The MPD Equations in Analytic Perturbative Form

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Abstract A longstanding approach to the dynamics of extended objects in curved space-time is given by the Mathisson-Papapetrou-Dixon (MPD) equations of motion for the so-called "pole-dipole approximation." This paper describes an analytic perturbation approach to the MPD equations via a power series expansion with respect to the particle's spin magnitude, in which the particle's kinematic and dynamical degrees of freedom are expressible in a completely general way to formally infinite order in the expansion parameter, and without any reference to pre-existent space-time symmetries in the background. An important consequence to emerge from the formalism is that the particle's squared mass and spin magnitudes can shift based on a classical analogue of "radiative corrections" due to spin-curvature coupling, whose implications are investigated. It is explicitly shown how to solve for the linear momentum and spin angular momentum for the spinning particle, up to second order in the expansion. As well, this paper outlines two distinct approaches to address the study of many-body dynamics of spinning particles in curved space-time. The first example is the spin modification of the Raychaudhuri equation for worldline congruences, while the second example is the computation of neighbouring worldlines with respect to an arbitrarily chosen reference worldline.

1 Introduction

It is an undeniable fact that, for all practical purposes, macroscopic astrophysical objects in the Universe possess classical spin angular momentum during their formation and long-term evolution. In addition, it has been known for a long time that relativistic astrophysical objects, such as black holes and neutron stars with spin, also theoretically exist within the theory of General Relativity, with strong observational evidence to effectively confirm their presence within the Universe. Further to this, it becomes very relevant to consider, for example, the physical consequences involving the spin interaction of relativistic systems, such as the orbital dynamics of rapidly

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spinning neutron stars around supermassive black holes expected to exist in the centres of most galaxies. Therefore, it is vital to properly understand the dynamics of extended bodies in curved space-time that incorporates classical spin.

The first attempt to introduce classical spin within the framework of General Relativity was presented by Mathisson [1], who demonstrated the existence of equations of motion involving an interaction due to the direct coupling of the Riemann curvature tensor with the moving particle's spin. Besides the so-called "pole-dipole approximation" for the case of a spinning point dipole as the leading order spin-dependent term alongside the mass monopole term, Mathisson's formalism includes a non-trivial set of higher-order multipole moment contributions that depend upon prior knowledge of the object's matter-energy tensor field. Several other contributions over the years also appear, most notably from Papapetrou [2] whose treatment of the problem was to describe the spinning object confined to within a space-time world tube that contains its centre-of-mass worldline, such that the associated matter field is given compact support. Other contributions, such as from Pirani [3], Tulczyjew [4], Madore [5], and several more have adopted competing perspectives on the treatment of higher-order multipole moments. Some years later, Dixon [6, 7] introduced what is arguably the most complete treatment of this problem by providing a fully self-consistent description of all multipole moment contributions to infinite order for the dynamics of extended bodies in curved space-time.

When considering the inspiral orbital motion of an equal-mass spinning binary neutron star system, for example, it is reasonable to demand an understanding of the higher-order multipole moment contributions when strong tidal disruptions are expected to occur to put stresses within the neutron stars' internal structure. However, for situations involving extreme mass-ratio systems, such as neutron stars or solar-mass black holes in orbit around supermassive black holes, it is justifiable to truncate the multipole moment expansion of the full equations of motion and focus exclusively on the pole-dipole approximation. Indeed, for most practical calculations involving an extreme mass ratio system, the essential physics of the dynamical motion is well-captured, provided that the spinning object's dimensions are sufficiently small compared to the background space-time's local radius of curvature in order for this justification to be well-motivated. Taking such an approach leads to what are commonly known as the Mathisson-Papapetrou-Dixon (MPD) equations of motion.

The application of the MPD equations for a variety of interesting astrophysical situations is evident within the literature. For obvious reasons, it is very natural to make use of the MPD equations to describe the dynamics of a spinning particle around spinning black holes, as described by the Kerr metric [8–12] to showcase the impact of spin-curvature interactions involving black hole spin on the spinning particle's long-term orbital motion. In so doing, the particle then becomes a sensitive probe of space-time curvature due to the MPD equations, from which it becomes possible to learn a great deal about the properties of the background source. From numerical simulations in particular, it is possible to explore the limits of stability for the MPD equations, such that deterministic chaos may result under extreme circumstances [12–15]. Furthermore, it is possible within this framework to derive predictions of gravitational wave generation [16, 17] expected to occur from spin-induced deviations away from geodesic motion.

Besides the numerically driven treatments of the MPD equations, there also exist more formal analytic studies within the literature [18–20]. The direct precursor to the analytic perturbative approach presented in this article was first developed by Chicone, Mashhoon, and Punsly (CMP) [21], with a focus upon the study of spinning clumps of plasma in the formation of astrophysical jets columnated along a Kerr black hole's axis of symmetry. For future reference, this approach to the MPD equations is hereafter known as the CMP approximation. This approach was then applied by Mashhoon and Singh [22] to identify analytic expressions for the leading-order perturbations for the circular orbit of a spinning particle around a Kerr black hole. Given the somewhat basic assumptions within the CMP approximation, the analysis is remarkably successful in reproducing the spinning particle's kinematic properties when compared with the full MPD equations. This is self-evident for situations where, for spin magnitude s and mass m, with radial distance r away from the background source, the spinning particle's Møller radius [22, 23] is confined to $s/m < 10^{-3}\,r$. This agreement, however, begins to break down when $s/(mr) \sim 10^{-2} - 10^{-1}$ for $r = 10\,M$, where M is the Kerr black hole mass. It suggests strongly that higher-order spin-curvature coupling terms are necessary to more completely account for the orbital motions when compared with the numerical treatment of the MPD equations.

On the basis of this observation, a full generalization of the CMP approximation was introduced by Singh [24] to account for the need to have higher-order analytic contributions within the perturbation treatment. As will be shown within this paper, several attractive features emerge with the additional terms incorporated. First, the formalism in terms of a power series expansion with respect to the particle's spin magnitude is extended to formally *infinite order*. Second, it is background independent and accommodates for *arbitrary motion* of the particle without any recourse to space-time symmetries within the metric. As such, it follows that this generalization is very robust and can be applied to a wide variety of astrophysical situations, examples of which are the modelling of globular clusters and other many-body dynamical systems in curved space-time. Explicit applications of this perturbation approach were presented by Singh to model the circular motion of a spinning test particle around a Kerr black hole [25] and that of a radially accreting or radiating Schwarzschild black hole, as described by the Vaidya metric [26]. Third, the formalism allows for the existence of a classical analogue for "radiative corrections" that shift the particle's overall squared mass and spin magnitude with respect to some "bare mass" and "bare spin," respectively. Such an observation under extreme conditions may well yield the ability to analytically determine the transition from stable to chaotic motion, which can be compared to existing studies [12–15] that involve numerical methods.

This paper presents the analytic perturbation treatment of the MPD equations based on the following outline. It begins in Sect. 2 with a general presentation of the MPD equations and its properties. Following that, Sect. 3 presents the perturbation approach based upon the CMP approximation. Afterwards, Sect. 4 gives a description of how to explicitly obtain the linear momentum and spin angular momentum terms for a spinning test particle, up to second order in the perturbation expansion. Prior to the concluding remarks, Sect. 5 presents two lines of study for a many-body extension of the MPD equations that can incorporate the analytic perturbation approach. The first one concerns spin-induced modifications of the Raychaudhuri equation for a congruence of worldlines, while the second one involves the computation of neighbouring worldlines for spinning particles determined with respect to a reference spinning particle. In order to account for the anticipated spin-spin interactions to appear within this many-body extension, a novel approach is employed in the form of introducing a spin-induced torsion field satisfying the Riemann-Cartan U_4 geometry along the reference particle's worldline, details of which are explained in due course.

Throughout this paper, the metric convention is $+2$ signature with geometric units of $G = c = 1$ utilized. The Riemann tensor and its contractions are defined in accordance with Misner et al. [27].

2 The MPD Equations in General Form

2.1 Equations of Motion

Assuming the pole-dipole approximation for a spinning pointlike object, the MPD equations for its linear four-momentum $P^\mu(\tau)$ and spin tensor $S^{\alpha\beta}(\tau)$ are described by

$$\frac{DP^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta}\, u^\nu\, S^{\alpha\beta}\,, \tag{1}$$

$$\frac{DS^{\alpha\beta}}{\mathrm{d}\tau} = P^\alpha\, u^\beta - P^\beta\, u^\alpha\,, \tag{2}$$

where (1) is the spin-curvature force equation due to the coupling of the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ with $S^{\alpha\beta}(\tau)$, plus the particle's four-velocity vector $u^\mu(\tau) = \mathrm{d}x^\mu(\tau)/\mathrm{d}\tau$ with affine parametrization τ, while (2) is the wedge product of the four-velocity with the four-momentum. An important consequence of the MPD equations is that the particle's four-momentum necessarily precesses around the centre-of-mass worldline. The affine parameter τ is formally left unspecified, but it is always possible to have it identified with proper time, such that $u^\mu\, u_\mu = -1$. When going beyond the pole-dipole approximation, (1) and (2) each have additional terms of the form $\mathcal{F}^\mu$ and $\mathcal{T}^{\alpha\beta}$ [18, 22], respectively. These contributions in turn require detailed knowledge of the spinning object's energy-momentum tensor $T^{\mu\nu}$ [6–8, 22], satisfying covariant conservation $\nabla_\nu\, T^{\mu\nu} = 0$. For the purposes of this paper, however, the expressions for (1) and (2) are sufficient.

2.2 Constraint Equations

By themselves, the MPD equations (1) and (2) are underdetermined. They require additional equations to specify the system, up to a parametrization constraint for τ. Specifically, it is necessary to determine some type of relationship involving the spin tensor that allows for the spinning particle's evolution to be understandable. Arguably, the most "reasonable" spin condition to be utilized, following Tulczyjew [4] and Dixon [6, 7] is to enforce orthogonality between the spin tensor and the linear four-momentum, such that

$$S^{\alpha\beta}\, P_\beta = 0\,. \tag{3}$$

It follows from use of both (3) and the MPD equations that the mass and spin magnitudes m and s, defined according to

$$m^2 = -P_\mu\, P^\mu\,, \tag{4}$$

$$s^2 = \frac{1}{2}\, S_{\mu\nu}\, S^{\mu\nu}\,, \tag{5}$$

are both *constants of the motion* [21].

In addition to this fact, use of (3) gives rise to determining the four-velocity u^μ with respect to P^μ and $S^{\alpha\beta}$ according to Tod et al. [10], such that

$$u^\mu = -\frac{P\cdot u}{m^2}\left[P^\mu + \frac{1}{2}\,\frac{S^{\mu\nu}\, R_{\nu\gamma\alpha\beta}\, P^\gamma\, S^{\alpha\beta}}{m^2 + \frac{1}{4}\, R_{\alpha\beta\rho\sigma}\, S^{\alpha\beta}\, S^{\rho\sigma}}\right]. \tag{6}$$

As noted above, the only remaining undetermined quantity is $P\cdot u$, which relates the particle's internal clock with respect to τ. While solving for $P\cdot u$ in terms of the normalization equation $u^\mu\, u_\mu = -1$ is obviously possible, it is also possible to impose a different constraint on τ, since the choice for $P\cdot u$ has no effect on the locus of the particle's worldline. For example, if it is desirable to measure time evolution in terms of a distant observer's clock, then letting $u^0 = \mathrm{d}t/\mathrm{d}\tau = 1$ and solving for $P\cdot u$ is a perfectly valid procedure to follow.

All these equations presented above will become useful for deriving the analytic perturbative approach to the MPD equations, based upon the CMP approximations to be presented next [24].

3 Analytic Perturbation Approach to the MPD Equations

3.1 CMP Approximation

The approximation of the MPD equations first introduced by Chicone et al. [21, 22] is based upon the assumption that $P^\mu - m\, u^\mu = E^\mu$ is small, where E^μ is the spin-curvature force. Further to this, the Møller radius ρ [21–23] is chosen to be a small quantity, in terms of $\rho = s/m \ll r$, where r is the distance from the particle to the source of the gravitational background. When taken together, the CMP approximation emerges as a series expansion to first-order in s, in the form

$$\frac{DP^{\mu}}{d\tau} \approx -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}\, u^{\nu}\, S^{\alpha\beta}\,, \tag{7}$$

$$\frac{DS^{\alpha\beta}}{d\tau} \approx 0\,. \tag{8}$$

According to (8), the spin tensor is parallel transported and the spin condition (3) becomes

$$S^{\alpha\beta}\, u_{\beta} \approx 0\,, \tag{9}$$

which agrees with the Pirani condition [3] for orthogonality between the spin tensor and the particle's four-velocity.

It is surprising to note that the CMP approximation is remarkably accurate for modelling the motion of a spinning test particle in curved space-time, with spin orientation as given in Fig. 1. This is illustrated very well when applied to circular motion around a Kerr black hole [22] and compared to the full MPD equations for $s/(mr) \sim 10^{-3}$ and $r = 10\,M$. However, when pushed to the more extreme situation of $s/(mr) \sim 10^{-2}$–10^{-1} for the same choice of r, a breakdown appears, as evidenced by Fig. 2 with the lack of any spin-induced modulation of the particle's τ-dependent radial position using the CMP approximation compared to the full MPD equations. This observation justifies the need for a generalization of the CMP approximation to be given below [24].

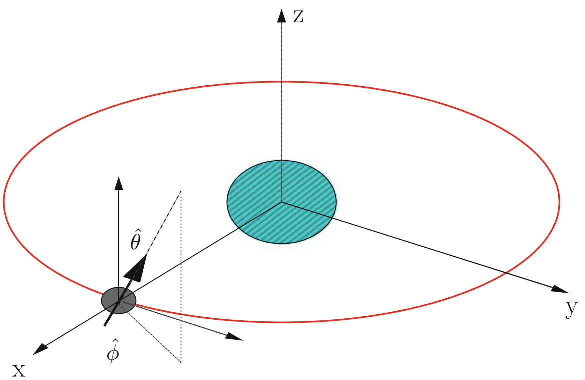

Fig. 1 Initial spin orientation $(\hat{\theta}, \hat{\phi})$ for a spinning particle in circular motion around a *black hole*. The observer is located along the x-axis

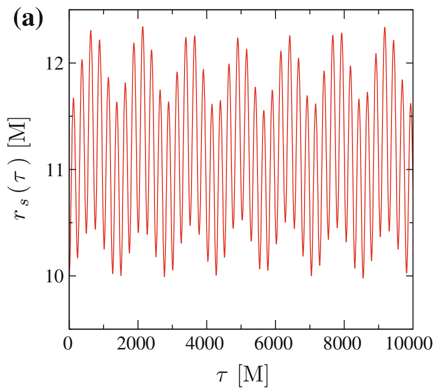

Fig. 2 The orbital radial co-ordinate $r_s(\tau)$ for $s/(mr) = 10^{-1}$, where $m = 10^{-2}\,M$, $r = 10\,M$, $\hat{\theta} = \hat{\phi} = \pi/4$, and $a = 0.50$. **a** $s/(mr) = 10^{-1}$ (MPD). **b** $s/(mr) = 10^{-1}$ (Linear)

3.2 Generalization of the CMP Approximation

To develop the generalization of the CMP approximation [24, 25], the linear momentum and spin angular momentum are expressed in terms of a power series expansion in the form

$$P^\mu(\varepsilon) \equiv \sum_{j=0}^{\infty} \varepsilon^j \, P^\mu_{(j)} \,, \tag{10}$$

$$S^{\mu\nu}(\varepsilon) \equiv \varepsilon \sum_{j=0}^{\infty} \varepsilon^j \, S^{\mu\nu}_{(j)} \;=\; \sum_{j=1}^{\infty} \varepsilon^j \, S^{\mu\nu}_{(j-1)} \,. \tag{11}$$

For (10) and (11), ε is an expansion parameter that is identified with s, with $P^\mu_{(j)}$ and $S^{\mu\nu}_{(j-1)}$ as the respective jth-order contributions in ε of the linear momentum and spin angular momentum. For the zeroth-order expressions in ε, they correspond to the dynamics of a spineless particle in geodesic motion. In addition, it is also assumed that the four-velocity is described as

$$u^\mu(\varepsilon) \equiv \sum_{j=0}^{\infty} \varepsilon^j \, u^\mu_{(j)} \,. \tag{12}$$

When substituting (10), (11), and (12) into the MPD equations expressed as

$$\frac{DP^\mu(\varepsilon)}{d\tau} = -\frac{1}{2} \, R^\mu{}_{\nu\alpha\beta} \, u^\nu(\varepsilon) \, S^{\alpha\beta}(\varepsilon) \,, \tag{13}$$

$$\frac{DS^{\alpha\beta}(\varepsilon)}{d\tau} = 2\,\varepsilon \, P^{[\alpha}(\varepsilon) \, u^{\beta]}(\varepsilon) \,, \tag{14}$$

where for consistency an extra factor of ε is introduced in (14), the jth-order expressions of the MPD equations become

$$\frac{DP^{\mu}_{(j)}}{d\tau} = -\frac{1}{2}\, R^{\mu}{}_{\nu\alpha\beta} \sum_{k=0}^{j-1} u^{\nu}_{(j-1-k)}\, S^{\alpha\beta}_{(k)}\,, \tag{15}$$

$$\frac{DS^{\alpha\beta}_{(j-1)}}{d\tau} = 2 \sum_{k=0}^{j-1} P^{[\alpha}_{(j-1-k)}\, u^{\beta]}_{(k)}\,. \tag{16}$$

For $P^{\mu}_{(0)} = m_0\, u^{\mu}_{(0)}$, where

$$m_0^2 \equiv -P^{(0)}_{\mu}\, P^{\mu}_{(0)}\,, \tag{17}$$

it follows that the zeroth-order term in ε is

$$\frac{DP^{\mu}_{(0)}}{d\tau} = 0\,, \tag{18}$$

while the respective first-order terms for the linear momentum and spin are then the CMP approximation, since

$$\frac{DP^{\mu}_{(1)}}{d\tau} = -\frac{1}{2}\, R^{\mu}{}_{\nu\alpha\beta}\, u^{\nu}_{(0)}\, S^{\alpha\beta}_{(0)}\,, \tag{19}$$

$$\frac{DS^{\alpha\beta}_{(0)}}{d\tau} = 0\,. \tag{20}$$

To complete the generalization, it is necessary to determine $u^{\mu}_{(j)}$ as a function of the linear and spin angular momentum expansion terms. To do this requires use of the supplementary equations (3)–(6), with important consequences to follow. From the spin condition (3), it is straightforward to show in terms of (10) and (11) that

$$P^{(0)}_{\mu}\, S^{\mu\nu}_{(j)} = -\sum_{k=1}^{j} P^{(k)}_{\mu}\, S^{\mu\nu}_{(j-k)}\,, \qquad j \geq 1 \tag{21}$$

for the (j + 1)th-order contribution in ε, where the first-order perturbation in ε is

$$P^{(0)}_{\mu}\, S^{\mu\nu}_{(0)} = 0\,. \tag{22}$$

It is possible to identify a classical analogue for a *bare mass* m_0 defined by (17) and a *bare spin* s_0, according to

$$s_0^2 \equiv \frac{1}{2}\, S^{(0)}_{\mu\nu}\, S^{\mu\nu}_{(0)}\,, \tag{23}$$

in analogy with the radiative corrections identified with the bare mass of quantum field theory, given that the m and s are dependent upon P^μ and $S^{\mu\nu}$, which are represented by (10) and (11), respectively. Therefore, it follows that the total mass and spin magnitudes can be identified as "radiative corrections" to m_0 and s_0, such that

$$m^2(\varepsilon) = m_0^2 \left(1 + \sum_{j=1}^{\infty} \varepsilon^j \, \bar{m}_j^2\right), \tag{24}$$

$$s^2(\varepsilon) = \varepsilon^2 \, s_0^2 \left(1 + \sum_{j=1}^{\infty} \varepsilon^j \, \bar{s}_j^2\right), \tag{25}$$

where

$$\bar{m}_j^2 = -\frac{1}{m_0^2} \sum_{k=0}^{j} P_\mu^{(j-k)} \, P_{(k)}^\mu, \tag{26}$$

$$\bar{s}_j^2 = \frac{1}{s_0^2} \sum_{k=0}^{j} S_{\mu\nu}^{(j-k)} \, S_{(k)}^{\mu\nu}, \tag{27}$$

are dimensionless jth-order corrections to m_0^2 and s_0^2, respectively. Since the m^2 and s^2 are already constant within the exact set of MPD equations, it follows that $\bar{m}_j^2$ and $\bar{s}_j^2$ must be individually constant for each order of ε.

With the four-velocity parametrized by ε in the form

$$u^\mu(\varepsilon) = -\frac{P \cdot u}{m^2(\varepsilon)} \left[P^\mu(\varepsilon) + \frac{1}{2} \, \frac{S^{\mu\nu}(\varepsilon) \, R_{\nu\gamma\alpha\beta} \, P^\gamma(\varepsilon) \, S^{\alpha\beta}(\varepsilon)}{m^2(\varepsilon) \, \Delta(\varepsilon)} \right], \tag{28}$$

$$\Delta(\varepsilon) \equiv 1 + \frac{1}{4\, m^2(\varepsilon)} \, R_{\mu\nu\alpha\beta} \, S^{\mu\nu}(\varepsilon) \, S^{\alpha\beta}(\varepsilon), \tag{29}$$

it can be expanded out once the yet undetermined scalar product $P \cdot u$ is specified. A particularly useful choice is

$$P \cdot u \equiv -m(\varepsilon), \tag{30}$$

which leads to

$$\begin{aligned} u_\mu(\varepsilon) \, u^\mu(\varepsilon) &= -1 + \frac{1}{4\, m^6(\varepsilon) \, \Delta^2(\varepsilon)} \, \tilde{R}_\mu(\varepsilon) \, \tilde{R}^\mu(\varepsilon) \\ &= -1 + O(\varepsilon^4), \end{aligned} \tag{31}$$

where

$$\tilde{R}^{\mu}(\varepsilon) \equiv S^{\mu\nu}(\varepsilon) R_{\nu\gamma\alpha\beta} \, P^{\gamma}(\varepsilon) \, S^{\alpha\beta}(\varepsilon) \,. \tag{32}$$

Therefore, to at least third-order in ε, it is shown from (31) that τ is the *proper time* for parameterization of the particle's centre-of-mass worldline in accordance with (30). A resulting tedious but otherwise straightforward calculation shows from substituting (10), (11), and (24) into (28) that the spinning particle's four-velocity is

$$\begin{aligned}
u^{\mu}(\varepsilon) \;=\; \sum_{j=0}^{\infty} \varepsilon^{j} \, u^{\mu}_{(j)} \;=\; & \frac{P^{\mu}_{(0)}}{m_0} + \varepsilon \left[\frac{1}{m_0} \left(P^{\mu}_{(1)} - \frac{1}{2} \, \bar{m}_1^2 \, P^{\mu}_{(0)} \right) \right] \\
& + \varepsilon^2 \left\{ \frac{1}{m_0} \left[P^{\mu}_{(2)} - \frac{1}{2} \, \bar{m}_1^2 \, P^{\mu}_{(1)} - \frac{1}{2} \left(\bar{m}_2^2 - \frac{3}{4} \, \bar{m}_1^4 \right) P^{\mu}_{(0)} \right] \right. \\
& \left. + \frac{1}{2m_0^3} \, S^{\mu\nu}_{(0)} \, R_{\nu\gamma\alpha\beta} \, P^{\gamma}_{(0)} \, S^{\alpha\beta}_{(0)} \right\} \\
& + \varepsilon^3 \left\{ \frac{1}{m_0} \left[P^{\mu}_{(3)} - \frac{1}{2} \, \bar{m}_1^2 \, P^{\mu}_{(2)} - \frac{1}{2} \left(\bar{m}_2^2 - \frac{3}{4} \, \bar{m}_1^4 \right) P^{\mu}_{(1)} \right. \right. \\
& \left. - \frac{1}{2} \left(\bar{m}_3^2 - \frac{3}{2} \, \bar{m}_1^2 \, \bar{m}_2^2 + \frac{5}{8} \, \bar{m}_1^6 \right) P^{\mu}_{(0)} \right] \\
& \left. + \frac{1}{2m_0^3} \, R_{\nu\gamma\alpha\beta} \left[\sum_{n=0}^{1} S^{\mu\nu}_{(1-n)} \sum_{k=0}^{n} P^{\gamma}_{(n-k)} \, S^{\alpha\beta}_{(k)} - \frac{3}{2} \, \bar{m}_1^2 \, S^{\mu\nu}_{(0)} \, P^{\gamma}_{(0)} \, S^{\alpha\beta}_{(0)} \right] \right\} \\
& + O(\varepsilon^4) \,,
\end{aligned} \tag{33}$$

which satisfies (31) to third order in ε.

At this point, the worldline of the spinning particle can be explicitly determined by simple integration of (33), since it provides an iterative means to explicitly evaluate $u^{\mu}(\varepsilon)$ order by order in ε, based upon solving (15) and (16) in terms of quantities defined for lower orders of ε.

3.3 Perturbations of the Møller Radius and Stability Implications for Particle Motion

An important study with regard to the analytic perturbative approach to the MPD equations involves the study of transitions from stable to chaotic motion for spinning particles in curved space-time. From strictly numerical considerations [12–15], there is ample evidence to expect that there exists a critical point when, for sufficiently large s, the motion becomes chaotic. Indeed, when considering many-body particle dynamics for astrophysical systems, the importance of detecting chaotic dynamics is very relevant. The pertinent question then concerns how to identify from presumably knowable and independent astrophysical parameters, such as the mass and spin of each constituent particle, the precise combination of factors that can trigger the transition from stable to chaotic motion.

It is understood from, for example, studying the case of a single MPD particle in circular orbit around a Kerr black hole [22] that the relevant parameter for determining the relative strength of the spin-curvature interaction to deviate from strictly geodesic motion is the Møller radius $\rho = s/m$. When $\rho \ll r$ for the particle orbit's radius of curvature, the capacity for entering an unstable region of phase space is minimized. For the choice of astrophysically realistic spins for MPD particles under numerical simulations [14, 15], it is shown that the particle motion never approaches instability.

Until now within this formalism, it is unclear how to extract useful analytic information about stability properties of MPD particle motion. Given the fact that the mass and spin parameters can shift in magnitude according to (24) and (25), this will have an impact upon the strength of the Møller radius, which in turn impacts upon the capacity for the spin-curvature interaction to induce unstable motion under potentially realistic physical situations. Therefore, determining the perturbations in ρ in analytic form may provide useful insight in determining the precise conditions to generate the transition from stable to chaotic motion for a single MPD particle.

It is straightforward to show from using (24) and (25) that

$$\begin{aligned}\rho(\varepsilon) &= \frac{s(\varepsilon)}{m(\varepsilon)} \\ &= \frac{s_0}{m_0}\left\{\varepsilon + \varepsilon^2\left[\frac{1}{2}\left(\bar{s}_1^2 - \bar{m}_1^2\right)\right]\right. \\ &\quad \left. + \varepsilon^3\left[\frac{1}{2}\left(\bar{s}_2^2 - \bar{m}_2^2\right) - \frac{1}{4}\bar{s}_1^2\,\bar{m}_1^2 - \frac{1}{8}\left(\bar{s}_1^4 - 3\,\bar{m}_1^4\right)\right] + O(\varepsilon^4)\right\}, \qquad (34)\end{aligned}$$

where the "radiative corrections" due to the second- and third-order contributions in ε cause a shift away from the zeroth-order Møller radius $\rho_0 = s_0/m_0$ defined by the "bare mass" m_0 and "bare spin" s_0. What is important to note—perhaps not surprisingly—is that the corrections $\bar{m}_j^2$ and $\bar{s}_j^2$ counteract each other's effects, at least on a formal level. The precise nature of the shift from ρ_0 to ρ requires determining (34) in reference to a specific background and appropriately defined initial conditions. However, it is clearly evident that variations in the Møller radius are precisely identifiable, which can then be correlated with the computation of Lyapunov exponents and Poincaré maps, for example, while performing stability analysis studies using standard techniques available.

4 Evaluation of the Linear Momentum and Spin Angular Momentum Terms

4.1 Local Fermi Co-ordinate Frame

With the formalism behind the analytic perturbative approach to the MPD equations now established, it is possible to explicitly evaluate the linear momentum and spin angular momentum expansion terms [25] in (10) and (11). As noted earlier, this is accomplished in an iterative fashion by solving the first-order perturbation terms with respect to zeroth-order quantities, followed by solving the second-order and higher terms with respect to lower-order quantities. It becomes a very straightforward process to achieve this by framing the problem in terms of the tetrad formalism and Fermi normal co-ordinates [22], with the obvious benefit of expressing all contributions with respect to a freely falling worldline in a locally flat background. The metric deviations in Fermi normal co-ordinates to leading order are then proportional to the projected Riemann curvature tensor evaluated on the worldline.

To proceed, first start with an orthonormal tetrad frame $\lambda^{\mu}{}_{\hat{\alpha}}$ with the orthogonality condition

$$\eta_{\hat{\alpha}\hat{\beta}} = g_{\mu\nu}\, \lambda^{\mu}{}_{\hat{\alpha}}\, \lambda^{\nu}{}_{\hat{\beta}}\,, \tag{35}$$

such that the tetrad satisfies parallel transport in the general space-time background described by co-ordinates x^{μ}, with

$$\frac{D\lambda^{\mu}{}_{\hat{\alpha}}}{\mathrm{d}\tau} = 0\,, \tag{36}$$

and where $\hat{\alpha}$ denote indices for the Fermi co-ordinates $X^{\hat{\alpha}}$ on a locally flat tangent space. The Riemann curvature tensor in the Fermi frame is then described by

$$^{F}R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = R_{\mu\nu\rho\sigma}\, \lambda^{\mu}{}_{\hat{\alpha}}\, \lambda^{\nu}{}_{\hat{\beta}}\, \lambda^{\rho}{}_{\hat{\gamma}}\, \lambda^{\sigma}{}_{\hat{\delta}}\,. \tag{37}$$

4.2 Zeroth-Order Contributions to the Linear Momentum and Spin Angular Momentum

By ensuring that $\lambda^{\mu}{}_{\hat{0}} = u^{\mu}_{(0)}$ in the usual fashion and making use of (35) along with the leading-order spin condition $P^{(0)}_{\mu}\, S^{\mu\nu}_{(0)} = 0$, it becomes evident that

$$P^{\mu}_{(0)} = \lambda^{\mu}{}_{\hat{\alpha}}\, P^{\hat{\alpha}}_{(0)} \;=\; m_0\, \lambda^{\mu}{}_{\hat{0}}\,, \tag{38}$$

$$S^{\mu\nu}_{(0)} = \lambda^{\mu}{}_{\hat{\imath}}\, \lambda^{\nu}{}_{\hat{\jmath}}\, S^{\hat{\imath}\hat{\jmath}}_{(0)}\,, \tag{39}$$

where $P^{\hat{\alpha}}_{(0)} = m_0 \, \delta^{\hat{\alpha}}{}_{\hat{0}}$ and $S^{\hat{\imath}\hat{\jmath}}_{(0)}$ is a constant-valued spatial antisymmetric tensor, with components to be determined from initial conditions.

4.3 First-Order Perturbation Terms

For the linear momentum, the first-order perturbation in ε is very straightforward to evaluate. With (19) and (36), it is obvious that $DP^{\mu}_{(1)}/\mathrm{d}\tau = \lambda^{\mu}{}_{\hat{\alpha}} \left(\mathrm{d}P^{\hat{\alpha}}_{(1)}/\mathrm{d}\tau\right)$, resulting in

$$\frac{\mathrm{d}P^{\hat{\alpha}}_{(1)}}{\mathrm{d}\tau} = -\frac{1}{2} \, {}^{F}R^{\hat{\alpha}}{}_{\hat{0}\hat{\imath}\hat{\jmath}} \, S^{\hat{\imath}\hat{\jmath}}_{(0)} \,. \tag{40}$$

This is immediately integrable, leading to

$$P^{\mu}_{(1)} = -\frac{1}{2} \, \lambda^{\mu}{}_{\hat{k}} \int \left({}^{F}R^{\hat{k}}{}_{\hat{0}\hat{\imath}\hat{\jmath}} \, S^{\hat{\imath}\hat{\jmath}}_{(0)}\right) \mathrm{d}\tau \,. \tag{41}$$

Because of the tetrad formalism employed within this evaluation, it follows immediately upon contracting (41) with $P^{(0)}_{\mu}$ that

$$\bar{m}_1^2 = 0 \tag{42}$$

for a general space-time background. This immediately leads to simplified expressions for (33) and (34), expressed as

$$\begin{aligned}
u^{\mu}(\varepsilon) \; &= \; \sum_{j=0}^{\infty} \varepsilon^{j} \, u^{\mu}_{(j)} \; = \; \frac{1}{m_0} \left[P^{\mu}_{(0)} + \varepsilon \, P^{\mu}_{(1)}\right] \\
&\quad + \varepsilon^2 \left\{ \frac{1}{m_0} \left[P^{\mu}_{(2)} - \frac{1}{2} \, \bar{m}_2^2 \, P^{\mu}_{(0)}\right] + \frac{1}{2m_0^3} \, S^{\mu\nu}_{(0)} \, R_{\nu\gamma\alpha\beta} \, P^{\gamma}_{(0)} \, S^{\alpha\beta}_{(0)} \right\} \\
&\quad + \varepsilon^3 \left\{ \frac{1}{m_0} \left[P^{\mu}_{(3)} - \frac{1}{2} \, \bar{m}_2^2 \, P^{\mu}_{(1)} - \frac{1}{2} \, \bar{m}_3^2 \, P^{\mu}_{(0)}\right] \right. \\
&\quad \left. + \; \frac{1}{2m_0^3} \, R_{\nu\gamma\alpha\beta} \sum_{n=0}^{1} S^{\mu\nu}_{(1-n)} \sum_{k=0}^{n} P^{\gamma}_{(n-k)} \, S^{\alpha\beta}_{(k)} \right\} + O(\varepsilon^4)
\end{aligned} \tag{43}$$

and

$$\rho(\varepsilon) = \frac{s_0}{m_0} \left\{ \varepsilon + \varepsilon^2 \left[\frac{1}{2} \, \bar{s}_1^2\right] + \varepsilon^3 \left[\frac{1}{2} \left(\bar{s}_2^2 - \bar{m}_2^2\right) - \frac{1}{8} \, \bar{s}_1^4\right] + O(\varepsilon^4) \right\} . \tag{44}$$

In contrast to the linear momentum to first-order, the corresponding expression for the spin tensor is considerably more complicated to evaluate. It is important first to note that, following (16) for $j = 2$,

$$\frac{D S^{\mu\nu}_{(1)}}{\mathrm{d}\tau} = 0 \,. \tag{45}$$

When $S^{\mu\nu}_{(1)}$ is described in terms of its tetrad projection, it is determined that

$$\begin{aligned} S^{\mu\nu}_{(1)} &= \lambda^{\mu}{}_{\hat{\alpha}}\, \lambda^{\nu}{}_{\hat{\beta}}\, S^{\hat{\alpha}\hat{\beta}}_{(1)} \\ &= 2\,\lambda^{[\mu}{}_{\hat{0}}\, \lambda^{\nu]}{}_{\hat{j}}\, S^{\hat{0}\hat{j}}_{(1)} + \lambda^{\mu}{}_{\hat{\imath}}\, \lambda^{\nu}{}_{\hat{j}}\, S^{\hat{\imath}\hat{j}}_{(1)} \,. \end{aligned} \tag{46}$$

By using the spin condition relation (21) for $j = 1$ and the tetrad projection for (41), it is shown that

$$S^{\hat{0}\hat{j}}_{(1)} = -\frac{1}{m_0}\, P^{(1)}_{\hat{\imath}}\, S^{\hat{\imath}\hat{j}}_{(0)} \,. \tag{47}$$

Regarding the components $S^{\hat{\imath}\hat{j}}_{(1)}$ in (46), they can be formally determined with respect to the "radiative correction" term $\bar{s}_1^2$ according to (27), such that

$$S^{\hat{\imath}\hat{j}}_{(1)} = \frac{1}{4}\, \bar{s}_1^2\, S^{\hat{\imath}\hat{j}}_{(0)} \,. \tag{48}$$

At this point, a problem emerges with this approach, in that (46) still depends upon $\bar{s}_1^2$, which remains an *undetermined parameter* to be computed. Given the discovery that $\bar{m}_1^2 = 0$ for the first-order mass shift correction, it would be tempting to simply make the same assumption about the corresponding spin shift correction. However, since $\bar{s}_1^2$ only needs to be *covariantly constant*, such an assumption is not well justified. This leads to the requirement of explicitly evaluating (45) independently of the tetrad in terms of the six-dimensional matrix differential equation[1]

$$\frac{D S^{(1)}_{\mu\nu}}{\mathrm{d}\tau} = \frac{\mathrm{d} S^{(1)}_{\mu\nu}}{\mathrm{d}\tau} + 2\, u^{\alpha}_{(0)}\, \Gamma^{\beta}{}_{\alpha[\mu}\, S^{(1)}_{\nu]\beta} \;=\; 0 \tag{49}$$

for a column vector of $\left(S^{(1)}_{\mu\nu}\right) \equiv \left(S^{(1)}_{01}, S^{(1)}_{02}, S^{(1)}_{03}, S^{(1)}_{12}, S^{(1)}_{23}, S^{(1)}_{31}\right)$. The solution to (49) is then used to evaluate the tetrad projected spin tensor $S^{\hat{\alpha}\hat{\beta}}_{(1)}$ and $\bar{s}_1^2$.

[1] A previous approach was reported in [25, 26] using a three-dimensional column vector of purely spatial components of $S^{(1)}_{\mu\nu}$ in combination with the spin condition to algebraically evaluate the remaining three components. In retrospect, this previous treatment is not viable in comparison to the approach presented in this paper. Therefore, this new perspective replaces the older one presented in [25, 26].

4.4 Second-Order Perturbation Terms

Once the challenge of evaluating the first-order perturbation term for the spin tensor is overcome, the second-order perturbation terms are easily determined. For example, from solving (15) for $j = 2$, the second-order linear momentum term is

$$P^{\mu}_{(2)} = -\frac{1}{2}\,\lambda^{\mu}{}_{\hat{\alpha}} \int \left(\frac{1}{m_0}\, {}^F R^{\hat{\alpha}}{}_{\hat{j}\hat{k}\hat{l}}\, P^{\hat{j}}_{(1)}\, S^{\hat{k}\hat{l}}_{(0)} + {}^F R^{\hat{\alpha}}{}_{\hat{0}\hat{\gamma}\hat{\beta}}\, S^{\hat{\gamma}\hat{\beta}}_{(1)} \right) \mathrm{d}\tau\,, \tag{50}$$

which is the sum of terms with tetrad projections in both along the worldline and orthogonal to it. Therefore, it follows from (26) for $j = 2$ that $\bar{m}_2^2 \neq 0$ in general. Similarly, for the spin angular momentum, the corresponding second-order expression for (16) is

$$\frac{DS^{\mu\nu}_{(2)}}{\mathrm{d}\tau} = \frac{1}{m_0^3}\, P^{[\mu}_{(0)}\, S^{\nu]\sigma}_{(0)}\, R_{\sigma\gamma\alpha\beta}\, P^{\gamma}_{(0)}\, S^{\alpha\beta}_{(0)}\,, \tag{51}$$

which when integrated leads to

$$S^{\mu\nu}_{(2)} = \frac{1}{m_0}\, \lambda^{[\mu}{}_{\hat{0}}\, \lambda^{\nu]}{}_{\hat{\imath}} \int S^{\hat{\imath}\hat{\jmath}}_{(0)}\, {}^F R_{\hat{\jmath}\hat{0}\hat{k}\hat{l}}\, S^{\hat{k}\hat{l}}_{(0)}\, \mathrm{d}\tau\,. \tag{52}$$

As well, $\bar{s}_2^2 \neq 0$ in general. Further evaluation to higher orders in ε also suggests no obvious challenges to overcome, so the procedure can be extended indefinitely to any desired order.

5 Moving Towards an MPD Many-Body Description of Classical Spinning Particle Dynamics

5.1 Research Motivations

It would be very interesting to develop an extension of the analytic perturbative approach for the MPD equations that can be applied towards a many-body description of classical spinning particle dynamics. This is important to understand because virtually every composite astrophysical system, such as spiral galaxies, globular clusters, galactic superclusters, accretion disks around black holes, and so forth are all built in terms of components with spin angular momentum incorporated. These systems are often modelled using only many-body Newtonian gravitation, which may be a reasonable assumption under certain predetermined circumstances, but seems inadequate to address the full range of physical consequences that may be experienced due to relativistic effects that are necessarily excluded.

The essential intent for investigating a many-body description of classical spinning particles in curved space-time is to better understand the interaction dynamics between two or more spinning particles in relative motion with respect to one another. Given the interest shown in understanding the transition from stable to chaotic motion for a single particle, this consideration becomes even more relevant than before when dealing with a many-body system in curved space-time, in which the possibility of instabilities may emerge under astrophysically realistic conditions. As well, by compressing the many-body system into a very small volume of space, it seems possible to investigate the continuum limit for spinning matter subject to the MPD equations as applied to each unit cell endowed with spin.

This section presents the first stage of some long-term research [28] on understanding the extension of the MPD equations to incorporate many-body spinning particle systems. For now, it involves understanding the evolution of a single neighbouring worldline due to spin, determined with respect to some reference worldline in the following two contexts:

1. **Raychaudhuri Equation**:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\,\theta^2 - \sigma^{\alpha\beta}\,\sigma_{\alpha\beta} + \omega^{\alpha\beta}\,\omega_{\alpha\beta} - R_{\alpha\beta}\,u^\alpha\,u^\beta\,, \tag{53}$$

2. **Generalized Jacobi Equation** for tidal dynamics:

$$\frac{D^2\xi^\mu}{\mathrm{d}\tau^2} + R^\mu{}_{\alpha\beta\gamma}\,u^\alpha\,\xi^\beta\,u^\gamma + \cdots = 0\,, \tag{54}$$

where θ is the expansion, $\sigma_{\alpha\beta}$ is the shear, and $\omega^{\alpha\beta}$ is the vorticity associated with a congruence of worldlines in (53), and where the unspecified extra terms in (54) refer to higher-order contributions involving the covariant time derivative of the separation vector ξ^μ between neighbouring worldlines.

In both contexts, there lies an important physical challenge to understand, which concerns the fact that the spinning particles moving along separate worldlines must *necessarily interact* with each other, which then significantly impacts upon their evolution in space-time. To put it another way, it is not reasonable to expect that one can just make multiple copies of MPD particles that are distributed throughout a given curved space-time background and have their worldlines determined *solely* by the background curvature. This issue can be best appreciated by appealing to the analogy of magnetostatics involving the attractive force felt by two parallel currents in a steady state, or conversely the repulsive force felt by two anti-parallel currents. Therefore, it is necessary to find an avenue that allows for the individual spinning particles to have non-trivial interactions.

5.2 Locally Induced Einstein-Cartan Interaction

In order to address the challenge of incorporating the interactions between two spinning particles that are separately subject to forces and torques as described by the MPD equations, it is useful to consider the situation in terms of the overall kinematics first without necessarily modifying the equations themselves.[2]

To this end, recall (28) and (29) with parametrization constraint (30) for the four-velocity $u^\mu(\varepsilon)$. Then the corresponding acceleration $a^\mu(\varepsilon)$ for the spinning particle, expanded out in ε, would formally read as

$$\begin{aligned} a^\mu(\varepsilon) &= \frac{Du^\mu(\varepsilon)}{\mathrm{d}\tau} \\ &= \varepsilon\left[u^\alpha_{(0)}\nabla_\alpha\, u^\mu_{(1)} + u^\alpha_{(1)}\nabla_\alpha\, u^\mu_{(0)}\right] \\ &\quad + \varepsilon^2\left[u^\alpha_{(0)}\nabla_\alpha\, u^\mu_{(2)} + u^\alpha_{(1)}\nabla_\alpha\, u^\mu_{(1)} + u^\alpha_{(2)}\nabla_\alpha\, u^\mu_{(0)}\right] + O(\varepsilon^3)\,. \end{aligned} \tag{55}$$

Consider now a situation involving two spinning particles propagating in a curved space-time background. Because both particles are subject to forces and torques generated by the MPD equations, their worldlines are not geodesics. At the same time, because they are independent particles with distinct physical properties to them, their responses to the spin-curvature interaction are obviously expected to be distinct.

Now choose one of these two worldlines to act as a reference with respect to the other one. For the particle on that reference worldline, its propagation with respect to the background space-time can be determined in accordance with Fermi-Walker transport [29], such that

$$\frac{D_{(FW)}u^\mu}{\mathrm{d}\tau} = u^\alpha\,\nabla^{(FW)}_\alpha\, u^\mu \;=\; 0\,, \tag{56}$$

where

$$\Gamma^\mu_{(FW)\alpha\beta} = \Gamma^\mu_{(0)\alpha\beta} + C^\mu_{(FW)\alpha\beta}\,, \tag{57}$$

$$C^\mu_{(FW)\alpha\beta} = a^\mu\, g^{(0)}_{\alpha\beta} - \delta^\mu{}_\beta\, a_\alpha\,, \tag{58}$$

and $\Gamma^\mu_{(0)\alpha\beta}$ is the metric connection for the space-time background.

At this point, it is important to make the following observation. From the perspective of an observer co-moving with the reference particle, it would appear from

[2] As presented, the MPD equations implicitly assume that all interactions are "local" in their description, at least within the context of General Relativity. If evidence emerges to show that gravitational interactions are necessarily "non-local" by some defining criterion, then it would follow that many-body spinning particle interactions as described here must incorporate non-locality. By implication, an appropriate modification of the MPD equations would then be well-motivated.

(56) that the particle is propagating along a *geodesic* in a *perturbed space-time background* due to particle acceleration generated by the MPD equations. Further to this perspective, it is self-evident from (58) that the Fermi-Walker connection (57) *is not symmetric* in its lower two indices, yielding an antisymmetric contribution. Therefore, it follows that $C^{\mu}_{(FW)\alpha\beta}$ is interpreted as the contortion tensor, with an induced torsion field generated by the reference particle that gets encoded within a new curvature tensor determined by (57).

The significance of this approach is that, in the instant of proper time as determined for the reference particle, the neighbouring particle is now subject to an MPD spin-curvature interaction *involving this new curvature tensor with induced torsion* due to the reference particle's acceleration. Furthermore, the geometry is now an instantaneous Riemann-Cartan U_4 [30], with the acceleration-dependent terms giving rise to what will now be called a *locally induced Einstein-Cartan (EC) interaction*.

Further details about the locally induced EC interaction and its subsequent application for the rest of this paper are deferred to a future publication [28]. However, it is worthwhile to show its presence in terms of the metric tensor of space-time in local Fermi normal co-ordinates $(T, \mathbf{X})$.[3] To linear order in the reference particle's acceleration, with $\mathbf{a}_j = a_j$ from the induced EC interaction labelled with boldface to distinguish these acceleration terms from others that arise elsewhere, it is shown that

$$g^{(FW)}_{00} = -1 - 2\, a_k\, X^k - \left[R^{(FW)}_{l0m0}(T) + \frac{\mathrm{d}}{\mathrm{d}T}\left(\mathbf{a}_{(l,m)}\Delta T\right)\right] X^l\, X^m\,, \tag{59}$$

$$g^{(FW)}_{(0j)} = \frac{2}{3}\, R^{(FW)}_{0(kl)j}(T)\, X^k\, X^l\,, \tag{60}$$

$$g^{(FW)}_{(ij)} = \delta_{ij} + \frac{1}{3}\, R^{(FW)}_{i(kl)j}(T)\, X^k\, X^l\,, \tag{61}$$

$$g^{(FW)}_{[0j]} = \mathbf{a}_j\, \Delta T\,, \tag{62}$$

$$g^{(FW)}_{[ij]} = \mathbf{a}_i\, X^j - \mathbf{a}_j\, X^i\,, \tag{63}$$

with the metric tensor having now both symmetric and antisymmetric components, in which the latter set are dependent upon $\mathbf{a}_j$.

It is very important to point out that, while the induced EC interaction is a purely frame-dependent effect akin to a fictitious force, the new curvature tensor based upon (57) is perfectly well-defined because (58) is a tensor. For this reason, the explicit time-dependent contributions in (59) and especially (62) are labelled with ΔT, to symbolize that these contributions are ephemeral and can only persist within a limited time scale. In terms of the physical interpretations that can be ascribed to the EC interaction, it can be understood as something analogous to the Lense-Thirring effect that causes the precession of a nearby gyroscope due to the spin rotation of a massive body. This also fits well with an interpretation of Einstein-Cartan theory in terms of defects within space-time in like manner to that of classical defect theory concerning

[3]For ease of notation, the indices are left unhatted.

internal stresses within an elastic medium [31]. From such an interpretation, the induced torsion due to the reference particle causes a space-time defect that then has to be restored to its original state once the particle leaves that space-time region.

5.3 Spin Modifications of the Raychaudhuri Equation

When using the spin-perturbed space-time background to compute the Raychaudhuri equation, it takes the form

$$\frac{\mathrm{d}\theta^{(FW)}}{\mathrm{d}\tau} = -\frac{1}{3}\,\theta^2 - \sigma^{\alpha\beta}\,\sigma_{\alpha\beta} + \omega^{\alpha\beta}\,\omega_{\alpha\beta} - R^{(FW)}_{\alpha\beta}\,u^\alpha\,u^\beta \\ - 2\,a^\alpha\,u^\beta\left(\sigma_{\alpha\beta} - \omega_{\alpha\beta}\right)\,, \tag{64}$$

where

$$\theta^{(FW)} = \theta \;=\; \nabla_\alpha\,u^\alpha\,, \tag{65}$$

$$\sigma^{(FW)}_{\alpha\beta} = \sigma_{\alpha\beta} + a_{(\alpha}\,u_{\beta)} \;=\; \left[\nabla_{(\beta}\,u_{\alpha)} - \frac{1}{3}\,\theta\,h_{\alpha\beta}\right] + a_{(\alpha}\,u_{\beta)}\,, \tag{66}$$

$$\omega^{(FW)}_{\alpha\beta} = \omega_{\alpha\beta} + a_{[\alpha}\,u_{\beta]} \;=\; \nabla_{[\beta}\,u_{\alpha]} + a_{[\alpha}\,u_{\beta]}\,. \tag{67}$$

are the modified forms of the expansion, shear, and vorticity with $h_{\alpha\beta} = g^{(0)}_{\alpha\beta} + u_\alpha\,u_\beta$, such that $u^\alpha\,h_{\alpha\beta} = 0$ and $h^\alpha{}_\alpha = 3$, while

$$R^{(FW)}_{\alpha\beta}\,u^\alpha\,u^\beta = R^{(0)}_{\alpha\beta}\,u^\alpha\,u^\beta - \frac{3}{2}\,a_\alpha\,a^\alpha \tag{68}$$

shows that the modified gravitational term includes the spin-dependent acceleration with opposite sign compared to the background Ricci term. This is interesting because it shows that the spin-curvature interaction causes a defocussing of the congruences. In particular, such a result may have relevance for cosmology as a way to account for a small repulsion that is currently attributed to dark energy, assuming that a sufficiently large MPD spin-curvature force can be generated due to the coupling of galactic spin with the cosmological space-time background. More details need to be examined to determine whether this possibility is physically realistic.

5.4 Generalized Jacobi Equation and Computation of Neighbouring Worldlines

The other approach towards finding a many-body description of spinning particles is to use the generalized Jacobi equation as a computational tool for determining a neighbouring worldline with respect to a reference worldline. Since this approach [32, 33] specifically computes neighbouring geodesics with respect to a reference geodesic in a general space-time background, it can also be applied for the case of the perturbed space-time background as described by (57).

Suppose that σ is the parameter identifying a family of worldlines $x^\mu(\tau;\sigma)$ in a given space-time background, with the reference worldline as $x^\mu(\tau;0) = x^\mu(\tau)$. Then the neighbouring worldline is related to the reference in terms of a Taylor expansion with respect to $\sigma = 0$, such that

$$\begin{aligned} x^\mu(\tau;\sigma) &= x^\mu(\tau) + \frac{\partial x^\mu(\tau)}{\partial\sigma}\,\sigma + \frac{1}{2}\,\frac{\partial^2 x^\mu(\tau)}{\partial\sigma^2}\,\sigma^2 + O(\sigma^3) \\ &= x^\mu(\tau) + \xi^\mu(\tau)\,\sigma \\ &\quad + \frac{1}{2}\left[k^\mu(\tau) - \Gamma^\mu_{(FW)\alpha\beta}\,\xi^\alpha(\tau)\,\xi^\beta(\tau)\right]\sigma^2 + O(\sigma^3)\,, \end{aligned} \tag{69}$$

where

$$\xi^\mu(\tau) = \frac{\partial x^\mu(\tau)}{\partial\sigma} \tag{70}$$

is the separation vector between the neighbouring worldlines and

$$k^\mu(\tau) = \frac{D_{(FW)}\xi^\mu(\tau)}{\mathrm{d}\sigma}\,. \tag{71}$$

The challenge is to solve for (70) and (71) for determining (69) to the desired order of the expansion in σ.

This study is currently a work in progress [28]. Nonetheless, it is still possible to outline the basic approach to solve for (70) and (71) necessary to obtain (69). This first point is to assume, just as in (10) and (11), that (70) and (71) can be written in the form

$$\xi^\mu(\varepsilon) \equiv \sum_{j=0}^{\infty} \varepsilon^j\,\xi^\mu_{(j)}\,, \tag{72}$$

$$k^\mu(\varepsilon) \equiv \sum_{j=0}^{\infty} \varepsilon^j\,k^\mu_{(j)}\,. \tag{73}$$

Focussing only on the first-order contribution in (69), the next step is to determine the generalized Jacobi equation (54) in order to determine the proper time-dependence of ξ^μ. This is best accomplished by solving the equation of motion for a test particle in a local Fermi frame [34, 35], in which the right-hand side is the acceleration of the neighbouring particle, denoted by the subscript "[1]" in what follows below. By again describing the background space-time in terms of the Fermi-Walker connection (57) and the locally induce EC interaction, it can be shown that the generalized Jacobi equation is

$$\frac{D^2_{(FW)}\xi^\mu}{\mathrm{d}\tau^2} + R^\mu_{(FW)\alpha\beta\gamma}\, u^\alpha\, \xi^\beta\, u^\gamma$$
$$+ 2\left[\left(\delta^\mu{}_\nu + u^\mu\, u_\nu\right) + \frac{D_{(FW)}\xi^\mu}{\mathrm{d}\tau}\, u_\nu\right] R^\nu_{(FW)\alpha\beta\gamma}\, \frac{D_{(FW)}\xi^\alpha}{\mathrm{d}\tau}\, \xi^\beta$$
$$\times \left(u^\gamma + \frac{1}{3}\,\frac{D_{(FW)}\xi^\gamma}{\mathrm{d}\tau}\right) + \Omega^\mu_{accel.} + \Omega^\mu_{EC} \;=\; 0\,, \tag{74}$$

where

$$\Omega^\mu_{accel.} \;=$$
$$\left(\delta^\mu{}_\nu + u^\mu\, u_\nu\right)\left\{\left(a^\nu - a^\nu_{[1]}\right) + \frac{D_{(FW)}a^\nu_{[1]}}{\mathrm{d}\sigma}\right.$$
$$+ \frac{D_{(FW)}\xi^\nu}{\mathrm{d}\tau}\left[\xi_\alpha\,\frac{D_{(FW)}a^\alpha}{\mathrm{d}\tau} + \frac{D_{(FW)}\xi_\alpha}{\mathrm{d}\tau}\left(a^\alpha_{[1]} - 2\,a^\alpha + \frac{D_{(FW)}a^\alpha}{\mathrm{d}\sigma}\right)\right.$$
$$+ \left(a^{[1]}_\alpha + \frac{D_{(FW)}a^{[1]}_\alpha}{\mathrm{d}\sigma}\right)\frac{D_{(FW)}\xi^\alpha}{\mathrm{d}\tau}\,\frac{D_{(FW)}\xi_\beta}{\mathrm{d}\tau}\,\frac{D_{(FW)}\xi^\beta}{\mathrm{d}\tau}$$
$$\left. - \; 2\,\xi^\alpha\,\nabla^{(FW)}_{(\alpha}\, a_{\beta)}\,\frac{D_{(FW)}\xi^\beta}{\mathrm{d}\tau}\right]$$
$$\left. + \left(a^\nu_{[1]} + \frac{D_{(FW)}\xi^\nu}{\mathrm{d}\sigma}\right)\frac{D_{(FW)}\xi_\alpha}{\mathrm{d}\tau}\,\frac{D_{(FW)}\xi^\alpha}{\mathrm{d}\tau}\right\}$$
$$- \; 2\, g^{\mu\sigma}_{(0)}\left(R^{(FW)}_{\alpha\beta\gamma\sigma} - R^{(FW)}_{\gamma\sigma\alpha\beta}\right)\frac{D_{(FW)}\xi^\alpha}{\mathrm{d}\tau}\,\xi^\beta\, u^\gamma$$
$$- \; 8\,\xi^\gamma\left[\delta^\mu{}_\alpha\,\nabla^{(FW)}_{[\beta}\, a_{\gamma]} - g^{\mu\sigma}_{(0)}\, g^{(0)}_{\gamma\alpha}\,\nabla^{(FW)}_{[\beta}\, a_{\sigma]}\right]\frac{D_{(FW)}}{\mathrm{d}\tau}\left(\xi^{(\alpha}\, u^{\beta)}\right) \tag{75}$$

and

$$\Omega^\mu_{EC} \;=$$
$$\left(\delta^\mu{}_\nu + u^\mu\, u_\nu\right)\left\{\xi_\alpha\,\nabla^{[\alpha}_{(FW)}\,\mathbf{a}^{\nu]} - \left(1 + \Delta\tau\,\frac{D_{(FW)}}{\mathrm{d}\tau}\right)\left(\mathbf{a}^\nu - \frac{D_{(FW)}\mathbf{a}^\nu}{\mathrm{d}\sigma}\right)\right.$$
$$+ \frac{D_{(FW)}\xi^\nu}{\mathrm{d}\tau}\left[\left(2\,\xi^\beta\,\nabla^{(FW)}_{[\alpha}\,\mathbf{a}_{\beta]}\right.\right.$$
$$\left.\left. - \;\mathbf{a}_\alpha - 2\,\Delta\tau\,\frac{D_{(FW)}}{\mathrm{d}\tau}\xi^\beta\,\nabla^{(FW)}_{(\alpha}\,\mathbf{a}_{\beta)}\right)\frac{D_{(FW)}\xi^\alpha}{\mathrm{d}\tau}\right]$$
$$\left. + \left(\mathbf{a}^\nu\,\frac{D_{(FW)}\xi_\alpha}{\mathrm{d}\tau} - \xi^\nu\,\frac{D_{(FW)}\mathbf{a}_\alpha}{\mathrm{d}\tau}\right)\frac{D_{(FW)}\xi^\alpha}{\mathrm{d}\tau}\right\}\,, \tag{76}$$

with the neighbouring particle's acceleration already Lie transported for immediate evaluation. It is worth recalling that the neighbouring particle's acceleration is determined by use of the MPD equations, but with use of the modified Riemann curvature tensor $R^{(FW)}_{\mu\nu\alpha\beta}$ instead of $R^{(0)}_{\mu\nu\alpha\beta}$ for the background space-time.

The final step required is to substitute (72) and all other expansions of ε into (74), to then solve for each order of ε. The difficult part of the task is to evaluate the zeroth-order contribution for ξ^μ, which applies for geodesics, but once that contribution is determined, all the others can be obtained without difficulty through simple integration, at least in principle. It remains to be seen [28] whether there are any further technical challenges in determining (69), but it appears that the conceptual difficulties are overcome.

6 Conclusion

This paper illustrates the foundations and possible applications for the analytic perturbative approach to the MPD equations. It should be apparent that the approach is powerful, in that it is both background independent and applicable to any conceivable type of motion for the spinning particle. Furthermore, the outline of future work on many-body particle dynamics shows its potential to model many-body astrophysical systems that can be compared with observations. Nonetheless, it should also be noted that the formalism is always subject to refinements. For example, it would be useful to expand upon this perturbative approach to incorporate the quadrupole moment interaction of these otherwise pointlike particles, which would undoubtedly contribute to non-trivial motion under more extreme physical situations, such as orbital dynamics near the event horizon of a black hole. As well, the formalism does not take into account backreaction effects due to the gravitational self-force, which seems relevant for addressing more sophisticated considerations than outlined in this paper. These modifications do not seem very difficult to incorporate within the existing formalism. In its present form, however, there is already a lot of possibilities for useful applications to follow.

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