\dot{H}^{\mu}_{1p}\|_{L^2(\Omega)}.$$

4.12 The Linearized Gauss Equations

In view of the notation adopted in Sect. 4.8.1, the linearized Gauss equations become

$$\sum_{\mu} \left(\dot{H}^{\mu}_{ik} H^{\mu}_{jl} + H^{\mu}_{ik} \dot{H}^{\mu}_{jl} - \dot{H}^{\mu}_{il} H^{\mu}_{jk} - H^{\mu}_{il} \dot{H}^{\mu}_{jk} \right) = \dot{R}_{ijkl}. \tag{4.12.1}$$

The system (4.12.1) consists of 6 equations in the 18 components $\dot{H}^{\mu}_{ij}$. We say H^{μ}_{ij} is *non-degenerate* in the neighbourhood of $x = 0$ when 6 of the components of $\dot{H}^{\mu}_{ij}$ can be solved in terms of the remaining 12 components and $\dot{R}_{ijkl}$. A *sufficient condition* for non-degeneracy is provided by [BGY83, Theorem F] which establishes non-degeneracy when at least one component of the Riemann curvature tensor R_{ijkl} is non-zero.

Accordingly, let us assume that the set H^{ν}_{ij} is non-degenerate in a neighbourhood of $x = 0$. This implies that the vector

$$\dot{U} = \begin{bmatrix} \dot{U}^4 \\ \dot{U}^5 \\ \dot{U}^6 \end{bmatrix}, \tag{4.12.2}$$

where U^{μ}, defined in (4.7.12), can be written as

$$\dot{U} = C\dot{H} + D\dot{R}. \tag{4.12.3}$$

In this relation, $\dot{H}$ denotes the distinguished 12 components of the set $\dot{H}^{\mu}_{ij}$, and $\dot{R}$ denotes the 6 non-trivial elements corresponding to the perturbed Riemann curvature tensor. It follows that $\dot{U} \in \mathbb{R}^{36}$, $\dot{H} \in \mathbb{R}^{12}$, $\dot{R} \in \mathbb{R}^{6}$, and therefore in (4.12.3), C represents a 36×12 matrix, while D represents a 36×6 matrix.

4.13 The Closed Symmetric System for the Linearized Problem and Quasi-linear Problem

With reference to the symmetrized and linearized Codazzi equations (4.8.13), let us set

$$B_0 = \begin{bmatrix} B_0^4 & 0 & 0 \\ 0 & B_0^5 & 0 \\ 0 & 0 & B_0^6 \end{bmatrix},$$

$$B_l = \begin{bmatrix} B_l^4 & 0 & 0 \\ 0 & B_l^5 & 0 \\ 0 & 0 & B_l^6 \end{bmatrix},$$

$$\dot{Q} = \begin{bmatrix} \dot{Q}^4 \\ \dot{Q}^5 \\ \dot{Q}^6 \end{bmatrix}, \qquad \dot{S} = \begin{bmatrix} \dot{S}^4 \\ \dot{S}^5 \\ \dot{S}^6 \end{bmatrix}, \tag{4.13.1}$$

and use this notation to write (4.8.13) as

$$B_0 \nabla_1 \dot{U} + B_l \nabla_l \dot{U} + \dot{Q} + \dot{S} = 0. \tag{4.13.2}$$

Observe that since $\dot{Q}$ depends linearly on the sets $\dot{H}^{\mu}_{ij}$ and $\dot{A}^{\mu}_{\nu m}$ as given in (4.8.11), we may introduce matrices $E,\ F$ to represent the dependence by

$$\dot{Q} = E\dot{U} + F\dot{A}, \tag{4.13.3}$$

where $\dot{U} \in \mathbb{R}^{36},\ \dot{A} \in \mathbb{R}^{6}$, E is a 36×36 matrix, and F is a 36×6 matrix.

Upon substitution of (4.12.3) in (4.13.3) we obtain

$$\begin{aligned} \dot{Q} &= E\left(C\dot{H} + D\dot{R}\right) + F\dot{A} \\ &= G\dot{H} + J\dot{R} + F\dot{A}, \end{aligned} \tag{4.13.4}$$

where $G = EC$ is a 36×12 matrix, and $J = ED$ is a 36×36 matrix. In consequence, the system (4.13.2) has the form

$$B_0 \nabla_1 \left(C\dot{H} + D\dot{R}\right) + B_l \nabla_l \left(C\dot{H} + D\dot{R}\right) + G\dot{H} + J\dot{R} + F\dot{A} + \dot{S} = 0, \tag{4.13.5}$$

which after rearrangement becomes

$$\begin{aligned} B_0 C \nabla_1 \dot{H} \; &+ B_l C \nabla_l \dot{H} + (B_0 \nabla_1 C + B_l \nabla_l C + G)\,\dot{H} \\ &+ B_0 \nabla_1 (D\dot{R}) + B_l \nabla_l (D\dot{R}) + J\dot{R} + F\dot{A} + \dot{S} = 0. \end{aligned} \quad (4.13.6)$$

We multiply (4.13.6) on left by the 12×36 matrix C^T to obtain the equivalent but compact form

$$\mathcal{A}_0 \nabla_1 \dot{H} + \mathcal{A}_l \nabla_l \dot{H} + \mathcal{B}\dot{H} + C^T F \dot{A} + \Lambda = 0, \quad (4.13.7)$$

where

$$\begin{aligned} \mathcal{A}_0 &= C^T B_0 C, \\ \mathcal{A}_l &= C^T B_l C, \\ \mathcal{B} &= C^T\,(B_0 \nabla_1 C + B_l \nabla_l C + G)\,, \\ \Lambda &= C^T\,\left(B_0 \nabla_1 (D\dot{R}) + B_l \nabla_l (D\dot{R}) + J\dot{R} + \dot{S}\right). \end{aligned}$$

The linearized Ricci equations (4.11.2) and (4.11.3) with

$$\dot{A}^{\nu}_{\mu 2}(-L, x_2, x_3) = \dot{A}^{\nu}_{\mu 3}(-L, x_2, x_3) = 0$$

next give

$$\dot{A}^{\nu}_{\mu 1} = 0, \quad (4.13.8)$$

$$\begin{aligned} \dot{A}^{\nu}_{\mu l}(x_1, x_2, x_3) = \int_{-L}^{x_1} &\left\{ \dot{g}^{pq} \left(H^{\mu}_{1p} H^{\nu}_{lq} - H^{\mu}_{lp} H^{\nu}_{1q} \right) \right. \\ &\left. + g^{pq} \left(\dot{H}^{\mu}_{1p} H^{\nu}_{lq} + H^{\mu}_{1p} \dot{H}^{\nu}_{lq} - \dot{H}^{\mu}_{1p} H^{\nu}_{lq} - H^{\mu}_{1p} \dot{H}^{\nu}_{lq} \right) \right\} dx_1', \\ & l = 2, 3. \end{aligned} \quad (4.13.9)$$

Insertion of (4.13.8) and (4.13.9) into (4.13.7) yields a symmetric system of 12 equations in the 12 unknowns $\dot{H}$ which are weakly non-local due to (4.13.9). The relations (4.11.4) and (4.11.5), however, indicate that the non-locality is very weak.

Remark 4.13.1 (Non-linear problem) The derivation just described is for the linearized system, but examination of the individual steps in the argument shows that for the non-linear problem the same procedure also yields a quasi-linear system of 12 equations.

4.14 The Weak Form of the Closed System

The purpose of previous sections is to formulate the theory in a manner suitable for proofs of existence and uniqueness in the embedding problem, which are developed in this Section.

Define the linear operator $\mathcal{L}$ in terms of the general variable $\widehat{H}$ by

$$\mathcal{L}\widehat{H} = \mathcal{A}_0 \nabla_1 \widehat{H} + \mathcal{A}_l \nabla_l \widehat{H} + \mathcal{B}\widehat{H} + C^T F \dot{A}, \tag{4.14.1}$$

where $\dot{A}$ is defined by (4.13.8) and (4.13.9).

We wish to consider the weak form of equations associated with the operator $\mathcal{L}$. For this purpose, let $(\cdot, \cdot)$ denote the inner product on the space $L^2(\Omega, \mathbb{R}^{12})$ and let the function $V \in C_0^\infty(\Omega, \mathbb{R}^{12})$. The weak form of the equation

$$\mathcal{L}\widehat{H} = -\Lambda$$

is then given by

$$(\mathcal{L}^* V, \widehat{H}) = -(V, \Lambda), \tag{4.14.2}$$

where $\mathcal{L}^*$ is the adjoint operator to $\mathcal{L}$. We conclude from (4.14.2) that $(\mathcal{L}^* V, \widehat{H})$ defines a bilinear form on $H_0^1(\Omega, \mathbb{R}^{12})$.

The proofs of existence and uniqueness rely upon the Lax-Milgram theorem (see, for example, [Yos65]) stated here for convenience.

Theorem 4.14.1 (Lax-Milgram Theorem). *Let X be a Hilbert space and $\mathcal{C}(\chi, \psi)$ a (possibly complex) bilinear functional defined on the product space $X \times X$. Let $\|\cdot\|_X$ and $(\cdot, \cdot)_X$ denote the norm and inner product on X. Suppose that*

$$(i) \quad |\mathcal{C}(\chi, \psi)| \le \gamma \|\chi\|_X \|\psi\|_X, \qquad \textit{(boundedness)}$$
$$(ii) \quad \mathcal{C}(\chi, \chi) \ge \delta \|\chi\|_X^2, \qquad \textit{(coerciveness)}$$

for positive constants δ, γ. Then there exists a uniquely determined bounded linear operator T with bounded inverse T^{-1} such that whenever $\chi, \psi \in X$ there holds

$$\mathcal{C}(\chi, T\psi) = (\chi, \psi)_X,$$

$$\|T\|_X \le \delta^{-1}, \qquad \|T^{-1}\|_X \le \gamma.$$

To apply the Lax-Milgram theorem to the weak equation (4.14.2), we set $X = H_0^1(\Omega, \mathbb{R}^{12})$, and let $\mathcal{C}(\chi, \psi) = (\mathcal{L}^* \chi, \psi)$. Note, however, that Condition (i) holds but not Condition (ii). To overcome this difficulty, we introduce additional terms to (4.14.2) that regularize the equation. Let $\epsilon > 0$ be an arbitrary positive constant. Then the regularized problem is given by

$$(\mathcal{L}^* V, \widehat{H}) + \epsilon(\partial V, \partial \widehat{H}) = -(V, \Lambda) - \epsilon(\partial V, \partial \Lambda), \tag{4.14.3}$$

in which we employ the notation

$$\left(\partial V^1, \partial V^2\right) = \int_\Omega \sum_{j=1}^{3} \partial_j V^1 \cdot \partial_j V^2 \, dx,$$

where we recall the $(, \cdot,)$ denotes the Euclidean inner product in $\mathbb{R}^{12}$. We now let the bilinear form $\mathcal{C}_\epsilon$ be defined by the expression on the left of (4.14.3). Upon assuming the weaker coerciveness estimate

$$(\mathcal{L}^* \widehat{H}, \widehat{H}) \geq \delta_1 \|\widehat{H}\|^2_{L^2(\Omega, \mathbb{R}^{12})} \tag{4.14.4}$$

for some positive constant δ_1, we have

$$\begin{aligned} &(i) \quad |\mathcal{C}_\epsilon(V, \widehat{H})| \leq \gamma \|V\|_X \|\widehat{H}\|_X, \\ &(ii) \quad \mathcal{C}_\epsilon(\widehat{H}, \widehat{H}) \geq \delta_1 \|\widehat{H}\|^2_{L^2(\Omega, \mathbb{R}^{12})} + \epsilon(\partial \widehat{H}, \partial \widehat{H}). \end{aligned}$$

The Lax-Milgram theorem clearly applies to the regularized problem and shows that a solution $\widehat{H}_\epsilon = T_\epsilon \Lambda$ exists to (4.14.3) and satisfies

$$(\mathcal{L}^* V, \widehat{H}_\epsilon) + \epsilon(\partial V, \partial \widehat{H}_\epsilon) = -(V, \Lambda) - \epsilon(\partial V, \partial \Lambda), \tag{4.14.5}$$

or alternatively

$$(\mathcal{L}^* V, \widehat{H}_\epsilon) - \epsilon(\partial^2 V, \widehat{H}_\epsilon) = -(V, \Lambda) - \epsilon(\partial V, \partial \Lambda), \tag{4.14.6}$$

for all $V \in H^1_0(\Omega, \mathbb{R}^{12})$. Accordingly, on setting $V = \widehat{H}_\epsilon$ in (4.14.5), we obtain

$$(\mathcal{L}^* \widehat{H}_\epsilon, \widehat{H}_\epsilon) + \epsilon(\partial \widehat{H}_\epsilon, \partial \widehat{H}_\epsilon) = -(\widehat{H}_\epsilon, \Lambda) - \epsilon(\partial \widehat{H}_\epsilon, \partial \Lambda). \tag{4.14.7}$$

The first term on the left of (4.14.7) may be bounded from below using assumption (4.14.4), while terms on the right may be bounded from above using the Cauchy-Schwarz inequality. These operations lead to the bounds

$$\begin{aligned} \delta_1 \|\widehat{H}_\epsilon\|^2_{L^2(\Omega, \mathbb{R}^{12})} + \epsilon \|\partial \widehat{H}_\epsilon\|^2_{L^2(\Omega, \mathbb{R}^{12})} \leq \|\widehat{H}_\epsilon\|_{L^2(\Omega, \mathbb{R}^{12})} \|\Lambda\|_{L^2(\Omega, \mathbb{R}^{12})} \\ + \epsilon \|\partial \widehat{H}_\epsilon\|_{L^2(\Omega, \mathbb{R}^{12})} \|\partial \Lambda\|_{L^2(\Omega, \mathbb{R}^{12})}. \end{aligned} \tag{4.14.8}$$

The arithmetic-geometric mean inequality in the form

$$ab \leq \frac{1}{3} a^2 + \frac{3}{4} b^2,$$

applied to terms on the right then yields

$$\delta_1 \|\widehat{H}_\epsilon\|^2_{L^2(\Omega,\mathbb{R}^{12})} + \epsilon\left(\frac{2}{3}\|\partial\widehat{H}_\epsilon\|^2_{L^2(\Omega,\mathbb{R}^{12})} - \frac{1}{2}\|\Lambda\|^2_{L^2(\Omega,\mathbb{R}^{12})}\right)$$
$$\leq \|\widehat{H}_\epsilon\|_{L^2(\Omega,\mathbb{R}^{12})}\|\Lambda\|_{L^2(\Omega,\mathbb{R}^{12})} + \frac{\epsilon}{4}\|\partial\Lambda\|^2_{L^2(\Omega,\mathbb{R}^{12})}$$
$$\leq \frac{\delta_1}{2}\|\widehat{H}_\epsilon\|^2_{L^2(\Omega,\mathbb{R}^{12})} + \frac{1}{2\delta_1}\|\Lambda\|^2_{L^2(\Omega,\mathbb{R}^{12})} + \frac{\epsilon}{4}\|\partial\Lambda\|^2_{L^2(\Omega,\mathbb{R}^{12})},$$

which after rearrangement gives

$$\frac{\delta_1}{2}\|\widehat{H}_\epsilon\|^2_{L^2(\Omega,\mathbb{R}^{12})} \leq \frac{1}{2\delta_1}\|\Lambda\|^2_{L^2(\Omega,\mathbb{R}^{12})} + \frac{\epsilon}{4}\|\partial\Lambda\|^2_{L^2(\Omega,\mathbb{R}^{12})}. \tag{4.14.9}$$

We conclude that $\widehat{H}_\epsilon$ is bounded independently of ϵ when $\Lambda \in H^1_0(\Omega)$, and consequently $\widehat{H}_\epsilon$ has a weakly convergent subsequence (also denoted by $\widehat{H}_\epsilon$) so that

$$\widehat{H}_\epsilon \rightharpoonup \widehat{H}, \qquad \text{in } L^2(\Omega, \mathbb{R}^{12}).$$

We now pass to the limit as $\epsilon \to 0$ in (4.14.6) and for all $V \in C^\infty_0(\Omega)$ obtain the relation

$$(\mathcal{L}^* V, \widehat{H}) = -(V, \Lambda),$$

which proves the existence of a weak solution $\widehat{H}$. Its uniqueness follows from the coercivity assumption (4.14.4).

Let us summarize the result in the following theorem.

Theorem 4.14.2 *Suppose the operator $\mathcal{L}$ defined by (4.14.1) satisfies the coercivity condition*

$$(\mathcal{L}^*\widehat{H}, \widehat{H}) \geq \delta_1 \|\widehat{H}\|^2_{L^2(\Omega,\mathbb{R}^{12})}$$

for some $\delta_1 > 0$. Then the weak form of the linearized isometric embedding problem (4.14.2) has a unique solution for all $\Lambda \in H^1_0(\Omega)$.

The next step is to apply Theorem 4.6.1 to the system (4.14.2), (4.14.6) and (4.14.7). Assume first that the (undotted) embedding is perturbed in a small neighbourhood of the point $x = 0$ chosen as the origin of a system of normal coordinates where the Christoffel symbols Γ^q_{ij} vanish. When the small neigbourhood is taken to be the box $-L \leq x_i \leq L$, $i = 1, 2, 3$, the quantity $\dot{A}$, defined by (4.13.8) and (4.13.9) that satisfies the bounds (4.11.4) and (4.11.5), becomes negligible in the box and do not enter into the coercivity computations. Accordingly, we have

Theorem 4.14.3 *When the quadratic form*

$$\dot{H}^T\,(-\partial_1\mathcal{A}_0 - \partial_l\mathcal{A}_l + \mathcal{B})\,\dot{H} \tag{4.14.10}$$

is positive-definite (or negative-definite) at $x = 0$ *there exists a unique weak solution to the linearized isometric embedding equations (4.14.2), (4.13.8), and (4.13.9).*

The parameters entering into the 12×12 symmetric coefficient matrix

$$-\partial_1 \mathcal{A}_0 - \partial_l \mathcal{A}_l + \frac{1}{2}\left(\mathcal{B}^T + \mathcal{B}\right) \tag{4.14.11}$$

are H^μ_{ij}, $\partial_1 \mathcal{A}_0$, $\partial_1 \mathcal{A}_1$, $\partial_2 \mathcal{A}_2$, $\partial_3 \mathcal{A}_3$, $A^\nu_{\mu 2}$, $A^\nu_{\mu 3}$ all evaluated at $x = 0$. In consequence, the classical chain rule may be applied to $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ to show that the parameters in the coefficient matrix reduce to H^μ_{ij}, $\partial_l H^\mu_{il}$, $A^\nu_{\mu 2}$, $A^\nu_{\mu 3}$ evaluated at $x = 0$. We therefore conclude that

(i) The Gauss relations provide 12 independent H^μ_{ij}.
(ii) The differentiated Gauss relations provide 15 independent $\partial_l H^\mu_{ij}$. (See, for example, Poole [Poo10].)
(iii) There are 6 independent $A^\nu_{\mu 2}$, $A^\nu_{\mu 3}$.

Hence there are $12+15+6 = 33$ free parameters entering into the 12×12 matrix (4.14.11) resulting in considerable simplification.

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