A Numerical Approach to the Solution of Dynamic Boundary Value Problems for Fluid-Saturated Solids

V.A. Osinov and C. Grandas-Tavera

Abstract A dynamic boundary value problem for a fluid-saturated solid can be represented as two coupled boundary value problems for one-phase media. This allows us to solve the problem with a commercial computer program without a built-in procedure for the solution of dynamic problems with non-zero permeability, provided that the user is able to establish the required coupling between the two problems. This approach has been implemented in the present paper with the computer program Abaqus/Standard using the dynamic analysis for one-phase media as a built-in procedure without the need to construct a user-defined finite element.

Keywords Fluid-saturated solid · Dynamic problem

1 Motivation

Dynamic boundary value problems for fluid-saturated porous solids are dealt with in various branches of mechanics, including soil and rock mechanics. However, most commercial computer codes do not provide a built-in solution procedure for such problems. In particular, this is true for the widely used finite-element program Abaqus. As far as Abaqus is concerned, these limitations can be overcome by the implementation of the subroutine `UEL` which allows us to construct a user-defined finite element with the desired degrees of freedom. The construction of a user-defined element requires a thorough knowledge of the finite-element method and entails careful debugging. In this regard, alternative ways of solving the problem without resort to the `UEL` subroutine may be useful (e.g. [1]).

This paper presents a numerical approach to the solution of the dynamic initial boundary value problem for a two-phase porous medium. The approach itself is not related to a specific discretization method such as the finite-element or finite-difference method. It consists in the representation of the boundary value

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problem for a two-phase medium as two coupled problems for one-phase media. Using this approach, the original problem for a two-phase medium can be solved with a computer program which solves boundary value problems for one-phase media, provided that the user is able to establish the required coupling between the two problems and to solve them concurrently. The proposed method has been implemented with Abaqus/Standard. Since Abaqus performs the dynamic analysis for one-phase media as a built-in procedure, a user-defined finite element is not needed.

2 Governing Equations

Assuming that the solid phase of a two-phase medium is much stiffer than the skeleton, the total stress is represented as the sum of the effective stress $\boldsymbol{\sigma}$ (compressive stresses are negative) and an isotropic stress $-p_f\mathbf{I}$, where p_f is the fluid pressure (positive for compression) and $\mathbf{I}$ is the unit tensor. We write the dynamic equations in the small-strain approximation neglecting the convective terms and replacing the material time derivatives with the partial ones. The equations of motion for the solid and fluid phases without mass forces are [2, 3]

$$\operatorname{div}\boldsymbol{\sigma} + (n-1)\operatorname{grad} p_f + \xi(\mathbf{v}_f - \mathbf{v}_s) = (1-n)\varrho_s \frac{\partial \mathbf{v}_s}{\partial t}, \tag{1}$$

$$-n \operatorname{grad} p_f - \xi(\mathbf{v}_f - \mathbf{v}_s) = n\varrho_f \frac{\partial \mathbf{v}_f}{\partial t}, \tag{2}$$

where $\mathbf{v}_s$, $\mathbf{v}_f$ are the velocities of the skeleton and the fluid, ϱ_s, ϱ_f are the densities of the solid and fluid phases, and n is the porosity of the skeleton. The coefficient ξ is inversely proportional to the permeability of the skeleton: $\xi = \varrho_f g n^2/k$, where k is the permeability (m/s), and g is the acceleration due to gravity.

A constitutive equation for the skeleton can be written in the general form

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{F}\,(\mathbf{D}_s, \boldsymbol{\sigma}, S), \tag{3}$$

where

$$\mathbf{D}_s = \frac{1}{2}\left[\operatorname{grad}\mathbf{v}_s + (\operatorname{grad}\mathbf{v}_s)^T\right] \tag{4}$$

is the stretching tensor of the skeleton, and the tensor-valued function $\mathbf{F}$ corresponds to an elasticity or plasticity model. In the latter case, Eq. (3) may contain a set $S = (S_1, \ldots, S_m)$ of additional state variables which have their own evolution equations denoted here by a function M:

$$\frac{\partial S}{\partial t} = M\,(\mathbf{D}_s, \boldsymbol{\sigma}, S). \tag{5}$$

Pore pressure changes are determined by the constitutive equation (neglecting grad n)

$$\frac{\partial p_f}{\partial t} = -K_f \operatorname{tr} \mathbf{D}_f - K_f \frac{(1-n)}{n} \operatorname{tr} \mathbf{D}_s, \tag{6}$$

where

$$\mathbf{D}_f = \frac{1}{2}\left[\operatorname{grad} \mathbf{v}_f + \left(\operatorname{grad} \mathbf{v}_f\right)^T\right] \tag{7}$$

is the stretching tensor of the fluid phase, and K_f is the compression modulus of the fluid.

Let us introduce new velocities $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and stresses $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$ defined as

$$\mathbf{v}^{(1)} = a_{11}\mathbf{v}_s + a_{12}\mathbf{v}_f, \tag{8}$$

$$\mathbf{v}^{(2)} = a_{21}\mathbf{v}_s + a_{22}\mathbf{v}_f, \tag{9}$$

$$\boldsymbol{\sigma}^{(1)} = a_{11}\boldsymbol{\sigma} + (n-1)\left(a_{11} + \frac{\varrho_s}{\varrho_f} a_{12}\right) p_f \mathbf{I}, \tag{10}$$

$$\boldsymbol{\sigma}^{(2)} = a_{21}\boldsymbol{\sigma} + (n-1)\left(a_{21} + \frac{\varrho_s}{\varrho_f} a_{22}\right) p_f \mathbf{I}, \tag{11}$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are constant coefficients. Provided that

$$a_{11}a_{22} - a_{12}a_{21} \neq 0, \tag{12}$$

transformation (8)–(11) from $\mathbf{v}_s$, $\mathbf{v}_f$, $\boldsymbol{\sigma}$, $p_f\mathbf{I}$ to $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$ can be inverted to give

$$\mathbf{v}_s = \frac{a_{22}}{a_0}\mathbf{v}^{(1)} - \frac{a_{12}}{a_0}\mathbf{v}^{(2)}, \tag{13}$$

$$\mathbf{v}_f = -\frac{a_{21}}{a_0}\mathbf{v}^{(1)} + \frac{a_{11}}{a_0}\mathbf{v}^{(2)}, \tag{14}$$

$$\boldsymbol{\sigma} = \kappa_1\boldsymbol{\sigma}^{(1)} + \kappa_2\boldsymbol{\sigma}^{(2)}, \tag{15}$$

$$p_f\mathbf{I} = \frac{\varrho_f}{(1-n)\varrho_s a_0}\left(a_{21}\boldsymbol{\sigma}^{(1)} - a_{11}\boldsymbol{\sigma}^{(2)}\right), \tag{16}$$

where

$$a_0 = a_{11}a_{22} - a_{12}a_{21}, \tag{17}$$

$$\kappa_1 = \frac{1}{a_0}\left(a_{22} + \frac{\varrho_f}{\varrho_s} a_{21}\right), \tag{18}$$

$$\kappa_2 = -\frac{1}{a_0}\left(a_{12} + \frac{\varrho_f}{\varrho_s} a_{11}\right). \tag{19}$$

In numerical calculations, p_f can be taken as a diagonal component of the right-hand side of (16), or as

$$p_f = \frac{\varrho_f}{3(1-n)\varrho_s a_0}\left(a_{21}\mathrm{tr}\,\boldsymbol{\sigma}^{(1)} - a_{11}\mathrm{tr}\,\boldsymbol{\sigma}^{(2)}\right). \tag{20}$$

Substituting (13)–(16) into (1), (2), we obtain

$$\mathrm{div}\,\boldsymbol{\sigma}^{(1)} + b_{11}\xi\mathbf{v}^{(1)} + b_{12}\xi\mathbf{v}^{(2)} = (1-n)\varrho_s\frac{\partial\mathbf{v}^{(1)}}{\partial t}, \tag{21}$$

$$\mathrm{div}\,\boldsymbol{\sigma}^{(2)} + b_{21}\xi\mathbf{v}^{(1)} + b_{22}\xi\mathbf{v}^{(2)} = (1-n)\varrho_s\frac{\partial\mathbf{v}^{(2)}}{\partial t}, \tag{22}$$

where

$$b_{11} = \frac{(a_{21}+a_{22})}{a_0}\left(-a_{11} + \frac{(1-n)\varrho_s}{n\varrho_f}a_{12}\right), \tag{23}$$

$$b_{12} = \frac{(a_{11}+a_{12})}{a_0}\left(a_{11} - \frac{(1-n)\varrho_s}{n\varrho_f}a_{12}\right), \tag{24}$$

$$b_{21} = \frac{(a_{21}+a_{22})}{a_0}\left(-a_{21} + \frac{(1-n)\varrho_s}{n\varrho_f}a_{22}\right), \tag{25}$$

$$b_{22} = \frac{(a_{11}+a_{12})}{a_0}\left(a_{21} - \frac{(1-n)\varrho_s}{n\varrho_f}a_{22}\right). \tag{26}$$

Equation (21) can be viewed as the equation of motion of a one-phase medium with the stress tensor $\boldsymbol{\sigma}^{(1)}$, the velocity vector $\mathbf{v}^{(1)}$ and the density $(1-n)\varrho_s$. Similarly, Eq. (22) can be viewed as the equation of motion of a one-phase medium with the stress tensor $\boldsymbol{\sigma}^{(2)}$, the velocity vector $\mathbf{v}^{(2)}$ and the same density $(1-n)\varrho_s$. Each equation contains two additional terms that play the role of a mass force. One term is proportional to the velocity of the same medium, while the other is proportional to the velocity of the other medium (a coupling term).

Defining the stretching tensors

$$\mathbf{D}^{(1)} = \frac{1}{2}\left[\mathrm{grad}\,\mathbf{v}^{(1)} + \left(\mathrm{grad}\,\mathbf{v}^{(1)}\right)^T\right], \tag{27}$$

$$\mathbf{D}^{(2)} = \frac{1}{2}\left[\mathrm{grad}\,\mathbf{v}^{(2)} + \left(\mathrm{grad}\,\mathbf{v}^{(2)}\right)^T\right], \tag{28}$$

and using the transformation relations, we can write constitutive equations for $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$ in terms of $\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)}$, $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$:

$$\frac{\partial\boldsymbol{\sigma}^{(1)}}{\partial t} = a_{11}\frac{\partial\boldsymbol{\sigma}}{\partial t} + (n-1)\left(a_{11} + \frac{\varrho_s}{\varrho_f}a_{12}\right)\frac{\partial p_f}{\partial t}\,\mathbf{I}, \tag{29}$$

$$\frac{\partial \boldsymbol{\sigma}^{(2)}}{\partial t} = a_{21}\frac{\partial \boldsymbol{\sigma}}{\partial t} + (n-1)\left(a_{21} + \frac{\varrho_s}{\varrho_f}\,a_{22}\right)\frac{\partial p_f}{\partial t}\,\mathbf{I}, \tag{30}$$

where

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{F}\left(\frac{a_{22}}{a_0}\,\mathbf{D}^{(1)} - \frac{a_{12}}{a_0}\,\mathbf{D}^{(2)},\ \kappa_1\boldsymbol{\sigma}^{(1)} + \kappa_2\boldsymbol{\sigma}^{(2)},\ S\right), \tag{31}$$

$$\frac{\partial p_f}{\partial t} = \frac{K_f}{a_0}\left(a_{21} - \frac{(1-n)}{n}\,a_{22}\right)\mathrm{tr}\,\mathbf{D}^{(1)} - \frac{K_f}{a_0}\left(a_{11} - \frac{(1-n)}{n}\,a_{12}\right)\mathrm{tr}\,\mathbf{D}^{(2)}. \tag{32}$$

Equation (29) expresses the rate of $\boldsymbol{\sigma}^{(1)}$ as a function of the stretching tensor $\mathbf{D}^{(1)}$ of the same medium, $\boldsymbol{\sigma}^{(1)}$ itself and the state variables S. As distinct from conventional constitutive equations, it contains external terms with $\mathbf{D}^{(2)}$ and $\boldsymbol{\sigma}^{(2)}$. The same holds for Eq. (30): it expresses the rate of $\boldsymbol{\sigma}^{(2)}$ as a function of $\mathbf{D}^{(2)}$, $\boldsymbol{\sigma}^{(2)}$, S and contains external terms with $\mathbf{D}^{(1)}$ and $\boldsymbol{\sigma}^{(1)}$. In this way the original system (1)–(3), (6) for a two-phase medium is split into two systems, namely (21), (29) and (22), (30), for two one-phase media. The two systems are coupled with each other through the mass-force terms in the equations of motion and the additional terms in the constitutive equations. In the case of a plastic skeleton, the two systems are supplemented by Eq. (5) written as

$$\frac{\partial S}{\partial t} = M\left(\frac{a_{22}}{a_0}\,\mathbf{D}^{(1)} - \frac{a_{12}}{a_0}\,\mathbf{D}^{(2)},\ \kappa_1\boldsymbol{\sigma}^{(1)} + \kappa_2\boldsymbol{\sigma}^{(2)},\ S\right). \tag{33}$$

Thus, we have shown that a boundary value problem for a two-phase medium can be represented as two boundary value problems for one-phase media to be solved in parallel. Note that the splitting of the original problem into two coupled one-phase problems and the corresponding numerical implementation could be performed only for the particular trivial case $\mathbf{v}^{(1)} = \mathbf{v}_s$, $\mathbf{v}^{(2)} = \mathbf{v}_f$ with $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 0$ in (8), (9). The reason why this may be insufficient is possible numerical instability which may depend not only on the discretization method but also on the choice of the transformation coefficients a_{ij}. For this reason, it is advantageous to develop a numerical algorithm for the general case.

3 Boundary Conditions

The two boundary value problems for one-phase media are coupled not only through the additional terms in the governing equations but also through boundary conditions. In what follows, $\boldsymbol{\tau}$, $\boldsymbol{\tau}^{(1)}$, $\boldsymbol{\tau}^{(2)}$ will denote the traction vectors corresponding to the stress tensors $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$. Vectors $\mathbf{n}$, $\mathbf{p}$, $\mathbf{q}$ will denote an orthonormal basis, where $\mathbf{n}$ is the outer normal unit vector and $\mathbf{p}$, $\mathbf{q}$ are tangential unit vectors on the boundary. The projections of a vector on $\mathbf{n}$, $\mathbf{p}$, $\mathbf{q}$ will be written with the subscripts n, p, q. For example, $v_{sn} = \mathbf{v}_s \cdot \mathbf{n}$, $v_{sp} = \mathbf{v}_s \cdot \mathbf{p}$. Known quantities in the boundary conditions will be marked with a tilde.

The following boundary conditions may be prescribed (in combination) at the boundary of a two-phase media: impermeable boundary, velocity of the solid phase, effective-stress vector, fluid pressure. These conditions can be written for the two one-phase media with the help of the transformation formulae of the previous section:

- impermeable boundary, $v_{fn} - v_{sn} = 0$,

$$(a_{21} + a_{22})v_n^{(1)} - (a_{11} + a_{12})v_n^{(2)} = 0, \tag{34}$$

- given velocity of the solid phase, $\mathbf{v}_s = \tilde{\mathbf{v}}$,

$$\frac{a_{22}}{a_0}\,\mathbf{v}^{(1)} - \frac{a_{12}}{a_0}\,\mathbf{v}^{(2)} = \tilde{\mathbf{v}}, \tag{35}$$

- given effective-stress vector, $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}}$,

$$\kappa_1 \boldsymbol{\tau}^{(1)} + \kappa_2 \boldsymbol{\tau}^{(2)} = \tilde{\boldsymbol{\tau}}, \tag{36}$$

- given fluid pressure, $p_f = \tilde{p}_f$,

$$\frac{\varrho_f}{(1-n)\varrho_s a_0}\left(a_{21}\tau_n^{(1)} - a_{11}\tau_n^{(2)}\right) = \tilde{p}_f. \tag{37}$$

The numerical solution of problems formulated originally for infinite domains may require non-reflecting boundary conditions prescribed on an artificial boundary in order to imitate the infinite domain. In this study we consider non-reflecting boundary conditions for plane normally incident waves in a two-phase medium with an isotropic elastic skeleton. These boundary conditions are (see Eqs. (44), (71), (72) in [4])

$$\tau_n = -\frac{\lambda + 2\mu}{c_1}\, v_{sn}, \tag{38}$$

$$\tau_p = -\frac{\mu}{c_2}\, v_{sp}, \tag{39}$$

$$\tau_q = -\frac{\mu}{c_2}\, v_{sq}, \tag{40}$$

$$p_f = \frac{K_f}{n c_1}\left[(1-n)v_{sn} + n v_{fn}\right], \tag{41}$$

where

$$c_1 = \sqrt{\left(\lambda + 2\mu + \frac{K_f}{n}\right)\frac{1}{\varrho}}, \tag{42}$$

$$c_2 = \sqrt{\frac{\mu}{\varrho}}, \tag{43}$$

$$\varrho = (1-n)\varrho_s + n\varrho_f, \tag{44}$$

and λ, μ are the Lamé constants of the skeleton.

Equations (38)–(41) written for the two one-phase media become

$$\kappa_1 \tau_n^{(1)} + \kappa_2 \tau_n^{(2)} = -\frac{\lambda + 2\mu}{c_1 a_0}\left(a_{22} v_n^{(1)} - a_{12} v_n^{(2)}\right), \tag{45}$$

$$\kappa_1 \tau_p^{(1)} + \kappa_2 \tau_p^{(2)} = -\frac{\mu}{c_2 a_0}\left(a_{22} v_p^{(1)} - a_{12} v_p^{(2)}\right), \tag{46}$$

$$\kappa_1 \tau_q^{(1)} + \kappa_2 \tau_q^{(2)} = -\frac{\mu}{c_2 a_0}\left(a_{22} v_q^{(1)} - a_{12} v_q^{(2)}\right), \tag{47}$$

$$a_{21}\tau_n^{(1)} - a_{11}\tau_n^{(2)} = \frac{K_f(1-n)\varrho_s}{n\varrho_f c_1}\left\{\left[(1-n)a_{22} - n a_{21}\right] v_n^{(1)} - \left[(1-n)a_{12} - n a_{11}\right] v_n^{(2)}\right\}. \tag{48}$$

The correct formulation of a boundary value problem for a fluid-saturated solid in the general three-dimensional case requires four scalar boundary conditions. A boundary value problem for a one-phase solid involves three scalar boundary conditions. Hence, two problems for one-phase media would require six boundary conditions (three for each problem). Nevertheless, the two boundary value problems for one-phase media considered here are well-posed with four boundary conditions, namely those derived from the four original boundary conditions for the two-phase medium. This is because the two problems are not independent but coupled and must be solved concurrently. In this connection, difficulties may arise in numerical implementation when using available software with limited possibilities of modification. The computer program that solves the boundary value problem for a one-phase medium will demand three boundary conditions from the user, so that we will need six boundary conditions altogether to be able to use the software. It might seem that a simple way to overcome this difficulty is to use two of the four boundary conditions in both problems and thus to gain six boundary conditions. However, this would lead to a wrong numerical solution (see Appendix for detail).

The required additional boundary conditions can be obtained from Eq. (16) which shows that the tensor on the right-hand side is spherical. This gives two boundary conditions

$$a_{21}\tau_p^{(1)} - a_{11}\tau_p^{(2)} = 0, \tag{49}$$

$$a_{21}\tau_q^{(1)} - a_{11}\tau_q^{(2)} = 0. \tag{50}$$

If boundary conditions (49), (50) are used in the first boundary value problem, they represent conditions imposed on $\tau_p^{(1)}$, $\tau_q^{(1)}$ with given $\tau_p^{(2)}$, $\tau_q^{(2)}$. If (49), (50) are used in the second boundary value problem, then $\tau_p^{(1)}$, $\tau_q^{(1)}$ are known and the conditions are imposed on $\tau_p^{(2)}$, $\tau_q^{(2)}$.

Plane-strain and axisymmetric problems involve three scalar boundary conditions for a fluid-saturated medium and two scalar boundary conditions for a one-phase medium. In the implementation of the present method, two boundary conditions are required for each one-phase medium, and hence, one additional boundary condition in the form (49), (50) for the tangential stress component is needed.

4 Numerical Implementation with Abaqus

The proposed approach has been implemented with the finite-element program Abaqus/Standard. Each of the two boundary value problems is solved in a separate Abaqus job. The two problems have the same mesh and element type. They have independent input files, user subroutines and do not share variables in memory. The first boundary value problem (BVP1) is described by Eqs. (21), (29), while the second one (BVP2) is described by Eqs. (22), (30). The two BVPs are coupled with each other through the mass-force terms in the equations of motion (21), (22), the additional terms in the constitutive Eqs. (29), (30), and also through boundary conditions.

The coupling algorithm is presented schematically in Fig. 1. Each BVP is solved within a time increment between times t_k and t_{k+1} using the required coupling quantities of the other BVP at time t_k. The calculation cycle of each BVP consists of the numerical integration over the increment to obtain the solution at time t_{k+1}, a waiting phase if the solution of the other BVP has not yet been completed, and a phase for reading the data of the other BVP calculated for time t_{k+1}. This solution procedure can be made possible with the user subroutine `UEXTERNALDB` in each BVP. The subroutine allows the user to introduce waiting and reading phases immediately before proceeding to the next time increment. Quantities necessary for the coupling are read by each BVP from the Abaqus result file (`*.fil`) of the other BVP during the reading phase and saved in a global Fortran module accessible to all user subroutines. The global module also contains quantities of the same BVP necessary for the calculation of the next increment, e.g. velocities.

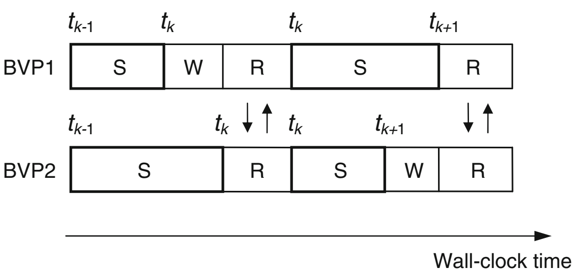

Fig. 1 Concurrent solution of two boundary value problems. Notation: *S* solving for a time increment, *W* waiting for the results of the other problem, *R* reading the results of the other problem

The mass-force terms $b_{ij}\xi\mathbf{v}^{(j)}$ in the equations of motion (21), (22) are treated as an inhomogeneously distributed load. This load is computed at the integration points with the help of the user subroutine `DLOAD`. Velocities available at the nodal points are interpolated to obtain values at the integration points. Constitutive Eqs. (29), (30) are implemented in the user subroutines `UMAT`. The global module of BVPj contains the deformation fields of BVPi for the last two times. This allows the `UMAT` subroutine of BVPj to calculate the external term $\mathbf{D}^{(i)}$ in the constitutive equations. The global module of BVPj also contains the external stress $\boldsymbol{\sigma}^{(i)}$ of BVPi which may appear in the constitutive equation of BVPj. Additional state variables governed by Eq. (33) can be calculated in either of the BVPs and stored in the global module. In the numerical example presented below in Sect. 5, the constitutive equation for the solid skeleton is linearly elastic and has no additional state variables.

Boundary conditions (34), (35) for velocity are implemented with the user subroutine `DISP`. Boundary conditions (36), (37), (49), (50) for tractions are implemented with the user subroutine `UTRACLOAD`. This subroutine is also used to implement non-reflecting boundary conditions (45)–(48). According to the scheme shown in Fig. 1, a boundary condition written for velocity or traction for time t_{k+1} contains a given quantity (with a tilde) for time t_{k+1} and known quantities for time t_k taken from the global module.

It should be mentioned that a somewhat similar approach to the solution of dynamic problems for two-phase media was implemented in [1] with Abaqus/Explicit. The problem was solved with a single Abaqus job using two identical meshes: one for the fluid phase and the other one for the skeleton. This corresponds to $\mathbf{v}^{(1)} = \mathbf{v}_s$, $\mathbf{v}^{(2)} = \mathbf{v}_f$. The mass-force terms $b_{ij}\xi\mathbf{v}^{(j)}$ were introduced with the help of connector elements acting between the collocated nodes. In relation to the algorithm described in [1], two advantages of the present approach can be noticed. First, the use of connector elements requires sophisticated programming prior to the solution of the problem. Second, for a given time integration method (explicit or implicit), the calculation of one time increment with a double mesh as in [1] requires more time than the calculation of one increment with two Abaqus jobs operating in parallel with single meshes.

5 Numerical Verification

The present approach implemented with Abaqus/Standard has been tested on the solution of a two-dimensional plane-strain dynamic problem for an elastic medium for the domain shown in Fig. 2. The medium is at rest at $t = 0$ with zero stresses and pore pressure. Boundary conditions for $t > 0$ (BC1, BC2 and BC3) are prescribed as follows.

- BC1 represents an impermeable boundary with a given velocity of the solid phase

$$v_{s1}(t) = 0, \tag{51}$$

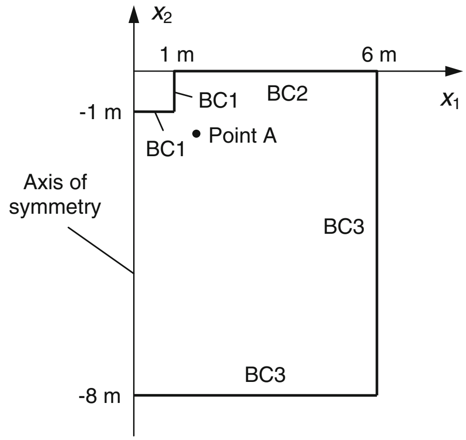

Fig. 2 Computational domain of the boundary value problem

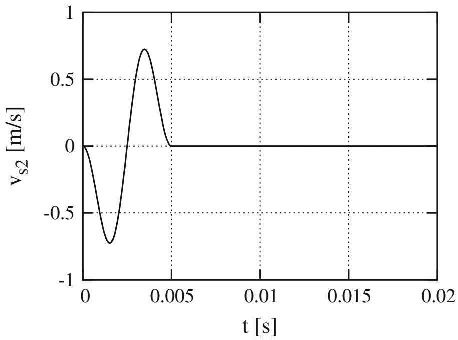

Fig. 3 Boundary condition (52) for the vertical velocity

$$v_{s2}(t) = \begin{cases} v_{amp} \; \sin(2\pi t/t_0) \; \sin(\pi t/t_0) & \text{if } t \le t_0, \\ 0 & \text{if } t > t_0, \end{cases} \tag{52}$$

where $v_{amp} = -0.3\pi$ m/s and $t_0 = 5 \times 10^{-3}$ s. The function $v_{s2}(t)$ is shown in Fig. 3. This boundary condition imitates the prescribed motion of a rigid foundation on a half space $x_2 < 0$.

- BC2 corresponds to a free surface with zero pore pressure and zero traction.
- BC3 describes a non-reflecting boundary as discussed in Sect. 3.

The solid skeleton is assumed to be linearly elastic and isotropic with the Lamé constants λ and μ. The parameters of the medium are shown in Table 1. The compression modulus of the pore fluid is taken to be equal to the modulus of pure water and is much higher than the shear modulus of the skeleton (a situation typical of soils). The size of the finite elements in the mesh varies between 3 and 9 cm. The time step is equal to 10^{-5} s.

Figures 4 and 5 show the solution to this boundary value problem as a function of time at Point A with the coordinates $x_1 = 1.5$ m, $x_2 = -1.5$ m, see Fig. 2. For

Table 1 Parameters of the medium

λ (MPa)	μ (MPa)	K_f (MPa)	k (m/s)	ϱ_s (kg/m^3)	ϱ_f (kg/m^3)	n
120	80	2200	10^{-2}	2650	1000	0.4

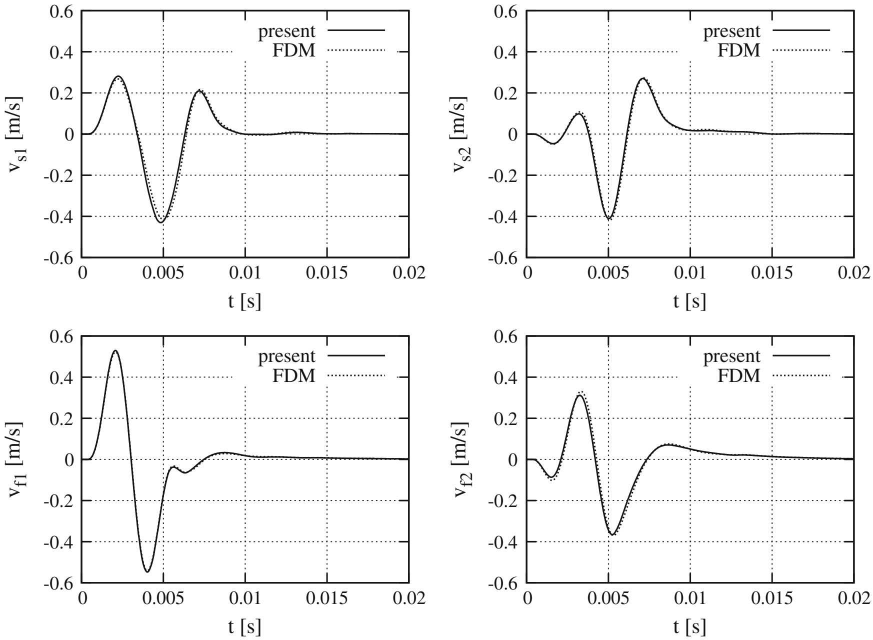

Fig. 4 Solutions obtained with the present approach and with the finite-difference method (FDM). Velocity components at Point A (see Fig. 2) are shown as functions of time

comparison, the figures also show the solution obtained independently by the finite-difference method for the original system (1)–(3), (6) without the decomposition into two coupled problems.

Calculations with the present method have shown that numerical solutions may exhibit instabilities in the form of spurious oscillations increasing with time. Stability has been found to depend on the time integration method and on the choice of the coefficients a_{ij}, see (8)–(11). Abaqus/Standard offers two implicit time integration schemes: the Hilber-Hughes-Taylor and the backward Euler schemes. Numerical experiments with the boundary value problem presented in this section revealed instability of the Hilber-Hughes-Taylor scheme for all combinations of a_{ij} that were tested. The solution shown in Figs. 4 and 5 could be obtained only with the backward Euler method. The dependence of stability of the backward Euler method on the coefficients a_{ij} is shown in Fig. 6 for some combinations of the coefficients. In the stable cases, there is no dependence of the solutions on the choice of a_{ij}. Note that stability of the proposed method in its general form presented in Sect. 2 may depend

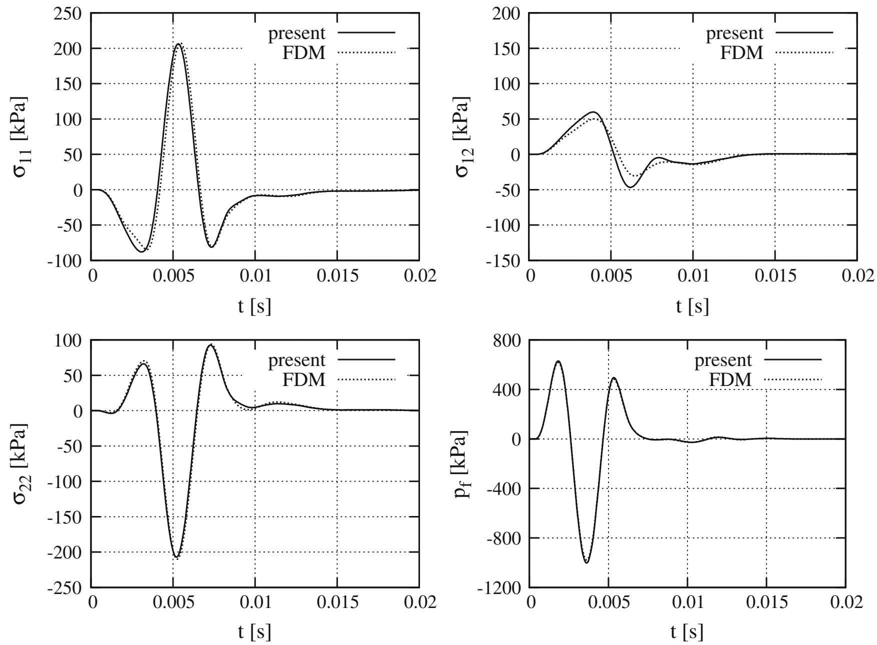

Fig. 5 The same as in Fig. 4 for the stresses

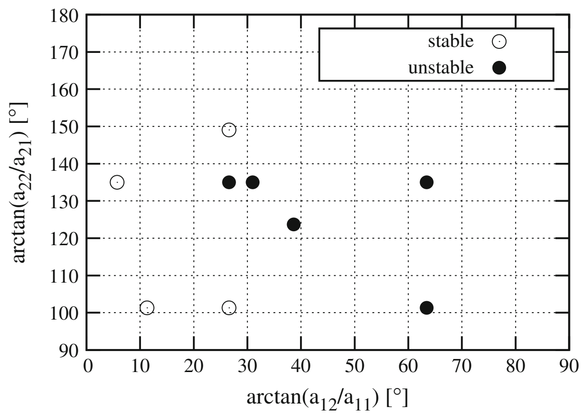

Fig. 6 Stable and unstable combinations of the coefficients a_{ij}

on the numerical implementation of the coupling and the solution method for the one-phase problems. The present findings apply to the algorithm described in Sect. 4.

6 Concluding Remarks

The system of dynamic equations for a porous fluid-saturated solid can be written as two coupled systems, each of them describing the deformation of a one-phase medium. This decomposition offers the possibility of solving boundary value problems for two-phase fluid-saturated solids with a computer program which can perform a dynamic analysis for one-phase solids only (as, for instance, the commercial program Abaqus). The necessary condition for the numerical implementation of this approach is the ability to establish the required coupling between the two problems within the available software and to solve them concurrently. The approach has been realized in the present study with the finite-element program Abaqus/Standard and verified by the comparison of the solution with that obtained independently by a different method.

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Appendix

As mentioned in Sect. 3, the use of the same boundary condition in the two coupled boundary value problems would lead to a wrong numerical solution. Here we will show this fact by a simple example.

Write a boundary condition in the form

$$g^{(1)}(t) + g^{(2)}(t) = f(t), \tag{53}$$

where the unknown terms $g^{(1)}$, $g^{(2)}$ contain quantities of the first and the second boundary value problems, and $f(t)$ is a given function. Obviously, two unknown functions $g^{(1)}(t)$, $g^{(2)}(t)$ cannot be found from one Eq. (53). They are determined not only by this boundary condition but also from the solution of the two boundary value problems. Suppose that boundary condition (53) is used in both boundary value problems. Let the numerical scheme be such that the values of $g^{(1)}$, $g^{(2)}$ at time $t + \Delta t$ are calculated as

$$g^{(1)}(t + \Delta t) = f(t + \Delta t) - g^{(2)}(t), \tag{54}$$

$$g^{(2)}(t + \Delta t) = f(t + \Delta t) - g^{(1)}(t), \tag{55}$$

where $g^{(1)}(t)$, $g^{(2)}(t)$ at time t are known, and Eqs. (54), (55) are used as boundary conditions in the first and the second boundary value problems, respectively. As a result of the numerical solution, we will find two functions $g^{(1)}$, $g^{(2)}$ from one Eq. (53). Considering the limit $\Delta t \to 0$, it is easy to see that the functions $g^{(1)}$, $g^{(2)}$

obtained in such a way approximate the solution

$$g^{(1)}(t) = \frac{1}{2} f(t) - \frac{1}{2} f(0) + g^{(1)}(0), \tag{56}$$

$$g^{(2)}(t) = \frac{1}{2} f(t) - \frac{1}{2} f(0) + g^{(2)}(0) \tag{57}$$

to Eq. (53), where $g^{(1)}(0)$, $g^{(2)}(0)$, $f(0)$ are given initial values satisfying (53). The numerical scheme with

$$g^{(2)}(t + \Delta t) = f(t + \Delta t) - g^{(1)}(t + \Delta t) \tag{58}$$

instead of (55) gives a different solution to (53):

$$g^{(1)}(t) = f(t) - f(0) + g^{(1)}(0), \tag{59}$$

$$g^{(2)}(t) = g^{(2)}(0). \tag{60}$$

We see that the use of one boundary condition twice results in the numerical determination of two unknown functions involved in the boundary condition independently of the solutions to the two boundary value problems.

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