Chapter 9

Axisymmetric Equations in Cylindrical Coordinates

If an idea's worth once, it's worth having twice.

Tom Stoppard

9.1 Domain $0 \le r < \infty, -\infty < z < \infty$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, z, t), \tag{9.1}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.3}$$

$$\lim_{r\to\infty} T(r, z, t) = 0, \qquad \lim_{z\to\pm\infty} T(r, z, t) = 0. \tag{9.4}$$

The solution:

$$\begin{aligned} T(r, z, t) = & \int_{-\infty}^{\infty}\int_0^{\infty} f(\rho, \zeta)\,\mathcal{G}_f(r, z, \rho, \zeta, t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_{-\infty}^{\infty}\int_0^{\infty} F(\rho, \zeta)\,\mathcal{G}_F(r, z, \rho, \zeta, t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} \Phi(\rho, \zeta, \tau)\,\mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{9.5}$$

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Using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r with ξ being the transform variable, and the exponential Fourier transform (2.20) with respect to the space coordinate z with η being the transform variable, we obtain the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \cos[(z-\zeta)\eta]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.6}$$

9.2 Domain $0 \le r < \infty,\ 0 < z < \infty$

9.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.7}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.8}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.9}$$

$$z=0: \quad T = g(r,t), \tag{9.10}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.11}$$

The solution:

$$\begin{aligned} T(r,z,t) &= \int_0^\infty\int_0^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^\infty\int_0^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^t\int_0^\infty\int_0^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty g(\rho,\tau)\, \mathcal{G}_g(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau. \end{aligned} \tag{9.12}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the sin-Fourier transform (2.25) with respect to the space coordinate z:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \times J_0(r\xi)\, J_0(\rho\xi)\, \sin(z\eta)\, \sin(\zeta\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.13}$$

Fundamental solution to the Dirichlet problem

$$\mathcal{G}_g(r,z,\rho,t) = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \times J_0(r\xi)\, J_0(\rho\xi)\, \sin(z\eta)\, \xi\, \eta\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.14}$$

The solution for $\rho = 0$ [160]

$$\mathcal{G}_g(r,z,0,t) = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \times J_0(r\xi)\, \sin(z\eta)\, \xi\, \eta\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{9.15}$$

will be analyzed in detail. Passing to polar coordinates in the (η,ξ)-plane ($\eta = \sigma\cos\vartheta$, $\xi = \sigma\sin\vartheta$), (9.15) is rewritten as

$$\mathcal{G}_g = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty \sigma^3 E_{\alpha,\alpha}(-a\sigma^2 t^\alpha)\, \mathrm{d}\sigma \times \int_0^{\pi/2} J_0(r\sigma\sin\vartheta)\, \sin(z\sigma\cos\vartheta)\, \sin\vartheta\, \cos\vartheta\, \mathrm{d}\vartheta. \tag{9.16}$$

Substitution $x = \sin\vartheta$ gives

$$\mathcal{G}_g = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty \sigma^3 E_{\alpha,\alpha}(-a\sigma^2 t^\alpha)\, \mathrm{d}\sigma \int_0^1 J_0(r\sigma x)\, \sin(z\sigma\sqrt{1-x^2})\, x\, \mathrm{d}x. \tag{9.17}$$

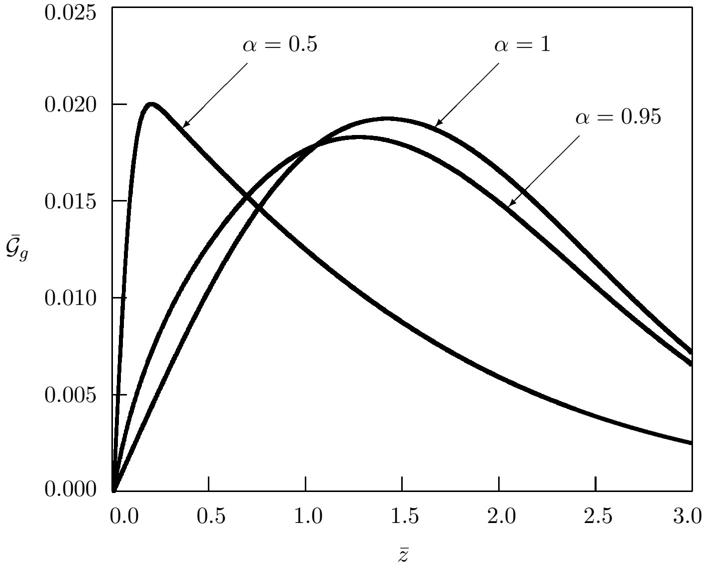

Figure 9.1: Dependence of the fundamental solution to the Dirichlet problem for a half-space on distance z for $\rho = 0$, $r = 0$, $0 \le \alpha \le 1$ [160]

With taking into account the integral (A.39) from the Appendix we obtain

$$\mathcal{G}_g = \frac{2ag_0 t^{\alpha-1} z}{\pi\,(r^2+z^2)} \int_0^\infty E_{\alpha,\alpha}(-a\sigma^2 t^\alpha) \times \left[\frac{\sin\left(\sigma\sqrt{r^2+z^2}\right)}{\sqrt{r^2+z^2}} - \sigma\cos\left(\sigma\sqrt{r^2+z^2}\right) \right] \sigma\, \mathrm{d}\sigma. \tag{9.18}$$

Dependence of the nondimensional solution $\overline{\mathcal{G}}_g = at^{\alpha+1}\,\mathcal{G}_g/(2\pi g_0)$ on nondimensional spatial coordinate $\bar{z} = z\big/\left(\sqrt{a}t^{\alpha/2}\right)$ is shown in Figs. 9.1 and 9.2 for $r = 0$. As the numerical values of $\bar{\mathcal{G}}_g$ for $0 < \alpha \le 1$ and $1 < \alpha \le 2$ are widely different, the typical results for $0 < \alpha \le 1$ and $1 < \alpha \le 2$ are presented in two figures using different scales.

Constant boundary value of a function in a local area. Consider time-fractional diffusion equation (9.1) with zero source, zero initial conditions and the constant boundary value of a function in the area $0 \le r < R$:

$$z = 0: \quad T = \begin{cases} T_0, & 0 \le r < R, \\ 0, & R < r < \infty. \end{cases} \tag{9.19}$$

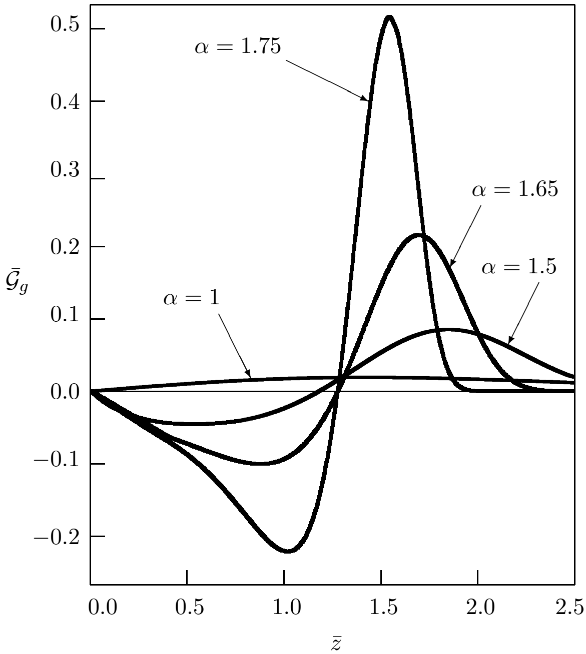

Figure 9.2: Dependence of the fundamental solution to the Dirichlet problem for a half-space on distance z for $\rho = 0$, $r = 0$, $1 \leq \alpha \leq 2$ [160]

The integral transforms allow us to obtain [160]

$$T = \frac{2T_0 R}{\pi} \int_0^\infty \int_0^\infty \left\{1 - E_\alpha[-a(\xi^2 + \eta^2)t^\alpha]\right\}$$
$$\times\ J_0(r\xi)\, J_1(R\xi)\, \frac{\eta}{\xi^2 + \eta^2}\, \sin(z\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.20}$$

Introducing polar coordinates in the (η, ξ)-plane and substituting $x = \sin\vartheta$, we arrive at

$$T = \frac{2T_0 R}{\pi} \int_0^\infty \left[1 - E_\alpha(-a\sigma^2 t^\alpha)\right] \mathrm{d}\sigma$$
$$\times \int_0^1 J_0(r\sigma x)\, J_1(R\sigma x)\, \sin(z\sigma\sqrt{1 - x^2})\, \mathrm{d}x. \tag{9.21}$$

Taking into account the integral (A.42) from Appendix leads to the expression for solution in the case $r = 0$:

$$T = T_0\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) - \frac{2T_0}{\pi}\int\limits_0^\infty E_\alpha(-a\sigma^2 t^\alpha)$$

$$\times\left[\sin(z\sigma) - \frac{z}{\sqrt{z^2+R^2}}\sin(\sqrt{z^2+R^2}\,\sigma)\right]\frac{1}{\sigma}\,\mathrm{d}\sigma. \tag{9.22}$$

Helmholtz equation ($\alpha \to 0$)

Using (A.6), we obtain

$$T = T_0 R\int\limits_0^\infty J_0(r\xi)\,J_1(R\xi)\,\mathrm{e}^{-\,z\sqrt{\xi^2+1/a}}\mathrm{d}\xi. \tag{9.23}$$

This equation can be simplified in the case $r = 0$ accounting for the appropriate integral (A.38)

$$T = T_0\left(\mathrm{e}^{-z/\sqrt{a}} - \frac{z}{\sqrt{z^2+R^2}}\mathrm{e}^{-\sqrt{z^2+R^2}/\sqrt{a}}\right). \tag{9.24}$$

Classical diffusion equation ($\alpha = 1$)

Taking into account (A.26), the solution reads [141]

$$T = \frac{T_0 R}{2}\int\limits_0^\infty J_0(r\xi)\,J_1(R\xi)\left[\mathrm{e}^{-z\xi}\mathrm{erfc}\left(\frac{z}{2\sqrt{at}} - \sqrt{at}\xi\right)\right.$$

$$\left. + \,\mathrm{e}^{z\xi}\mathrm{erfc}\left(\frac{z}{2\sqrt{at}} + \sqrt{at}\xi\right)\right]\mathrm{d}\xi. \tag{9.25}$$

In the case $r = 0$ after some algebra we get (see (A.15))

$$T = T_0\left[\mathrm{erfc}\left(\frac{z}{2\sqrt{at}}\right) - \frac{z}{\sqrt{z^2+R^2}}\,\mathrm{erfc}\left(\frac{\sqrt{z^2+R^2}}{2\sqrt{at}}\right)\right]. \tag{9.26}$$

Wave equation ($\alpha = 2$)

For wave equation the solution has the simple form for $r = 0$:

a) $\sqrt{a}t \le R$

$$T = \begin{cases} T_0, & 0 \le z < \sqrt{a}t \\ 0, & \sqrt{a}t < z < \infty, \end{cases} \tag{9.27}$$

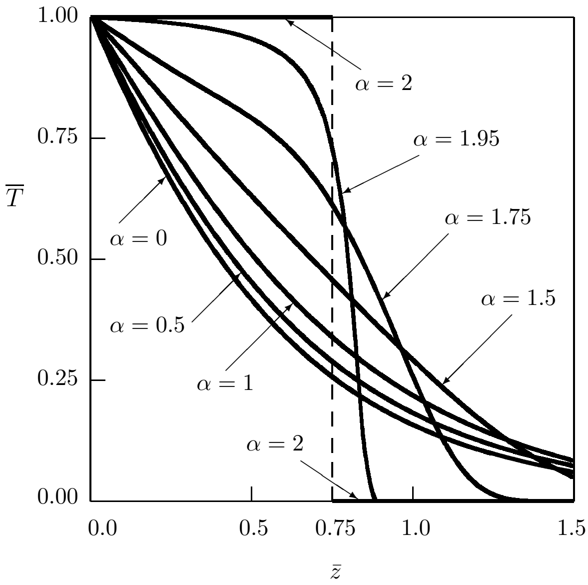

Figure 9.3: Solution to the Dirichlet problem with constant boundary value of a function in a local area of a half-space ($r = 0$, $\kappa = 0.75$) [160]

b) $\sqrt{a}t > R$

$$T = \begin{cases} T_0\left(1 - \dfrac{z}{\sqrt{z^2 + R^2}}\right), & 0 \le z < \sqrt{at^2 - R^2}, \\ T_0, & \sqrt{at^2 - R^2} < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty. \end{cases} \tag{9.28}$$

Dependence of nondimensional solution $\overline{T} = T/T_0$ on nondimensional space coordinate $\bar{z} = z/R$ is shown in Figs. 9.3 and 9.4 for typical values $\kappa = \sqrt{a}t^{\alpha/2}/R = 0.75$ and $\kappa = 1.5$, respectively.

9.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, z, t), \tag{9.29}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.30}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.31}$$

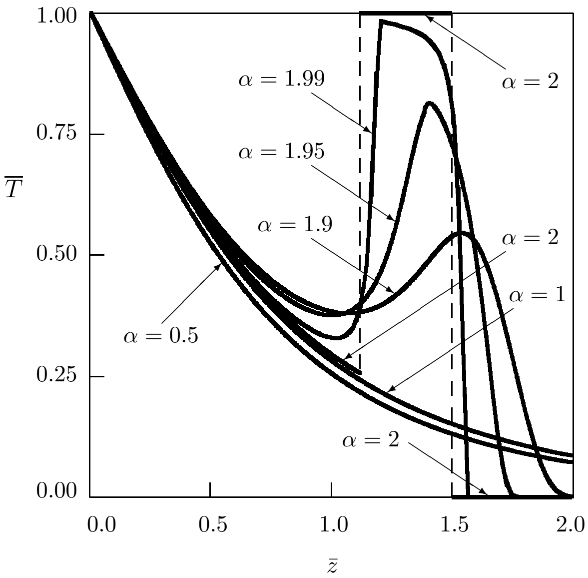

Figure 9.4: Solution to the Dirichlet problem with constant boundary value of a function in a local areal of a half-space ($r = 0$, $\kappa = 1.5$) [160]

$$z = 0: \quad -\frac{\partial T}{\partial z} = g(r, t), \tag{9.32}$$

$$\lim_{r\to\infty} T(r, z, t) = 0, \quad \lim_{z\to\infty} T(r, z, t) = 0. \tag{9.33}$$

The solution:

$$\begin{aligned} T(r, z, t) = & \int_0^\infty \int_0^\infty f(\rho, \zeta)\, \mathcal{G}_f(r, z, \rho, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\ & + \int_0^\infty \int_0^\infty F(\rho, \zeta)\, \mathcal{G}_F(r, z, \rho, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\ & + \int_0^t \int_0^\infty \int_0^\infty \Phi(\rho, \zeta, \tau)\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\tau \\ & + \int_0^t \int_0^\infty g(\rho, \tau)\, \mathcal{G}_g(r, z, \rho, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau. \end{aligned} \tag{9.34}$$

The Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the cos-Fourier transform (2.37) with respect to the space coordinate z lead to the following fundamental solutions

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$
$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta)\, \cos(\zeta\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.35}$$

Fundamental solution to the mathematical Neumann problem. The diffusion-wave equation in a half-space is considered under zero initial conditions and the following boundary condition:

$$z = 0: \quad -\frac{\partial \mathcal{G}_m}{\partial z} = g_0 \frac{\delta(r-\rho)}{r}\, \delta(t). \tag{9.36}$$

The solution

$$\mathcal{G}_m(r,z,\rho,t) = \frac{2ag_0t^{\alpha-1}}{\pi}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right]$$
$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{9.37}$$

in the case $\rho = 0$ [160]

$$\mathcal{G}_m(r,z,0,t) = \frac{2ag_0t^{\alpha-1}}{\pi}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] J_0(r\xi)\, \cos(z\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{9.38}$$

can be simplified. Passing to polar coordinates in the (η,ξ)-plane ($\eta = \sigma\cos\vartheta$, $\xi = \sigma\sin\vartheta$)

$$\mathcal{G}_m = \frac{2ag_0t^{\alpha-1}}{\pi}\int_0^\infty \sigma^2 E_{\alpha,\alpha}(-a\sigma^2t^\alpha)\, \mathrm{d}\sigma$$
$$\times \int_0^{\pi/2} J_0(r\sigma\sin\vartheta)\, \cos(z\sigma\cos\vartheta)\, \sin\vartheta\, \mathrm{d}\vartheta, \tag{9.39}$$

rewriting (9.39) as

$$\mathcal{G}_m = \frac{2ag_0t^{\alpha-1}}{\pi}\int_0^\infty \sigma^2 E_{\alpha,\alpha}(-a\sigma^2t^\alpha)\, \mathrm{d}\sigma$$
$$\times \int_0^1 J_0(r\sigma x)\, \cos(z\sigma\sqrt{1-x^2})\, \frac{x}{\sqrt{1-x^2}}\, \mathrm{d}x, \tag{9.40}$$

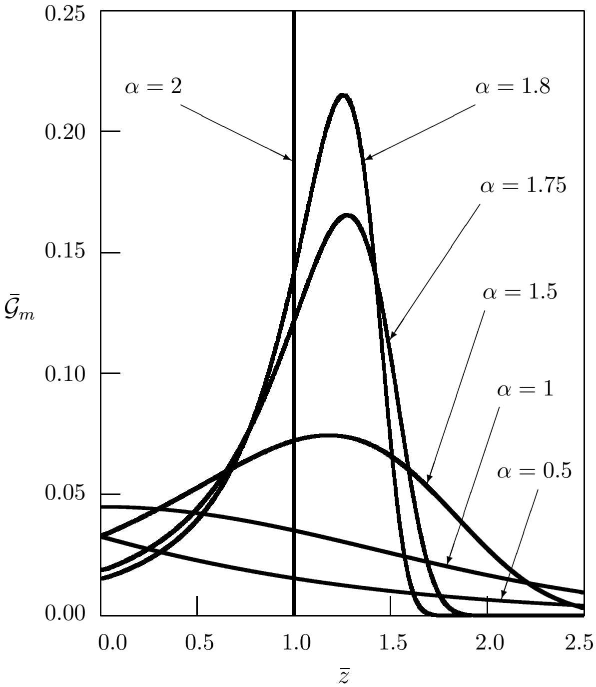

Figure 9.5: Dependence of fundamental solution to the mathematical Neumann problem for a half-space on distance z for $\rho = 0$, $r = 0$ [160]

and taking into account (A.40), we get

$$\mathcal{G}_m = \frac{2ag_0 t^{\alpha-1}}{\pi\sqrt{r^2+z^2}} \int_0^\infty \sigma E_{\alpha,\alpha}(-a\sigma^2 t^\alpha) \sin\left(\sqrt{r^2+z^2}\,\sigma\right) \mathrm{d}\sigma. \tag{9.41}$$

Dependence of nondimensional solution $\overline{\mathcal{G}}_m = \sqrt{a}t^{1+\alpha/2}\,\mathcal{G}_m/(2\pi g_0)$ on space coordinate $\bar{z} = z/(\sqrt{a}t^{\alpha/2})$ for $r = 0$ is shown in Fig. 9.5.

Constant boundary value of the normal derivative of a function in a local area. The time-fractional diffusion equation with zero source and zero initial conditions is considered under Neumann boundary condition with constant value of the normal derivative of a function in the domain $0 \le r < R$:

$$z = 0: \quad -\frac{\partial T}{\partial z} = \begin{cases} g_0, & 0 \le r < R, \\ 0, & R < r < \infty. \end{cases} \tag{9.42}$$

The integral transforms allow us to obtain

$$T = \frac{2g_0 R}{\pi} \int_0^\infty \left[1 - E_\alpha(-a\sigma^2 t^\alpha)\right] \frac{1}{\sigma}\, \mathrm{d}\sigma \times \int_0^1 J_0(r\sigma x)\, J_1(R\sigma x)\, \frac{\cos(z\sigma\sqrt{1-x^2})}{\sqrt{1-x^2}}\, \mathrm{d}x. \tag{9.43}$$

Equation (9.43) is simplified for $r = 0$:

$$T = g_0 \left(\sqrt{z^2+R^2} - z\right) - \frac{2g_0}{\pi} \int_0^\infty E_\alpha(-a\sigma^2 t^\alpha) \times \left[\cos(z\sigma) - \cos(\sqrt{z^2+R^2}\sigma)\right] \frac{1}{\sigma^2}\, \mathrm{d}\sigma. \tag{9.44}$$

Let us analyze several particular cases corresponding to the standard equations.

Helmholtz equation ($\alpha \to 0$)

$$T = g_0 R \int_0^\infty J_0(r\xi)\, J_1(R\xi)\, \mathrm{e}^{-\, z\sqrt{\xi^2+1/a}} \frac{1}{\sqrt{\xi^2+1/a}}\, \mathrm{d}\xi. \tag{9.45}$$

This equation is simplified for $r = 0$:

$$T = g_0\sqrt{a}\left(\mathrm{e}^{-z/\sqrt{a}} - \mathrm{e}^{-\sqrt{z^2+R^2}/\sqrt{a}}\right). \tag{9.46}$$

Classical diffusion equation ($\alpha = 1$)

The solution for the standard heat conduction equation was obtained by Parkus [141] and has the following form:

$$T = \frac{g_0 R}{2} \int_0^\infty J_0(r\xi)\, J_1(R\xi) \left[\mathrm{e}^{-z\xi} \mathrm{erfc}\left(\frac{z}{2\sqrt{at}} - \sqrt{at}\xi\right) - \mathrm{e}^{z\xi} \mathrm{erfc}\left(\frac{z}{2\sqrt{at}} + \sqrt{at}\xi\right)\right] \frac{1}{\xi}\, \mathrm{d}\xi. \tag{9.47}$$

In the case $r = 0$ we have:

$$T = g_0 \left\{ \sqrt{z^2+R^2}\, \mathrm{erfc}\left(\frac{\sqrt{z^2+R^2}}{2\sqrt{at}}\right) - z\, \mathrm{erfc}\left(\frac{z}{2\sqrt{at}}\right) + 2\sqrt{\frac{at}{\pi}} \left[\exp\left(-\frac{z^2}{4at}\right) - \exp\left(-\frac{z^2+R^2}{4at}\right)\right] \right\}. \tag{9.48}$$

Wave equation ($\alpha = 2$)

$$T = \begin{cases} g_0 R \displaystyle\int_z^{\sqrt{a}t} F(x,r,z)\,\mathrm{d}x, & 0 < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty, \end{cases} \tag{9.49}$$

where

$$F(x,r,z) = \begin{cases} 1, & r + \sqrt{x^2 - z^2} < R, \\ \dfrac{1}{\pi} \arccos\left(\dfrac{x^2 + r^2 - z^2 - 1}{2r\sqrt{x^2 - z^2}} \right), & \\ & |r - \sqrt{x^2 - z^2}| < R < r + \sqrt{x^2 - z^2}, \\ 0, & R < |r - \sqrt{x^2 - z^2}|. \end{cases} \tag{9.50}$$

The solution simplifies for $r = 0$:

a) $\sqrt{a}t \le R$

$$T = \begin{cases} g_0(\sqrt{a}t - z), & 0 \le z < \sqrt{a}t \\ 0, & \sqrt{a}t < z < \infty, \end{cases} \tag{9.51}$$

b) $\sqrt{a}t > R$

$$T = \begin{cases} g_0\left(\sqrt{z^2 + R^2} - z\right), & 0 \le z < \sqrt{at^2 - R^2}, \\ g_0(\sqrt{a}t - z), & \sqrt{at^2 - R^2} < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty. \end{cases} \tag{9.52}$$

Dependence of nondimensional solution $\overline{T} = T/(Rg_0)$ on nondimensional space coordinate $\bar{z} = z/R$ is shown in Fig. 9.6 and Fig. 9.7 for $r = 0$ with $\kappa = \sqrt{a}t^{\alpha/2}/R = 0.75$ and $\kappa = 1.5$, respectively. The plot of $\overline{T}$ versus $\bar{r} = r/R$ at the boundary $z = 0$ is depicted in Fig. 9.8 for $\kappa = 0.75$.

Fundamental solution to the physical Neumann problem. In this case the time-fractional diffusion-wave equation with zero source and zero initial conditions is considered under the physical Neumann boundary condition with the given flux at the boundary:

$$\begin{aligned} z = 0: &\quad -D_{RL}^{1-\alpha} \frac{\partial \mathcal{G}_p}{\partial z} = g_0 \, \frac{\delta(r - \rho)}{r} \, \delta(t), \quad 0 < \alpha \le 1, \\ z = 0: &\quad -I^{\alpha-1} \frac{\partial \mathcal{G}_p}{\partial z} = g_0 \, \frac{\delta(r - \rho)}{r} \, \delta(t), \quad 1 < \alpha \le 2. \end{aligned} \tag{9.53}$$

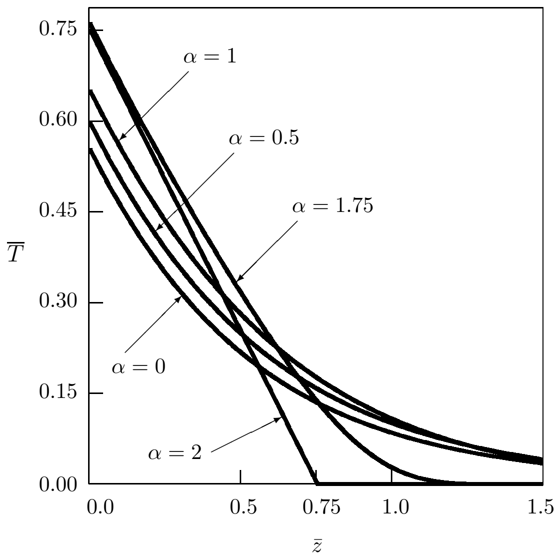

Figure 9.6: Dependence of solution on distance z for $r = 0$ (the Neumann boundary condition with constant normal derivative of a function in a local area of a half-space); $\kappa = 0.75$ [160]

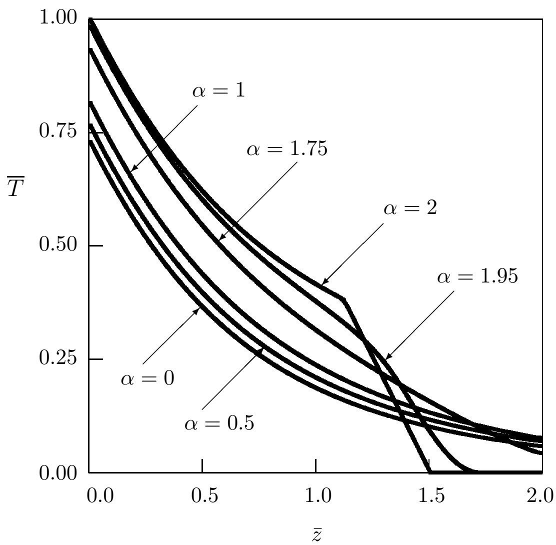

Figure 9.7: Dependence of solution on distance z for $r = 0$ (the Neumann boundary condition with constant normal derivative of a function in a local area of a half-space); $r = 0$, $\kappa = 1.5$ [160]

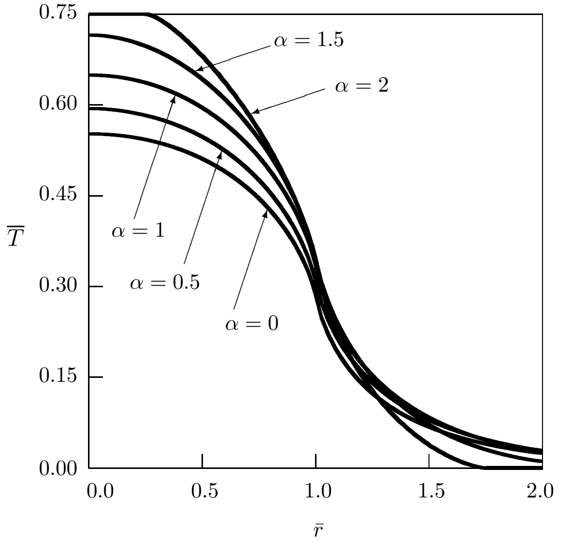

Figure 9.8: Dependence of solution on distance r for $z = 0$ (the Neumann boundary condition with constant normal derivative of a function in a local area of a half-space); $\kappa = 0.75$ [160]

The solution reads:

$$G_p(r, z, \rho, t) = \frac{2ag_0}{\pi} \int_0^\infty \int_0^\infty E_\alpha \left[-a\left(\xi^2 + \eta^2\right) t^\alpha\right] J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta)\, \xi \, \mathrm{d}\xi \, \mathrm{d}\eta. \tag{9.54}$$

For $\rho = 0$ the solution simplifies and after passing to the polar coordinates in the (η, ξ)-plane we arrive at

$$G_p(r, z, 0, t) = \frac{2ag_0}{\pi} \int_0^\infty E_\alpha\left(-a\sigma^2 t^\alpha\right) \frac{\sin\left(\sqrt{r^2 + z^2}\sigma\right)}{\sqrt{r^2 + z^2}} \sigma \, \mathrm{d}\sigma. \tag{9.55}$$

Constant boundary value of the heat flux in a local area. Of particular interest is the problem with the constant boundary value of the heat flux in the area $0 < r < R$:

$$z = 0: \quad -D_{RL}^{1-\alpha} \frac{\partial T}{\partial z} = \begin{cases} g_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \qquad 0 < \alpha \le 1,$$

$$z = 0: \quad -I^{\alpha-1} \frac{\partial T}{\partial z} = \begin{cases} g_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \qquad 1 < \alpha \le 2. \tag{9.56}$$

The solution has the following form:

$$T = \frac{2ag_0 Rt}{\pi} \int_0^\infty \int_0^\infty E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] J_0(r\xi)\, J_1(R\xi)\, \cos(z\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.57}$$

For $r = 0$ we get

$$T = \frac{2ag_0 t}{\pi} \int_0^\infty E_{\alpha,2}\left(-a\sigma^2 t^\alpha\right)\left[\cos(z\sigma) - \cos\left(\sqrt{z^2+R^2}\,\sigma\right)\right] \mathrm{d}\sigma. \tag{9.58}$$

In particular, for the standard wave equation ($\alpha = 2$):

a) $\sqrt{a}t \le R$

$$T = \begin{cases} \sqrt{a} g_0, & 0 < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty; \end{cases} \tag{9.59}$$

b) $\sqrt{a}t > R$

$$T = \begin{cases} 0, & 0 < z < \sqrt{a}t - R, \\ \sqrt{a} g_0, & \sqrt{a}t - R < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty. \end{cases} \tag{9.60}$$

Dependence of nondimensional solution $\overline{T} = t^{\alpha-1}T/(Rg_0)$ on nondimensional spatial coordinate $\bar{z} = z/R$ is shown in Fig. 9.9 for $r = 0$, whereas the plot of $\overline{T}$ versus $\bar{r} = r/R$ at the boundary $z = 0$ is depicted in Fig. 9.10. In both cases $\kappa = 0.75$.

9.2.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.61}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.62}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.63}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + HT = g(r,t), \tag{9.64}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.65}$$

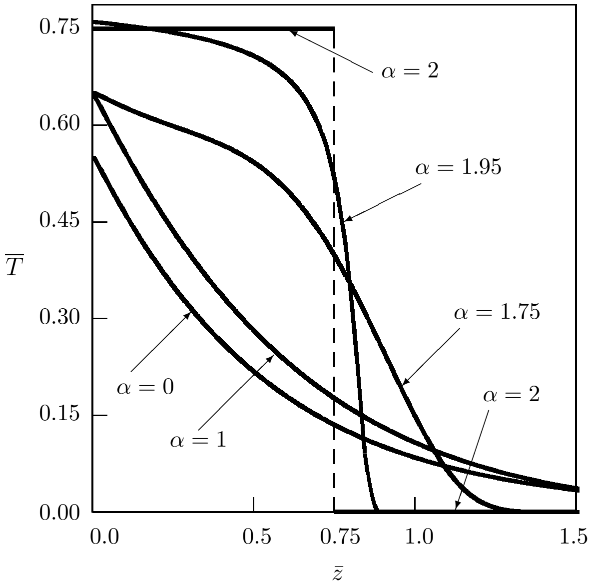

Figure 9.9: Dependence of solution on distance z for $r = 0$ (the Neumann boundary condition with the constant flux in a local area of a half-space); $\kappa = 0.75$

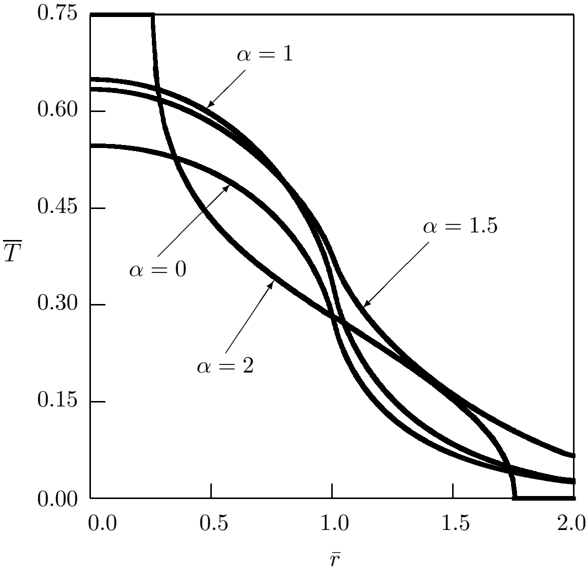

Figure 9.10: Dependence of solution on distance r for $z = 0$ (the Neumann boundary condition with the constant flux in a local area of a half-space); $\kappa = 0.75$

The solution:

$$
\begin{aligned}
T(r,z,t) = & \int_0^\infty \int_0^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
& + \int_0^\infty \int_0^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
& + \int_0^t \int_0^\infty \int_0^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
& + \int_0^t \int_0^\infty g(\rho,\tau)\, \mathcal{G}_g(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau. \qquad (9.66)
\end{aligned}
$$

The Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the sin-cos-Fourier transform (2.40) with respect to the space coordinate z lead to the following fundamental solutions:

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{2}{\pi} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\
& \times \frac{J_0(r\xi)\, J_0(\rho\xi)}{\eta^2+H^2} [\eta\cos(z\eta) + H\sin(z\eta)] \\
& \times [\eta\cos(\zeta\eta) + H\sin(\zeta\eta)]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \qquad (9.67)
\end{aligned}
$$

Fundamental solution to the mathematical Robin problem. The solution was obtained in [179] and reads:

$$
\begin{aligned}
\mathcal{G}_m(r,z,\rho,t) = & \frac{2 a g_0 t^{\alpha-1}}{\pi} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\
& \times \frac{J_0(r\xi)\, J_0(\rho\xi)}{\eta^2+H^2} [\eta\cos(z\eta) + H\sin(z\eta)]\, \xi\, \eta\, \mathrm{d}\xi\, \mathrm{d}\eta. \qquad (9.68)
\end{aligned}
$$

Consider several particular cases of the solution (9.68).

Classical diffusion equation ($\alpha = 1$)

$$\mathcal{G}_m(r,z,\rho,t) = \frac{g_0}{2t}\exp\left(-\frac{r^2+\rho^2+z^2}{4at}\right) I_0\left(\frac{r\rho}{2at}\right)\left\{\frac{1}{\sqrt{\pi a t}}\right.$$
$$\left. -H\exp\left[\left(\sqrt{at}H+\frac{z}{2\sqrt{at}}\right)^2\right]\operatorname{erfc}\left(\sqrt{at}H+\frac{z}{2\sqrt{at}}\right)\right\}. \tag{9.69}$$

Subdiffusion with $\alpha = 1/2$

$$\mathcal{G}_m(r,z,\rho,t) = \frac{g_0}{2\sqrt{\pi}t}\int_0^\infty \exp\left(-u^2-\frac{r^2+\rho^2+z^2}{8a\sqrt{tu}}\right) I_0\left(\frac{r\rho}{4a\sqrt{tu}}\right)$$
$$\times\left\{\frac{1}{\sqrt{2\pi a u}t^{1/4}} - H\exp\left[\left(\sqrt{2au}Ht^{1/4}+\frac{z}{2\sqrt{2au}t^{1/4}}\right)^2\right]\right.$$
$$\left.\times\operatorname{erfc}\left(\sqrt{2au}Ht^{1/4}+\frac{z}{2\sqrt{2au}t^{1/4}}\right)\right\}\mathrm{d}u. \tag{9.70}$$

Wave equation ($\alpha = 2$)

Under assumption $\rho = 0$, we obtain

$$\mathcal{G}_m(r,z,0,t) = \begin{cases} \dfrac{\sqrt{a}g_0}{\sqrt{at^2-r^2}}\left[\delta\left(\sqrt{at^2-r^2}-z\right) - He^{-H\left(\sqrt{at^2-r^2}-z\right)}\right], \\ \qquad 0 < r < \sqrt{a}t, \quad 0 < z < \sqrt{at^2-r^2}, \\ 0, \qquad \sqrt{a}t < r < \infty, \quad \sqrt{at^2-r^2} < z < \infty. \end{cases} \tag{9.71}$$

Fundamental solution to the physical Robin problem. In this case the axisymmetric time-fractional diffusion-wave equation is considered under zero initial conditions and the physical Robin boundary condition

$$z=0: \quad -D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial z} + H\mathcal{G}_p = g_0\,\frac{\delta(r-\rho)}{r}\,\delta(t), \quad 0<\alpha\le 1,$$
$$z=0: \quad -I^{\alpha-1}\frac{\partial \mathcal{G}_p}{\partial z} + H\mathcal{G}_p = g_0\,\frac{\delta(r-\rho)}{r}\,\delta(t), \quad 1<\alpha\le 2. \tag{9.72}$$

The Laplace transform with respect to time t leads to the boundary condition

$$z=0: \quad s^{\alpha-1}H\mathcal{G}_p^* - \frac{\partial \mathcal{G}_p^*}{\partial z} = g_0\,\frac{\delta(r-\rho)}{r}\,s^{\alpha-1}. \tag{9.73}$$

Hence, the kernel (2.42) of the sin-cos-Fourier transform (2.40) with respect to the spatial coordinate z will depend on the Laplace transform variable s,

$$K(z,\eta,s) = \frac{\eta\cos(z\eta) + s^{\alpha-1}H\sin(z\eta)}{\sqrt{\eta^2 + (s^{\alpha-1}H)^2}}, \tag{9.74}$$

and in the transform domain we obtain

$$\widehat{\widetilde{\mathcal{G}}}_p^{\,*} = g_0\, J_0(\rho\xi)\, \frac{\eta}{\sqrt{\eta^2 + (s^{\alpha-1}H)^2}}\, \frac{s^{\alpha-1}}{s^\alpha + a(\xi^2+\eta^2)}. \tag{9.75}$$

Inversion of the Laplace transform in (9.75) depends on the value of α. For $0 < \alpha \le 1$ we have [179]

$$\begin{aligned}\mathcal{G}_p(r,z,\rho,t) = \frac{2ag_0}{\pi}\int_0^\infty\int_0^\infty\int_0^t \xi\, J_0(r\xi)\, J_0(\rho\xi)\, \tau^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)\tau^\alpha\right] \\ \times\left\{(t-\tau)^{-\alpha}\, E_{2-2\alpha,1-\alpha}\left[-\frac{H^2}{\eta^2}(t-\tau)^{2-2\alpha}\right]\cos(z\eta) + \frac{H}{\eta}\,(t-\tau)^{1-2\alpha}\right. \\ \left. + E_{2-2\alpha,2-2\alpha}\left[-\frac{H^2}{\eta^2}(t-\tau)^{2-2\alpha}\right]\sin(z\eta)\right\}\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned} \tag{9.76}$$

whereas for $1 < \alpha \le 2$ we get

$$\begin{aligned}\mathcal{G}_p(r,z,\rho,t) = \frac{2ag_0}{\pi}\int_0^\infty\int_0^\infty\int_0^t \xi\, J_0(r\xi)\, J_0(\rho\xi)\, E_{\alpha}\left[-a\left(\xi^2+\eta^2\right)\tau^\alpha\right] \\ \times\left\{\frac{\eta^2}{H^2}(t-\tau)^{2\alpha-3}\, E_{2\alpha-2,2\alpha-2}\left[-\frac{\eta^2}{H^2}(t-\tau)^{2\alpha-2}\right]\cos(z\eta) + \frac{\eta}{H}\,(t-\tau)^{\alpha-2}\right. \\ \left. \times E_{2\alpha-2,\alpha-1}\left[-\frac{H^2}{\eta^2}(t-\tau)^{2-2\alpha}\right]\sin(z\eta)\right\}\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned} \tag{9.77}$$

9.3 Domain $0 \le r < \infty$, $0 < z < L$

9.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, z, t), \tag{9.78}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.79}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.80}$$

$$z = 0: \quad T = g_1(r, t), \tag{9.81}$$

$$z = L: \quad T = g_2(r, t), \tag{9.82}$$

$$\lim_{r\to\infty} T(r, z, t) = 0. \tag{9.83}$$

The solution:

$$\begin{aligned} T(r, z, t) = &\int_0^L \int_0^\infty f(\rho, \zeta)\, \mathcal{G}_f(r, z, \rho, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\ &+ \int_0^L \int_0^\infty F(\rho, \zeta)\, \mathcal{G}_F(r, z, \rho, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\ &+ \int_0^t \int_0^L \int_0^\infty \Phi(\rho, \zeta, \tau)\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty g_1(\rho, \tau)\, \mathcal{G}_{g1}(r, z, \rho, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty g_2(\rho, \tau)\, \mathcal{G}_{g2}(r, z, \rho, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau. \end{aligned} \tag{9.84}$$

The problem is solved using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the finite sin-Fourier transform (2.44) with respect to the space coordinate z. For the fundamental solutions we obtain

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L}\sum_{k=1}^{\infty}\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times\; J_0(r\xi)\, J_0(\rho\xi)\, \sin(z\eta_k)\, \sin(\zeta\eta_k)\, \xi\, \mathrm{d}\xi, \tag{9.85}$$

where

$$\eta_k = \frac{k\pi}{L}. \tag{9.86}$$

The fundamental solutions to the first and second Dirichlet problems under zero initial conditions are calculated as

$$\mathcal{G}_{g1}(r,z,\rho,t) = \frac{a g_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\bigg|_{\zeta=0}, \tag{9.87}$$

$$\mathcal{G}_{g2}(r,z,\rho,t) = -\frac{a g_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\bigg|_{\zeta=L}. \tag{9.88}$$

9.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.89}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.90}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.91}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_1(r,t), \tag{9.92}$$

$$z=L: \quad \frac{\partial T}{\partial z} = g_2(r,t), \tag{9.93}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.94}$$

The solution:

$$
\begin{aligned}
T(r,z,t) &= \int_0^L \int_0^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\
&+ \int_0^L \int_0^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\
&+ \int_0^t \int_0^L \int_0^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\tau, \\
&+ \int_0^t \int_0^\infty g_1(\rho,\tau)\, \mathcal{G}_{g1}(r,z,\rho,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau, \\
&+ \int_0^t \int_0^\infty g_2(\rho,\tau)\, \mathcal{G}_{g2}(r,z,\rho,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau.
\end{aligned}
\tag{9.95}
$$

Using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the finite cos-Fourier transform (2.48) with respect to the space coordinate z, we get the fundamental solutions to the first and second Cauchy problems and to the source problem:

$$
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L} {\sum_{k=0}^{\infty}}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha \left[-a\left(\xi^2+\eta_k^2\right) t^\alpha\right] \\ w_0 t\, E_{\alpha,2} \left[-a\left(\xi^2+\eta_k^2\right) t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha} \left[-a\left(\xi^2+\eta_k^2\right) t^\alpha\right] \end{pmatrix}
$$
$$
\times\, J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta_k)\, \cos(\zeta\eta_k)\, \xi \, \mathrm{d}\xi, \tag{9.96}
$$

where

$$
\eta_k = \frac{k\pi}{L}. \tag{9.97}
$$

The fundamental solutions to the first and second mathematical and physical Neumann problems under zero initial conditions are calculated as

$$
\mathcal{G}_{m1}(r,z,\rho,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.98}
$$

$$
\mathcal{G}_{m2}(r,z,\rho,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=L}, \tag{9.99}
$$

$$
\mathcal{G}_{p1}(r,z,\rho,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.100}
$$

$$
\mathcal{G}_{p2}(r,z,\rho,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.101}
$$

9.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.102}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.103}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.104}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + HT = g_1(r,t), \tag{9.105}$$

$$z=L: \quad \frac{\partial T}{\partial z} + HT = g_2(r,t), \tag{9.106}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.107}$$

The solution:

$$\begin{aligned}
T(r,z,t) =& \int_0^L \int_0^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^L \int_0^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^t \int_0^L \int_0^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau, \\
&+ \int_0^t \int_0^\infty g_1(\rho,\tau)\,\mathcal{G}_{g1}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
&+ \int_0^t \int_0^\infty g_2(\rho,\tau)\,\mathcal{G}_{g2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{9.108}$$

Using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z, we get the fundamental solutions to the first and second Cauchy problems and to the source problem:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L}\sum_{k=1}^{\infty}\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)\, J_0(\rho\xi)}{\eta_k^2+H^2+\frac{2H}{L}}\left[\eta_k\cos(z\eta_k)+H\sin(z\eta_k)\right]$$

$$\times \left[\eta_k\cos(\zeta\eta_k)+H\sin(\zeta\eta_k)\right]\xi\,\mathrm{d}\xi, \tag{9.109}$$

where η_k are the positive roots of the transcendental equation

$$\tan(L\eta_k) = \frac{2H\eta_k}{\eta_k^2-H^2}. \tag{9.110}$$

The fundamental solutions to the first and second mathematical Robin problems under zero initial conditions have the following form:

$$\mathcal{G}_{m1}(r,z,\rho,t) = \frac{a g_0}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.111}$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{a g_0}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.112}$$

9.4 Domain $0 \le r < R$, $-\infty < z < \infty$

9.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(r,z,t), \tag{9.113}$$

$$t=0: \quad T=f(r,z), \qquad 0<\alpha\le 2, \tag{9.114}$$

$$t=0: \quad \frac{\partial T}{\partial t}=F(r,z), \qquad 1<\alpha\le 2, \tag{9.115}$$

$$r=R: \quad T=g(z,t), \tag{9.116}$$

$$\lim_{z\to\pm\infty} T(r,z,t)=0. \tag{9.117}$$

The solution:

$$T(r,z,t) = \int_{-\infty}^{\infty}\int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta + \int_{-\infty}^{\infty}\int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta + \int_0^t\int_{-\infty}^{\infty}\int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau + \int_0^t\int_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \tag{9.118}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.96) with respect to the radial coordinate r with ξ_n being the transform variable, and the exponential Fourier transform (2.20) with respect to the space coordinate z with η being the transform variable lead to the following fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{1}{\pi R^2}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} p_0\,E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \times \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{[J_1(R\xi_k)]^2}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta, \tag{9.119}$$

where ξ_k are the positive roots of the transcendental equation

$$J_0(R\xi_k) = 0. \tag{9.120}$$

The fundamental solution to the Dirichlet problem under zero initial conditions is expressed as

$$\mathcal{G}_g(r,z,\zeta,t) = \frac{a g_0 t^{\alpha-1}}{\pi R}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \times \frac{\xi_k\,J_0(r\xi_k)}{J_1(R\xi_k)}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta. \tag{9.121}$$

Constant boundary value of temperature in a local area. In this case the time-fractional heat conduction equation in a long cylinder is considered under zero initial conditions and the following boundary condition:

$$r = R: \quad T = \begin{cases} T_0, & |z| < l, \\ 0, & |z| > l. \end{cases} \tag{9.122}$$

The solution of this problem was obtained in [191]:

$$T(r,z,t) = \frac{2T_0}{\pi R} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 + \eta^2)\, \eta\, J_1(R\xi_k)} \times \left\{1 - E_\alpha \left[-a\left(\xi_k^2 + \eta^2\right) t^\alpha\right]\right\} \sin(l\eta)\, \cos(z\eta)\, \mathrm{d}\eta. \tag{9.123}$$

It should be emphasized that the relation [212]

$$\frac{2}{R} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 - \beta^2)\, J_1(R\xi_k)} = \frac{J_0(r\beta)}{J_0(R\beta)}$$

for $\beta = i\eta$ can be rewritten as

$$\frac{2}{R} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 + \eta^2)\, J_1(R\xi_k)} = \frac{I_0(r\eta)}{I_0(R\eta)}.$$

Hence, (9.123) takes the form

$$T(r,z,t) = \frac{T_0}{\pi} \int_{-\infty}^{\infty} \frac{I_0(r\eta)}{I_0(R\eta)}\, \frac{\sin(l\eta)\, \cos(z\eta)}{\eta}\, \mathrm{d}\eta - \frac{2T_0}{\pi R} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 + \eta^2)\, J_1(R\xi_k)} \times E_\alpha \left[-a\left(\xi_k^2 + \eta^2\right) t^\alpha\right] \frac{\sin(l\eta)\, \cos(z\eta)}{\eta}\, \mathrm{d}\eta. \tag{9.124}$$

At the boundary surface $r = R$, the first integral in (9.124) satisfies the boundary condition (9.122), whereas the second one equals zero.

9.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.125}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.126}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.127}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(z,t), \tag{9.128}$$

$$\lim_{z \to \pm\infty} T(r,z,t) = 0. \tag{9.129}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_{-\infty}^{\infty}\int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_{-\infty}^{\infty}\int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^t\int_{-\infty}^{\infty}\int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t\int_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{9.130}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.100) with respect to the radial coordinate r, and the exponential Fourier transform (2.20) with respect to the space coordinate z lead to the following fundamental solutions:

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) = & \frac{p_0}{\pi R^2}\int_{-\infty}^{\infty} E_\alpha\left(-a\eta^2 t^\alpha\right)\cos[(z-\zeta)\eta]\,\mathrm{d}\eta \\ & + \frac{p_0}{\pi R^2}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta \end{aligned} \tag{9.131}$$

or, taking into account (2.186), as

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) = & \frac{p_0}{R^2\sqrt{a}t^{\alpha/2}}\,M\left(\frac{\alpha}{2};\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right) \\ & + \frac{p_0}{\pi R^2}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta, \end{aligned} \tag{9.132}$$

where ξ_k are the positive roots of the transcendental equation

$$J_1(R\xi_k) = 0. \tag{9.133}$$

Similarly,

$$\mathcal{G}_F(r,z,\rho,\zeta,t)=\frac{w_0 t}{\pi R^2}\int\limits_{-\infty}^{\infty}E_{\alpha,2}\left(-a\eta^2t^\alpha\right)\cos[(z-\zeta)\eta]\,\mathrm{d}\eta$$

$$+\frac{w_0 t}{\pi R^2}\sum_{k=1}^{\infty}\int\limits_{-\infty}^{\infty}E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta \qquad (9.134)$$

or (see (2.187))

$$\mathcal{G}_F(r,z,\rho,\zeta,t)=\frac{w_0 t^{1-\alpha/2}}{R^2\sqrt{a}}\,W\left(-\frac{\alpha}{2},2-\frac{\alpha}{2};-\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)$$

$$+\frac{w_0 t}{\pi R^2}\sum_{k=1}^{\infty}\int\limits_{-\infty}^{\infty}E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta \qquad (9.135)$$

and

$$\mathcal{G}_\Phi(r,z,\rho,\zeta,t)=\frac{q_0 t^{\alpha-1}}{\pi R^2}\int\limits_{-\infty}^{\infty}E_{\alpha,\alpha}\left(-a\eta^2t^\alpha\right)\cos[(z-\zeta)\eta]\,\mathrm{d}\eta$$

$$+\frac{q_0 t^{\alpha-1}}{\pi R^2}\sum_{k=1}^{\infty}\int\limits_{-\infty}^{\infty}E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos[(z-\zeta)\eta]\mathrm{d}\eta \qquad (9.136)$$

or, taking into accout (2.188),

$$\mathcal{G}_\Phi(r,z,\rho,\zeta,t)=\frac{q_0 t^{\alpha/2-1}}{R^2\sqrt{a}}\,W\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)$$

$$+\frac{q_0 t^{\alpha-1}}{\pi R^2}\sum_{k=1}^{\infty}\int\limits_{-\infty}^{\infty}E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta. \qquad (9.137)$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$\mathcal{G}_m(r,z,\zeta,t)=\frac{aRg_0}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \qquad (9.138)$$

$$\mathcal{G}_p(r,z,\zeta,t)=\frac{aRg_0}{p_0}\,\mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\rho=R}. \qquad (9.139)$$

9.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.140}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.141}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.142}$$

$$r = R: \quad \frac{\partial T}{\partial r} + HT = g(z,t), \tag{9.143}$$

$$\lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.144}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_{-\infty}^{\infty}\int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_{-\infty}^{\infty}\int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^t\int_{-\infty}^{\infty}\int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t\int_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{9.145}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.104) with respect to the radial coordinate r, and the exponential Fourier transform (2.20) with respect to the space coordinate z lead to the following fundamental solutions:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{1}{\pi R^2}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\ & \times \frac{\xi_k^2}{\xi_k^2+H^2}\,\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos[(z-\zeta)\eta)]\,\mathrm{d}\eta, \end{aligned} \tag{9.146}$$

where ξ_k are the positive roots of the transcendental equation

$$\xi_k J_1(R\xi_k) = H\,J_0(R\xi_k). \tag{9.147}$$

The fundamental solution to the mathematical Robin problem under zero initial conditions is calculated as

$$\mathcal{G}_m(r, z, \zeta, t) = \frac{aRg_0}{q_0}\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t)\Big|_{\rho=R}. \tag{9.148}$$

9.5 Domain $0 \le r < R,\ 0 < z < \infty$

9.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, z, t), \tag{9.149}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.150}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.151}$$

$$r = R: \quad T = g_1(z, t), \tag{9.152}$$

$$z = 0: \quad T = g_2(r, t), \tag{9.153}$$

$$\lim_{z\to\infty} T(r, z, t) = 0. \tag{9.154}$$

The solution:

$$\begin{aligned} T(r, z, t) = &\int_0^\infty\int_0^R f(\rho, \zeta)\, \mathcal{G}_f(r, z, \rho, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^\infty\int_0^R F(\rho, \zeta)\, \mathcal{G}_F(r, z, \rho, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^t\int_0^\infty\int_0^R \Phi(\rho, \zeta, \tau)\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty g_1(\zeta, \tau)\, \mathcal{G}_{g_1}(r, z, \zeta, t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t\int_0^R g_2(\rho, \tau)\, \mathcal{G}_{g_2}(r, z, \rho, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau. \end{aligned} \tag{9.155}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.96) with respect to the radial coordinate r, and the sin-Fourier transform (2.25) with respect to the space coordinate z allow us to obtain the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{\pi R^2} \sum_{k=1}^{\infty} \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_1(R\xi_k)]^2} \sin(z\eta)\, \sin(\zeta\eta)\, \mathrm{d}\eta, \tag{9.156}$$

where ξ_k are the positive roots of the transcendental equation

$$J_0(R\xi_k) = 0. \tag{9.157}$$

The fundamental solutions to the Dirichlet problems under zero initial conditions are expressed as

$$\mathcal{G}_{g_1}(r,z,\zeta,t) = -\frac{aRg_{01}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \rho}\right|_{\rho=R}, \tag{9.158}$$

$$\mathcal{G}_{g_2}(r,z,\rho,t) = \frac{ag_{02}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\right|_{\zeta=0}. \tag{9.159}$$

9.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.160}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.161}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.162}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g_1(z,t), \tag{9.163}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.164}$$

$$\lim_{z\to\infty} T(r,z,t) = 0. \tag{9.165}$$

The solution:

$$\begin{aligned} T(r,z,t) &= \int_0^\infty \int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^\infty \int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^t \int_0^\infty \int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t \int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.166}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.100) with respect to the radial coordinate r, and the cos-Fourier transform (2.37) with respect to the space coordinate z allow us to obtain the fundamental solutions.

The fundamental solution to the first Cauchy problem under zero Neumann boundary condition is

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) &= \frac{4p_0}{\pi R^2} \int_0^\infty E_\alpha\left(-a\eta^2 t^\alpha\right)\,\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta \\ &+ \frac{4p_0}{\pi R^2} \sum_{k=1}^\infty \int_0^\infty E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ &\times \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos(z\eta)\cos(\zeta\eta)\,\mathrm{d}\eta \end{aligned} \tag{9.167}$$

or

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) &= \frac{p_0}{R^2\sqrt{a}t^{\alpha/2}}\left[M\left(\frac{\alpha}{2};\frac{z+\zeta}{\sqrt{a}t^{\alpha/2}}\right)+M\left(\frac{\alpha}{2};\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)\right] \\ &+ \frac{4p_0}{\pi R^2} \sum_{k=1}^\infty \int_0^\infty E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ &\times \frac{J_0(r\xi_n)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos(z\eta)\cos(\zeta\eta)\mathrm{d}\eta, \end{aligned} \tag{9.168}$$

where ξ_k are the positive roots of the transcendental equation

$$J_1(R\xi_k) = 0. \tag{9.169}$$

The fundamental solution to the second Cauchy problem under zero Neumann boundary condition is

$$\begin{aligned} \mathcal{G}_F(r,z,\rho,\zeta,t) &= \frac{4w_0 t}{\pi R^2} \int_0^\infty E_{\alpha,2}\left(-a\eta^2 t^\alpha\right)\, \cos(z\eta)\, \cos(\zeta\eta)\, \mathrm{d}\eta \\ &+ \frac{4w_0 t}{\pi R^2} \sum_{k=1}^\infty \int_0^\infty E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \\ &\times \frac{J_0(r\xi_k) J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2} \cos(z\eta) \cos(\zeta\eta) \mathrm{d}\eta \end{aligned} \tag{9.170}$$

or

$$\begin{aligned} \mathcal{G}_F(r,z,\rho,\zeta,t) &= \frac{w_0 t^{1-\alpha/2}}{R^2\sqrt{a}} \left[W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{z+\zeta}{\sqrt{a}t^{\alpha/2}}\right)\right. \\ &\left. + W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)\right] + \frac{4w_0 t}{\pi R^2} \sum_{k=1}^\infty \int_0^\infty E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \\ &\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\, \cos(z\eta)\, \cos(\zeta\eta)\, \mathrm{d}\eta. \end{aligned} \tag{9.171}$$

The fundamental solution to the source problem under zero Neumann boundary condition is

$$\begin{aligned} \mathcal{G}_\Phi(r,z,\rho,\zeta,t) &= \frac{4q_0 t^{\alpha-1}}{\pi R^2} \int_0^\infty E_{\alpha,\alpha}\left(-a\eta^2 t^\alpha\right)\, \cos(z\eta)\, \cos(\zeta\eta)\, \mathrm{d}\eta \\ &+ \frac{4q_0 t^{\alpha-1}}{\pi R^2} \sum_{k=1}^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \\ &\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\, \cos(z\eta)\, \cos(\zeta\eta)\, \mathrm{d}\eta \end{aligned} \tag{9.172}$$

or

$$\mathcal{G}_{\Phi}(r,z,\rho,\zeta,t) = \frac{q_0 t^{\alpha/2-1}}{R^2\sqrt{a}} \left[W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{z+\zeta}{\sqrt{a}t^{\alpha/2}}\right)\right.$$

$$\left. + W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)\right] + \frac{4q_0 t^{\alpha-1}}{\pi R^2} \sum_{k=1}^{\infty} \int_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^{\alpha}\right]$$

$$\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2} \cos(z\eta)\cos(\zeta\eta)\,\mathrm{d}\eta. \tag{9.173}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$\mathcal{G}_{m_1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.174}$$

$$\mathcal{G}_{m_2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.175}$$

$$\mathcal{G}_{p_1}(r,z,\zeta,t) = \frac{aRg_{01}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.176}$$

$$\mathcal{G}_{p_2}(r,z,\rho,t) = \frac{ag_{02}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}. \tag{9.177}$$

9.5.3 Robin boundary condition

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.178}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.179}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.180}$$

$$r=R: \quad \frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.181}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.182}$$

$$\lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.183}$$

The solution:

$$\begin{aligned}
T(r,z,t) &= \int_0^\infty \int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^\infty \int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^t \int_0^\infty \int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int_0^t \int_0^\infty g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int_0^t \int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \qquad (9.184)
\end{aligned}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.104) with respect to the radial coordinate r, and the sin-cos-Fourier transform (2.40), (2.42) with respect to the space coordinate z result in the following fundamental solutions:

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} &= \frac{4}{\pi R^2} \sum_{k=1}^{\infty} \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\
&\times \frac{\xi_k^2}{\xi_k^2+H_1^2}\, \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\, \frac{\eta\cos(z\eta)+H_2\sin(z\eta)}{\eta^2+H_2^2} \\
&\times \Big[\eta\cos(\zeta\eta)+H_2\sin(\zeta\eta)\Big]\,\mathrm{d}\eta. \qquad (9.185)
\end{aligned}$$

The fundamental solutions to the first and second mathematical Robin problems under zero initial conditions are expressed as

$$\mathcal{G}_{m1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \qquad (9.186)$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}. \qquad (9.187)$$

9.6 Domain $0 \le r < R$, $0 < z < L$

9.6.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.188}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.189}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.190}$$

$$r = R: \quad T = g_1(z,t), \tag{9.191}$$

$$z = 0: \quad T = g_2(r,t), \tag{9.192}$$

$$z = L: \quad T = g_3(r,t). \tag{9.193}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_0^L \int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^L \int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^t \int_0^L \int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t \int_0^L g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t \int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ & + \int_0^t \int_0^R g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.194}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.96) with respect to the radial coordinate r with ξ_k being the transform variable, and the finite sin-Fourier transform (2.44) with respect to the space coordinate z with η_m being the transform variable allow us to obtain the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{R^2 L} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha \left[-a\left(\xi_k^2+\eta_m^2\right) t^\alpha\right] \\ w_0 t\, E_{\alpha,2} \left[-a\left(\xi_k^2+\eta_m^2\right) t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha} \left[-a\left(\xi_k^2+\eta_m^2\right) t^\alpha\right] \end{pmatrix}$$

$$\times\ \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_1(R\xi_k)]^2}\ \sin(z\eta_m)\ \sin(\zeta\eta_m), \tag{9.195}$$

where ξ_k are the positive roots of the transcendental equation

$$J_0(R\xi_k) = 0 \quad \text{and} \quad \eta_m = \frac{m\pi}{L}. \tag{9.196}$$

The fundamental solutions to the Dirichlet problems under zero initial conditions are expressed as

$$\mathcal{G}_{g1}(r,z,\zeta,t) = -\frac{aRg_{01}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \rho}\right|_{\rho=R}, \tag{9.197}$$

$$\mathcal{G}_{g2}(r,z,\rho,t) = \frac{ag_{02}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\right|_{\zeta=0}, \tag{9.198}$$

$$\mathcal{G}_{g3}(r,z,\rho,t) = -\frac{ag_{03}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\right|_{\zeta=L}. \tag{9.199}$$

9.6.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.200}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.201}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.202}$$

$$r=R: \quad \frac{\partial T}{\partial r} = g_1(z,t), \tag{9.203}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.204}$$

$$z=L: \quad \frac{\partial T}{\partial z} = g_3(r,t). \tag{9.205}$$

The solution:

$$
\begin{aligned}
T(r,z,t) = & \int\limits_0^L \int\limits_0^R f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
& + \int\limits_0^L \int\limits_0^R F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
& + \int\limits_0^t \int\limits_0^L \int\limits_0^R \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
& + \int\limits_0^t \int\limits_0^L g_1(\zeta,\tau)\, \mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
& + \int\limits_0^t \int\limits_0^R g_2(\rho,\tau)\, \mathcal{G}_{g_2}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau \\
& + \int\limits_0^t \int\limits_0^R g_3(\rho,\tau)\, \mathcal{G}_{g_3}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau. \qquad (9.206)
\end{aligned}
$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.100) with respect to the radial coordinate r with ξ_k being the transform variable, and the finite cos-Fourier transform (2.48) with respect to the space coordinate z with η_m being the transform variable lead to the fundamental solutions:

$$
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{R^2 L} \sum_{k=0}^{\infty} {\sum_{m=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha \left[-a\left(\xi_k^2+\eta_m^2\right) t^\alpha\right] \\ w_0 t\, E_{\alpha,2} \left[-a\left(\xi_k^2+\eta_m^2\right) t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha} \left[-a\left(\xi_k^2+\eta_m^2\right) t^\alpha\right] \end{pmatrix}
$$

$$
\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \cos(z\eta_m)\, \cos(\zeta\eta_m), \qquad (9.207)
$$

where ξ_k are the nonnegative roots of the transcendental equation

$$
J_1(R\xi_k) = 0 \quad \text{and} \quad \eta_m = \frac{m\pi}{L}. \qquad (9.208)
$$

Recall that Eq. (9.208) has also the zero root $\xi_0 = 0$.

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$
\mathcal{G}_{m1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \qquad (9.209)
$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.210}$$

$$\mathcal{G}_{m3}(r,z,\rho,t) = \frac{a g_{03}}{q_0}\, \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}, \tag{9.211}$$

$$\mathcal{G}_{p1}(r,z,\zeta,t) = \frac{a R g_{01}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.212}$$

$$\mathcal{G}_{p2}(r,z,\rho,t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.213}$$

$$\mathcal{G}_{p3}(r,z,\rho,t) = \frac{a g_{03}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.214}$$

9.6.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.215}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.216}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.217}$$

$$r = R: \quad \frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.218}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.219}$$

$$z = L: \quad \frac{\partial T}{\partial z} + H_2 T = g_3(r,t). \tag{9.220}$$

The solution:

$$\begin{aligned} T(r,z,t) = &\int_0^L \int_0^R f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ &+ \int_0^L \int_0^R F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ &+ \int_0^t \int_0^L \int_0^R \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\tau \end{aligned}$$

$$+ \int_0^t \int_0^L g_1(\zeta,\tau)\, \mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^R g_2(\rho,\tau)\, \mathcal{G}_{g_2}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^R g_3(\rho,\tau)\, \mathcal{G}_{g_3}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau. \tag{9.221}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.104) with respect to the radial coordinate r with ξ_k being the transform variable, and the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z with η_m being the transform variable result in the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{R^2 L} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{\xi_k^2}{\xi_k^2+H_1^2} \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \frac{\eta_m \cos(z\eta_m) + H_2 \sin(z\eta_m)}{\eta_m^2 + H_2^2 + \frac{2H_2}{L}}$$

$$\times \Big[\eta_m \cos(\zeta\eta_m) + H_2 \sin(\zeta\eta_m)\Big], \tag{9.222}$$

where ξ_k and η_m are the nonnegative roots of the transcendental equations

$$\xi_k\, J_1(R\xi_k) = H_1 J_0(R\xi_k) \quad \text{and} \quad \tan(L\eta_m) = \frac{2H_2\eta_m}{\eta_m^2 - H_2^2}. \tag{9.223}$$

The fundamental solutions to the mathematical Robin problems under zero initial conditions are expressed as

$$\mathcal{G}_{m1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.224}$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.225}$$

$$\mathcal{G}_{m3}(r,z,\rho,t) = \frac{ag_{03}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.226}$$

9.7 Domain $R < r < \infty$, $-\infty < z < \infty$

9.7.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.227}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.228}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.229}$$

$$r = R: \quad T = g(z,t), \tag{9.230}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.231}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_{-\infty}^{\infty}\int_{R}^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_{-\infty}^{\infty}\int_{R}^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^t\int_{-\infty}^{\infty}\int_{R}^{\infty} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t\int_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{9.232}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.117) with respect to the radial coordinate r with ξ being the transform variable, and the exponential Fourier transform (2.20) with respect to the space coordinate z with η being the transform variable:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\ & \times \cos[(z-\zeta)\eta]\,\frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)} \\ & \times \Big[J_0(\rho\xi)Y_0(R\xi) - Y_0(\rho\xi)J_0(R\xi)\Big]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{9.233}$$

The fundamental solution to the Dirichlet problem under zero initial conditions has the form

$$\mathcal{G}_g(r,z,\zeta,t) = -\frac{a g_0 t^{\alpha-1}}{\pi^2} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^{\alpha}\right]$$

$$\times \frac{J_0(r\xi)Y_0(R\xi)-Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)} \cos[(z-\zeta)\eta]\, \xi \,\mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.234}$$

9.7.2 Neumann boundary condition

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.235}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.236}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.237}$$

$$r=R: \quad -\frac{\partial T}{\partial r} = g(z,t), \tag{9.238}$$

$$\lim_{r\to\infty} T(r,z,t)=0, \qquad \lim_{z\to\pm\infty} T(r,z,t)=0. \tag{9.239}$$

The solution:

$$T(r,z,t) = \int\limits_{-\infty}^{\infty}\int\limits_{R}^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$

$$+ \int\limits_{-\infty}^{\infty}\int\limits_{R}^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$

$$+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{R}^{\infty} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \tag{9.240}$$

The Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.119) with respect to the radial coordinate r and the exponential Fourier transform (2.20) with respect to the space coordinate z result in the following fundamental solution:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \cos[(z-\zeta)\eta]\, \frac{J_0(r\xi)Y_1(R\xi)-Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}$$

$$\times \Big[J_0(\rho\xi)Y_1(R\xi)-Y_0(\rho\xi)J_1(R\xi)\Big]\, \xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{9.241}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\begin{pmatrix} \mathcal{G}_m(r,z,\zeta,t) \\ \mathcal{G}_p(r,z,\zeta,t) \end{pmatrix} = \frac{ag_0}{\pi^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \begin{pmatrix} t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)Y_1(R\xi)-Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}\, \cos[(z-\zeta)\eta]\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{9.242}$$

9.7.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.243}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.244}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.245}$$

$$r=R: \quad -\frac{\partial T}{\partial r} + HT = g(z,t), \tag{9.246}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.247}$$

The solution:

$$T(r,z,t) = \int_{-\infty}^{\infty}\int_{R}^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$

$$+ \int_{-\infty}^{\infty}\int_{R}^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$

$$+ \int_0^t \int_{-\infty}^{\infty} \int_R^{\infty} \Phi(\rho, \zeta, \tau)\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} g(\zeta, \tau)\, \mathcal{G}_g(r, z, \zeta, t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau. \tag{9.248}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.121) with respect to the radial coordinate r and the exponential Fourier transform (2.20) with respect to the space coordinate z:

$$\begin{pmatrix} \mathcal{G}_f(r, z, \rho, \zeta, t) \\ \mathcal{G}_F(r, z, \rho, \zeta, t) \\ \mathcal{G}_\Phi(r, z, \rho, \zeta, t) \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{Y_0(r\xi)[\xi J_1(R\xi) + H J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi) + H Y_0(R\xi)]}{[\xi J_1(R\xi) + H J_0(R\xi)]^2 + [\xi Y_1(R\xi) + H Y_0(R\xi)]^2}$$

$$\times \Big\{ Y_0(r\xi)[\xi J_1(R\xi) + H J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi) + H Y_0(R\xi)] \Big\}$$

$$\times \cos[(z-\zeta)\eta]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.249}$$

The fundamental solution to the mathematical Robin problem under zero initial condition is expressed as

$$\mathcal{G}_g(r, z, \zeta, t) = \frac{a g_0 t^{\alpha-1}}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \cos[(z-\zeta)\eta]$$

$$\times \frac{Y_0(r\xi)[\xi J_1(R\xi) + H J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi) + H Y_0(R\xi)]}{[\xi J_1(R\xi) + H J_0(R\xi)]^2 + [\xi Y_1(R\xi) + H Y_0(R\xi)]^2} \xi \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.250}$$

9.8 Domain $R < r < \infty$, $0 < z < \infty$

9.8.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) + \Phi(r, z, t), \tag{9.251}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.252}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.253}$$

$$r = R: \quad T = g_1(z,t), \tag{9.254}$$

$$z = 0: \quad T = g_2(r,t), \tag{9.255}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.256}$$

The solution:

$$\begin{aligned}
T(r,z,t) &= \int_0^\infty \int_R^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
&+ \int_0^\infty \int_R^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
&+ \int_0^t \int_0^\infty \int_R^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_0^\infty g_1(\zeta,\tau)\, \mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_R^\infty g_2(\rho,\tau)\, \mathcal{G}_{g_2}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau.
\end{aligned} \tag{9.257}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.117) with respect to the radial coordinate r, and the sin-Fourier transform (2.25) with respect to the space coordinate z we obtain

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} &= \frac{2}{\pi} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\
&\times \sin(z\eta)\, \sin(\zeta\eta)\, \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)} \\
&\times \Big[J_0(\rho\xi)Y_0(R\xi) - Y_0(\rho\xi)J_0(R\xi)\Big]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta.
\end{aligned} \tag{9.258}$$

The fundamental solution to the first Dirichlet problem under zero initial conditions has the following form

$$\begin{aligned}
\mathcal{G}_{g1}(r,z,\zeta,t) &= -\frac{4ag_{01}t^{\alpha-1}}{\pi^2} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\
&\times \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)}\, \sin(z\eta)\, \sin(\zeta\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta,
\end{aligned} \tag{9.259}$$

whereas the fundamental solution to the second Dirichlet problem is calculated as

$$\mathcal{G}_{g_2}(r,z,\rho,t) = \frac{a g_{02}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta} \right|_{\zeta=0}. \tag{9.260}$$

9.8.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.261}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.262}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.263}$$

$$r=R: \quad -\frac{\partial T}{\partial r} = g_1(z,t), \tag{9.264}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.265}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.266}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_0^\infty \int_R^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^\infty \int_R^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^t \int_0^\infty \int_R^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t \int_0^\infty g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t \int_R^\infty g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.267}$$

The Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.119) with respect to the radial coordinate r and the cos-Fourier transform (2.37) with respect to the space coordinate z result in the following fundamental solution:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \cos(z\eta)\, \cos(\zeta\eta)\, \frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}$$

$$\times \Big[J_0(\rho\xi)Y_1(R\xi) - Y_0(\rho\xi)J_1(R\xi)\Big]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.268}$$

The fundamental solutions to the first and second mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\begin{pmatrix} \mathcal{G}_{m1}(r,z,\zeta,t) \\ \mathcal{G}_{p1}(r,z,\zeta,t) \end{pmatrix} = \frac{4ag_{01}}{\pi^2} \int_0^\infty \int_0^\infty \begin{pmatrix} t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}\, \cos(z\eta)\, \cos(\zeta\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.269}$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.270}$$

$$\mathcal{G}_{p2}(r,z,\rho,t) = \frac{ag_{02}}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\zeta=0}. \tag{9.271}$$

9.8.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.272}$$

$$t=0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.273}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.274}$$

$$r=R: \quad -\frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.275}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.276}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.277}$$

The solution:

$$
\begin{aligned}
T(r,z,t) &= \int_0^{\infty}\int_R^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^{\infty}\int_R^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^{t}\int_0^{\infty}\int_R^{\infty} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\infty} g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_R^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \qquad (9.278)
\end{aligned}
$$

The Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.121) with respect to the radial coordinate r and the sin-cos-Fourier transform (2.40), (2.42) with respect to the space coordinate z give:

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} &= \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\
&\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+H_1J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi)+H_1Y_0(R\xi)]}{[\xi J_1(R\xi)+H_1J_0(R\xi)]^2+[\xi Y_1(R\xi)+H_1Y_0(R\xi)]^2} \\
&\times \Big\{Y_0(r\xi)[\xi J_1(R\xi)+H_1J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi)+H_1Y_0(R\xi)]\Big\} \\
&\times \frac{\eta\cos(z\eta)+H_2\sin(z\eta)}{\eta^2+H_2^2}\Big[\eta\cos(\zeta\eta)+H_2\sin(\zeta\eta)\Big]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (9.279)
\end{aligned}
$$

The fundamental solution to the first mathematical Robin problem under zero initial conditions has the form

$$
\begin{aligned}
\mathcal{G}_{m1}(r,z,\zeta,t) &= \frac{4ag_{01}t^{\alpha-1}}{\pi^2}\int_0^{\infty}\int_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\
&\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+H_1J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi)+H_1Y_0(R\xi)]}{[\xi J_1(R\xi)+H_1J_0(R\xi)]^2+[\xi Y_1(R\xi)+H_1Y_0(R\xi)]^2} \\
&\times \frac{\eta\cos(z\eta)+H_2\sin(z\eta)}{\eta^2+H_2^2}\Big[\eta\cos(\zeta\eta)+H_2\sin(\zeta\eta)\Big]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (9.280)
\end{aligned}
$$

The fundamental solution to the second mathematical Robin problem is calculated as

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{a g_{02}}{q_0}\,\mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}. \tag{9.281}$$

9.9 Domain $R < r < \infty$, $0 < z < L$

9.9.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.282}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.283}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.284}$$

$$r=R: \quad T = g_1(z,t), \tag{9.285}$$

$$z=0: \quad T = g_2(r,t), \tag{9.286}$$

$$z=L: \quad T = g_3(r,t), \tag{9.287}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.288}$$

The solution:

$$\begin{aligned}
T(r,z,t) &= \int_0^L \int_R^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^L \int_R^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int_0^t \int_0^L \int_R^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int_0^t \int_0^L g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int_0^t \int_R^\infty g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau
\end{aligned}$$

$$+\int_0^t \int_R^\infty g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \tag{9.289}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.117) with respect to the radial coordinate r, and the finite sin-Fourier transform (2.44) with respect to the space coordinate z we obtain

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L}\sum_{m=1}^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \sin(z\eta_m)\,\sin(\zeta\eta_m)\,\frac{J_0(r\xi)Y_0(R\xi)-Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)}$$

$$\times\Big[J_0(\rho\xi)Y_0(R\xi)-Y_0(\rho\xi)J_0(R\xi)\Big]\,\xi\,\mathrm{d}\xi, \tag{9.290}$$

where $\eta_m = m\pi/L$.

The fundamental solution to the first Dirichlet problem under zero initial conditions has the form

$$\mathcal{G}_{g1}(r,z,\zeta,t) = -\frac{4ag_{01}t^{\alpha-1}}{\pi L}\sum_{m=1}^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right]$$

$$\times\frac{J_0(r\xi)Y_0(R\xi)-Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)}\,\sin(z\eta_m)\,\sin(\zeta\eta_m)\,\xi\,\mathrm{d}\xi, \tag{9.291}$$

whereas the fundamental solutions to the second and third Dirichlet problems are calculated as

$$\mathcal{G}_{g2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial\zeta}\bigg|_{\zeta=0}, \tag{9.292}$$

$$\mathcal{G}_{g3}(r,z,\rho,t) = -\frac{ag_{03}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial\zeta}\bigg|_{\zeta=L}. \tag{9.293}$$

9.9.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(r,z,t), \tag{9.294}$$

$$t=0:\quad T=f(r,z),\qquad 0<\alpha\le 2, \tag{9.295}$$

$$t=0:\quad \frac{\partial T}{\partial t}=F(r,z),\qquad 1<\alpha\le 2, \tag{9.296}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g_1(z,t), \tag{9.297}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.298}$$

$$z = L: \quad \frac{\partial T}{\partial z} = g_3(r,t), \tag{9.299}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.300}$$

The solution:

$$\begin{aligned}
T(r,z,t) &= \int_0^L \int_R^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
&+ \int_0^L \int_R^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
&+ \int_0^t \int_0^L \int_R^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_0^L g_1(\zeta,\tau)\, \mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_R^\infty g_2(\rho,\tau)\, \mathcal{G}_{g_2}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau \\
&+ \int_0^t \int_R^\infty g_3(\rho,\tau)\, \mathcal{G}_{g_3}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau.
\end{aligned} \tag{9.301}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.119) with respect to the radial coordinate r, and the finite cos-Fourier transform (2.48) with respect to the space coordinate z, we obtain

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} &= \frac{2}{L} \sum_{m=0}^{\infty}{}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix} \\
&\times \cos(z\eta_m)\, \cos(\zeta\eta_m)\, \frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi) + Y_1^2(R\xi)} \\
&\times \Big[J_0(\rho\xi)Y_1(R\xi) - Y_0(\rho\xi)J_1(R\xi)\Big]\, \xi\, \mathrm{d}\xi,
\end{aligned} \tag{9.302}$$

where $\eta_m = m\pi/L$.

The fundamental solutions to the first mathematical and physical Neumann problems under zero initial conditions have the form

$$\begin{pmatrix} \mathcal{G}_{m1}(r,z,\zeta,t) \\ \mathcal{G}_{p1}(r,z,\zeta,t) \end{pmatrix} = \frac{4ag_{01}}{\pi L} \sum_{m=0}^{\infty}{}' \int_0^{\infty} \begin{pmatrix} t^{\alpha-1} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^{\alpha}\right] \\ E_{\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^{\alpha}\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)Y_1(R\xi)-Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)} \cos(z\eta_m)\cos(\zeta\eta_m)\,\mathrm{d}\xi, \tag{9.303}$$

whereas the fundamental solutions to the second and third mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.304}$$

$$\mathcal{G}_{m3}(r,z,\rho,t) = \frac{ag_{03}}{q_0}\,\mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}, \tag{9.305}$$

$$\mathcal{G}_{p2}(r,z,\rho,t) = \frac{ag_{02}}{p_0}\,\mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.306}$$

$$\mathcal{G}_{p3}(r,z,\rho,t) = \frac{ag_{03}}{p_0}\,\mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.307}$$

9.9.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.308}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.309}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.310}$$

$$r=R: \quad -\frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.311}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.312}$$

$$z=L: \quad \frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.313}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.314}$$

The solution:

$$\begin{aligned} T(r,z,t) &= \int_0^L \int_R^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^L \int_R^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^t \int_0^L \int_R^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t \int_0^L g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t \int_R^\infty g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &+ \int_0^t \int_R^\infty g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.315}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.121) with respect to the radial coordinate r, and the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z we get

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} &= \frac{2}{L} \sum_{m=1}^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix} \\ &\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+H_1 J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi)+H_1 Y_0(R\xi)]}{[\xi J_1(R\xi)+H_1 J_0(R\xi)]^2 + [\xi Y_1(R\xi)+H_1 Y_0(R\xi)]^2} \\ &\times \Big\{ Y_0(r\xi)[\xi J_1(R\xi)+H_1 J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi)+H_1 Y_0(R\xi)] \Big\} \\ &\times \frac{\eta_m \cos(z\eta_m) + H_2 \sin(z\eta_m)}{\eta_m^2 + H_2^2 + 2H_2/L} \Big[\eta_m \cos(\zeta\eta_m) + H_2 \sin(\zeta\eta_m)\Big]\,\xi\,\mathrm{d}\xi, \end{aligned} \tag{9.316}$$

where η_m are the positive roots of the transcendental equation

$$\tan(L\eta_m) = \frac{2H_2\eta_m}{\eta_m^2 - H_2^2}. \tag{9.317}$$

The fundamental solution to the first mathematical Robin problem under zero initial conditions has the form

$$\mathcal{G}_{m1}(r, z, \zeta, t) = \frac{4ag_{01}t^{\alpha-1}}{\pi L} \sum_{m=1}^{\infty} \int_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2 + \eta_m^2\right)t^{\alpha}\right]$$

$$\times \frac{Y_0(r\xi)[\xi J_1(R\xi) + H_1 J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi) + H_1 Y_0(R\xi)]}{[\xi J_1(R\xi) + H_1 J_0(R\xi)]^2 + [\xi Y_1(R\xi) + H_1 Y_0(R\xi)]^2}$$

$$\times \frac{\eta_m \cos(z\eta_m) + H_2 \sin(z\eta_m)}{\eta_m^2 + H_2^2 + 2H_2/L}\Big[\eta_m \cos(\zeta\eta_m) + H_2 \sin(\zeta\eta_m)\Big]\,\xi\,\mathrm{d}\xi, \tag{9.318}$$

the fundamental solutions to the second and third mathematical Robin problems are calculated as

$$\mathcal{G}_{m2}(r, z, \rho, t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_{\Phi}(r, z, \rho, \zeta, t)\Big|_{\zeta=0}, \tag{9.319}$$

$$\mathcal{G}_{m3}(r, z, \rho, t) = \frac{ag_{03}}{q_0}\,\mathcal{G}_{\Phi}(r, z, \rho, \zeta, t)\Big|_{\zeta=L}. \tag{9.320}$$