Chapter 17
Mathematical Modelling Tasks and the Mathematical Thinking of Students

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Abstract This chapter describes a study that aims to illustrate how the interaction between mathematical modelling tasks and students' mathematical thinking takes place. We based the study upon David Tall's theory about mathematical thinking regarding mathematical modelling as a pedagogical way to teach mathematics. The research took place in the last year of a university course, with education students undertaking a mathematics degree. A qualitative approach was used with an interpretative analysis to infer points from data collected through audio recordings, video, quizzes, and written data. We have concluded that mathematical modelling can provide, and require the development of cognitive processes that promote interactions between "elementary" and "advanced" mathematical thinking through the cognitive processes of representation, abstraction and generalization.

17.1 Introduction

One of the main purposes of learning and teaching mathematics relates to the construction of mathematical knowledge by students. This construction is present in numerous studies in the field of mathematical education focused on the development of students' mathematical thinking. Many students demonstrate understanding difficulties when it comes to mathematics. According to Sfard (1991), there must be something different in mathematical thinking:

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> Indeed, since in its inaccessibility mathematics seems to surpass all the other scientific disciplines, there must be something really special and unique in the kind of thinking involved in constructing a mathematical universe. (p. 2)

A study about mathematical thinking is presented in this chapter using David Tall's theory, which traces an association between *elementary mathematical thinking* and *advanced mathematical thinking*. According to Tall (2002), the importance of advanced mathematical thinking studies lies in investigating how mathematical development takes place among people, as well as how people's minds work. This sort of investigation can lead the researcher to developing ways to improve mathematical lessons to help students develop advanced mathematical thinking. In this chapter, we have considered mainly the work of Dreyfus (2002) and Tall (1995, 2002) to address the cognitive theory of mathematical thinking and cognitive processes related to such a process.

We describe a research study reported in Palharini (2010) that took place in the field of mathematical modelling, under the perspective of mathematical education. The research was developed in the final year of graduation at the university with 19 students who were undertaking a mathematics degree to teach mathematics in middle and secondary years of school, from which we analyzed the work of three students. In this context, we considered the insertion of modelling tasks into the discussion regarding their potential for the development of mathematical thinking.

The aim of this research is to answer the question: Does student engagement in modelling tasks promote the development of advanced mathematical thinking? As mathematical thinking is something that happens in people's minds, during mathematical tasks, we view mathematical modelling tasks, following Almeida and Dias (2004) and Almeida and Ferruzzi (2009), under a cognitive perspective described by Kaiser and Sriraman (2006) and used by Borromeo Ferri (2006).

17.2 Theoretical Assumptions

17.2.1 Transitions from Elementary Mathematical Thinking into Advanced Mathematical Thinking

Seeking to understand mathematical development in individuals, and how cognitive development of mathematical thinking occurs, we studied the work of Tall and Dreyfus regarding the cognitive theory of mathematical thinking, which these authors proposed. To speak of mathematical development of an individual, it is necessary to understand how people deal with their daily tasks, in particular those involving mathematical processes. To execute such tasks, we have to think in a mathematical way. Mathematics education deals with the theory regarding mathematical thinking in terms of "elementary mathematical thinking" and "advanced mathematical thinking". According to Tall (1995), it is possible to visualize the cognitive growth from elementary to advanced mathematical thinking:

> The cognitive growth from elementary to advanced mathematical thinking in the individual may therefore be hypothesised to start from "perception of" and "action on" objects in the external world, building through two parallel developments—one visuospatial to verbal-deductive, the other successive process-to-concept encapsulations using manipulable symbols—leading to a use of all of this to inspire creative thinking based on formally defined objects and systematic proof. (p. 3)

Both advanced and elementary mathematical thinking involve cognitive processes that consist of a large array of interacting component processes (Dreyfus 2002). According to Tall (2002), it is necessary to study mental processes that are referred to, in this chapter, as *cognitive processes,* associated with thought. *Representation* and *abstraction* are the most important processes of mathematical thinking.

In addition, representation includes three sub-processes: *visualization*, *switching representations* or *translating*, and *modelling*. Abstraction includes *generalization* and *synthesis*. All of these processes take place in people's minds as cognitive actions performed without perception of such performance. However, in what situations may educators perceive these processes? Are there moments in people's lives in which such processes appear and, if so, can they be refined? Is it possible to create learning environments or scenarios to explore the emergence of such processes in people's mathematical tasks? These are some of the questions permeating the research development, presented in this chapter.

In this context, we used a *theory of three mathematical worlds* formulated by Tall (2004b): "Distinct but interrelated worlds of mathematical thinking each with its own sequence of development of sophistication [. . .] that in total spans the range of growth from the mathematics of new-born babies to the mathematics of research mathematicians" (p. 1). The ideas of the three mathematical worlds are discussed by Tall (2004a, b, 2006) and Gray and Tall (1994).

According to Tall (2004a) the name of the first mathematical world is the "conceptual-embodied world," or "embodied world," and it "grows out of our perceptions of the world, as well as it consists of our thinking of things that we perceive and sense, not only in the physical world, but in our own mental world of meaning" (p. 2). In the said world, things happen by reflection and by the use of increasingly sophisticated language. Here, people can focus on aspects of sensory experience. It is common for the emergence of representation processes to take place, such as visualization, since it is necessary to see objects, manipulate them and think of encapsulating their most important features, as a graphical representation or diagrams, such as a triangle.

The second world is the "proceptual-symbolic world," or simply, "proceptual world." According to Tall (2004a), "it is the world of symbols that we use for calculation and manipulation in arithmetic, algebra, calculus and so on . . . These begin with actions (such as pointing and counting) that are encapsulated as concepts by using symbols" (p. 2). In this world, the use of a symbolic representation is necessary (one that cannot exist without mental representations), as well as processes of abstraction, generalization or synthesis, used to deal with symbols, switching representations or translating, or modelling.

The third world, named by Tall (2004a) as the "formal-axiomatic world," or "formal world," is "based on properties, expressed in terms of formal definitions that are used as axioms to specify mathematical structures (such as group, field, vector space, topological space and so on)" (p. 3). Here, mathematical thinking processes in individuals relate to each other. Mathematical proof is used, and there are increasing mathematical languages in a sophisticated way, using the logic and rigor of mathematics.

During students' engagement with mathematical tasks they move through the Three Worlds of Mathematics. In this movement a few interactions take place between elementary and advanced mathematical thinking. To analyze these interactions it is possible to consider the emergence of cognitive processes of mathematical thinking as studied by Tall (2002) and Dreyfus (2002). Considering the importance for teachers to be conscious of these processes to comprehend some of the difficulties their students face, we will present mathematical modelling as a pedagogical way to teach mathematics from a cognitive perspective.

17.2.2 Mathematical Modelling

According to Bean (2003, p. 2), "all forms of 'modelling' require interpretive and creative thinking". When dealing with thought and cognitive processes, a cognitive perspective of mathematical modelling is required. As to modelling perspectives, Kaiser and Sriraman (2006) described, among others, a cognitive perspective, a kind of meta-perspective, in which the research objectives and its psychological goals are, respectively:

> a) analysis of cognitive processes taking place during modelling processes and understanding of these cognitive processes [. . .] b) promotion of mathematical thinking processes by using models as mental images or even physical pictures or by emphasising modelling as mental process such as abstraction or generalization. (p. 304)

Following Kaiser and Sriraman (2006), our research considers the analysis of the emergence of cognitive processes during modelling tasks. When a researcher views mathematical modelling from a cognitive perspective, it is necessary to see students' work in class. In this context, we understand mathematical modelling as a pedagogical way to teach mathematics that suggests an approach through mathematics of a problematic situation, not essentially mathematically-related, according to Almeida and Dias (2004).

To illustrate elements regarding the way that students' mathematical thinking involved in modelling tasks takes place, we used modelling tasks where, from a real situation, students need to translate between reality and mathematics, make assumptions, choose variables, deduce a mathematical model, interpret it and validate it according to the real situation. To indicate tasks for the classroom where teachers can provide students with mathematical thinking processes, we analyze modelling tasks.

17.3 Methodology

This chapter describes a research, reported in Palharini (2010), that was developed in the final year of graduation at the university with 19 students that were undertaking a mathematics degree to teach mathematics in middle and secondary years of school. We analyzed the work of three students who will be referred to as student A, B and C. Data were collected during a school year while students were engaged in modelling tasks. Data collection included audio and video recording of the classes, students' written and verbal response to quizzes and written data about students' solution for modelling tasks. By considering all the procedures in the research we used a qualitative approach following Bogdan and Biklen (1994), and an interpretative analysis regarding students' mathematical thinking processes while they move through the Three Worlds of Mathematics, considering David Tall's theory.

For the development of modelling tasks in class, during the school year, we used a gradual introduction, as described as moments by Almeida and Dias (2004). These moments were used as an approach to introduce modelling tasks and to teach mathematical modelling to students. Firstly, the teacher brings to the classroom mathematical modelling tasks already developed. Containing a real situation and some initial data related with the situation, in this task there is a problem to solve with necessary information to do it. Students, with the support of a teacher, re-create the said task. In the second moment, the teacher, with the students, defines a problem to study from a set of information about something real (here students are responsible for the formulation of hypotheses, development of mathematical model and other modelling stages). Finally, the students themselves create a modelling task, including choosing a theme from the real world, the definition of a problem, the definition of variables and hypotheses, the deduction of the mathematical model, its interpretation and validation in order to answer the initial problem. We analyzed four tasks, and each student participated in the development of two of those (Table 17.1).

The data collected supports two types of analysis undertaken:

1. Specific analysis occurred where we analysed responses of all three students to the *Diazepam in the Body Task*. Responses from the three students were then analysed for the second task each engaged with (as indicated in Table 17.1).
2. Expanded analysis, involved considering the involvement of each of the three students in the two tasks, in order to consider elements relating to what mathematical thinking was involved in mathematical modelling tasks.

Firstly we analyzed qualitatively data collected from all tasks (type 1). In the expanded analysis we conducted an interpretative analysis following the theoretical background.

Table 17.1 Student involvement in the various modelling tasks

Modelling task	Students involved		
	A	B	C
Diazepam in the body	✓	✓	✓
World records	✓	✗	✗
Ethanol production vs the dynamics of vehicle production	✗	✓	✗
Quantitative analysis of nicotine and cadmium accumulated in a smoker's body	✗	✗	✓

17.4 Results and Discussion

To illustrate elements that show us the emergence of advanced mathematical thinking we try to observe and report mathematical thinking processes that appeared during the modelling task development. We observe such processes when students transit through the three mathematical worlds mentioned by Tall (2004b). In all tasks, we can see students creating and using tables followed by use of graphical representations to understand the real problem. To do this it is necessary to create a mental representation of the problem, for example, for the real problem stated in the *Diazepam in the Body Task.*

Diazepam in the Body Task

Diazepam is a medication indicated for patients with anxiety attacks, somatic and psychological anxiety related to the treatment of psychiatric disorders in the relief of muscle spasm due to trauma, such as injury and inflammation. It is a prescription drug (black label) and its improper use or use for long stretches of time can be harmful.

Information package leaflet:

Elimination: the Diazepam plasma concentration/time curve is biphasic: an initial rapid and intense distribution phase, with a half-life that can reach 3 hours and a prolonged terminal elimination phase (half-life of 20–50 hours). The terminal half-life elimination (t1/2b) of active metabolic nordiazepam is about 100 hours. Diazepam and its metabolic effects are primarily excreted in the urine, predominantly in conjugated form. The clearance of Diazepam is 20–30 ml/min.

Thus the *half-life* (Diazepam elimination) is given as follows: In the early stage it can reach 3 hours. In the final stage it can last 20–50 hours.

Treatment should be administered at a starting dose not exceeding 10 mg and the duration should be as short as possible not exceeding 2–3 months including the period of gradual withdrawal of the drug.

From the context, students studied the situation to estimate the concentration of Diazepam in the body over a period of time. During modelling tasks to initiate mathematical development a simplification is necessary, an abstraction of a non-mathematical situation to a mathematical situation. It starts with insights of patterns and regularities that denote a data mathematical behaviour. Such insight is called by Dreyfus (2002) an abstraction process. In this sense, we infer that students develop advanced mathematical thinking, since abstraction use denotes a possible occurrence of advanced mathematical thinking.

In Fig 17.1 we present how students used graphical representation or table representation to obtain an algebraic model that describes the problem of diazepam concentration in the body with numbered arrows 1, 2 and 3 representing the translations that students made to answer the question. In Fig. 17.1 it is possible to note the use of objects, such as graphs and tables that allow the students to see the behaviour of the data. This allows us to infer that students used the cognitive process characterised by Dreyfus (2002) as visualization and that students' thinking transits through the conceptual-embodied world, since such objects are considered by Tall (2004b) as embodied objects. Furthermore, the use of symbols (translations 1 and 3), make us believe that student thinking transits through the conceptual-embodied and proceptual-symbolic world. In this transit the use of three kinds of representation to deal with mathematical concepts is observed, as differential equations and exponential functions, which were possible from a table given to the students (Fig. 17.1). The transitions between representations are from the table to an algebraic representation, using data provided by the table (see translation 1), from an algebraic to a graphical representation, using the model developed (see translation 2), and last from a graphical to an algebraic representation, where students apply the model to estimate when a Diazepam dose of 10 mg is eliminated from the body. We can consider these transitions being possible by the use of representation processes, as characterised by Dreyfus (2002).

The same process took place in the tasks *World Records*, *Ethanol Production vs The Dynamics of Vehicle Production* and *Quantitative Analysis of Nicotine and Cadmium Accumulated in a Smoker's Body* that was developed by the students from choosing the subject to answer the initial question related to this subject, according to the suggestion of the third moment of modelling referred to in Sect. 17.3.

In the formulation of the mathematical model students use mathematical symbols and mathematical operations, theorems and definitions, such as the concept of ordinary differential equations, concept of limits, Cauchy's theorem, the concepts of monotone and convergent sequences, exponential function, and others. These actions operated by students are related to their transit through the proceptual-symbolic and axiomatic-formal world (Tall 2004b), and with the emergence of representation processes. The student written data from the *World Records Task* reveals how the student moves in the axiomatic-formal world, and probably the use of advanced mathematical thinking:

> The data form a decreasing sequence, that is, (R_n) forms a monotonically decreasing sequence. The human body has its limits, so it is impossible for one person to complete the 100 m swimming in one second; we do not know the value of the limit, but it exists and so (R_n) is limited. Given these observations, we believe that the data during the year leads to a sequence (R_n) monotonically decreasing and also limited. Hence, by a theorem of analysis (all limited monotone sequences are convergent), we can say that this is a convergent

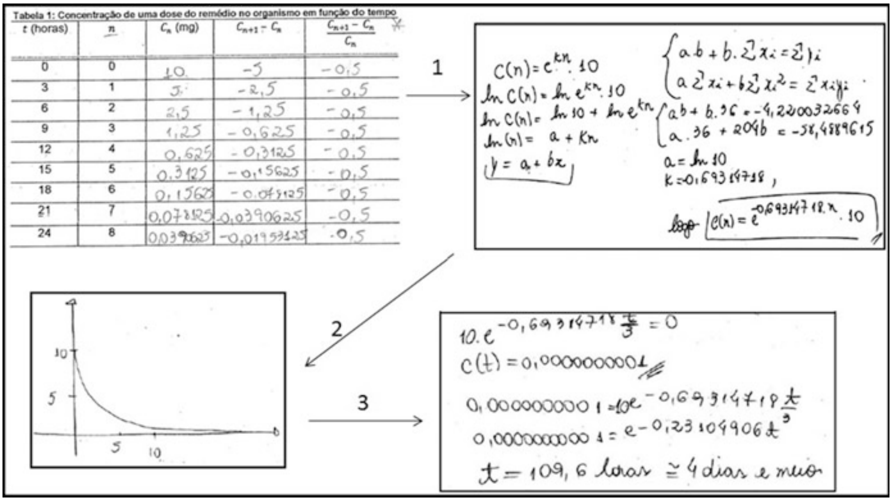

Fig. 17.1 Representations used by student B during the *Diazepam in the Body Task*

sequence. As the sequence is convergent, then there exists the limit of the sequence of records. That is, there is a real number L such that lim $R_n = L$ (Written data of the student A in the *World Records Task*).

By processing symbols and using objects related to the formal mathematics there is an interaction between the student's transit in the symbolic and axiomatic worlds. This interaction implies an interaction of thoughts, an interaction between elementary and advanced mathematical thinking. In this context, students use procedures consistent with elementary mathematics and subsequently advanced mathematics, based on abstract entities that should be used by deductions and formal definitions.

When students are constructing a mathematical model their focus is in obtaining a symbolic representation which initially requires a mental representation. There is the use of various representations, switching representations (as seen in Fig. 17.1), and the generalization process is also required. This is what we can see in Fig. 17.2, where students make a generalization to estimate the nicotine concentration in the body of a smoker.

What Fig. 17.2 shows is how student C, from the amount of nicotine in one cigar, developed a mathematical model, $\mathrm{C}(t) = 0,8 \cdot \frac{1-e^{-0,346573632t}}{1-e^{-0,346573632}}$, to eliminate the amount of nicotine of various cigars of the smoker's body in *t* hours. Whilst student C uses symbols related to the symbolic world, by working with mathematical concepts and deduction based on formal properties, he/she transits into the symbolic and axiomatic worlds. In this context, when students generalize from a simple case to a more general case, they use the process of generalization, related to the abstraction processes, in interaction with the representation processes. According to Dreyfus (2002, p. 34), "if a student develops the ability of making abstractions, consciously, of the mathematical situations, he/she reaches a level of advanced mathematical

t	Nicotine concentration in the Body
0	$C(0) = 0{,}8e^{0}$
1	$C(0) + C(1) = 0{,}8 + 0{,}8e^{-0{,}346573632.1} = 0{,}8(1 + e^{-0{,}346573632})$
2	$C(0) + C(1) + C(2) = 0{,}8(1 + e^{-0{,}346573632}) + 0{,}8e^{-0{,}346573632.2} =$
	$= 0{,}8(1 + e^{-0{,}346573632} + e^{-0{,}346573632.2})$
3	$C(0) + C(1) + C(2) + C(3) =$
	$= 0{,}8(1 + e^{-0{,}346573632} + e^{-0{,}346573632.2}) + 0{,}8e^{-0{,}346573632.3} =$
	$= 0{,}8(1 + e^{-0{,}346573632} + e^{-0{,}346573632.2} + e^{-0{,}346573632.3})$
…	…
m	$C(0) + C(1) + C(2) + \cdots + C(m) =$
	$0{,}8\left(\underbrace{1 + e^{-0{,}346573632.1} + e^{-0{,}346573632.2} + e^{-0{,}346573632.3} + \cdots + e^{-0{,}346573632.m}}_{sum\ of\ terms\ of\ a\ geometric\ progression}\right)$

Fig. 17.2 Generalization by student C in the *Quantitative Analysis of Nicotine Accumulated in a Smoker's Body Task*

thinking". In the same way, interactions between abstraction and representation processes may represent the occurrence of advanced mathematical thinking.

17.5 Final Remarks

Considering modelling tasks where, from a real situation, students need to translate between reality and mathematics, make assumptions, choose variables, deduce a mathematical model, interpret it and validate it according to the real situation, from our research it is possible to conclude that to solve problems students need to use different objects of the three worlds. During students' engagement with modelling tasks we could see that they move through the Three Worlds of Mathematics and in these movements interactions between elementary and advanced mathematical thinking took place. Modelling tasks allow students to complete a cycle of development, from one world to another, non-linearly, using cognitive processes related to mathematical thinking. Advanced mathematical thinking occurs when multiple processes interact. So, guided by data analysis, we can assert that students' engagement in modelling tasks promote the development of advanced mathematical thinking. We can understand the transitions between the mathematical worlds as a refinement of thoughts. In such refinement, students can develop mathematical thinking from "elementary" to "advanced" and from "advanced" to "elementary" thinking. According to our study it is possible to see that in situations with modelling tasks educators can perceive and study mathematical thinking processes and, by knowing these, it is possible to create learning environments or scenarios to explore the emergence of such processes.

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