Spin and Quadrupole Contributions to the Motion of Astrophysical Binaries

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Abstract Compact objects in general relativity approximately move along geodesics of spacetime. It is shown that the corrections to geodesic motion due to spin (dipole), quadrupole, and higher multipoles can be modeled by an extension of the point mass action. The quadrupole contributions are discussed in detail for astrophysical objects like neutron stars or black holes. Implications for binaries are analyzed for a small mass ratio situation. There quadrupole effects can encode information about the internal structure of the compact object, e.g., in principle they allow a distinction between black holes and neutron stars, and also different equations of state for the latter. Furthermore, a connection between the relativistic oscillation modes of the object and a dynamical quadrupole evolution is established.

1 Introduction

The problem of motion is among of the most fundamental ones in general relativity. As a part of the present proceedings on "Equations of Motion in Relativistic Gravity" this does probably not require any explanations. The problem is addressed using multipolar approximation schemes, the most prominent are due to Mathisson [1, 2] and Dixon [3], and another one is due to Papapetrou [4]. These particular methods have in common that equations of motion for extended bodies are derived from the conservation of energy-momentum. In the present contribution, the focus lies on theoretical models for compact stars and black holes based on point-particle actions. There equations of motions follow from a variational principle instead of conservation of energy-momentum. These point-particle actions were probably first discussed in general relativity by Westpfahl [5] for the case of a pole-dipole particle and later generalized by Bailey and Israel [6] to generic multipoles.

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However, without further justification, it is not obvious how a point-particle action relates to an actual extended body. Most important is the effacing principle [7], which indicates that a nonrotating star can be represented by a point mass up to a high order within the post-Newtonian approximation. (More details on the use of point-masses for self-gravitating bodies within this approximation can be found in other contributions to these proceedings, see, e.g., the contribution by G. Schäfer) This suggests that extensions of the point-mass action can serve as models for extended bodies, even in the self-gravitating case. A similar conclusion arises from the framework of effective field theory applied to gravitating compact bodies [8]. Indeed, the effective action belonging to a compact body naturally takes on the form of a point-particle action, which puts previous works on similar actions [5, 6] into a different light. This provides enough motivation for us to further elaborate the action approach of [6] in Sect. 2, where it is combined with useful aspects of more recent literature [9–13]. An application to the post-Newtonian approximation of self-gravitating extended bodies is omitted, because various formalisms exist for it and the aim is to highlight aspects that are independent of (and hopefully useful for) all of them.

It is worth mentioning that the effective field theory framework offers a machinery which can be used, at least in principle, to *compute* the effective point-particle action from a complete microphysical description of the extended body. In practice, however, this procedure is not viable for realistic astrophysical objects and one must be satisfied with a more phenomenological construction of the effective action. This is in fact analogous to other situations in physics. For instance, it is usually admitted that thermodynamic potentials can be derived from a microscopic description. Yet an explicit calculation is often too complicated, or the microscopic description is even unclear. But a phenomenological construction of thermodynamic potentials or equations of state is usually possible. This analogy is further elaborated in Sect. 5. There an adiabatic quadrupole deformation due to spin [14] is discussed. An application to a binary system in the extreme mass ratio case is given. Quadrupole deformation due to an external gravitational fields is discussed in Sects. 4, 5 and 6, both in an adiabatic [15–17] and a dynamical situation [18, 19].

A main critique against point-particles arises from the fact that Dirac delta distributions are ill-defined sources for the nonlinear Einstein equations. But the situation changes once one softens the Dirac delta using regularization techniques. It is then possible to solve the field equations iteratively within some approximation, like the post Newtonian one. If one regards the chosen regularization prescription as a part of the phenomenological model, then point-particles must be accepted as viable sources in general relativity (at least for applications within approximation schemes). This point is further stressed in Sect. 6.4. It is important that a weak field approximation for the point-particle mimics the field of the actual self-gravitating extended body away from the source. (This is precisely the criterion for the phenomenological construction of the effective point-particle source.) Hence, though one applies the effective source to weak field approximations, e.g., to compute predictions for a binary, strong-field effects from the interior of the bodies are taken into account.

The signature of spacetime is taken to be $+2$. Units are such that the speed of light c is equal to one. The gravitational constant is denoted by G. We are going to utilize three different frames, denoted by different indices. Greek indices refer to the coordinate frame, lower case Latin indices from the beginning of the alphabet belong to a local Lorentz frame, and upper case Latin indices from the beginning of the alphabet denote the so called body-fixed Lorentz frame. Round and square brackets are used for index symmetrization and antisymmetrization, respectively, e.g., $A^{(\mu\nu)} \equiv \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu})$. The convention for the Riemann tensor is

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\beta,\alpha} - \Gamma^{\mu}{}_{\nu\alpha,\beta} + \Gamma^{\rho}{}_{\nu\beta}\Gamma^{\mu}{}_{\rho\alpha} - \Gamma^{\rho}{}_{\nu\alpha}\Gamma^{\mu}{}_{\rho\beta}. \tag{1}$$

2 Point-Particle Actions

Action principles for spinning point particles have a long tradition, see, e.g., [5, 6, 9–13, 20–24]. In this section, the advantages from several of these references are brought together. Our approach is most similar to [6]. Compared to the presentation in [11], a simpler (manifestly covariant) variation technique is applied and the transition to tetrad gravity is discussed at a later stage. This makes the derivation more transparent.

2.1 Manifestly Covariant Variation

Before we start to formulate the action principle, let us introduce a useful notation due to DeWitt [25], see also [12, Appendix A]. One can define a linear operator $G^{\nu}{}_{\mu}$ such that the covariant derivative ∇_α and the Lie derivative $\mathcal{L}_\xi$ read

$$\nabla_\alpha := \partial_\alpha + \Gamma^{\mu}{}_{\nu\alpha}G^{\nu}{}_{\mu}, \qquad \mathcal{L}_\xi := \xi^\mu\partial_\mu - (\partial_\nu\xi^\mu)G^{\nu}{}_{\mu}. \tag{2}$$

For instance, $G^{\nu}{}_{\mu}$ operates on a tensor $T_\alpha{}^\beta$ as $G^{\nu}{}_{\mu}T_\alpha{}^\beta := -\delta^\nu_\alpha T_\mu{}^\beta + \delta^\beta_\mu T_\alpha{}^\nu$. That is, $G^{\nu}{}_{\mu}$ is a linear operator that acts on the spacetime indices of a tensor. Notice that $G^{\nu}{}_{\mu}$ does not act on indices of the body-fixed frame. Further, $G^{\nu}{}_{\mu}$ obeys a product rule like a differential operator. Similarly, we can construct a covariant differential D and a covariant variation Δ of quantities defined along a worldline z^α by

$$D := d + \Gamma^{\mu}{}_{\nu\alpha}(dz^\alpha)G^{\nu}{}_{\mu}, \qquad \Delta := \delta + \Gamma^{\mu}{}_{\nu\alpha}(\delta z^\alpha)G^{\nu}{}_{\mu}. \tag{3}$$

For scalars the contributions from the connection vanish. Notice that a variation of the worldline δz^α is not manifestly covariant if the component values of tensors defined on the worldline are held fixed. The variation Δ instead parallel transports to the varied worldline. When it is applied to a tensor *field* taken at the worldline,

e.g., $T_\alpha{}^\beta(z)$, then the variation δ splits into a part due to the shift of the worldline δz^ρ and a part coming from the variation of the field itself. Let us denote the latter part by $\delta_z T_\alpha{}^\beta := (\delta T_\alpha{}^\beta)(z)$, so we have

$$\delta \equiv \delta_z + (\delta z^\rho)\partial_\rho, \qquad \Delta \equiv \delta_z + (\delta z^\alpha)\nabla_\alpha, \qquad \text{(for fields)}. \tag{4}$$

For instance, the metric compatibility of ∇_α then leads to

$$\Delta g_{\mu\nu} = \delta_z g_{\mu\nu}. \tag{5}$$

2.2 Action Principle

We envisage an action principle localized on a worldline $z^\rho(\lambda)$. Here λ is an arbitrary parameter, not necessarily identical to the proper time τ. (Let us require that the action is invariant under reparametrizations of the worldline.) We further assume that the action is varied with respect to a "body-fixed" frame defined by Lorentz-orthonormal basis vectors $\Lambda_A{}^\mu(\lambda)$ labeled by A,

$$\Lambda_A{}^\mu \Lambda_B{}^\nu g_{\mu\nu} \equiv \eta_{AB}, \qquad \Lambda_A{}^\mu \Lambda_B{}^\nu \eta^{AB} \equiv g^{\mu\nu}. \tag{6}$$

Now, stars are in general differentially rotating and it is difficult to interpret a body-fixed frame. Such a frame is thus rather an abstract element of our theoretical model, inspired by the Newtonian theory of rigid bodies (see, e.g., [11, Sect. 3.1.1]).

The constraint (6) implies that $\Lambda_A{}^\mu$ and $g_{\mu\nu}$ are in general not independent and one should take special care when both are varied at the same time. In order to address this problem, we split the variation $\Delta\Lambda_A{}^\nu$ as

$$\Lambda^{A\mu}\Delta\Lambda_A{}^\nu = \Lambda^{A[\mu}\Delta\Lambda_A{}^{\nu]} + \frac{1}{2}\Delta(\Lambda^{A\mu}\Lambda_A{}^\nu) = \Delta\Theta^{\mu\nu} - \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}\delta_z g_{\alpha\beta}. \tag{7}$$

where we used $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$ and (5). In the last step, we also introduced the abbreviation

$$\Delta\Theta^{\mu\nu} := \Lambda^{A[\mu}\Delta\Lambda_A{}^{\nu]}, \tag{8}$$

which is similar to the antisymmetric variation symbol used in [22], see also [13, Eq. (2.7)]. The independence of $\Delta\Theta^{\mu\nu}$ from the metric variation $\delta_z g_{\alpha\beta}$ will be made more manifest in Sect. 2.4. For now let us just appeal to the fact that the 6 degrees of freedom of the antisymmetric symbol $\Delta\Theta^{\mu\nu}$ exactly matches the degrees of freedom of a Lorentz frame (3 boosts and 3 rotations). Thus $\Delta\Theta^{\mu\nu}$ corresponds to the independent variation of the body-fixed Lorentz frame.

Let us consider an action that is as generic as possible,

$$W = W_F + W_M, \qquad W_F[g_{\mu\nu}, \ldots] = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R + \ldots, \tag{9}$$

$$W_M[g_{\mu\nu}, z^\rho, \Lambda_A{}^\mu, \ldots] = \int d\lambda\, L_M(g_{\mu\nu}, u^\mu, \Lambda_A{}^\mu, \Omega^{\mu\nu}, \phi_I), \tag{10}$$

Here ϕ_I collectively denotes other dependencies of the Lagrangian L_M and the dots denote other fields, like the electromagnetic one. (In this section I is a multi-index that may comprise any sort of spacetime, Lorentz, or label indices.) Notice that fields like $g_{\mu\nu}$ are taken at the worldline position z^ρ in W_M. The 4-velocity u^μ and the angular velocity $\Omega^{\mu\nu}$ are defined by

$$u^\mu := \frac{dz^\mu}{d\lambda}, \qquad \Omega^{\mu\nu} := \Lambda^{A\mu}\frac{D\Lambda_A{}^\nu}{d\lambda}, \tag{11}$$

Notice that $\Omega^{\mu\nu}$ is antisymmetric due to (6) and $Dg_{\mu\nu}/d\lambda = 0$.

2.3 Variation

For the sake of deriving equations of motion, we may assume $\delta\lambda = 0$. Then the variation can be commuted with ordinary or partial λ-derivatives. Furthermore, the Lagrangian L_M is a scalar and we can make use of $\delta L_M \equiv \Delta L_M$ to write its variation in a manifestly covariant manner,

$$\delta L_M = p_\mu \Delta u^\mu + \frac{1}{2}S_{\mu\nu}\Delta\Omega^{\mu\nu} + \frac{\partial L_M}{\partial \Lambda_A{}^\mu}\Delta\Lambda_A{}^\mu + \frac{\partial L_M}{\partial g_{\mu\nu}}\Delta g_{\mu\nu} + \frac{\partial L_M}{\partial \phi_I}\Delta\phi_I, \tag{12}$$

where we have defined the linear momentum p_μ and spin $S_{\mu\nu} = -S_{\nu\mu}$ as generalized momenta belonging to the velocities u^μ and $\Omega^{\mu\nu}$,

$$p_\mu := \frac{\partial L_M}{\partial u^\mu}, \qquad S_{\mu\nu} := 2\frac{\partial L_M}{\partial \Omega^{\mu\nu}}. \tag{13}$$

It should be noted that (12) can be checked using a usual variation δ together with the identity (21), but here it is a simple consequence of the chain rule for Δ. Obviously this method nicely organizes the Christoffel symbols.

The 5 individual terms in (12) are transformed as follows:

- The 1st term of (12) is evaluated with the help of

$$\Delta u^\mu \equiv \delta u^\mu + \Gamma^\mu{}_{\alpha\beta}u^\alpha\delta z^\beta = \frac{D\delta z^\mu}{d\lambda}. \tag{14}$$

- The 2nd term of (12) requires the most work. In order to evaluate $\Delta\Omega^{\mu\nu}$, we need to commute Δ with the covariant differential D contained in $\Omega^{\mu\nu}$, Eq. (11). The definitions in (3) lead to

$$[\Delta, D] = [(\delta_z \Gamma^{\mu}{}_{\nu\alpha}) - (\delta z^{\beta}) R^{\mu}{}_{\nu\alpha\beta}](dz^{\alpha}) G^{\nu}{}_{\mu}. \tag{15}$$

Notice the analogy to the commutator of covariant derivatives, which also gives rise to curvature. It is useful to derive intermediate commutators first, for instance

$$[G^{\nu}{}_{\mu}, G^{\beta}{}_{\alpha}] = \delta^{\beta}_{\mu} G^{\nu}{}_{\alpha} - \delta^{\nu}_{\alpha} G^{\beta}{}_{\mu}. \tag{16}$$

Next, we express $\delta_z \Gamma^{\mu}{}_{\nu\alpha}$ in (15) with the help of

$$\delta\Gamma^{\nu}{}_{\beta\alpha} = \frac{1}{2} g^{\nu\rho} \left[\nabla_{\beta} \delta g_{\alpha\rho} + \nabla_{\alpha} \delta g_{\beta\rho} - \nabla_{\rho} \delta g_{\alpha\beta} \right]. \tag{17}$$

Now it is straightforward to evaluate $\Delta\Omega^{\mu\nu}$. In the result, we replace $\Delta\Lambda_A{}^{\mu}$ using (7), make use of

$$\frac{D\delta_z g_{\rho\sigma}}{d\lambda} = u^{\alpha} (\nabla_{\alpha} \delta g_{\rho\sigma})(z), \tag{18}$$

and finally arrive at

$$\begin{aligned} \Delta\Omega^{\mu\nu} = {} & \frac{D(\Delta\Theta^{\mu\nu})}{d\lambda} + 2\Omega_{\alpha}{}^{[\mu} \Delta\Theta^{\nu]\alpha} + R^{\mu\nu}{}_{\alpha\beta} u^{\alpha} \delta z^{\beta} \\ & + \Omega^{\alpha[\mu} g^{\nu]\beta} \delta_z g_{\alpha\beta} + g^{\beta[\mu} g^{\nu]\rho} u^{\alpha} (\nabla_{\beta} \delta g_{\rho\alpha})(z). \end{aligned} \tag{19}$$

- Before proceeding to the 3rd term of (12), let us recall the transformation property of a tensor under an infinitesimal coordinate transformation $x^{\mu'} = x^{\mu} - \xi^{\mu}$,

$$\phi_{I'} - \phi_I = -(\partial_{\nu} \xi^{\mu}) G^{\nu}{}_{\mu} \phi_I, \quad \text{e.g.,} \quad u^{\mu'} - u^{\mu} = -u^{\nu} \partial_{\nu} \xi^{\mu}. \tag{20}$$

The Lagrangian is a scalar and thus invariant, but it depends on tensors which transform. As $\partial_{\nu}\xi^{\mu}$ is quite arbitrary, the invariance of the Lagrangian L_M leads to the identity

$$p_{\mu} u^{\nu} + S_{\mu\alpha} \Omega^{\nu\alpha} + \frac{\partial L_M}{\partial \Lambda_A{}^{\mu}} \Lambda_A{}^{\nu} - 2 \frac{\partial L_M}{\partial g_{\nu\alpha}} g_{\mu\alpha} + \frac{\partial L_M}{\partial \phi_I} G^{\nu}{}_{\mu} \phi_I \equiv 0. \tag{21}$$

We eliminate the partial derivative of L_M with respect to $\Lambda_A{}^{\mu}$ using this relation and we replace $\Delta\Lambda_A{}^{\mu}$ using (7) to arrive at

$$\frac{\partial L_M}{\partial \Lambda^{A\mu}}\Delta\Lambda^{A\mu} = \frac{1}{2}\left[p^\mu u^\nu - S_\alpha{}^\mu \Omega^{\nu\alpha} + (G^{\mu\nu}\phi_I)\frac{\partial L_M}{\partial \phi_I} - 2\frac{\partial L_M}{\partial g_{\mu\nu}}\right]\delta_z g_{\mu\nu} + \left[p_\mu u_\nu - S_{\alpha\mu}\Omega_\nu{}^\alpha - (G_{\mu\nu}\phi_I)\frac{\partial L_M}{\partial \phi_I}\right]\Delta\Theta^{\mu\nu}. \tag{22}$$

- In the 4th term of (12) we use (5).
- The 5th term of (12) is not touched for now, as this requires a specialization of ϕ_I. This is discussed in the Sect. 3.1.

All these transformations are now applied to (12). Furthermore, we insert a unity in the form of

$$1 \equiv \int d^4x\, \delta_{(4)}, \qquad \delta_{(4)} := \delta(x^\mu - z^\mu), \tag{23}$$

into the terms containing field variations of type δ_z. This allows one to rewrite these variations at the spacetime point x^μ and perform partial integrations. Notice that $\delta_{(4)}$ has compact support for finite λ-intervals, so these partial integrations do not require assumptions on field variations at the spatial boundary. Finally, (12) turns into

$$\begin{aligned}\delta L_M = \int d^4x \Bigg[& p^\mu u^\nu \delta_{(4)} + (G^{\mu\nu}\phi_I)\frac{\partial L_M}{\partial \phi_I}\delta_{(4)} - \nabla_\alpha(S^{\alpha\mu}u^\nu\delta_{(4)})\Bigg]\frac{\delta g_{\mu\nu}(x)}{2} \\ & + \frac{\partial L_M}{\partial \phi_I}\Delta\phi_I + \left[p_\mu u_\nu - (G_{\mu\nu}\phi_I)\frac{\partial L_M}{\partial \phi_I} - \frac{1}{2}\frac{DS_{\mu\nu}}{d\lambda}\right]\Delta\Theta^{\mu\nu} \\ & + \left[\frac{1}{2}S_{\alpha\beta}R^{\alpha\beta}{}_{\rho\mu}u^\rho - \frac{Dp_\mu}{d\lambda}\right]\delta z^\mu + \frac{d}{d\lambda}\left[p_\mu \delta z^\mu + \frac{1}{2}S_{\mu\nu}\Delta\Theta^{\mu\nu}\right].\end{aligned} \tag{24}$$

Notice that the covariant and ordinary derivatives with respect to λ are identical for the last term.

2.4 Metric Versus Tetrad Gravity

The separation of metric and body-fixed-frame variations by means of (7) is an elegant trick to derive equations of motion. However, if further calculations at the level of the action are performed, one often needs an explicit split between gravitational and body-fixed-frame degrees of freedom. This can be achieved by introducing a tetrad gravitational field $e_a{}^\mu(x)$, that is, a field of Lorentz-orthonormal basis vectors labeled by a and defined at every spacetime point x^ρ,

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} \equiv \eta_{ab}, \qquad e_a{}^\mu e_b{}^\nu \eta^{ab} \equiv g^{\mu\nu}. \tag{25}$$

Tetrad gravity replaces the metric by virtue of the latter relation and regards $e_a{}^\mu$ as the fundamental gravitational field in the variation principle. This allows us to split $\Lambda_A{}^\mu$ as

$$\Lambda_A{}^\mu = \Lambda_A{}^a e_a{}^\mu(z), \tag{26}$$

where $\Lambda_A{}^a$ is now just a usual (flat-spacetime) Lorentz matrix

$$\Lambda_A{}^a \Lambda_B{}^b \eta_{ab} \equiv \eta_{AB}, \qquad \Lambda_A{}^a \Lambda_B{}^b \eta^{AB} \equiv \eta^{ab}. \tag{27}$$

Thus $\Lambda_A{}^a$ is independent of the gravitational field $e_a{}^\mu$ and the announced manifest split is indeed given by (26).

Based on this split, we can understand the meaning of $\Delta\Theta^{\mu\nu}$ in more detail. As $\Lambda_A{}^a$ is a usual Lorentz matrix, we can follow Ref. [22] and describe its independent variations by an antisymmetric symbol $\delta\theta^{ab} := \Lambda^{Aa}\delta\Lambda_A{}^b$. Then $\Delta\Theta^{\mu\nu}$ reads explicitly

$$\Delta\Theta^{\mu\nu} = e_a{}^\mu e_b{}^\mu \delta\theta^{ab} + \left(\Gamma^{[\nu\mu]}{}_\alpha + e^{a[\mu}\partial_\alpha e_a{}^{\nu]}\right)\delta z^\alpha + e^{a[\mu}\delta_z e_a{}^{\nu]}. \tag{28}$$

One can be even more explicit and write $\delta\theta^{ab}$ as a linear combination of six independent variations of angle variables parameterizing Λ^{Aa}, see [22, Sect. 3.A]. Anyway, $\Delta\Theta^{\mu\nu}$ is in fact a linear combination of the independent frame variations $\delta\theta^{ab}$ with other variations. Now, it is legitimate to regard $\Delta\Theta^{\mu\nu}$, δz^α, and $\delta e_a{}^\mu$ as independent variations instead of $\delta\theta^{ab}$, δz^α, and $\delta e_a{}^\mu$. This just corresponds to a linear recombining of the equations of motion following from the variation. Equation (28) shows that this recombination manifestly removes noncovariant terms related to the δz^α-variation and an antisymmetric part of the energy-momentum tensor due to $e^{a[\mu}\delta_z e_a{}^{\nu]}$ (the symmetric part arises from $e^{a(\mu}\delta e_a{}^{\nu)} = \frac{1}{2}\delta g^{\mu\nu}$ as usual). All of this is important for the next section, where equations of motion are deduced from (24).

In the next step it is possible to return to metric gravity by a partial gauge fixing of the tetrad. For instance, a possible gauge condition is to require that the matrix $(e_{a\mu})$ is symmetric (in spite of the different nature of its indices). Then $e_a{}^\mu$ is given by the matrix square-root of the metric. This gauge choice leads to the same conclusions as in [9, Sect. IV.B], where a more direct construction was followed. In the end, the partially gauge-fixed tetrad is a function of the metric, so we have obtained a metric gravity theory. It might look like the introduction of a tetrad field accompanied by an enlarged gauge group of gravity is just extra baggage. However, more gauge freedom is important for applications, as gauges can and should be adopted to the problem at hand. For instance, for an ADM-like canonical formulation of spinning particles, it is a wise choice to adopt the Schwinger time-gauge for the tetrad [10]. Further, some subtle aspects of the consistency of the theory can be analyzed more easily within tetrad gravity (e.g., the algebra of gravitational constraints, because after reduction to metric gravity the gravitational field momentum receives complicated corrections [10]). Spinning particles should always be coupled to tetrad gravity in the first place.

3 Equations of Motion

In order to draw conclusions from (24), one must further specialize the so far arbitrary ϕ_I. The assumptions we are going to introduce in the following are not the least restrictive, but already allow important insights on the structure of the equations of motion.

3.1 Further Assumptions

Let us assume from now on that the ϕ_I can be split into two groups. We denote by ϕ_I^{field} the part that contains spacetime fields (functions of x), so its variation $\Delta\phi_I^{\text{field}}$ can be evaluated using (4). The second group ϕ_I^{wl} contains variables defined on the worldline only (functions of λ) and its first order derivatives $\dot{\phi}_I^{\text{wl}}$, where $\dot{} := D/d\lambda$. Most importantly, we assume that the $\delta\phi_I^{\text{wl}}$ correspond to independent variational degrees of freedom, like Lagrange multipliers or the dynamical multipoles introduced in Sect. 6. Without loss of generality, one can then assume that the ϕ_I^{wl} carry indices of the body-fixed frame instead of spacetime indices, so that $G^{\nu}{}_{\mu}\phi_I^{\text{wl}} = 0$.

Notice that our assumptions do not allow time (i.e., λ) derivatives of u^{μ} and $\Omega^{\mu\nu}$ as part of the ϕ_I^{wl}. If such accelerations would appear in subleading contributions of the Lagrangian (within some approximation scheme), then one can often remove them by a redefinition of variables [26]. Further, acceleration-dependent Lagrangians are often problematic due to Ostrogradsky instability. For these reasons, we also assumed that at most first-order time derivatives of the ϕ_I^{wl} appear in L_M. However, our assumptions here are not entirely exhaustive. For instance, a concrete situation for which our assumptions should be relaxed in the future is discussed at the end of Sect. 5.1.

3.2 Equations of Motion for Linear Momentum and Spin

With these assumptions, we have

$$\frac{\partial L_M}{\partial \phi_I}\Delta\phi_I = \frac{\partial L_M}{\partial \phi_I^{\text{field}}}\left[\delta_z\phi_I^{\text{field}} + (\delta z^{\alpha})\nabla_{\alpha}\phi_I^{\text{field}}\right] + \left[\frac{\partial L_M}{\partial \phi_I^{\text{wl}}} - \frac{d\psi^I_{\text{wl}}}{d\lambda}\right]\delta\phi_I^{\text{wl}} + \frac{d}{d\lambda}\left[\psi^I_{\text{wl}}\delta\phi_I^{\text{wl}}\right], \tag{29}$$

where we used that the worldline variables do not carry spacetime indices and we introduced their canonical generalized momenta,

$$\psi^I_{\text{wl}} := \frac{\partial L_M}{\partial \dot{\phi}_I^{\text{wl}}}. \tag{30}$$

The second line leads to the usual Euler-Lagrange equations for the worldline degrees of freedom ϕ_I^{wl}, which are discussed in Sect. 6.4. Let us focus on the other terms for now. Using the arbitrariness and independence of δz^μ and $\Delta\Theta^{\mu\nu}$, we can read off the equations of motion for the linear momentum and the spin from (24) and (29),

$$\frac{Dp_\mu}{d\lambda} = \frac{1}{2} S_{\alpha\beta} R^{\alpha\beta}{}_{\rho\mu} u^\rho + (\nabla_\mu \phi_I^{\mathrm{field}}) \frac{\partial L_M}{\partial \phi_I^{\mathrm{field}}}, \tag{31}$$

$$\frac{DS_{\mu\nu}}{d\lambda} = 2p_{[\mu} u_{\nu]} - 2(G_{[\mu\nu]} \phi_I^{\mathrm{field}}) \frac{\partial L_M}{\partial \phi_I^{\mathrm{field}}}. \tag{32}$$

The total λ-derivative in the last line of (24) was ignored here. (Here we assume that the variation vanishes at the end points of the worldline). The energy-momentum tensor density $\sqrt{-g}T^{ab}$ is simply given by the coefficient in front of $\delta g_{\mu\nu}/2$ in (24). However, an explicit determination requires yet another specialization of ϕ_I^{field}, because the fields can dependent on the metric. In the absence of ϕ_I^{field}, one immediately recovers the result of Tulczyjew [27].

3.3 Quadrupole

Let us now explore the case that $\phi_I^{\mathrm{field}} = \{R_{\mu\nu\alpha\beta}\}$. It is useful to introduce an abbreviation for the corresponding partial derivative of the Lagrangian,

$$J^{\mu\nu\alpha\beta} := -6 \frac{\partial L_M}{\partial R_{\mu\nu\alpha\beta}}. \tag{33}$$

The conventional factor of -6 is motivated by comparing (31), now reading

$$\frac{Dp_\mu}{d\lambda} = \frac{1}{2} S_{\alpha\beta} R^{\alpha\beta}{}_{\rho\mu} u^\rho - \frac{1}{6} \nabla_\mu R_{\nu\rho\beta\alpha} J^{\nu\rho\beta\alpha}, \tag{34}$$

with the corresponding result of Dixon at the quadrupolar approximation level. An identification of $J^{\mu\nu\alpha\beta}$ with Dixon's reduced quadrupole moment is tempting, as this makes (34) formally identical to Dixon's result. It is important that $J^{\mu\nu\alpha\beta}$ inherits the symmetries of the Riemann tensor. From these symmetries and the properties of the operator $G^\mu{}_\nu$, we obtain

$$J^{\sigma\rho\alpha\beta} G^\mu{}_\nu R_{\sigma\rho\alpha\beta} = -4 J^{\sigma\rho\alpha\beta} \delta^\mu_\sigma R_{\nu\rho\alpha\beta} = -4 J^{\mu\rho\alpha\beta} R_{\nu\rho\alpha\beta}. \tag{35}$$

This simplifies (32) to

$$\frac{DS_{\mu\nu}}{d\lambda} = 2p_{[\mu} u_{\nu]} + \frac{4}{3} R_{\alpha\beta\rho[\mu} J_{\nu]}{}^{\rho\beta\alpha}, \tag{36}$$

and formally agrees with Dixon's spin equation of motion, too. Finally, the energy-momentum tensor agrees with the explicit result in [28] (in the present conventions, see (5.3) in [11]). This derives from

$$\delta R^{\mu}{}_{\nu\alpha\beta} = \nabla_\alpha \delta\Gamma^{\mu}{}_{\nu\beta} - \nabla_\beta \delta\Gamma^{\mu}{}_{\nu\alpha}, \tag{37}$$

which must be further expanded using (17) and then leads to

$$\frac{\partial L_M}{\partial R_{\mu\nu\alpha\beta}} \delta_z R_{\mu\nu\alpha\beta} = \int d^4x \left[-\frac{1}{3} J^{\mu\rho\alpha\beta} R^{\nu}{}_{\rho\alpha\beta} \delta_{(4)} - \frac{2}{3} \nabla_\beta \nabla_\alpha (J^{\mu\alpha\beta\nu} \delta_{(4)}) \right] \frac{\delta g_{\mu\nu}}{2}. \tag{38}$$

This is the contribution coming from the first term in (29). Another contribution arises from the second term in the first row of (24), which is evaluated using (35). Collecting all terms in front of $\delta g_{\mu\nu}$ in (24), we can read off the energy momentum tensor density as

$$\begin{aligned} \sqrt{-g} T^{\mu\nu} = \int d\lambda \Big[& u^{(\mu} p^{\nu)} \delta_{(4)} - \nabla_\alpha (S^{\alpha(\mu} u^{\nu)} \delta_{(4)}) \\ & + \frac{1}{3} R_{\beta\alpha\rho}{}^{(\mu} J^{\nu)\rho\alpha\beta} \delta_{(4)} - \frac{2}{3} \nabla_\beta \nabla_\alpha (J^{\mu(\alpha\beta)\nu} \delta_{(4)}) \Big]. \end{aligned} \tag{39}$$

3.4 Other Multipoles

Dixon's moments are essentially defined as integrals over the energy-momentum tensor of the extended body. Though these definitions can be applied to self-gravitating bodies, the derivation of the equations of motions based on these definitions only succeeds for test-bodies [3]. It was shown in [29] (see also the corresponding contribution by A. Harte in these proceedings) using methods for self-force calculations that for self-gravitating objects the equations of motions are still of the same form, but the multipole moments must be renormalized. The multipoles arising from the effective action should therefore be related to these renormalized moments. For self-gravitating bodies, one can not in general calculate the moments in the equations of motion using Dixon's integral formulas any more. In the language of effective field theory, the multipoles are calculated through a "matching" procedure instead, which will be explained in Sect. 5.

Other gravitational multipoles can be incorporated by including symmetrized covariant derivatives of the curvature in ϕ_I^{field}. Similarly, electromagnetic multipoles arise from an analogous construction based on the Faraday tensor $F_{\mu\nu}$. A quite exhaustive case is therefore

$$\phi_I^{\text{field}} = \{R_{\mu\nu\alpha\beta}, \nabla_\rho R_{\mu\nu\alpha\beta}, \nabla_{(\sigma}\nabla_{\rho)} R_{\mu\nu\alpha\beta}, \ldots, F_{\mu\nu}, \nabla_{(\rho} F_{\mu)\nu}, \nabla_{(\sigma}\nabla_\rho F_{\mu)\nu}, \ldots\}. \tag{40}$$

Notice that the commutation of covariant derivatives results in curvature terms and that, e.g., $3\nabla_\alpha F_{\mu\nu} = 2\nabla_{(\alpha}F_{\mu)\nu} - 2\nabla_{(\alpha}F_{\nu)\mu}$, which can be checked using $\nabla_{[\alpha}F_{\mu\nu]} = 0$. Again the partial derivatives of L_M with respect to the ϕ_I^{field} can be called multipole moments. However, these multipoles and also p_μ are probably not unique, because L_M in not unique. For instance, contractions of covariant derivatives with u^μ can be written as λ-derivatives and one can partially integrate them. Notice that Dixon's multipole moments have the same symmetries as ours, but satisfy additional orthogonality relations to a timelike vector defined on the worldline.

It is also possible to include a term proportional to $A_\mu u^\mu$ in the Lagrangian, as this combination transforms into a total λ-derivative under a gauge transformation of the electromagnetic potential A_μ. It just leads to the well-known Lorentz force. However, in the present approach a part of the Lorentz force is hidden in the definition of p_μ, making the equations of motion not manifestly gauge-invariant.

4 Symmetries, Transformations, and Conditions

In this section we discuss symmetries, conservations laws, various transformations of the action, and conditions it must fulfill.

4.1 Symmetries and Conserved Quantities

Action principles have the advantage that one can easily derive conserved quantities from the Noether theorem [30]. Here we are going to consider only symmetry transformations where the fields are not transformed. Further, we assume $\delta\lambda = 0$, so the variational formula (24) together with (29) is still valid.

On the one hand, we require that the Lagrangian transforms under such a symmetry into a total derivative

$$\delta L_M = \frac{dK}{d\lambda}, \tag{41}$$

without making use of the equations of motion. On the other hand, if we assume that the equations of motion hold, then only the total time derivatives from (24) with (29) inserted contribute to δL_M. These total derivatives are located in the last lines of (24) and (29). (The first lines of (24) and (29) vanishes because fields are not transformed here.) We therefore have the conservation law

$$\frac{d}{d\lambda}\left[p_\mu \delta z^\mu + \frac{1}{2}S_{\mu\nu}\Delta\Theta^{\mu\nu} + \psi^I_{\text{wl}}\delta\phi^{\text{wl}}_I - K\right] = 0. \tag{42}$$

A simple example is given by the global symmetry under a change of the body-fixed frame. In order to make things even more simple, we assume that L_M does not

depend on $\Lambda_A{}^\mu$ and on the $\dot{\phi}_I^{\rm wl}$, so that $\psi^I_{\rm wl} = 0$. But L_M still implicitly depends on $\Lambda_A{}^\mu$ through $\Omega^{\mu\nu}$. A constant infinitesimal Lorentz transformation of the body-fixed frame then reads

$$\delta z^a = 0, \qquad \delta\Lambda^{A\mu} = \omega^{AB}\Lambda_B{}^\mu, \tag{43}$$

where ω^{AB} is a constant infinitesimal antisymmetric matrix. Obviously $\Omega^{\mu\nu}$ is invariant under this transformation, so (41) is fulfilled with $K = 0$. Further, we have $\Delta\Theta^{ab} = \Lambda_A{}^a\Lambda_B{}^b\omega^{AB}$ and (42) reads

$$\frac{1}{2}\omega^{AB}\frac{d}{d\lambda}\left[S_{\mu\nu}\Lambda_A{}^\mu\Lambda_B{}^\nu\right] = 0. \tag{44}$$

As ω^{AB} is arbitrary, we see that the components of the spin in the body-fixed frame $S_{AB} \equiv S_{\mu\nu}\Lambda_A{}^\mu\Lambda_B{}^\nu$ are constant. A corollary of this fact is that the spin length S is constant, where $2S^2 = S_{AB}S^{AB} = S_{\mu\nu}S^{\mu\nu}$.

The next important example is a symmetry of the spacetime described by a Killing vector field ξ^μ, $\mathcal{L}_\xi g_{\mu\nu} = 0$. (Notice that also $\mathcal{L}_\xi R_{abcd} = 0$ etc.) Other fields entering L_M are assumed to be invariant under this symmetry, too, e.g., $\mathcal{L}_\xi F_{\mu\nu} = 0$. We consider an infinitesimal shift of the worldline coordinate

$$\delta z^\mu = \epsilon\xi^\mu, \qquad \Delta\Lambda_A{}^\nu = -\epsilon\mathcal{L}_\xi\Lambda_A{}^\nu = \epsilon\Lambda_A{}^\mu\nabla_\mu\xi^\nu, \qquad \delta\phi_I^{\rm wl} = 0, \tag{45}$$

where ϵ is an infinitesimal constant and we assume parallel transport of $\Lambda_A{}^\mu$ along ξ^ν, i.e., $\xi^\nu\nabla_\nu\Lambda_A{}^\mu = 0$. Notice that the fields are not transformed, but their symmetry along ξ^μ is important. Recall that $-\epsilon\mathcal{L}_\xi$ generates an infinitesimal coordinate transformation. Therefore, L_M is invariant under this transformation if all the variables it depends on, including the fields, would be transformed by $-\epsilon\mathcal{L}_\xi$. But the shift (45) only applies to z^μ, $\Lambda_A{}^\mu$, and $\phi_I^{\rm wl}$, so the result of (45) on L_M is exactly opposite to the case when all variables *except* z^μ, $\Lambda_A{}^\mu$, and $\phi_I^{\rm wl}$ are transformed. These variables are all the fields, so the δL_M produced by (45) can be obtained by transforming all the fields using $+\epsilon\mathcal{L}_\xi$. But the fields were assumed to be invariant under this transformation. Hence we have argued that (45) is a symmetry of the action, $\delta L_M = 0$, and $K = 0$. Combining (42) and (45), we find the conserved quantity

$$E_\xi := p_\mu\xi^\mu + \frac{1}{2}S^{\mu\nu}\nabla_\mu\xi_\nu = \text{const}, \tag{46}$$

where $\Delta\Theta^{\mu\nu} = \epsilon\nabla^{[\mu}\xi^{\nu]}$ was used. It is interesting that this covers to all multipole orders. This was also shown in [31, p. 210] based on the equations of motion.

A special kind of conserved quantities that is not covered by the Noether theorem here are mass-like quantities. We will see later on that masses enter the action as parameters and are therefore constant by assumption.

4.2 Legendre Transformations

Before proceeding, it is worth to point out that of course not every conceivable Lagrangian L_M is acceptable. Some choices are mathematically inconsistent or physically unacceptable for other reasons. Some Lagrangians L_M are technically more difficult to handle and it makes sense to assume simplifying conditions for L_M for a first study. One such assumption we make here is that the relation between spin and angular velocity is a bijection. Notice that this relation is fixed by L_M through our definition of $S_{\mu\nu}$ in (13). A violation of our assumption can have the interesting implication that the spin supplementary condition follows from (13), see [22], but we will not considering this scenario here. The supplementary conditions are discussed in the next section.

With this assumption on the relation between spin and angular velocity, we can solve for $\Omega^{\mu\nu}$ in terms of $S^{\mu\nu}$ (and probably other variables). This allows a Legendre transformation in $\Omega^{\mu\nu}$, i.e.,

$$W_M\left[e_a{}^\mu, z^\rho, \Lambda_A{}^\mu, S_{\mu\nu}, \ldots\right] = \int d\lambda \left[\frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + R_M\left(g_{\mu\nu}, u^\mu, \Lambda_A{}^\mu, S_{\mu\nu}, \phi_I\right)\right]. \tag{47}$$

It is important that the spin is varied independently now. Notice that this notion of Legendre transformation is unusual in mechanics, as $\Omega^{\mu\nu}$ is not a time derivative, but a combination of time derivatives. Still Legendre transformations are applicable in much more generic situations, which is heavily used, i.e., in thermodynamics. The function R_M establishes the connection to Routhian approaches [32–34]. The Routhian is a mixture of a Hamiltonian and a Lagrangian. Here it is essentially the sum of R_M and the connection part in $\frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}$. Notice that therefore the Routhian is not manifestly covariant and covariance only becomes apparent at the level of the equations of motion. In contrast, in our construction R_M is manifestly covariant.

A consequence of reparametrization invariance is that L_M must be a homogeneous function of degree one in all (first-order) λ-derivatives. For our assumptions in Sect. 3.1 this applies only to u^μ, $\Omega^{\mu\nu}$, and $\dot{\phi}^{\text{wl}}_I$, so Eulers theorem on homogeneous functions reads

$$L_M = \frac{\partial L_M}{\partial u^\mu} u^\mu + \frac{\partial L_M}{\partial \Omega^{\mu\nu}} \Omega^{\mu\nu} + \frac{\partial L_M}{\partial \dot{\phi}^{\text{wl}}_I} \dot{\phi}^{\text{wl}}_I = p_\mu u^\mu + \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + \psi^I_{\text{wl}} \dot{\phi}^{\text{wl}}_I. \tag{48}$$

This is a consequence of reparametrization-gauge invariance, so in this sense it is analogous to (21), which follows from coordinate-gauge invariance. Let us proceed with a Legendre transformation in u^μ. This is more subtle, as the relation between u^μ and p_μ can not be a bijection. To see this, first notice that (48) can be interpreted as a constraint on the component $p_\mu u^\mu$ of p_μ. This can be formulated as the famous mass-shell constraint

$$p_\mu p^\mu + \mathcal{M}^2 = 0, \tag{49}$$

where $\mathcal{M}$ is called dynamical mass and usually depends on the dynamical variables. Thus the momentum only has three independent components. On the other hand, u^μ has four independent components: three physical and one gauge degree of freedom due to reparametrization invariance in λ. (If we would choose λ to be the proper time, then just 3 components are independent. But the constraint $u^\mu u_\mu = -1$ makes the variational principle more subtle.) That is, the constraint (49) produces a mismatch in degrees of freedom between u^μ and p_μ, so they can not be connected by a bijection. However, the Legendre transformation can in fact be generalized to the case where constraints appear. One can perform the Legendre transformation "as usual" if all constraints are added to the action using Lagrange multipliers [35, 36]. Here we need one Lagrange multiplier α for (49), which together with the three independent components of p_μ provides a total of four independent variables. This exactly matches the four independent degrees of freedom of u^μ. The Lagrange multiplier α isolates the reparametrization-gauge degree of freedom, while p_μ represents the physical degrees of freedom. Again we require L_M to be such that no pathologies for this "constraint" Legendre transformation arise.

Finally, the result of the transformation is

$$W_M\left[{e_a}^\mu, z^\rho, p_\mu, \alpha, S_{\mu\nu}, {\Lambda_A}^\mu, \phi_I^{\mathrm{wl}}, \psi^I_{\mathrm{wl}}, \ldots\right] = \int d\lambda\left[p_\mu u^\mu + \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} + \psi^I_{\mathrm{wl}}\dot{\phi}_I^{\mathrm{wl}} - \frac{\alpha}{2}(p_\mu p^\mu + \mathcal{M}^2)\right], \tag{50}$$

where

$$\mathcal{M} = \mathcal{M}\left(g_{\mu\nu}, p_\mu, {\Lambda_A}^\mu, S_{\mu\nu}, \phi_I^{\mathrm{wl}}, \psi^I_{\mathrm{wl}}, \phi_I^{\mathrm{field}}\right). \tag{51}$$

We assume that we can also Legendre transform in the $\dot{\phi}_I^{\mathrm{wl}}$ without giving rise to further constraints or pathologies. From the variations of p_μ, $S_{\mu\nu}$, and ψ^I_{wl}, we obtain

$$u^\mu = \alpha p^\mu + \frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial p_\mu}, \qquad \Omega^{\mu\nu} = \alpha\frac{\partial\mathcal{M}^2}{\partial S_{\mu\nu}}, \qquad \dot{\phi}_I^{\mathrm{wl}} = \frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial\psi^I_{\mathrm{wl}}}. \tag{52}$$

These are just the inverses to variable transformations used in the Legendre transformations. Because we did not touch the variables $g_{\mu\nu}$, ϕ_I^{wl}, ${\Lambda_A}^\mu$ and ϕ_I^{field}, it is clear that

$$\frac{\partial L_M}{\partial g_{\mu\nu}} \equiv -\frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial g_{\mu\nu}}, \qquad \frac{\partial L_M}{\partial {\Lambda_A}^\mu} \equiv -\frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial {\Lambda_A}^\mu}, \tag{53}$$

$$\frac{\partial L_M}{\partial \phi_I^{\mathrm{wl}}} \equiv -\frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial \phi_I^{\mathrm{wl}}}, \qquad \frac{\partial L_M}{\partial \phi_I^{\mathrm{field}}} \equiv -\frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial \phi_I^{\mathrm{field}}}. \tag{54}$$

The Lagrange multiplier α is determined by choosing a normalization for u^μ, which corresponds to a gauge choice for λ. For a given dynamical mass function $\mathcal{M}$, one can then evaluate the equations of motion (31) and (32).

Coming back to the plan outlined in the introduction, we have the option to construct either L_M, R_M, or $\mathcal{M}$ in a phenomenological manner. Let us explore the last option here, e.g., because it promotes both p_μ and $S_{\mu\nu}$ to dynamical variables, which are probably easier to identify in realistic situations compared to u^μ and $\Omega^{\mu\nu}$. Further, it is suggestive that the mass $\mathcal{M}$ of the object as a function of the dynamical variables completely determines the macroscopic dynamics of the body. This situation is analogous to a thermodynamic potential (like the internal energy) describing the large-scale behavior of a thermodynamic system. This is the first indication that thinking in terms of thermodynamic analogies is very useful here.

4.3 Supplementary Conditions

The model for spinning bodies developed up to now comprises too many degrees of freedom. We expect three rotational degrees of freedom instead of six provided by the Lorentz frame ${\Lambda_A}^\mu$. Similarly, the spin should only have three independent components, too. It is suggestive to impose that the time direction of the body-fixed frame is aligned to a (to be defined) rest frame described by a unit time-like vector r^μ, and that the spin only has spatial components in this rest frame,

$$ {\Lambda_0}^\mu = r^\mu, \qquad S_{\mu\nu} r^\nu = 0. \tag{55} $$

One can also envision different time-like vectors in each of these conditions. However, using the same vector seems to fit well to the interpretation of r^μ as a rest frame. The condition on the spin is usually called spin supplementary condition. Two specific options are $r^\mu = u^\mu/\sqrt{-u_\nu u^\nu}$ or $r^\mu = p^\mu/\sqrt{-p_\nu p^\nu}$. The latter condition is usually considered as the best choice, as it uniquely fixes the representative worldline of the extended object [37–39] (if Dixon's definitions for the multipoles are used). A more detailed discussion of supplementary conditions is given in the contributions by D. Giulini, L.F. Costa and J. Natário. But notice that in flat space the choice of this condition can be related to the choice of the representative worldline for the extended body. In curved spacetime this relation could not be established yet. From a careful perspective one should therefore reckon that different spin supplementary conditions may lead to inequivalent models. As long as the relation to the choice of center is not clarified for curved spacetimes, one must regard this condition as a constitutive relation of the model. For this reason, one should also avoid conditions which are not manifestly covariant.

The most straightforward way to implement (55) into a given action is to add these conditions using Lagrange multipliers. In general, this will modify the dynamics by constraint forces. As in classical mechanics, one requires that (55) is preserved in time, which should fix the Lagrange multipliers. This can lead to inconsistencies,

in which case one should revise the action or the choice for r^μ. It can also lead to further constraints, which we regard as unphysical here as they further reduces the number of independent variables (we want exactly three rotational degrees of freedom). Similarly, if some of the Lagrange multipliers remain undetermined, then the degrees of freedom are increased, which we also regard as unphysical. The last possibility is that the Lagrange multipliers are uniquely fixed by requiring that (55) is preserved. In the end, we can insert this solution for the Lagrange multipliers into the action. In this way we obtain an action without Lagrange multipliers which preserves (55).

4.4 Conditions on the Dynamical Mass

Having this said, we can try to directly construct an action which preserves (55). This approach is in fact very natural here. For instance, one can make an ansatz for $\mathcal{M}^2$ and use this requirement to fix some of the coefficients. The first condition in (55) can be written as $\eta_{0A} = \Lambda_{A\mu} r^\mu$ and is preserved in time if

$$0 = \frac{Dr^\mu}{d\lambda} + \Omega^{\mu\nu} r_\nu, \qquad \text{where } \Omega^{\mu\nu} = \alpha \frac{\partial \mathcal{M}^2}{\partial S_{\mu\nu}}. \tag{56}$$

Using (21) and (53), we can write the spin equation of motion (32) in the form

$$\frac{DS_{\mu\nu}}{d\lambda} = 2S_{\alpha[\mu}\Omega_{\nu]}{}^{\alpha} + \alpha \frac{\partial \mathcal{M}^2}{\partial \Lambda_A{}^{\alpha}} \delta^{[\mu}_{\alpha} \Lambda_A{}^{\nu]}. \tag{57}$$

With the help of this equation, we see that the spin supplementary condition is preserved in time if it holds (56) and additionally

$$0 = \frac{\partial \mathcal{M}^2}{\partial \Lambda_A{}^{\alpha}} \delta^{[\mu}_{\alpha} \Lambda_A{}^{\nu]} r_\nu. \tag{58}$$

This condition is often trivially fulfilled, namely when $\mathcal{M}^2$ does not explicitly depend on $\Lambda_A{}^\mu$. We are going to construct a simple example now, in order to show that functions $\mathcal{M}$ exist which are consistent with all of our requirements.

4.5 A Simple Construction of the Dynamical Mass

Instead of constructing an action which fulfills (56) and (58) for a specific choice of r^μ, one can look at a specific action and construct a r^μ such that the requirements (56) and (58) are fulfilled. Let us consider a simple example where $\mathcal{M}^2$ is a nonconstant

analytic function f depending only on $S^2 := S^{\mu\nu}S_{\mu\nu}/2$, i.e., $\mathcal{M}^2 = f(S^2)$. It is clear that (58) is fulfilled, because there is no explicit dependence on the body-fixed frame. We still need to satisfy (56), which reads explicitly

$$0 = \frac{Dr^\mu}{d\lambda} + \alpha f' S^{\mu\nu} r_\nu = \frac{Dr^\mu}{d\lambda}. \tag{59}$$

That is, the vector r^μ must be parallel transported along the worldline. This does indeed characterize a suitable spin supplementary condition, which was first discussed in [40, Sect. 3.4]. Although, with this condition, r^μ lacks an immediate interpretation as a rest frame, the numerical results in [40] show that it leads to similar predictions as the choice $r^\mu = p^\mu/\sqrt{-p_\nu p^\nu}$. Further discussions on this supplementary condition are given in other contributions to these proceedings.

For the case of a black hole, the laws of black hole dynamics [41, Box 33.4] suggests that

$$\mathcal{M}^2_{\mathrm{BH}} = f(S^2) = m_0^2 + \frac{S^2}{(2Gm_0)^2}. \tag{60}$$

where m_0 is the constant irreducible mass related to the horizon area. We can now have a look at the angular velocity with respect to asymptotic time, so we have $\alpha = \mathcal{M}^{-1}$ (for a body at rest). Evaluating (52), we find agreement with what is usually identified as the angular velocity of the horizon. This is a nice check for the consistency of the interpretation of our variables. Notice that the laws of black hole dynamics owe their name to their similarity to the laws of thermodynamics. Further, an action principle similar to the one presented here can be used to derive the so called first law of black hole *binary* dynamics [13]. Again we encounter the thermodynamic character of the approach.

For objects other than black holes, we can derive $\mathcal{M}$ from the moment of inertia. One usually defines the moment of inertia $I(S^2)$ as the proportionality factor between spin and angular velocity, which can be read off from (52). Again we have $\alpha = \mathcal{M}^{-1} = f^{-1/2}$, so (52) leads to the differential equation $I^{-1} = f^{-1/2} f'$. Its solution reads

$$\mathcal{M}^2 = f(S^2) = \left[m_0 + \int_0^{S^2} \frac{dx}{2I(x)} \right]^2, \tag{61}$$

where the irreducible mass m_0 enters as an integration constant. For neutron stars, the function $I(S^2)$ can be obtained numerically, e.g., using the RNS code [42, 43]. Alternatively, one can numerically compute the gravitating mass $\mathcal{M}$ directly as a function of S for a fixed number of baryons in the star. It would be interesting to see if both methods lead to compatible results. It should be noted that both black holes and neutron stars posses a quadrupole (and other multipoles) when they are spinning, which was neglected here. It will be included in the next section.

Interestingly, it is implied by [44] that for the pole-dipole case one can construct a $\mathcal{M}^2$ such that (56) and (58) are fulfilled for $r^\mu = p^\mu/\sqrt{-p_\nu p^\nu}$ without the need for approximations or truncations of $\mathcal{M}^2$. Then, however, $\mathcal{M}^2$ is not solely dependent on S. The details on this are left for a future work.

5 Spin-Induced Quadrupole

In this section, we are going to develop a simple phenomenological model for $\mathcal{M}^2$ describing the spin-induced quadrupole of a star. This is the quadrupole of a star arising from a deformation away from spherical symmetry due to rotation. We start with a reasonable ansatz for $\mathcal{M}^2$. The main idea for this ansatz is to include all possible covariant (general coordinate invariant) terms up to a certain power in spin and curvature. The unknown coefficients in this ansatz are then fixed by comparing to the Kerr metric and to numerical solutions for the gravitational field of a rotating neutron star. One should emphasize that a truncation of $\mathcal{M}^2$ requires negligibly small interaction energies, not small multipoles.

5.1 Construction of the Action

We are going to include in our ansatz the quadratic order in spin and second order derivatives of the metric. This means that we include terms linear in the curvature and covariant derivatives of the curvature are not allowed. This implies that we exclude λ-derivatives of the curvature for now, but this restriction will be loosened below. Symbolically we have $\phi_I^{\text{field}} = \{R_{\mu\nu\alpha\beta}\}$, which according to Sect. 3.4 implies that we neglect interaction terms involving octupole and higher multipoles. Finally, let us assume the absence of further worldline degrees of freedom in this section, or $\phi_I^{\text{wl}} = \emptyset$, so we have no need for a dependence of $\mathcal{M}^2$ on $\Lambda_A{}^\mu$. [Then (58) is already fulfilled.]

The main task is to collect all possible interaction terms. One must take care of including only independent terms, which can by tricky due to the symmetries of the Riemann tensor. A procedure for this was applied to the construction of effective Lagrangians or Routhians in [8, 34, 45]. Instead, we are going to construct $\mathcal{M}^2$ directly, but the arguments are essentially the same. We will follow a different approach to implement the spin supplementary condition, too, by making an ansatz for r^μ around the case

$$r^\mu = \frac{p^\mu}{\sqrt{-p_\nu p^\nu}} + \mathcal{O}(R_{\mu\nu\alpha\beta}). \tag{62}$$

As a first simplification, one can replace $R_{\mu\nu\alpha\beta}$ by its tracefree version, the Weyl tensor $C_{\mu\nu\alpha\beta}$. The traces are given by $R_{\mu\nu} := g^{\alpha\beta}R_{\mu\alpha\nu\beta}$ and $R := g^{\mu\nu}R_{\mu\nu}$, which

are related to the energy momentum tensor $T^{\mu\nu}$ through Einstein's gravitational field equations

$$R^{\mu\nu} = 8\pi G\left(T^{\mu\nu} - \frac{1}{2}T^{\alpha}{}_{\alpha}g_{\mu\nu}\right). \tag{63}$$

The energy momentum tensor can contain contributions from fields penetrating the compact object, like electromagnetic or dark matter fields. We assume that these can be neglected, i.e., the bodies are mainly interacting via the gravitational field. But in the case of self-gravitating bodies, the energy momentum tensor also includes a singular contribution from the point-particle (39) itself. Let us assume that these singular self-interactions can be dropped. Then we can effectively make use of the vacuum field equations $R^{\mu\nu} = 0$ at the particle location, so we have $R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta}$. However, in general one is not allowed to use field equations at the level of the action. But in the current context this is essentially a valid procedure, as it is equivalent to a field redefinition in the action, see [8, 26], or [46, Appendix A]. Without loss of generality, we can therefore restrict to $\phi_I^{\text{field}} = \{C_{\mu\nu\alpha\beta}\}$ in the quadrupole case.

The most important and most obvious requirement on the allowed interaction terms in $\mathcal{M}^2$ is general coordinate invariance. Further restrictions on the terms and transformations identifying equivalent terms (equivalent within our truncation) are:

1. In four spacetime dimensions, the Weyl tensor can be split into an electric $E_{\mu\nu}$ and a magnetic part $B_{\mu\nu}$ with respect to a time-like unit vector. Choosing this vector to be r^{μ}, it holds

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}r^{\alpha}r^{\beta}, \qquad B_{\mu\nu} = \frac{1}{2}\eta_{\mu\alpha\rho\sigma}C_{\nu\beta}{}^{\rho\sigma}r^{\alpha}r^{\beta}, \tag{64}$$

   where $\eta_{\mu\nu\alpha\beta}$ is the volume form. These tensors have the properties

$$E_{\mu\nu} = E_{\nu\mu}, \qquad E_{\mu\nu}g^{\mu\nu} = 0, \qquad E_{\mu\nu}r^{\nu} = 0, \tag{65}$$
$$B_{\mu\nu} = B_{\nu\mu}, \qquad B_{\mu\nu}g^{\mu\nu} = 0, \qquad B_{\mu\nu}r^{\nu} = 0. \tag{66}$$

   These properties make $E_{\mu\nu}$ and $B_{\mu\nu}$ much easier to handle compared to $C_{\mu\nu\alpha\beta}$.
2. We include only terms invariant under parity transformations. In this respect it is important to notice that $B_{\mu\nu}$ is of odd parity. We conclude that any terms with an odd number of magnetic Weyl tensors must also include an odd number of volume forms $\eta_{\mu\nu\alpha\beta}$. Due to the antisymmetry of $\eta_{\mu\nu\alpha\beta}$, it will always be contracted with both indices of the spin at the current level of truncation. Then we can rewrite all terms involving $\eta_{\mu\nu\alpha\beta}$ in terms of the dual of the spin tensor

$$*S_{\alpha\beta} := \frac{1}{2}S^{\mu\nu}\eta_{\mu\nu\alpha\beta}. \tag{67}$$

Notice that we have the identity [22, Eq. (A.7)]

$$S_{\mu\alpha} * S^{\alpha\nu} = -\frac{1}{4}\delta^{\nu}_{\mu} S_{\alpha\beta} * S^{\alpha\beta}. \tag{68}$$

As a consequence, it holds $B_{\mu\nu} S^{\mu}{}_{\alpha} * S^{\alpha\nu} = 0$. It is customary to define a spin vector $S^{\mu} := r_{\nu} * S^{\nu\mu}$.

3. It should be noticed that one is in general not allowed to neglect terms involving the combination $S_{\mu\nu} r^{\nu}$, though these numerically vanish due to the spin supplementary condition (55): A variation of these terms can lead to nonvanishing contributions to the equations of motion. Instead, terms in the action which are at least quadratic in $S_{\mu\nu} r^{\nu}$ can be neglected, as their contributions to the equations of motion are at least linear in the spin supplementary condition and thus always vanish.
4. The quadrupole interaction terms can be simplified using the leading order truncation of the mass-shell constraint $p_{\mu} p^{\mu} + \mathcal{M}^2 = 0$. (For the ansatz in (70) given below, this implies that we can set $p_{\mu} p^{\mu} \approx -\mu^2$ in the higher order terms of $\mathcal{M}^2$.) This transformation does in fact just correspond to a redefinition of the Lagrange multiplicator α, and the idea is therefore similar to the field redefinitions mentioned above [26].
5. Time derivatives of p_{μ} and $S_{\mu\nu}$ can also be removed by redefinitions of variables, which follows from the ideas in [26] and is again analogous to the mentioned field redefinitions. Besides that, the absence of higher order time derivatives was already assumed in Sect. 3.1.

The last point also shows that our ansatz will automatically cover time derivatives of $E_{\mu\nu}$ and $B_{\mu\nu}$ of arbitrary order. As we work at linear level in the curvature, the time derivatives can always be removed from the curvature through partial integration. After this transformation, all time derivatives finally apply to p_{μ} and $S_{\mu\nu}$ only, which can be removed by virtue of the argument 5 above. This suggests that these terms belong to the quadrupole level, too, although time derivatives of the fields are in fact covariant derivatives $D/d\lambda = u^{\mu}\nabla_{\mu}$. This is of course related to the ambiguity of the multipoles pointed out in Sect. 3.4. At linear level in the curvature, one can assume that the covariant derivatives are projected orthogonal to r^{μ}, because $r^{\mu}\nabla_{\mu} \approx u^{\mu}\nabla_{\mu} = D/d\lambda$ up to higher order terms, which can be partially integrated.

The first point mentioned above suggests to include just $E_{\mu\nu}$ and $B_{\mu\nu}$ in ϕ_I^{field}. However, this is currently not possible, because we assumed in Sect. 3.1 that the ϕ_I^{field} contain just fields, but r^{μ} in (64) is only defined on the worldline. For instance, one would have to clarify the meaning of $\nabla_{\alpha} r^{\mu}$ arising from $\nabla_{\alpha} E_{\mu\nu}$ in (31). For simplicity, let us stick to $\phi_I^{\text{field}} = \{C_{\mu\nu\alpha\beta}\}$ here, but have in mind that $\mathcal{M}^2$ depends on $C_{\mu\nu\alpha\beta}$ only through the combinations $E_{\mu\nu}$ and $B_{\mu\nu}$. The equations of motion are initially expressed in terms the quadrupole moment related to $C_{\mu\nu\alpha\beta}$,

$$\tilde{J}^{\mu\nu\alpha\beta} := -6\frac{\partial L_M}{\partial C_{\mu\nu\alpha\beta}}. \tag{69}$$

see (33), but these are at once related to the moments belonging to $E_{\mu\nu}$ and $B_{\mu\nu}$ through the chain rule. The interpretation of the latter moments as quadrupoles is much more obvious than for (69), as $E_{\mu\nu}$ and $B_{\mu\nu}$ are symmetric tracefree spatial tensors in the rest frame defined by r^μ. These moments can be called electric and magnetic quadrupoles, respectively. They match the quadrupole degrees of freedom of the gravitational field outside the body [47], in contrast to (33), which in general is not tracefree. This approach to define electric and magnetic quadrupoles was briefly discussed in [11]. An explicit split into electric and magnetic quadrupoles at the level of the equations of motion was performed in [48].

5.2 Ansatz

The most general ansatz for $\mathcal{M}^2$ now reads

$$\mathcal{M}^2 = \mu^2 + C_{BS^2p} B_{\mu\nu} S^\mu S^{\nu\alpha} \hat{p}_\alpha + C_{ES^2} E_{\mu\nu} S^{\mu\alpha} S^\nu{}_\alpha + \mathcal{O}(E^2, B^2, S^3), \tag{70}$$

where we introduced the abbreviation $\hat{p}_\mu = p_\mu/\mu$. We assume that μ, C_{BS^2p}, and C_{ES^2} are constants. Remember that within the curvature terms we can set $r^\mu \approx \hat{p}^\mu$, which is due to (62) and point 4 of the last section. Notice that μ must be a function of the constant spin length, $\mu^2 = f(S^2)$, cf. (61). Otherwise the Legendre transformation would be problematic. Consistent with our truncation, we may write

$$\mu^2 = m_0^2 + \frac{m_0}{I_0} S^2 + \mathcal{O}(S^4), \tag{71}$$

where $I_0 \equiv I(0)$ is the moment of inertia in a slow rotation limit $S \to 0$. Furthermore, the constants C_{BS^2p} and C_{ES^2} will in general depend on μ and S. This is further discussed below.

Next, we want to check if (56) is fulfilled. Notice that (56) is required to hold at linear order in spin only. For this purpose, let us make an ansatz for r^μ to linear order in S,

$$r^\mu = C_{rp} \hat{p}^\mu + G C_{rBS} B^{\mu\nu} S_\nu + \mathcal{O}(E^2, B^2, S^2). \tag{72}$$

The normalization $r^\mu r_\mu = -1$ leads to $C_{rp} = 1$. Inspecting (56), we see that most of the contributions from the C_{rBS}-term are shifted to higher orders, namely quadratic level in spin: This is due to $\Omega = \mathcal{O}(S)$ and $\dot{S}_\nu = \mathcal{O}(S^2)$. The only C_{rBS}-term linear in spin contains $\dot{B}^{\mu\nu}$. Though this is a derivative of the curvature, it is not of higher order, because we realized that our ansatz effectively also covers λ-derivatives of the curvature. We conclude that $C_{rBS} = 0$, or $r^\mu \approx \hat{p}^\mu$. For calculating $\dot{r}^\mu$ in (56), it is useful to rewrite (31) as

$$\frac{Dp_\mu}{d\lambda} = \alpha (E_{\alpha[\mu} S_{\nu]}{}^\alpha - B_{\alpha[\mu} r_{\nu]} S^\alpha) \left[2p^\nu + \frac{\partial \mathcal{M}^2}{\partial p_\nu} \right] - \frac{\alpha}{2} (\nabla_\mu \phi_I^{\text{field}}) \frac{\partial \mathcal{M}^2}{\partial \phi_I^{\text{field}}}. \tag{73}$$

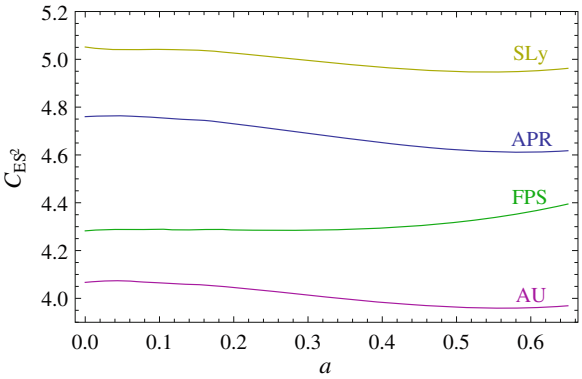

Fig. 1 The coefficient C_{ES^2} as a function of the dimensionless spin $a = S/G\mu^2$, where μ is identified with the gravitating mass and is given by $\mu = 1.4M_\odot$ here. The data points were generated using the RNS code [42, 43], where a multipole extraction according to [49] was used. The labels SLy, APR, FPS, and AU refer to equations of state considered in [50]

Finally, the condition (56) is fulfilled to linear order in spin if $C_{BS^2p} = 2$ in our ansatz (70). The condition (58) is of course also fulfilled. In summary, we must have

$$C_{rp} = 1, \qquad C_{rBS} = 0, \qquad C_{BS^2p} = 2, \tag{74}$$

while C_{ES^2} is not determined by basic principles, but depends on the specific object. Instead of fixing r^μ algebraically like in (72), it would be interesting to view (56) as an evolution equation for r^μ in the future, analogous to (59) in the pole-dipole case.

In Fig. 1 the numerical value of C_{ES^2} is shown as a function of the spin length for fixed mass μ but different neutron star models. It is apparent from the plot that C_{ES^2} is approximately independent of the spin length. However, one should be careful and check this assumption for the specific case of interest. This determination of C_{ES^2} is actually a simple example of a matching procedure. The quadrupole moment J of the effective point particle is parametrized through the ansatz (70) as $J \sim C_{ES^2}S^2$. This is compared (or matched) to the quadrupole moment of a numeric neutron star spacetime computed with the RNS code [42, 43]. Here the quadrupole moment is identified through the exterior spacetime. This means that the effective point particle mimics the exterior spacetime of a numerically constructed neutron star model, which depends crucially on strong field effects in the interior. This makes C_{ES^2} an interesting indicator for both the neutron star equation of state and strong-field modifications of gravity. For black holes, a comparison with the Kerr metric leads to $C_{ES^2} = 1$.

Finally, we come back to the thermodynamic analogy to our approach. The quadrupole relation $J \sim C_{ES^2}S^2$ can be viewed as a simple (idealized) "equation of state" relating the macroscopic variables J and S. As in the case of the ideal gas, this model can be improved to meet the required accuracy. This can be done systematically here by extending the ansatz (70) to higher orders.

5.3 Application

As an application for the spin-induced quadrupole constructed in the last section, we consider the case of a test particle moving in a Kerr spacetime. This test particle can be characterized as a pole-dipole-quadrupole particle. We aim at an estimate for the relevance of the spin-squared contributions, so we may consider a specific orbital configuration that simplifies the discussion. This is obviously a circular orbit in the equatorial plane of the Kerr geometry. Let us further assume that the spin of the test body is aligned with the rotation axis of the background spacetime.

In the absence of a quadrupole, these orbits can be constructed in a simple manner, which was first used in [51]. This method is in fact still applicable for the considered quadrupole model [52]. It requires that conserved quantities, spin supplementary condition, and constraints on the orbital configuration are enough to uniquely fix the 10 dynamic variables contained in p_μ and $S^{\mu\nu}$. This is just an algebraic calculation, in contrast to solving the differential equations of motion. A numeric study for Schwarzschild spacetime is given in [53].

The spin supplementary condition ($S^{\mu\nu}p_\nu = 0$) contains three independent equations. The constraint on the orbit provides three further independent conditions: one due to equatorial orbits ($p^\theta = 0$) and two due to spin alignment ($S^{\mu\theta} = 0$). So we need to identify $10 - 3 - 3 = 4$ conserved quantities in order to solve for p_μ and $S^{\mu\nu}$ algebraically. Three conserved quantities were already identified in Sect. 4.1. These are the spin-length $S := \sqrt{\frac{1}{2}S_{ab}S^{ab}}$ and the quantities derived from the two Killing vectors of Kerr spacetime (∂_t and ∂_θ) through (46). Well call the latter two the energy $E := E_{\partial_t}$ and total angular momentum $J_\phi := E_{-\partial_\phi}$ of the particle. The last remaining conserved quantity is just the mass-like parameter μ, which in the action approach is constant by assumption. However, one should remember that (70) is truncated and thus only approximately valid. One can equivalently say that μ is only conserved approximately, corresponding to the truncation of (70). This point of view was taken in [52].

Now we are in a position to solve for p_a and S^{ab}. Most important is the equation for p^r. After some algebra [52], one finds that $(p^r)^2$ is given by a polynomial of second order in E. We denote the roots of this polynomial by U_+ and U_-, i.e.,

$$\left(p^r\right)^2 \propto (E - U_+)(E - U_-). \tag{75}$$

For p^r to be a real number, we need to have both $E \leq U_+$ and $E \leq U_-$, or both $E \geq U_+$ and $E \geq U_-$. It turns out that the important relation is just $E \geq U_+$ for the most relevant part of the parameter space. This justifies to call U_+ effective potential: The test body can only move in the region where $E \geq U_+$ and its turning points are given by $E = U_+$, because then $p^r = 0$ (which implies $u^r = 0$, see [52]). Therefore the minimum of U_+ as a function of r defines circular orbits. This completes our construction.

The various contributions to the dimensionless binding energy $e := E/\mu - 1$ are plotted in Fig. 2 for the case of a very rapidly rotating (small) black hole in a

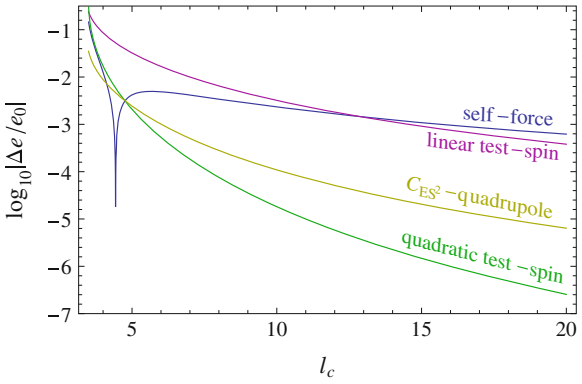

Fig. 2 Various corrections to the binding energy e for a maximally spinning (small) black hole in a Schwarzschild background. Here $l_c := J_\phi - S$ is the *orbital* angular momentum. The mass ratio is formally taken to be $q = 1$ in the plot, though the result are only valid for $q \ll 1$. The *curves* can be scaled to the case of interest ($q \lesssim 10^{-2}$): self-force and linear spin effects scale as $\propto q$, the others as $\propto q^2$

Schwarzschild background. A comparison with recent results for the conservative part of the self-force [54] is also included. In a Kerr background, the last stable circular orbit can be very close to the horizon, so that the discussed effects can be some orders of magnitude stronger. The reader is referred to [52] for a more complete discussion.

6 Dynamical Quadrupole and Tidal Forces

For the model developed in the last section, the quadrupole adiabatically follows the spin evolution. Thus, the quadrupole is not an independent dynamical variable. In this section, we are going to investigate dynamical quadrupoles, but restrict to the nonspinning case for simplicity.

6.1 Basic Idea

We have already discovered that the dynamical mass $\mathcal{M}$ plays a role similar to a thermodynamic potential. From this perspective, one can compare the variables it depends on, like p_μ and $S_{\mu\nu}$, to thermodynamic state variables. Noticing that p_μ and $S_{\mu\nu}$ are the monopole and dipole moment, a natural extension is to introduce dynamical "state" variables for other multipoles, too. A possible motivation arises from the realization that stars have oscillation modes and that these modes can be excited by tidal forces from an external time-dependent gravitational field. This phenomenon is

well understood in Newtonian gravity [55], see also [56–58] and references therein. If one wants to capture it by our approach, one obviously must introduce dynamical worldline variables corresponding to these oscillation modes. Suitable point-particle actions were already discussed in [45, 59], though with applications to absorption or binary systems in mind.

The key to find a model for dynamical multipoles is to understand the reaction of the multipoles to external fields. We focus here on the response of the quadrupole to external tidal fields. In fact, we will encode the quadrupole dynamics in terms of a response function. This function can equivalently be called the propagator of the quadrupole [45], which better highlights the fact that it is a necessary ingredient for deriving predictions using perturbative calculations, e.g., in the post-Newtonian approximation. A third possible naming is correlation function between quadrupole and external field. This better accentuates the parallels to statistical mechanics or thermodynamics. The idea is that if one would be able to model the correlations of the most important multipoles among each other and with external fields, then one can in principle predict the motion of extended objects (with complicated internal structure) to any desired precision.

It is important to notice that the multipole moments of a compact object can be defined through their exterior field. The response functions of the multipoles to externally applied tidal fields can therefore be obtained by analyzing the gravitational field outside of the body. The final goal is to extract these functions from numerical simulations of a *single* compact object. However, for a first simpler investigation one can restrict to linear perturbations of nonrotating compact objects. The unperturbed metric in the exterior is then just the Schwarzschild one. Because this metric is static and spherically symmetric, its linear perturbations can then be decomposed into Fourier basis in the time direction and spherical harmonic basis $Y^{lm}(\theta, \phi)$ in angular directions. Then their radial dependence is described by the famous Zerilli [60] or Regge-Wheeler [61] equations for electric- or magnetic-parity-type perturbations, respectively. The Zerilli equation can be transformed into the simpler Regge-Wheeler form [62], so we can focus just on the latter one. It reads

$$\frac{d^2X}{dr_*^2} + \left[\left(1 - \frac{R_S}{r}\right)\frac{l(l+1) - \frac{3R_S}{r}}{r^2} + \omega^2\right] X = 0, \tag{76}$$

where ω is the frequency of the perturbation, l is the angular momentum quantum number, r is radial coordinate in the Regge-Wheeler gauge, R_S is the Schwarzschild radius (representing the mass of the body), $r_* = r + R_S \log(r/R_S - 1)$ is the tortoise radial coordinate, and X denotes the Regge-Wheeler master function. Given some boundary values for X at the surface of the body (which result from a solution to the more complicated interior perturbation equations), it is straightforward to integrate this equation numerically. The question is how one can decompose X into external (applied) tidal field and multipolar field generated by the body in response to the external field. This is a complicated problem in the general relativistic case. Let us therefore start with the Newtonian theory in order to get a better understanding of the problem [19].

6.2 Newtonian Case

The Newtonian case can be obtained as a weak field and slow motion approximation of general relativity. That is, we have to set $R_S = 0$ (weak field) and $\omega = 0$ (slow motion) in (76). The perturbation of the Newtonian potential Φ_{pert} can be reconstructed as

$$\Phi_{\text{pert}} = -\frac{1}{2\pi}\int d\omega \sum_{lm} e^{i\omega t} Y^{lm} \frac{1}{2}\left[\frac{d}{dr} + \frac{l(l+1)}{2r}\right] X_{lm\omega}, \tag{77}$$

where the $X_{lm\omega}$ are solutions to the Newtonian limit of (76) for all values of the parameters l, m, and ω.

The generic solution to the Newtonian limit of (76) reads

$$X = C_1 r^{l+1} + C_2 r^{-l}, \tag{78}$$

where C_1 and C_2 are integration constants. The r^{l+1} part diverges asymptotically, which means that its source is located at infinity. Therefore, C_1 is the strength of the external field. Similarly, the r^{-l} part is singular at the origin and emanates from the compact body, so C_2 describes the l-polar field of the body. The frequency-domain response $\tilde{F}_l$ of the multipoles to external fields is then proportional to the ratio of C_2 and C_1. In the conventions used in [18, 19], it holds

$$\tilde{F}_l(\omega) = \frac{l(l-1)}{G(l+1)(l+2)(2l-1)!!}\frac{C_2}{C_1} \tag{79}$$

This response must in general be computed numerically. The first step is to numerically solve the interior problem of a perturbed body, including the interior gravitational field perturbation. Then the gravitational field is matched to (78) at the surface, which leads to numeric values for the integration constants and thus for the response (79). This response can in general acquire a complicated frequency dependence through the internal dynamics. Usually one defines normal oscillation modes by requiring that the body keeps up a multipolar field without external excitation, i.e., for $C_1 = 0$. Therefore the response (79) has a pole at normal mode frequencies.

In the case of linear perturbations of a nonrotating barotropic star, the response turns out to be quite simple. For the quadrupolar case $l = 2$, the outcome is shown in Fig. 3. In fact, the form of the response can even be computed analytically and reads [19]

$$\tilde{F}_l = \sum_n \frac{I_{nl}^2}{\omega_{nl}^2 - \omega^2}. \tag{80}$$

This is just the sum of response functions of harmonic oscillators with resonance frequencies (poles) at ω_{nl}. Here n labels the type and overtone number of the oscillation

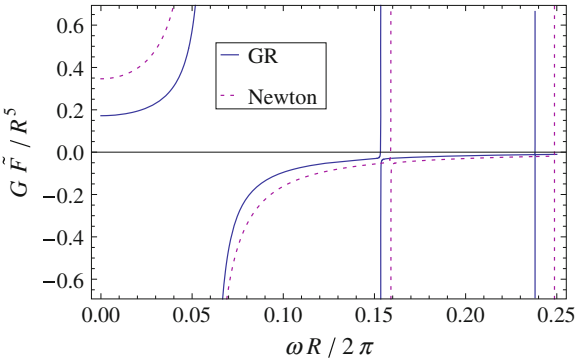

Fig. 3 Response function of the quadrupole, $l = 2$, for a one solar mass star. The equation of state is a polytrope with index 1 and such that the radius R is 17.7 km in the Newtonian case or 15.7 km in the relativistic case

modes. The constants I_{nl} are the so called overlap integrals, which here simply take the role of coupling constants between the oscillators and the external driving forces. As a consequence, the internal dynamics can be captured by an effective action through just a set of harmonic oscillators, which are coupled to the tidal force of the gravitational field [19] (with coupling constants I_{nl}). By fitting the numeric result for $\tilde{F}_l$ to (80), one can extract the constants ω_{nl} and I_{nl}.

It is worth to point out that the presented Newtonian setup is simple enough to perform explicitly the effective field theory procedure of integrating out small scales, see [19]. This turns a compact fluid configuration into a point particle on macroscopic scales.

6.3 Relativistic Case at Zero Frequency

Let us now return to the relativistic case, but restrict to even parity and the adiabatic case $\omega = 0$. The connection between the relativistic tidal constants defined in [15–17, 63] and the response function is given by a Taylor-expansion,

$$\frac{\tilde{F}_l(\omega)}{l!} = \mu_l + i\lambda_l\omega + \mu_l'\omega^2 + \mathcal{O}(\omega^3), \tag{81}$$

see [19]. Here the constants μ_l are named after the astronomer A.E.H. Love, who introduced them for tidal effects in the Earth-Moon system. A dimensionless version of the Love numbers μ_l is often defined as

$$k_l = \frac{(2l-1)!!}{2R^{2l+1}} G\mu_l, \tag{82}$$

where R is the radius of the star. The λ_l-term in (81) is related to absorption [45] and μ_2' was introduced in [63].

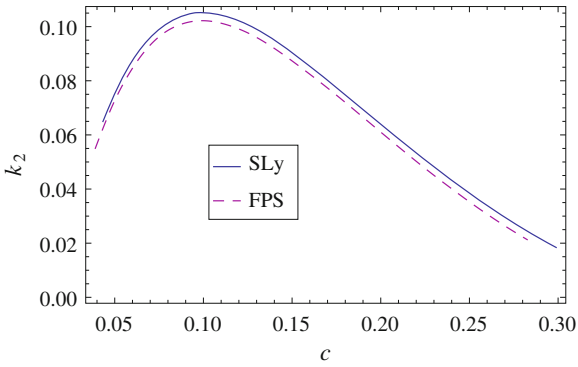

Fig. 4 Dimensionless quadrupolar Love number k_2 as a function of the compactness $c = R_S/2R$ for two different equations of state (SLy, FPS)

It remains to define how the response should be computed in the adiabatic relativistic case. First, we again solve (76), this time for $\omega = 0$, and find an analytic result in terms of the Gauss hypergeometric function ${}_2F_1$,

$$X = C_1 r^{l+1}\, {}_2F_1(-l-2, 2-l, -2l; R_S/r) + C_2 r^{-l}\, {}_2F_1(l-1, l+3, 2(l+1); R_S/r), \tag{83}$$

see, e.g., [64]. Again we can obtain numeric values for the integration constants by solving the perturbation equations inside the body and then match the gravitational field to (83) at the surface. In the limit of $1/r \to 0$ the hypergeometric functions are equal to 1, so (83) turns into (78). This implies that the interpretation of the integration constants as magnitudes of external field and response is still valid. The even-parity response in the adiabatic case $\omega = 0$ then follows from (79) as before. A plot of the outcome in terms of the dimensionless Love number k_2 is given in Fig. 4. An extension of the application from Sect. 5.3 to adiabatic tidal deformations can be found in [52].

For integer values of l, the hypergeometric functions in (83) turn into polynomials (which possibly contain logarithms). Then one might worry that the exponents on r from the two independent solutions in (83) can overlap and spoil an unique identification of external field and response. However, this is avoided by examining X for generic values of l, in the sense of an analytic continuation. This is in spirit similar to working in generic dimension, as done in [64].

6.4 Relativistic Case for Generic Frequency

We now turn our attention to the case of generic frequency in the even parity sector [18]. One can still solve (76) analytically [65], this time in terms of a series involving hypergeometric functions. We write the generic solution schematically as

$$X = A_1 X^l_{\rm MST} + A_2 X^{-l-1}_{\rm MST}, \tag{84}$$

where we denote the solution from [65] by a subscript MST.

Note here that $X^l_{\rm MST} \sim r^l$ and $X^{-l-1}_{\rm MST} \sim r^{-l}$, which means that (79) essentially still works. Of course, one has to take into account the normalization of the $X_{\rm MST}$ in order to rewrite the C_i in (79) in terms of the A_i. This introduces complicated ω-dependent corrections into (79). These are computed through a matching of the asymptotic field of the extended body to the field of the point-particle model. The details of this procedure can be found in [18]. The basic steps are as follows:

- The field of the effective theory is obtained from an inhomogeneous version of (76) with a point particle source. It is understood that the post-Minkowskian approximation is applied, as this removes the singular point of (76) at the Schwarzschild radius. The explicit form of the source term derives from (39).
- The solution to the inhomogeneous equation is constructed from the homogeneous solution (84) using the method of variation of parameters. This method involves integrals over products of singular source and the $X_{\rm MST}$. The integration constants just represent a generic solution to the homogeneous solution that can always be added.
- Here the integration constants must be restricted further. Due to the singular behavior of the differential equation at $r = 0$, the homogeneous solution might actually not be homogeneous at $r = 0$. But the externally applied field is homogeneous everywhere, including $r = 0$. The restriction of the integration constants is therefore equivalent to the identification of the external part of the field and the part generated by the particle.
- Notice that an l-pole source involves l partial derivatives of a delta distribution. This suggests to identify the self-field by $X^{-l-1}_{\rm MST} \sim r^{-l}$ and the external field by $X^l_{\rm MST} \sim r^l$ for dimensional reasons. Here the idea of analytic continuation in l is again crucial.
- The integrals arising in the variation of parameters are actually singular. This is not surprising, as the self-field of point-particles always leads to this kind of problem. A regularization method must be introduced.

These steps lead to a refined (frequency dependent) version of (79) expressing the response function in terms of A_1 and A_2. The final step is again to obtain numeric values for A_1 and A_2 for an actual (extended) neutron star.

The result for the general relativistic response function is shown in Fig. 3. It can still be fitted by (80) very well. This implies that the internal dynamics can be approximated by a set of harmonic oscillators. Restricting to the quadrupolar level $l = 2$ for simplicity, this translates to a dynamical mass of the form

$$\mathcal{M}^2 \approx \mu^2 + \mu \sum_n \left(\psi_n{}^{AB} \psi_{nAB} + \omega_n^2 \phi_{nAB} \phi_n{}^{AB} + 2 I_n E^{AB} \phi_{nAB} \right), \tag{85}$$

where the internal dynamical variables ϕ_{nAB} and $\psi_n{}^{AB}$ only have spatial components in the body-fixed frame ($\phi_{n0B} = 0 = \psi_n{}^{0B}$) and are symmetric tracefree in the

indices A and B. The dynamical equations for the quadrupolar worldline variables can be extracted from (52), (29), and (54),

$$\dot{\phi}_n{}^{AB} = \frac{\alpha}{2}\frac{\partial \mathcal{M}^2}{\partial \psi_{nAB}}, \qquad \dot{\psi}_{nAB} = -\frac{\alpha}{2}\frac{\partial \mathcal{M}^2}{\partial \phi_n{}^{AB}}. \tag{86}$$

In the linear perturbation regime, the contributions of the internal dynamical variables are small compared to μ^2. The index n still labels the type of the oscillation mode. The mass quadrupole is the coefficient in front of E^{AB}, i.e., $Q_{AB} := \sum_n I_n \phi_{nAB}$. Now the frame enters through $E^{AB} = \lambda^A{}_\mu \lambda^B{}_\nu E^{\mu\nu}$, so we need to check if (58) is fulfilled. Using $\Lambda_0{}^\mu = r^\mu$ and $E_{\mu\nu} r^\nu = 0$ it is easy to see that this is the case. In fact, (58) is always fulfilled if the time direction of the body-fixed frame $\Lambda_0{}^\mu$ drops out of the action.

Some final remarks on the problem of regularization of point particles are in order. It was shown already in [59] that the quadrupole diverges at order ω^2 in dimensional regularization. It is therefore not surprising that poles appear in the generalization of (79) at order ω^2, which must be subtracted within some renormalization scheme. At the same time, the poles give rise to an explicit appearance of a renormalization scale parameter, which in a sense parametrizes the ambiguity in the choice of the renormalization scheme. An important point is that this scale parameter is in fact fixed by the requirement that the response function has an asymptotic behavior for $\omega \to \infty$ compatible with (80). Different regularization and renormalization schemes will in general lead to slightly different numeric values for this scale parameter. However, within a given scheme its value can be uniquely matched and is therefore not ambiguous. In this sense, the regularization and renormalization scheme is a part of the phenomenological model.

7 Conclusions

We considered point-particle models for extended bodies in gravity, in particular for black holes and neutron stars. The multipoles of the point particles are adjusted such that their field predicted from a weak field approximation matches an exact/numerical solution for the extended object in question. This incorporates strong field effects from the interior of the extended object in the model. This is of particular importance when binary systems are considered using weak field approximations, e.g., for gravitational wave source modeling or pulsar timing.

Therefore, point-particle actions are far more powerful than what was probably envisioned when they were first investigated [5, 6]. The resulting equations of motion are similar to Dixon's results. Here we developed astrophysical realistic models for the multipoles in these equations. The latest development is the inclusion of oscillation modes in relativistic tidal interaction of neutron stars.

An interesting topic not discussed here are universal relations for various neutron star properties. Here "universal" refers to an approximate independence among various proposed realistic equations of state. In [66, 67] universal relations between the dimensionless moment of inertia $I/G^2\mu^3$, the quadrupolar Love number $\mu_2/G^4\mu^5$, and the quadrupole constant C_{ES^2} were found and coined I-Love-Q relations. Further investigations, also including higher multipoles, followed shortly afterwards [68–73]. This indicates that coefficients in (70) arising at higher orders are actually not independent, but are (approximately) fixed by universality. (For black holes, in fact all coefficients are fixed, which is guaranteed by the no hair theorem.) This makes the expansion (70) a meaningful tool to study the impact of the equation of state on observations, as predictions of the effective model are then parametrized by only a small set of constants.

The most interesting development for the future is probably the description of oscillation modes for rotating bodies, which can be tried in a slow rotation approximation. It is also interesting to investigate if universal properties hold for the ingredients of the response function, e.g., for the overlap integrals.

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