Chapter 4
Quantifiers Are Logical Constants, but Only Ambiguously

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Abstract Why is it crucial to categorize quantifiers as logical constants? After clarifying the importance of the issue in a larger context of logical theories, the paper investigates the following question: Why do we encounter a more contentious debate on the logical constancy of quantifiers than in the case of sentential connectives? Starting from the intuitive and naive rationale and moving to more complicated arguments for the well-accepted view that quantifiers are logical constants, I identify two tiers of the meanings assigned to quantifiers: At the first-level the interpretation is changing, and at the meta-level a constant meaning is assigned. I claim the tension arises from this ambiguous nature of the quantifier-semantics and illustrate the effects of the tension both in relevant literature and in a non-classical logic where the interpretation of the universal quantifier includes the empty domain. The double features of the universal quantifier – varying at one level and constant at another level – could raise skepticism toward the debate on logical constants itself, and in turn, toward the Tarskian analysis of logical consequence whose success heavily relies on the clear-cut status of logical constants.

Are quantifiers logical constants? Many have said "yes,"[1] but for different reasons, and a few have said "no" or have been skeptical of a definite answer. The question involves two debated topics in the philosophy of logic – quantifiers and logical constants. One might easily say "It all depends on what we mean by 'logical constants'." This is not incorrect. The response acknowledges that the concept of

[1] "[I]t is generally agreed that signs for negation, conjunction, disjunction, conditionality, and the first-order quantifiers should count as logical constants, ... " (MacFarlane [5]).

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logical constants is quite tricky and sometimes controversial, while many might think we know what quantifiers are. Yes, we can identify a given symbol as a quantifier, since it is a syntactic entity after all. However, as we will see below, how this syntactic piece is assigned its semantics doubly confuses our question "Are quantifiers logical constants?"

The paper is not about logical constants in general. First of all, I do not propose any necessary and sufficient features of logical constants. Moreover, I do not intend to get into an even bigger picture and make a claim for a position on the spectrum running from the essential nature of logical constants to the skepticism about the logical constant debate itself. Instead of making any direct claim about logical constants, the paper will focus on the way quantifiers have been addressed in the discussions about logical constants. More specifically, by exploring the way quantifiers are handled in formal semantics, I identify interesting and important features of quantifiers that the existing literature has overlooked and uncover our ambiguous stance regarding their status as logical constants. Our discussions, I strongly suspect, will only confirm how murky our concept of logical constants is, and at the same time will provide us with a new angle on existing literature on the topic.

The first section presents the background story for logical constants. Not being a survey of different views on the issue, it is the setting which helps us to understand the origin of the topic and, hence, to realize what issues are at stake in the logical constant debate. I will reveal and highlight certain assumptions one needs to accept when he/she gets into the discourse of logical constants. In the second section, starting with the most uncontroversial view that sentential connectives are logical constants, we will examine their properties as logical constants. These properties are cited and applied in the third section when we take on quantifiers. Are we fixing the meaning of quantifiers as we are for connectives? That story is not that simple. When the story is unfolded, we will see the complexity and the myth involved in our logical constants talk, quantifier talk, and beyond. By pointing out the difficulty in categorizing quantifiers either as logical constants or as non-logical constants, I would like to show how fragile and unclear our grasp of logical constants is. Nonetheless as seen in our preliminary section, when one subscribes to the most prominent logical theory, that is, the Tarskian view, he/she is not in a position to abandon the logical constant project. Our quantifier talk might, I suspect and hope, provide an occasion to re-think the Tarskian project.

4.1 Preliminaries

Logic is the study of valid reasoning, and a logical theory, exploring theoretical issues surrounding validity, presents its own coherent picture for important questions like the following: What makes one argument valid and another not? Does it have something to do with the meanings of the words in an argument, but nothing else? Do we need to conquer the meaning of meanings in order to conquer validity? (If so, would it be a hopeless project from the beginning?) What is the relation among logical, necessary, and *a priori* truth? No relation, whatsoever? Any one of these questions is controversial enough to consume philosophers' lives generation after generation. For better or for worse, we agree that logical constants have been quite often at the center of these hotly debated topics.

Up-front I would like to say that controversies surrounding logical constants, some believe, are a symptom of a wrong direction taken by the entire logical validity/logical truth project.[2] Then, the next step is to ask the following question:

(Q) Can we talk about validity (and logical truth) without talking about logical constants?

Answers differ. Let me start our preliminary discussions with question (Q) in order to show how logical constants have been a major topic in some logical theories while logical constants do not get in the picture at all for some other theories. A contrast in ways to explain the validity/non-validity of arguments (1) and (2) nicely illustrates why some say "no" and some say "yes" to question (Q):

(1) Every man is mortal.
Socrates is a man.

Socrates is mortal.

(2) Every man is mortal.
Socrates is mortal.

Socrates is a man.

A desideratum: We need to convince the reader that in the case of (1), it is impossible that the premises are true while the conclusion is false, and in the case of (2), it is possible to have true premises and a false conclusion. In spite of the common goal, how to think about impossibility/possibility cannot be more different in answer (A) versus answer (B):

Answer (A):
Imagine a world where every man is mortal and Socrates is a man. Then, in that world, it must be the case that Socrates is mortal. Hence, (1) is valid. On the other hand, I can imagine a world where every man is mortal and Socrates is mortal, but Socrates is a dog. Hence, (2) is not valid.

Answer (B):
Validity mainly has something to do with meanings, but not with how the world looks. We do not have to know the meaning of every word in a given argument, but only certain kinds of words. In the case of (1), 'man' could mean dog or cat, 'Socrates' could refer to Plato, etc. As long as 'every' has the meaning as we know, the truth of the premises guarantees the truth of the conclusion. On the other hand, in argument (2), when we interpret 'man' as cat, our conclusion is false while the premises are true. Hence, (2) is not valid.

There are many ways to point out differences between (A) and (B), and let's focus on the nature of a counterexample each side provides to assure that (2) is not valid. Approach (A) presents a possible *world* as a counter example, say w, such that the premises are true and the conclusion is false in w. On the other hand, (B) introduces a possible *interpretation* as a counterexample, say I, according to which the conclusion turns out to be false and the premises are true in the actual world.

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[2] Etchemendy [2, Ch. 9].

That is, (A) fixes the interpretations/meanings of words and varies over worlds and (B) does the opposite. Etchemendy identifies the crucial differences between these two approaches and names (A) *representational* and (B) *interpretational*.[3] It is not my intention to evaluate one over another, but to draw out the essence of each position.

The representational approach is quite intuitive since the modal nature of the impossibility/possibility (of the true premises and a false conclusion) for arguments (1) and (2) is directly transferred to an impossible/possible world scenario. We can easily see this method ends up equating logical truth with necessary truth. Some might welcome this result and some might not. Another big obstacle for the representational approach is how to systematize it. We have modal intuitions, and for simple arguments like (1) and (2) the intuition does the job. However, if logical validity and logical truth are built on our modal intuition, we would feel somewhat backwards or question-begging. Whether a sentence is logically true seems to be independent of our world-views. If a (formal) theory is given to us, we might be able to test out whether the theory fits (some of) our modal intuitions, but as time-honored related philosophical controversies show us, we cannot come up with a formal system which utilizes modal intuition only.[4] First of all, intuition differs from one person to another. Second, searching for all possible worlds is next to impossible! Third, the judgment of possible/impossible worlds assumes our understanding of the meanings of a sentence or words. Here we run into another cart-before-horse scene: A logical theory which incorporates semantics as an important component is supposed to explain the mechanism of the birth of the meaning of a complex unit out of the meanings of simpler units, and a representational theory assumes this miraculous process happens (which is true) and tells us whether there is a possible world where these sentences are true. The more we think about it the more likely approach (A) belongs to metaphysics rather than to logic. Recalling that our project is to explain logical validity/non-validity, we start seeing the shortcomings of approach (A).

Sure enough, in answer (B) we do not find (heavy) modal terminology. The stake seems to be meaning or interpretation. While answer (A) demands us to imagine all possible worlds, answer (B) asks us to consider all possible interpretations. Everything seems to be taken care of in terms of interpretations/meanings. Hence, the success of this method would mean a successful reduction of modality to non-modality. Any misgiving we raised above against (A) would disappear here. That is exactly what Tarski's celebrated analysis of logical consequence aimed for. Many have believed that Tarski's work fits the bill and that model theory, which grew out of Tarski's analysis, is the end of the story about validity and logical truth.

There has been an exception to this consensus, though. At the end of the twentieth century, Etchemendy presented a book-length argument to show that Tarski's project fails in modal-reduction (conceptually) and could be saved extensionally only

[3] Etchemendy [2, Chs. 2 & 4].

[4] This is one of the reasons why we have various modal systems.

thanks to other extra assumptions and the weakness of a first-order language. Not surprisingly, there has been a strong skepticism toward Etchemendy's criticism against Tarski's project. Again, I am not entertaining this on-going debate in this paper, but would like to get to the heart of Tarski's project since that will lead us to the main topic of the section – the birth of logical constants.

How does Tarski's idea propose to analyze away modality? He does not provoke any possible world talk here, but only different assignments for certain terms in given sentences:

> [W]e have the concept of the *satisfaction of a sentential function* by single objects or by a sequence of objects. ... The intuitive meaning of such phrases as: *John and Peter satisfy the condition 'X and Y are brothers', or the triple numbers, 2, 3, and 5 satisfies the equation x+y=z",* can give rise to no doubts. ... One of the concepts which can be defined in terms of the concept of satisfaction is the concept of *model*. ... Let L be any class of sentences. We replace all extra-logical constants which occur in the sentences belonging to L by corresponding variables, ... [W]e obtain class L' of sentential functions. An arbitrary sequence of objects which satisfies every sentential function of the class L' will be called a *model* or *realization of the class K of sentences.* ... In terms of these concepts we can define the concept of logical consequence as follows:
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> *The sentence X follows* logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.* [5]

For a given sentence, say α, we replace "extra-logical constants" with variables and make its sentential function α'. We call a sequence of objects which satisfies α' a model of α. Instead of 'every possible world' in the representational approach, Tarski presents 'every model', which is every possible interpretation. To put it simply, Tarski's logical consequence test is performed by allowing various (re)interpretations of words in an argument.

However, clearly, not all of the words should get re-interpreted to get the right result. If so, no argument would be valid. In the case of argument (1) above, if 'every' is re-interpreted as 'some' and 'mortal' as 'rich,' we find a counter example with interpretations which make the two premises true but the conclusion false. We want to say that this is not a genuine counterexample, by blocking the re-interpretation of the word 'every.' On the other hand, if we take every word to be a logical constant, argument (2) would be valid, which is not correct. Hence, approach (B) could be successful *only if* we would realize that there are certain (not all) words that we are not allowed to reinterpret. These are logical constants. The existence of logical constants is a necessary component of Tarski's analysis of logical consequence.

It is important to note that logical constants are an issue only in the context of an interpretational, not of a representational, theory. In the case of the representational account the meanings of words are fixed, and we change worlds. Every word should have its constant meaning. In the case of the interpretational account, we need to rule out two extreme scenarios: Either all of the word meanings are fixed or none of them is fixed. If we allow none of the words in an argument to change its meaning,

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[5] Tarski [10, pp. 416–17].

then many non-valid arguments would be judged to be valid (like argument (2) above). If we allow all of them to get re-interpreted, then no argument would be valid (except when a conclusion is the same as a premise). Here is a rationale for a selection of logical constants.

What is so special about ‘every’ as opposed to ‘man,’ ‘mortal,’ and ‘Socrates’ in the above arguments? Could we come up with necessary and sufficient features of logical constants? Or, is it a relative or a pragmatic decision to decide which vocabulary items are logical constants?[6] Without attempting to tackle any of these questions, I would like to start with the most simple and uncontroversial collection of logical constants, that is, sentential connectives.

For the rest of the paper, I assume the following: (i) Approach (A) and approach (B) are fundamentally different, and so we need to choose one or the other (possibly neither), but not both. (ii) Only in the case of the interpretational approach, logical constants have become an important issue. Hence, our discussions will focus on Tarskian-style interpretational analysis. It is time to examine closely how meaning-(non)assignment takes place in interpretational semantics.

4.2 Semantics of Connectives

Let’s take up the simplest logic – sentential logic – to see how sentential connectives are categorized as logical constants. There are three kinds of basic syntactic entities in sentential languages – sentential symbols, sentential connectives, and parentheses. After formal syntax tells us which strings are grammatically acceptable, formal semantics assigns meanings to those grammatical strings, i.e. sentences. Examining how meanings are assigned to various kinds of vocabulary might shed light on why we think connectives are logical constants.

The goal of formal semantics is to define the logical consequence relation in a rigorous way, so we need to come up with a systematic algorithm to assign T or F to any given sentence. To rule out meaning-ambiguity, a mathematical tool – the function – is adopted so that one and only one meaning is attached to a given sentence. It is important to note that there are two kinds of semantic functions at work in the case of sentential logic. One is a semantic function, say v, which assigns meaning, that is, T or F, to each sentence symbol, and the other is a semantic function, say C, for each sentential connective which defines the meaning of a connective. First of all, the two functions, v and C, are different in their domains;

[6]Tarski himself was quite agnostic about the issue in his paper: “Perhaps it will be possible to find important objective arguments which will enable us to justify the traditional boundary between logical and extra-logical expressions. But I also consider it to be quite possible that investigations will bring no positive results in this direction, so that we shall be compelled to regard such concepts as ‘logical consequence’, ‘analytical statement’, and ‘tautology’ as relative concepts which must, on each occasion, be related to a definite, although in greater or less degree arbitrary, division of terms into logical and extra-logical.” (Tarski [10, p. 420])

one is a set of sentential symbols and the other a set of n-tuples of truth values. More importantly, I would like to draw attention to a hierarchical difference between v and C: (i) Function v assigns T or F to each sentence symbol. (ii) Given function v, function C is defined in terms of semantic values assigned by the semantic function v. Hence, let me call semantic function v a first-level function and semantic function C a meta-function over function v.

Another major difference between v and C is illustrated by the following simple example: Do we know the meaning of a sentence symbol, say A_1? No, until semantic function v is given to us. The meaning could be either T or F. How about sentence $(A_1 \,\&\, A_1)$? No again, until semantic function v is given to us. An interesting point is that all we need to know for the meanings of both A_1 and $(A_1 \,\&\, A_1)$ is function v, which I call first-level, precisely because we know the semantic function for '&', say $C_{\&}$. That is, the meanings of sentential connectives are assumed to be constant throughout a sentential logical system. This is the essence of our practice where we take connectives to be logical constants. As we will see below, the meaning-constancy of connectives is implemented both in the semantic function for sentences and other logical concepts, i.e. logical consequence, logical truth, and logical equivalence.

With varying semantic functions v and fixed semantic functions C,[7] a truth-table provides a decisive algorithm for the meaning of any arbitrary sentence, say α. Suppose n sentence-type symbols occur in α. Then, 2^n many possible v-functions are enumerated, and for each row the computation of the meaning of α is mechanically performed without any additional piece of information when (or since) semantic functions for connectives in α are assumed. A truth-table visually exhibits two important aspects which distinguish two kinds of semantic functions. (i) Functions v and C are of different levels: Given a row of truth-values, that is, a v-function, C-functions are applied to the values of the given v-function. (ii) While v-functions are different from row to row, C-functions are constant throughout the entire truth-table. The semantic function for sentences being recursively defined as an extension of function v, we may compute, given v, the meaning of any arbitrary sentence. That is, having v-functions as basic functions for the recursion, an extension of v-functions, i.e. $\bar{v}$, is defined in terms of C-functions.

Let's see how v-functions and C-functions feature in the following important logical concepts:

Sentence α is a *logical consequence* of a set of sentences Γ if and only if every v-function that makes every member of Γ true also makes α true.
α is a *logical truth* if and only if every v-function makes α true.
α is *logically equivalent* to β if and only if every v-function that makes α true also makes β true, and vice versa.

Throughout the three definitions, we may make the following quite intriguing observations about the two kinds of semantic functions we have been talking about:

[7] I do not want to say 'constant semantic functions' to avoid a confusion with constant functions in general use, e.g. $g(\mathbf{x}) = n$.

(i) The three logical concepts are defined in terms of certain conditions holding for every v-function.
(ii) C-functions are not mentioned in any definition.

What ultimately matters for these important logical concepts? What decides the extension of these concepts? Item (i) tells us no specific v-function matters at all. Item (ii) is even more puzzling: Does it tell us C-functions do not matter?

On the contrary, C-functions are assumed in every definition since the truth of a sentence is defined recursively in terms of C-functions as we said before. It would be awkward to say 'some C-functions' or 'every C-function.' There is one and only one semantic function for each connective while there are uncountably many v-functions. The examination of the above definitions tells us that for these logical concepts what matters is C-functions, not v-functions. In this sense, sentential logic is a logic of connectives.

So far we have explored the features of sentential connectives to account for the rationale for calling them logical constants. Let us step back from our common practice and raise the following questions: How constant are they? What justifies their constant meanings? Is it possible to change C-functions? Yes. Then we are doing a non-classical logic. There are many kinds of non-classical logic, and changing C-functions forms the basis of one of them.[8]

The meaning of the conditional symbol has been a source of dissatisfaction with the way classical logic handles sentential connectives. It is the beginning of C. I. Lewis' strict implication proposal as opposed to material implication, which opened the door for modal concepts into a logical system. Realizing Lewis' strict implication cannot avoid a similar paradox with conditionals, some logicians decided to implement relevance between propositions P and Q when we talk about the implication from P to Q. Hence, the meaning of the conditional symbol is modified: Relevance logic has its own C-functions (which are different from classical C-functions) and they remain constant throughout their system. Note that one and the same sentence[9] would get different truth values between classical and relevance logic. That is, the meanings of logical constants are not constant from logic to logic. Therefore, what we mean by logical 'constants' is logic-relative: Given a logic, C-functions remain the same.

On the other hand, even though C-functions are logic-relative, we do not rewrite the definitions of logical concepts from logic to logic. Note that the extensions of these concepts change, though. That is, in one logic, sentence α is a logical consequence of Γ while not in another logic. That does not mean we are changing the concept of logical consequence or logical truth *per se*. The reason why these concepts remain the same while their extensions change (thanks to changes in C-functions) is that the role of logical constants in the logical concepts has not changed from logic to logic. Many of us have believed that there has been a strong consensus

[8]In the case of many-valued logic, changing the range of v-functions takes place.

[9]I am talking about a sentence as a syntactic object, not a proposition.

about the robust role of logical constants in doing logic. This is one of the reasons why the topic of logical constants has enjoyed its special and unique status in the philosophy of logic. When we move to quantified logic, however, things become tricky and unstable. Is it because of quantifiers? Or, is it because we do not have as clear a concept of logical constants as we believe we have? We will revisit these questions at the end.

4.3 Semantics of Quantifiers

4.3.1 Domain of Discourse

Quantifiers have been classified as logical constants, with some exceptions.[10] In the previous section, we located a distinction between logical constants and other syntactic units in how meanings are assigned to them. So, let's examine how semantic entities are attached to first-order logic vocabulary and check whether the main features we identified for connectives as logical constants also hold for quantifiers.

In addition to parentheses and sentential connectives, first-order languages have the following list of vocabulary: quantifier(s), predicates, names (optional), variables, and function symbols (optional). Unlike sentential languages, these syntactic units require heterogeneous kinds of semantic entities. An object is assigned for a name, a set of n-tuples of objects for an n-place predicate, and an n-ary operation over an n-tuple of objects for an n-ary function symbol. Informally, a sentence Pc_1c_2 is true if and only if the ordered pair of an object denoted by c_1 and an object denoted by c_2 is a member of the set of ordered pairs denoted by predicate P. Let me call these heterogeneous semantic assignments "p-functions."[11]

How about quantifier $\forall$? A non-empty set of objects is assigned to it. Let me call it a "Q-function." One way to address the main question of the paper (i.e. whether quantifiers are logical constants) is to raise the following question: Are there fundamental differences between p-functions and a Q-function? In the previous section, we identified the following four semantic factors which distinguish the logical constants (i.e. sentential connectives) and the other units (i.e. sentential symbols). First, the semantic functions for logical constants (C-functions in the above) are meta-operations over the basic semantic functions (v-functions) which assign meanings for basic vocabulary (sentence symbols in the case of sentential logic). The second aspect, which has been more the focus in the literature, is that C-

[10]Enderton [1, pp. 69–70] puts quantifier symbol $\forall$ into the parameter group, along with predicate symbols, constant symbols, and function symbols. Etchemendy, who is against the logical constants debate itself, experiments both with fixed and various meanings for $\forall$. (Etchemendy [2, Chs. 5 & 8].)

[11]I call them "p-functions," meaning semantic functions for parameters, as opposed to for logical constants.

functions are fixed while v-functions are not. Third, the meaning-constancy of C-functions plays a crucial role for logical concepts (like logical consequence, logical truth, and logical equivalence): the extension of these concepts is determined by the C-functions. Fourth, when we change C-functions, we move from one logic to another.

Let's compare p-functions with a Q-function in terms of these four aspects. First, is there a hierarchical difference? Is a Q-function a meta-operation over p-functions? Obviously not. Nonetheless, there seems to be a level-difference between a Q-function versus p-functions: A Q-function delineates the boundary of the other kinds of semantic entities. That is, the ranges of p-functions are determined by the Q-function: Names are interpreted as objects within the set which is the value of the Q-function. Similar things are going on with the interpretations of predicates and function symbols. Let's recall that the interpretation of $\forall$ is the domain of discourse. Even though there is no meta-structure between the Q-function and p-functions (as we found between C-functions and v-functions in the case of sentential languages), we can see what Q-function assigns serves as the background frame within which p-functions are situated. One may say that the Q-function is more fundamental than the rest of the semantic functions.

How about the second aspect of logical constants, that is, their meanings being fixed? Is the Q-function fixed as C-functions are for connectives? The truth of a sentence is determined by the interpretations of its components, and semantic functions are doing the job. In order to know the truth-value of sentence $\forall xRxx$, we need to know the interpretations of both the quantifier $\forall$ and the predicate R. Is the interpretation of quantifier $\forall$ fixed throughout a given logic? The Q-function assigns a set of objects, say set A, to $\forall$, and p-function for R assigns a set of paired orders of objects; hence the interpretation of R is a subset of the set $A \times A$. The only constraint for classical logic is that set A should be non-empty, but by no means we are talking about a fixed set; hence, there is no fixed meaning. Recall that in the case of sentential logic as long as we get the truth-values of α and β, we may compute the truth-value of sentence $(\alpha \ \& \ \beta)$ since the C-function for the connective & is fixed throughout a given logical system. That is not the case with the sentence $\forall xRxx$. As we noted in the previous section, the fixed meanings of connectives seem to support our common practice and belief that they are logical constants. On the other hand, the interpretation of $\forall$ is changing from one set to another (as long as it is not the empty set) since the domain of discourse is changing from discourse to discourse. I suspect this is the main reason why Enderton classifies $\forall$ as a parameter as opposed to a logical symbol.[12] Then, how can we justify the claim that quantifiers are logical constants? Why do many of us consider quantifiers to be logical constants? Keeping this question in mind, let's move to the third feature of logical constants on our list.

[12]Our logical constants are subsets of Enderton's logical symbols. Logical symbols are either those whose interpretations are fixed (i.e. connectives and the identity symbol) or those which are not interpreted (i.e. parentheses, commas, and variables); Enderton [1, pp. 69–70].

The next item on our checklist is the role logical constants play in logical consequence, logical truth, and logical equivalence. In the case of sentential logic, no semantic function other than C-functions matters in deciding on the extensions of these logical concepts. Given C-functions, we may decide whether α is logical consequence of Γ, whether α is logically true, or α and β are logically equivalent. No specific v-function is needed, since those logical concepts demand certain conditions hold for 'every' v-function. Does the Q-function in first-order logic play a similar role as C-functions do in the case of sentential logic? If so, we could define the logical concepts we examined before in the following way:

Sentence α is a *logical consequence* of a set of sentences Γ if and only if
every p-function that makes every member of Γ true also makes α true.
α is a *logical truth* if and only if every p-function makes α true.
α is *logically equivalent* to β if and only if
every p-function that makes α true also makes β true, and vice versa.

An easy counter example: Suppose a Q-function interprets $\forall$ as set {Tom, Mary}. Then, for a unary predicate R, there are four possible p-functions. They are:

$p_1(R) = \emptyset$.
$p_2(R) = \{$ Tom $\}$.
$p_3(R) = \{$ Mary $\}$.
$p_4(R) = \{$ Tom, Mary $\}$.

Then, the following statements would be true according to the above (pseudo) definitions:

- $\forall xRx$ is a logical consequence of $\sim \forall x\ \forall y((Rx\&Ry) \rightarrow x = y)$.
- $\forall x \forall y((Rx\&Ry) \rightarrow x = y) \ \vee \ \forall xRx$ is logically true.
- $\forall xRx$ is logically equivalent to $\sim\forall x\forall y((Rx\&Ry) \rightarrow x = y)$.

The main reason why we are getting wrong results is that throughout the definitions we fixed the interpretation of a quantifier as the domain which has only two members. Obviously, we need to revise the definitions so that certain conditions hold for every Q-function as well as every p-function.[13]

Sentence α is a *logical consequence* of a set of sentences Γ if and only if
every Q-function and every p-function that makes every member of Γ
true also makes α true.
α is a *logical truth* if and only if every Q-function and every p-function make α true.
α is *logically equivalent* to β if and only if
every Q-function and every p-function that makes α true also makes β
true, and vice versa.

In these definitions, there would be no reason to make a distinction between Q-functions and p-functions. According to classical model theory, a model is defined as a set of both Q-functions and p-functions. Hence,

[13]Please note that in the revised definitions, $\forall xRx$ is not a logical consequence of $\sim \forall x\ \forall y((Rx\&Ry) \rightarrow x = y))$, $\forall x\forall y((Rx\&Ry) \rightarrow x = y) \vee \forall xRx$ is not logically true, and $\forall xRx$ is not logically equivalent to $\sim\forall x\forall y((Rx\&Ry) \rightarrow x = y)$.

Sentence α is a *logical consequence* of a set of sentences Γ if and only if every model that makes every member of Γ true also makes α true.
α is a *logical truth* if and only if every model makes α true.
α is *logically equivalent* to β if and only if every model that makes α true also makes β true, and vice versa.

We need a pause here: According to the above definitions, there seems to be no distinction between the interpretation of quantifiers and the interpretations of other non-logical symbols and neither of them should be fixed. None of them plays the same role as C-functions do in the case of sentential logic. Our question revisited: Why have we considered quantifiers to be logical constants?

Our last item is to check whether a change of Q-functions correlates with a change of logic. We have seen that relevance logic departs from classical logic by changing some C-functions.[14] As we have seen, Q-functions are not as fixed as C-functions are. On the other hand, there is one aspect of Q-functions related to this issue: classical logic gives a constraint for Q-functions, that is, the interpretation of $\forall$ cannot be the empty set. When we allow the empty set as an interpretation of $\forall$, we are engaging in a non-classical logic called free logic. Here there seems to be a similarity between C-functions and Q-functions. We will take up this aspect later in the paper.

The interpretations of connectives mark a clear difference from the interpretations of other symbols both in their level and constancy. Connectives are interpreted over the semantic values of components of a sentence. The interpretations of connectives are so constant that a change in their meanings amounts to a change of logic. Things are not as clear-cut in the case of quantifiers as they are for connectives. The interpretation of quantifiers sets up a boundary for the interpretations of other syntactic parameters. In that sense, Q-functions and p-functions seem to have a slight difference in their semantic categories, but not a first-level versus meta-level hierarchical difference. Moreover, when it comes down to the meaning-constancy issue, Q-functions do not behave at all as C-functions do. A domain (which is the interpretation of $\forall$) is not fixed, and hence we cannot tell whether a random sentence is true even when other symbols are interpreted. Here is a contrast between $(A_1 \& A_2)$ versus $\forall xRxx$.[15] Logical consequence, logical truth, and logical equivalence – all of them demand the domain not be fixed! There is one bottom line, though: The domain cannot be empty in classical logic. Hence, here is a tiny hint of constancy. When we include the empty domain, we are shifting from one logic to another.

[14]It might be slightly wrong to say that changing the meaning of the conditional symbol is the only motivation for relevance logic, but this is related to different views of logical consequence.

[15]If we know the truth values of A_1 and A_2, we may compute the truth value of $(A_1 \& A_2)$. On the other hand, even if we know the interpretation of the predicate R, we cannot tell whether the sentence $\forall xRxx$ is true unless we know the domain, i.e. the interpretation of the universal quantifier.

4.3.2 Looking for Constancy

So far our discussions suggest quantifiers do not behave like connectives in terms of semantic assignments. If the universal quantifier denotes different sets from one model to another (which is the case), we cannot say its meaning is fixed. Then, why has the universal quantifier been considered as a logical constant? *In what sense* do many of us believe that the meaning of the universal quantifier is fixed? I would like to go through three different levels of responses, two in this subsection and the third one in the next subsection.

Let me start with a very simple and naive intuition behind our common belief that quantifiers are logical constants and call it the 'natural language argument.'

Every man is mortal.
Socrates is a man.
―――――――――――――
Socrates is mortal.

To check the validity of the argument, we could change the meanings or references of 'mortal,' 'man,' or 'Socrates,' but not 'every.' If we changed the meaning of 'every' to the meaning of 'some,' this argument would not be valid. The sentence "Grass is green or grass is not green" is logically true as long as we fix the meanings of 'or' and 'not.' At this intuitive level, we can easily see why connectives and quantifiers have been grouped together as logical constants.

Let's examine how the intuition behind the natural language argument is transferred into a first-order language so that $\forall$ may be justified as a logical constant. However, our examinations of the formal semantics of connectives and quantifiers concluded that there is a big difference between these two kinds of vocabulary in how constant their assigned meanings are. For example, we do not know the truth of the sentence "Everybody is a sophomore" until we know the domain of the discourse, even when we know who are sophomores. On the other hand, we know the truth of the sentence "Grass is green and snow is white" as long as we know the truth-values of "grass is green" and "snow is white." Using the terminology of (classical) first-order semantics, the interpretation of the universal quantifier, which is a non-empty set, is not constant. So, when the domain has only single object, the sentence $\forall x \forall y\ x = y$ is true, and otherwise it is false. Then, in the above simple argument in what sense do we assume the meaning of 'every' is fixed, and hence, a logical constant? Obviously, if we change 'every' to 'some' or 'most,' the argument is not valid anymore. That does not mean that the extension of 'every' is fixed, but *something about* 'every' is fixed.

Where does that constancy appear in first-order semantics? Let's carefully compare the semantic clauses of '&' and '$\forall$':

$(\alpha\ \ \&\ \beta)$ is satisfied by truth function v iff
  α is satisfied by v <u>and</u> β is satisfied by v.
$\forall x \phi(x)$ is satisfied by a model iff
  for <u>every</u> object in the domain, say a, $\phi(x/\mathrm{a})$ is satisfied by the model.

As the natural language argument says, if we replace ‘every’ with ‘some’ in the above semantic clause, we do not get the right result. Let me call it the ‘semantic clause argument.’ The semantic clause for $\forall$ also seems to be very similar to the semantic clause for &. First of all, those clauses are not changing from one model to another; that is, things are fixed. The way $\forall$ is interpreted as ‘every’ is analogous to the way the semantic clauses are set up for connectives, e.g. $\sim$, &, $\vee$, etc. Like connectives, the universal quantifier is interpreted in a meta-language, and the meanings of their interpretations, say “it is not the case,” “and,” “or,” and “every,” are understood at a meta-level as if they are fixed.[16]

However, the similarity is so striking and obvious that we have overlooked the following interesting difference between $\forall$ and &, I claim. Let me make a contrast between a connective and a quantifier by rewriting the above semantic clauses in the following way[17]:

(α & β) is satisfied by a model, say $\mathfrak{I}$, if and only if
  α is satisfied by $\mathfrak{I}$ <u>and</u> β is satisfied by $\mathfrak{I}$.
$\forall x \phi(x)$ is satisfied by a model, say $\mathfrak{I}$, if and only if
  for <u>every</u> object in $\underline{\forall^{\mathfrak{I}}}$, say a, $\phi(x/\mathrm{a})$ is satisfied by $\mathfrak{I}$.

In the case of ‘&,’ its interpretation/meaning “and” is given in a meta-language <u>once</u>. On the other hand, in the case of ‘$\forall$,’ interpretations take place <u>twice</u>: Once with ‘every’ (like ‘and’ for &) and also with $\forall^{\mathfrak{I}}$ as the domain of discourse. While the former meaning does not change from model to model, just like ‘and’, the latter interpretation is changing from model to model. Notation $\forall^{\mathfrak{I}}$ tells us its variance. Hence, the former aspect accounts for the view that $\forall$ is a logical constant, while the latter explicitly shows us that $\forall$ is non-logical vocabulary, i.e. a parameter, as Enderton classifies it.

Before delving into the matter further, I would like to address one immediate concern. Is it legitimate to bring in two interpretations for one and the same piece of vocabulary? Wouldn’t it have caused an ambiguity? Interestingly, it does not cause an ambiguity, which turns out to be a mixed-blessing for our understanding of quantifiers. The two interpretations of $\forall$ come at two different levels – one as a set (varying from model to model at the first-level) and the other as ‘every’ (fixed at a meta-level). Hence, an ambiguity (as we usually worry about) does not take place. This might be one of the main reasons why no attention has been paid to the fact that the universal quantifier is interpreted *in two different ways* in its semantic clause.

[16]The flip side of the same story can be told in terms of inference rules. For each piece of vocabulary, we have inference rules and some think that the inference rules endow these logical vocabulary with its own meaning. If we take this proof-theoretic view, the point I am making here could be expressed in the following way: Both $\forall$ and other connectives are given their meanings when their own inference rules are introduced. (The relation between meanings and inference rules has been debated for decades – Do inference rules endow logical constants with meanings? Or, are inference rules justified in terms of the meanings which are given by semantic clauses of the logical constants? I have no intention to say anything about this directly, but let me emphasize that the point I am making here is consistent with both views.)

[17]I replace truth-function with model here.

Even though the two-tiered interpretation of ∀ has not caused a plain ambiguity, I suspect that the two levels of the ∀-semantics are a main source of the confusion and controversies we encounter in the literature on quantifiers, especially in the context of logical constants.

4.3.3 Dancing on Two Levels

Believing that quantifiers are logical constants as connectives are, many have attempted to find the common features between these two categories of vocabulary.[18] At the same time, realizing that quantifiers' constancy is different from connectives' constancy and requires more involved explanations, much literature has focused on quantifiers in the discussions of logical constants. Gila Sher's work on quantifiers is one of the most recent efforts in that direction. Sher, along other philosophers,[19] has defended Tarski's interpretational analysis of logical consequence. As we concluded in the first section, Tarski's logical consequence would not even get the right extension unless we get the right collection of logical constants. Unlike with connectives, quantifiers' interpretations do not easily convince us that they are constant. Being aware of the importance of and the difficulties involved with quantifiers as logical constants, Sher brings in more complicated and more sophisticated machinery than just appealing to 'every' in the semantic clause for a universal quantified formula cited above. Let me call this machinery the 'domain-size argument.'

Sharing the intuition that logical constants should be specific content-free, Sher wants to show they are *formal*. She defines the formality of logical constants in the following sprit of Mostowski: "A logical quantifier does not allow us to distinguish between different elements of [the universe]."[20] The next step is to extend this spirit to different domains of discourse but isomorphic to one another, and "being formal is being invariant under isomorphic structures."[21] Then, Sher draws our attention to the Homomorphism Theorem: If two models are isomorphic to each other, no first-order sentence can differentiate them. From that result, Sher concludes that ∀ is a logical constant. Since universal-quantified sentences turn out to be of the same truth-value in isomorphic models, the universal quantifier is not tied up with the content of a domain, but only with a formal character, mainly size, she argues.[22] Is this criterion convincing? As I explain below, Sher's contribution to the topic could

[18] Peacocke's [7] well-cited criterion is one prime example for this direction of effort.

[19] Refer to Gomez-Torrente [3] and Shapiro [8].

[20] Sher [9, p. 14], and refer to Mostowski [6] for more technical details.

[21] Sher [9, p. 53].

[22] MacFarlane [5] pushes the result further and makes things clearer: "Indeed, because cardinality is permutation-invariant, cardinality, every cardinality quantifier is included, . . . " (Section 5).

be better understood in the context of the two-tiered interpretations of $\forall$ I have identified and argued for.

There is a gap between the description of how quantified sentences are behaving among isomorphic models and the claim that quantifiers are logical constants. Leaving the logical constant agenda aside, the most immediate and innocent question we could raise about the Homomorphism Theorem is: Why do quantified sentences get the same semantic values in isomorphic models, even though their domains are different? It is not because quantifiers denote the same thing (which is not the case) or are constant (which is, at best, question-begging), but because our first-order languages are not expressive enough to reflect different contents of the domains. If so, the true import of the Homomorphism Theorem is the weakness of expressiveness of a first-order language. That is, if two models are isomorphic, there is no first-order sentence which distinguishes one from the other.

Things can get even shakier in light of the compactness theorem of first-order logic. Think about a non-standard model of arithmetic, say $\mathcal{M}'$ (where there is an object which is greater than any natural number), which is not isomorphic to a standard model $\mathcal{M}$. The Compactness Theorem takes us to the following proposition: Even though $\mathcal{M}$ and $\mathcal{M}'$ are not isomorphic to each other, for every first-order sentence, say α, α is true in $\mathcal{M}$ if and only if it is true in $\mathcal{M}'$. This result does not set us off to explore what that mysterious object is, but to convince us that our first-order arithmetic language is not expressive enough to pin down the world of numbers you and I normally conceive. So, being isomorphic (hence, each being of the same size in its domain) is a sufficient condition for the same semantic value of every quantified sentence, but not a necessary one. Then, we seem to be losing ground in tying up the constancy of a quantifier and the size of a domain. After all, it might be a test of the expressiveness of our language.

On the other hand, Sher's project could become more meaningful when the following issue is highlighted: Even when the domains of two models consist of different elements – hence, $\forall$ denotes different sets – universal quantified sentences turn out to be of the same semantic value as long as the models are isomorphic to each other. There must be *something* constant in the way $\forall$ plays a role in the semantics of quantified sentences. First of all, Sher realized that we need more or different explanations for $\forall$ as a logical constant than in the case of connectives, and mainly focused on quantifiers.[23] Let us recall that one level of interpretation of $\forall$ (say, the first-level interpretation) is a non-empty set. Given two different models,

[23]We should, therefore, not be surprised to run into the following comments on Sher's proposal: "Sher's proposal is somewhat awkward when understood as a general criterion for logical constancy since it has no straightforward application to sentential connectives." (Warmbrōd [11, p. 507]) Note her own definition of Tarskian logical constants: "*C* is a (Tarskian) *logical term* iff *C* is a truth-functional connective <u>or</u> *C* satisfies conditions of (A) to (E) above on logical constants [permutation-invariant especially among isomorphic structures]." (Sher [9, p. 56], and my underline.)

say $\mathfrak{I}$ and $\mathfrak{J}$, the interpretation of $\forall$ in $\mathfrak{I}$, say set I, and the interpretation of $\forall$ in $\mathfrak{J}$, say set J, are different from each other. Sher admits this varying aspect of the universal quantifier:

> Take the universal quantifier. In every model for a first-order logic the universal quantifier is interpreted as a singleton set (i.e., the set of the universe). But in a model with 10 elements it is a set of a set with 10 elements, whereas in a model with 9 elements it is a set of a set with 9 elements. Are these interpretations the same?
>
> I think that what distinguishes logical constants in Tarski's semantics is not the fact that their interpretation does not vary from model to model (it does!) but the fact that they are interpreted *outside* the system of models.[24]

Facing the fact that the universal quantifier's interpretation changes from model to model, Sher argues that the interpretation is done outside, that is, "given by *rules external to the system*."[25] I find this move somewhat puzzling, but at the same time encouraging. Why do we think the constancy of a term is guaranteed by the fact that its interpretation is given by external rules? Isn't every interpretation, not just the interpretations of logical constants, given outside the system? I would like to think what Sher meant by "outside the system of models" is the meta-level interpretation, not the first-level interpretation of the universal quantifier, according to my terminology. On the other hand, if we focus on the other level of the meaning of $\forall$ (say, the meta-level interpretation), that is, 'every,' then Sher's complicated argument would seem quite unnecessary. The meaning constancy of $\forall$ would have nothing to do with the truth-values of quantified sentences among isomorphic models, and the interpretation 'every' would apply to every model, regardless of the size of the model.

Sher's explanation of $\forall$ in terms of the Homomorphism Theorem is not well-connected with the rationale for connectives, and I agree with Warmbrōd that it is somewhat awkward and non-motivating to invoke a quite involved criterion for $\forall$ which is not needed for connectives. On the other hand, I am sympathetic with Sher's project in that it does not ignore a puzzle around $\forall$: If a domain is the interpretation of a quantifier, it cannot be constant, but $\forall$ seems to be closer to connectives than to predicates or function symbols in a logical theory.

Sher's project, I believe, is a good example to illustrate how philosophers have tried to grapple with the two different ways that $\forall$ is taken in its semantic clause. The first level's denotation is a set, and the second level's meta-meaning is 'every.' Sher seems to bridge these two realms and at the same time to find a formal feature of the meta-meaning 'every.' When two models are isomorphic – hence, each domain is of the same size – the difference in the elements of the two domains does not matter in terms of the semantic value of a quantified sentence. Then the meaning of a universal quantifier is more than just its denotation. Instead, there is something constant at a meta-level which guarantees the same semantic value among different models. Sher and some commentators identify this constant feature as the size of

[24] Sher [9, pp. 48–49].

[25] Sher [9, p. 49].

a model. However, the Löwenheim-Skolem Theorem shows us that different sized models could share the constant feature as well, that is, the same semantic value of a first-order sentence among non-isomorphic models. I do not think the same semantic value in different models is due to the role of logical constants, but rather it has something to do with the limit of the expressiveness of first-order logic. Therefore, Sher's proposal, an ambitious and earnest attempt to tackle one of the time-honored controversies as formally as possible, is short of convincing us why quantifiers are logical constants. In the serious defense of the Tarskian analysis, we witness a tension between the two-tiered approaches of the universal quantifier.

Let me visit another place where the tension has become apparent by presenting a fundamental difference between the constancy of connectives and the constancy of the universal quantifier. When we modify classical interpretations of logical constants, and thereby create a non-classical logic, we see a different pattern between connectives and quantifiers in terms of their semantics and inference rules.

In the case of connectives, we pointed out the relation between their assigned meanings and classical/non-classical logic. For example, when we change the meaning of a conditional symbol, a non-classical logic, relevance logic, is born. The semantic clause for $\rightarrow$ is different and a change in inference rules involving $\rightarrow$ occurs. Let us examine what happens if we change the meaning of a quantifier.

When we allow the empty domain as an interpretation of a universal quantifier, we are departing from classical logic to free logic or inclusive logic. According to our account of the two-tiered interpretations of $\forall$, the inclusion of the empty domain means a change in the first-level interpretation. The meta-level meaning 'every' does not seem to change. Note that "free logic is formal logic whose quantifiers are interpreted in the usual way."[26] Nonetheless we need new inference rules in this new inclusive logic.

It is important to make a contrast between relevance logic and free/inclusive logic in terms of what changes and what does not. In the case of relevance logic, a change in the meaning of the conditional symbol (which is expressed in the semantic clause of $(\alpha \rightarrow \beta)$) yields a change in inference rules.[27] Accordingly, we concluded that the meanings of logical constants are constant in a given logic, but could change from logic to logic. That is, their meanings are logic-relative. At the same time we noted that the role of connectives as logical constants is constant from logic to logic, and we therefore do not need to change the definitions of logical concepts – logical consequence, logical truth, and logical equivalence. In the case of free/inclusive logic, only one level (i.e. the first level) of change in terms of the universal quantifier takes place, and its inference rules change. Facing a comparison between connectives and quantifiers in terms of non-classical logics, we have choices to make: (i) We identify the meaning of the universal quantifier at the first level, that is, the domain of discourse. When we include the non-empty domain, a non-classical logic is born. (ii) We take the meta-level interpretation 'every' as

[26] Holt [4, §1.1].

[27] Some might say that a change in inference rules yields a change in meaning.

the meaning of the universal quantifier. Since free/inclusive logic interprets the universal quantifier as ‘every’ as well, this non-classical logic has nothing to do with a change in the meanings of logical constants.

I claim both views have their own dilemma or more stories to fill in. Position (i) admits that quantifiers are not logical constants since the first-level interpretation is not fixed at all due to different domains of discourse. Considering how much it is accepted that quantifiers are logical constants, I do not think this position would be welcomed by many of us. Position (ii) needs to tell us more about why inference rules involving quantifiers need to change when we do not change the meaning of quantifiers. Especially from a proof-theoretic point of view, it is an inference rule that assigns meanings to logical vocabulary. Then, in this case, the inference rules involving the universal quantifier change but its meaning stays the same. Interestingly enough, both positions end up concluding that free/inclusive logic is not a case where a change in logical constants is the origin of their departure from classical logic. If so, free/inclusive logic stands on a different footing from relevance or intuitionist logic even though both are non-classical logics.

4.4 Concluding Remarks

It is not my view that we should make up our mind about the logical constancy of quantifiers, but that the decision would be a necessary component when we accept the Tarskian standard model-theoretic approach of logical consequence. When we take the logical constant talk seriously (as many do), things become trickier when it comes to quantifiers. To our surprise, we could not find much in common between sentential connectives (which are taken to be logical constants without any controversy) and quantifiers. Then do we need a different kind of rationale for quantifiers? If so, what is it? Our natural language argument intuitively shows the meanings of quantifiers need to be fixed in order to get the right extension of logical consequence. When the formal semantic clause cashes in this intuition, we realize that two different kinds of meaning are assigned to the universal quantifier – a set of objects at the first level and ‘every’ at a meta-level. A more sophisticated argument for the logical constancy of quantifiers (i.e. the domain size argument) can be viewed as an effort to resolve a tension between these two different levels of the interpretation of quantifiers. The tension is spotted in the case of a non-classical logic in which the empty domain is included as the domain of discourse: The meta-meaning ‘every’ remains the same and so does the semantic clause of the universal quantified formulas, while the inference rules involving the universal quantifier change.

Suppose I am right that there has been a double way to handle quantifiers. Am I claiming that it is wrong to do so? No, the paper is not about the normative semantics for quantifiers, but aims to be descriptive. I do not see anything wrong with the semantic clause for quantified formulas. Nevertheless the ambiguous feature of quantifier semantics has been overlooked in literature, and I argue that this ambiguity has led to more controversies around quantifiers than around connectives in terms of logical constancy. Does this show that the logical theory (that is, standard model-theory) which assumes the right extension of logical constants is wrong? Not exactly, but the ambiguous nature of quantifiers should give us pause about the entire project whose success hinges on the successful account of logical constants. It would be interesting to inquire further into whether the double feature of quantifiers is reflected in the representational approach. In the case of representational semantics, the meanings are fixed and worlds are changing. It is easy to fix the meaning of 'snow' or 'white' (as we know) and to vary worlds where snow is white or snow is not white. How about 'every' or 'everything'? What does it mean to fix the meaning of 'every'? Does it mean that the domain is whatever we have in the actual world? I would leave the matter here for further work, and just say that the double nature of quantifiers has helped us to look into the nature of logical constants and to get to the bottom of opposing logical views in a fresh way.

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