# Pricing Strategies for Maximizing Viral Advertising in Social Networks

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**Abstract.** Viral advertising in social networks is playing an important role for the promotions of new products, ideas and innovations. It usually starts from a set of initial adopters and spreads via social links to become viral. Given a limited budget, one central problem in studying viral advertising is *influence maximization*, in which one needs to target a set of initial adopters such that the number of users accepting the advertising afterwards is maximized. To solve this problem, previous works assume that each user has a fixed cost and will spread the advertising as long as the provider offers a benefit that is equal to the cost. However, the assumption is oversimplified and far from real scenarios. In practice, it is crucial for the provider to understand how to incentivize the initial adopters.

In this paper, we propose the use of concave probability functions to model the user valuation for sharing the advertising. Under the new pricing model, we show that it is NP-hard to find the optimal pricing strategy. Due to the hardness, we then propose a discrete greedy pricing strategy which has a constant approximation performance guarantee. We also discuss how to discretize the budget to provide a good trade-off between the performance and the efficiency. Extensive experiments on different data sets are implemented to validate the effectiveness of our algorithm in practice.

**Keywords:** Viral advertising · Influence maximization · Social networks

## 1 Introduction

The emergence and proliferation of online social networks such as Facebook, Twitter and Google+ have greatly boosted the spread of information. People are actively engaged in the social networks and generating contents at an ever-increasing rate. Viral advertising, which utilizes information diffusion for the promotions of new products, ideas and innovations, has attracted enormous attentions from companies and providers. Compared with TVs, newspapers and radios which broadcast advertising, viral advertising in social networks has the effect of "word-of-mouth" which is considered to be more trustworthy. Moreover, the information diffusion between users can spread across multiple links and trigger large cascades of adoption.


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To start a cascade of viral advertising, one would first incentivize a set of initial adopters and let them spread the advertising further. For example, popular e-commerce platforms like Amazon, e-Bay and JD[1] all encourage people to share information of the products they buy in their social networks for advertising. In JD, users could get vouchers if they comment and share the products information. Suppose there is a limited budget, a question arising naturally is: How should we distribute the budget, so that the spread of viral advertising can be maximized. The process can be described as a two-step **pricing strategy**: we first offer each user a discriminative price as incentive; The users who accept the incentive should then share the advertising in social networks.

JD's strategy is to offer each user uniform price, regardless of their valuations and influence. Alternatively, we may allocate the budget proportionally so that users with high influence can be allocated high price. In general, to determine who should be offered price and how much should be offered to each of them, it is crucial to understand both user valuations for being initial adopters and the information diffusion process. From the perspective of users, their valuations for being initial adopters usually depend on the price offered. For the advertisers, they may expect more users with high influence to share and spread the information.

A similar problem that has been extensively studied is *influence maximization*. The problem aims to select the most influential set of users as initial adopters to maximize the spread of influence. However, the underlying assumption that each user has an inherent *constant value* for being initial adopter may not be reasonable. In practice, users' decisions are often not deterministic: They may decline or ignore the offered price, leading to unpredictability of the information spread. In comparison, in this work, we address that the user decisions for being initial adopters are probabilistic rather than constant. Given the probability distribution, we study optimal pricing strategies to maximize the viral advertising under some well-studied diffusion models.

### 1.1 Our Results

The main contributions of this paper are:

- We introduce a concave probabilistic model in which users' values are distributed according to some concave functions. The model is practical and can characterize different users' preferences for sharing the advertising in social networks.
- We formalize the optimization problem and show that it can be reduced to NP-hard quadratic programming problem. Due to the hardness, we propose an approximate discrete greedy algorithm with near optimal result. We further analyze how to discretize the budget to provide a good tradeoff between the performance and efficiency.

[1] JD(http://www.jd.com) is the largest online direct sales company in China.

– Extensive experiments on different data sets are implemented to validate the effectiveness of our algorithm. Our algorithm significantly outperforms other algorithms in almost all cases. In addition, we evaluate the discrete greedy algorithm with respect to different granularity. The results reveal that our algorithm can converge asymptotically.

### 1.2 Related Work

Our work has a strong tie with the problem *influence maximization*, which was first proposed by Domingos and Richardson [11,19]. Later, Kempe et al. [14] formulated the problem as a discrete optimization problem, and proposed greedy algorithm with hill-climbing strategies to find the influential nodes. Due to the monotonicity and submodularity of the information diffusion process, the algorithm can be proved to achieve constant approximation ratio. Following their work, extensive researches [8–10,15,21] have studied algorithmic improvement of the spread of influence in social networks. Despite a lot of progress in choosing which nodes to select, the problem of how to incentivize the initial nodes is often neglected.

Another thread of our work is inspired by the problem of *revenue maximization* which was first introduced by Hartline et al. [12]. In order to influence many buyers to buy a product, a seller could first offer some popular buyers discounts. The problem then studies marketing strategies like how large the discounts be and in what sequence should the selling happen. Specifically, the work assumed that the willingness that each user may pay for a product is drawn from some given probability distributions. A lot of following works have studied revenue maximization in social networks. Some of them have considered Nash Equilibrium between users for purchasing one product. Different pricing strategies were designed to maximize the revenue, i.e., uniform pricing [5,7], discriminative pricing [3,22], iterative pricing [1] etc.

In a recent work of Singer [21], the author considered auction based influence maximization in which each user can bid a cost for being an initial adopter. To make sure that each user declares the true cost, they designed incentive compatible mechanisms. However, the mechanism requires extra step for each user to bid a cost and may be cumbersome to implement in practice. Comparatively, we adopt a pricing strategy with a more natural way for incentivizing each user. In [10], Demaine et al. proposed partial incentives in social networks to influence people. The pricing and influence models in our paper generalize the models in [10] and we further analyze the discrete setting of the problem.

## 2 Preliminaries

We model the social network as a directed (undirected) graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where node set $\mathcal{N}$ represents users and edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ represents social relationships between them. Given a limited budget $B$, a pricing strategy is to distribute the budget among users to maximize the spread of the viral advertising. In this section, we first introduce the pricing model and information diffusion models respectively. Then, we will formulate our optimization problem and establish its hardness.

### 2.1 Pricing Model

After distributing the budget, we use the pricing model to characterize users' valuations for being initial adopters, i.e., how much price is it need to incentivize a user? In *influence maximization*, users are assumed to have some *constant value* for being initial adopters. However in general cases, the users' decisions are often not deterministic. In comparison, we address that users have different valuations and the valuations are dependent on the allocated price.

In this paper, we propose a probabilistic model in which the user values are drawn from some prior known distributions: $\mathcal{F} = \{F_i | i \in \mathcal{N}\}$. Suppose that user $i$ is offered a price $p_i$ as incentive, then $F_i(p_i)$ is the probability that $i$ will accept the price for being an initial adopter. Literatures like [13,17] have observed that the marginal gain of user satisfaction decreases as the price increases, which arises naturally in practical situations as *diminishing returns*. Accordingly, we also regard the cumulative functions $F_i$ as *concave functions*, i.e. for any $x$ and $y$ in the interval and for any $t \in [0, 1]$, $F_i(tx+(1-t)y) \geq tF_i(x)+(1-t)F_i(y)$. Fig. 1 presents some examples of the possible distribution functions. In Fig. 1(a), the user's value is distributed uniformly in the range of $[0, \tau]$, where $\tau$ is a constant threshold. In Fig. 1(b), the valuation is drawn from $F_i(p_i) = \sqrt{\frac{p_i}{d_i+1}}$, where $d_i$ is the degree of user $i$.

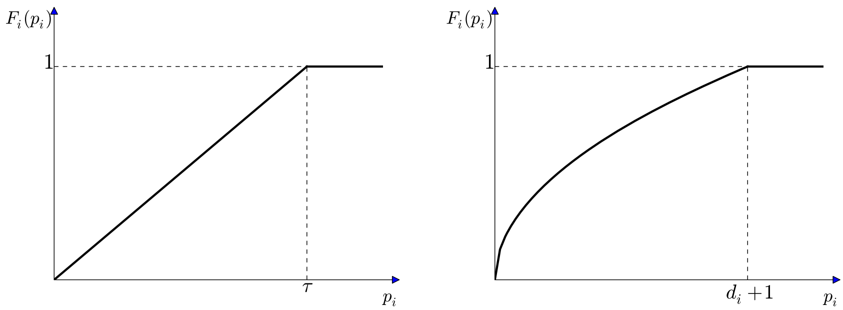



**Fig. 1.** Examples of user valuation distribution function

### 2.2 Information Diffusion Models

After distributing the budget, a set of initial adopters $S$ will share the advertising to trigger a cascade. The advertising will spread in the social networks as a piece of information. In an information cascade, we say that a node is *active* if it adopts the information, otherwise it is called *inactive*. Initially, only the users in $S$ are active. The information then spreads via social links to influence more users. The number of active users after the cascade stops is the spread of the viral advertising, or the influence of $S$, denoted as $\sigma(S)$. Modeling the information diffusion process has been extensively studied [2,4,6,14,16]. We introduce some of the most widely used models here.

*Coverage model.* In the Coverage model [21], each user $i$ is associated with the set of its neighbors $N(i)$, which is also the influence of user $i$. The information will not further spread. So the influence of the $S$ will be $\sigma(S) = |\bigcup_{i \in S} N(i)|$.

*Independent Cascade model.* In IC model [14], the information starts from $S$ as follows: At step $t$, the newly activated nodes in $S_t$ try to activate their neighbors independently. Each active node $u$ succeeds in activating its neighbor $v$ with probability $\mu_{(u,v)}$. The newly activated nodes are added into set $S_{t+1}$. The process continues until $S_t = \emptyset$.

*Linear Threshold model.* In LT model [14], for each neighbor $w$ a node $v$ associates a weight $\omega_{v,w} \geq 0$ where $\sum_{w \in N(v)} \omega_{v,w} \leq 1$, and chooses some threshold $\theta_v \in [0,1]$ uniformly at random. The node $v$ is activated at time step $t$ if $\sum_{w \in N_t(v)} \omega_{v,w} \geq \theta_v$, where $N_t(v)$ denotes the neighbors of $v$ that are active at time step $t$.

### 2.3 Problem and Optimization Objective

Given a budget $B$, a pricing strategy will identify a price vector $\mathbf{p} = (p_1, p_2, ..., p_n), (\sum_i^n p_i = B)$, where $p_i \in \mathbb{R}_{\geq 0}$ is the price offered to user $i$. We use $\mathbf{p_{-i}}$ as the price list offered to users except $i$. There are two ways for a user $i$ to get active: On one hand, $i$ may accept the offered price $p_i$ with probability $F_i(p_i)$; On the other hand, $i$ could be influenced by other users with probability $q(S, i)$, where $q(S, i)$ is a reachability function of the probability that the nodes in $S$ could influence $i$ under some diffusion models. The function can also be written as $q(S(\mathbf{p_{-i}}), i)$, representing the probability that users accepted $\mathbf{p_{-i}}$ could influence $i$. Thus, the probability $w_i(\mathbf{p})$ that $i$ could get active after the cascade, can be formulated as:

$$w_i(\mathbf{p}) = 1 - (\underbrace{1 - F_i(p_i)}_{i\ does\ not\ take\ p_i})(\underbrace{1 - q(S(\mathbf{p_{-i}}), i)}_{i\ is\ not\ influenced}) \tag{1}$$

Given the probability that each user gets active, the objective can be formulated as an optimization problem in which the goal is to maximize overall expected number of active users after the cascade:

$$\begin{aligned} \max \ f(\mathbf{p}) &= \sum_{i \in \mathcal{N}} w_i(\mathbf{p}) \\ \text{s.t.} : \ &\sum_{i \in \mathcal{N}} p_i \leq B \end{aligned} \tag{2}$$

**Hardness.** Now we show that the above optimization problem is NP-hard. In particular, it is NP-hard even if the social network forms a line structure.

**Theorem 1.** *Identifying the optimal pricing strategy is NP-hard even when the social network forms line structure.*

*Proof.* Consider an instance of the quadratic programming problem:

$$\min \ g(\mathbf{p}) = \frac{1}{2}\mathbf{pQp^T} + \mathbf{cp^T}$$
$$\text{s.t.}: \sum_{i=1}^{n} p_i \leq B$$

where $Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$ and $c = (-2, -2, ..., -2, -1)$. Since $\mathbf{pQp^T} = 2\sum_{i=2}^{n} p_i p_{i-1}$, $\mathbf{Q}$ is indefinite. According to [20], the quadratic programming function $g(\mathbf{p})$ is NP-hard. We show that the programming can be viewed as a special case of the pricing function $f(\mathbf{p})$.

Given the instance of the quadratic programming problem, we define a corresponding instance of the pricing function under the Coverage model where the social network $G$ forms a line structure as presented in Figure 2.

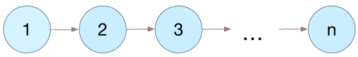



**Fig. 2.** Line structure of the social network

In this case, the probability that a user gets active with pricing vector $\mathbf{p}$ is

$$w_i(\mathbf{p}) = \begin{cases} p_1 & i = 1 \\ 1 - (1 - p_i)(1 - p_{i-1}) & i \geq 2 \end{cases}$$

In this way, $f(\mathbf{p}) = 2\sum_{i=1}^{n} p_i - \sum_{i=2}^{n} p_i p_{i-1}$ is the negative of $g(\mathbf{p})$. So maximizing function $f(\mathbf{p})$ is equivalent to minimizing the function $g(\mathbf{p})$, which is also NP-hard.

## 3 A Continuous Greedy Process

Due to the hardness result from Theorem 1, no polynomial algorithm exists for the optimization problem unless $P = NP$. To motivate our approximation algorithm, in this section, we present a continuous greedy process with constant approximation ratio. Even though the process takes infinite steps for implementation, it provides analytic picture behind our main algorithm. Intuitively, the process allocates the budget smoothly to the users with high influence and low valuation. It can be formally regarded as a particle $p(t)$ starting from $p(0) = 0$ and following a certain flow over a unit time interval:

$$\frac{d\mathbf{p}}{dt} = v(p)$$

where $v(p)$ is defined as

$$v(p) = \arg\max_{i \in \mathcal{N}} \left( \frac{f(\mathbf{p} + p_i) - f(\mathbf{p})}{p_i} \right) \quad (p_i = \epsilon \rightarrow 0) \tag{3}$$

until $t = 1$.

At each interval, we allocate a small unit of budget ($\epsilon \rightarrow 0$) to the user with maximal marginal gain. Since the marginal gain of each user is a function of the allocated budget, we have to run the process continuously to get the local optimum. This process provides good approximation :

**Lemma 1.** *Let the optimal value be $OPT$, the continuous greedy process has $1 - 1/e$ approximation ratio, i.e., $f(p(1)) \geq (1 - 1/e)OPT$.*

The proof of Lemma 1 borrows idea from the problem of maximization of *smooth submodular functions*. For the completeness of the proof, we first introduce the concept of *smooth submodular function*.

**Definition 1.** *A function $f : [0,1]^X \rightarrow \mathbb{R}$ is* ***smooth monotone submodular*** *if*

- *$f \in C_2([0,1]^X)$, i.e., it has second partial derivatives everywhere.*
- *For each $j \in X$, $\frac{\partial f}{\partial y_j} \geq 0$ everywhere (monotonicity).*
- *For any $i, j \in X$ (possibly equal), $\frac{\partial f}{\partial y_i \partial y_j} \leq 0$ everywhere (submodularity).*

In Definition 1, $\frac{\partial f}{\partial y_i \partial y_j} \leq 0$ indicates that a function is smooth submodular if it is concave along any non-negative direction. Examining the objective formulation 2, it can be easily proved that $f$ satisfies the above conditions so is smooth submodular. Vondrák et al. [23] showed that the continuous greedy process as presented above can achieve $1 - 1/e$ approximation ratio to maximize a smooth submodular function, which concludes Lemma 1.

## 4 Pricing Strategies for Viral Advertising

Following the idea of the continuous greedy process, in this section, we consider discretizing the budget so that the process can run in polynomial time. Our main result is a discrete greedy algorithm with constant approximation ratio. We further discuss how the discrete granularity affects the result of the algorithm.

### 4.1 A Discrete Greedy Strategy

In our discrete pricing strategy, we first divide the budget $B$ into $m$ equal pieces, denoted as $\mathcal{B} = \{b_1, b_2, ..., b_m\}$, where $b_j = B/m$ for all $j \in \{0, 1, 2, ..., m\}$. In this setting, the prices offered to each user are disjoint subsets from $\mathcal{B}$, namely $A_i \subseteq \mathcal{B}, A_i \cap A_j = \varnothing$ for all $i, j \in \mathcal{N}$. Thus, the pricing strategy can be regarded as a mapping problem in which $m$ elements of the budget set are to be allocated to $n$ users. The objective is then a set function where the input is a subset from the ground set: For the user set $\mathcal{N}$ and pricing set $\mathcal{B}$, define the new ground set as $X = \mathcal{N} \times \mathcal{B}$. By associating the variable $a_{ij}$ with user-price pairs, i.e., $a_{ij}$ means assigning price $b_j$ to user $i$, the ground set $X = \{a_{11}, a_{12}, ..., a_{1m}, a_{21}, a_{22}..., a_{2m}, ..., a_{n1}, a_{n2}, ..., a_{nm}\}$ is the set of all price elements that can be chosen. Note that each piece of price is replicated $n$ times in the set $X$, e.g. the price $b_j$ is replicated as $a_{1j}, a_{2j}, ..., a_{nj}$, only one of them can be chosen for the solution. Slightly abusing notations, we use $\sigma(A) : 2^X \to \mathbb{R}_+$ as the expected number of active users after the cascade by choosing price set $A = \bigcup_{i \in \mathcal{N}} A_i, |A| = m$. The problem of finding the optimal pricing strategy in discrete setting can be described as choosing a set $A$ ($|A| = m, A \subseteq X$) of $m$ elements from $X$ to maximize the the expected number of active users $\sigma(A)$, with the constraint that only one of $\{a_{1j}, a_{2j}, ..., a_{nj}\}$ can be chosen for all $j \in \{1, 2, ..., m\}$.

**Coverage Model.** We begin by maximizing the objective function under Coverage model. There are two reasons we first consider the Coverage model. On one hand, the diffusion process under the Coverage model can be regarded as information exposures to users, i.e, a user is influenced if and only if one of his/her neighbors is an initial adopter; Moreover, the Coverage model is simplistic and exhibits similar properties as the IC and LT model.

To optimize the spread of viral advertising, we first prove that the pricing function $\sigma(\cdot)$ is monotone and submodular in the discrete setting. A function $\sigma(\cdot)$ is submodular if for any element $a \in X$, for all $V \subseteq T$, there is $\sigma(V \cup \{a\}) - \sigma(V) \geq \sigma(T \cup \{a\}) - \sigma(T)$. Submodularity implies that the marginal gain of choosing an element decreases as the number of chosen elements increases.

**Theorem 2.** *The pricing function $\sigma(\cdot) : 2^X \to \mathbb{R}_+$ is monotone submodular under Coverage model if the cumulation distribution functions of user valuation functions $F_i(\cdot) : \mathbb{R}_+ \to [0, 1], i \in \mathcal{N}$ are non-decreasing concave functions.*

*Proof.* **Monotonicity** Obviously, by adding a new pricing element $a_{ij}$ to the outcome set $A$, namely assigning the $j$th piece of price to user $i$, the probability that user $i$ could get active will not decrease. Accordingly, the probability that user $i$ influences other users $q(\{i\}, \cdot)$ will also not decrease, which concludes that $\sigma(\cdot)$ is non-decreasing monotone.

**Submodularity** For submodularity, let $\delta(a_{ij})$ denote the marginal gain of $\sigma(\cdot)$ by adding the element $a_{ij}$. It can be formulated as the sum of increased probability from all users:

$$\delta(a_{ij}) = \sigma(A^j \cup \{a_{ij}\}) - \sigma(A^j) = \delta_i(a_{ij}) + \sum_{k \in \mathcal{N} \setminus \{i\}} \delta_k(a_{ij}) \tag{4}$$

where $A^j$ is set of first $j-1$ price elements that have been allocated, and $\delta_i(a_{ij})$ is the *marginal gain* of user $i$. As the class of submodular functions is closed under non-negative linear combinations, we only need to prove that the function for each user is submodular respectively.

For user $i$, by adding a price element $a_{ij}$, the increased probability is:

$$\begin{aligned}\delta_i(a_{ij}) &= w_i(A^j \cup \{a_{ij}\}) - w_i(A^j) \\ &= (F_i(A^j \cup \{a_{ij}\}) - F_i(A^j))(1 - q(S(A^j), i))\end{aligned}$$

Since submodularity is the discrete analog of concavity, there is $F_i(V \cup \{a_{ij}\}) - F_i(V) \geq F_i(T \cup \{a_{ij}\}) - F_i(T)$ if $V \subseteq T$. Meanwhile, $1 - q(S(A^j), i)$ does not change with $a_{ij}$, we can conclude that $w_i(V \cup \{a_{ij}\}) - w_i(V) \geq w_i(T \cup \{a_{ij}\}) - w_i(T)$ if $V \subseteq T$, which indicates that $w_i(\cdot)$ is submodular.

Similarly, for any other user $k$, the increased probability can be formulated as:

$$\delta_k(a_{ij}) = (q(S(A^j \cup \{a_{ij}\}), k) - q(S(A^j), k))(1 - F_k(A^j))$$

In the Coverage model, $q(S, i) = 1$ means $i$ is a neighbor of $S$, and 0 otherwise. According to the monotonicity of $S(\cdot)$, $S(V) \subseteq S(T)$ if $V \subseteq T$. So by adding an element $a_{ij}$, the smaller set $S(V \cup \{a_{ij}\}) - S(V)$ is more likely to influence a node $k$, i.e., $q(S(V \cup \{a_{ij}\}), k) - q(S(V), k) \geq q(S(T \cup \{a_{ij}\}), k) - q(S(T), k)$. So $w_k(\cdot)$ is also submodular.

Summing up the increased probabilities of all users, we have $\sigma(V \cup \{a_{ij}\}) - \sigma(V) \geq \sigma(T \cup \{a_{ij}\}) - \sigma(T)$ if $V \subseteq T$, showing that $\sigma(\cdot)$ is submodular. □

According to the work of Nemhauser et al. [18], finding a set $A$ of with uniform matroid $(m)$ to maximize the monotone submodular function is NP-hard and a greedy algorithm with hill-climbing strategy approximates the optimal to a factor of $1 - 1/e$. Let the optimal value in the discrete setting be $OPT_d$, we can conclude our main theorem as:

**Theorem 3.** *A greedy hill-climbing strategy can achieve* $1 - 1/e$ *approximation ratio of the optimal pricing strategy in the discrete setting, i.e.,* $f(A) \geq (1 - 1/e)OPT_d$.

Following Theorem 3, we now present our greedy pricing strategy in Algorithm 1. In each step of the algorithm, we greedily choose the user that has the largest marginal gain by allocating a piece of price.

**IC Model and LT Model.** As for the IC model and LT model, we show that the pricing function $\sigma$ is still monotone and submodular. By adopting a different diffusion model, we have a different diffusion function $q$. Recall that

**Algorithm 1.** *DiscreteGreedy*$(\mathcal{G}, \mathcal{B})$

```
A ← ∅;
for j ← 1 to m do
    s ← 0, max ← −1;
    for i ∈ N do
        δ(a_ij) ← σ(A^j ∪ {a_ij}) − σ(A^j);
        if δ(a_ij) > max then
            s ← i, max ← δ(a_ij);
    A ← A ∪ {a_sj};
return A
```

by adding a new pricing element $a_{ij}$, the marginal gain of $\sigma(\cdot)$ is the sum of the increased probability from all users. For user $i$, the function $\sigma_i(\cdot)$ is still submodular since the reachability function remains the same. For any other user $k$ $(k \neq i)$, the marginal gain is $\delta_k(a_{ij}) = (q(S(A^j \cup \{a_{ij}\}), k) - q(S(A^j), k))(1 - F_k(A^j))$. Since $S(\cdot)$ is monotone, $\sigma(\cdot)$ is submodular if and only if the reachability function $q(S, k)$ is submodular. In [14], Kempe et al. have already showed that the functions are submodular under the IC and LT model [14]. So the pricing function $\sigma$ is also submodular.

**Lemma 2.** *For submodular influence functions like IC and LT model, the pricing function $\sigma$ is also monotone and submodular, and Algorithm 1 approximates the optimal to a factor of* $1 - 1/e$.

### 4.2 How to Choose $m$?

Despite the near optimal results from Algorithm 1, for more general situations, we would like to know how it approximates the optimal value $OPT$. In this section, we will discuss how to choose the discrete granularity $m$ to achieve a good tradeoff between the performance and efficiency.

Apparently, by increasing $m$ to $\infty$, the discrete greedy algorithm is close to the continuous greedy process which also takes infinite steps. To the other extreme, if $m$ decreases to 1, the algorithm is simply choosing one user to allocate. To achieve a good trade-off between the performance and the algorithm efficiency, we focus on deriving the gap between the continuous process $f(p(1))$ and the discrete greedy strategy $f(A)$, since the continuous greedy process approximates well to the optimal value $OPT$.

**Lemma 3.** *When $m \geq O(n)$, $f(A) \geq (1 - 1/e - o(1))OPT$ with high probability.*

*Proof.* After allocating first $j-1$ pieces of the total budget ($A_j = \frac{j-1}{m}B$), the continuous process will get a price vector $\mathbf{p_j}$ and the greedy pricing strategy will get a set of $A^{j+1}$. Observe the marginal gain by increasing price $a = B/m$. For the continuous process, the marginal gain is $\delta(\mathbf{a}^*) = f(\mathbf{p_{j+1}}) - f(\mathbf{p_j})$, where $\mathbf{a}^* = \mathbf{p_{j+1}} - \mathbf{p_j}$ and $|\mathbf{a}^*| = B/m$. For the discrete greedy strategy, we choose

$a_{ij}$ at the $j$th step, where $i = arg\max_{i \in \mathcal{N}}(f(A^j \cup \{a_{ij}\}) - f(A^j))$. Consider the marginal gain $\delta(a_{ij}) = f(\mathbf{p_j} + a_{ij}) - f(\mathbf{p_j})$. Taking Taylor series and bounding the lower items, we have:

$$\delta(\mathbf{a}^*) = \delta(a_{ij}) + R(\xi) \tag{5}$$

where $\xi = (1-c)\mathbf{a}^* + ca_{ij}$, $c \in [0,1]$. For $R(\xi)$, there is:

$$\begin{aligned} R(\xi) &= \frac{\partial \delta}{\partial \xi}(\mathbf{a}^* - a_{ij}) = \sum_{k \in \mathcal{N}} \frac{\partial \delta}{\partial \xi_\mathbf{k}} \mathbf{a}^*_\mathbf{k} - \frac{\partial \delta}{\partial \xi_\mathbf{i}} a_{ij} \\ &\leq (\frac{\partial f}{\partial \mathbf{p_j}} - \frac{\partial f}{\partial \mathbf{p_{j+1}}})a - \frac{\partial \delta}{\partial \xi_\mathbf{i}} a = Ca \end{aligned} \tag{6}$$

where $C$ is a constant. So when $a \leq O(1/n)$, namely $m \geq n$, the error $R(\xi)$ is within $O(1/n)$, which concludes that $\delta(\mathbf{a}^*) \approx \delta(a_{ij})$. Accordingly, there is $f(A) \geq (1 - 1/e - o(1))OPT$ with high probability. □

# 5 Evaluations

In addition to the provable performance guarantee, in this section, we conduct extensive simulations to evaluate our DiscreteGreedy pricing strategy. We first compare our proposed algorithm with several intuitive heuristics. Then we will observe how the discrete granularity of the budget affects the results of the DiscreteGreedy strategy.

## 5.1 Experiment Setup

Our experiments are conducted on 3 real social network data sets. The first is CondMat data set which contains $23,133$ nodes and $93,497$ undirected edges. The nodes represent authors and an undirected edge $(i,j)$ represents coauthor relationships between them. The second is Youtube data set which contains $560,123$ nodes and $1,976,329$ undirected edges. The nodes are users and the edges indicates friendship relationships between them. The last data set is Weibo data set which has $877,391$ nodes and $1,419,850$ directed edges. Weibo is a Twitter-like mirco blog in China. The directed edge $(i,j)$ means that user $i$ is following $j$. All 3 data sets exhibit small world, high clustering complex network structural features.

Since our algorithm is applicable to any forms of concave functions of user valuations, we manually set the distributions in the experiments. We first consider a uniform symmetric setting in which $F_i(p_i) = \frac{p_i}{\tau}$. The user values are distributed uniformly in the range $(0,\tau]$. The second distribution function we consider is $F_i(p_i) = \sqrt{\frac{p_i}{\tau}}$. The function also has a threshold $\tau$ and the acceptance probability is proportional to the square root of the offered price. We also consider differential valuation functions where the user valuation is a function of their degree: $F_i(p_i) = \frac{r+d_i+1}{r+p_i}\frac{p_i}{d_i+1}$. The parameter $r$ is used to control the shape of the curve: a larger value of $r$ indicates a steeper curve. The threshold here is $d_i + 1$.

For the information diffusion models, since all of them exhibit monotone submodular properties, we mainly conduct experiments under the Coverage model. In this model, the influence of user $i$ is the set of users that follows $i$ in Weibo data set. In CondMat and Youtube data set, it is the set of friends\coauthors of user $i$. We also considered the IC and LT model. As the information diffusion process under these two models are stochastic processes, we need to take Monte Carlo methods to generate fixed graphs and take the average influence of different probability results. In the IC model, without loss of generality, we assume uniform diffusion probability $\mu$ on each edge and assign $\mu$ to be 0.01. In the LT model, the weight of a directed edge $(i, j)$ is $\frac{1}{d_j}$, so the sum of the weights will not exceed 1.

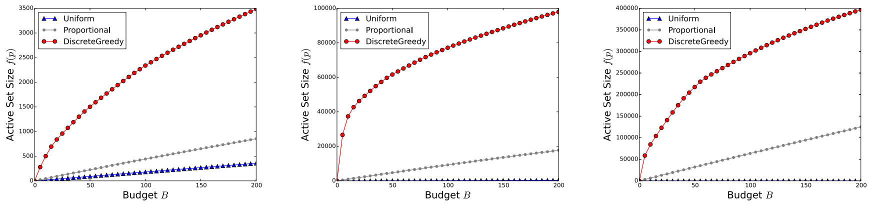
**Comparison Methods.** In our DiscreteGreedy algorithm, we discretize the budget into $O(n)$ pieces in our DiscreteGreedy pricing strategy to obtain near optimal pricing solutions. We compare our greedy algorithm with the following 3 heuristics:

– *Uniform*: The Uniform pricing strategy adopts a simple idea that all users get the same price $p_i = \frac{B}{n}$, regardless of users' influence and valuation.
– *Proportional*: In the Proportional pricing strategy, the price offered to user $i$ is proportional to $d_i$, i.e., $p_i = \frac{d_i}{\sum_{j \in \mathcal{N}} d_j} B$.
– *FullGreedy*: We use the greedy algorithm in *influence maximization* to select the initial adopters and offer each of them the threshold price.

### 5.2 Results

**The Spread of Viral Advertising.** In Fig. 3 to 5, we separately present the results w.r.t different valuation functions under the Coverage model. The $x$-axis represents the total budget that is allocated. The $y$-axis is the active set size of the users after cascade, or the spread of the viral advertising.

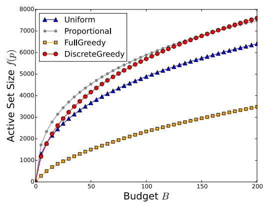
We first evaluate the viral advertising spread when user values are distributed as $F_i(p_i) = \frac{p_i}{\tau}$. In this case, since the valuation function grows linearly to the threshold $\tau$, the marginal gain of a user will remain the same in the range $p_i \in (0, \tau]$. Therefore, the FullGreedy algorithm has the same result as the DiscreteGreedy algorithm, which will not be presented here. Obviously from Fig. 3(a) to 3(c), our DiscreteGreedy algorithm outperforms other heuristics significantly. The results of the Proportional strategy grow almost linearly as the allocated budget increases. There are two reasons for this: First, the probability for a user to accept the price is linear as $F$; Second, there are not much overlap between the influenced users in this case. There is a large gap between the DiscreteGreedy and the Proportional strategies, indicating that allocating the budget more concentratedly may have better result. In the Uniform algorithm, the price offered to each user is quite low. So few users might accept to be initial adopters, leading to poor performance of the algorithm in all 3 data sets. This illustrates that differential pricing strategy is necessary for the viral advertising.



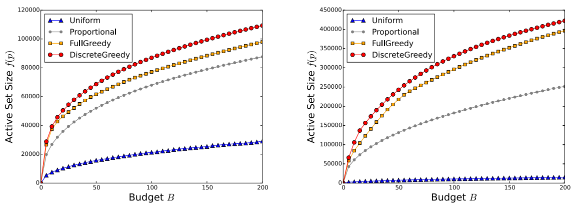

(a) CondMat (b) Youtube (c) Weibo

**Fig. 3.** The spread of viral advertising with valuation function $F_i(p_i) = \frac{p_i}{\tau}$

In Fig. 4, we set the pricing function as $F_i(p_i) = \sqrt{\frac{p_i}{\tau}}$. In this case, the users are more likely to be initial adopters with small amount of price. The user who has the maximal marginal gain at beginning may not hold as the allocated budget increases. The FullGreedy allocates each user full prices $\tau$, so it has a worse result than the DiscreteGreedy strategy. Both the Proportional and the Uniform algorithm perform better than in Fig. 4, due to the reason that users with low allocated price also have a higher probability to be active. However, the performance of the 3 heuristics vary distinctly in different data sets.



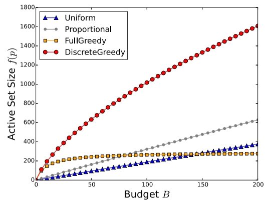

(a) CondMat



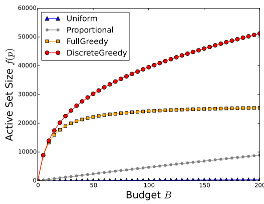

(b) Youtube (c) Weibo

**Fig. 4.** The spread of viral advertising with valuation function $F_i(p_i) = \sqrt{\frac{p_i}{\tau}}$

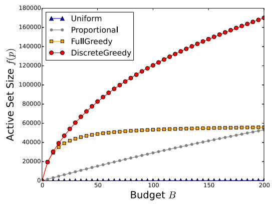
Fig. 3 presents the results with the valuation distribution function as $F_i(p_i) = \frac{r+d_i+1}{r+p_i}\frac{p_i}{d_i+1}$. In this function, users have different valuation distribution functions which is related with their degree. For a user with degree $d_i$, the threshold will be $d_i + 1$. We set $r = 10$ so the valuation function has a steeper curve than the above two functions. In this way, the FullGreedy strategy performs even worse since the marginal gain of each user decreases very quickly when the allocated price increases. Note that the threshold is actually not necessary in practice. The FullGreedy strategy is infeasible if users don't have a threshold. In the Proportional algorithm, as the price offered increases proportionally with users' degree, the probability that different users accept the price are almost the same. Therefore, it still grows linearly in this case. The Uniform strategy is quite inefficient in large data sets such as Youtube and Weibo.




(a) CondMat




(b) Youtube

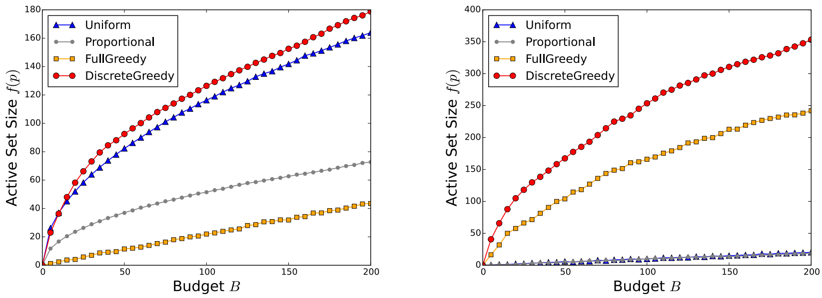



(c) Weibo

**Fig. 5.** The spread of viral advertising with valuation function $F_i(p_i) = \frac{r+d_i+1}{r+p_i}\frac{p_i}{d_i+1}$

In Figure 6(a) and Figure 6(b), we fix the user valuation function as $F_i(p_i) = \sqrt{\frac{p_i}{\tau}}$ and conduct the experiments under the IC model and LT model. As the information diffusion processes are stochastic, we simulate the graph massive times (1000) and conduct the experiments on a smaller sample of the Weibo data set. As shown in Fig 6, our DiscreteGreedy strategy still has the best performance.

**The Accuracy.** Finally, we run the DiscreteGreedy strategy in CondMat data set w.r.t different granularity. We discretize the budget into $m$ pieces where $m$ ranges from 1 to $10n$. The results are presented in Table 1. We start from $m = 1$ ($n/m = 23,333$) and gradually increase $m$ to $n$ ($n/m \approx 1$). Apparently, the active set size converges when $m$ approaches $n$, i.e. $n/m \approx 1$. When user valuations are linear functions, i.e., when $F_i(p_i) = \frac{p_i}{\tau}$, the DiscreteGreedy strategy is likely to pay the current selected user full price. So the results can converge when the granularity is the threshold $\tau$. In other cases, though the active set




(a) IC model (b) LT model

**Fig. 6.** The spread of advertising in the Weibo data set in the IC and LT model

size converges slower, the results can stabilize when $m \to n$. When $m > n$, the result is not likely to grow much.

**Table 1.** Active set size in CondMat data set w.r.t. different granularity

| $O(n/m)$ \ $F_i$ | $\frac{p_i}{5}$ | $\sqrt{\frac{p_i}{\tau}}$ | $\frac{r+d_i+1}{r+p_i} \frac{p_i}{d_i+1}$ |
|---|---|---|---|
| 23,133 | 281.00 | 281.00 | 277.14 |
| 1,000 | 3309.24 | 3620.32 | 1541.26 |
| 100 | 3452.58 | 5296.93 | 1605.29 |
| 10 | 3485.00 | 6939.77 | 1608.26 |
| 1 | 3485.00 | 7599.42 | 1609.14 |
| 0.1 | 3485.00 | 7600.15 | 1609.21 |

## 6 Conclusion

In this work, we studied optimal pricing strategies in a social network to maximize viral advertising with budget constraint. We formalized the pricing strategy as an optimization problem and established its hardness. A novel *DiscreteGreedy* algorithm with near optimal results was proposed, and the tradeoff between the performance and efficiency was discussed. Extensive evaluations showed that our DiscreteGreedy algorithm outperforms other intuitive heuristics significantly in almost all cases. Moreover, the DiscreteGreedy algorithm can converge to a hight accuracy if the budget is discretized properly.

For possible future works, we are interested in the following aspects. First, we would like to study the user valuation distributions empirically, i.e., how much price does it need to incentivize a user. Second, we aim to improve the algorithm efficiency, especially in the IC and LT models. Finally, we may consider other factors that influence users' decisions to implement real applications.

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