Chapter 13
Problem Solving Methods for Mathematical Modelling

Gilbert Greefrath

Abstract Providing a method for problem solving can support students working on modelling tasks. A few candidate methods are presented here. In a qualitative study, one of these problem solving methods was introduced to students in grades 4 and 6 (Germany), to be used in their work on modelling tasks. The students were observed as they worked and were subsequently interviewed. The results reveal differences between grades, and widely varying problem solving processes. The differences in written final solutions are considerably less pronounced.

13.1 Sub-competencies of Modelling Competencies

The scope of mathematics is twofold: it deals in abstract structures and ideas, and produces models which serve to describe the environment. In order to achieve this second objective, it is essential to integrate realistic problem settings and applications into mathematical teaching. Mathematical modelling has therefore been identified as one of the most crucial general-mathematical skills by the German educational standards in mathematics. Modelling competency is described by NISS et al. (2007, p. 12) as "the ability to identify relevant questions, variables, relations or assumptions in a real world situation, to translate these into mathematics and to interpret and validate the solution to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions", checking properties etcetera. Motivated by the great importance of this competency for mathematical learning, this chapter aims to present and discuss problem solving resources which can help students in their work on modelling problems in mathematics lessons.

Problem solving resources for modelling tasks are usually based on prototypical solutions of modelling problems, as illustrated by modelling cycles. These

G. Greefrath (✉)
Institute of Education in Mathematics and Information Technology, University of Münster, Fliednerstr. 21, 48149 Münster, Germany
e-mail: greefrath@uni-muenster.de

G.A. Stillman et al. (eds.), *Mathematical Modelling in Education Research and Practice*, International Perspectives on the Teaching and Learning of Mathematical Modelling, DOI 10.1007/978-3-319-18272-8_13

modelling cycles describe the persistent dialogue between reality and mathematics. Some cycles are less detailed, regrouping the process of constructing a model from a given situation into one single step. This step is, for example, labelled "modelling/mathematising" by Ortlieb (2004). Today, modelling usually refers to the cycle as a whole. Mathematising, like in Ortlieb's work, usually refers to the stage which starts on a real problem and ends with the completion of the mathematical model.

A model of the mathematical modelling cycle, elaborated by Blum and his colleagues in the DISUM[1] project, tackles the process from a cognitive perspective, aiming to describe as precisely as possible how students approach a modelling task (see Fig. 13.1). The main departure from the above-mentioned model crafted by Ortlieb is the development of the so-called situation model (Blum and Leiß 2007). The process of model creation is considered in more detail, representing in one initial, additional step the process by which the model's author extracts a mental representation from an initial situation. This expansion was mainly motivated by research into reading comprehension (see, e.g., Kintsch and Greeno 1985).

The capacity to execute each individual step in the modelling cycle can be considered a sub-competency of modelling (cf. Maaß 2006). Blum's modelling cycle provides one way of characterising the individual sub-competencies involved in modelling, as in Table 13.1. Working mathematically (step 4 in Fig. 13.1) is not included as a sub-competency, as it is not specific to modelling competency. Working from other cyclic models of modelling (e.g., Greefrath et al. 2013: Kaiser and Sriraman 2006), other sets of sub-competencies with differing emphasis could conceivably be used. The model of Blum (Fig. 13.1) differentiates the sub-competencies in a clear and detailed way.

Explicitly dividing the modelling process into separate stages is one possible way to reduce complexity for both teachers and students, making it possible to produce more suitable tasks. In particular, this approach to modelling allows individual sub-competencies to be taught separately, enabling a comprehensive modelling competency to develop over time. The insight afforded by the division of the modelling process into several independent sections can be used to create problem solving methods for students, and thus to provide students with problem solving resources during their modelling tasks.

13.2 Problem Solving Methods

Problem solving methods can support students' work on tasks. They are one possible form of problem solving resources. These resources exist in different types and forms. Resources can, for example, help to motivate students to further investigation, providing them with feedback which enables them to judge whether

[1] The DISUM Project (Didactical intervention modes for mathematics teaching oriented towards self-regulation and directed by tasks) led by W. Blum, R. Messner and R. Pekrun.

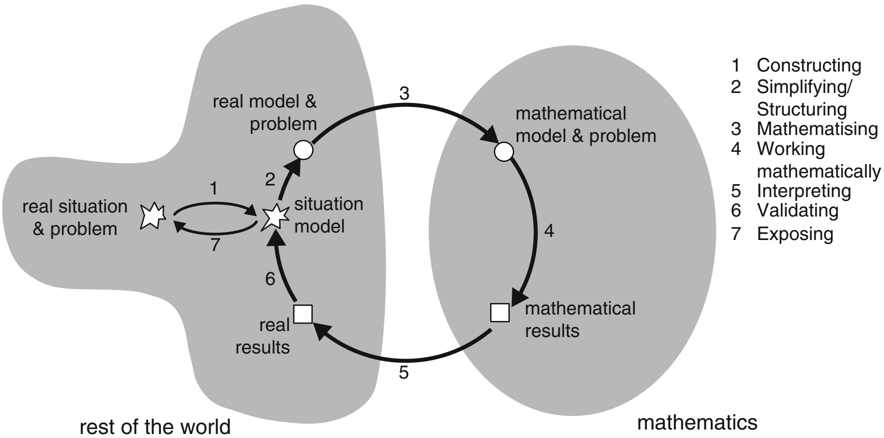

Fig. 13.1 Modelling cycle (Blum and Borromeo Ferri 2009, p. 46)

Table 13.1 Sub-competencies involved in modelling

Sub-competency	Indicator
Constructing	Students construct their own mental model from a given problem, and thus formulate an understanding of their task.
Simplifying	Students identify relevant and irrelevant information from a physical problem.
Mathematizing	Students translate specific, simplified physical situations into mathematical models (e.g., terms, equations, figures, diagrams, functions).
Interpreting	Students relate results obtained from manipulation within the model to the physical situation, and thus obtain physical results.
Validating	Students judge the physical results obtained in terms of plausibility.
Exposing	Students relate the results obtained in the situational model to the real situation, and thus obtain an answer to the problem.

their solution or solution strategy was correct or successful. They can also provide content related hints or information which helps to solve the task. The following categories of resources exist: motivational resources, feedback resources, general strategic resources, content-related strategic resources and content-related resources. Within each individual category, a further distinction between direct and indirect resources can be made. Direct resources explicitly address the student, a specific section of the problem setting, or a concrete mathematical concept. Indirect resources, on the other hand, address the whole class, the task as a whole, or less concrete mathematical concepts (Zech 1998, pp. 315ff.). The problem solving methods presented hereafter are indirect general strategic resources, as although they do refer to general technical problem solving and modelling methods, they do not provide any concrete and contextual indications relating to the content of the task.

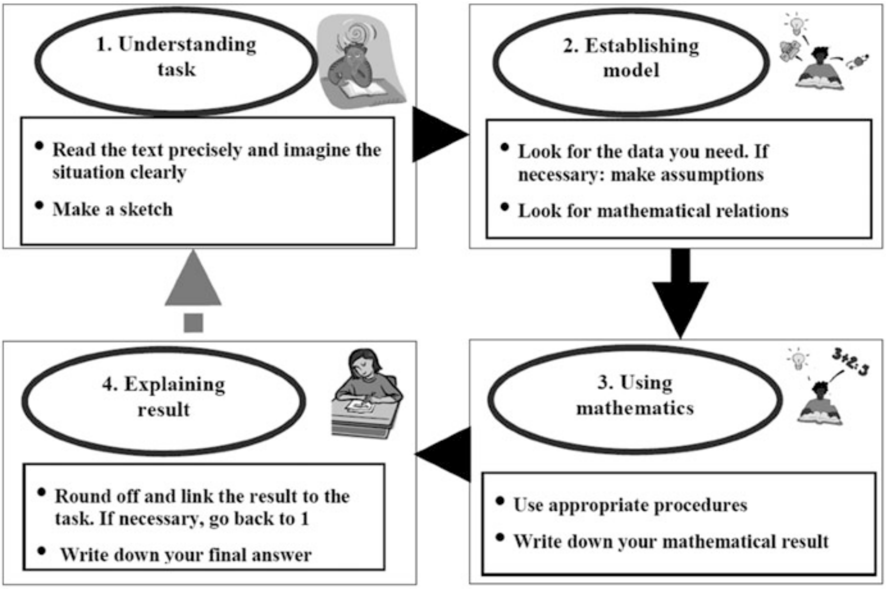

Fig. 13.2 Problem solving method as described by Blum (Blum and Borromeo Ferri, 2009, p. 54)

In the DISUM project, Blum developed a problem solving method for students based on a simplified modelling cycle (see Fig. 13.2). This method comprises four stages, named *understanding the task, establishing the model, using mathematics* and *explaining the results*. Each stage is explained to students with two explicative bullet points. The introduction of the method to students can be supplemented with additional example tasks, allowing them to practise its application.

Böhm (2010) developed a problem solving method in three stages. The first stage aims to understand the problem (*"Problem verstehen"*) to obtain a mental representation of the task at hand. In the second stage, (*"Mathematik ins Spiel bringen und rechnen"*), the problem is mathematised and manipulated mathematically. In the third stage, (*"Resultat auf das Problem beziehen und prüfen"*), the results obtained by mathematical manipulation are related to the problem, and validated. Zöttl et al. (2010) use a similar problem solving method in the KOMMA[2] project. Their method consists of the phases *understanding the task, calculating*, and *explaining the result*.

An alternative problem solving method was presented by Greefrath and Leuders (2013). In the development of this method, the authors considered the stages of problem solving as set out by Polya. In his book *How to solve it*, Polya compiled a catalogue of heuristic questions designed to help manipulation in problem solving tasks. The process of problem solving is divided into the following stages:

[2] KOMMA (KOMpendium MAthematik) is a project aiming at the implementation and evaluation of a learning environment for mathematical modelling.

Understanding the problem, devising a plan, carrying out the plan and *looking back* (Polya 1973). Schoenfeld (1985) added to this by describing some of the stages in more detail. He introduced a distinction at the end of the problem solving process between verification and transition. His suggested problem solving method for students beginning secondary-level education therefore contains five steps, and can be used for both modelling and problem solving tasks:

- Understanding the problem: rephrase it in the students' own words
- Choosing an approach: describe assumptions and plan future calculations
- Performing: perform calculations and manipulations
- Explaining the results
- Checking: check results, calculations and approach

It has been demonstrated that the problem solving methods presented by Blum (Blum and Borromeo Ferri 2009) and Böhm (2010) are structured like a cycle, whereas the problem solving methods described by Zöttl et al. (2010) and Greefrath and Leuders (2013) are structured more linearly, because there is no explicit link to the starting point. This reflects that the first methods are more closely fitted to the modelling cycle, whereas the second set of methods were more closely modelled on the process of problem solving.

There exist some interesting empirical results on students working with problem solving methods. Maaß (2004) used a simplified modelling cycle as a problem solving method. In a qualitative study of grade 7 and 8 school students in Germany, she examined classroom modelling competency, and was able to identify adequate modelling competency in a large subset of the students at the end of the study. Concerning the implementation of the problem solving method, she writes that the students felt they were aided by knowledge about the modelling process and the cycle representation. Böhm (2010) views the three-stage problem solving method as an intermediate tool for the classroom. It should be developed further at a later point, including additional steps, so as not to prevent additional learning later on. The problem solving method is part of a comprehensive programme which aims at fostering modelling-related competencies.

Also in two big research projects in Germany, problem solving methods are used. In the KOMMA project, Zöttl et al. (2010) used structured examples (Reiss and Renkl 2002) of problems solved, consisting of the problem statement and a set of solution steps. Positive results were achieved in the domain of mathematical reasoning and proof by implementing these structured solved examples in the classroom. In a study by Schukajlow et al. (2010) during the DISUM project, significant differences in students' modelling performance could be demonstrated when using the Blum problem solving method (Blum and Borromeo Ferri 2009) applied to Pythagoras' theorem. Lessons that included the problem solving method were found to be the more effective form of teaching and learning. Also, students exposed to the method were more acutely aware of the cognitive strategies they were using, that is the method itself.

However, problem solving methods of this kind are also suspected to bear disadvantages. Meyer and Voigt (2010), for example, have argued that problem solving methods are aimed towards understanding final solutions quickly rather than towards understanding the actual problem solving process, and are little more than additional subject matter for students to learn. Franke and Ruwisch (2010) point out that it can be too much to ask students with little experience in problem solving to search for solutions while simultaneously maintaining a general overview of the process as a whole when using a method.

13.3 Study Design

In a qualitative study, three pairs of German school students in each of grades 4 and 6 (10 and 12 years old), in Grundschule (primary education) and Realschule (secondary education) respectively, were observed while working with the problem solving method set out by Greefrath and Leuders (2013), and subsequently interviewed. The first task (*Teeth Brushing Task*) aimed to achieve an understanding of the method by asking students to arrange the described steps into the correct causal order. This task was given identically to all participating students.

Teeth Brushing Task

Pia is attempting to answer the following question:

How much water could I save in a year while brushing my teeth?

To find an answer, Pia went through several steps:

- I should measure the amount, and add it up.
- Five cupfuls of water flowed through the tap. Five cups are about 1 l. There are 365 days in a year, so that makes 365 l.
- When I brush my teeth, I need water to wash out my mouth. But if I let the water run while I'm brushing, I'm wasting water. How much water runs through the tap over a whole year?
- 365 l is equal to 36 buckets of fresh water that could be saved in a year. That seems about right, because over the course of a year the tap will have been running for many minutes in total.
- So, over a whole year, I could save 365 l of fresh water by turning the tap off while I brush.

Put the steps of Pia's solution in the correct order. Label each step with the name that fits best: understanding the problem, choosing an approach, performing, explaining the results, checking.

The second task involved using the problem solving method. The secondary school students were presented with the method set out by Greefrath and Leuders (2013) as described above, along with the following problem setting:

Use this method to answer the following question: How much fluid do I drink in a week?

The primary school students were given a photograph of a student their age standing next to an oversized flip-flop, and asked to determine the height of a (fictional) person whom the flip-flop might fit.

Flip-Flop Task

Determine the height of a (fictional) person whom the flip-flop might fit.

The observations and interviews were filmed. The videos were transcribed in full and analysed based on Grounded Theory (Strauss and Corbin 1990). The transcripts were analysed line-by-line, and each individual text passage was assigned one of the steps from the problem solving method (understanding, choosing, performing, explaining, checking). The different stages were assigned by two people working independently in order to achieve the highest possible levels of inter-rater reliability in coding.

13.4 Results

Although the first 6th grade pair followed the given problem solving method closely, their attention was particularly focused on the concrete example from the first task. The students mixed up the phases of choosing and performing, and referred alternatively to the problem statement, example, and method. The method did stimulate metacognitive processes, but did not help to optimise the process of answering the question. Even though observation showed that the two students were not successful in following the given method closely, upon being asked whether the method had helped them, they responded as follows: "I thought it was good that we could always look at it. If we had had to do it from nothing, I think it would have been harder. Instead, we had some sort of reference."

The second 6th grade pair followed the given method closely. Some difficulties were only revealed when they were asked to label each phase. A content-related error occurred in the last part of the manipulations. Plausibility considerations did not suffice to disclose the error. This pair of students demonstrated much more clearly the procedure which is hoped to be achieved by providing the problem solving method. Interestingly, one of the two students remarked in the final interview that he saw a similarity to the advice usually given for word problems: question, calculations, results. Overall, the use of this kind of method was met with a very positive reception from the students. The written final solution of the pair reproduced clearly the sequence and structure of the method.

Understanding: How much do you drink in a week?
Choosing: I drink about 2 l every day, so I would calculate 2 L · 7.
Performing: 2 L · 7 = 14 L
Explaining: Each week I drink about 14 L, and each day about 2 L.
Checking: That's equal to 9 · 1.5 l bottles and one 0.5 l bottle.

The third 6th grade pair followed the individual phases of the method. They completed all of the phases, except the last one. However, they followed the example given in the first task more closely than the method itself. This led to difficulties in abstracting from the example in places. The *choosing* and *performing* steps were performed according to the method. The next two steps were, however, mixed up and cut short. This pair also made positive statements about the method, and described the explanations included as important:

Interviewer: Did you find the method helpful?
Student 1: So, I think that if the explanations weren't there, with only 'understanding', 'choosing', I wouldn't have been able to do it. It says here also 'write in your own words', 'describe assumptions' and 'plan calculations'. That helped.
Interviewer: What about if you had had to do the question without a method? Would you have done it in the same way?
Student 2: I would have found it harder, and maybe needed more time to do it.

This pair also provided a written final solution that was structured according to the given method. Details concerning the interview can be found in Hilmer (2012).

The grade 4 student pairs managed to solve the first task quite easily, putting the steps in order. In the second task, no pairs used the method all the way through to the last step. Two pairs managed nevertheless to provide a good answer. A graphical representation (Fig. 13.3) of the typical answering process of a grade 4 pair shows that the students very quickly abandoned the structure of the method, jumping back and forth between the different stages. Still, the method helped a few

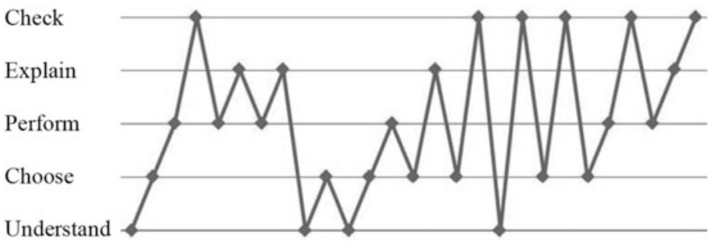

Fig. 13.3 Visualisation of an answering process typical of a grade 4 pair

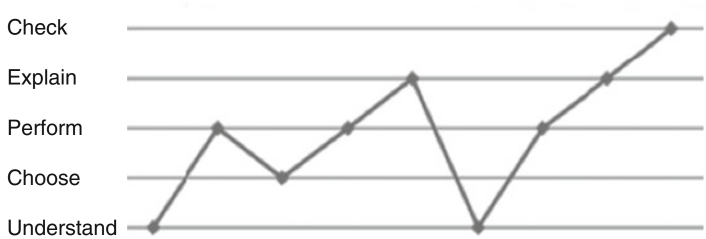

Fig. 13.4 Visualisation of an answering process typical of a grade 6 pair

of the students to make a good start to their answer, and these students worked more independently thereafter. (For details see Förderer 2013.) A typical answering process of a grade 6 pair (Fig. 13.4) shows that the students used the structure of the method, not often jumping back and forth between the different stages.

13.5 Discussion and Conclusion

The results of this study show that students' final solutions can be influenced by providing a method for problem solving. The influence on the written solution is more pronounced than on the solving processes that each pair of students went through, which present considerable differences – from each other and from the model solving process. This is reminiscent of the individual modelling routes that were described by Borromeo Ferri (2007). One aspect that is clearly important is the way that the method is established in lessons. In this study, the method was only presented rapidly, and then used directly afterwards. Clearly, there are students who have difficulty manipulating this type of method, and others who already benefit from a short introduction, helping them to use the method themselves. The results obtained from the students could plausibly be influenced by a more extensive introduction to the method. Whether or not the method had any lasting effect on the problem solving process, all pairs of 6th grade students interviewed recorded positive impressions, and felt supported by its presence. This reinforces some aspects of the findings recorded by Schukajlow et al. (2010). Providing full examples of solved problems, like in the first task of this study, seems to engage some students in a clearly positive way. But there are more factors to control, for example to investigate the effect of metacognitive beliefs (Garofalo and Lester 1985). A more detailed study of school students' problem solving processes seems essential, as providing a method certainly produced a visible impact on the problem solving process. The written final solutions of the students seem, however, to bear little relation to the way the method was in fact used as the problem was being answered.

References

Blum, W., & Borromeo Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45–58.

Blum, W., & Leiß, D. (2007). How do students' and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: Education, engineering and economics* (pp. 222–231). Chichester: Horwood.

Böhm, U. (2010). "Aller Anfang ist schwer, Modellieren lernen umso mehr!?"– Erste Schritte auf dem Weg zur langfristigen Förderung von Modellierungskompetenzen im Mathematikunterricht. In R. Bruder & A. Eichler (Eds.), *Materialien für einen realitätsbezogenen Mathematikunterricht Band 15* (pp. 1–14). Hildesheim: Franzbecker.

Borromeo Ferri, R. (2007). Personal experiences and extra-mathematical knowledge as an influence factor on modelling routes of students. *Proceedings of the fifth congress of the European society for research in mathematics education* (pp. 2080–2089), Larnaca.

Förderer, M. (2013). Modellieren mit Hilfe eines Lösungsplans – Analyse der Bearbeitung einer Fermi-Aufgabe in der 4. Klasse. Unpublished master's thesis, University of Münster.

Franke, M., & Ruwisch, S. (2010). *Didaktik Sachrechnens in der Grundschule*. Heidelberg: Spektrum Akademischer Verlag.

Garofalo, J., & Lester, F. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*(3), 163–176.

Greefrath, G., & Leuders, T. (2013). Verbrauch im Haushalt – Schätzen und Überschlagen. In S. Prediger, B. Barzel, S. Hussmann, & T. Leuders (Eds.), *mathewerkstatt* (pp. 5–22). Berlin: Cornelsen.

Greefrath, G., Kaiser, G., Blum, W., & Borromeo Ferri, R. (2013). Mathematisches Modellieren – Eine Einführung in theoretische und didaktische Hintergründe. In R. Borromeo Ferri, G. Greefrath, & G. Kaiser (Eds.), *Mathematisches Modellieren für Schule und Hochschule. Theoretische und didaktische Hintergründe* (pp. 11–37). Wiesbaden: Springer Spektrum.

Hilmer, A. (2012). *Modellieren mithilfe eines Lösungsplans – Analyse der Bearbeitung eines Fermiproblems in der 6. Klasse*. Unpublished master's thesis, University of Münster.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM, 38*(3), 302–310.

Kintsch, W., & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review, 92*(1), 109–129.

Maaß, K. (2004). *Mathematisches Modellieren im Unterricht. Ergebnisse einer empirischen Studie*. Hildesheim: Franzbecker.

Maaß, K. (2006). What are modelling competencies? *ZDM, 38*(2), 113–142.

Meyer, M., & Voigt, J. (2010). Rationale Modellierungsprozesse. In B. Brandt, M. Fetzer, & M. Schütte (Eds.), *Auf den Spuren Interpretativer Unterrichtsforschung in der Mathematikdidaktik* (pp. 117–148). Münster: Waxmann.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 3–32). New York: Springer.

Ortlieb, C. P. (2004). Mathematische Modelle und Naturerkenntnis. *Mathematica Didactica, 27* (1), 23–40.

Polya, G. (1973). *How to solve it. A new aspect of mathematical method*. Princeton: Princeton University Press.

Reiss, K., & Renkl, A. (2002). Learning to prove: The idea of heuristic examples. *Zentralblatt für Didaktik der Mathematik, 34*(1), 29–35.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic.

Schukajlow, S., Krämer, J., Blum, W., Besser, M., Brode, R., Leiss, D., & Messner, R. (2010). Lösungsplan in Schülerhand: zusätzliche Hürde oder Schlüssel zum Erfolg? *Beiträge zum Mathematikunterricht*, Bd. 2, 771–774.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park: Sage.

Zech, F. (1998). *Grundkurs Mathematikdidaktik*. Weinheim und Basel: Beltz.

Zöttl, L., Ufer, S., & Reiss, K. (2010). Modelling with heuristic worked examples in the KOMMA learning environment. *Journal für Mathematik-Didaktik, 31*(1), 143–165.