An overview on bounded elements in some partial algebraic structures

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Abstract The notion of bounded element is fundamental in the framework of the spectral theory. Before implanting a spectral theory in some algebraic or topological structure it is needed to establish which are its bounded elements. In this paper, we want to give an overview on bounded elements of some particular algebraic and topological structures, summarizing our most recent results on this matter.

1 Introduction

Though the notion of bounded element has been considered, in different forms, within the theory regarding the structure of (topological) *-algebras, it is not so for the algebraic structures that do not possess a multiplication or possess just a partial one. Indeed, for (topological) *-algebras, the notion of bounded element is strictly linked to the operation of multiplication. In 1965, Allan wanted to construct a spectral theory for locally convex algebras. He judged natural to mimic the spectral theory of a closed operator on a Banach space: it is well known that if A is a closed operator on a Banach space $\mathscr{B}$, then its spectrum is the set of the complex numbers λ such that the operator $A - \lambda I$ has no bounded inverse. It became fundamental for him, therefore, to fix the concept of bounded element for a locally convex algebra. He defined (see [1, Def. 2.1]) *bounded* those and only those elements a of the locally convex algebra $\mathfrak{A}[\tau]$ for which there exists a complex number $\lambda \neq 0$ such that the set $\{(\lambda x)^n;\ n = 1, 2, \ldots\}$ is a bounded subset of $\mathfrak{A}[\tau]$. This definition does not apply to the algebraic structures we will examine in this overview: in general, neither a partial *-algebra nor a C*-inductive locally convex space possesses an everywhere defined multiplication, hence powers of a given element need not be defined.

Another notion of bounded element of a *-algebra is due to Vidav [11, Definition in Section 2] and involves a convex pointed cone P of positive elements of the algebra (which are all and only those elements that can be written as the finite sums

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of elements of the form a^*a, with $a \in \mathfrak{A}$): an element $a \in \mathfrak{A}$ is bounded if there exists a positive number ξ such that $e \in \mathfrak{A}$ the identity of $\mathfrak{A}$.

In order to extend the notion of bounded element to our case we have at disposal more than one possibility: we can define bounded elements taking into account both the topological structure and the algebraic structure of the set where we pick elements.

The paper is organized as follows. In Section 2 we will summarize the definitions and results we gave in [2] about bounded elements in a *-semisimple topological partial *-algebra: in that paper we considered the elements that are bounded with respect to a sufficient family $\mathfrak{M}$ of invariant positive sesquilinear (ips) forms (see also [4]) and elements that are bounded with respect to some positive cone, hence defined in purely algebraic terms. The outcome is that, under appropriate conditions, order bounded elements reduce to $\mathfrak{M}$-bounded ones. In Section 3, in the setting of C*-inductive locally convex spaces, we consider both bounded elements defined starting from the C*-inductive structure and those we have defined by means of an order cone and finally prove the equivalence of the two different notions we have given in [5].

2 Bounded elements in *-semisimple partial *-algebras

This section summarizes the results showed in [2] by J-P. Antoine, C. Trapani, and the author. We refer to that paper for the proofs and further readings. Before going forth, let us recall, for convenience of the reader, the main definitions we need.

A *partial *-algebra* $\mathfrak{A}$ is a complex vector space with conjugate linear involution * and a distributive partial multiplication $\cdot$, defined on a subset $\Gamma \subset \mathfrak{A}\times\mathfrak{A}$, satisfying the property that $(x, y) \in \Gamma$ if, and only if, $(y^*, x^*) \in \Gamma$ and $(x \cdot y)^* = y^* \cdot x^*$. From now on we will write simply xy instead of $x \cdot y$ whenever $(x, y) \in \Gamma$. For every $y \in \mathfrak{A}$, the set of left (resp. right) multipliers of y is denoted by $L(y)$ (resp. $R(y)$), i.e., $L(y) = \{x \in \mathfrak{A} : (x, y) \in \Gamma\}$ (resp. $R(y) = \{x \in \mathfrak{A} : (y, x) \in \Gamma\}$). We denote by $L\mathfrak{A}$ (resp. $R\mathfrak{A}$) the space of universal left (resp. right) multipliers of $\mathfrak{A}$ (for more details, we refer to [3]).

In general, a partial *-algebra is not associative, but in several situations a weaker form of associativity holds. More precisely, we say that $\mathfrak{A}$ is *semi-associative* if $y \in R(x)$ implies $yz \in R(x)$, for every $z \in R\mathfrak{A}$, and

$$(xy)z = x(yz).$$

The partial *-algebra $\mathfrak{A}$ has a unit if there exists an element $e \in \mathfrak{A}$ such that $e = e^*$, $e \in R\mathfrak{A} \cap L\mathfrak{A}$ and $xe = ex = x$, for every $x \in \mathfrak{A}$.

Let $\mathfrak{A}$ be a partial *-algebra. We assume that $\mathfrak{A}$ is a locally convex Hausdorff vector space under the topology τ defined by a (directed) set $\{p_\alpha\}_{\alpha\in\mathscr{I}}$ of seminorms. Assume that (cl) for every $x \in \mathfrak{A}$, the linear map $\mathsf{L}_x : R(x) \mapsto \mathfrak{A}$ with $\mathsf{L}_x(y) = xy$, $y \in R(x)$, is closed with respect to τ, in the sense that, if $\{y_\alpha\} \subset R(x)$ is a net such that $y_\alpha \to y$ and $xy_\alpha \to z \in \mathfrak{A}$, then $y \in R(x)$ and $z = xy$.

in this case, $\mathfrak{A}$ is said to be a *topological partial *-algebra*. If the involution $x \mapsto x^*$ is continuous, we say that $\mathfrak{A}$ is a **-topological* partial *-algebra.

Starting from the family of seminorms $\{p_\alpha\}_{\alpha \in \mathscr{I}}$, we can define a second topology τ^* on $\mathfrak{A}$ by introducing the set of seminorms $\{p^*_\alpha(x)\}_{\alpha \in \mathscr{I}}$, where

$$p^*_\alpha(x) = \max\{p_\alpha(x), p_\alpha(x^*)\}, \quad x \in \mathfrak{A}.$$

The involution $x \mapsto x^*$ is automatically τ^*-continuous. By (cl) it follows that, for every $x \in \mathfrak{A}$, both maps L_x, $\mathsf{R} = (\mathsf{L}_{x^*})^*$ are τ^*-closed. Hence, $\mathfrak{A}[\tau^*]$ is a *-topological partial *-algebra.

Let $\mathscr{H}$ be a complex Hilbert space and $\mathscr{D}$ a dense subspace of $\mathscr{H}$. We denote by $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ the set of all (closable) linear operators X such that $D(X) = \mathscr{D}$, $D(X^*) \supseteq \mathscr{D}$. The set $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger := X^* \upharpoonright \mathscr{D}$ and the *(weak)* partial multiplication $X_1 \,\Box\, X_2 := {X_1}^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$), that is, whenever $X_2 \mathscr{D} \subset D({X_1}^{\dagger *})$ and $X_1^* \mathscr{D} \subset D(X_2^*)$.

It is easy to check that $X_1 \in L^{\mathrm{w}}(X_2)$ if and only if there exists $Z \in \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ such that

$$\langle X_2 \xi | X_1^\dagger \eta \rangle = \langle Z \xi | \eta \rangle, \quad \forall \xi, \eta \in \mathscr{D}. \tag{1}$$

In this case $Z = X_1 \,\Box\, X_2$. $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ is neither associative nor semi-associative. If I denotes the identity operator of $\mathscr{H}$, $I_{\mathscr{D}} := I \upharpoonright \mathscr{D}$ is the unit of the partial *-algebra $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$. We will indicate by t_s the *strong topology* on $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$, defined by the seminorms

$$p_\xi(X) = \|X\xi\|, \quad X \in \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H}),\ \xi \in \mathscr{D}.$$

Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with locally convex topology τ. Then a subspace $\mathfrak{B}$ of $R\mathfrak{A}$ is called a *multiplication core* [2, Definition 2.3] if

(d_1) $e \in \mathfrak{B}$ if $\mathfrak{A}$ has a unit e;
(d_2) $\mathfrak{B} \cdot \mathfrak{B} \subseteq \mathfrak{B}$;
(d_3) $\mathfrak{B}$ is τ^*-dense in $\mathfrak{A}$;
(d_4) for every $b \in \mathfrak{B}$, the map $x \mapsto xb$, $x \in \mathfrak{A}$, is τ-continuous;
(d_5) one has $b^*(xc) = (b^*x)c$, $\forall\, x \in \mathfrak{A}, b, c \in \mathfrak{B}$.

$\mathfrak{A}[\tau]$ is called $\mathfrak{A}_0$-*regular* if it possesses a multiplication core $\mathfrak{A}_0$ which is a *-algebra and, for every $b \in \mathfrak{A}_0$, the map $x \mapsto bx$, $x \in \mathfrak{A}$, is τ-continuous [4, Def. 4.1].

A **-representation* of a partial *-algebra $\mathfrak{A}$ in the Hilbert space $\mathscr{H}$ is a linear map $\pi : \mathfrak{A} \to \mathscr{L}^\dagger(\mathscr{D}(\pi), \mathscr{H})$ such that: (i) $\pi(x^*) = \pi(x)^\dagger$, for every $x \in \mathfrak{A}$;

(ii) $x \in L(y)$ in $\mathfrak{A}$ implies $\pi(x) \in L^{w}(\pi(y))$ and $\pi(x) \square \pi(y) = \pi(xy)$. The subspace $\mathcal{D}(\pi)$ is called the *domain* of the *-representation π. The *-representation π is said to be *bounded* if $\overline{\pi(x)} \in \mathcal{B}(\mathcal{H})$ for every $x \in \mathfrak{A}$. We will denote by $\text{Rep}_c(\mathfrak{A})$ the set of all (τ, t_s)-continuous *-representations of $\mathfrak{A}$. Let φ be a positive sesquilinear form on $D(\varphi) \times D(\varphi)$, where $D(\varphi)$ is a subspace of $\mathfrak{A}$. Then we have

$$\varphi(x, y) = \overline{\varphi(y, x)}, \quad \forall\, x, y \in D(\varphi), \tag{2}$$

$$|\varphi(x, y)|^2 \leqslant \varphi(x, x)\varphi(y, y), \quad \forall\, x, y \in D(\varphi). \tag{3}$$

We put

$$N_\varphi = \{x \in D(\varphi) : \varphi(x, x) = 0\}.$$

By (3), we have

$$N_\varphi = \{x \in D(\varphi) : \varphi(x, y) = 0, \quad \forall\, y \in D(\varphi)\},$$

and so N_φ is a subspace of $D(\varphi)$ and the quotient space $D(\varphi)/N_\varphi := \{\lambda_\varphi(x) \equiv x + N_\varphi; x \in D(\varphi)\}$ is a pre-Hilbert space with respect to the inner product

$$\langle \lambda_\varphi(x) | \lambda_\varphi(y) \rangle = \varphi(x, y), \quad x, y \in D(\varphi).$$

We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by completion of $D(\varphi)/N_\varphi$.

Our overview on bounded elements starts focusing on the so-called *-semisimple topological partial *-algebras. A topological partial *algebras. $\mathfrak{A}[\tau]$ is called **-semisimple* [2, Definition 3.5] if, for every $x \in \mathfrak{A} \setminus \{0\}$ there exists $\pi \in \text{Rep}_c(\mathfrak{A})$ such that $\pi(x) \neq 0$ or, equivalently, if the **-radical* of $\mathfrak{A}$

$$\mathcal{R}^*(\mathfrak{A}) := \{x \in \mathfrak{A} : \pi(x) = 0, \text{ for all } \pi \in \text{Rep}_c(\mathfrak{A})\}$$

is equal to $\{0\}$.

A positive sesquilinear form φ on $\mathfrak{A} \times \mathfrak{A}$ is said to be *invariant*, and called an *ips-form*, if there exists a subspace $B(\varphi)$ of $\mathfrak{A}$ (called a *core* for φ) with the properties

(ips$_1$) $B(\varphi) \subset R\mathfrak{A}$;
(ips$_2$) $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$;
(ips$_3$) $\varphi(xa, b) = \varphi(a, x^* b)$, $\forall\, x \in \mathfrak{A}, \forall\, a, b \in B(\varphi)$;
(ips$_4$) $\varphi(x^* a, yb) = \varphi(a, (xy)b)$, $\forall\, x \in L(y), \forall\, a, b \in B(\varphi)$.

We will denote by $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ the set of *all* τ-continuous ips-forms with core $\mathfrak{B}$.

A family $\mathfrak{M}$ of continuous ips-forms on $\mathfrak{A} \times \mathfrak{A}$ is *sufficient* if $x \in \mathfrak{A}$ and $\varphi(x, x) = 0$, for every $\varphi \in \mathcal{M}$ imply $x = 0$.

Proposition 1. *Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with unit e. Let $\mathfrak{B}$ be a multiplication core. For an element $x \in \mathfrak{A}$ the following statements are equivalent.*

(i) $x \in \mathscr{R}^*(\mathfrak{A})$.
(ii) $\varphi(x,x) = 0$, *for every* $\varphi \in \mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$.

Remark 1. By Proposition 1, $\mathfrak{A}[\tau]$ is *-semisimple if, and only if, for some multiplication core $\mathfrak{B}$, the family $\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$ is *sufficient*.

If the family $\mathfrak{M}$ is sufficient, any larger family $\mathfrak{M}' \supset \mathfrak{M}$ is sufficient too. In this case, the maximal sufficient family (having $\mathfrak{B}$ as core) is obviously $\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$. Hence if a sufficient family $\mathfrak{M} \subseteq \mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$ exists, then $\mathfrak{A}[\tau]$ is *-semisimple.

We say that the *weak* multiplication $x \Box y$ is well defined (with respect to $\mathfrak{M}$) if there exists $z \in \mathfrak{A}$ such that:

$$\varphi(ya, x^*b) = \varphi(za, b), \ \forall\, a,b \in \mathfrak{B}, \forall\, \varphi \in \mathfrak{M}.$$

In this case, we put $x \Box y := z$ and the sufficiency of $\mathfrak{M}$ guarantees that z is unique. The weak multiplication $\Box$ clearly depends on $\mathfrak{M}$: the larger is $\mathfrak{M}$, the stronger is the weak multiplication, in the sense that if $\mathfrak{M} \subseteq \mathfrak{M}' \subseteq \mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$ and $x \Box y$ exists w.r. to $\mathfrak{M}'$, then $x \Box y$ exists with respect to $\mathfrak{M}$ too.

Since it may be difficult to identify in practice such a sufficient family of continuous ips-forms that guarantees the *-semisimplicity of $\mathfrak{A}[\tau]$, we examine in what sense ips-forms may be replaced by a special class of continuous linear functionals, called *representable*.

Definition 1. Let ω be a linear functional on $\mathfrak{A}$ and $\mathfrak{B}$ a subspace of $R\mathfrak{A}$. We say that ω is *representable* (with respect to $\mathfrak{B}$) if the following requirements are satisfied:

(r_1) $\omega(a^*a) \geq 0$ for all $a \in \mathfrak{B}$ ($\mathfrak{B}$-positiveness);
(r_2) $\omega(b^*(x^*a)) = \overline{\omega(a^*(xb))}, \ \forall\ a,b \in \mathfrak{B}, x \in \mathfrak{A}$;
(r_3) $\forall x \in \mathfrak{A}$ there exists $\gamma_x > 0$ such that $|\omega(x^*a)| \leq \gamma_x\, \omega(a^*a)^{1/2}$, for all $a \in \mathfrak{B}$.

We will denote by $\mathscr{R}_c(\mathfrak{A}, \mathfrak{B})$ the set of τ-continuous linear functionals that are representable (with respect to $\mathfrak{B}$).

In this case, one can prove that there exists a triple $(\pi_\omega^{\mathfrak{B}}, \lambda_\omega^{\mathfrak{B}}, \mathscr{H}_\omega^{\mathfrak{B}})$ such that

(a) $\pi_\omega^{\mathfrak{B}}$ is a *-representation of $\mathfrak{A}$ in $\mathscr{H}_\omega^{\mathfrak{B}}$;
(b) $\lambda_\omega^{\mathfrak{B}}$ is a linear map of $\mathfrak{A}$ into $\mathscr{H}_\omega^{\mathfrak{B}}$ with $\lambda_\omega^{\mathfrak{B}}(\mathfrak{B}) = \mathscr{D}(\pi_\omega^{\mathfrak{B}})$ and $\pi_\omega^{\mathfrak{B}}(x)\lambda_\omega^{\mathfrak{B}}(a) = \lambda_\omega^{\mathfrak{B}}(xa)$, for every $x \in \mathfrak{A}$, $a \in \mathfrak{B}$;
(c) $\omega(b^*(xa)) = \langle \pi_\omega^{\mathfrak{B}}(x)\lambda_\omega^{\mathfrak{B}}(a) | \lambda_\omega^{\mathfrak{B}}(b)\rangle$, for every $x \in \mathfrak{A}$, $a,b \in \mathfrak{B}$.

In particular, if $\mathfrak{A}$ has a unit e and $e \in \mathfrak{B}$, we have:

(a_1) $\pi_\omega^{\mathfrak{B}}$ is a cyclic *-representation of $\mathfrak{A}$ with cyclic vector ξ_ω;
(b_1) $\lambda_\omega^{\mathfrak{B}}$ is a linear map of $\mathfrak{A}$ into $\mathscr{H}_\omega^{\mathfrak{B}}$ with $\lambda_\omega^{\mathfrak{B}}(\mathfrak{B}) = \mathscr{D}(\pi_\omega^{\mathfrak{B}})$, $\xi_\omega = \lambda_\omega^{\mathfrak{B}}(e)$ and $\pi_\omega^{\mathfrak{B}}(x)\lambda_\omega^{\mathfrak{B}}(a) = \lambda_\omega^{\mathfrak{B}}(xa)$, for every $x \in \mathfrak{A}$, $a \in \mathfrak{B}$;
(c_1) $\omega(x) = \langle \pi_\omega^{\mathfrak{B}}(x)\xi_\omega | \xi_\omega \rangle$, for every $x \in \mathfrak{A}$.

For what we have already noted, it is interesting to identify a class of topological partial *-algebras for which representable linear functionals and ips-forms can be freely replaced by one another, since every representable linear functional comes (as for *-algebras with unit) from an ips-form. These partial *-algebras are called *fully representable*: a topological partial *-algebra $\mathfrak{A}[\tau]$, with multiplication core $\mathfrak{B}$ is fully representable if

(fr) $D(\overline{\varphi_\omega}) = \mathfrak{A}$, for every continuous linear functional ω on $\mathfrak{A}$ which is representable w.r. to the same core $\mathfrak{B}$.

The following definitions and results can be found in [2, Subsection 5.1].

Definition 2. Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and a sufficient family $\mathfrak{M}$ of continuous ips-forms with core $\mathfrak{B}$. An element $x \in \mathfrak{A}$ is called $\mathfrak{M}$-*bounded* if there exists $\gamma_x > 0$ such that

$$|\varphi(xa, b)| \leq \gamma_x \, \varphi(a,a)^{1/2} \varphi(b,b)^{1/2}, \ \forall \, \varphi \in \mathfrak{M}, \ a, b \in \mathfrak{B} \, .$$

A useful characterization of $\mathfrak{M}$-bounded elements is given by the following proposition.

Proposition 2. *Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. Then, an element $x \in \mathfrak{A}$ is $\mathfrak{M}$-bounded if, and only if, there exists $\gamma_x \in \mathbb{R}$ such that $\varphi(xa, xa) \leq \gamma_x^2 \, \varphi(a,a)$ for all $\varphi \in \mathfrak{M}$ and $a \in \mathfrak{B}$.*

If x, y are $\mathfrak{M}$-bounded elements and their weak product $x \Box y$ exists, then $x \Box y$ is also $\mathfrak{M}$-bounded.

2.1 Order bounded elements

Before giving the definition of order bounded element of a topological partial *-algebra $\mathfrak{A}[\tau]$ with unit and endowed multiplication core, we need to introduce an order structure in $\mathfrak{A}[\tau]$. We have done it by defining several order cones or wedges of $\mathfrak{A}[\tau]$.

2.1.1 Order structure of $\mathfrak{A}[\tau]$

Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. If $\mathfrak{A}[\tau]$ is *-semisimple, there is a natural order on $\mathfrak{A}$ defined by the family $\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$ or by any sufficient subfamily $\mathfrak{M}$ of $\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$, and this order can be used to define a different notion of *boundedness* of an element $x \in \mathfrak{A}$ [8, 10, 11].

Definition 3. Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra and $\mathfrak{B}$ a subspace of $R\mathfrak{A}$. A subset $\mathfrak{K}$ of $\mathfrak{A}_h := \{x \in \mathfrak{A} : \ x = x^*\}$ is called a $\mathfrak{B}$-*admissible wedge* if

(1) $e \in \mathfrak{K}$, if $\mathfrak{A}$ has a unit e;
(2) $x + y \in \mathfrak{K}, \ \forall \, x, y \in \mathfrak{K}$;
(3) $\lambda x \in \mathfrak{K}, \ \forall \, x \in \mathfrak{K}, \ \lambda \geq 0$;

(4) $(a^*x)a = a^*(xa) =: a^*xa \in \mathfrak{K},\ \forall\, x \in \mathfrak{K},\ a \in \mathfrak{B}$.

As usual, $\mathfrak{K}$ defines an order on the real vector space $\mathfrak{A}_h$ by $x \le y \Leftrightarrow y - x \in \mathfrak{K}$.

In the rest of this section, we will suppose that the partial *-algebras under consideration are *semi-associative*. Under this assumption, the first equality in (4) of Definition 3 is automatically satisfied.

Now, let us define a series of admissible cones with respect to some subspace of $R\mathfrak{A}$.

- Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. We put

$$\mathfrak{B}^{(2)} = \left\{ \sum_{k=1}^{n} x_k^* x_k,\ x_k \in \mathfrak{B},\ n \in \mathbb{N} \right\}.$$

  If $\mathfrak{B}$ is a *-algebra, this is nothing but the set (wedge) of positive elements of $\mathfrak{B}$. The $\mathfrak{B}$-*strongly positive* elements of $\mathfrak{A}$ are then defined as the elements of $\mathfrak{A}^+(\mathfrak{B}) := \overline{\mathfrak{B}^{(2)}}^{\tau}$. Since $\mathfrak{A}$ is semi-associative, the set $\mathfrak{A}^+(\mathfrak{B})$ of $\mathfrak{B}$-strongly positive elements is a $\mathfrak{B}$-admissible wedge.
- We also define

$$\mathfrak{A}^+_{\text{alg}} = \left\{ \sum_{k=1}^{n} x_k^* x_k,\ x_k \in R\mathfrak{A},\ n \in \mathbb{N} \right\},$$

  the set (wedge) of positive elements of $\mathfrak{A}$ and we put $\mathfrak{A}^+_{\text{top}} := \overline{\mathfrak{A}^+_{\text{alg}}}^{\tau}$. The semi-associativity implies that $R\mathfrak{A} \cdot R\mathfrak{A} \subseteq R\mathfrak{A}$ and then $\mathfrak{A}^+_{\text{top}}$ is $R\mathfrak{A}$-admissible.
- Let $\mathfrak{M} \subseteq \mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$. An element $x \in \mathfrak{A}$ is called $\mathfrak{M}$-*positive* if

$$\varphi(xa, a) \ge 0, \quad \forall \varphi \in \mathfrak{M},\ a \in \mathfrak{B}.$$

  It can be proved that an $\mathfrak{M}$-positive element is automatically hermitian. We denote by $\mathfrak{A}^+_{\mathfrak{M}}$ the set of all $\mathfrak{M}$-positive elements. Clearly $\mathfrak{A}^+_{\mathfrak{M}}$ is a $\mathfrak{B}$-admissible wedge.

As can be easily checked, the following inclusions hold

$$\mathfrak{A}^+(\mathfrak{B}) \subseteq \mathfrak{A}^+_{\text{top}} \subseteq \mathfrak{A}^+_{\mathfrak{M}}, \quad \forall\, \mathfrak{M} \subseteq \mathscr{P}_{\mathfrak{B}}(\mathfrak{A}). \tag{4}$$

Moreover, it can be proved that, if the family $\mathfrak{M}$ is sufficient, then $\mathfrak{A}^+_{\mathfrak{M}}$ is a cone, i.e., $\mathfrak{A}^+_{\mathfrak{M}} \cap (-\mathfrak{A}^+_{\mathfrak{M}}) = \{0\}$; this automatically implies that $\mathfrak{A}^+(\mathfrak{B})$ is a cone too.

Put $\mathfrak{A}^+_{\mathscr{P}} := \mathfrak{A}^+_{\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})}$. It can be proved that, if $\mathfrak{A}$ is a fully-representable *-semisimple *-topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$ and if $\mathfrak{A}[\tau]$ is a Fréchet space and the following property holds

(P) $y \in \mathfrak{A}$ and $\omega(a^*ya) \ge 0$, for every $\omega \in \mathscr{R}_c(\mathfrak{A}, \mathfrak{B})$ and $a \in \mathfrak{A}_0$, imply $y \in \mathfrak{A}^+(\mathfrak{B})$

then the chain of inclusions (4) collapses: $\mathfrak{A}^+(\mathfrak{B}) = \mathfrak{A}^+_{\mathscr{P}}$ (see [2, Propositions 5.13, 5.14 and Corollary 5.16]).

The following statement shows that $\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$-positivity is exactly what is needed if we want the order to be preserved under any continuous *-representation.

Proposition 3. *Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. Let $x \in \mathfrak{A}$. Then, the following are equivalent:*

1. $x \in \mathfrak{A}^+_{\mathscr{P}}$;
2. *the operator $\pi(x)$ is positive for every (τ, t_s)-continuous *-representation π with $\pi(e) = I_{\mathscr{D}(\pi)}$.*

2.1.2 Order bounded elements

Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. As we have seen in Section 2.1.1, $\mathfrak{A}[\tau]$ has several natural orders, all related to the topology τ. Each of them can be used to define *bounded* elements.

Let $x \in \mathfrak{A}$; put $\Re(x) = \frac{1}{2}(x + x^*)$, $\Im(x) = \frac{1}{2i}(x - x^*)$. Then $\Re(x), \Im(x) \in \mathfrak{A}_h$ (the set of self-adjoint elements of $\mathfrak{A}$) and $x = \Re(x) + i\Im(x)$.

Let now $\mathfrak{K}$ be an arbitrary $\mathfrak{B}$-admissible cone.

Definition 4. An element $x \in \mathfrak{A}$ is called $\mathfrak{K}$-*bounded* if there exists $\gamma \geq 0$ such that

$$\pm\Re(x) \leq \gamma e; \qquad \pm\Im(x) \leq \gamma e.$$

We denote by $\mathfrak{A}_b(\mathfrak{K})$ the family of $\mathfrak{K}$-bounded elements.

The following statements are easily checked.

(1) $\alpha x + \beta y \in \mathfrak{A}_b(\mathfrak{K}), \quad \forall x, y \in \mathfrak{A}_b(\mathfrak{K}),\ \alpha, \beta \in \mathbb{C}$.
(2) $x \in \mathfrak{A}_b(\mathfrak{K}) \Leftrightarrow x^* \in \mathfrak{A}_b(\mathfrak{K})$.

For $x \in \mathfrak{A}_h$, put

$$\|x\|_b := \inf\{\gamma > 0 : -\gamma e \leq x \leq \gamma e\}.$$

$\|\cdot\|_b$ is a seminorm on the real vector space $(\mathfrak{A}_b(\mathfrak{K}))_h$.

Let $\mathfrak{A}[\tau]$ be a *-semisimple topological partial *-algebra with multiplication core $\mathfrak{B}$. We can then specify the wedge $\mathfrak{K}$ as one of those defined above. Take first $\mathfrak{K} = \mathfrak{A}^+_{\mathfrak{M}}$, where $\mathfrak{M} = \mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$ is the sufficient family of all continuous ips-forms with core $\mathfrak{B}$. For simplicity, we write again $\mathscr{P} := \mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$, hence $\mathfrak{A}^+_{\mathscr{P}} := \mathfrak{A}^+_{\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})}$ and $\mathfrak{A}_b(\mathscr{P}) := \mathfrak{A}_b(\mathfrak{A}^+_{\mathscr{P}})$.

Proposition 4. *If $x \in \mathfrak{A}_b(\mathscr{P})$, then $\pi(x)$ is a bounded operator, for every (τ, t_s)-continuous *-representation of $\mathfrak{A}$. Moreover, if $x = x^*$, $\|\pi(x)\| \leq \|x\|_b$.*

Hence, as it is natural, the $\mathfrak{A}_b(\mathscr{P})$-bounded elements are those that are represented by a bounded operator in any (τ, t_s)-continuous *-representation of $\mathfrak{A}$.

The following theorem states the equivalence, under opportune hypothesis, of the notions of order bounded element and of element bounded with respect to a sufficient family of ips-forms.

Theorem 1. *Let $\mathfrak{A}[\tau]$ be a *-semisimple topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. For $x \in \mathfrak{A}$, the following statements are equivalent.*

(i) *x is $\mathscr{P}_{\mathfrak{B}}(\mathfrak{A})$-bounded;*
(ii) *$x \in \mathfrak{A}_b(\mathscr{P})$;*
(iii) *$\pi(x)$ is bounded, for every $\pi \in \mathrm{Rep}_c(\mathfrak{A})$, and*

$$\sup\{\|\overline{\pi(x)}\|,\ \pi \in \mathrm{Rep}_c(\mathfrak{A})\} < \infty.$$

Another possible choice for the order cone is, for instance, $\mathfrak{A}^+(\mathfrak{B})$. It is clear that $\mathfrak{A}_b(\mathfrak{A}^+(\mathfrak{B})) \subseteq \mathfrak{A}_b(\mathscr{P})$; it can be proved also that the two wedges coincide if $\mathfrak{A}[\tau]$ is a Fréchet space which is also a fully representable, semi-associative *-topological partial *-algebra, with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$ and the property (P) (see Subsection 2.1.1) holds.

3 Bounded elements for a C*-inductive locally convex space

In this section we recap what S. Di Bella, C. Trapani and the author have done in [5], i.e. extending the notion of bounded element to the case of C*-inductive locally convex spaces; for this reason, we refer to that paper for the proofs of every result we report on.

Before going forth, we recall the notions of directed system of C*-algebras and of C*-inductive locally convex space we introduced in [7].

Let $\mathfrak{A}$ be a vector space over $\mathbb{C}$. Let $\mathbb{F}$ be a set of indices directed upward and consider, for every $\alpha \in \mathbb{F}$, a Banach space $\mathfrak{A}_\alpha \subset \mathfrak{A}$ such that:

(I.1) $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$, if $\alpha \le \beta$;
(I.2) $\mathfrak{A} = \bigcup_{\alpha \in \mathbb{F}} \mathfrak{A}_\alpha$;
(I.3) $\forall \alpha \in \mathbb{F}$, there exists a C*-algebra $\mathfrak{B}_\alpha$ (with unit e_α and norm $\|\cdot\|_\alpha$) and a norm-preserving isomorphism of vector spaces $\phi_\alpha : \mathfrak{B}_\alpha \to \mathfrak{A}_\alpha$;
(I.4) $x_\alpha \in \mathfrak{B}_\alpha^+ \Rightarrow x_\beta = (\phi_\beta^{-1}\phi_\alpha)(x_\alpha) \in \mathfrak{B}_\beta^+$, for every $\alpha, \beta \in \mathbb{F}$ with $\beta \ge \alpha$.

We put $j_{\beta\alpha} = \phi_\beta^{-1}\phi_\alpha$, if $\alpha, \beta \in \mathbb{F},\ \beta \ge \alpha$.

If $x \in \mathfrak{A}$, there exist $\alpha \in \mathbb{F}$ such that $x \in \mathfrak{A}_\alpha$ and (a unique) $x_\beta \in \mathfrak{B}_\beta$ such that $x = \phi_\beta(x_\beta)$, for all $\beta \ge \alpha$.

Then, we put

$$j_{\beta\alpha}(x_\alpha) := x_\beta \qquad \text{if } \alpha \le \beta.$$

By (I.4), it follows easily that $j_{\beta\alpha}$ preserves the involution; i.e., $j_{\beta\alpha}(x_\alpha^*) = (j_{\beta\alpha}(x_\alpha))^*$.

The family $\{\mathfrak{B}_\alpha, j_{\beta\alpha}, \beta \geq \alpha\}$ is a *directed system of C*-algebras*, in the sense that:

(J.1) for every $\alpha, \beta \in \mathbb{F}$, with $\beta \geq \alpha, j_{\beta\alpha} : \mathfrak{B}_\alpha \to \mathfrak{B}_\beta$ is a linear and injective map; $j_{\alpha\alpha}$ is the identity of $\mathfrak{B}_\alpha$,
(J.2) for every $\alpha, \beta \in \mathbb{F}$, with $\alpha \leq \beta$, $\phi_\alpha = \phi_\beta j_{\beta\alpha}$.
(J.3) $j_{\gamma\beta} j_{\beta\alpha} = j_{\gamma\alpha}, \alpha \leq \beta \leq \gamma$.

We assume that, in addition, the $j_{\beta\alpha}$'s are Schwarz maps (see, e.g., [9]); i.e.,

(sch) $j_{\beta\alpha}(x_\alpha)^* j_{\beta\alpha}(x_\alpha) \leq j_{\beta\alpha}(x_\alpha^* x_\alpha), \quad \forall x_\alpha \in \mathfrak{B}_\alpha,\ \alpha \leq \beta.$

For every $\alpha, \beta \in \mathbb{F}$, with $\alpha \leq \beta, j_{\beta\alpha}$ is continuous [9] and, moreover,

$$\|j_{\beta\alpha}(x_\alpha)\|_\beta \leq \|x_\alpha\|_\alpha, \quad \forall x_\alpha \in \mathfrak{B}_\alpha.$$

An involution in $\mathfrak{A}$ is defined as follows. Let $x \in \mathfrak{A}$. Then $x \in \mathfrak{A}_\alpha$, for some $\alpha \in \mathbb{F}$, i.e., $x = \phi_\alpha(x_\alpha)$, for a unique $x_\alpha \in \mathfrak{B}_\alpha$. Put $x^* := \phi_\alpha(x_\alpha^*)$. Then if $\beta \geq \alpha$, we have

$$\phi_\beta^{-1}(x^*) = \phi_\beta^{-1}(\phi_\alpha(x_\alpha^*)) = j_{\beta\alpha}(x_\alpha^*) = (j_{\beta\alpha}(x_\alpha))^* = x_\beta^*.$$

It is easily seen that the map $x \mapsto x^*$ is an involution in $\mathfrak{A}$. Moreover, by the definition itself, it follows that every map ϕ_α *preserves the involution*; i.e., $\phi_\alpha(x_\alpha^*) = (\phi_\alpha(x_\alpha))^*$, for all $x_\alpha \in \mathfrak{B}_\alpha$, $\alpha \in \mathbb{F}$.

Definition 5. A locally convex vector space $\mathfrak{A}$, with involution *, is called a *C*-inductive locally convex space* if

(i) there exists a family $\{\{\mathfrak{B}_\alpha, \phi_\alpha\}, \alpha \in \mathbb{F}\}$, where $\mathbb{F}$ is a direct set and, for every $\alpha \in \mathbb{F}$, $\mathfrak{B}_\alpha$ is a C*-algebra and ϕ_α is a linear injective map of $\mathfrak{B}_\alpha$ into $\mathfrak{A}$, satisfying the above conditions (I.1)–(I.4) and (sch), with $\mathfrak{A}_\alpha = \phi_\alpha(\mathfrak{B}_\alpha), \alpha \in \mathbb{F}$;
(ii) $\mathfrak{A}$ is endowed with the locally convex inductive topology τ_{ind} generated by the family $\{\{\mathfrak{B}_\alpha, \phi_\alpha\}, \alpha \in \mathbb{F}\}$.

The family $\{\{\mathfrak{B}_\alpha, \phi_\alpha\}, \alpha \in \mathbb{F}\}$ is called *the defining system of* $\mathfrak{A}$. We notice that the involution is automatically continuous in $\mathfrak{A}[\tau_{\text{ind}}]$.

A C*-inductive locally convex space has a natural positive cone.

An element $x \in \mathfrak{A}$ is called *positive* if there exists $\gamma \in \mathbb{F}$ such that $\phi_\alpha^{-1}(x) \in \mathfrak{B}_\alpha^+$, $\forall \alpha \geq \gamma$.

We denote by $\mathfrak{A}^+$ the set of all positive elements of $\mathfrak{A}$.

Then,

(i) Every positive element $x \in \mathfrak{A}$ is hermitian; i.e., $x \in \mathfrak{A}_h := \{y \in \mathfrak{A} : y^* = y\}$.
(ii) $\mathfrak{A}^+$ is a nonempty convex pointed cone; i.e., $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$.
(iii) If $\alpha \in \mathbb{F}$ and $x_\alpha \in \mathfrak{B}_\alpha^+$, $\phi_\alpha(x_\alpha)$ is positive.

Moreover, every hermitian element $x = x^*$ is the difference of two positive elements, i.e. there exist $x^+, x^- \in \mathfrak{A}^+$ such that $x = x^+ - x^-$.

Now, let $\mathfrak{A}$ be a C*-inductive locally convex space with defining family of C*-algebras $\{\mathfrak{B}_\alpha; \alpha \in \mathbb{F}\}$ ($\mathbb{F}$ is an index set directed upward). There are also in this case several possibilities: the first one consists in taking elements that have *representatives* in every C*-algebra $\mathfrak{B}_\alpha$ of the family whose norms are uniformly bounded; the second one consists in taking into account the order structure of $\mathfrak{A}$, in the same spirit of the quoted papers of Vidav and Schmüdgen.

3.1 Bounded elements and the C-inductive structure of $\mathfrak{A}$*

In this section we will report definitions and results that can be found in [5], regarding bounded elements defined through the C*-inductive structure of the space.

Definition 6. Let $\mathfrak{A}$ be a C*-inductive locally convex space. An element $x \in \mathfrak{A}$ is called *bounded* if $x \in \mathfrak{A}_\alpha$, for every $\alpha \in \mathbb{F}$ and $\sup_{\alpha \in \mathbb{F}} \|x_\alpha\|_\alpha < \infty$.

The set of bounded elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}_b$.

It is easy to see that the set $\mathfrak{A}_b$ is a Banach space under the norm $\|x\|_b = \sup_{\alpha \in \mathbb{F}} \|x_\alpha\|_\alpha$.

In what follows we will consider *-representations of a C*-inductive locally convex space. We recall the basic definitions.

Let $\mathbb{F}$ be a set directed upward by $\leq$. A family $\{\mathscr{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$, where each $\mathscr{H}_\alpha$ is a Hilbert space (with inner product $\langle\cdot|\cdot\rangle_\alpha$ and norm $\|\cdot\|_\alpha$) and, for every $\alpha, \beta \in \mathbb{F}$, with $\beta \geq \alpha$, $U_{\beta\alpha}$ is a linear map from $\mathscr{H}_\alpha$ into $\mathscr{H}_\beta$, is called a *directed contractive system of Hilbert spaces* if the following conditions are satisfied

(i) $U_{\beta\alpha}$ is injective;
(ii) $\|U_{\beta\alpha}\xi_\alpha\|_\beta \leq \|\xi_\alpha\|_\alpha, \quad \forall \xi_\alpha \in \mathscr{H}_\alpha$;
(iii) $U_{\alpha\alpha} = I_\alpha$, the identity of $\mathscr{H}_\alpha$;
(iv) $U_{\gamma\alpha} = U_{\gamma\beta}U_{\beta\alpha}, \alpha \leq \beta \leq \gamma$.

A directed contractive system of Hilbert spaces defines a conjugate dual pair $(\mathscr{D}^\times, \mathscr{D})$ which is called the *joint topological limit* [6] of the directed contractive system $\{\mathscr{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ of Hilbert spaces.

Definition 7. Let $\mathfrak{A}$ be the C*-inductive locally convex space defined by the system $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\}, \alpha \in \mathbb{F}\}$ as in Definition 5. For each $\alpha \in \mathbb{F}$, let π_α be a *-representation of $\mathfrak{B}_\alpha$ in Hilbert space $\mathscr{H}_\alpha$. The collection $\pi := \{\pi_\alpha\}$ is said to be a *-representation of $\mathfrak{A}$ if

(i) for every $\alpha, \beta \in \mathbb{F}$ there exists a linear map $U_{\beta\alpha} : \mathscr{H}_\alpha \to \mathscr{H}_\beta$ such that the family $\{\mathscr{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ is a directed contractive system of Hilbert spaces;

(ii) the following equality holds

$$\pi_\beta(j_{\beta\alpha}(x_\alpha)) = U_{\beta\alpha}\pi_\alpha(x_\alpha)U^*_{\beta\alpha}, \quad \forall x_\alpha \in \mathfrak{B}_\alpha,\ \beta \geq \alpha. \tag{5}$$

In this case we write $\pi(x) = \varinjlim \pi_\alpha(x_\alpha)$ for every $x = (x_\alpha) \in \mathfrak{A}$ or, for short, $\pi = \varinjlim \pi_\alpha$.

The *-representation π is said to be *faithful* if $x \in \mathfrak{A}^+$ and $\pi(x) = 0$ imply $x = 0$ (of course, $\pi(x) = 0$ means that there exists $\gamma \in \mathbb{F}$ such that $\pi_\alpha(x_\alpha) = 0$, for $\alpha \geq \gamma$).

Remark 2. With this definition (which is formally different from that given in [7] but fully equivalent), $\pi(x)$, $x \in \mathfrak{A}$, is not an operator but rather a collection of operators. However, as it was shown in [7], $\pi(x)$ can be regarded as an operator acting on the joint topological limit $(\mathscr{D}^\times, \mathscr{D})$ of $\{\mathscr{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ (see [6]). The corresponding space of operators was denoted by $\mathsf{L}_\mathsf{B}(\mathscr{D}, \mathscr{D}^\times)$; it behaves in the very same way as the space $\mathfrak{L}_\mathsf{B}(\mathscr{D}, \mathscr{D}^\times)$ studied in [5, Section 3] and reduces to it when the family of Hilbert spaces is exactly $\{\mathscr{H}_A; A \in \mathscr{L}^\dagger(\mathscr{D})\}$. The main difference consists in the fact that the $\mathscr{H}_\alpha$'s need not be all subspaces of a certain Hilbert space $\mathscr{H}$.

Let $\pi = \varinjlim \pi_\alpha$ be a faithful representation. Then, for every $\alpha \in \mathbb{F}$, π_α is a faithful *-representation of $\mathfrak{B}_\alpha$.

As shown in [7, Proposition 3.16], if a C*-inductive locally convex space $\mathfrak{A}$ fulfills the following conditions

(r_1) if $x_\alpha \in \mathfrak{B}_\alpha$ and $j_{\beta\alpha}(x_\alpha) \geq 0$, $\beta \geq \alpha$, then $x_\alpha \geq 0$;
(r_2) $e_\beta \in j_{\beta\alpha}(\mathfrak{B}_\alpha)$, $\quad \forall \alpha, \beta \in \mathbb{F}, \beta \geq \alpha$;
(r_3) every positive linear functional $\omega = \varinjlim \omega_\alpha$ on $\mathfrak{A}$ satisfies the following property

- if $\alpha \in \mathbb{F}$ and $\omega_\beta(j_{\beta\alpha}(x^*_\alpha)j_{\beta\alpha}(x_\alpha)) = 0$, for some $\beta \geq \alpha$ and $x_\alpha \in \mathfrak{B}_\alpha$, then $\omega_\alpha(x^*_\alpha x_\alpha) = 0$;

then, $\mathfrak{A}$ admits a faithful representation. These conditions, in fact, guarantee that $\mathfrak{A}$ possesses sufficiently many positive linear functionals, in the sense that for every $x \in \mathfrak{A}^+$, $x \neq 0$, there exists a positive linear functional ω such that $\omega(x) > 0$ [7, Theorem 3.14].

The following theorem provides a relation between the bounded elements of $\mathfrak{A}$ and its *bounded* representations.

Theorem 2. *Let $\mathfrak{A}$ be a C*-inductive locally convex space and $x = (x_\alpha) \in \mathfrak{A}$.*

(i) If $x \in \mathfrak{A}_b$, then, for every representation $\pi = \varinjlim \pi_\alpha$ of $\mathfrak{A}$, one has

$$\sup_{\alpha \in \mathbb{F}} \|\pi_\alpha(x_\alpha)\|_{\alpha\alpha} < \infty,$$

where $\|\cdot\|_{\alpha\alpha}$ denote the norm of $\mathfrak{B}(\mathscr{H}_\alpha)$.

(ii) Conversely, if $\mathfrak{A}$ *admits a faithful *-representation* $\pi^f = \varinjlim \pi^f_\alpha$ *and*

$$\sup_{\alpha\in\mathbb{F}} \|\pi^f_\alpha(x_\alpha)\|_{\alpha\alpha} < \infty,$$

then $x \in \mathfrak{A}_b$.

3.2 Bounded elements and the order structure of $\mathfrak{A}$

Here we collect a series of definitions and results given in [5] about bounded elements of a C*-inductive locally convex space defined by an order cone. As before, we refer to that paper for the proofs.

The reader will immediately realize that the following definitions are very similar to those given in Subsection 2.1.2, however we search here a characterization of bounded elements that originates from the bounded elements of the C*-algebras that give raise to the C*-inductive locally convex space.

Let $\mathfrak{A}$ be a C*-inductive locally convex space. If $x \in \mathfrak{A}$, we put, as before,

$$\Re(x) = \frac{x + x^*}{2} \ \text{ and } \ \Im(x) = \frac{x - x^*}{2i}.$$

Both $\Re(x)$ and $\Im(x)$ are symmetric elements of $\mathfrak{A}$.

Assume that $\mathfrak{A}$ has an element $u = u^*$ such that $\|u_\alpha\|_\alpha \leq 1$, for every $\alpha \in \mathbb{F}$, and there exists $\gamma \in \mathbb{F}$ such that $u_\beta = j_{\beta\gamma}(e_\gamma)$ $\forall \beta \geq \gamma$ (e_γ is the unit of $\mathfrak{B}_\gamma$). For shortness we call the element u a *pre-unit* of $\mathfrak{A}$. It is not difficult to prove that the pre-unit $u \in \mathfrak{A}$, if any, is unique.

Definition 8. Let $\mathfrak{A}$ be a C*-inductive locally convex space with pre-unit u. We say that $x \in \mathfrak{A}$ is *order bounded (with respect to u)* if there exists $\lambda > 0$ such that

$$-\lambda u \leq \Re(x) \leq \lambda u \qquad -\lambda u \leq \Im(x) \leq \lambda u.$$

The following theorem shows that the notions of bounded element and of order bounded element we gave within the present section are equivalent.

Theorem 3. *Let* $\mathfrak{A}$ *be a C*-inductive locally convex space satisfying condition* (r_1). *Assume that* $\mathfrak{A}$ *has a pre-unit u. Then,* $x \in \mathfrak{A}_b$ *if, and only if, x has a representative for every* $\alpha \in \mathbb{F}$ *(i.e., for every* $\alpha \in \mathbb{F}$*, there exists* $x_\alpha \in \mathfrak{B}_\alpha$ *such that* $x = \phi_\alpha(x_\alpha)$*) and x is order bounded with respect u.*

Now, recalling that the set $\mathfrak{A}_b$ is a Banach space under the norm $\|x\|_b = \sup_{\alpha\in\mathbb{F}} \|x_\alpha\|_\alpha$, we can draw a consequence of Theorem 3.

Proposition 5. *Let* $x = x^* \in \mathfrak{A}_b$ *and put*

$$p(x) = \inf\{\lambda > 0; -\lambda u \leq x \leq \lambda u\}.$$

Then, $p(x) = \|x\|_b$.

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