Conic Scalarization Method in Multiobjective Optimization and Relations with Other Scalarization Methods*

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Abstract. The paper presents main features of the conic scalarization method in multiobjective optimization. The conic scalarization method guarantees to generate all proper efficient solutions and does not require any kind of convexity or boundedness conditions. In addition the preference and reference point information of the decision maker is taken into consideration by this method. In this paper, relations with other scalarization methods are investigated and it is shown that some efficient solutions computed by the Pascoletti-Serafini and the Benson's scalarization methods, can be obtained by the conic scalarization method.

Keywords: Conic scalarization method, Weighted sum method, Epsilon constraint method, Benson's method, Weighted Chebyshev method, Pascoletti-Serafini method, Multiobjective optimization, proper efficiency.

1 Introduction

In general, scalarization means the replacement of a multiobjective optimization problem by a suitable scalar optimization problem which is an optimization problem with a real valued objective function.

In this paper we give main features of the conic scalarization method. The conic scalarization method enables to completely characterize the whole set of efficient and properly efficient solutions of multiobjective problems without convexity and boundedness conditions.

In this paper we present theorems which establish relations between the conic scalarization and the Pascoletti-Serafini and the Benson's scalarization methods. It is shown that some efficient solutions computed by the Pascoletti-Serafini and the Benson's scalarization methods, can be obtained by the conic scalarization method.

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The rest of the paper is organized as follows. Section 2 gives some preliminaries. Main characteristics of the conic scalarization method are given in Section 3. In section 4, new relations between the conic scalarization method and Pascoletti-Serafini and Benson's scalarization methods are established. Finally, Section 5 draws some conclusions from the paper.

2 Preliminaries

We begin this section with standard definitions from multi-objective optimization.

Let $\mathbb{R}^n_+ := \{y = (y_1, ..., y_n) : y_i \geq 0, i = 1, \ldots, n\}$, and let $\mathbb{Y} \subset \mathbb{R}^n$ be a nonempty set.

Throughout the paper, $\mathbb{R}_+$ denotes the set of nonnegative real numbers. $\mathsf{cl}(\mathbb{Y})$, $\mathsf{bd}(\mathbb{Y})$, $\mathsf{int}(\mathbb{Y})$, and $\mathsf{co}(\mathbb{Y})$ denote the *closure*, the *boundary*, the *interior*, and the *convex hull* of a set $\mathbb{Y}$, respectively.

A nonempty subset $\mathbb{C}$ of $\mathbb{R}^n$ is called a *cone* if $y \in \mathbb{C}, \lambda \geq 0 \Rightarrow \lambda y \in \mathbb{C}$. Pointedness of $\mathbb{C}$ means that $\mathbb{C} \cap (-\mathbb{C}) = \{0_{\mathbb{R}^n}\}$.

We will assume that $\mathbb{R}^n$ is partially ordered by a convex pointed cone $\mathbb{C} \subset \mathbb{R}^n$.

Definition 1. *1. An element $y \in \mathbb{Y}$ is called a minimal element of $\mathbb{Y}$ (with respect to the ordering cone $\mathbb{C}$) if $(\{y\} - \mathbb{C}) \cap \mathbb{Y} = \{y\}$.*

2. *An element $y \in \mathbb{Y}$ is called a weakly minimal element of $\mathbb{Y}$ if $(\{y\} - \mathsf{int}(\mathbb{C})) \cap \mathbb{Y} = \emptyset$.*
3. *An element $y \in \mathbb{Y}$ is called a properly minimal element of $\mathbb{Y}$ in the sense of Benson [1] if y is a minimal element of $\mathbb{Y}$ and the zero element of $\mathbb{R}^n$ is a minimal element of $\mathsf{cl}(\mathsf{cone}(\mathbb{Y} + \mathbb{C} - \{y\}))$, where $\mathsf{cone}(\mathbb{Y}) := \{\lambda y : \lambda \geq 0, y \in \mathbb{Y}\}$.*
4. *An element $\overline{y} \in \mathbb{Y}$ is called a* properly minimal element *of $\mathbb{Y}$ in the sense of Henig [11] if it is a minimal element of $\mathbb{Y}$ with respect to some convex cone $\mathbb{K}$ with $\mathbb{C} \setminus \{0_{\mathbb{R}^n}\} \subset \mathrm{int}(\mathbb{K})$.*

Henig proved that in the case when the vector space is partially ordered by a closed pointed cone, the two definitions of proper efficiency given in Definition 1, are equivalent (see [11, Theorem 2.1]). Therefore, in the sequel we simply will use the notion of proper efficiency.

Consider a multiobjective optimization problem (in short MOP):

$$\min_{x \in \mathbb{X}} [f_1(x), ..., f_n(x)], \tag{1}$$

where $\mathbb{X}$ is a nonempty set of feasible solutions and $f_i : \mathbb{X} \to \mathbb{R}, i = 1, ..., n$ are real-valued functions. Let $f(x) = (f_1(x), \ldots, f_n(x))$ for every $x \in \mathbb{X}$ and let $\mathbb{Y} := f(\mathbb{X})$.

Definition 2. *A feasible solution $x \in \mathbb{X}$ is called efficient, weakly efficient or properly efficient solution of multi-objective optimization problem (1) if $y = f(x)$ is a minimal, weakly minimal or properly minimal element of $\mathbb{Y}$, respectively.*

Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. $\|y\|_1 = \sum_{i=1}^n |y_i|$, $\|y\|_2 = (y_1^2 + \cdots + y_n^2)^{1/2}$, and $\|y\|_\infty = \max\{|y_1|, \ldots, |y_n|\}$ denote the l_1, l_2 (Euclidean), and l_∞ norms of y, respectively.

Let $\mathbb{C}$ be a given cone in $\mathbb{R}^n$. Recall that the dual cone $\mathbb{C}^*$ of $\mathbb{C}$ and its quasi-interior $\mathbb{C}^\#$ are defined by

$$\mathbb{C}^* = \{w \in \mathbb{R}^n : w^T y \geq 0 \text{ for all } y \in \mathbb{C}\} \tag{2}$$

and

$$\mathbb{C}^\# = \{w \in \mathbb{R}^n : w^T y > 0 \text{ for all } y \in \mathbb{C} \setminus \{0\}\}, \tag{3}$$

respectively, where w^T denotes the transpose of vector w, and $w^T y = \sum_{i=1}^n w_i y_i$ is the scalar product of vectors $w = (w_1, \ldots w_n)$ and $y = (y_1, \ldots, y_n)$. The elements of these cones define monotone and strongly monotone linear functionals whose level sets (hyperplanes) are used to characterize support points of convex sets.

The following three cones called augmented dual cones of $\mathbb{C}$ were introduced in [14], and it was proven that the elements of these cones define monotone sublinear functionals with conical level sets. Due to this property, these functionals are used to generate efficient solutions of nonconvex multiobjective problems.

$$\mathbb{C}^{a*} = \{(w, \alpha) \in \mathbb{C}^\# \times \mathbb{R}_+ : w^T y - \alpha\|y\| \geq 0 \text{ for all } y \in \mathbb{C}\}, \tag{4}$$

$$\mathbb{C}^{a\circ} = \{(w, \alpha) \in \mathbb{C}^\# \times \mathbb{R}_+ : w^T y - \alpha\|y\| > 0 \text{ for all } y \in \text{int}(\mathbb{C})\}, \tag{5}$$

and

$$\mathbb{C}^{a\#} = \{(w, \alpha) \in \mathbb{C}^\# \times \mathbb{R}_+ : w^T y - \alpha\|y\| > 0 \text{ for all } y \in \mathbb{C} \setminus \{0\}\}, \tag{6}$$

where $\mathbb{C}$ is assumed to have a nonempty interior in the definition of $\mathbb{C}^{a\circ}$.

3 Conic Scalarization (CS) Method

The history of development of the CS method goes back to the paper [5], where Gasimov introduced a class of monotonically increasing sublinear functions on partially ordered real normed spaces and showed without convexity and boundedness assumptions that support points of a set obtained by using these functions are properly minimal in the sense of Benson [1]. The question of "can every properly minimal point of a set be calculated in a similar way", was answered only in the case when the objective space is partially ordered by a certain Bishop–Phelps cone. Since then, different theoretical and practical applications by using the suggested class of sublinear functions have been realized [3,6,7,8,9,12,13,14,17,20,22]. The theoretical fundamentals of the conic scalarization method in general form was firstly explained in [14]. The full description of the method is given in [15].

The idea of the CS method is very simple: choose preference parameters which consist of a weight vector $w \in \mathbb{C}^\#$ and a reference point $a \in \mathbb{R}^n$, determine an augmentation parameter $\alpha \in \mathbb{R}_+$ such that $(w, \alpha) \in \mathbb{C}^{a*}$ (or $(w, \alpha) \in \mathbb{C}^{a\circ}$, or

$(w, \alpha) \in \mathbb{C}^{a\#}$), where for a convenience the l_1-norm is used, and solve the scalar optimization problem:

$$\min_{x \in \mathbb{X}} \sum_{i=1}^{n} w_i(f_i(x) - a_i) + \alpha \sum_{i=1}^{n} |f_i(x) - a_i| \qquad (CS(w, \alpha, a))$$

The set of optimal solutions of this scalar problem will be denoted by $Sol(CS(w, \alpha, a))$. Reference point $a = (a_1, \ldots, a_n)$ may be identified by a decision maker in cases when she/he desires to calculate minimal elements that are close to some point. The CS method does not impose any restrictions on the ways for determining reference points. The reference point can be chosen arbitrarily.

The following theorem quoted from [15] explains main properties of solutions obtained by the conic scalarization method in the case when $\mathbb{C} = \mathbb{R}^n_+$. This special case for the cone determining the partial ordering, allows one to explicitly determine augmented dual cones which are used for choosing scalarizing parameters (w, α). For the general case of this theorem see [14, Theorem 5.4].

Theorem 1. *[15, Theorem 6] Let $a \in \mathbb{R}^n$ be a given reference point, and let $\mathbb{C} = \mathbb{R}^n_+$. Assume that $Sol(CS(w, \alpha, a)) \neq \emptyset$ for a given pair $(w, \alpha) \in \mathbb{C}^{a*}$. Then the following hold.*

(i) If

$$(w, \alpha) \in \mathbb{C}^{a\circ} = \{((w_1, \ldots, w_n), \alpha) : 0 \leq \alpha \leq w_i, w_i > 0, i = 1, \ldots, n \text{ and there exists } k \in \{1, \cdots, n\} \text{ such that } w_k > \alpha\},$$

then every element of $Sol(CS(w, \alpha, a))$ is a weakly efficient solution of (1).

(ii) If $Sol(CS(w, \alpha, a))$ consists of a single element, then it is an efficient solution (1).

(iii) If

$$(w, \alpha) \in \mathbb{C}^{a\#} = \{((w_1, \ldots, w_n), \alpha) : 0 \leq \alpha < w_i, i = 1, \ldots, n\},$$

then every element of $Sol(CS(w, \alpha, a))$ is a properly efficient solution of (1), and conversely, if $\overline{x}$ is a properly efficient solution of (1), then there exists $(w, \alpha) \in \mathbb{C}^{a\#}$ and a reference point $a \in \mathbb{R}^n$ such that $\overline{x}$ is a solution of $Sol(CS(w, \alpha, a))$.

The following theorem gives simple characterization of minimal elements.

Theorem 2. *[15, Theorem 7] Let $\mathbb{Y} \subset \mathbb{R}^n$ be a given nonempty set and let $\mathbb{C} = \mathbb{R}^n_+$. If $\overline{y}$ is a minimal element of $\mathbb{Y}$, then $\overline{y}$ is an optimal solution of the following scalar optimization problem:*

$$\min_{y \in \mathbb{Y}} \{ \sum_{i=1}^{n} (y_i - \overline{y}_i) + \sum_{i=1}^{n} |y_i - \overline{y}_i| \}. \tag{7}$$

By using assertions of Theorems 1 and 2, we arrive at the following conclusion. By solving the problem $(CS(w, \alpha, a))$ for "all" possible values of the augmentation parameter α between 0 and $\min\{w_1, \ldots, w_n\}$, one can calculate all the efficient solutions corresponding to the decision maker's preferences (the weighting vector $w = (w_1, \ldots, w_n)$ and the reference point a).

The following two remarks illustrate the geometry of the CS method.

Remark 1. It is clear that in the case when $\alpha = 0$ (or, if $f(\mathbb{X}) \subseteq \{a\} \pm \mathbb{C}$) the objective function of the scalar optimization problem $(CS(w, \alpha, a))$ becomes an objective function of the weighted sum scalarization method. The minimization of such an objective function over a feasible set enables to obtain only those efficient solutions x (if the corresponding scalar problem has a solution), for which the minimal vector $f(x)$ is a supporting point of the objective space with respect to some hyperplane

$$H(w) = \{y : w^T y = \beta\},$$

where $\beta = w^T f(x)$. It is obvious that minimal points which are not supporting points of the objective space with respect to some hyperplane, cannot be detected by this way. By augmenting the linear part in $(CS(w, \alpha, a))$ with the norm term (using a positive augmentation parameter α), the hyperplane $H(w)$ becomes a conic surface defined by the cone

$$S(w, \alpha) = \{y \in \mathbb{R}^n : w^T y + \alpha\|y\| \leq 0\}, \tag{8}$$

and therefore the corresponding scalar problem $(CS(w, \alpha, a))$ computes solution x, for which the corresponding vector $f(x)$ is a supporting point of the objective space with respect to this cone. The change of the α, leads to a different supporting cone. The supporting cone corresponding to some weight vector w becomes narrower as α increases, and the smallest cone (which anyway contains the ordering cone) is obtained when α equals its maximum allowable value (for example, $\min\{w_1, \ldots, w_n\}$, if $(w, \alpha) \in \mathbb{C}^{a\#}$). This analysis shows that by changing the α parameter, one can compute different minimal points of the problem corresponding to the same weight vector. And since the method computes supporting points of the decision space with respect to cones (if $\alpha \neq 0$), it becomes clear why this method does not require convexity and boundedness conditions and why it is able to find optimal points which cannot be detected by hyperplanes. Since the cases $\alpha = 0$, or $f(\mathbb{X}) \subseteq \{a\} \pm \mathbb{C}$ leads to the objective function of the weighted sum scalarization method, we can say that the CS method is a generalization of the weighted sum scalarization method.

Remark 2. It follows from the definition of augmented dual cone that $w^T y - \alpha\|y\| \geq 0$ for every $(w, \alpha) \in \mathbb{C}^{a*}$ and all $y \in \mathbb{C}$. Hence

$$\mathbb{C} \subset C(w, \alpha) = \{y \in \mathbb{R}^n : w^T y - \alpha\|y\| \geq 0\}, \tag{9}$$

where $C(w, \alpha)$ is known as the Bishop-Phelps cone corresponding to a pair $(w, \alpha) \in \mathbb{C}^{a*}$. It has been proved that, if $\mathbb{C}$ is a closed convex pointed cone having a weakly compact base, then

$$\mathbb{C} = \cap_{(w,\alpha)\in\mathbb{C}^{a*}} C(w,\alpha),$$

see [14, Theorems 3.8 and 3.9].

On the other hand, since $w^T y - \alpha\|y\| \geq 0$ for every $(w,\alpha) \in \mathbb{C}^{a*}$ and all $y \in \mathbb{C}$, then clearly $w^T y + \alpha\|y\| \leq 0$ for every $y \in -\mathbb{C}$. Thus we conclude that all the cones $S(w,\alpha) = \{y \in \mathbb{R}^n : w^T y + \alpha\|y\| \leq 0\}$ (see (8)) with $(w,\alpha) \in \mathbb{C}^{a*}$, contain the ordering cone $-\mathbb{C}$. Moreover if $(w,\alpha) \in \mathbb{C}^{a\#}$ then we have [14, Lemma 3.6]

$$-\mathbb{C} \setminus \{0\} \in int(S(w,\alpha)) = \{y \in \mathbb{R}^n : w^T y + \alpha\|y\| < 0\}. \quad (10)$$

Due to this property, the CS method guarantees to calculate "all" properly efficient solutions corresponding to the given weights and the given reference point. That is, every solution of the scalar problem $(CS(w,\alpha,a))$, is a properly efficient solutions of the multi-objective optimization problem (1), if $(w,\alpha) \in \mathbb{C}^{a\#}$, see Theorem 1 (iii).

In some cases for a given cone $\mathbb{C}$ and a given norm, there may be available to find a pair $(w,\alpha) \in \mathbb{C}^{a*}$ such that $\mathbb{C} = C(w,\alpha)$. For example if $\mathbb{C} = \mathbb{R}^n_+$ then

$$\mathbb{R}^n_+ = C(w^1,\alpha^1) = \{y \in \mathbb{R}^n : (w^1)^T y - \alpha^1\|y\|_1 \geq 0\}, \quad (11)$$

where $w^1 = (1,...,1) \in \mathbb{R}^n$, $\alpha^1 = 1$, and the l_1 norm is used in the definition (see [15, Lemma 4]). Similarly, $\mathbb{R}^n_-$ can be represented as a level set $S(w^1,\alpha^1)$ (see (8)) of the function

$$g_{(w^1,\alpha^1)}(y) = y_1 + \ldots + y_n + |y_1| + \ldots + |y_n| \quad (12)$$

in the form:

$$\mathbb{R}^n_- = S(w^1,\alpha^1) = \{(y_1,\ldots,y_n) \in \mathbb{R}^n : y_1 + \ldots + y_n + |y_1| + \ldots + |y_n| \leq 0\}. \quad (13)$$

Hence, it becomes clear that the presented scalarization method enables one to calculate minimal elements which are "supporting" elements of $f(\mathbb{X})$ with respect to the conic surfaces like $S(w,\alpha)$ (see (8)). In practice, one can divide the interval between 0 and $\min\{w_1,\ldots,w_n\}$ into several parts, and for all these values of the augmentation parameter α, the scalar problem $(CS(w,\alpha,a))$ can be solved for the same weights and the same reference point chosen. This will enable decision maker to compute different efficient solutions (if any) with respect to the same set of weights. The scalar problem $(CS(w,\alpha,a))$ is nonsmooth and nonconvex if the original problem is not convex. Such a problem can be solved by using some standard softwares (see, for example [9,12,22]), or special solution algorithms can be applied, see for example [6,8,16].

4 Relations with Other Methods

In this section we present theorems which establish relations between the CS, the Pascoletti-Serafini (PSS) and the Benson's (BS) scalarization methods. It is shown that some efficient solutions computed by the PSS and the BS methods, can be obtained by the CS method.

4.1 Relations between the Conic Scalarization (CS) and the Pascoletti-Serafini Scalarization (PSS) Methods

The method known as the Pascoletti-Serafini scalarization method, is studied in [10,18,21] by Tammer, Weidner, Winkler, Pascoletti and Serafini.

The scalar problem of the PSS method is defined as follows:

$$\begin{array}{rl} & minimize \quad t \qquad (PSS(a,r)) \\ \text{s.t.} & a + tr - f(x) \in \mathbb{C} \\ & x \in \mathbb{X}, t \in \mathbb{R}, \end{array}$$

where $a \in \mathbb{R}^n$ and $r \in \mathbb{C}$ are parameters of $(PSS(a,r))$. The problem $(PSS(a,r))$ can also be written in the form (see [4])

$$minimize \quad t \tag{14}$$

$$\text{s.t.} \qquad a + tr - \mathbb{C} \cap f(X) \neq \emptyset, \qquad t \in \mathbb{R}.$$

This problem can be interpreted in the following form. The ordering cone $\mathbb{C}$ is moved in direction $-r$ along the line $a + tr$ till the set $(a + tr - \mathbb{C}) \cap f(X)$ is reduced to the empty set. The smallest value $\bar{t}$ for which $(a + \bar{t}r - \mathbb{C}) \cap f(X) \neq \emptyset$ is the solution of (14). If the pair $(\bar{t}, \bar{x})$ is a solution of $(PSS(a,r))$ the element $\bar{y} = f(\bar{x})$ with $\bar{y} \in (a + \bar{t}r - \mathbb{C}) \cap f(X)$ will be characterized as a weakly minimal solution of (1).

Theorem 3. *Assume that $\mathbb{C}$ is a closed convex pointed cone with nonempty interior, and that $a \in \mathbb{R}^n$, $r \in int(\mathbb{C})$ and $(\bar{t}, \bar{x})$ is an optimal solution of $(PSS(a,r))$. Then, there exists a weight vector $\bar{w} = (\bar{w}_1, ...\bar{w}_n) \in \mathbb{C}^{\#}$ and an augmentation parameter $\bar{\alpha} \geq 0$ with $(\bar{w}, \bar{\alpha}) \in C^{a\circ}$ such that*

$$\min_{x \in \mathbb{X}} \bar{w}^T (f(x) - a) + \bar{\alpha} \| f(x) - a \| \leq \bar{t}.$$

Proof. Let $(w, \alpha) \in C^{a\circ}$. By definition of $C^{a\circ}$ (see also (9)), $\mathbb{C} \subset C(w, \alpha)$. Then problem $(PSS(a,r))$ can be written in the following form with possibly a broader set of feasible solutions:

$$\begin{array}{rl} & minimize \quad t \qquad (PSS_{C(w,\alpha)}(a,r)) \\ \text{s.t} & a + tr - f(x) \in C(w, \alpha) \\ & x \in \mathbb{X}, \end{array}$$

By definition of $C(w, \alpha)$, the inclusion $a + tr - f(x) \in C(w, \alpha)$ implies

$$w^T (a + tr - f(x)) - \alpha \| a + tr - f(x) \| \geq 0,$$

or

$$w^T (f(x) - a - tr) + \alpha \| f(x) - a - tr \| \leq 0.$$

Obviously,

$$\alpha (\| f(x) - a \| - \| tr \|) \leq \alpha \| f(x) - a - tr \|.$$

Then if we change the norm term by the left hand side in the above inequality, the set of feasible solutions of $(PSS_{C(w,\alpha)}(a,r))$ will again be extended:

$$w^T(f(x)-a)+\alpha\|f(x)-a\| \le tw^T r + |t|\alpha\|r\|. \tag{15}$$

In dependence on the sign of $\bar{t}$ we can consider only positive or only negative range for t in (15). If only negative (or only positive) values of t will be considered then the right hand side of (15) becomes $t(w^T r-\alpha\|r\|)$ (or $t(w^T r+\alpha\|r\|)$). Since $r \in int(C)$ and $(w,\alpha) \in C^{a\circ}$ we have $w^T r - \alpha\|r\| > 0$ (or $w^T r + \alpha\|r\| > 0$).

Thus, by dividing both sides of (15) with $w^T r-\alpha\|r\| > 0$ (or $w^T r+\alpha\|r\| > 0$) and denoting $\bar{w} = w/(w^T r - \alpha\|r\|)$ and $\bar{\alpha} = \alpha/(w^T r - \alpha\|r\|)$ (or $\bar{w} = w/(w^T r + \alpha\|r\|)$ and $\bar{\alpha} = \alpha/(w^T r + \alpha\|r\|)$), we obtain that the problem $(PSS_{C(w,\alpha)}(a,r))$ can be written (with a possibly broader feasible set) in the form:

$$minimize \quad t \tag{16}$$

$$\text{s.t} \qquad \bar{w}^T(f(x)-a)+\bar{\alpha}\|f(x)-a\| \le t \tag{17}$$

$$x \in \mathbb{X}, \tag{18}$$

This problem is equivalent to the following problem $(CS(\bar{w},\bar{\alpha},a))$:

$$\min_{x\in\mathbb{X}}[\bar{w}^T(f(x)-a)+\bar{\alpha}\|f(x)-a\|].$$

Since the set of feasible solutions of problem (16) - (18) is larger than the one of $(PSS(a,r))$, we obtain

$$\min_{x\in\mathbb{X}}[\bar{w}^T(f(x)-a)+\bar{\alpha}\|f(x)-a\|] \le \bar{t},$$

which completes the proof of theorem.

4.2 Relationship between the Conic Scalarization (CS) and the Benson's (BS) methods.

In this section we explain relationship between the BS and the CS methods. The idea of the BS method is to choose some initial feasible solution $x^0 \in X$ and, if it is not itself efficient, produce a dominating solution that is. To do so, nonnegative deviation variables $l_i = f_i(x^0) - f_i(x)$ are introduced, and their sum is maximized. This results in an x dominating x^0, if one exists, and the objective ensures that it is efficient, pushing x as far from x^0 as possible. The corresponding scalar problem for given x^0 is:

$$\begin{array}{c} \max \sum_{i=1}^n l_i \qquad (BS(x^0)) \\ s.t. \\ f_i(x^0) - l_i - f_i(x) = 0, \quad i=1,\ldots,n \\ l \geqq 0, x \in \mathbb{X}. \end{array}$$

Theorem 4. *Let $\bar{x}$ be an efficient solution to (1). Suppose that $\bar{x}$ is an optimal solution of Benson scalar problem $(BS(x^0))$ for a feasible solution $x^0 \in X$. Then $\bar{x}$ is an optimal solution of the conic scalar problem $CS(w^1, \alpha^1, f(x^0))$, where $w^1 = (1, \ldots, 1) \in \mathbb{R}^n$, $\alpha^1 = 1$, and the l_1 norm is used:*

$$min_{x \in X} \sum_{i=1}^{n} (f_i(x) - f_i(x^0)) + \sum_{i=1}^{n} |f_i(x) - f_i(x^0)|.$$

Proof. Since $f(x^0) \in f(X)$, $f(\bar{x}) \in f(x^0) - R^n_+$ and (see (13))

$$-R^n_+ = \{y : (w^1)^T y + \alpha^1 \|y\|_1 \leq 0\},$$

we have:

$$\sum_{i=1}^{n} (f_i(x) - f_i(x^0)) + \sum_{i=1}^{n} |f_i(x) - f_i(x^0)| = 0$$

for all $x \in X_0 = \{x \in X : f(x) \in f(x^0) - R^n_+\}$, and in particular for $x = \bar{x}$. Obviously,

$$\sum_{i=1}^{n} (f_i(x) - f_i(x^0)) + \sum_{i=1}^{n} |f_i(x) - f_i(x^0)| > 0$$

for all $x \in X \setminus X_0$ which completes the proof.

Theorem 5. *Let $\bar{x}$ be an optimal solution of Benson scalar problem $(BS(x^0))$ for a feasible solution $x^0 \in X$. Assume that $\bar{x}$ is a properly efficient solution to (1). Then there exists $\bar{\alpha} \in [0, 1)$ such that $\bar{x}$ is an optimal solution of the conic scalar problem $CS(w^1, \bar{\alpha}, f(\bar{x}))$, where $w^1 = (1, \ldots, 1) \in \mathbb{R}^n$ with the l_1 norm:*

$$min_{x \in X} \sum_{i=1}^{n} (f_i(x) - f_i(\bar{x})) + \bar{\alpha} \sum_{i=1}^{n} |f_i(x) - f_i(\bar{x})|. \tag{19}$$

Proof. We have:

$$-R^n_+ = \{y : (w^1)^T y + \alpha^1 \|y\|_1 \leq 0\},$$

and clearly

$$-\mathbb{R}^n_+ \setminus \{0\} \in int(\{y \in \mathbb{R}^n : (w^1)^T y + \alpha \|y\| \leq 0\}),$$

for every $\alpha \in [0, 1)$ (see (10)), where

$$int(\{y \in \mathbb{R}^n : (w^1)^T y + \alpha \|y\| \leq 0\}) = \{y \in \mathbb{R}^n : (w^1)^T y + \alpha \|y\| < 0\}).$$

Since $\bar{x}$ is a properly efficient solution to (1), there exists $\bar{\alpha} \in [0, 1)$ such that

$$\{f(\bar{x})\} + \{y \in \mathbb{R}^n : (w^1)^T y + \bar{\alpha} \|y\| \leq 0\} \cap f(X) = \{f(\bar{x})\}.$$

This leads

$$\{y \in \mathbb{R}^n : (w^1)^T (y - f(\bar{x})) + \bar{\alpha} \|y - f(\bar{x})\| \leq 0\} \cap f(X) = \{f(\bar{x})\}.$$

The last relation means that

$$(w^1)^T (f(x) - f(\bar{x})) + \bar{\alpha} \|f(x) - f(\bar{x})\| \geq 0$$

for every $x \in X$. which proves the theorem.

5 Conclusion

In this paper, the conic scalarization method is analyzed and main properties of solutions obtained by this method are explained. Additionally simple characterization of minimal elements is given. It has been emphasized that the conic scalarization method guarantee to generate the proper efficient solutions while it does not require any kind of convexity and/or boundedness assumptions. In addition the preference and reference point information of decision maker is taken into consideration by this method.

The paper also discussed relations between the conic scalarization method and Pascoletti-Serafini and Benson's scalarization methods. It has been shown that some solutions obtained by the Pascoletti-Serafini and Benson's scalarization methods, can also be obtained by the conic scalarization method.

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